Cassini's Identity

Prove by mathematical induction Cassini's identity, given by

$F_{n+1}F_{n-1}-F_n^2=(-1)^n$.

Solution

Base case: When $n=1$ we get $F_2F_0-F_1^2=-1$ for the left side and $(-1)^1=-1$ for the right side, so the equation is true for $n=1$.

Induction Step: Suppose $F_{n+1}F_{n-1}-F_n^2=(-1)^n$ is true for all $n=k$. Then

$$\begin{aligned}
F_{k+2}F_k-F_{k+1}^2 &= \left(F_k+F_{k+1}\right)F_k-F_{k+1}^2 \\
&= F_k^2+F_{k+1}\left(F_k-F_{k+1}^2\right) \\
&= F_k^2-F_{k+1}F_{k-1} \\
&= -(-1)^k \\
&= (-1)^{k-1}
\end{aligned}$$

So the equation is true for all $n=k+1$ and is therefore true for all positive integers.