Enumeration of (unique reduced alternating) knot diagrams

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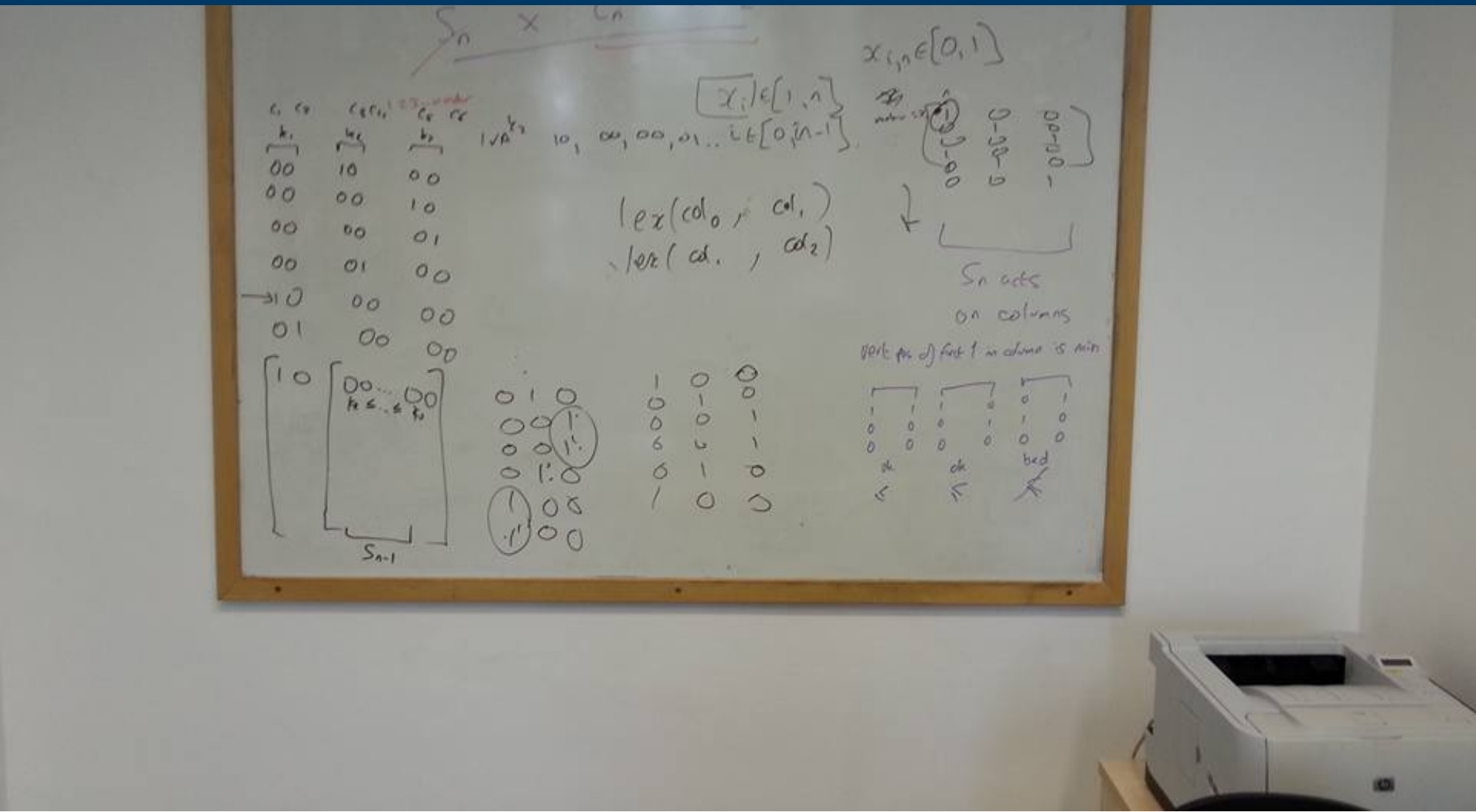
University of Glasgow



What is a knot?

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- A *knot* is an embedding of the circle in $\mathbb{R}^3$.
- An intuitive way to think about this is to consider a knot as a knotted piece of string with the ends glued together.

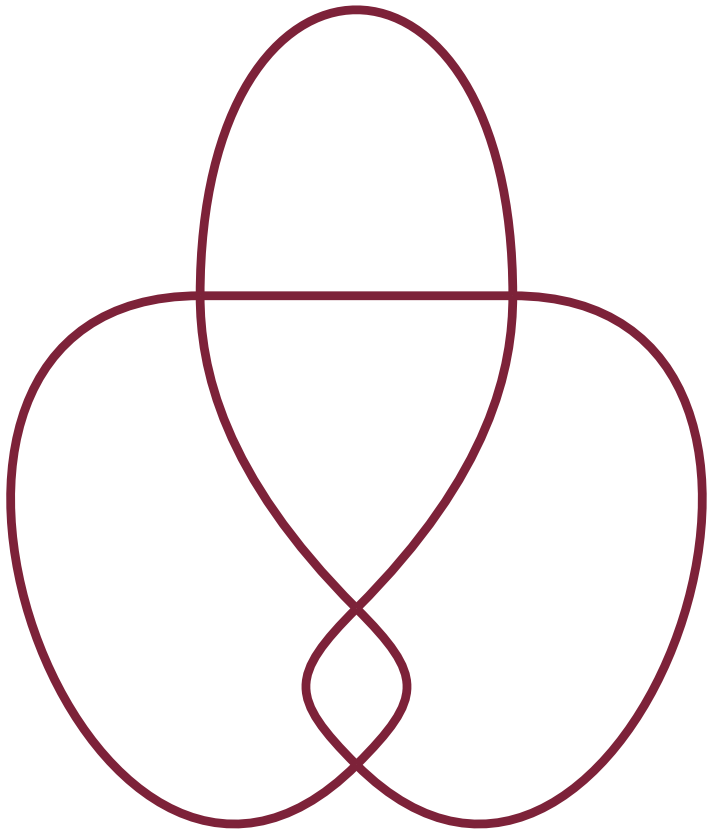
Drawing knots on paper

- A function $f : \mathbb{R}^3 \to \mathbb{R}^2$ where $f(x, y, z) = f(x, y)$, is called a *projection map*, and the image of a knot K under f is called the *projection* of K.
- Such a projection is often refered to as the *shadow* of K.

Drawing knots on paper

- Information regarding the orientation of arcs at crossings is given by leaving gaps in a knot's shadow..

Drawing knots on paper



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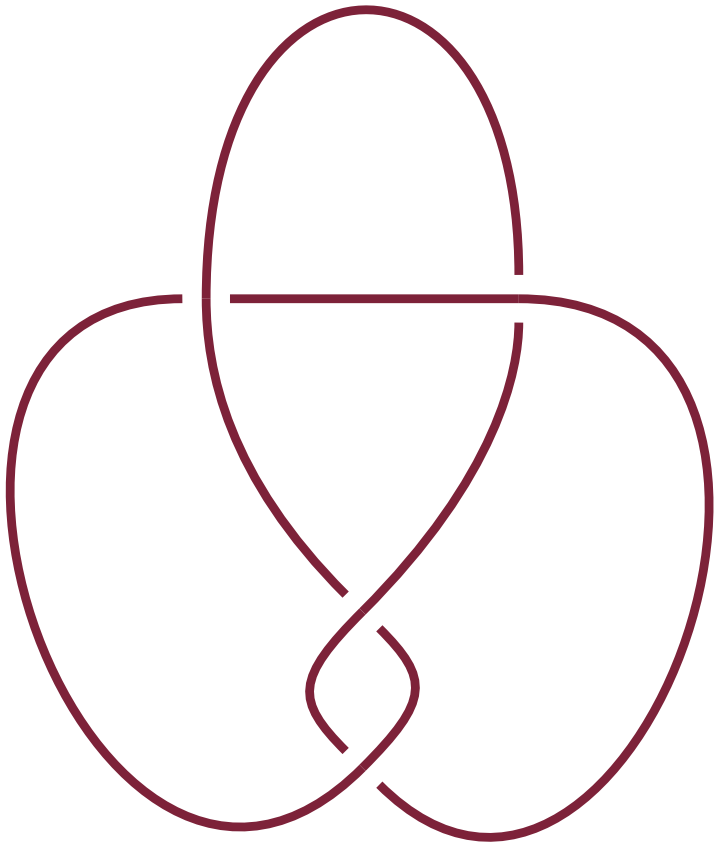

Representations of knots

- Knot diagrams are really just 4-valent planar graphs.
  - The vertices in the graph correspond to the crossings in the knot diagram.
  - The arcs between vertices correspond to arcs between crossings in the knot diagram.
  - The arcs are decorated with their orientation at their source and target crossings.
- Other data structures familiar to computer scientists can be used, linked lists of crossings were popular in the 1950's.

Representations of knots

- The representations used by topologists are typically also used for representing knots in a computer.
- Examples are Dowker-Thistlethwait codes (DT codes), **Gauss codes**, braid representatives, Conway notation, and many more.

Representing knots with Gauss codes

- The strategy for representing a given knot (with n crossings) by a Gauss code is as follows.
  1. Label the crossings with the numbers 1 to n.
  2. Pick a point on the knot.
  3. Pick a direction and walk around the knot, writing out a list of the numbers you come to (with a negative sign indicating that a crossing was visited on an under strand). Stop when each number appears twice once with each sign.

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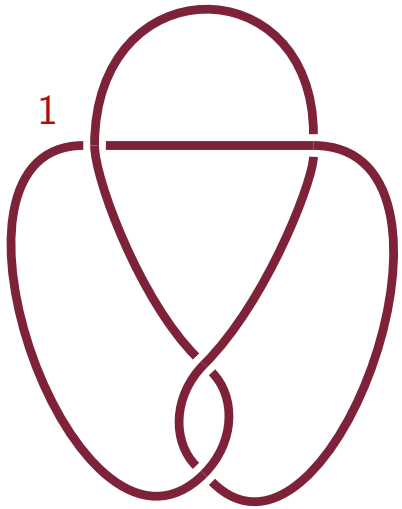

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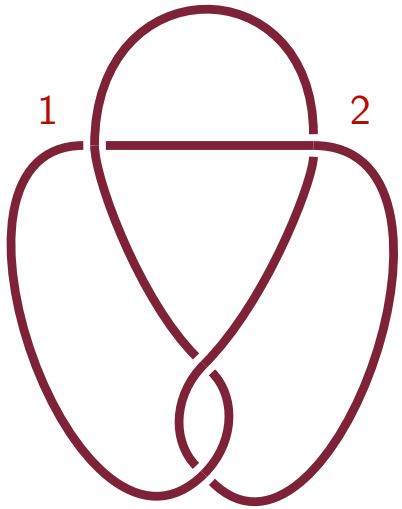

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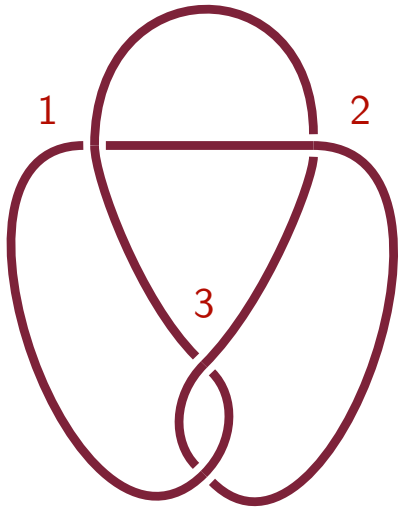

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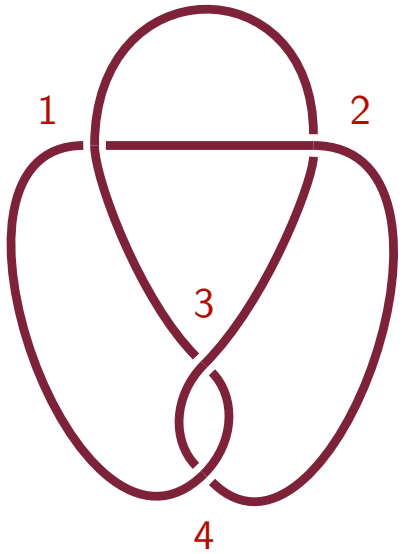

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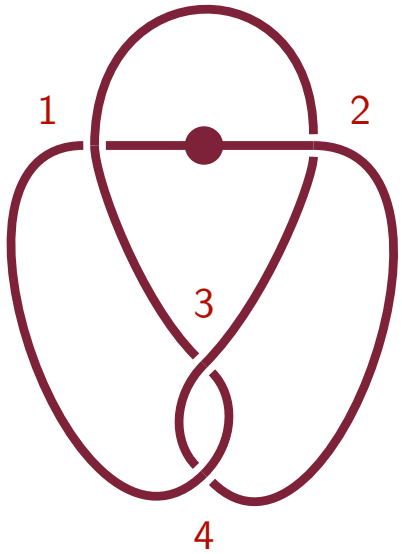

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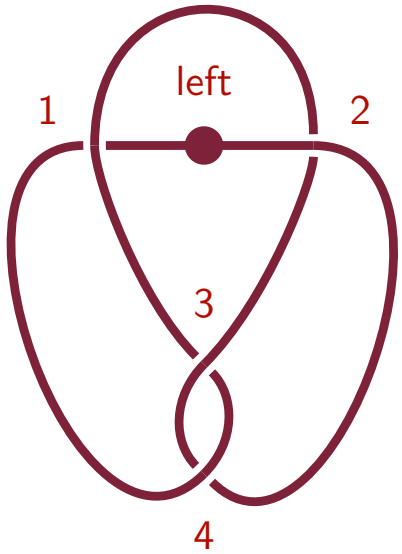

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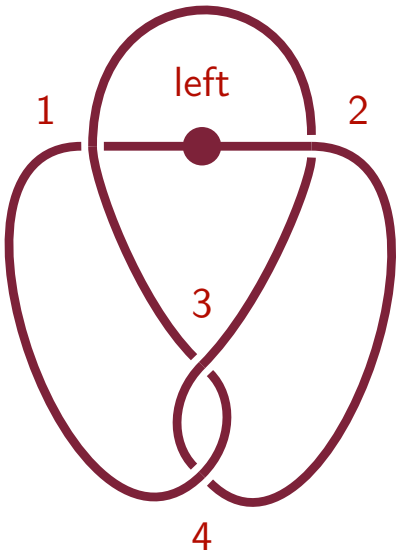

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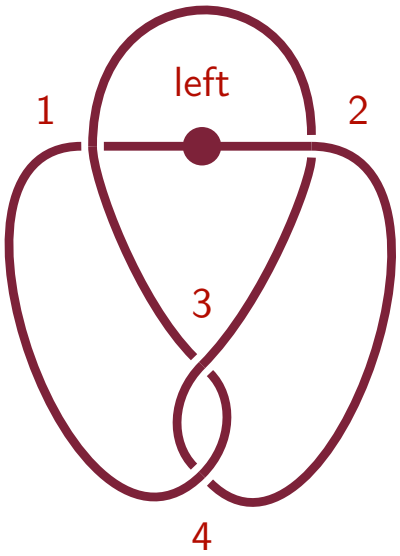

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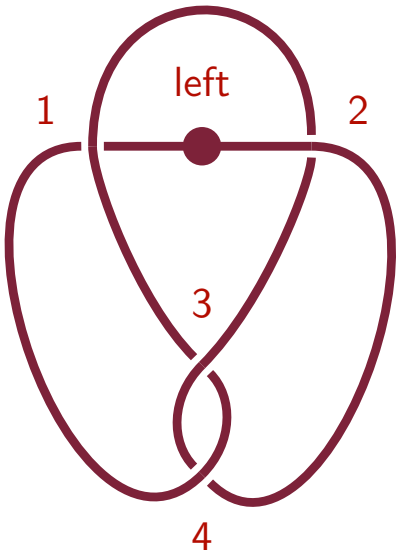

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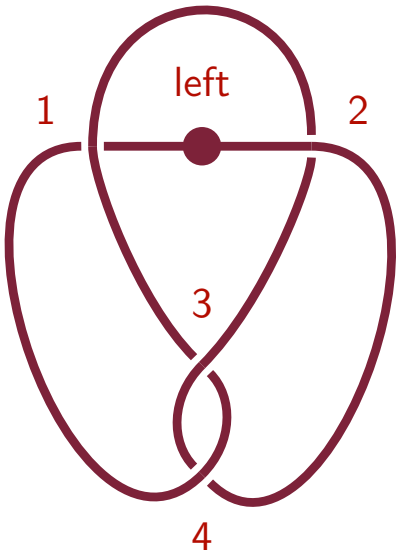

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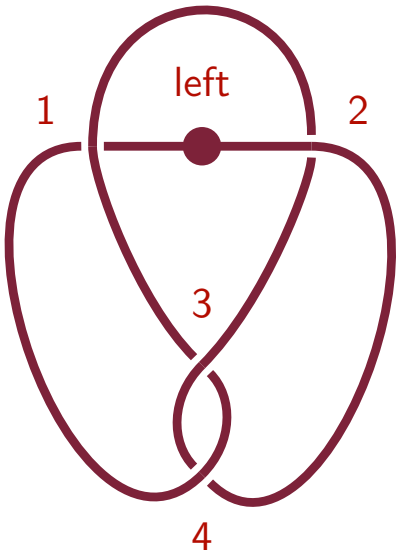

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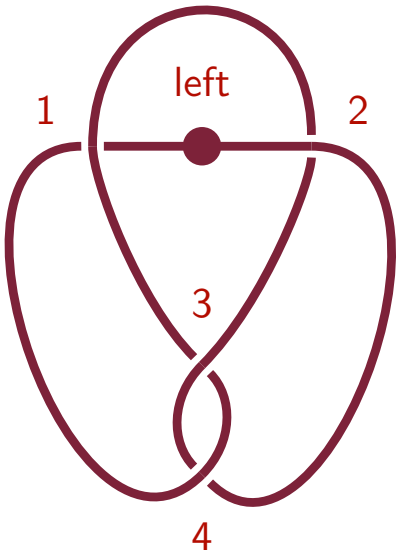

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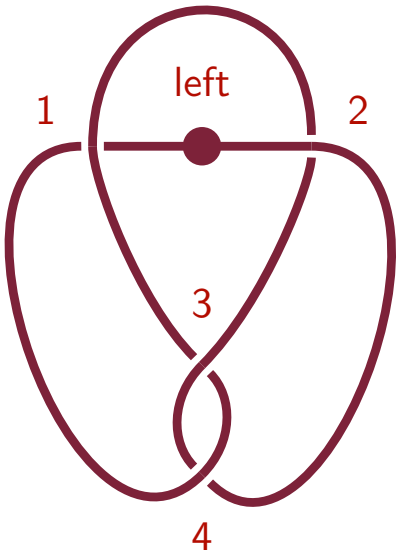

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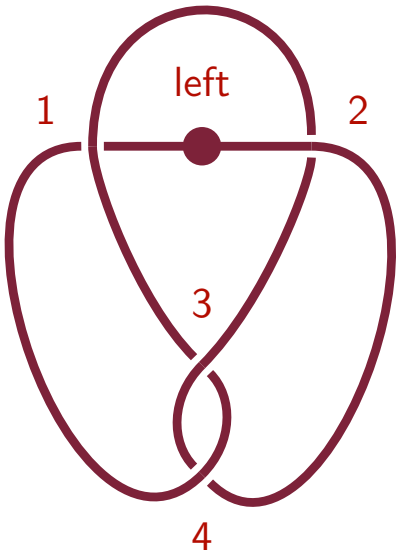

$$-1\ ,\ 4\ ,\ -3\ ,\ 1\ ,\ -2\ ,\ 3\ ,\ -4\ ,\ 2$$

Knot enumeration by Gauss codes

- There are three sensible questions to ask:
  - Do all knots give rise to a Gauss code?
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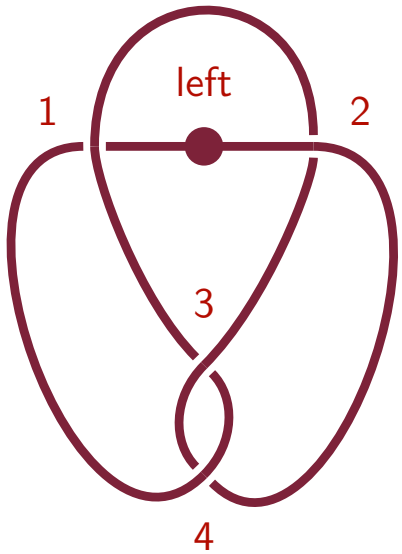
A candidate for a canonical Gauss code

- $-1, 4, -3, 1\ -2, 3, -4, 2$



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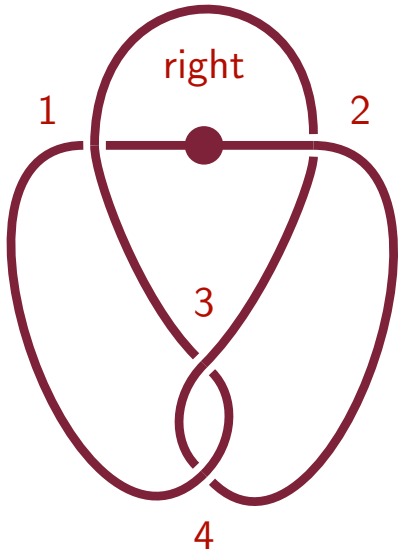
- $2, -4, 3, -2, 1, -3, 4, -1$



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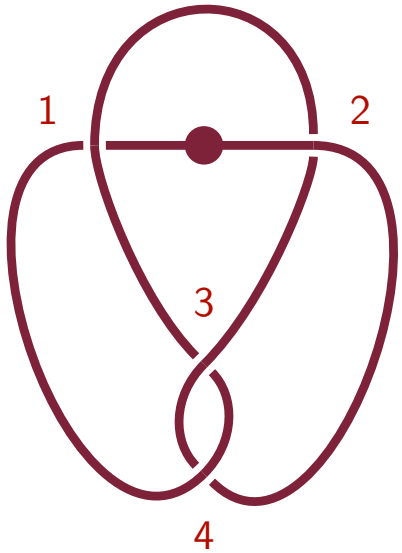

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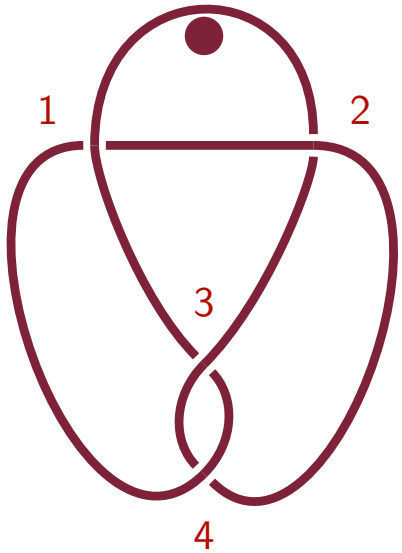
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- I wonder what this should be