

The Lucas Numbers

The Lucas numbers are closely related to the Fibonacci numbers and satisfy the same recursion relation $L_{n+1} = L_n + L_{n-1}$, but with starting values $L_1 = 1$ and $L_2 = 3$. The Lucas numbers are an example of a generalized Fibonacci sequence that satisfies $f_{n+1} = f_n + f_{n-1}$, with starting values $f_1 = p$ and $f_2 = q$.

(a) Using mathematical induction, prove that $f_{n+2} = F_n p + F_{n+1} q$.

Let F_n, f_n denote the n th Fibonacci number.

$$f_{n+2} = F_n p + F_{n+1} q, \text{ where } f_1 = p \text{ and } f_2 = q.$$

Proof

From the initial definition of Fibonacci numbers, we have:

$$F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 3$$

Proof by induction:

For all $n \in \mathbb{N}$, let $P(n)$ be the proposition:

$$f_{n+2} = F_n p + F_{n+1} q$$

Basis for the Induction

$P(0)$ is the case:

$$f_{0+2} = F_0 p + F_{0+1} q = 0 + (1)q = q = f_2$$

$P(1)$ is the case:

$$f_{1+2} = F_1 p + F_{1+1} q = p + q = f_3$$

Induction Hypothesis

We need to show that, if $P(k)$ is true, where $k > 1$, then it follows that $P(k+1)$ is true.

So the induction hypothesis is:

$$f_{k+2} = F_k p + F_{k+1} q$$

$$f_{k+3} = F_{k+1} p + F_{k+2} q$$

We need to show:

$$f_{k+3} = F_{k+1} p + F_{k+2} q$$

Induction Step

$$\begin{aligned}
 f_{k+3} &= f_{k+1} + f_{k+2} \\
 &= (F_{k+1}p + F_kq) + (F_kp + F_{k+1}q) \\
 &= (F_{k+1} + F_k)p + (F_k + F_{k+1})q \\
 &= F_{k+2}p + F_{k+3}q
 \end{aligned}$$

So $P(k+1)$ and the result follows by the Principle of Induction.

(b) Form a table containing the first 12 Fibonacci and Lucas numbers that can be referred to in the next two problems.

n	F_n	L_n
0	0	0
1	1	1
2	1	3
3	2	4
4	3	7
5	5	11
6	8	18
7	13	29
8	21	47
9	34	76
10	55	123
11	89	199
12	144	322

(c) Prove that $L_n = F_{n-1} + F_{n+1}$.

From the result in (a):

$$L_n = F_{n-1}(L_1) + F_n(L_2)$$

$$\begin{aligned}
&= F_{n-1}(1) + F_n(3) \\
&= 2F_{n-1} + (F_{n-1} + F_{n-2}) \\
&= F_{n-1} + (F_{n-1} + F_n) \\
&= F_{n-1} + F_{n+1}
\end{aligned}$$

(d) Prove that $F_n = \frac{1}{5}(L_{n-1} + L_{n+1})$.

$$\begin{aligned}
\frac{1}{5}(L_{n-1} + L_{n+1}) &= \frac{1}{5}((F_{n-2} + F_n) + (F_n + F_{n+2})) \\
&= \frac{1}{5}(F_{n-2} + 2F_n + F_n + F_{n+1}) \\
&= \frac{1}{5}(F_{n-2} + 3F_n + F_n + F_{n+1}) \\
&= F_n
\end{aligned}$$