MODULE 1

INTRODUCTION TO MULTILEVEL MODELS

OUTLINE

1. Introduction to Multilevel Data
2. Multiple Sources of Variability
3. Partitioning Variability With an Unconditional MLM
4. Data Analysis Example
5. Intraclass Correlation
6. Multilevel Latent Variable Specification

OUTLINE

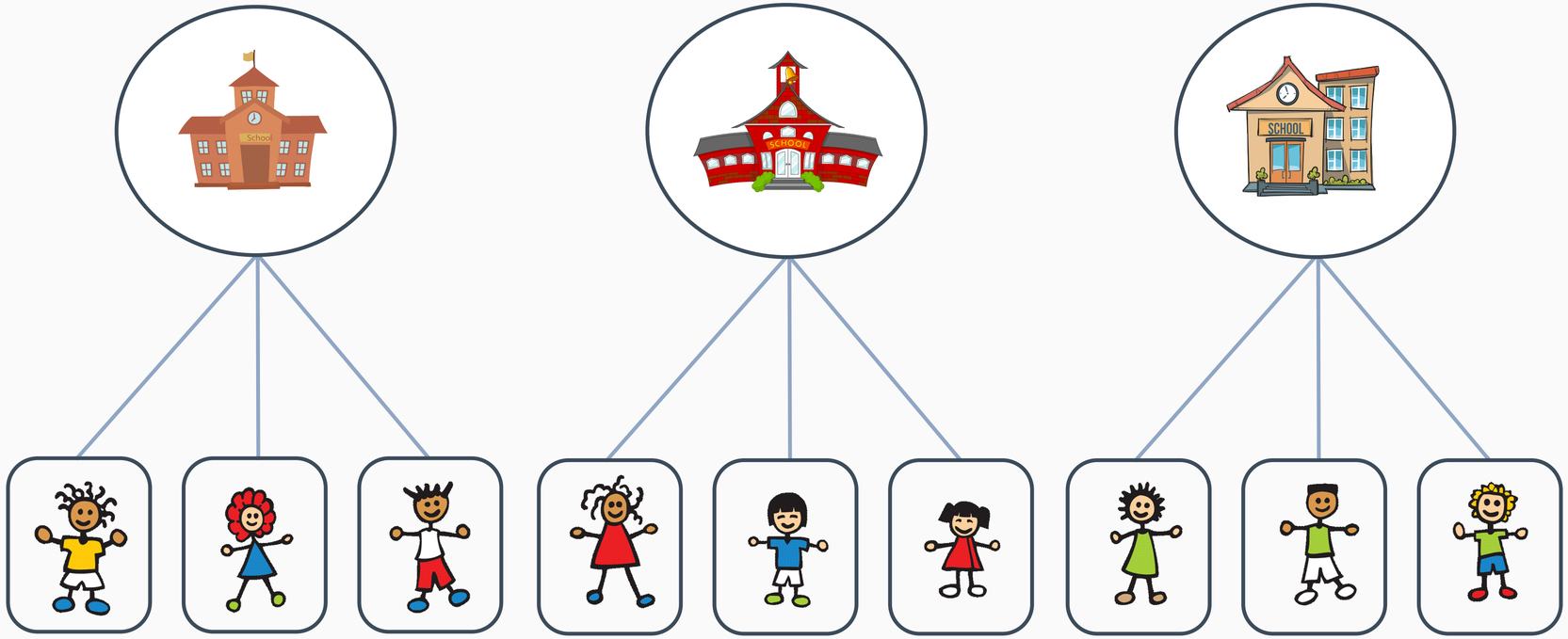
1. Introduction to Multilevel Data
2. Multiple Sources of Variability
3. Partitioning Variability With an Unconditional MLM
4. Data Analysis Example
5. Intraclass Correlation
6. Latent Variable Specification

WHAT ARE MULTILEVEL DATA?

- A unit of analysis is the whom or what of interest
- Multilevel data have multiple units of analysis, with lower-level units nested in higher-level units
- Units at both levels are a random sample from a larger population of units

PERSONS WITHIN ORGANIZATIONS

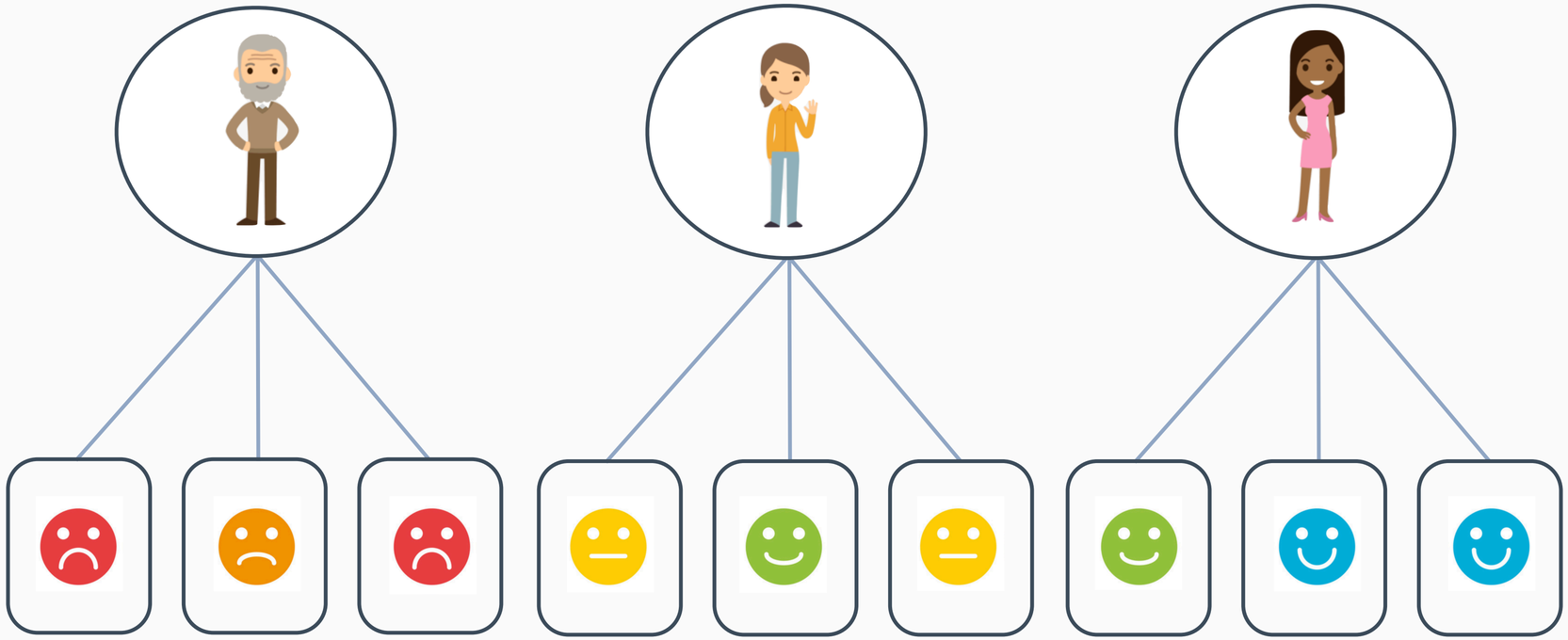
Level-2 (Between-Cluster) Units



Level-1 (Within-Cluster) Units

MEASUREMENTS WITHIN PERSONS

Level-2 (Between-Cluster) Units



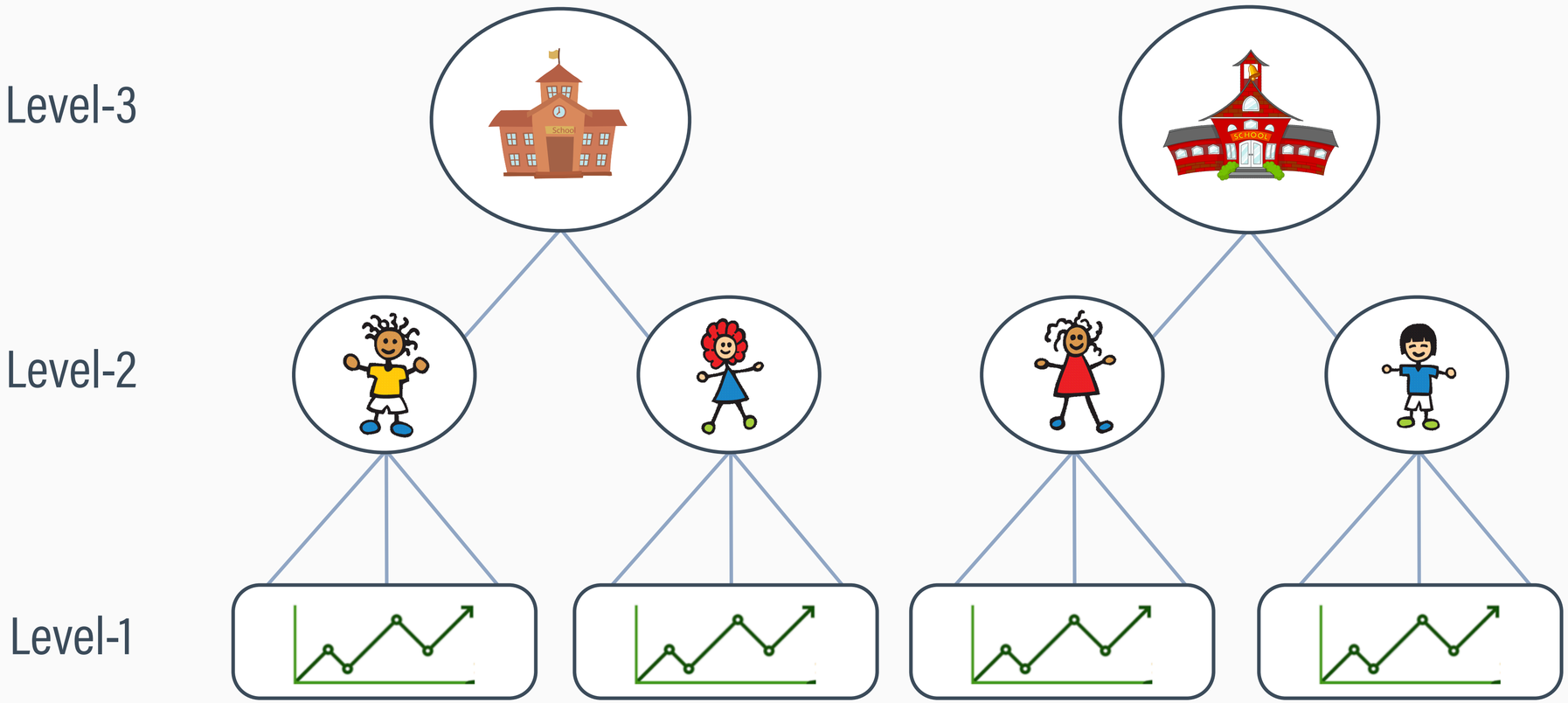
Level-1 (Within-Cluster) Units

OTHER EXAMPLES

Level-2 (Between-Cluster) Units

Level-1 (Within-Cluster) Units

THREE-LEVEL DATA STRUCTURE



MULTILEVEL MODELING TRADITIONS

- Mixed effects models from biostatistics (Laird & Ware, 1982)
- Hierarchical linear models in the social sciences (Raudenbush & Bryk, 2002)
- Multilevel structural equation models in the behavioral and social sciences (Muthén, & Asparouhov, 2008, 2009)

ALTERNATIVES TO MLM

- MLMs treat nonindependence (cluster-level variation) as a meaningful phenomenon to model and explain
- If variation due to clustering is viewed as a statistical nuisance, single-level regression with cluster-robust standard errors or dummy coding is an option
- McNeish, Stapleton, & Silverman (2017) and McNeish and Kelley (2019) describe alternatives

OUTLINE

1 Introduction to Multilevel Data

2 Multiple Sources of Variability

3 Partitioning Variability With an Unconditional MLM

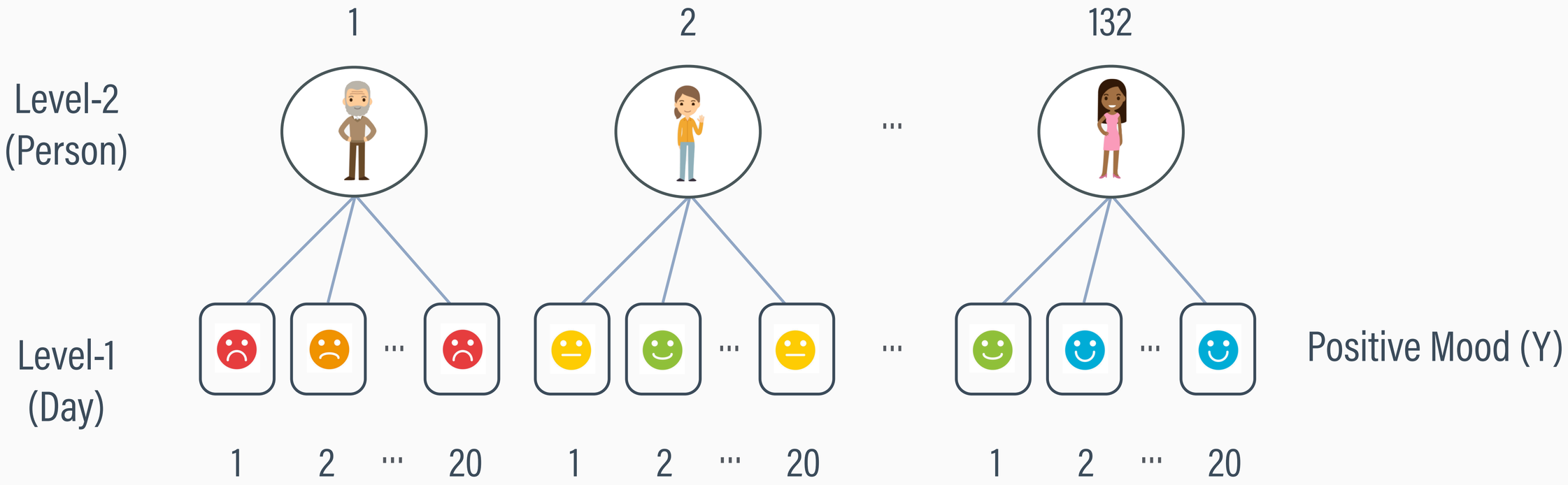
4 Data Analysis Example

5 Intraclass Correlation

6 Latent Variable Specification

DAILY DIARY APPLICATION

- $n_j = 20$ daily positive affect and sleep assessments nested within J = 132 chronic pain patients (N = 2680 data records)

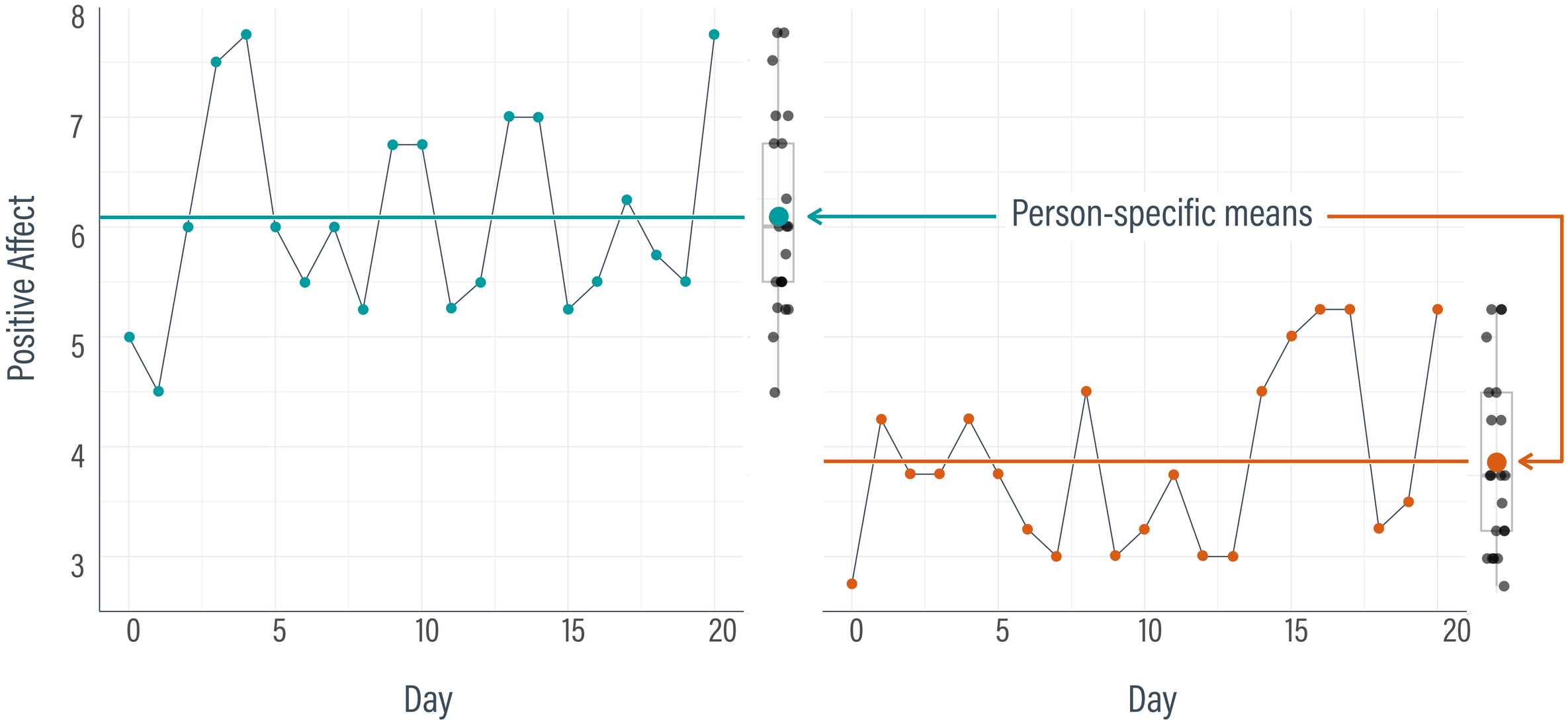


DATA STRUCTURE

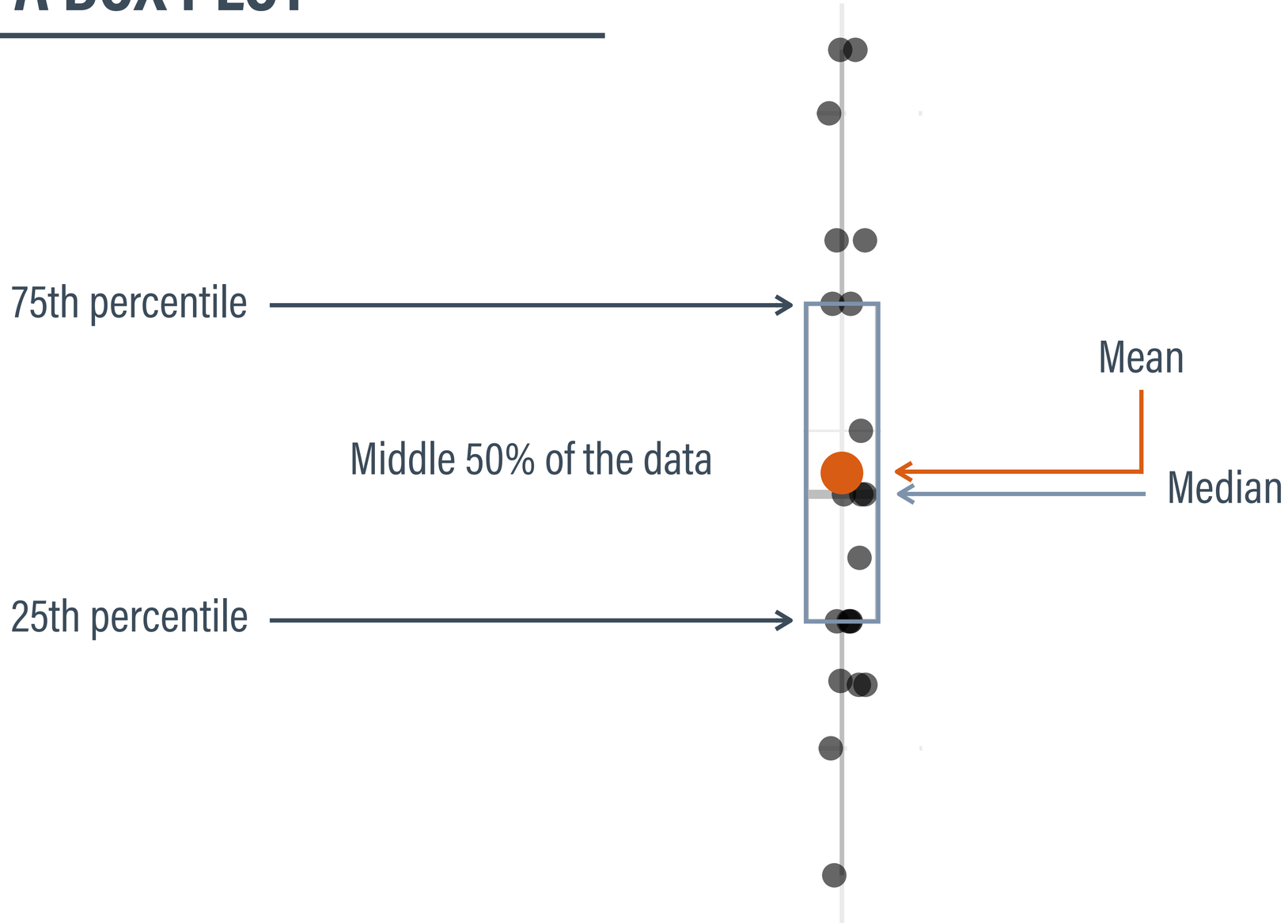
- Data in stacked or long format
- Each level-2 unit (person) has multiple rows, one per level-1 (daily) observation
- The i subscript indexes level-1 observations, and j indexes level-2 units

Row	i	j	Y_{ij}	
1	1	1	7.3	Person 1
2	2	1	2.5	
...	...	1	...	
20	20	1	6.3	
21	1	2	4.0	Person 2
22	2	2	4.0	
...	...	2	...	
40	20	2	4.4	
...	...	...	...	...
2621	1	132	3.3	Person 132
2622	2	132	4.8	
...	...	132	...	
2640	20	132	4.8	

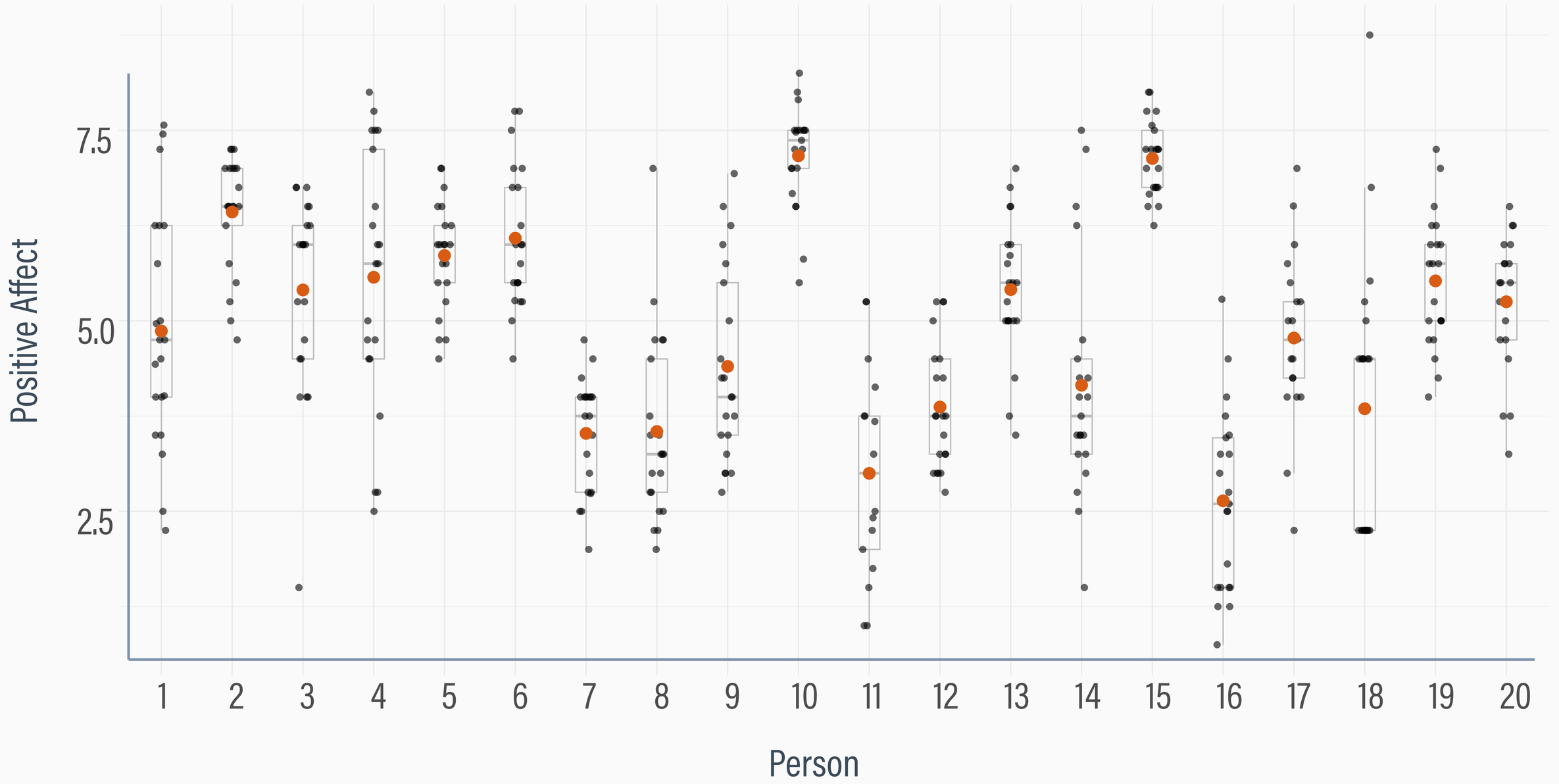
DATA FROM TWO PEOPLE



ANATOMY OF A BOX PLOT



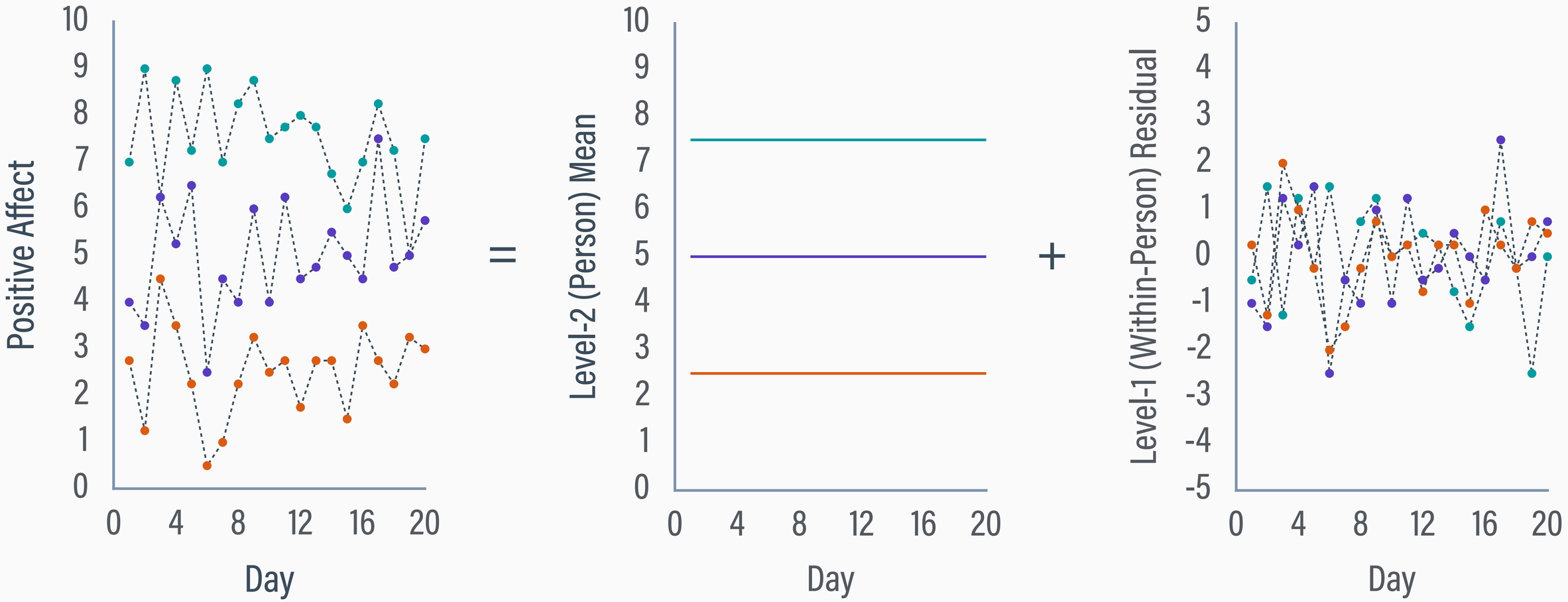
BOX PLOTS FOR 20 PARTICIPANTS



The data consist of up to 20 daily measurements nested within each person. In small groups of two or three, discuss two or three main features of the data that are apparent from the person-specific box plots.

TWO SOURCES OF VARIATION

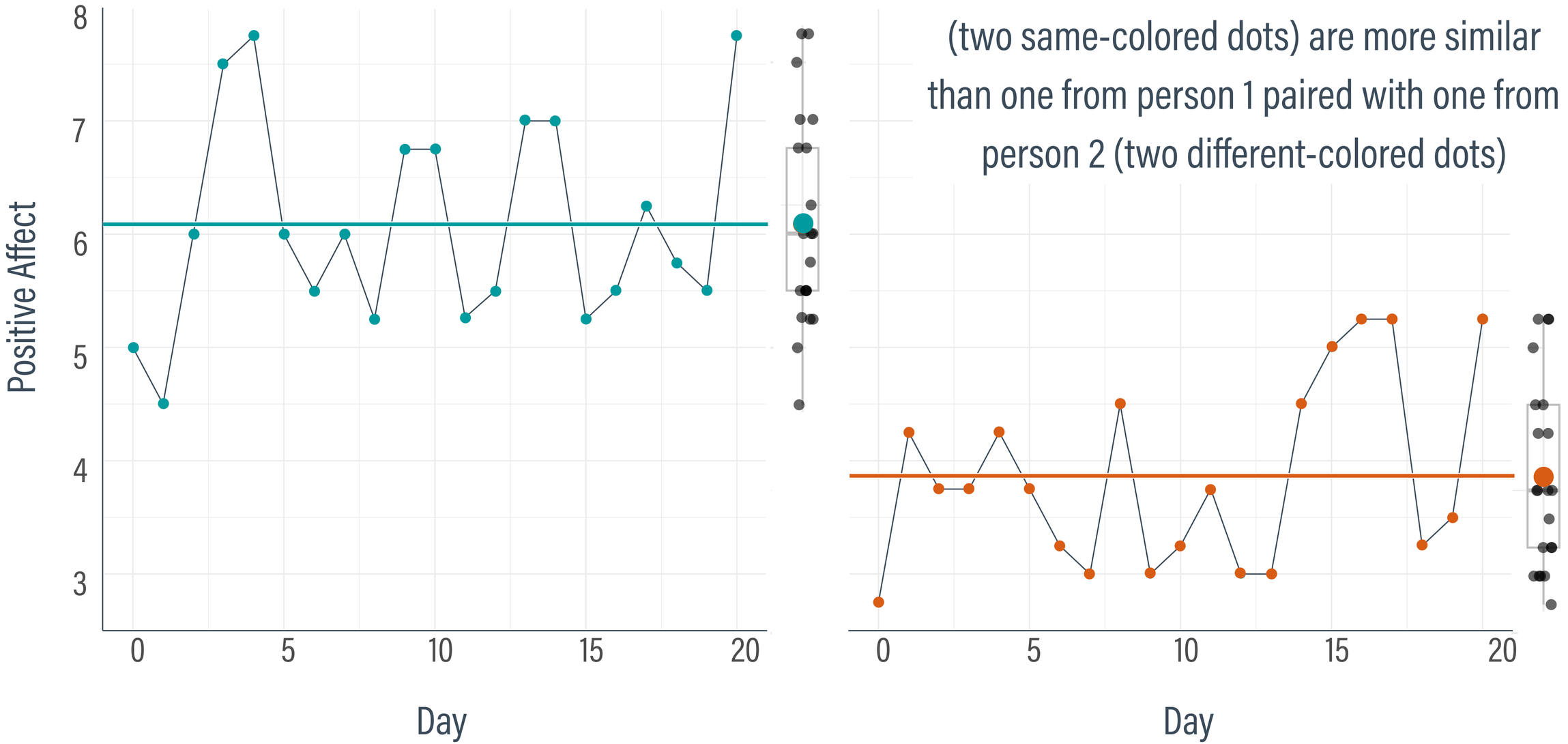
Score = Level-2 (Between-Cluster) Mean + Level-1 (Within-Cluster) Residual



STATISTICAL INDEPENDENCE

- Statistical independence is a key assumption requiring that each observation is unrelated to all other observations
- Multilevel data often (but not always) violate independence
- Any two level-1 observations (daily affect) from the same level-2 unit (person) are more similar than two observations from different level-2 units

STATISTICAL NONINDEPENDENCE



Two measurements from the same person (two same-colored dots) are more similar than one from person 1 paired with one from person 2 (two different-colored dots)

OUTLINE

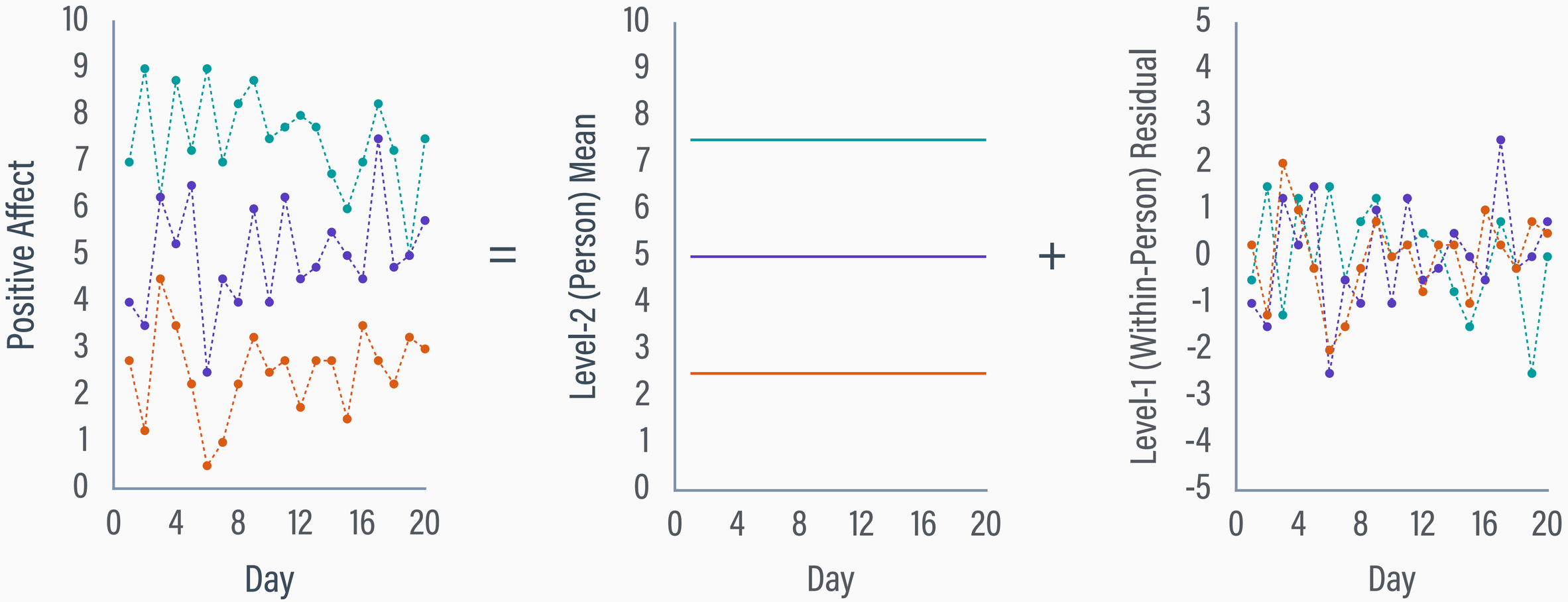
1. Introduction to Multilevel Data
2. Multiple Sources of Variability
3. Partitioning Variability With an Unconditional MLM
4. Data Analysis Example
5. Intraclass Correlation
6. Latent Variable Specification

MULTIPLE SOURCES OF VARIATION

- Raw scores combine two components that quantify difference aspects of the variable
- Between-cluster variation: Each level-2 unit has its own mean (e.g., a person's stable affect level across days)
- Within-cluster variation: Observations vary around that mean (e.g., daily mood fluctuations)

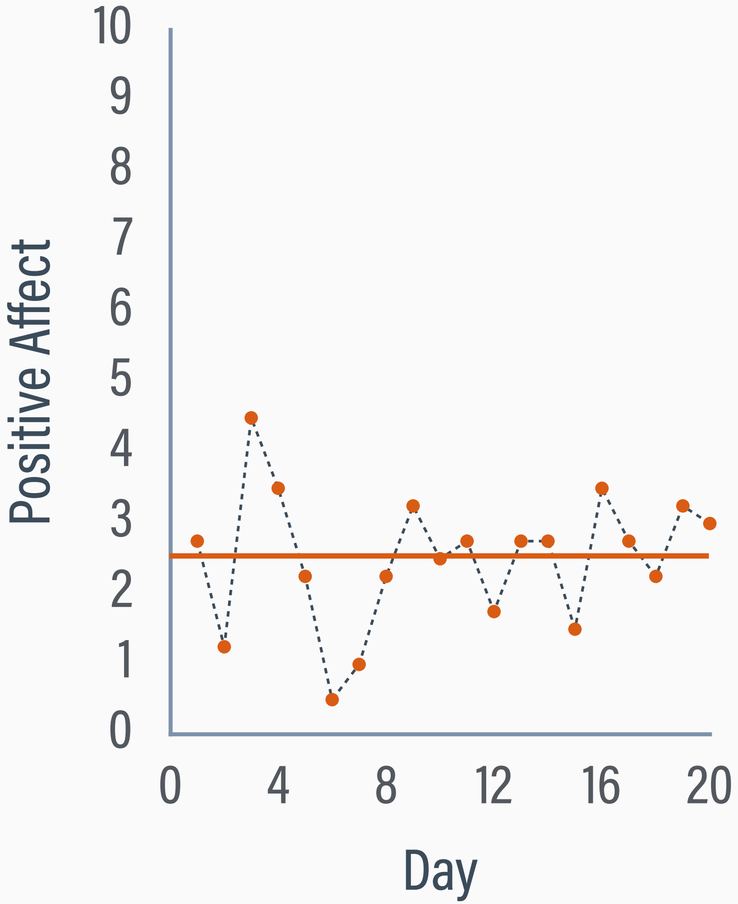
VARIATION SUMMARY

Raw Score = Level-2 (Between-Cluster) Mean + Level-1 (Within-Cluster) Residual



PERSON 1 DECOMPOSITION

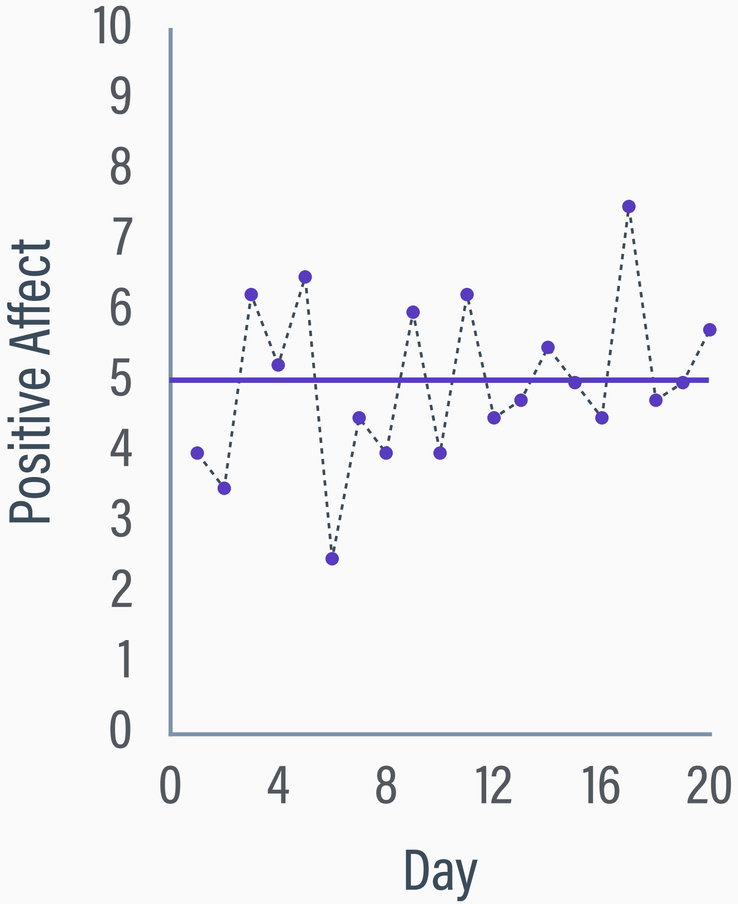
Score = Level-2 (Between-Cluster) Mean + Level-1 (Within-Cluster) Residual



Person	Day	Affect		Level-2 Mean		Level-1 Residual
1	1	2.75	=	2.5	+	0.25
1	2	1.25	=	2.5	+	−1.25
1	3	4.50	=	2.5	+	2.00
1	...	...	=	...	+	...
1	18	2.25	=	2.5	+	−0.25
1	19	3.25	=	2.5	+	0.75
1	20	3.00	=	2.5	+	0.50

PERSON 2 DECOMPOSITION

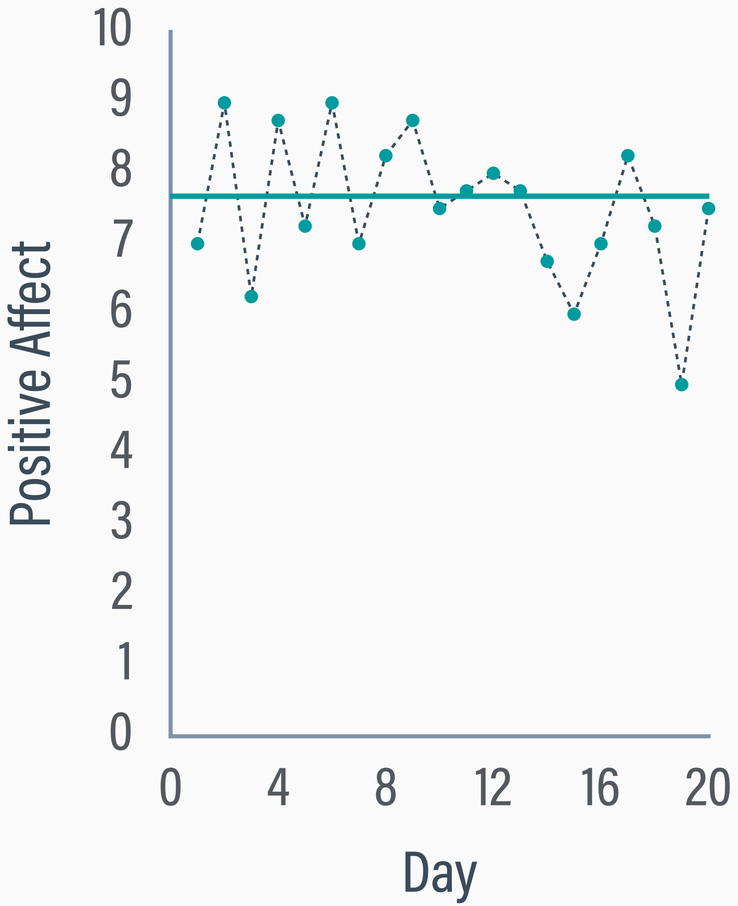
Score = Level-2 (Between-Cluster) Mean + Level-1 (Within-Cluster) Residual



Person	Day	Affect		Level-2 Mean		Level-1 Residual
2	1	4.00	=	5.0	+	−1.00
2	2	3.50	=	5.0	+	−1.50
2	3	6.25	=	5.0	+	1.25
2	...	...	=	...	+	...
2	18	4.75	=	5.0	+	−0.25
2	19	5.00	=	5.0	+	0
2	20	5.75	=	5.0	+	0.75

PERSON 3 DECOMPOSITION

Score = Level-2 (Between-Cluster) Mean + Level-1 (Within-Cluster) Residual



Person	Day	Affect		Level-2 Mean		Level-1 Residual
3	1	7.00	=	7.5	+	−0.50
3	2	9.00	=	7.5	+	1.50
3	3	6.25	=	7.5	+	−1.25
3	...	...	=	...	+	...
3	18	7.25	=	7.5	+	−0.25
3	19	5.00	=	7.5	+	−2.50
3	20	7.50	=	7.5	+	0

WITHIN-CLUSTER (LEVEL-1) MODEL

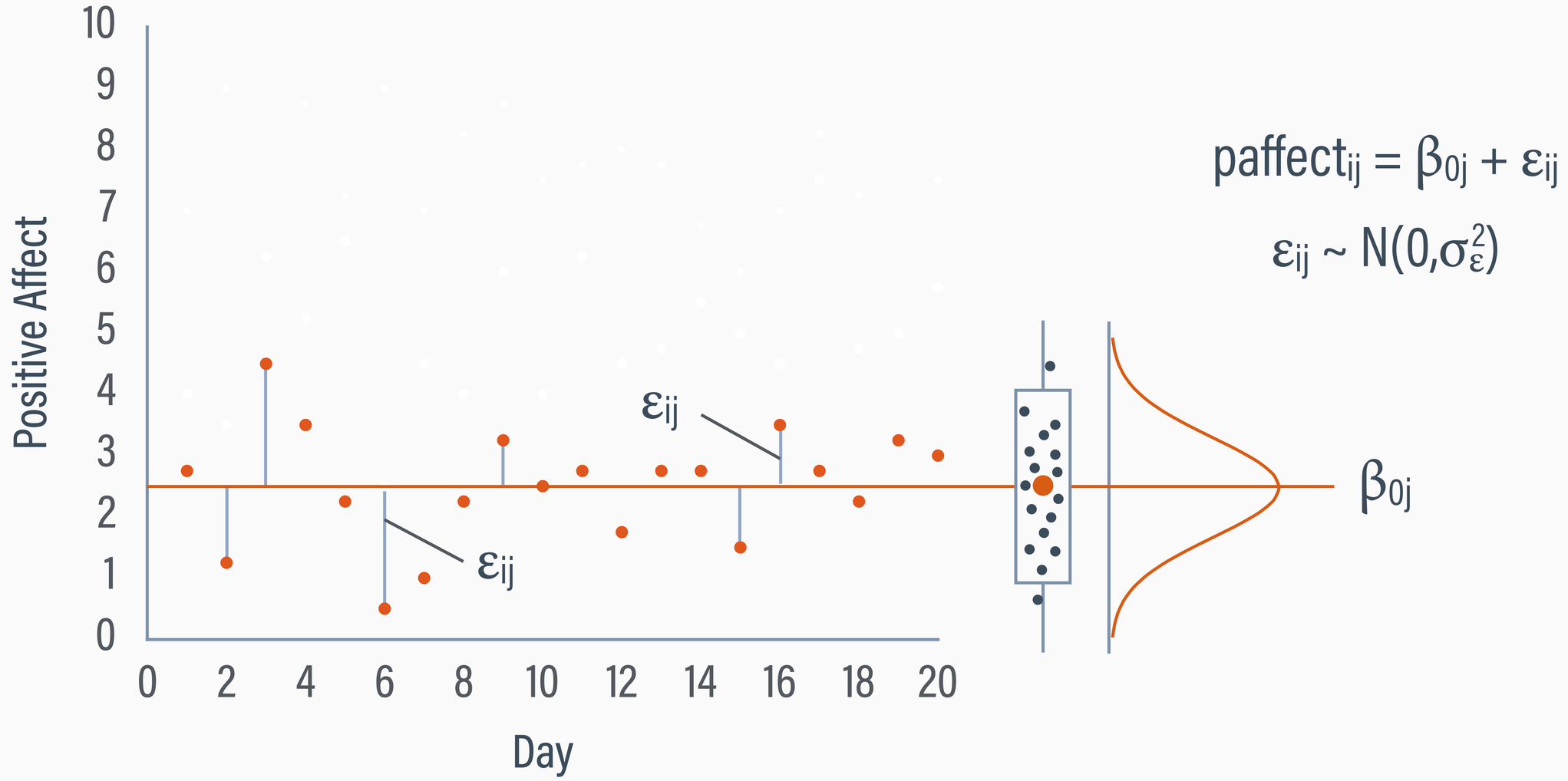
- Affect observation i for person j is the sum of their own level-2 affect mean (β_{0j}) and a within-person residual (ε_{ij})

$$paffect_{ij} = \beta_{0j} + \varepsilon_{ij}$$

- Assumption: residuals are normal with constant variation across all days (level-1 units) and persons (level-2 units)

$$\varepsilon_{ij} \sim N(0,\sigma^2_{\varepsilon})$$

WITHIN-CLUSTER MODEL



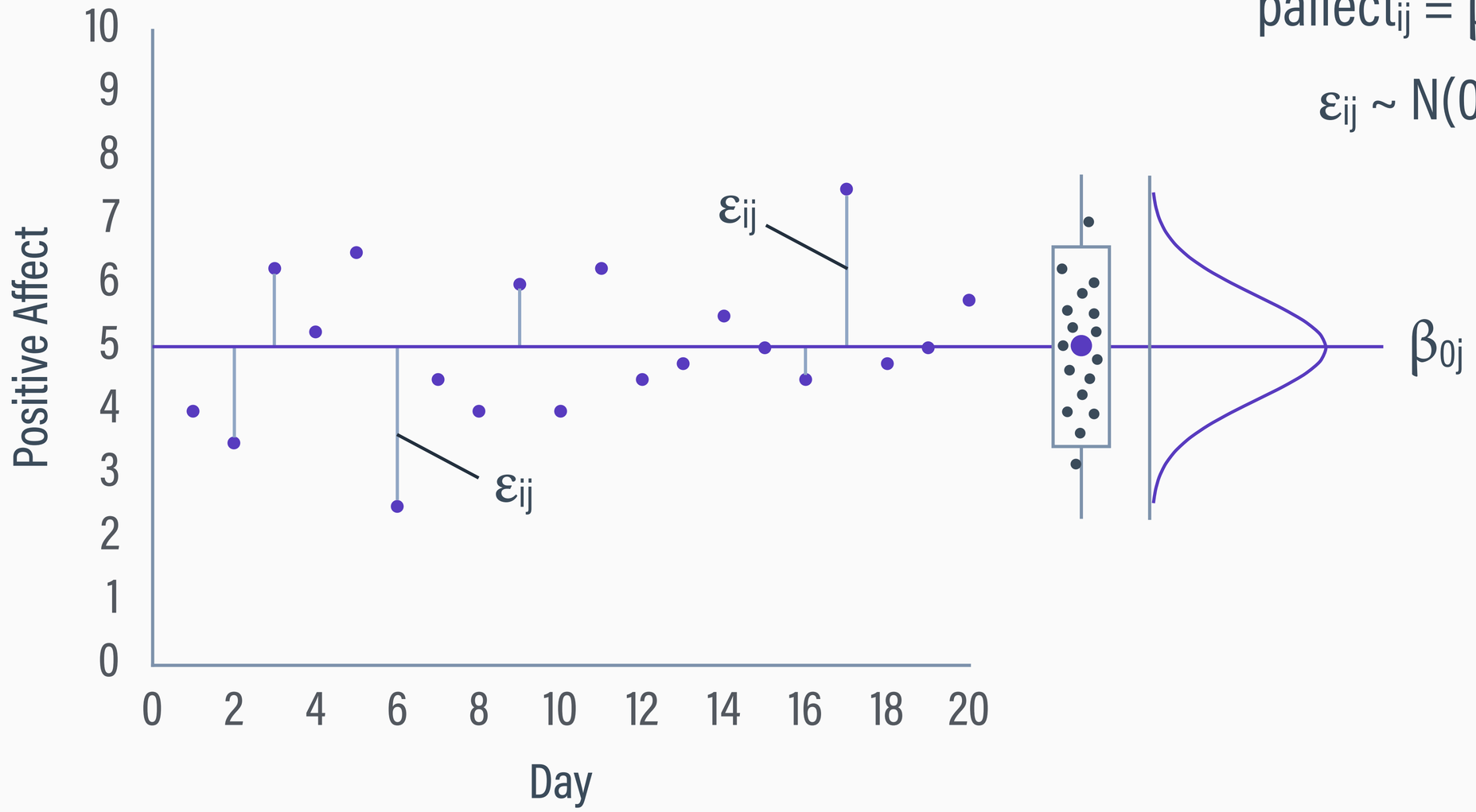
$$paffect_{ij} = \beta_{0j} + \varepsilon_{ij}$$

$$\varepsilon_{ij} \sim N(0,\sigma^2_{\varepsilon})$$

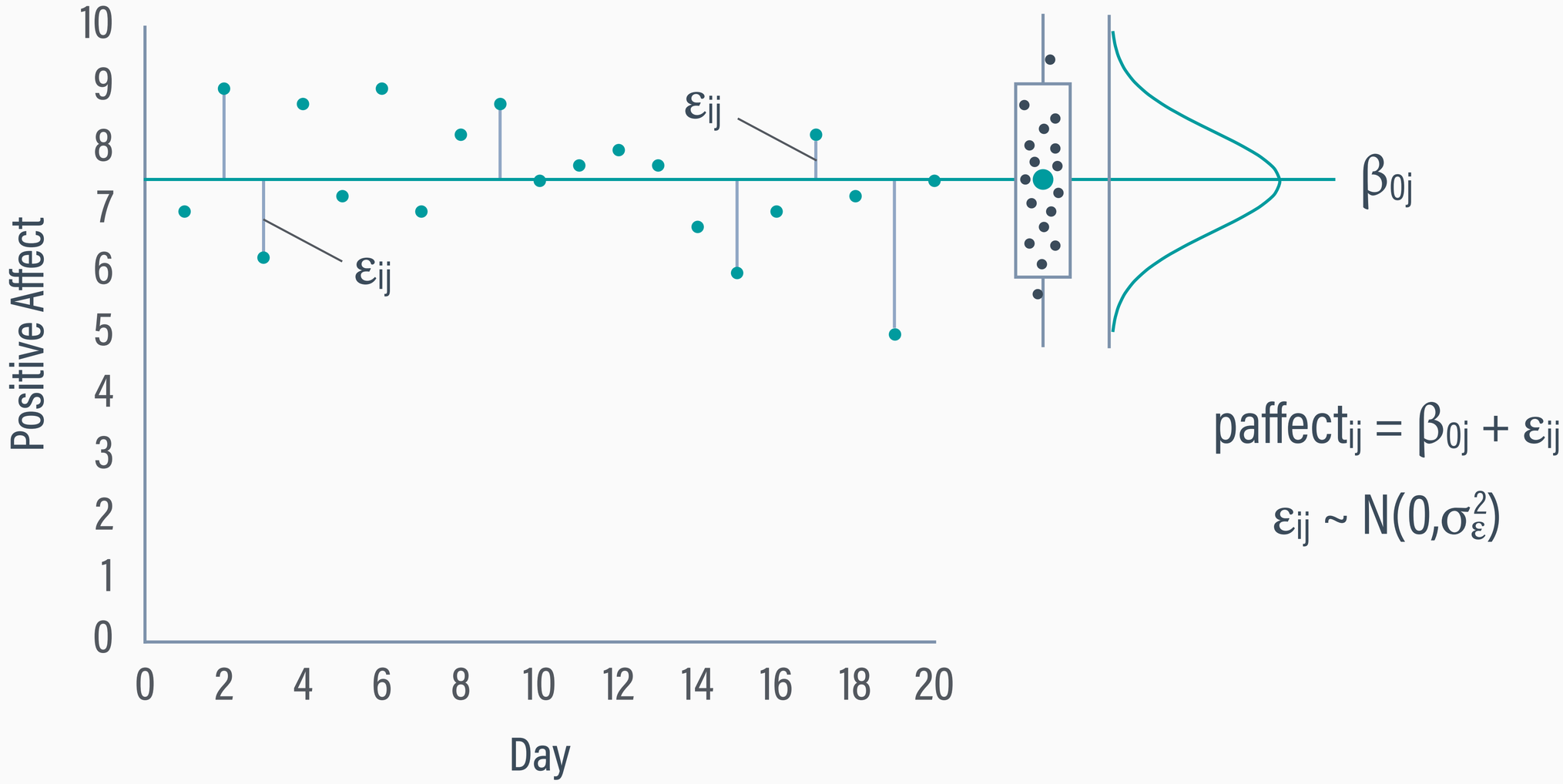
WITHIN-CLUSTER MODEL

$$paffect_{ij} = \beta_{0j} + \varepsilon_{ij}$$

$$\varepsilon_{ij} \sim N(0,\sigma^2_\varepsilon)$$



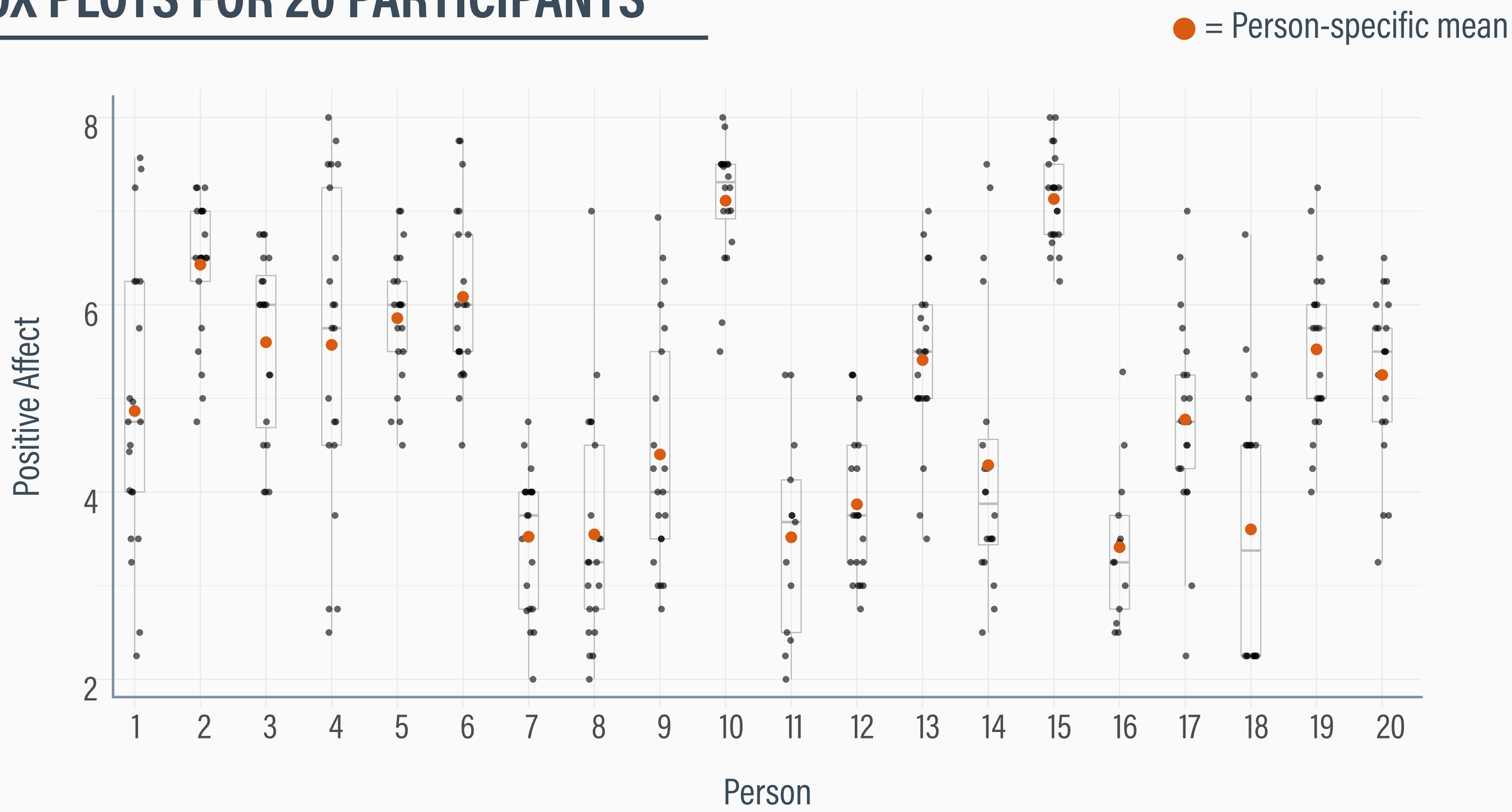
WITHIN-CLUSTER MODEL



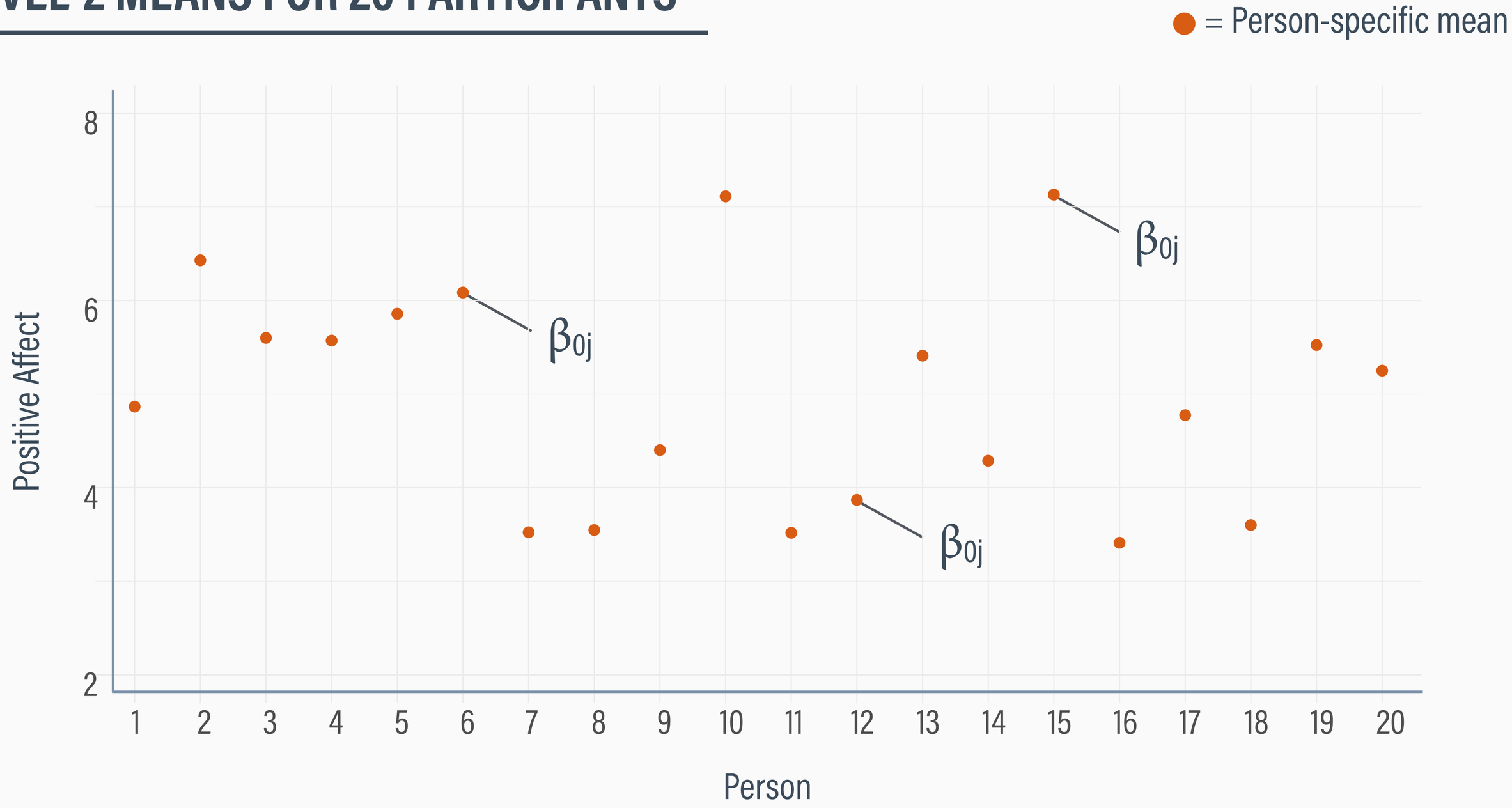
$$paffect_{ij} = \beta_{0j} + \varepsilon_{ij}$$

$$\varepsilon_{ij} \sim N(0,\sigma^2_{\varepsilon})$$

BOX PLOTS FOR 20 PARTICIPANTS



LEVEL-2 MEANS FOR 20 PARTICIPANTS



BETWEEN-CLUSTER (LEVEL-2) MODEL

- The affect mean for person j (β_{0j}) is the sum of the grand mean (γ_{00}) and a between-person residual (u_{0j})

$$\beta_{0j} = \gamma_{00} + u_{0j}$$

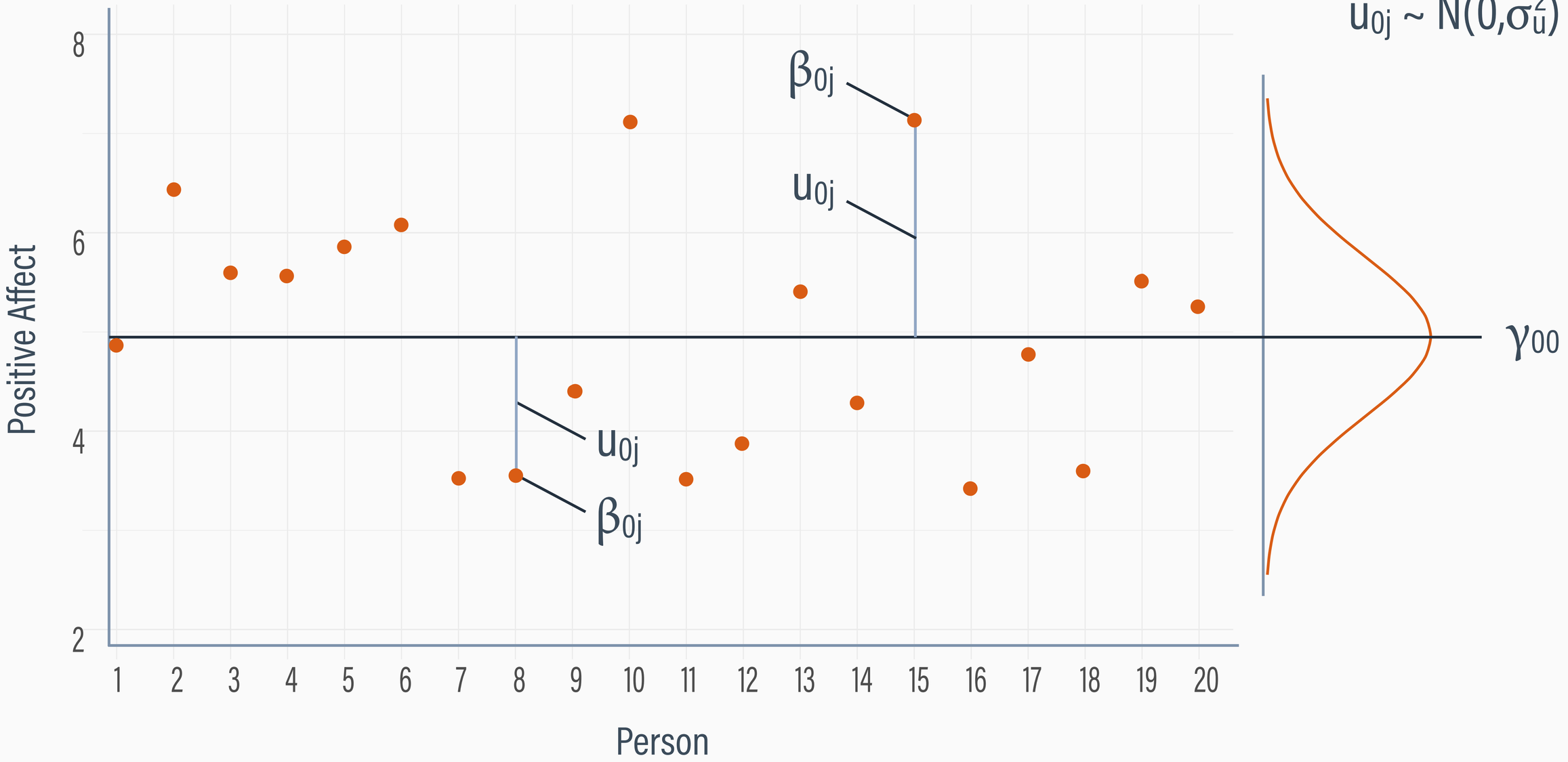
- Random intercept residuals are normal with constant variation across persons (level-2 units)

$$u_{0j} \sim N(0,\sigma_u^2)$$

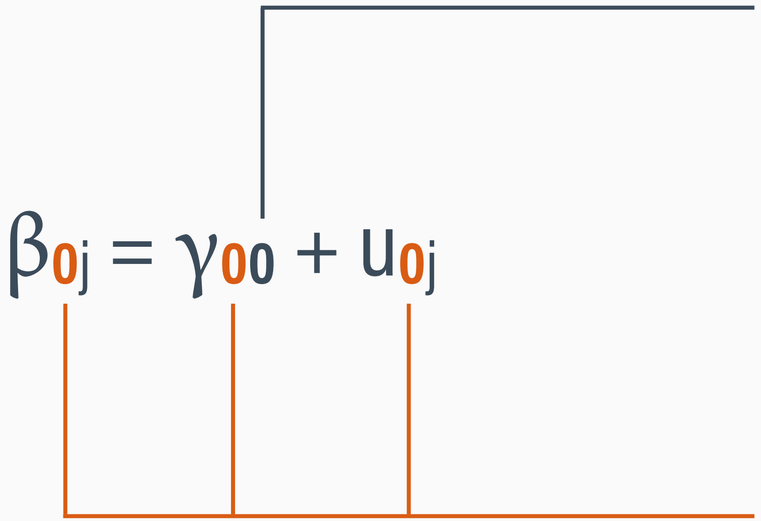
BETWEEN-CLUSTER MODEL

$$\beta_{0j} = \gamma_{00} + u_{0j}$$

$$u_{0j} \sim N(0,\sigma_u^2)$$



DECODING THE SUBSCRIPTS



The second subscript numbers the coefficients on the right side starting at 0 (the intercept)

The first subscript tells which level-2 outcome variable these terms are attached to (γ_{00} and u_{0j} belong in β_{0j}'s equation)

COMBINED-MODEL NOTATION

- Level-specific regression equations can be reduced into a single combined-model equation (Raudenbush & Bryk, 2002)

$$\beta_{0j} = \gamma_{00} + u_{0j}$$

$$paffect_{ij} = \beta_{0j} + \varepsilon_{ij}$$

- Replace the β_{0j} intercept in the level-1 equation with it's level-2 equation

$$\beta_{0j} = \gamma_{00} + u_{0j}$$

$$paffect_{ij} = \beta_{0j} + \varepsilon_{ij}$$

- Each daily observation equals the grand mean plus a level-2 and level-1 residual

$$paffect_{ij} = \gamma_{00} + u_{0j} + \varepsilon_{ij}$$

COMMON NOTATIONAL SYSTEMS

Combined-model equation =(Raudenbush & Bryk, 2002)

$$paffect_{ij} = \gamma_{00} + u_{0j} + \varepsilon_{ij}$$

Standard(ish) regression notation (Scott, Shrout, & Weinberg, 2013)

$$paffect_{ij} = \beta_0 + u_{0j} + \varepsilon_{ij}$$

Linear mixed model cluster-level matrix equation

$$\mathbf{y}_{ij} = \mathbf{X}_j\boldsymbol{\beta} + \mathbf{Z}_j u_j + \varepsilon_j = \begin{pmatrix} posaffect_{1j} \\ posaffect_{2j} \\ \dots \\ posaffect_{nj} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \dots \\ 1 \end{pmatrix} \beta_0 + \begin{pmatrix} 1 \\ 1 \\ \dots \\ 1 \end{pmatrix} u_{0j} + \begin{pmatrix} \varepsilon_{1j} \\ \varepsilon_{2j} \\ \dots \\ \varepsilon_{nj} \end{pmatrix}$$

FIXED FACTORS

- In ANOVA and regression models, mean differences are modeled using dummy codes

$$paffect_{ij} = \beta_0 + \beta_1(D_1) + \beta_2(D_2) + \ldots + \beta_{131}(D_{131}) + \varepsilon_{ij}$$

- In this framework, level-2 units comprise a fixed factor because they represent all possible groups of interest (e.g., all possible conditions, demographic groups, etc.)

RANDOM FACTORS

- In multilevel models, level-2 groups are treated as a sample from a larger population of clusters
- Mean differences across groups are captured by level-2 variables (the u_{0j} terms) rather than fixed dummy codes

$$paffect_{ij} = \beta_0 + u_{0j} + \varepsilon_{ij}$$

- The u_{0j} terms are called latent variables, random intercepts, and random effects

OUTLINE

1 Introduction to Multilevel Data

2 Multiple Sources of Variability

3 Partitioning Variability With an Unconditional MLM

4 Data Analysis Example

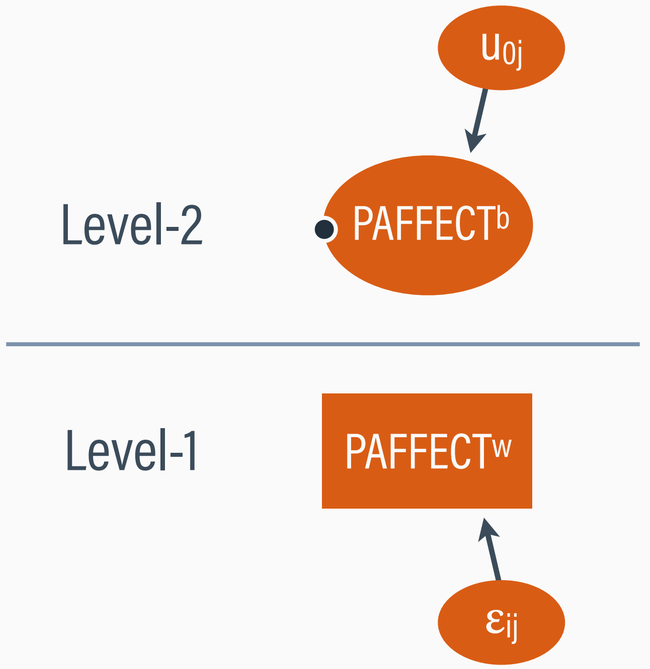
5 Intraclass Correlation

6 Latent Variable Specification

ANALYSIS OVERVIEW

- Between-cluster variation captures the magnitude of deviations between each person's average mood and the grand mean

- Within-cluster variation captures the magnitude of daily positive mood fluctuations around each person's level-2 average



BLIMP STUDIO SCRIPT 1.1

```
DATA: PainDiary.dat;
VARIABLES: Person Day PosAffect NegAffect Pain WorkGoal LifeGoal SleepQual Female Education
  Employment MarStatus NumDiagnose ActivityLevel PainAccept Catastrophize Stress Anxiety;
CLUSTERID: Person;    # level-2 identifier invokes random intercepts
MODEL: PosAffect ~ intercept | intercept;    # coefficients | random effects
SEED: 90291;
BURN: 10000;
ITER: 10000;
```

RBLIMP SCRIPT 1 (MODEL 1)

```
model1 <- rblimp(
        data = PainDiary,
        clusterid = 'Person',
        model = 'PosAffect ~ intercept | intercept',
        seed = 90291,
        burn = 10000,
        iter = 10000)
output(model1)
```

BLIMP OUTPUT

= level-2 estimate

= level-1 estimate

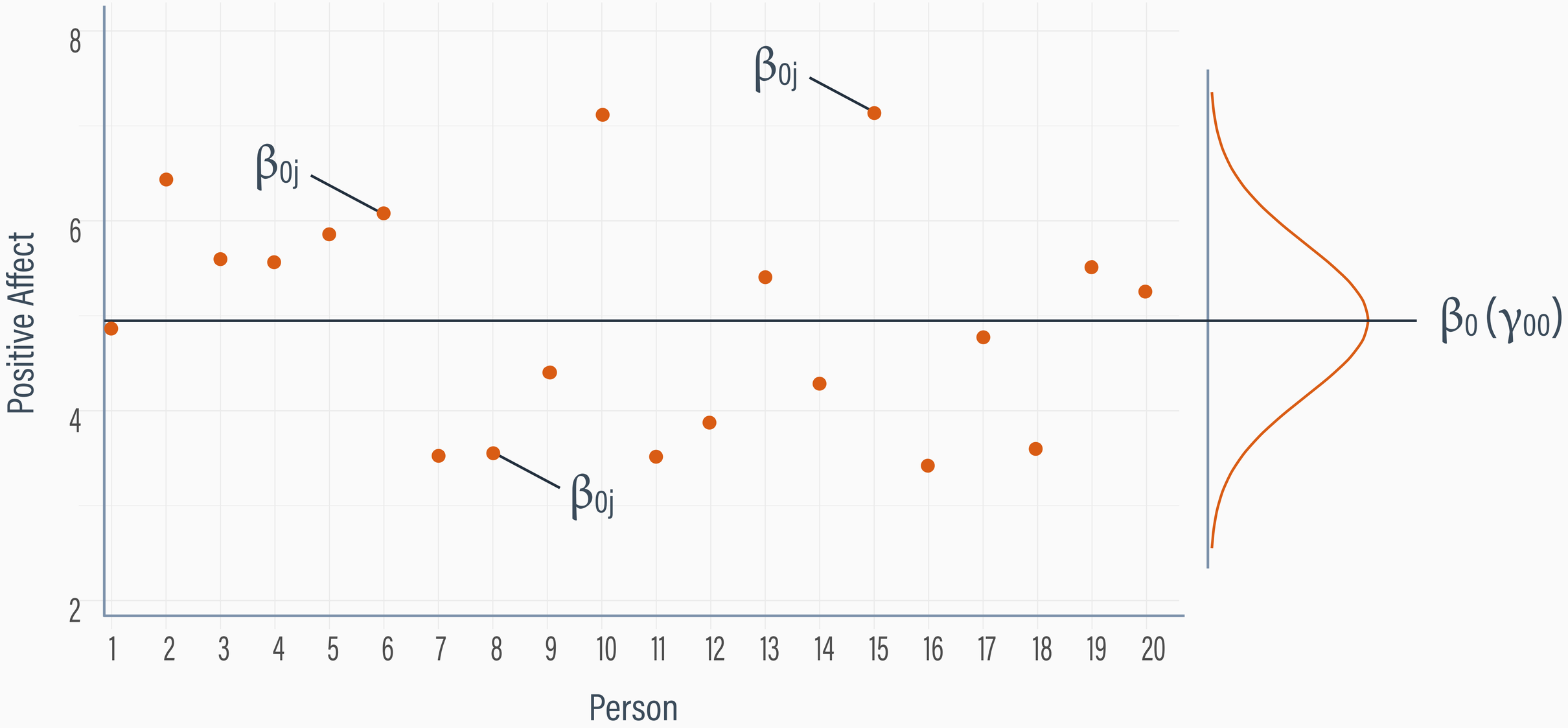
Outcome Variable: PosAffect

Parameters	Estimate	StdDev	2.5%	97.5%	ChiSq	PValue	N_Eff
Variances:							
L2 : Var(Intercept)	**2.526**	0.330	1.983	3.268	---	---	8536.828
Residual Var.	**1.396**	0.039	1.321	1.473	---	---	8929.462
Coefficients:							
Intercept	**5.032**	0.131	4.777	5.288	1482.655	0.000	170.776
Standard Deviations:							
L2 : SD(Intercept)	**1.589**	0.102	1.408	1.808	---	---	8536.495
Residual SD	**1.181**	0.017	1.149	1.214	---	---	8944.948
Proportion Variance Explained							
by Coefficients	0.000	0.000	0.000	0.000	---	---	nan
by Level-2 Random Intercepts	0.644	0.030	0.584	0.702	---	---	8392.176
by Level-1 Residual Variation	0.356	0.030	0.298	0.416	---	---	8392.176

FIXED EFFECT INTERPRETATIONS

- $paffect_{ij} = \beta_0 + u_{0j} + \varepsilon_{ij}$
- $\beta_0 = 5.03$ is the positive affect grand mean (the mean of the level-2 or person-level means)

FIXED INTERCEPT



RANDOM EFFECT INTERPRETATIONS

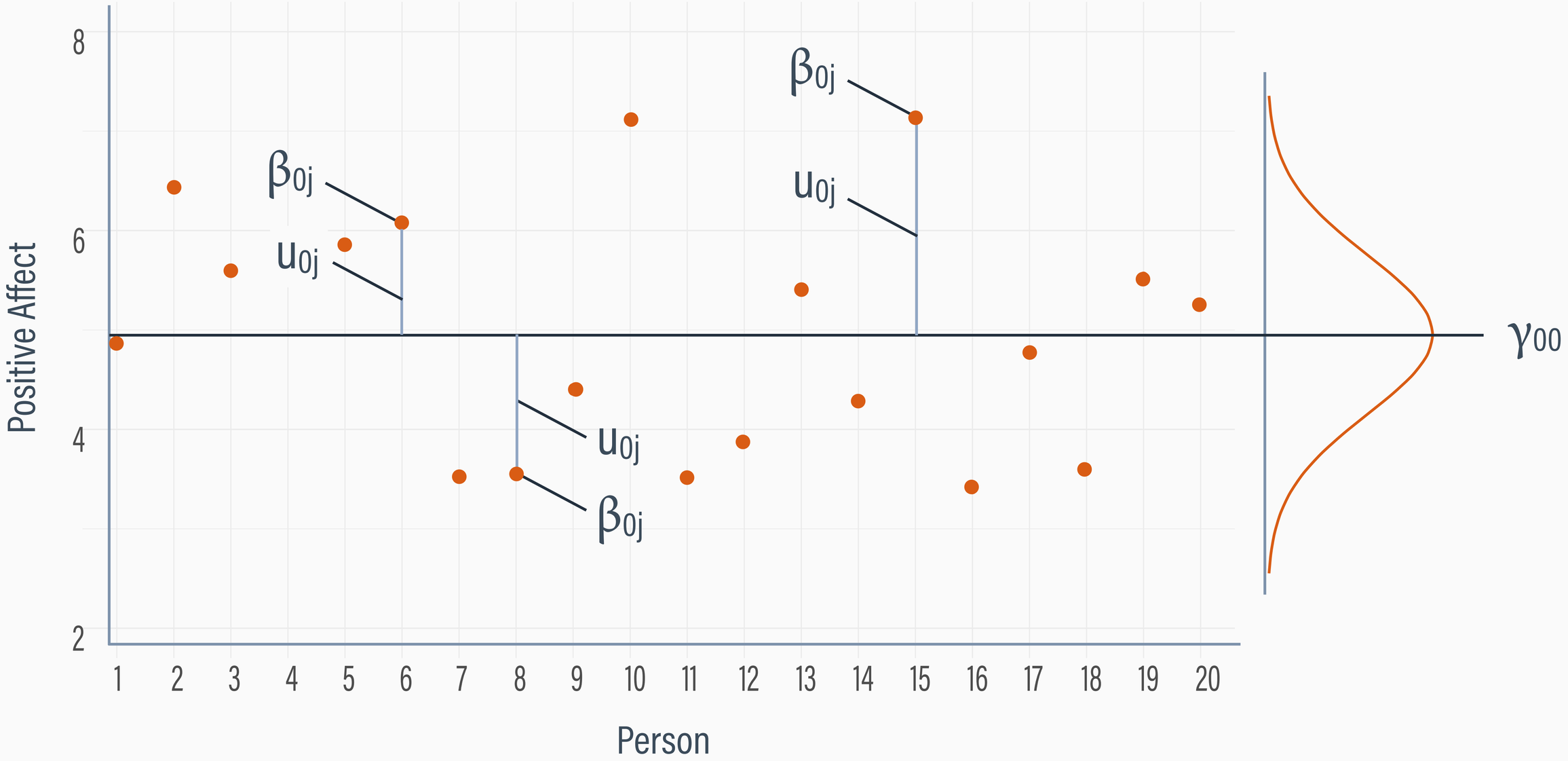
- $u_{0j} = \beta_{0j} - \beta_0$
- $var(u_{0j}) = 2.53$ is the average squared distance between the level-2 affect means and the grand mean
- $sd(u_{0j}) = 1.59$ is the average distance between the level-2 affect means and the grand mean

- $\varepsilon_{ij} = paffect_{ij} - \beta_{0j}$
- $var(\varepsilon_{ij}) = 1.40$ is the average squared distance between the level-1 affect observations and their level-2 means
- $sd(\varepsilon_{ij}) = 1.18$ is the average distance between the level-1 affect observations and their level-2 means

BETWEEN-CLUSTER VARIATION

$var(u_{0j}) = 2.53$

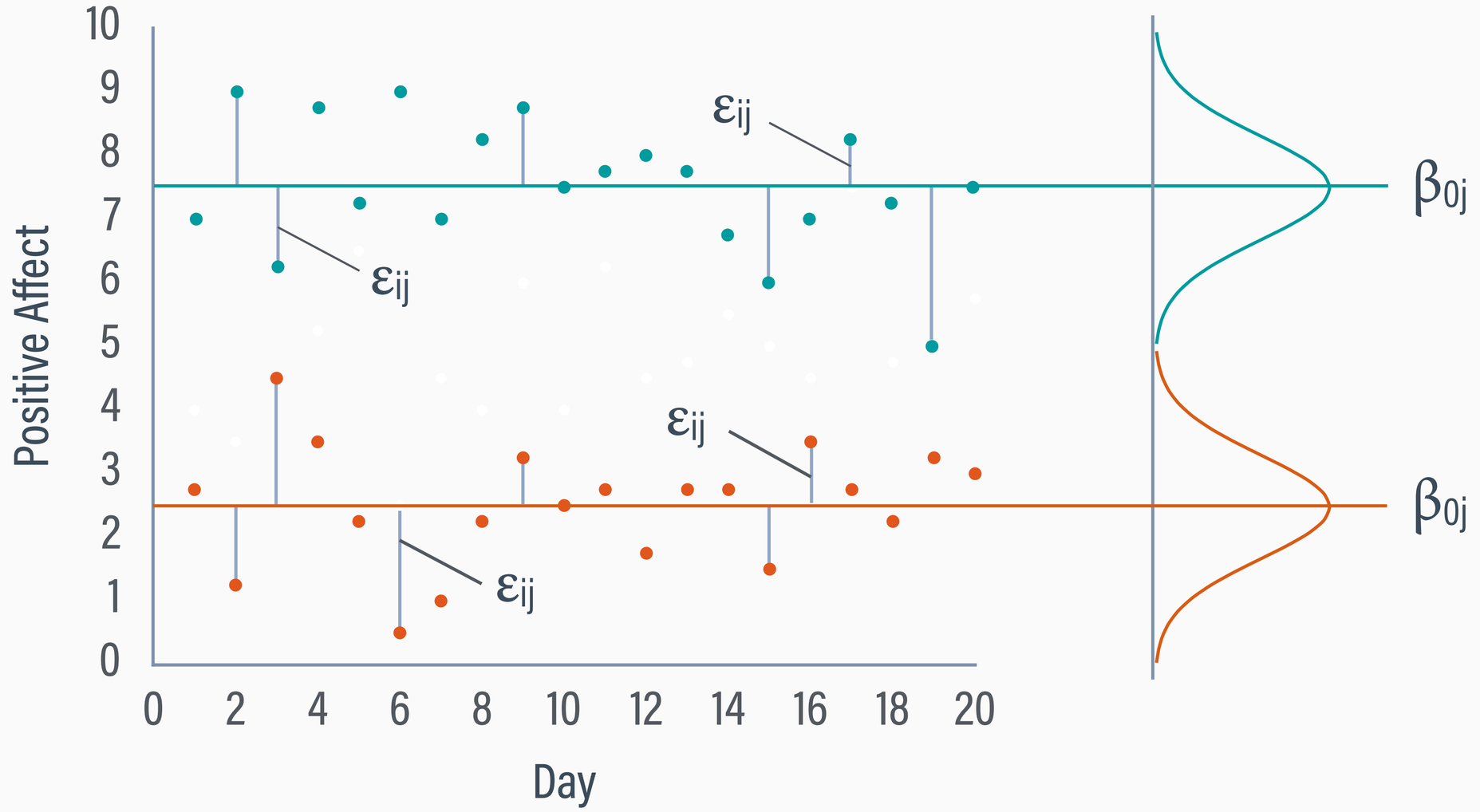
$sd(u_{0j}) = 1.59$



WITHIN-CLUSTER VARIATION

$\text{var}(\varepsilon_{ij}) = 1.40$

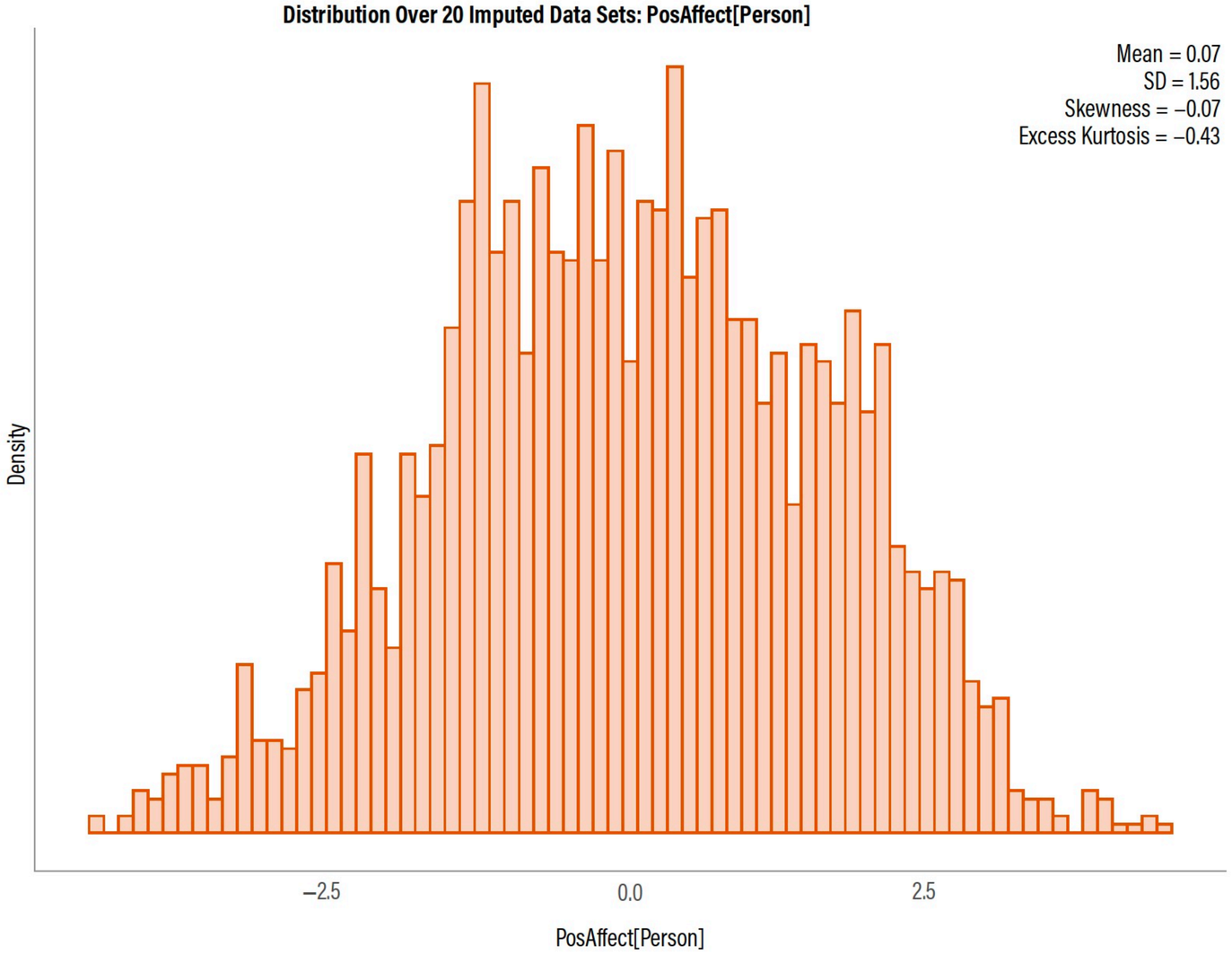
$\text{sd}(\varepsilon_{ij}) = 1.18$



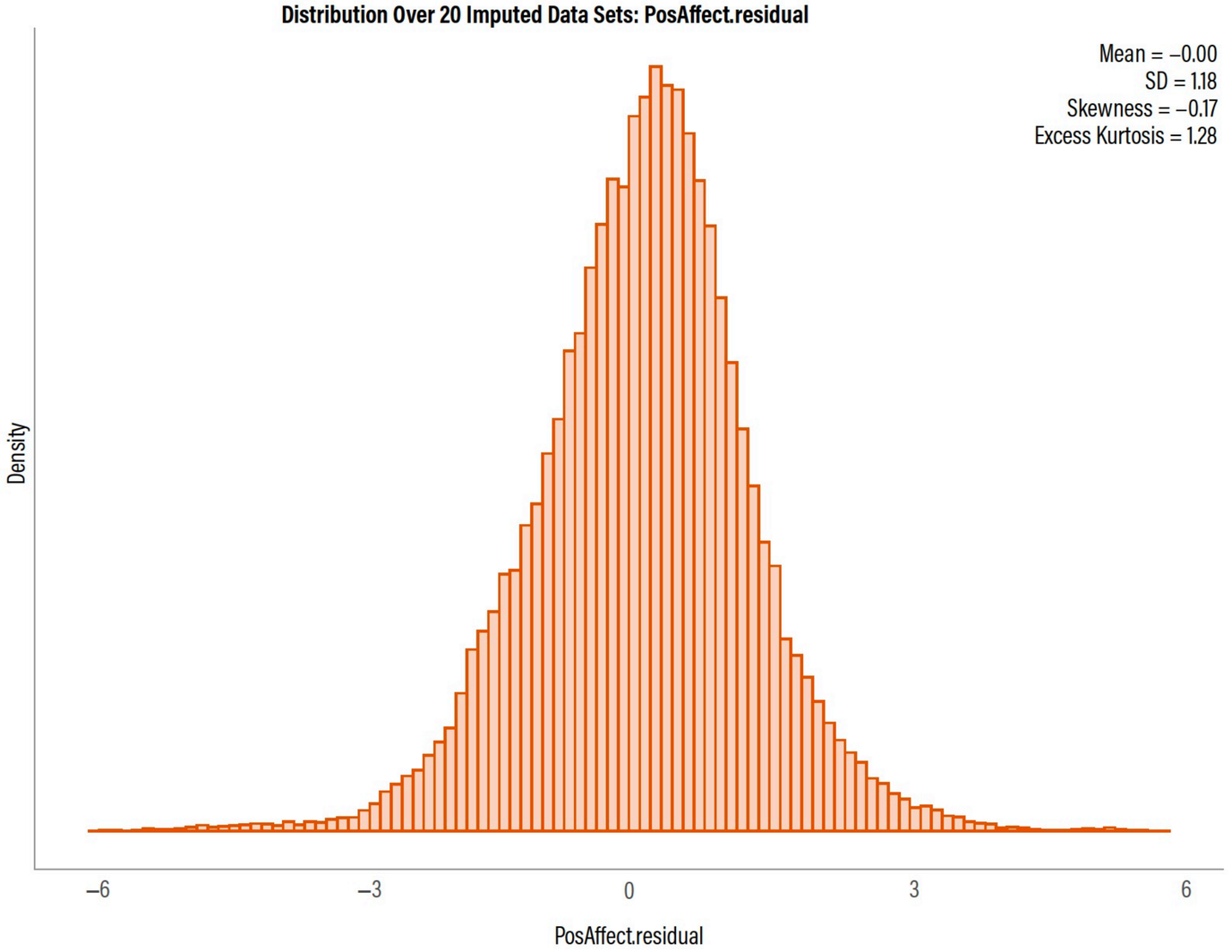
RBLIMP SCRIPT 2 (MODEL 2)

```
model2 <- rblimp(
        data = PainDiary,
        clusterid = 'Person',
        model = 'PosAffect ~ intercept | intercept',
        seed = 90291,
        burn = 10000,
        iter = 10000,
        nimps = 20)
output(model2)
univariate_plot(vars = c('PosAffect[Person]','PosAffect.residual'), model = model2, stats = T)
```

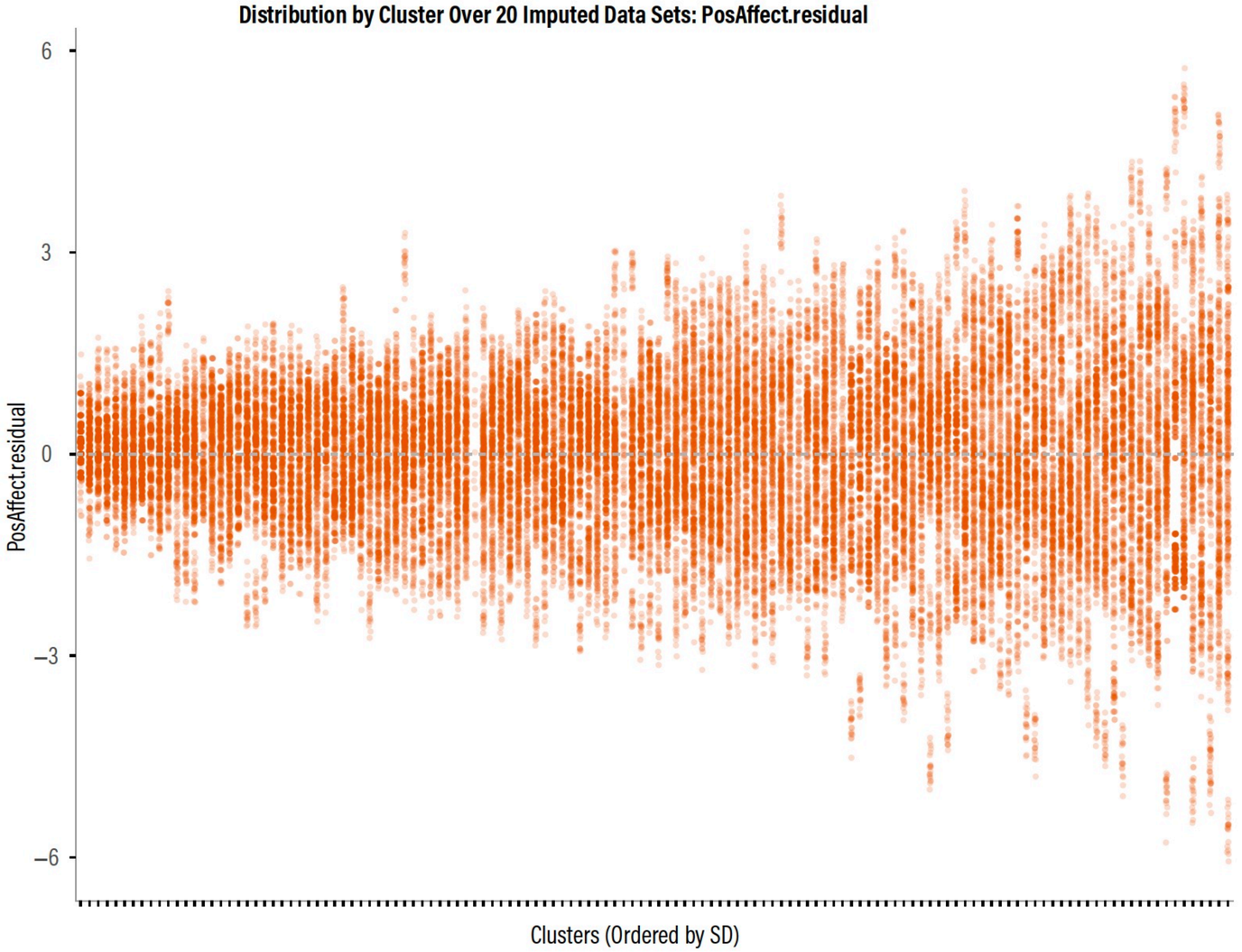
LEVEL-2 RESIDUALS



LEVEL-1 RESIDUALS



LEVEL-1 RESIDUALS BY CLUSTER



OUTLINE

1 Introduction to Multilevel Data

2 Multiple Sources of Variability

3 Partitioning Variability With an Unconditional MLM

4 Data Analysis Example

5 Intraclass Correlation

6 Latent Variable Specification

MODEL-PREDICTED VARIANCE

- Linear model parameters (e.g., regression, SEM, MLM) combine to predicted the data's variances and covariances
- The model-predicted variance of the outcome is the sum of its level-2 and level-1 variation
- That is, the model-predicted outcome variance is $\sigma_u^2 + \sigma_\varepsilon^2$

COVARIANCE ALGEBRA STEPS

- Covariance algebra shows that the model-predicted variance of the outcome is the sum of its level-2 and level-1 variation

$$\text{var}(Y_{ij}) = \text{cov}(Y_{ij}, Y_{ij})$$

$$= \text{cov}(\gamma_{00} + u_{0j} + \varepsilon_{ij},\ \gamma_{00} + u_{0j} + \varepsilon_{ij}) \quad \text{\# substitute Y equations}$$

$$= \text{cov}(u_{0j} + \varepsilon_{ij},\ u_{0j} + \varepsilon_{ij}) \quad \text{\# drop constants}$$

$$= \text{cov}(u_{0j}, u_{0j}) + \text{cov}(u_{0j}, \varepsilon_{ij}) + \text{cov}(\varepsilon_{ij}, u_{0j}) + \text{cov}(\varepsilon_{ij}, \varepsilon_{ij}) \quad \text{\# expand}$$

$$= \text{cov}(u_{0j}, u_{0j}) + \text{cov}(\varepsilon_{ij}, \varepsilon_{ij}) \quad \text{\# residuals at different levels are uncorrelated}$$

$$= \text{var}(u_{0j}) + \text{var}(\varepsilon_{ij}) \quad \text{\# the covariance of a variable with itself is a variance}$$

$$= \sigma_u^2 + \sigma_\varepsilon^2 \quad \text{\# total variance is the sum of within + between}$$

MODEL-PREDICTED COVARIANCE

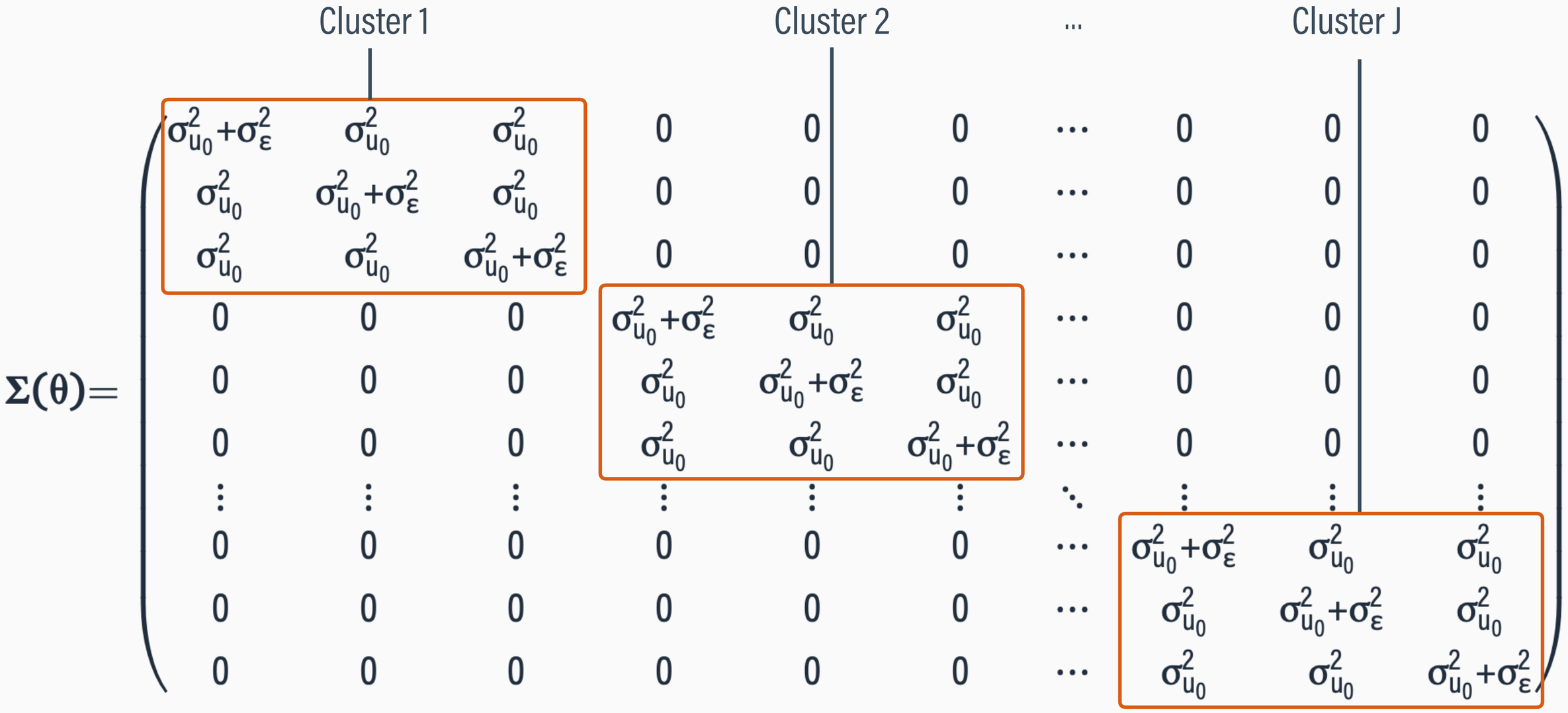
- The MLM address non-independence by allowing observations from the same cluster to correlate
- The model-predicted covariance (unstandardized correlation) between two observations from the same cluster is σ_a
- Observations from two different clusters are uncorrelated (i.e., observations are independent after controlling for clustering)

COVARIANCE ALGEBRA STEPS

- The covariance between two observations I and k from the same cluster j (Y_{ij} and Y_{kj}) equals the level-2 variance

$$
\begin{aligned}
&\text{cov}(Y_{ij}, Y_{kj}) \\
&\quad = \text{cov}(\gamma_{00} + u_{0j} + \varepsilon_{ij},\ \gamma_{00} + u_{0j} + \varepsilon_{kj}) && \text{\# substitute y equations} \\
&\quad = \text{cov}(u_{0j} + \varepsilon_{ij},\ u_{0j} + \varepsilon_{kj}) && \text{\# drop constants} \\
&\quad = \text{cov}(u_{0j}, u_{0j}) + \text{cov}(u_{0j}, \varepsilon_{kj}) + \text{cov}(\varepsilon_{ij}, u_0) + \text{cov}(\varepsilon_{ij}, \varepsilon_{kj}) && \text{\# expand} \\
&\quad = \text{cov}(u_{0j}, u_{0j}) + \text{cov}(\varepsilon_{ij}, \varepsilon_{kj}) && \text{\# residuals at different levels are uncorrelated} \\
&\quad = \text{cov}(u_{0j}, u_{0j}) && \text{\# within-cluster residuals are conditionally independent} \\
&\quad = \text{var}(u_{0j}) = \sigma^2_u && \text{\# model-predicted covariance due to clustering}
\end{aligned}
$$

MODEL-PREDICTED COVARIANCE MATRIX



INTRACLASS CORRELATION

- Cluster-level (level-2) mean differences are responsible for the non-independence of observations
- The intraclass correlation is the proportion of the total variation due to level-2 mean differences
- It is also the expected correlation between two observations from the same level-2 group

INTRACLASS CORRELATION

- The ICC is the expected correlation between two observations belonging to the same level-2 cluster (Y_{ij} and Y_{kj})

$$ICC = \frac{cov(Y_{ij}, Y_{kj})}{\sqrt{var(Y_{ij})var(Y_{kj})}} = \frac{\sigma_u^2}{\sqrt{(\sigma_u^2 + \sigma_\varepsilon^2)(\sigma_u^2 + \sigma_\varepsilon^2)}} = \frac{\sigma_{u_0}^2}{\sigma_u^2 + \sigma_\varepsilon^2} = \frac{\sigma_{Between}^2}{\sigma_{Total}^2}$$

- The ICC is also the proportion of the total variation due to level-2 cluster mean differences (like an R^2)

INTRACLASS CORRELATION EXAMPLE

- Level-2 (person-specific) mean differences account for 64% of the total variability in the positive affect scores

$$ICC = \frac{\sigma_u^2}{\sigma_u^2 + \sigma_\varepsilon^2} = \frac{2.52}{2.52 + 1.40} = .64$$

- Alternatively, the expected correlation between two daily mood observations from the same person is .64

BLIMP OUTPUT

= level-2 estimate

= level-1 estimate

Outcome Variable: PosAffect

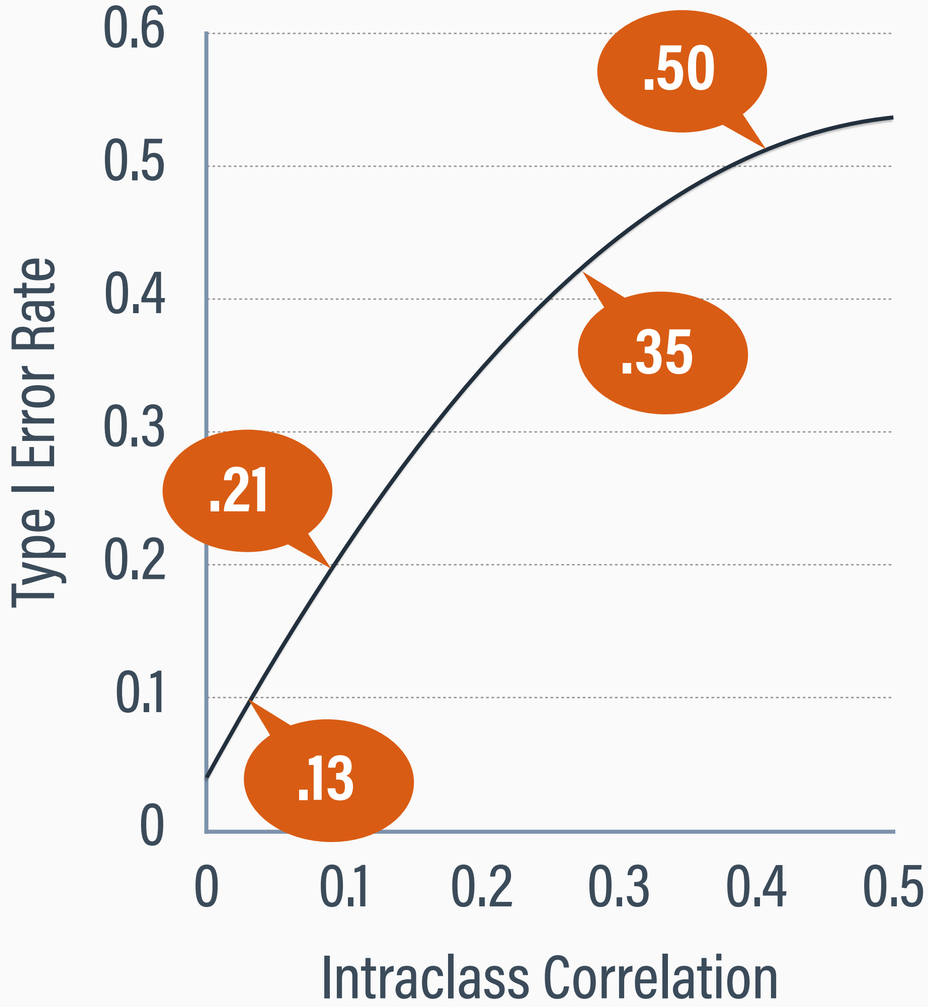
Parameters	Estimate	StdDev	2.5%	97.5%	ChiSq	PValue	N_Eff
Variances:							
L2 : Var(Intercept)	2.526	0.330	1.983	3.268	---	---	8536.828
Residual Var.	1.396	0.039	1.321	1.473	---	---	8929.462
Coefficients:							
Intercept	5.032	0.131	4.777	5.288	1482.655	0.000	170.776
Standard Deviations:							
L2 : SD(Intercept)	1.589	0.102	1.408	1.808	---	---	8536.495
Residual SD	1.181	0.017	1.149	1.214	---	---	8944.948
Proportion Variance Explained							
by Coefficients	0.000	0.000	0.000	0.000	---	---	nan
by Level-2 Random Intercepts	**0.644**	0.030	0.584	0.702	---	---	8392.176
by Level-1 Residual Variation	**0.356**	0.030	0.298	0.416	---	---	8392.176

BENCHMARKS FROM THE LITERATURE

Field	Nesting	ICC Range	References
Education	Students in schools	.10 to .25	Hedges & Hedberg, 2007, 2013; Sellström & Bremberg, 2006; Spybrook et al., 2011
Clinical psychology	Clients in therapists	.01 to .06	Baldwin et al. (2011); Kivlighan et al. (2020)
Diary/longitudinal	Measurements in persons	.40 to .60	Arend & Schäfer (2019); Bolger & Laurenceau (2013); Singer & Willett (2003)
Prevention/public health	Persons in neighborhoods	.02 to .07	Murray & Short (1995); Pals, Beaty, Posner, & Bull (2009)
Prevention/public health	Persons in countries	.04 to .19	Masood & Reidpath (2016)

CONSEQUENCES OF IGNORING CLUSTERING

- A non-zero ICC implies that the N data points are redundant (we have fewer than N independent pieces of information)
- Failing to account for clustering overstates the N, which attenuates SEs
- Type I error rates increase with the ICC



OUTLINE

1. Introduction to Multilevel Data
2. Multiple Sources of Variability
3. Partitioning Variability With an Unconditional MLM
4. Data Analysis Example
5. Intraclass Correlation
6. Latent Variable Specification

BLIMP STUDIO SCRIPT 1.2

```
DATA: PainDiary.dat;
VARIABLES: Person Day PosAffect NegAffect Pain WorkGoal LifeGoal SleepQual Female Education
  Employment MarStatus NumDiagnose ActivityLevel PainAccept Catastrophize Stress Anxiety;
CLUSTERID: Person;
LATENT: Person = beta0j;    # define level-2 intercept latent variable
MODEL:
beta0j ~ intercept;    # level-2 regression equation (β0j = γ00 + u0j)
PosAffect ~ intercept@beta0j;    # level-1 equation (paffectij = β0j + εij)
PARAMETERS:
icc = beta0j.totalvar / (beta0j.totalvar + PosAffect.totalvar);
BURN: 10000;
ITER: 20000;
SEED: 90291;
```

RBLIMP SCRIPT 1 (MODEL 2)

```
model2 <- rblimp(
        data = PainDiary,
        clusterid = 'Person',
        latent = 'Person = beta0j',
        model = '
          beta0j ~ intercept;
          PosAffect ~ intercept@beta0j',
        parameters = 'icc = beta0j.totalvar / (beta0j.totalvar + PosAffect.totalvar)',
        seed = 90291,
        burn = 10000,
        iter = 20000)
output(model2)
```

LEVEL-1 OUTPUT

= level-2 estimate

= level-1 estimate

Outcome Variable: PosAffect

Parameters	Estimate	StdDev	2.5%	97.5%	ChiSq	PValue	N_Eff
Variances:							
Residual Var.	**1.395**	0.039	1.321	1.475	---	---	18142.100
Coefficients:							
beta0j	@ 1.000	---	---	---	---	---	---
Proportion Variance Explained							
by Coefficients	0.000	0.000	0.000	0.000	---	---	nan
by Residual Variation	1.000	0.000	1.000	1.000	---	---	nan

LEVEL-2 OUTPUT

= level-2 estimate

= level-1 estimate

Latent Variable: beta0j

Parameters	Estimate	StdDev	2.5%	97.5%	ChiSq	PValue	N_Eff
Variances:							
Residual Var.	**2.526**	0.331	1.984	3.280	---	---	17551.323
Coefficients:							
Intercept	**5.033**	0.141	4.751	5.310	1267.313	0.000	18611.557
Proportion Variance Explained							
by Coefficients	0.000	0.000	0.000	0.000	---	---	nan
by Residual Variation	1.000	0.000	1.000	1.000	---	---	nan

INTRACLASS CORRELATION

GENERATED PARAMETERS:

Summaries based on 20000 iterations using 2 chains.

Parameters	Estimate	StdDev	2.5%	97.5%	ChiSq	PValue	N_Eff
icc	**0.644**	0.030	0.585	0.703	462.557	0.000	17246.138