

Fibonacci Addition Formula

Using the relation $Q^n Q^m = Q^{n+m}$, prove the Fibonacci addition formula

$$F_{n+m} = F_{n-1}F_m + F_n F_{m+1}.$$

Solution

Using $Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$, we write $Q^n Q^m = Q^{n+m}$ as

$$\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} \begin{bmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{bmatrix} = \begin{bmatrix} F_{n+m+1} & F_{n+m} \\ F_{n+m} & F_{n+m-1} \end{bmatrix}.$$

The first element leads to $F_{n+1}F_{m+1} + F_n F_m = F_{n+m+1}$.

Substituting $n \rightarrow n+1$ and $m \rightarrow m$, we get $F_{n+m} = F_{n-1}F_m + F_n F_{m+1}$.