# Practical numerical simulations: Assignment 3

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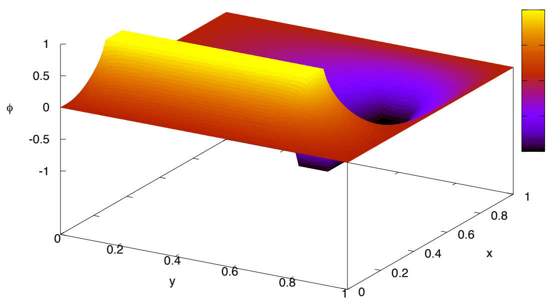
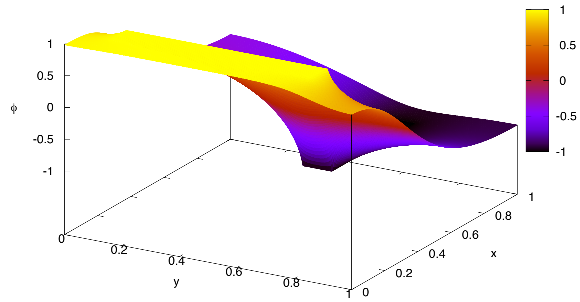



(a) *Dirichlet data for 100x100 grid*




(b) *Von-Neumann data for 100x100 grid*

This assignment dealt with simulating a field which obeyed the Laplace equation.

$$\nabla^2\phi = 0 \tag{1}$$

We represent this problem by discretising a square nxn 2d grid with positions (x,y) ranging from 0 to n. Then using the Taylor series and some linear algebra we arrive at an iterative scheme to apply (off the edges of the grid).

$$\phi(x,y)_{i+1} = \frac{\omega}{4}[\phi(x+1,y)_i + \phi(x-1,y)_{i+1} + \phi(x-1,y)_{i+1} + \phi(x,y+1)_i] + (1-\omega)\phi(x,y)_i \tag{2}$$

Where 0< $\omega < 2$, is known as the successive over-relaxation parameter. The scheme is $\mathcal{O}(\mathrm{h}^2)$, and we have h=$\frac{1}{n}$

In C++ a Field class was used with void functions which updated the field based on whether the field was to obey Dirichlet or Von Neumann boundary conditions. The constraints on the grid were handled here also, ie. the values of region A, B and the boundary.

The convergence criterion imposed in the code was to sum absolute values in the left-most column of the grid which was not the boundary, at each iteration, and compare successive values of this summation. When these values differed by less than 1x$10^{-9}$ we stop the iteration. This is potentially excessive but prevented getting 'stuck' in a solution which just returned the initial value (or something close to it) of the field, when using a criterion dependant only on one position, for instance.

## 1 Dirichlet

Dirichlet boundary conditions constrain the field to be fixed on the boundary, $\phi_{boundary}$=0, in our case.

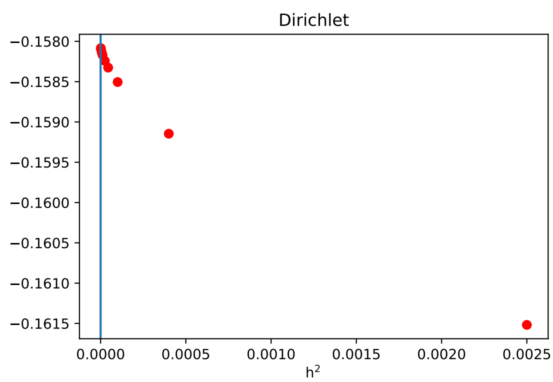



Figure 2: $\phi(3/10, 6/10)$ = ***-0.158***

We were tasked with finding the value of $\phi(3/10, 6/10)$, accounting for finite-h artefacts we plot a sequence of values of $\phi(3/10, 6/10)$ for progressively larger grids, thus progressively lower h. Then, finding the intercept we arrive at a value correct to 3sf of **-0.158**

## 2 Von Neumann

The boundary of our grid now must have the derivative of $\phi$ normal to the boundary equal to zero. This means, to continue with $\mathcal{O}(h^2)$ accuracy, we update values on the boundaries as below.

$$\phi(0, y)_{i+1} = \frac{4\phi(h, y)_i - \phi(2h, y)_i}{3} \tag{3}$$

Following the same procedure we again calculate a value for $\phi(3/10, 6/10)$.

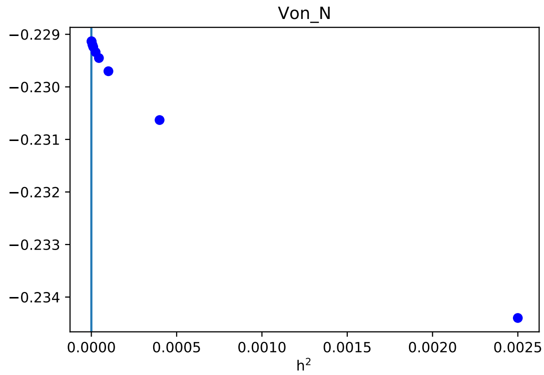



Figure 3: $\phi(3/10, 6/10)$ = ***-0.229***

Now, analysing the optimum choice of $\omega$ for these systems, we see that the Dirichlet field is much faster to converge than Von-Neumann. We count the number of sweeps through the grid, and plot $\log_{10}$(count) vs $\omega$, seen in figure 4.. To get the total number of updates of individual sites, add $log_{10}(n^2)$ to the values on the y-axis.

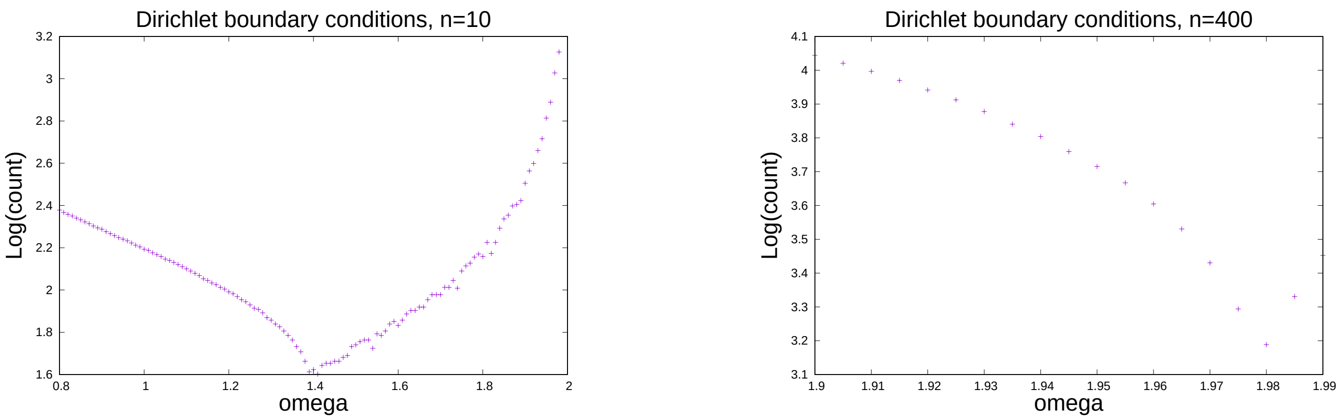



Figure 4

Optimum values of $\omega$ for varying values of n for both Dirichlet and von Neumann conditions are seen in table 1.

| n | Dirichlet | | | Von Neumann | | |
|---|---|---|---|---|---|---|
| | optimum $\omega$ | log(sweeps) | log(updates) | optimum $\omega$ | log(sweeps) | log(updates) |
| 10 | 1.41 | 1.6 | 3.6 | 1.88 | 2.24 | 4.24 |
| 50 | 1.835 | 2.33 | 5.83 | 1.97 | 2.985 | 6.383 |
| 100 | 1.925 | 2.6 | 6.6 | 1.985 | 3.27 | 7.27 |
| 200 | 1.955 | 2.877 | 7.479 | 1.99 | 3.59 | 8.192 |
| 400 | 1.98 | 3.188 | 8.688 | 1.99 | 3.99 | 9.194 |

Table 1:

According to Wolfram (`http://mathworld.wolfram.com/SuccessiveOverrelaxationMethod.html`) it is not usually possible to calculate the optimum SOR without checking explicity, further they suggested an heuristic $\omega = 2 - \mathcal{O}(h)$. This seems true for von Neumann but not Dirichlet data.

**Click *here* to see videos showing the evolution of these fields for a 200x200 grid.**