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LETTER TO THE EDITOR

Simple formula for the average Gaunt factor

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Abstract. A simple analytic formula for the free-free Gaunt factor averaged over a Maxwellian distribution is obtained by the integration of the asymptotic expansion of the classical expression for the Gaunt factor, in which the incident energy is replaced by a mean one. A comparison with the exact quantum mechanical calculation shows that the formula approximates the average free-free Gaunt factor within 1% for $kT/Z^2\,\mathrm{Ryd}\leqslant 0.3$ and $0.4(kT/Z^2\,\mathrm{Ryd})^{3/2}\leqslant \hbar\omega/Z^2\,\mathrm{Ryd}\leqslant 0.3$. The average bound-free Gaunt factor is also considered.

Non-relativistic quantum mechanics gives the following expression for the free-free Gaunt factor (Sommerfeld 1953)

$$g_{\mathrm{ff}}(E', E)=\frac{3^{1/2}\pi}{(\mathrm{e}^{2\pi\eta'}-1)(1-\mathrm{e}^{-2\pi\eta})}\,x\frac{\mathrm{d}}{\mathrm{d}x}|F(\mathrm{i}\eta', \mathrm{i}\eta, 1; x)|^2 \tag{1}$$

where $\eta'=Z(\mathrm{Ryd}/E')^{1/2}$, $\eta=Z(\mathrm{Ryd}/E)^{1/2}$, $E'-E=\hbar\omega$ is the photon energy, $x=-4\eta\eta'/(\eta-\eta')^2$ and F is the hypergeometric function. Nearly twenty approximations to (1) are known (Brussaard and van de Hulst 1962). The classical radiation theory (Landau and Lifshitz 1959) yields a simpler formula

$$g_{\mathrm{ff}}(E', E)=\frac{3^{1/2}\pi}{4}\,\mathrm{i}\nu H^{(1)}_{\mathrm{i}\nu}(\mathrm{i}\nu)H^{(1)\prime}_{\mathrm{i}\nu}(\mathrm{i}\nu) \tag{2}$$

where $\nu=\frac{1}{2}\sigma\eta'^3$, $\sigma=\hbar\omega/Z^2\,\mathrm{Ryd}$, $H^{(1)}_{\alpha}(\beta)$ is the Hankel function of first kind and $H^{(1)\prime}_{\alpha}(\beta)=\mathrm{d}H^{(1)}_{\alpha}(\beta)/\mathrm{d}\beta$. For $\nu\gg 1$ we have the Kramers approximation $g_{\mathrm{ff}}=1$. To average (1) or (2) over an equilibrium energy distribution (T is the temperature)

$$G_{\mathrm{ff}}(\omega)=\frac{1}{kT}\int_0^{\infty}\mathrm{e}^{-E/kT}g_{\mathrm{ff}}(E', E)\,\mathrm{d}E \tag{3}$$

numerical integration is required. The expressions for the average Gaunt factor obtained by the analytical integration are known for the Born approximation (η', $\eta\ll 1$) and for the long-wavelength limit ($\nu\ll 1$) in the classical approximation (Brussaard and van de Hulst 1962, Bekefi 1966). In the Kramers approximation, apparently, $G_{\mathrm{ff}}=1$.

In this letter a simple formula for the average Gaunt factor is presented. It is obtained from the asymptotic expansion of (2), where the incident energy is replaced by its mean value.

The expression (2) corresponds to the limit η, $\eta'\to\infty$ in (1), while $\xi=\eta-\eta'$ is finite (Biedenharn 1956). The quantities ξ and ν are related by $\xi=\nu 2\eta'^2/\eta(\eta+\eta')$ and in the classical limit $\xi\to\nu$. Formula (2) is valid for $\eta'\gg 1$ and $\nu\approx\xi\ll\eta'$, i.e.

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$\hbar\omega \ll E'$ (the classical theory does not take into account the effect of radiation on the velocity of an electron). The validity region of the classical formula can be expanded if one symmetrises (2) by either replacing ν by ξ or introducing a mean value of the energy E_c. The symmetrisation and the ways of forming mean values have been discussed by Ter-Martirosyan (1952), Biedenharn and Thaler (1956), Naccache (1972) and Heim *et al* (1989). We set $E_c = E + s\hbar\omega$ where $0 \leqslant s \leqslant 1$. The value $s = \frac{1}{2}$ corresponds to the symmetry with respect to E and E'. Thus

$$\nu = \frac{\sigma}{2(\eta^{-2} + s\sigma)^{3/2}}. \tag{4}$$

The classical formula (2) has the asymptotic expansion

$$g_{\mathrm{ff}}(E', E) = 1 + \frac{0.21775}{\nu^{2/3}} - \frac{0.01312}{\nu^{4/3}} + \ldots. \tag{5}$$

Table 1 shows that (5) approximates (2) within 1% for values of ν as low as 0.4 (the values of (2) are taken from the paper by Grant (1958)).

Inserting (5) and (4) into (3), we get

$$G_{\mathrm{ff}}(\omega) = 1 + 0.3457\frac{\tau + s\sigma}{\sigma^{2/3}} - 0.0331\frac{2\tau(\tau + s\sigma) + (s\sigma)^2}{\sigma^{4/3}} \tag{6}$$

where $\tau = kT/Z^2$ Ryd. This formula is valid for

$$\tau^{3/2} \leqslant \sigma \ll 1. \tag{7}$$

If $s = \frac{1}{2}$ the first two terms of (6) coincide with those obtained by integrating the asymptotic expansion of the quantum mechanical formula in the limit $\eta' \gg \eta \gg 1$ (Grant 1958).

In figure 1 the values of (6) for $s = 0, \frac{1}{2}, 1$ are plotted together with those of the rigorous quantum mechanical calculation by Karzas and Latter (1961) at $\sigma/\tau = 1$, i.e.

Table 1. Comparison of the asymptotic expansion (5) with the classical Gaunt factor (2).

	$\nu = 0.3$	0.4	0.5	0.6	0.8	1.0	1.2	1.6	2.0
Equation (2)	1.396	1.344	1.306	1.276	1.232	1.201	1.180	1.151	1.132
Equation (5)	1.421	1.357	1.313	1.280	1.235	1.205	1.183	1.152	1.132

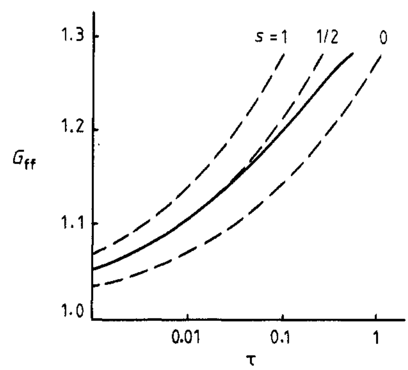

Figure 1. Average free-free Gaunt factor for $\sigma/\tau = 1$ plotted against τ: ——, quantum mechanical calculation by Karzas and Latter (1961); - - - -, calculation from (6) with $s = \frac{1}{2}$.

in the case when $\hbar\omega \sim E'$. The $s=\frac{1}{2}$ curve agrees with the quantum mechanical calculation for $\tau \ll 1$, while the $s=0$ and $s=1$ curves differ significantly from that. Thus the symmetrisation of the classical formula allows the region of validity to be extended up to $\sigma \sim \tau \ll 1$. However, the discrepancy between the quantum mechanical values and (6) at $s=\frac{1}{2}$ grows when τ increases. It can be reduced, if we define s in dependence on τ, such that $s \to \frac{1}{2}$ as $\tau \to 0$. We put here

$$s=\tfrac{1}{2}-\tfrac{1}{2}\tau^{1/2}. \tag{8}$$

The comparison with the exact numerical calculation by Peach (1970) in table 2 shows that the formula (6) with s defined by (8) is correct within 1% error for $\tau \leqslant 0.3$ and $0.4\tau^{3/2} \leqslant \sigma \leqslant 0.3$ according to (7). The values corresponding to $s=\frac{1}{2}$ are given in parentheses.

Table 2. Average free-free Gaunt factor. The upper number is the exact quantum mechanical value (Peach 1970), the following number is calculated from (6) with s from (8), the lower number in parentheses is calculated from (6) at $s=\frac{1}{2}$.

T/Z^2, K \ σ	0.05	0.10	0.15	0.20	0.25	0.30
4 000	1.114	1.104	1.103	1.103	1.104	1.105
	1.113	1.104	1.105	1.107	1.110	1.113
	(1.122)	(1.116)	(1.118)	(1.122)	(1.126)	(1.130)
10 000	1.191	1.154	1.141	1.135	1.131	1.129
	1.190	1.152	1.139	1.133	1.131	1.130
	(1.204)	(1.170)	(1.160)	(1.156)	(1.156)	(1.156)
25 000	1.345	1.261	1.225	1.205	1.192	1.184
	1.343	1.259	1.222	1.200	1.187	1.177
	(1.362)	(1.285)	(1.253)	(1.235)	(1.225)	(1.218)
48 000	1.493	1.370	1.314	1.281	1.259	1.256
	1.459	1.382	1.328	1.292	1.267	1.248
	(1.478)	(1.413)	(1.366)	(1.336)	(1.316)	(1.301)

It should be noted, in conclusion, that (5) with (4) can be analytically continued to the free-bound transitions ($E<0$). The sum of the free-bound continua over high principal quantum numbers, say, $n \geqslant N$, is usually replaced by an integral (Griem 1964). The corresponding average Gaunt factor $G_{\mathrm{bf}}(\omega)$ can be obtained by taking the integral in (3) from $E=-\hbar\omega_N$ to $E=0$ where $\hbar\omega_N=\min\{\hbar\omega, I_N\}$ and I_N is the ionisation energy of the Nth level, and multiplying it by the normalisation factor $(\mathrm{e}^{\hbar\omega_N/kT}-1)^{-1}$. Thus

$$G_{\mathrm{bf}}(\omega, N)=1+\frac{0.3457}{\sigma^{2/3}}\left(\tau+s\sigma-\frac{\sigma_N}{1-\mathrm{e}^{-\sigma_N/\tau}}\right)$$
$$-\frac{0.0331}{\sigma^{4/3}}\left[2\tau(\tau+s\sigma)+(s\sigma)^2-\frac{\sigma_N(2\tau+2s\sigma-\sigma_N)}{1-\mathrm{e}^{-\sigma_N/\tau}}\right]$$

where $\sigma_N=2\hbar\omega_N/Z^2$ Ryd. Next we can consider an average total Gaunt factor

$G_t(\omega, N)$ corresponding to all lower states from $E=-\hbar\omega_N$ to $E=\infty$. It is

$$G_t(\omega, N)=1+0.3457\frac{\tau+s\sigma-\sigma_N}{\sigma^{2/3}}-\frac{2\tau(\tau+s\sigma-\sigma_N)+(s\sigma-\sigma_N)^2}{\sigma^{4/3}}.$$

It is obvious that G_t reduces to G_{ff} when $I_N \to 0$.

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