Neighbour Swapping

(a) For each number in the string, allow it to either stay fixed or change places with one of its neighbours. Define a_n to be the number of different strings that can be formed. Here are the first four values of n:

n	*strings*	a_n
1	1	1
2	12, 21	2
3	123, 132, 213	3
4	1234, 1243, 1324, 2134, 2143	5

Prove that $a_n = F_{n+1}$.

Proof

The set of strings can be separated into two subsets:

A: strings that start with 1.
B: strings where 1 and 2 are interchanged.

For set A, the remaining $n - 1$ numbers form a_{n-1} strings. For set B, the remaining $n - 2$ numbers form a_{n-2} strings.

Therefore, the total number of string is given by $a_n = a_{n-1} + a_{n-2}$. Since $a_1 = 1 = F_2$ and $a_2 = 2 = F_3$, it follows that $a_n = F_{n+1}$.

(b) Consider a problem similar to that above, but now allow the first 1 to change places with the last n, as if the string lies on a circle. Suppose $n \geq 3$, and define b_n as the number of different strings that can be formed. Show that $b_n = L_n$, where L_n is the nth Lucas number.

Again, the set of strings can be separated into two subsets:

A: strings that start with 1.
B: strings where 1 and 2 are interchanged.

For set A, the number of strings is $a_n = F_{n+1}$. For set B, the number of strings is $a_{n-2} = F_{n-1}$. It follows that $a_n = F_{n+1} + F_{n-1}$, which is the nth Lucas number. Therefore, $b_n = L_n$.

Source: Chasnow, Jeffery R. *Fibonacci Numbers and the Golden Ratio.* 2016. bookboon.com.