A Scalable Algorithm for Sparse and Robust Portfolios

The Sparse Markowitz Model [1]

$$\min_{\boldsymbol{x}\in\mathbb{R}^n} \frac{1}{2\gamma}\boldsymbol{x}^\top\boldsymbol{x}+\frac{\sigma}{2}\boldsymbol{x}^\top\boldsymbol{\Sigma}\boldsymbol{x}-\boldsymbol{\mu}^\top\boldsymbol{x}, \tag{1}$$

$$\text{s.t. } \boldsymbol{l}\leq \boldsymbol{A}\boldsymbol{x}\leq \boldsymbol{u}, \boldsymbol{e}^\top\boldsymbol{x}=1, \boldsymbol{x}\geq \boldsymbol{0}. \tag{2}$$

- Where $||\boldsymbol{x}||_0 \leq k$, to reduce transaction fees, prevent indexing.
- Where γ, σ ensure robustness against errors in the return vector $\boldsymbol{\mu}$, the covariance matrix $\boldsymbol{\Sigma}$.

Main Contributions

- A **tractable** nonlinear transformation which decouples the discrete, continuous.
- A **scalable** cutting-plane method which solves real-world problems instances, including the S&P 500, Wilshire 5000.
- A **generalizable** approach which solves other problems, e.g., sparse regression with non-negativity constraints [2].

Overview of the Approach

- Introduce new variables $\hat{x}_i = z_i x_i$, $\boldsymbol{z}\in\{0,1\}^n$, $\boldsymbol{Z}=\mathrm{Diag}(\boldsymbol{z})$.
- Perform a non-linear reformulation of (1) into:

$$\min_{\boldsymbol{x}\in\mathbb{R}^n, \boldsymbol{z}\in\{0,1\}^n} \frac{1}{2\gamma}\boldsymbol{x}^\top\boldsymbol{Z}\boldsymbol{x}+\frac{\sigma}{2}\boldsymbol{x}^\top\boldsymbol{Z}\boldsymbol{\Sigma}\boldsymbol{Z}\boldsymbol{x}-\boldsymbol{\mu}^\top\boldsymbol{Z}\boldsymbol{x}. \tag{4}$$

- This is equivalent to $(\boldsymbol{A})$, via duality.

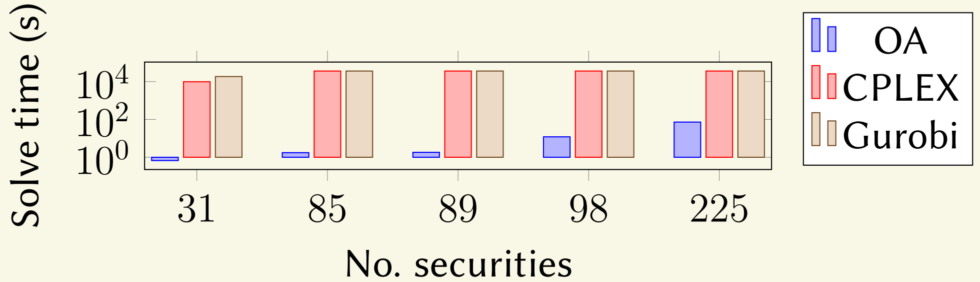
A Saddle-Point Reformulation

$$(\boldsymbol{A}):\ \min_{\boldsymbol{z}\in\{0,1\}^n}\max_{\boldsymbol{w}\in\mathbb{R}^n} \frac{-\gamma}{2}\sum_i z_i w_i^2+\Phi(\boldsymbol{w}) \text{ s.t. } \sum_{i=1}^n z_i \leq k. \tag{3}$$

- $\Phi(\boldsymbol{w})$ is concave in dual vars.
- Subgradients of $f(\boldsymbol{z})$ are given by $\frac{\partial f(\boldsymbol{z})}{\partial z_i}=\frac{-\gamma}{2}w_i^2$.
- Relax problem to $\boldsymbol{z}\in[0,1]^n$, exchange max, min operators, take dual with respect to $\boldsymbol{z}$.
- Yields a tractable, high-quality QCQP lower bound on $f(\boldsymbol{z}^*)$.

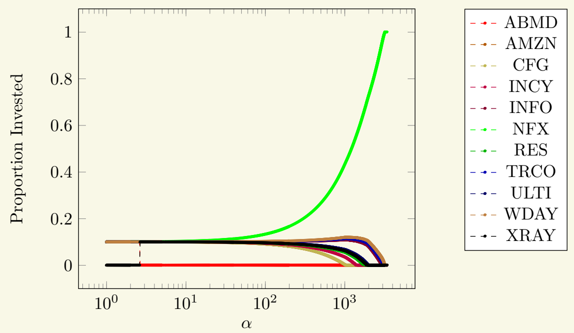
Comparison With State-of-the-Art

- Speed comparison for the OR-library test-set problems [3].



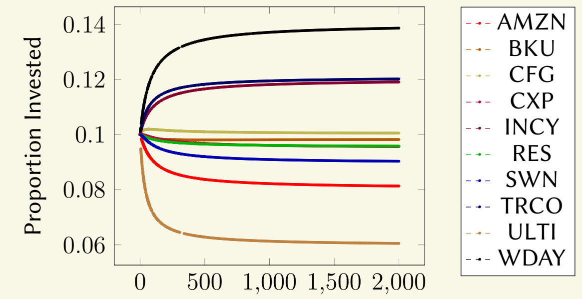
- We obtain a 4 **order of magnitude** speedup.

Sensitivity to μ



- Sensitivity to $\boldsymbol{\mu}_{\text{new}}=\alpha\boldsymbol{\mu}$ for the Russell 1000 with $k=10$.
- The optimal $\boldsymbol{z}^*$ changes once as we vary α.

Sensitivity to γ



- Sensitivity to γ for the Russell 1000 with $k=10$.
- The optimal $\boldsymbol{z}^*$ does not change as we vary γ.

The Edge of Our Approach

Set of securities	Sparsity	Min Time (s)	Max Time (s)
S&P 500	$k=50$	105 s	374 s
S&P 500	$k=200$	169 s	370 s
Russell 1000	$k=50$	1,732 s	3,320 s
Russell 1000	$k=200$	3,647 s	4642 s
Wilshire 5000	$k=50$	18 s	457 s
Wilshire 5000	$k=200$	18 s	104 s

- We used a different solver for the Wilshire 5000's QCQP bound; this improved solve times.

Bibliography

[1] D. Bertsimas and R. Cory-Wright, "A scalable algorithm for sparse and robust portfolios," *Oper. Res., Under Review*, 2018.

[2] L. Breiman, "Better subset regression using the nonnegative garrote," *Techno.*, vol. 37, no. 4, pp. 373–384, 1995.

[3] T.-J. Chang *et al.*, "Heuristics for cardinality constrained portfolio optimisation," *Comp. Oper. Res.*, vol. 27, no. 13, pp. 1271–1302, 2000.

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