Continued Fractions for Square Roots

(a) Starting with $\sqrt{2} = 1 + \left(\sqrt{2} - 1\right)$, find a recursive definition for $\sqrt{2}$ and use it to derive its continued fraction.

(b) Use the same trick as above to find the continued fraction for $\sqrt{3}$.

Solution

(a)

We have

$$\begin{aligned}\sqrt{2} &= 1 + \left(\sqrt{2} - 1\right) \\ &= 1 + \frac{1}{1 + \sqrt{2}}\end{aligned}$$

which can be iterated as follows:

$$\begin{aligned}\sqrt{2} &= 1 + \frac{1}{1 + \sqrt{2}} \\ &= 1 + \cfrac{1}{2 + \cfrac{1}{1 + \sqrt{2}}} \\ &= 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{1 + \sqrt{2}}}}\end{aligned}$$

on so on, such that $\sqrt{2} = [1;\bar{2}]$

(b)

We have

$$\begin{aligned}\sqrt{3} &= 1 + \left(\sqrt{3} - 1\right) \\ &= 1 + \frac{2}{1 + \sqrt{3}}\end{aligned}$$

which can be iterated as follows:

$$
\begin{aligned}
\sqrt{3} &= 1 + \cfrac{2}{1+\sqrt{3}} \\
&= 1 + \cfrac{2}{2 + \cfrac{2}{1+\sqrt{3}}} \\
&= 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{2}{1+\sqrt{3}}}}
\end{aligned}
$$

on so on, such that $\sqrt{3} = [1;\ 1, 2]$