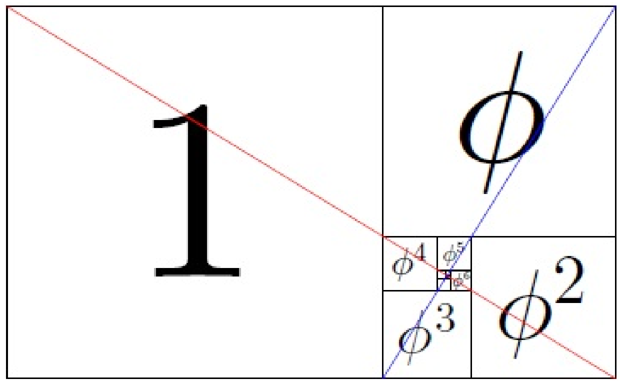
The Eye of God

Prove that the accumulation point of all the spiralling squares (called the eye of God by the author Clifford Pickover) is the intersection point of the diagonal lines of the two largest golden rectangles. After fixing an origin, find the coordinates of this intersection point.



Solution

From $\Phi = 1 + \phi = 1/\phi$, the two diagonal lines can be determined to be

$$y = -\phi x + 1 \text{ (larger diagonal)}$$

$$x = \Phi x - \Phi \text{ (smaller diagonal)}$$

If we can show that the diagonal lines passing through rectangle with side length ϕ^4 pass through subsequent smaller copies of this rectangle, then the lines must eventually intersect at the accumulation point. The longer diagonal passes through $\left(1, \phi^2\right)$ and $\left(1 + \phi^3, \phi^3\right)$. We need to show that these points satisfy $y = -\phi x + 1$. To show this, we'll use the following relationship:

$$\phi^2 = -\phi + 1$$

Next, we substitute the x values into the equation for the diagonal line. For $x = 1$, $y = \phi^2$, we have

$$\begin{aligned} y &= -\phi + 1 \\ &= \phi^2. \end{aligned}$$

For $x = 1 + \phi^3$, $y = \phi^3$, we have

$$y = -\phi\left(1 + \phi^3\right) + 1$$

$$= 1 - \phi - \phi^4$$

$$= 1 - \phi - (1 - \phi)^2$$

$$= 1 - \phi - 1 + 2\phi - \phi^2$$

$$= \phi(1 - \phi)$$

$$= \phi^3.$$

Next we show that the shorter diagonal passes through $(1 + \phi^3, \phi^2)$ and $(1 + \phi^4, \phi^3)$ by satisfying $x = \Phi x - \Phi$. For $x = 1 + \phi^3$, we have

$$y = \Phi(1 + \phi^3) - \Phi$$

$$= \phi^2.$$

For $x = 1 + \phi^4$, we have

$$y = \Phi(1 + \phi^4) - \Phi$$

$$= \phi^3.$$

This completes the proof. Equating the two values of y gives

$$-\phi x + 1 = \Phi x - \Phi,$$

which is solved for $x = (5 + 3\sqrt{5})/10$. The value of y can then be solved as $y = (5 - \sqrt{5})/10$. The numberical values can be approximated as $(x, y) \approx (1.1708, 0.2764)$.