Enumeration of (unique reduced alternating) knot diagrams

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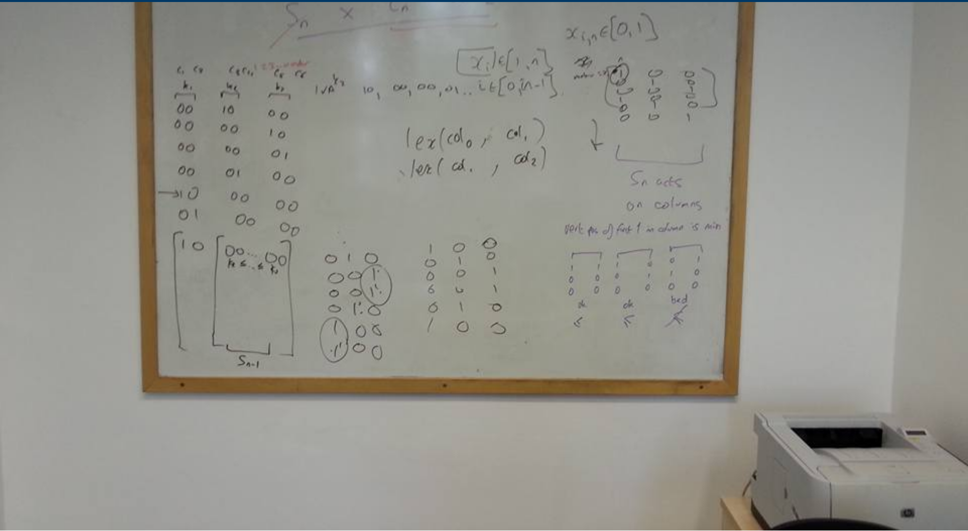
University of Glasgow



What is a knot?

- A *knot* is an embedding of the circle in $\mathbb{R}^3$.

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- A *knot* is an embedding of the circle in $\mathbb{R}^3$.
- An intuitive way to think about this is to consider a knot as a knotted piece of string with the ends glued together.

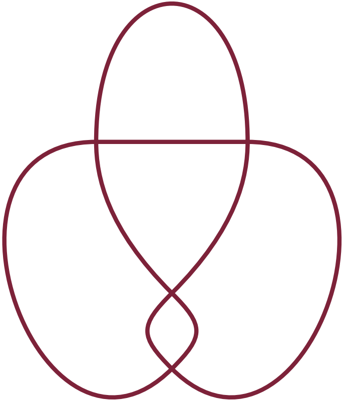
Drawing knots on paper

- A function $f : \mathbb{R}^3 \to \mathbb{R}^2$ where $f(x, y, z) = f(x, y)$, is called a *projection map*, and the image of a knot K under f is called the *projection* of K.
- Such a projection is often refered to as the *shadow* of K.

Drawing knots on paper

- Information regarding the orientation of arcs at crossings is given by leaving gaps in a knot's shadow..

Drawing knots on paper



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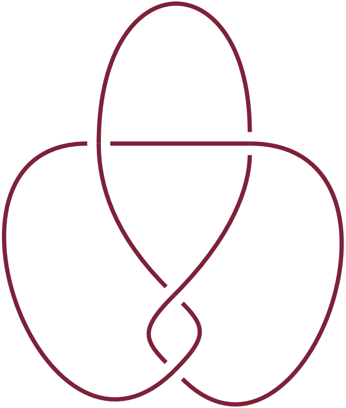

Representations of knots

- Knot diagrams are really just 4-valent planar graphs.
  - The vertices in the graph correspond to the crossings in the knot diagram.
  - The arcs between vertices correspond to arcs between crossings in the knot diagram.
  - The arcs are decorated with their orientation at their source and target crossings.
- Other data structures familiar to computer scientists can be used, linked lists of crossings were popular in the 1950's.

Representations of knots

- The representations used by topologists are typically also used for representing knots in a computer.
- Examples are Dowker-Thistlethwait codes (DT codes), **Gauss codes**, braid representatives, Conway notation, and many more.

Representing knots with Gauss codes

- The strategy for representing a given knot (with n crossings) by a Gauss code is as follows.
  1. Label the crossings with the numbers 1 to n.
  2. Pick a point on the knot.
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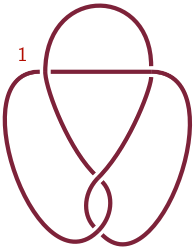

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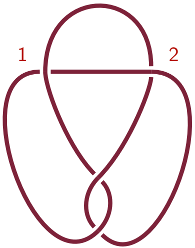

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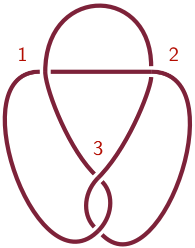

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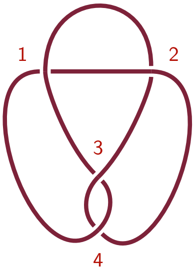

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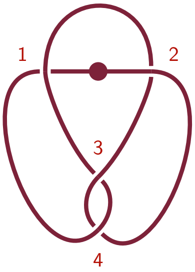

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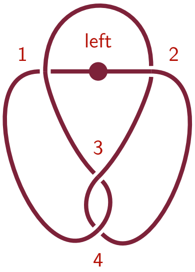

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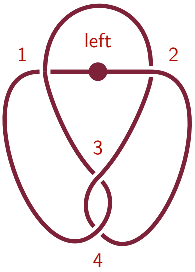

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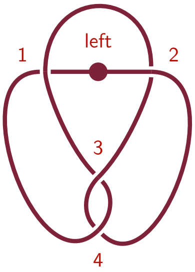

$-1\ \ , 4$

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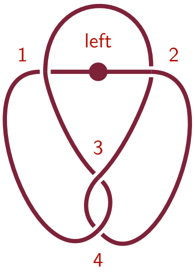

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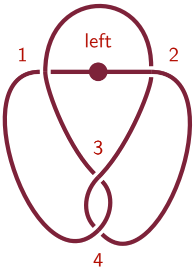

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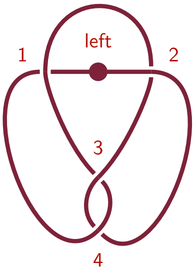

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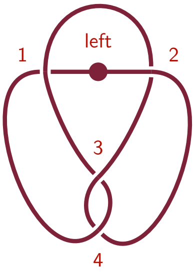

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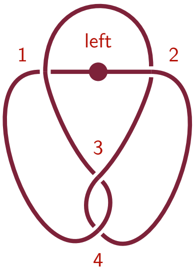

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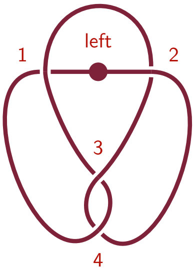

-1 , 4 , -3 , 1 , -2 , 3 , -4 , 2

Knot enumeration by Gauss codes

- There are three sensible questions to ask:
  - Do all knots give rise to a Gauss code?
  - Can two Gauss codes correspond to the same knot?
  - Do all Gauss codes represent a knot?

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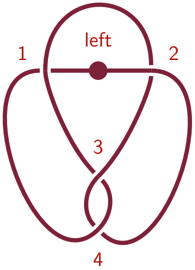
A candidate for a canonical Gauss code

$-1, 4, -3, 1, -2, 3, -4, 2$



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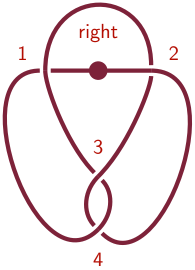
$-1, 4, -3, 1, -2, 3, -4, 2$

$2, -4, 3, -2, 1, -3, 4, -1$



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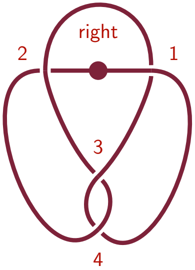
$$-1, 4, -3, 1, -2, 3, -4, 2$$
$$2, -4, 3, -2, 1, -3, 4, -1$$
$$1, -4, 3, -1, 2, -3, 4, -2$$



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$2, -4, 3, -2, 1, -3, 4, -1$

$1, -4, 3, -1, 2, -3, 4, -2$

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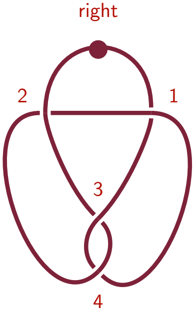

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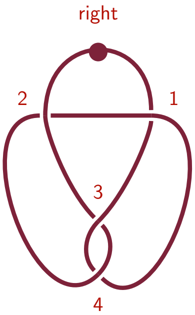

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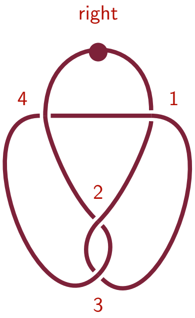
- For which the diagram now shows the labelling giving rise to the signed code:

  -1, 2, -3, 1, -4, 3, -2, 4.



A candidate for a canonical Gauss code

- Is this new?

A candidate for a canonical Gauss code

- Is this new? Kinda.

A candidate for a canonical Gauss code

- Is this new? Kinda.
- Topologists don't seem to care about labelling a knot in a canonical manner when determining a knot's Gauss code.
- But such a lexicographically minium representation of the unsigned Gauss code is necissary in the input of an algorithm which follows.

A candidate for a canonical Gauss code

Name	Gauss Notation
3_1	−1, 3, −2, 1, −3, 2
4_1	1, −4, 3, −1, 2, −3, 4, −2
5_1	−1, 4, −2, 5, −3, 1, −4, 2, −5, 3
5_2	−1, 5, −2, 1, −3, 4, −5, 2, −4, 3
6_1	−1, 4, −3, 1, −5, 6, −2, 3, −4, 2, −6, 5
6_2	−1, 4, −3, 1, −2, 6, −5, 3, −4, 2, −6, 5
6_3	1, −6, 2, −1, 4, −5, 6, −2, 3, −4, 5, −3

Symmetries involved in Gauss code enumeration

- The choices involved in each stage of obtaining a Gauss code from a given knot lead to rich symmetries in the problem of enumerating knots by Gauss codes.
- As we have seen many Gauss codes represent the same knot.
- As an example case: the smallest prime knot, the *trefoil*, is represented by WORK THIS OUT!

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