Binet's Formula for the Lucas Numbers

Determine the analogue to Binet's formula for the Lucas numbers, defined as

$L_{n+1} = L_n + L_{n-1}$

with the initial values $L_1 = 1$ and $L_2 = 3$. Again it will be simpler to define the value of L_0 and use it and L_1 as the initial values.

Answer:

The general solution to the Fibonacci recursion relation is

$$L_n = c_1 \phi^n + c_2 (-\varphi)^n$$

By substituting intital values for the Lucas sequence we get

$$c_1 + c_2 = 2,$$
$$c_1 \phi + c_2 \varphi = 1.$$

Multiply the first equation by φ and add it to the second equation:

$$c_1 (\phi + \varphi) = 2\varphi + 1.$$

$2\varphi + 1 = \phi + \varphi = \sqrt{5}$, so $c_1 = 1$ and $c_2 = 1$. The solution is then

$$L_n = \phi^n + (-\varphi)^n.$$