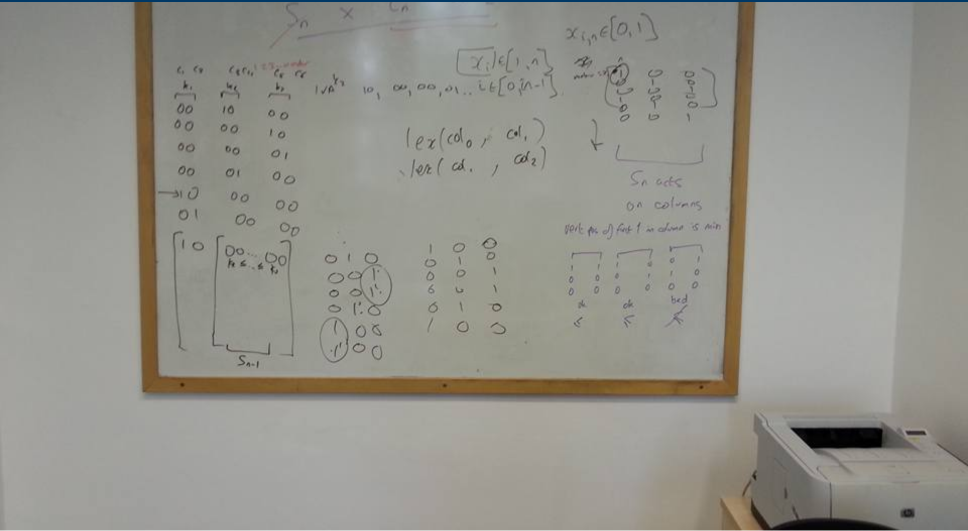
Enumeration of (unique reduced alternating) knot diagrams

Ciaran McCreesh, Alice Miller, Patrick Prosser, **Craig Reilly**, James Trimble

University of Glasgow



What is a knot?

- A *knot* is an embedding of the circle in $\mathbb{R}^3$.

What is a knot?

- A *knot* is an embedding of the circle in $\mathbb{R}^3$.
- An intuitive way to think about this is to consider a knot as a knotted piece of string with the ends glued together.

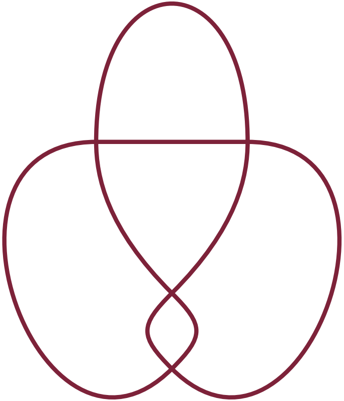
Drawing knots on paper

- A function $f : \mathbb{R}^3 \to \mathbb{R}^2$ where $f(x, y, z) = f(x, y)$, is called a *projection map*, and the image of a knot K under f is called the *projection* of K.
- Such a projection is often refered to as the *shadow* of K.

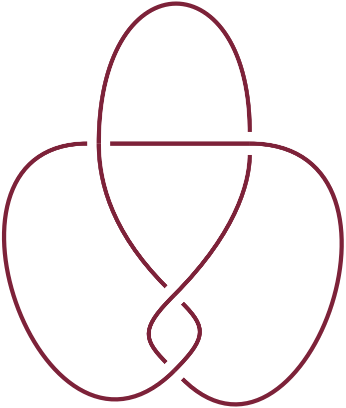
Drawing knots on paper

- Information regarding the orientation of arcs at crossings is given by leaving gaps in a knot's shadow..

Drawing knots on paper



Drawing knots on paper



Representations of knots

- Knot diagrams are really just 4-valent planar graphs.
  - The vertices in the graph correspond to the crossings in the knot diagram.
  - The arcs between vertices correspond to arcs between crossings in the knot diagram.
  - The arcs are decorated with their orientation at their source and target crossings.
- Other data structures familiar to computer scientists can be used, linked lists of crossings were popular in the 1950's.

Representations of knots

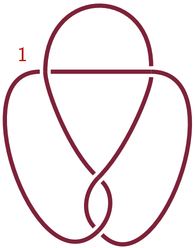
- The representations used by topologists are typically also used for representing knots in a computer.
- Examples are Dowker-Thistlethwait codes (DT codes), **Gauss codes**, braid representatives, Conway notation, and many more.

Representing knots with Gauss codes

- The strategy for representing a given knot (with n crossings) by a Gauss code is as follows.
  1. Label the crossings with the numbers 1 to n.
  2. Pick a point on the knot.
  3. Pick a direction and walk around the knot, writing out a list of the numbers you come to (with a negative sign indicating that a crossing was visited on an under strand). Stop when each number appears twice once with each sign.

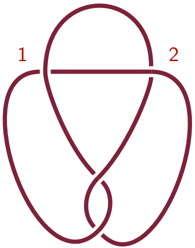
Representing knots with Gauss codes

- The strategy for representing a given knot (with n crossings) by a Gauss code is as follows.
  1. Label the crossings with the numbers 1 to n.
  2. Pick a point on the knot.
  3. Pick a direction and walk around the knot, writing out a list of the numbers you come to (with a negative sign indicating that a crossing was visited on an under strand). Stop when each number appears twice once with each sign.



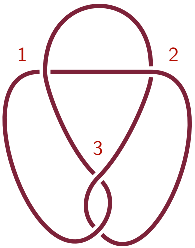
Representing knots with Gauss codes

- The strategy for representing a given knot (with n crossings) by a Gauss code is as follows.
  1. Label the crossings with the numbers 1 to n.
  2. Pick a point on the knot.
  3. Pick a direction and walk around the knot, writing out a list of the numbers you come to (with a negative sign indicating that a crossing was visited on an under strand). Stop when each number appears twice once with each sign.



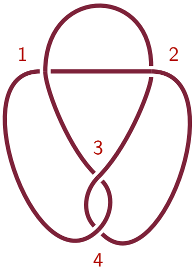
Representing knots with Gauss codes

- The strategy for representing a given knot (with n crossings) by a Gauss code is as follows.
  1. Label the crossings with the numbers 1 to n.
  2. Pick a point on the knot.
  3. Pick a direction and walk around the knot, writing out a list of the numbers you come to (with a negative sign indicating that a crossing was visited on an under strand). Stop when each number appears twice once with each sign.



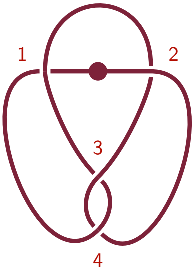
Representing knots with Gauss codes

- The strategy for representing a given knot (with n crossings) by a Gauss code is as follows.
  1. Label the crossings with the numbers 1 to n.
  2. Pick a point on the knot.
  3. Pick a direction and walk around the knot, writing out a list of the numbers you come to (with a negative sign indicating that a crossing was visited on an under strand). Stop when each number appears twice once with each sign.



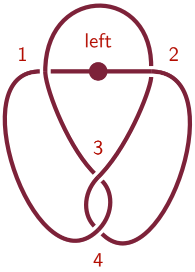
Representing knots with Gauss codes

- The strategy for representing a given knot (with n crossings) by a Gauss code is as follows.
  1. Label the crossings with the numbers 1 to n.
  2. Pick a point on the knot.
  3. Pick a direction and walk around the knot, writing out a list of the numbers you come to (with a negative sign indicating that a crossing was visited on an under strand). Stop when each number appears twice once with each sign.



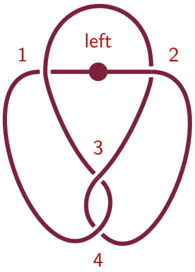
Representing knots with Gauss codes

- The strategy for representing a given knot (with n crossings) by a Gauss code is as follows.
  1. Label the crossings with the numbers 1 to n.
  2. Pick a point on the knot.
  3. Pick a direction and walk around the knot, writing out a list of the numbers you come to (with a negative sign indicating that a crossing was visited on an under strand). Stop when each number appears twice once with each sign.



Representing knots with Gauss codes

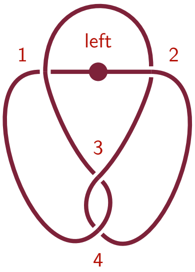
- The strategy for representing a given knot (with n crossings) by a Gauss code is as follows.
  1. Label the crossings with the numbers 1 to n.
  2. Pick a point on the knot.
  3. Pick a direction and walk around the knot, writing out a list of the numbers you come to (with a negative sign indicating that a crossing was visited on an under strand). Stop when each number appears twice once with each sign.



-1

Representing knots with Gauss codes

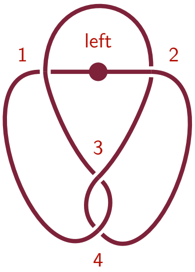
- The strategy for representing a given knot (with n crossings) by a Gauss code is as follows.
  1. Label the crossings with the numbers 1 to n.
  2. Pick a point on the knot.
  3. Pick a direction and walk around the knot, writing out a list of the numbers you come to (with a negative sign indicating that a crossing was visited on an under strand). Stop when each number appears twice once with each sign.



-1 , 4

Representing knots with Gauss codes

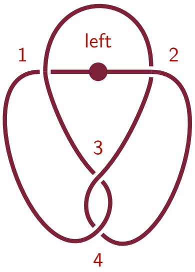
- The strategy for representing a given knot (with n crossings) by a Gauss code is as follows.
  1. Label the crossings with the numbers 1 to n.
  2. Pick a point on the knot.
  3. Pick a direction and walk around the knot, writing out a list of the numbers you come to (with a negative sign indicating that a crossing was visited on an under strand). Stop when each number appears twice once with each sign.



$-1\ ,\ 4\ ,\ -3$

Representing knots with Gauss codes

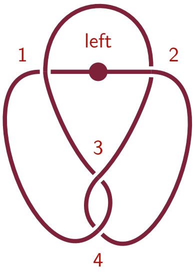
- The strategy for representing a given knot (with n crossings) by a Gauss code is as follows.
  1. Label the crossings with the numbers 1 to n.
  2. Pick a point on the knot.
  3. Pick a direction and walk around the knot, writing out a list of the numbers you come to (with a negative sign indicating that a crossing was visited on an under strand). Stop when each number appears twice once with each sign.



$-1\ ,4\ ,-3\ ,1$

Representing knots with Gauss codes

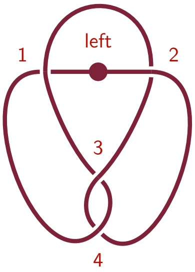
- The strategy for representing a given knot (with n crossings) by a Gauss code is as follows.
  1. Label the crossings with the numbers 1 to n.
  2. Pick a point on the knot.
  3. Pick a direction and walk around the knot, writing out a list of the numbers you come to (with a negative sign indicating that a crossing was visited on an under strand). Stop when each number appears twice once with each sign.



$-1\ ,4\ ,-3\ ,1\ ,-2$

Representing knots with Gauss codes

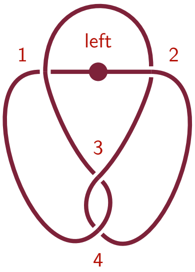
- The strategy for representing a given knot (with n crossings) by a Gauss code is as follows.
  1. Label the crossings with the numbers 1 to n.
  2. Pick a point on the knot.
  3. Pick a direction and walk around the knot, writing out a list of the numbers you come to (with a negative sign indicating that a crossing was visited on an under strand). Stop when each number appears twice once with each sign.



$-1\ , 4\ , -3\ , 1\ , -2\ , 3$

Representing knots with Gauss codes

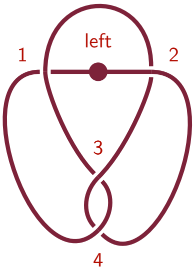
- The strategy for representing a given knot (with n crossings) by a Gauss code is as follows.
  1. Label the crossings with the numbers 1 to n.
  2. Pick a point on the knot.
  3. Pick a direction and walk around the knot, writing out a list of the numbers you come to (with a negative sign indicating that a crossing was visited on an under strand). Stop when each number appears twice once with each sign.



-1 , 4 , -3 , 1 , -2 , 3 , -4

Representing knots with Gauss codes

- The strategy for representing a given knot (with n crossings) by a Gauss code is as follows.
  1. Label the crossings with the numbers 1 to n.
  2. Pick a point on the knot.
  3. Pick a direction and walk around the knot, writing out a list of the numbers you come to (with a negative sign indicating that a crossing was visited on an under strand). Stop when each number appears twice once with each sign.



$-1\ ,4\ ,-3\ ,1\ ,-2\ ,3\ ,-4\ ,2$

Knot enumeration by Gauss codes

- There are three sensible questions to ask:
  - Do all knots give rise to a Gauss code?
  - Can two Gauss codes correspond to the same knot?
  - Do all Gauss codes represent a knot?

Knot enumeration by Gauss codes

- There are three sensible questions to ask:
  - Do all knots give rise to a Gauss code? Yes.
  - Can two Gauss codes correspond to the same knot?
  - Do all Gauss codes represent a knot?

Knot enumeration by Gauss codes

- There are three sensible questions to ask:
    - Do all knots give rise to a Gauss code? Yes. Trivially.
    - Can two Gauss codes correspond to the same knot?
    - Do all Gauss codes represent a knot?

Knot enumeration by Gauss codes

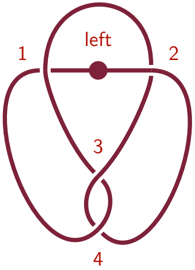
- There are three sensible questions to ask:
  - Do all knots give rise to a Gauss code? Yes. Trivially.
  - Can two Gauss codes correspond to the same knot? Yes.
  - Do all Gauss codes represent a knot?

Knot enumeration by Gauss codes

- There are three sensible questions to ask:
  - Do all knots give rise to a Gauss code? Yes. Trivially.
  - Can two Gauss codes correspond to the same knot? Yes.
  - Do all Gauss codes represent a knot? No!

A candidate for a canonical Gauss code

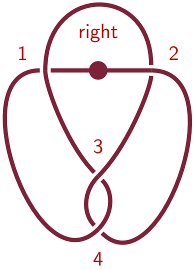
$-1, 4, -3, 1\ -2, 3, -4, 2$



A candidate for a canonical Gauss code

-1, 4, -3, 1 -2, 3, -4, 2

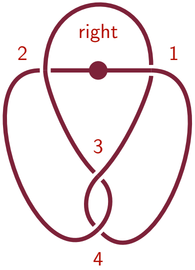
- 2, -4, 3, -2, 1, -3, 4, -1



A candidate for a canonical Gauss code

$-1, 4, -3, 1 -2, 3, -4, 2$

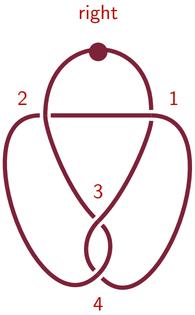
- $2, -4, 3, -2, 1, -3, 4, -1$
- $1, -4, 3, -1, 2, -3, 4, -2$



A candidate for a canonical Gauss code

$-1, 4, -3, 1 -2, 3, -4, 2$

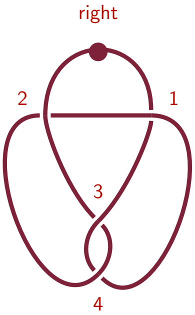
- $2, -4, 3, -2, 1, -3, 4, -1$
- $1, -4, 3, -1, 2, -3, 4, -2$
- $-1, 3, -4, 1 -2, 4, -3, 2$



A candidate for a canonical Gauss code

$-1, 4, -3, 1\ -2, 3, -4, 2$

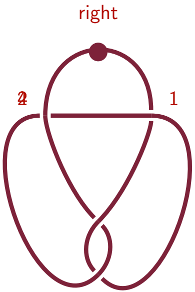
- $2, -4, 3, -2, 1, -3, 4, -1$
- $1, -4, 3, -1, 2, -3, 4, -2$
- $-1, 3, -4, 1\ -2, 4, -3, 2$
- Our candidate is a code with the lexicographically minimum shadow (where a shadow corresponds to an unsigned code). For this knot the shadow is:
- 1, 2, 3, 1, 4, 3, 2, 4.



A candidate for a canonical Gauss code

$-1, 4, -3, 1 -2, 3, -4, 2$

- $2, -4, 3, -2, 1, -3, 4, -1$
- $1, -4, 3, -1, 2, -3, 4, -2$
- $-1, 3, -4, 1 -2, 4, -3, 2$
- Our candidate is a code with the lexicographically minimum shadow (where a shadow corresponds to an unsigned code). For this knot the shadow is:
- 1, 2, 3, 1, 4, 3, 2, 4.
- For which the diagram now shows the labelling giving rise to the signed code:
- $-1, 2, -3, 1, -4, 3, -2, 4.$



A candidate for a canonical Gauss code

- Is this new?

A candidate for a canonical Gauss code

- Is this new? Kinda.

A candidate for a canonical Gauss code

- Is this new?
- Topologists don't seem to care about labelling a knot in a canonical manner when determining a knot's Gauss code.
- But such a lexicographically minium representation of a the shadow code is necissary in the input of an algorithm which follows.

A candidate for a canonical Gauss code

- Is this new?
- Topologists don't seem to care about labelling a knot in a canonical manner when determining a knot's Gauss code.
- But such a lexicographically minium representation of a the shadow code is necissary in the input of an algorithm which follows.

```
Name      Gauss Notation
3_1       -1, 3, -2, 1, -3, 2
4_1             1, -4, 3, -1, 2, -3, 4, -2
5_1            -1, 4, -2, 5, -3, 1, -4, 2, -5, 3
5_2       -1, 5, -2, 1, -3, 4, -5, 2, -4, 3
6_1       -1, 4, -3, 1, -5, 6, -2, 3, -4, 2, -6, 5
6_2       -1, 4, -3, 1, -2, 6, -5, 3, -4, 2, -6, 5
6_3        1, -6, 2, -1, 4, -5, 6, -2, 3, -4, 5, -3
```

Symmetries involved in Gauss code enumeration

- stuff