The Golden Ratio and the Ratio of Fibonacci Numbers

(a) Define x_n to be the nth rational approximation to x obtained from its continued fraction, where, for example, $x_0 = [a_0;\ ]$, $x_1 = [a_0; a_1]$, and $x_2 = [a_0; a_1, a_2]$. Using $\Phi = [1; \overline{1}]$, verify that Φ_0, Φ_1, Φ_2, and Φ_3 are just the ratios of consecutive Fibonacci numbers.

(b) Prove by induction that $\Phi_n = F_{n+2}/F_{n+1}$

Solution

(a)

We have

$$\Phi_0 = [1;\ ] = 1 = \frac{1}{1} = \frac{F_2}{F_1},$$

$$\Phi_1 = [1; 1] = 1 + \frac{1}{1} = 2 = \frac{2}{1} = \frac{F_3}{F_2},$$

$$\Phi_2 = [1; 1, 1] = 1 + \cfrac{1}{1 + \cfrac{1}{1}} = \frac{3}{2} = \frac{F_4}{F_3},$$

$$\Phi_3 = [1; 1, 1, 1] = 1 + \cfrac{1}{1 + \cfrac{1}{\cfrac{1}{1 + \cfrac{1}{1}}}} = \frac{5}{3} = \frac{F_5}{F_4}.$$

(b)

Base Case: (a) shows the equality is true for $n = 0, 1, 2,$ and 3.

Induction Step: Suppose that the equality is true for positive integers $n = k$. then

$$\begin{aligned}
\frac{F_{k+3}}{F_{k+2}} &= \frac{F_{k+1} + F_{k+2}}{F_{k+2}} \\
&= 1 + \frac{F_{k+1}}{F_{k+2}} \\
&= 1 + \frac{1}{\Phi_k} \\
&= \Phi_{k+1}
\end{aligned}$$

So $\Phi_n = F_{n+2}/F_{n+1}$ is true for $n = k + 1$ and, by induction, is true for all positive integers.