The Lucas Numbers

The Lucas numbers are closely related to the Fibonacci numbers and satisfy the same recursion relation $L_{n+1} = L_n + L_{n-1}$, but with starting values $L_1 = 1$ and $L_2 = 3$. The Lucas numbers are an example of a generalized Fibonacci sequence that satisfies $f_{n+1} = f_n + f_{n-1}$, with starting values $f_1 = p$ and $f_2 = q$.

(a) Using mathematical induction, prove that $f_{n+2} = F_n p + F_{n+1} q$.

Let F_n, f_n denote the nth Fibonacci number.

$f_{n+2} = F_n p + F_{n+1} q$, where $f_1 = p$ and $f_2 = q$.

Proof

From the initial definition of Fibonacci numbers, we have:

$F_0 = 0$, $F_1 = 1$, $F_2 = 1$, $F_3 = 3$

Proof by induction:

For all $n \in N$, let $P(n)$ be the proposition:

$f_{n+2} = F_n p + F_{n+1} q$

Basis for the Induction

$P(0)$ is the case:

$f_{0+2} = F_0 p + F_{0+1} q = 0 + (1)q = q = f_2$

$P(1)$ is the case:

$f_{1+2} = F_1 p + F_2 q = p + q = f_3$

Induction Hypothesis

We need to show that, if $P(k)$ is true, where $k > 1$, then it follows that $P(k+1)$ is true.

So the induction hypothesis is:
$f_{k+2} = F_k p + F_{k+1} q$
$f_{k+3} = F_{k+1} p + F_{k+2} q$

We need to show:
$f_{k+3} = F_{k+1} p + F_{k+2} q$

Induction Step

$$f_{k+3} = f_{k+1} + f_{k+2}$$

$$= \left(F_{k-1}p + F_k q\right) + \left(F_k p + F_{k+1} q\right)$$

$$= \left(F_{k-1} + F_k\right)p + \left(F_{k+1} + F_k\right)q$$

$$= F_{k+1}p + F_{k+2}q$$

So $P(k+1)$ and the results follows by the Principle of Induction.

(b) Form a table containing the first 12 Fibonacci and Lucas numbers that can be referred to in the next two problems.

n	F_n	L_n
0	0	0
1	1	1
2	1	3
3	2	4
4	3	7
5	5	11
6	8	18
7	13	29
8	21	47
9	34	76
10	55	123
11	89	199
12	144	322

(c) Prove that $L_n = F_{n-1} + F_{n+1}$.

From the result in (a):

$$L_n = F_{n-1}\left(L_1\right) + F_n\left(L_2\right)$$

$$= F_{n-1}(1) + F_n(3)$$

$$= 2F_{n-1} + \left(F_{n-1} + F_{n-2}\right)$$

$$= F_{n-1} + \left(F_{n-1} + F_n\right)$$

$$= F_{n-1} + F_{n+1}$$

(d) Prove that $F_n = \dfrac{1}{5}\left(L_{n-1} + L_{n+1}\right).$

$$\begin{aligned}
\frac{1}{5}\big(L_{n-1} & = \frac{1}{5}\left(\left(F_{n-2} + F_n\right)\right.\\
+ L_{(n} & \qquad + \left(F_n\right.\\
& \qquad \left.\left. + F_{n+2}\right)\right)\\[2mm]
& = \frac{1}{5}\left(F_{n-2} + 2F_n\right.\\
& \qquad + + F_n\\
& \qquad \left. + F_{n+1}\right)\\[2mm]
& = \frac{1}{5}\left(F_{n-2} + 3F_n\right.\\
& \qquad + + F_n\\
& \qquad \left. + F_{n+1}\right)\\[2mm]
& = F_n
\end{aligned}$$