Linearization of Powers of the Golden Ratio

The golden ratio φ and the golden ratio conjugate ϕ are defined as

Assume n is a positive integer and $F_0 = 0$.

(a) Using $\phi^2 = \phi + 1$, prove by mathematical induction the following linearization of powers of the golden ratio:

$\boldsymbol{\phi^n = F_n\phi + F_{n-1}}$.

Base case: For $n = 1$, $\phi^1 = F_1\phi + F_{1-1} = (1)\phi + F_{0-1} = (1)\phi + 0$, so we get $\phi = \phi$, which is true.

Induction step: Suppose the equality is true for a positive integer $n = k$.

$$\phi^k = \phi\phi^k$$

$$= \phi\left(F_k\phi + F_{k-1}\right)$$

$$= \phi^2 F_k + \phi F_{k-1}$$

$$= (\phi + 1)F_k + \phi F_{k-1}$$

$$= \phi\left(F_k + F_{k-1}\right) + F_k$$

$$= F_{k+1}\phi + F_k$$

Therefore, $\phi^n = F_n\phi + F_{n-1}$ is true for $n = k + 1$ and thus for all positive integers.

(a) Using $\varphi^2 = -\varphi + 1$, prove by mathematical induction the following linearization of powers of the golden ratio:

$\boldsymbol{(-\varphi)^n = -F_n\varphi + F_{n-1}}$.

Base case: For $n = 1$, $(-\varphi)^1 = -F_1\varphi + F_{1-1} = -\varphi + F_{0-1} = -\varphi + 0$, so we get $-\varphi = -\varphi$, which is true.

Induction step: Suppose the equality is true for a positive integer $n = k$.

$$
\begin{aligned}
(-\varphi)^{k+1} &= -\varphi(-\varphi)^k \\
&= -\varphi\left(-F_k\varphi + F_{k-1}\right) \\
&= \varphi^2 F_k - \varphi F_{k-1} \\
&= (-\varphi + 1)F_k - \varphi F_{k-1} \\
&= -\varphi\left(F_k + F_{k-1}\right) + F_k \\
&= -F_{k+1}\varphi + F_k
\end{aligned}
$$

Therefore, $(-\varphi)^n = -F_n\varphi + F_{n-1}$ is true for $n = k + 1$ and thus for all positive integers.