Fibonacci Numbers with Negative Indices

The Fibonacci numbers can be extended to zero and negative indices using the relation $F_n = F_{n+2} - F_{n+1}$. Determine F_0 and find a general formula for F_{-n} in terms of F_n. Prove your result using mathematical induction.

Solution

Formula

Let F_n be the n^{th} Fibonacci number where $F_0 = 0$, $F_1 = 1$, $F_2 = 1$.

From the relation

$$F_n = F_{n+2} - F_{n+1}$$

we have:

$$F_0 = F_2 - F_1$$
$$F_0 = 1 - 1 = 0$$
$$F_{-1} = 1 - 0 = 1$$
$$F_{-2} = 0 - 1 = -1$$
$$F_{-3} = 1 - (-1) = 2$$
$$F_{-4} = -1 - 2 = -3$$

The formula is:

$$F_{-n} = (-1)^{n+1} F_n$$

Proof

For all $n \in \mathbb{N}_{>0}$, let $P(n)$ be the proposition:

$$F_{-n} = (-1)^{n+1} F_n$$

$P(1)$ holds since $F_{-1} = 1 - 0 = 1 = (-1)^{1+1} F_1$
$P(2)$ holds since $F_{-2} = 0 - 1 = -1 = (-1)^{2+1} F_2$

Induction Hypoethesis

If $P(k)$ and $P(k-1)$ are true, where $k > 1$, then $P(k+1)$ is true.

$$F_{-(k+1)} \quad = (-1)^k F_{k-1} - (-1)^{k+1} F_k$$

$$= (-1)^k F_{k-1} + (-1)^k F_k$$

$$= (-1)^k \left(F_{k-1} + F_k\right)$$

$$= (-1)^k \left(F_{k+1}\right)$$

$$= (-1)^{k+2} \left(F_{k+1}\right)$$

So $P(k) \wedge P(k-1) \Rightarrow P(k+1)$ and the result follows by induction.