1. Express as concisely and accurately as you can the relationship between $b|a$ and a/b.

   $b|a$ means b divides a or $a = bn$ for some integer n. This is equivalent to a/b or a divided by b with no remainder. So $b|a$ iff $a/b \in \mathbb{Z}$

2. Determine whether each of the following is true or false and prove your answer.

   (a) $0|7$ False. $0|7 \to 7/0 \in \mathbb{Z}$. But $7/0$ is undefined.

   (b) $9|0$ True. $9|0 \to 0/9 \in \mathbb{Z}$. $0/9 = 0 \in \mathbb{Z}$.

   (c) $0|0$ False. $0|0 \to 0/0 \in \mathbb{Z}$. But $0/0$ is undefined.

   (d) $1|1$ True. $1|1 \to 1/1 \in \mathbb{Z}$. $1/1 = 1 \in \mathbb{Z}$.

   (e) $7|44$ False. $7|44 \to 44/7 \in \mathbb{Z}$. But $(\nexists n \in \mathbb{Z})(44 = 7n)$.

   (f) $7|(-42)$ True. $7|(-42) \to -42/7 \in \mathbb{Z}$. $-42/7 = -6 \in \mathbb{Z}$.

   (g) $(-7)|(-49)$ True. $(-7)|(-42) \to -49/-7 \in \mathbb{Z}$. $-49/-7 = 7 \in \mathbb{Z}$.

   (h) $(-7)|(-56)$ True. $(-7)|(-56) \to -56/-7 \in \mathbb{Z}$. $-56/-7 = 8 \in \mathbb{Z}$.

   (i) $(\forall n \in \mathbb{Z})(1|n)$ True. $(\forall n \in \mathbb{Z})(1|n) \to (\forall n \in \mathbb{Z})(n/1) \in \mathbb{Z}$. $n/1 = n \in \mathbb{Z}$.

   (j) $(\forall n \in \mathbb{N})(n|0)$ True. $(\forall n \in \mathbb{N})(n|0) \to (\forall n \in \mathbb{Z})(0/n) \in \mathbb{Z}$. $0/n = 0 \in \mathbb{Z}$.

   (k) $(\forall n \in \mathbb{Z})(n|0)$ True. $(\forall n \in \mathbb{Z})(n|0) \to (\forall n \in \mathbb{Z})(0/n) \in \mathbb{Z}$. $0/n = 0 \in \mathbb{Z}$.

3. Prove all the parts of the theorem in the lecture, giving the basic properties of divisibility. Namely, show that for any integers a, b, c, d with $a \neq 0$

   (a) $a|0$, $a|a$.
   $a|0 \to 0/a \in \mathbb{Z}$ (True). $a|a \to a/a = 1 \in \mathbb{Z}$ (True).

   (b) $a|1$ if and only if $a = \pm 1$
   $a|1 \to 1/a \in \mathbb{Z} \to a = \pm 1$. $a = \pm 1 \to 1|1$.

   (c) If $a|b$ and $c|d$, then $ac|bd$ for $c \neq 0$.
   $a|b$ and $c|d \to b/a \in \mathbb{Z}$ and $d/c \in \mathbb{Z}$. So $b = am$ and $d = cn$ for integers m, n. $a = \frac{b}{m}$ and $c = \frac{d}{n}$. This means $ac = \frac{bd}{mn}$. It follows that $ac|bd$.

   (d) If $a|b$ and $b|c$, then $a|c$ for $a \neq 0$.
   $a|b$ and $b|c \to b = am$ and $c = bn$ for integers m, n. So $a = \frac{(c/n)}{m}$ and $am = \frac{c}{n}$. $a = \frac{b}{m}$ and $c = \frac{d}{n}$. This means $ac = \frac{bd}{mn}$ and $amn = c$. $a|c$ follows since mn is an integer.

   (e) $a|b$ and $b|a$ if and only if $a = \pm b$.
   $b = am$ and $a = bn \to (1 = mn$ and $1 = n/m) \to m = n \to [(b = a$ and $a = b)$ or $(b = -a$ and $a = -b)]$. This proves the first conditional. Now assume $a = \pm b$. $b|a$ since $\pm 1 \in \mathbb{Z}$. $a = \pm b \to b = \pm a$, so $b = (a)(1)$ or $b = (a)(-1)$. But this implies $a|b$ since $\pm 1 \in \mathbb{Z}$, which proves the second conditional.

   (f) If $a|b$ and $b \neq 0$ then $|a| \leq |b|$.
   $a|b$, so $b = am$ for some integer m. Now assume $|b| < |a|$. Then $m = \frac{|b|}{|a|}$, which means $0 < m < 1$. But this means m is not an integer, which is a contradiction. Therefore, $|a| \leq |b|$.

   (g) If $a|b$ and $a|c$, then $a|bx + cy$ for integers x, y.
   If $a|b$ and $a|c$, then $b = ax$ and $c = ay$ for integers x, y. So $x = \frac{b}{a}$ and $y = \frac{c}{a}$. $bx = b\frac{b}{a}$ and $cy = c\frac{c}{a}$. $bx + cy = \frac{b^2+c^2}{a}$. But $_2 + c^2$ is an integer, so $a|bx + cy$.