1. Prove that the intersection of two intervals is again an interval. Is the same true for unions?

   Assuming a non-empty intersection. A set S is an interval if $(\forall a, b \in S)(\forall y \in \mathbb{R})(a \leq y \leq b) \rightarrow y \in S$. Let S and T be intervals, where $a, b \in S \cap T$ and $a \leq y \leq b$. $y \in S$ and $y \in T$, so $S \cap T$ is an interval.

   The same is not true for unions. Consider $S = (a, b) \cup (b, c)$. This cannot be an interval since b cannot be in S.

2. Taking $\mathbb{R}$ as the universal set, express the following as simply as possible in terms of intervals and unions of intervals.

   (a) $[1, 3]'$ $(-\infty, 1) \cup (3, \infty)$
   (b) $(1, 7]'$ $(-\infty, 1] \cup (7, \infty)$
   (c) $(5, 8]'$ $(-\infty, 5] \cup (8, \infty)$
   (d) $(3, 7) \cup [6, 8]$ $(3, 8]$
   (e) $(-\infty, 3)' \cup (6, \infty)$ $[3, \infty)$
   (f) $\{\pi\}'$ $(-\infty, \pi) \cup (\pi, \infty)$
   (g) $(1, 4] \cap [4, 10]$ $\{4\}$
   (h) $(1, 2) \cap [2, 3]$ $\{\}$
   (i) A', where $A = (-\infty, 5] \cup (7, \infty)$ $(5, 7]$

3. Prove that if a set A of integers/rationals/reals has an upper bound, then it has infinitely many different upper bounds.

   An upper bound of A is an element m such that $\forall n \in A, m \geq n$. Let U be the set consisting of the upper bounds of A. Assume U is finite. Then there is a largest element $u \in U$. But there is a $v \in \mathbb{R}$ such that $v > u$, making v an upper bound not in U. This is a contradiction. Therefore, A has infinitely many different upper bounds.

4. Prove that if a set A of integers/rationals/reals has a least upper bound, then it is unique.

   Assume x is the lub and assume a non-unique lub. This means $\exists y \neq x$ such that y is a lub. If $y > x$ then y is not a lub. If $y < x$ then x is not a lub. This is a contradiction. Therefore, the lub is unique.

5. Let A be a set of integers, rationals, or reals. Prove that b is the least upper bound of A iff:

   (a) $(\forall a \in A)(a \leq b)$; and
   (b) whenever $c < b$ there is an a such that $a > c$. .

   For the first part of the bi-conditional: Assume b is the lub of A and $\exists a \in A$ such that $a > b$. Then b is not the least upper bound. This is a contradiction. Now, assume $c < b$ and there is not an $a \in A$ such that $a > c$. Then c is an upper bound and is less than the lub. This is a contradiction.
   For the second part of the bi-conditional: Assume $(\forall a \in A)(a \leq b)$. So b is in A. If b is not the lub, then $\exists c, a \leq c < b$. If b is in A then it is not the case that $a \leq b$. If b is not in A then there is a contradiction. Now, assume whenever $c < b$ there is an $a \in A$ such that $a > c$. If b is not the lub, there is a $c < b$ such that c is the lub. If $a > c$ then a cannot be in A. This is a contradiction Therefore, both conditionals are proven.

6. The following variant of the above characterization is often found. Show that b is the lub of A iff:

   (a) $(\forall a \in A)(a \leq b)$; and
   (b) $(\forall \epsilon > 0)(\exists a \in A)(a > b - \epsilon)$

   (a) is the same as the previous question. For (b): Assume b is the lub of A and $(\exists \epsilon > 0)(\nexists a \in A)(a > b - \epsilon)$. This would imply $b - \epsilon$ is the lub of A, which is a contradiction. Now, assume $(\forall \epsilon > 0)(\exists a \in A)(a > b - \epsilon)$ and b is not the lub. This implies $\exists \epsilon$ such that $b - \epsilon)$ is the lub, so $(\forall a \in A)(a \leq b - \epsilon)$. This is a contradiction, thus completing proof of the biconditional for (b).

7. Give an example of a set of integers that has no upper bound. The set of integers defined by the natural numbers has no upper bound.

8. Show that any finite set of integers/rationals/reals has a least upper bound.

   Since A is finite, A has a maximum element, say b. So $(\forall a \in A)(a \leq b)$. This is the definition of least upper bound.

9. Intervals: What is lub (a, b)? What is lub $[a, b]$? What is max (a, b)? What is max $[a, b]$?

   b, b, none, b.

10. Let $A = \{|x - y|, x, y \in (a, b)\}$. Prove that A has an upper bound. What is lub A?

    $|x - y| \leq |a| + |b|$, so $|a| + |b|$ is an upper bound.

11. Define the notion of a *lower bound* of a set of integers/rationals/reals.

    b is a lower bound of A if and only if $(\forall a \in A)(a \geq b)$.

12. Define the notion of a *greatest lower bound* (glb) of a set of integers/rationals/reals by analogy with our original definition of lub.

    Let B be the set of all lower bounds of A. b is the greatest lower bound of A if and only if $(\forall x \in B)(b \geq x)$.

13. State and prove the analog of question 5 for greatest lower bounds.

    Question skipped: this would follow the same proof methodology as 5, but for greatest lower bound.

14. State and prove the analog of question 6 for greatest lower bounds.

    Question skipped: this would follow the same proof methodology as 6, but for greatest lower bound.

15. Show that the Completeness Property for the real number system could equally well have been defined by the statement, "Any nonempty set of reals that has a lower bound has a greatest lower bound"

    Let A be a non-empty set $\in \mathbb{R}$ that is bounded above. The lub defined on the complement of A (A') is now the greatest lower bound of A.

16. The integers satisfy the Completeness Property, but for a trivial reason. What is that reason?

    Subsets of integers are always closed intervals.