Sum of Lucas Numbers Squared

Show that the sum over the first n Lucas numbers squared is given by

$$\sum_{i=1}^{n} L_i^2 = L_n L_{n+1} - 2.$$

Solution

$$\begin{aligned} L_n L_n &= L_n(L_n + L_{n-1}) \\ &+ \\ &= L_n^2 + L_{n-1}L_n \\ &= L_n^2 + L_{n-1}(L_{n-1} + L_{n-2}) \\ &= L_n^2 + L_{n-1}^2(L_{n-2}L_{n-1}) \\ &\vdots \quad \vdots \\ &= L_n^2 + L_{n-1}^2 + \ldots + L_2^2 + L_1 L_2) \end{aligned}$$

$L_1 = 1$ and $L_2 = 3$, so $L_1 L_2 = L_1^2 + 2$ which proves the identity since $L_1 L_2 - 2 = L_1^2$.