A Unified Framework for Mixed-Integer Optimization: Nonlinear Formulations and Scalable Algorithms

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Preprint available at: ryancorywright.github.io

Motivation: Best Subset Selection

In high-dimensional regression, desirable to fit a parsimonious model which uses at most k features. Achieved via best subset selection

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A problem with logical structure: $x = 0$ *if* $z = 0$ for z binary.

A Family of Problems With Logical Constraints

Logical on/off structure appears in many important problems!

Central problems in optimization/statistics have logical relations between continuous variables $\boldsymbol{x}$, binary variables $\boldsymbol{z}$: $\boldsymbol{x} = \mathbf{0}$ *if* $\boldsymbol{z} = \mathbf{0}$.

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- Many others: **Sparse Portfolio Selection, Network Design, Unit Commitment, Scheduling**.

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Optimization textbooks propose an exact approach to on/off constraints:

- Big-M approach: rewrite "$x = 0$ if $z = 0$" as $-Mz \leq x \leq Mz$.
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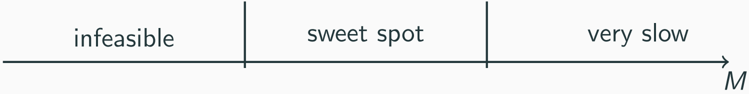

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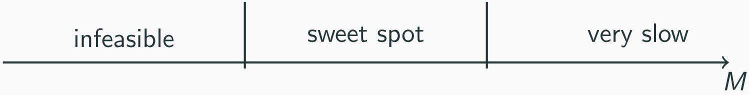
- 

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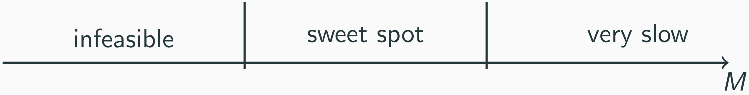
- 

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State-of-the-art methods are either heuristics or too slow.
Alternatives are needed.

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- We show that adding a $\frac{1}{2\gamma}\|\boldsymbol{x}\|_2^2$ **ridge regularizer** to the objective is a viable and often more scalable alternative to big-M.

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- By using strong duality, we derive a saddle-point reformulation, which is **exactly** solvable via an outer-approximation procedure.
- We obtain provably **near-optimal solutions in polynomial time** by solving a Boolean relaxation efficiently.
- Our approach **is scalable**: it solves sparse regression problems with $100,000$s of covariates, sparse portfolio selection problems with 1000s of securities, network design problems with 100s of nodes.

The Unified Framework

A Mixed-Integer Nonlinear Program With Logical Constraints

$$\min_{\boldsymbol{z}\in\mathcal{Z},\ \boldsymbol{x}\in\mathbb{R}^n} \boldsymbol{c}^\top \boldsymbol{z} + \underbrace{g(\boldsymbol{x})}_{\text{convex function}} + \underbrace{\Omega(\boldsymbol{x})}_{\text{regularizer}} \quad \text{s.t. } \underbrace{x_i = 0 \text{ if } z_i = 0}_{\text{logical constraint}},\ \forall i,$$

where:

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- $\mathcal{Z} \subseteq \{0,1\}^n$ constrains $\boldsymbol{z}$, e.g., cardinality constraint $\boldsymbol{e}^\top \boldsymbol{z} \leq k$.
- We model convex constraints $\boldsymbol{x} \in \mathcal{X}$ via $g(\boldsymbol{x}) = +\infty$ if $\boldsymbol{x} \notin \mathcal{X}$.
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    1. A big-M penalty: $\Omega(\boldsymbol{x}) = 0$ if $\|\boldsymbol{x}\|_\infty \le M$ and $+\infty$ otherwise.
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All six problems on the second slide fit into this framework!

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- Allows us to solve all six problems using the same piece of code.

Simplifying the Problem

We rewrite the problem as

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\end{aligned}$$

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which proves $f(z)$ is convex!

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The Saddle-Point Reformulation

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where

- $h(\cdot)$ is concave; the Fenchel conjugate of $g(\cdot)$ (up to a minus sign).
- $\Omega^\star(\alpha) = M|\alpha|$ for the big-M penalty.
- $\Omega^\star(\alpha) = \gamma/2\alpha^2$ for the ridge penalty.

So What?

The Outer-Approximation Method

Our saddle-point representation:

$$\min_{\boldsymbol{z}\in\mathcal{Z}} f(\boldsymbol{z}) = \min_{\boldsymbol{z}\in\mathcal{Z}} \max_{\boldsymbol{\alpha}} \ \boldsymbol{c}^\top \boldsymbol{z} + h(\boldsymbol{\alpha}) - \sum_i z_i \Omega^\star(\alpha_i)$$

lends itself to a tractable outer-approximation method.

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- Iteratively adding cuts, minimizing piecewise linear underestimator in Julia/CPLEX minimizes $f(\boldsymbol{z})$. Using Branch-and-Cut with lazy constraints solves entire problem using one branch-and-bound tree.
- As will see in numerical results, solves very large-scale problems.

The Outer Approximation Process

We solve the problem

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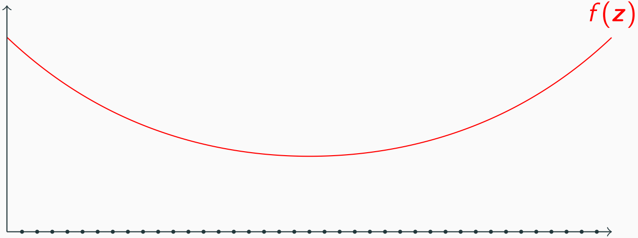
by iteratively minimizing a piecewise linear underestimator of f.



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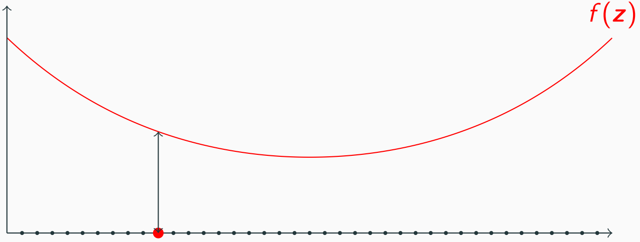

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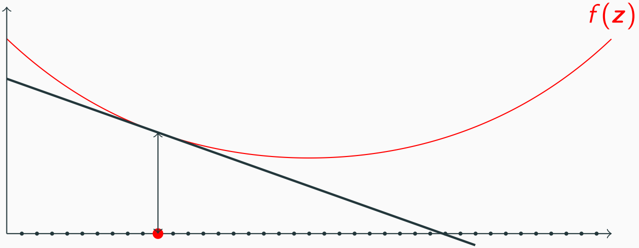

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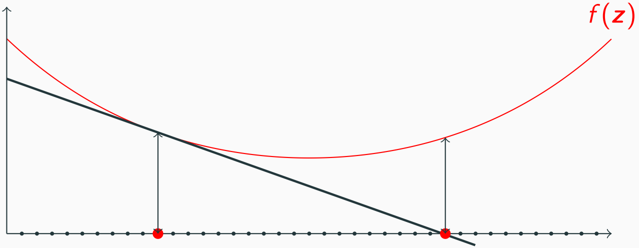

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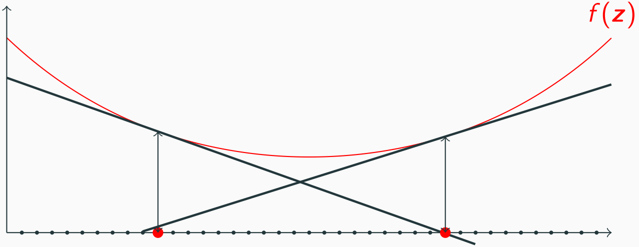

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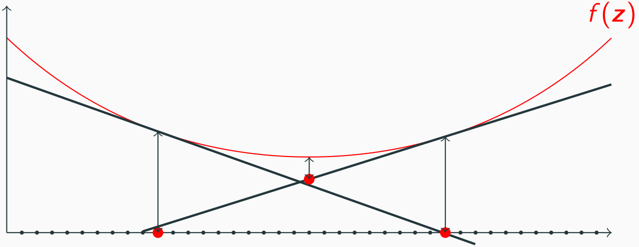

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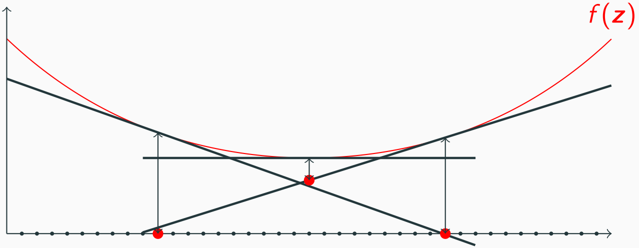

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- Solve by sub-gradient descent method, or transform to SOCP.
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- Randomly rounding relaxation $\boldsymbol{z}^\star$ according to $z_i \sim \mathrm{Bernoulli}(z_i^\star)$ gives a Boolean vector $\boldsymbol{z}$. How good is it?
- Let $\boldsymbol{z}$ be a random rounding of $\boldsymbol{z}^\star$. Then,

$$0 \leq f(\boldsymbol{z}) - f(\boldsymbol{z}^\star) \leq \epsilon$$

with probability at least

$$1 - |\mathcal{R}| \exp\left(\frac{-\epsilon^2}{\kappa}\right)$$

  - $|\mathcal{R}|$ is number of strictly fractional entries in $\boldsymbol{z}^\star$.
  - κ is a function of $|\mathcal{R}|$, problem data.

How does the approach perform on real data?

Sparse Empirical Risk Minimization Scalability

- For regression $f(\boldsymbol{z})$ is closed form, scales to $100,000$s of features.
- For classification, $f(\boldsymbol{z})$ is cheap, scales to $10,000$s of features.
- Outer-approximation algorithm is more accurate than ElasticNet, MCP, SCAD, and runtimes are comparable to Lasso.
- Code available: github.com/jeanpauphilet.

Sparse Portfolio Selection Scalability

Solves sparse portfolio selection problems with 1000s of securities[2].

Reference	Solution method	Size (no. securities)
Frangioni and Gentile ('09)	Perspective cut+SDP	400
Bonami and Lejeune ('09)	Nonlinear Branch-and-Bound	200
Gao and Li ('13)	SOCP relaxation Branch-and-Bound	300
Cui et al. ('13)	SOCP relaxation Branch-and-Bound	300
Zheng et. al. ('14)	SDP Branch-and-Bound	400
Frangioni et. al. ('16)	Aprox. Proj. Perspective Cut	400
Bertsimas and C-W ('18)	OA with γ-regularization	$3,200$

[2] See Bertsimas & Cory-Wright ('18), for an in-depth implementation.

Network Design Scalability

- $f(\boldsymbol{z})$ obtained by solving a quadratic program.
- Approach solves problems with 100s of nodes.
- Objective value 5% better than CPLEX for small problems, 40% better for large problems.

Main Messages and Highlights

- Don't feel married to big-M! We provide a **non-linear alternative** which often scales as well or better: substituting xz for x and adding a ridge regularizer.

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- By using strong duality, we derive a saddle-point reformulation, which is **exactly** solvable via an outer-approximation procedure.
- Our approach: outer-approximation+warm-start+random rounding **is scalable**.

Thanks for listening!

Questions?

Preprint available at: ryancorywright.github.io

Supplementary Material

Selected References

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