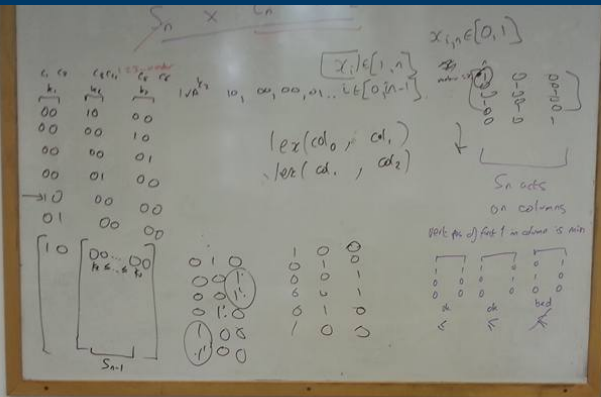


Enumeration of (unique reduced alternating) knot diagrams

Ciaran McCreesh, Alice Miller, Patrick Prosser,
Craig Reilly, James Trimble



University of Glasgow



What is a knot?

- A *knot* is an embedding of the circle in \mathbb{R}^3 .

What is a knot?

- A *knot* is an embedding of the circle in \mathbb{R}^3 .
- An intuitive way to think about this is to consider a knot as a knotted piece of string with the ends glued together.

Drawing knots on paper

- A function $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ where $f(x, y, z) = f(x, y)$, is called a *projection map*, and the image of a knot K under f is called the *projection* of K .
- Such a projection is often referred to as the *shadow* of K .

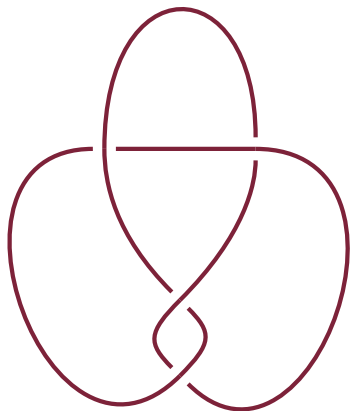
Drawing knots on paper

- Information regarding the orientation of arcs at crossings is given by leaving gaps in a knot's shadow..

Drawing knots on paper



Drawing knots on paper



Representations of knots

- Knot diagrams are really just 4-valent planar graphs.
 - The vertices in the graph correspond to the crossings in the knot diagram.
 - The arcs between vertices correspond to arcs between crossings in the knot diagram.
 - The arcs are decorated with their orientation at their source and target crossings.
- Other data structures familiar to computer scientists can be used, linked lists of crossings were popular in the 1950's.

Representations of knots

- The representations used by topologists are typically also used for representing knots in a computer.
- Examples are Dowker-Thistlethwait codes (DT codes), **Gauss codes**, braid representatives, Conway notation, and many more.

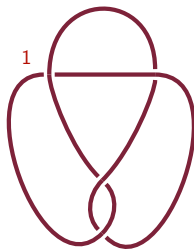
Representing knots with Gauss codes

- The strategy for representing a given knot (with n crossings) by a Gauss code is as follows.
 - 1 Label the crossings with the numbers 1 to n .
 - 2 Pick a point on the knot.
 - 3 Pick a direction and walk around the knot, writing out a list of the numbers you come to (with a negative sign indicating that a crossing was visited on an under strand). Stop when each number appears twice once with each sign.



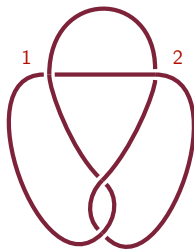
Representing knots with Gauss codes

- The strategy for representing a given knot (with n crossings) by a Gauss code is as follows.
 - 1 Label the crossings with the numbers 1 to n .
 - 2 Pick a point on the knot.
 - 3 Pick a direction and walk around the knot, writing out a list of the numbers you come to (with a negative sign indicating that a crossing was visited on an under strand). Stop when each number appears twice once with each sign.



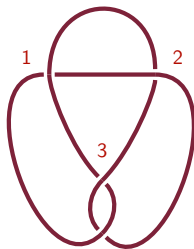
Representing knots with Gauss codes

- The strategy for representing a given knot (with n crossings) by a Gauss code is as follows.
 - 1 Label the crossings with the numbers 1 to n .
 - 2 Pick a point on the knot.
 - 3 Pick a direction and walk around the knot, writing out a list of the numbers you come to (with a negative sign indicating that a crossing was visited on an under strand). Stop when each number appears twice once with each sign.



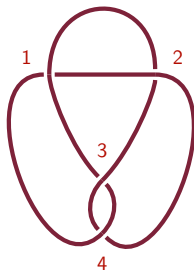
Representing knots with Gauss codes

- The strategy for representing a given knot (with n crossings) by a Gauss code is as follows.
 - 1 Label the crossings with the numbers 1 to n .
 - 2 Pick a point on the knot.
 - 3 Pick a direction and walk around the knot, writing out a list of the numbers you come to (with a negative sign indicating that a crossing was visited on an under strand). Stop when each number appears twice once with each sign.



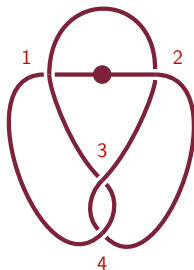
Representing knots with Gauss codes

- The strategy for representing a given knot (with n crossings) by a Gauss code is as follows.
 - 1 Label the crossings with the numbers 1 to n .
 - 2 Pick a point on the knot.
 - 3 Pick a direction and walk around the knot, writing out a list of the numbers you come to (with a negative sign indicating that a crossing was visited on an under strand). Stop when each number appears twice once with each sign.



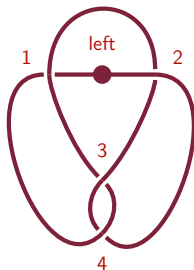
Representing knots with Gauss codes

- The strategy for representing a given knot (with n crossings) by a Gauss code is as follows.
 - 1 Label the crossings with the numbers 1 to n .
 - 2 Pick a point on the knot.
 - 3 Pick a direction and walk around the knot, writing out a list of the numbers you come to (with a negative sign indicating that a crossing was visited on an under strand). Stop when each number appears twice once with each sign.



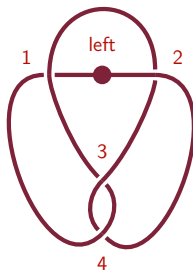
Representing knots with Gauss codes

- The strategy for representing a given knot (with n crossings) by a Gauss code is as follows.
 - 1 Label the crossings with the numbers 1 to n .
 - 2 Pick a point on the knot.
 - 3 Pick a direction and walk around the knot, writing out a list of the numbers you come to (with a negative sign indicating that a crossing was visited on an under strand). Stop when each number appears twice once with each sign.



Representing knots with Gauss codes

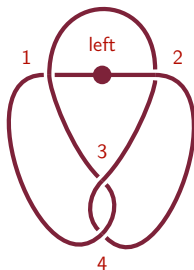
- The strategy for representing a given knot (with n crossings) by a Gauss code is as follows.
 - 1 Label the crossings with the numbers 1 to n .
 - 2 Pick a point on the knot.
 - 3 Pick a direction and walk around the knot, writing out a list of the numbers you come to (with a negative sign indicating that a crossing was visited on an under strand). Stop when each number appears twice once with each sign.



—1

Representing knots with Gauss codes

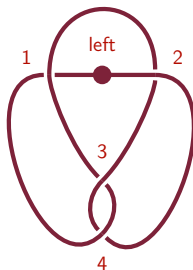
- The strategy for representing a given knot (with n crossings) by a Gauss code is as follows.
 - 1 Label the crossings with the numbers 1 to n .
 - 2 Pick a point on the knot.
 - 3 Pick a direction and walk around the knot, writing out a list of the numbers you come to (with a negative sign indicating that a crossing was visited on an under strand). Stop when each number appears twice once with each sign.



$-1, 4$

Representing knots with Gauss codes

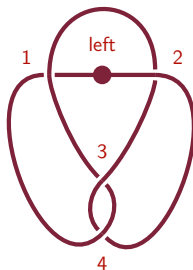
- The strategy for representing a given knot (with n crossings) by a Gauss code is as follows.
 - 1 Label the crossings with the numbers 1 to n .
 - 2 Pick a point on the knot.
 - 3 Pick a direction and walk around the knot, writing out a list of the numbers you come to (with a negative sign indicating that a crossing was visited on an under strand). Stop when each number appears twice once with each sign.



$-1, 4, -3$

Representing knots with Gauss codes

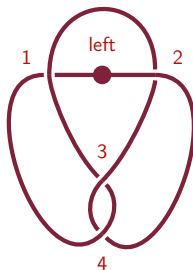
- The strategy for representing a given knot (with n crossings) by a Gauss code is as follows.
 - 1 Label the crossings with the numbers 1 to n .
 - 2 Pick a point on the knot.
 - 3 Pick a direction and walk around the knot, writing out a list of the numbers you come to (with a negative sign indicating that a crossing was visited on an under strand). Stop when each number appears twice once with each sign.



$-1, 4, -3, 1$

Representing knots with Gauss codes

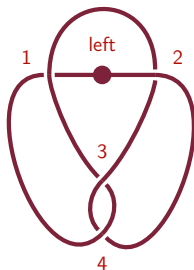
- The strategy for representing a given knot (with n crossings) by a Gauss code is as follows.
 - 1 Label the crossings with the numbers 1 to n .
 - 2 Pick a point on the knot.
 - 3 Pick a direction and walk around the knot, writing out a list of the numbers you come to (with a negative sign indicating that a crossing was visited on an under strand). Stop when each number appears twice once with each sign.



-1 , 4 , -3 , 1 , -2

Representing knots with Gauss codes

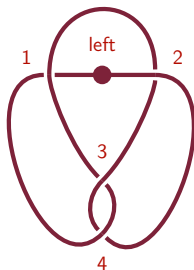
- The strategy for representing a given knot (with n crossings) by a Gauss code is as follows.
 - 1 Label the crossings with the numbers 1 to n .
 - 2 Pick a point on the knot.
 - 3 Pick a direction and walk around the knot, writing out a list of the numbers you come to (with a negative sign indicating that a crossing was visited on an under strand). Stop when each number appears twice once with each sign.



-1 , 4 , -3 , 1 , -2 , 3

Representing knots with Gauss codes

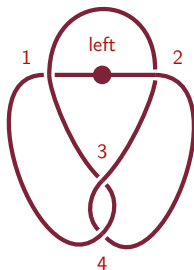
- The strategy for representing a given knot (with n crossings) by a Gauss code is as follows.
 - 1 Label the crossings with the numbers 1 to n .
 - 2 Pick a point on the knot.
 - 3 Pick a direction and walk around the knot, writing out a list of the numbers you come to (with a negative sign indicating that a crossing was visited on an under strand). Stop when each number appears twice once with each sign.



$-1, 4, -3, 1, -2, 3, -4$

Representing knots with Gauss codes

- The strategy for representing a given knot (with n crossings) by a Gauss code is as follows.
 - 1 Label the crossings with the numbers 1 to n .
 - 2 Pick a point on the knot.
 - 3 Pick a direction and walk around the knot, writing out a list of the numbers you come to (with a negative sign indicating that a crossing was visited on an under strand). Stop when each number appears twice once with each sign.



$-1, 4, -3, 1, -2, 3, -4, 2$

Knot enumeration by Gauss codes

- There are three sensible questions to ask:
 - Do all knots give rise to a Gauss code?
 - Can two Gauss codes correspond to the same knot?
 - Do all Gauss codes represent a knot?

Knot enumeration by Gauss codes

- There are three sensible questions to ask:
 - Do all knots give rise to a Gauss code? Yes.
 - Can two Gauss codes correspond to the same knot?
 - Do all Gauss codes represent a knot?

Knot enumeration by Gauss codes

- There are three sensible questions to ask:
 - Do all knots give rise to a Gauss code? Yes. Trivially.
 - Can two Gauss codes correspond to the same knot?
 - Do all Gauss codes represent a knot?

Knot enumeration by Gauss codes

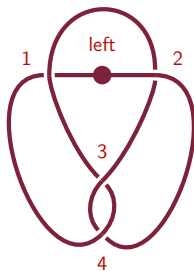
- There are three sensible questions to ask:
 - Do all knots give rise to a Gauss code? Yes. Trivially.
 - Can two Gauss codes correspond to the same knot? Yes.
 - Do all Gauss codes represent a knot?

Knot enumeration by Gauss codes

- There are three sensible questions to ask:
 - Do all knots give rise to a Gauss code? Yes. Trivially.
 - Can two Gauss codes correspond to the same knot? Yes.
 - Do all Gauss codes represent a knot? No!

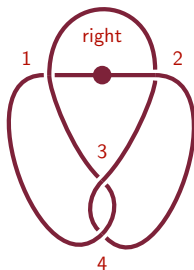
A candidate for a canonical Gauss code

$-1, 4, -3, 1 -2, 3, -4, 2$



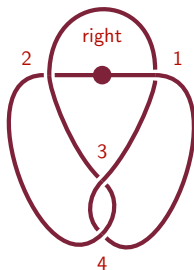
A candidate for a canonical Gauss code

$-1, 4, -3, 1 -2, 3, -4, 2$
 $2, -4, 3, -2, 1, -3, 4, -1$



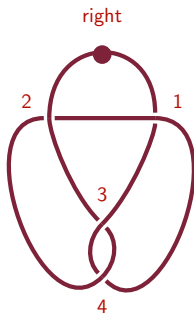
A candidate for a canonical Gauss code

$-1, 4, -3, 1 -2, 3, -4, 2$
 $2, -4, 3, -2, 1, -3, 4, -1$
 $1, -4, 3, -1, 2, -3, 4, -2$



A candidate for a canonical Gauss code

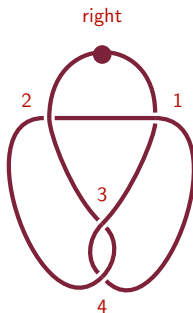
$-1, 4, -3, 1 -2, 3, -4, 2$
 $2, -4, 3, -2, 1, -3, 4, -1$
 $1, -4, 3, -1, 2, -3, 4, -2$
 $-1, 3, -4, 1 -2, 4, -3, 2$



A candidate for a canonical Gauss code

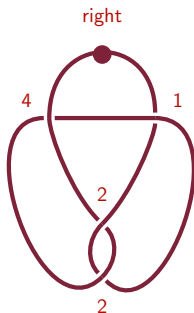
- Our candidate is a code with the lexicographically minimum shadow (where a shadow corresponds to an unsigned code). For this knot the shadow is:

1, 2, 3, 1, 4, 3, 2, 4.



A candidate for a canonical Gauss code

- Our candidate is a code with the lexicographically minimum shadow (where a shadow corresponds to an unsigned code). For this knot the shadow is:
1, 2, 3, 1, 4, 3, 2, 4.
- For which the diagram now shows the labelling giving rise to the signed code:
-1, 2, -3, 1, -4, 3, -2, 4.



A candidate for a canonical Gauss code

- Is this new?

A candidate for a canonical Gauss code

- Is this new? Kinda.

A candidate for a canonical Gauss code

- Is this new? Kinda.
- Topologists don't seem to care about labelling a knot in a canonical manner when determining a knot's Gauss code.
- But such a lexicographically minimum representation of a the shadow code is necessary in the input of an algorithm which follows.

A candidate for a canonical Gauss code

Name	Gauss Notation
3_1	-1, 3, -2, 1, -3, 2
4_1	1, -4, 3, -1, 2, -3, 4, -2
5_1	-1, 4, -2, 5, -3, 1, -4, 2, -5, 3
5_2	-1, 5, -2, 1, -3, 4, -5, 2, -4, 3
6_1	-1, 4, -3, 1, -5, 6, -2, 3, -4, 2, -6, 5
6_2	-1, 4, -3, 1, -2, 6, -5, 3, -4, 2, -6, 5
6_3	1, -6, 2, -1, 4, -5, 6, -2, 3, -4, 5, -3
7_1	-1, 5, -2, 6, -3, 7, -4, 1, -5, 2, -6, 3, -7, 4
7_2	-1, 7, -2, 1, -3, 6, -4, 5, -7, 2, -5, 4, -6, 3
7_3	1, -6, 2, -7, 5, -1, 3, -4, 6, -2, 7, -5, 4
7_4	1, -5, 6, -7, 2, -1, 3, -4, 7, -6, 5, -2, 4
7_5	-1, 7, -2, 1, -3, 5, -4, 6, -7, 2, -6, 3, -5, 4
7_6	-1, 4, -5, 3, -6, 5, -4, 2, -7, 6, -3, 1, -5, 4
7_7	1, 4, -2, 1, -2, 7, -6, 2, -4, 2, -5, 6, -3, 4

Symmetries involved in Gauss code enumeration

- stuff