The Lucas Numbers

The Lucas numbers are closely related to the Fibonacci numbers and satisfy the same recursion relation $L_{n+1} = L_n + L_{n-1}$, but with starting values $L_1 = 1$ and $L_2 = 3$. The Lucas numbers are an example of a generalized Fibonacci sequence that satisfies $f_{n+1} = f_n + f_{n-1}$, with starting values $f_1 = p$ and $f_2 = q$.

(a) Using mathematical induction, prove that $f_{n+2} = F_n p + F_{n+1} q$.

Let F_n, f_n denote the nth Fibonacci number.

$f_{n+2} = F_n p + F_{n+1} q$, where $f_1 = p$ and $f_2 = q$.

Proof

From the initial definition of Fibonacci numbers, we have:

$F_0 = 0$, $F_1 = 1$, $F_2 = 1$, $F_3 = 3$

Proof by induction:

For all $n \in N$, let $P(n)$ be the proposition:

$f_{n+2} = F_n p + F_{n+1} q$

Basis for the Induction

$P(0)$ is the case:

$f_{0+2} = F_0 p + F_{0+1} q = 0 + (1) q = q = f_2$

$P(1)$ is the case:

$f_{1+2} = F_1 p + F_2 q = p + q = f_3$

Induction Hypothesis

We need to show that, if $P(k)$ is true, where $k > 1$, then it follows that $P(k+1)$ is true.

So the induction hypothesis is:
$f_{k+2} = F_k p + F_{k+1} q$
$f_{k+3} = F_{k+1} p + F_{k+2} q$

We need to show:
$f_{k+3} = F_{k+1} p + F_{k+2} q$

Induction Step

$$f_{k+3} = f_{k+1} + f_{k+2}$$

$$\begin{aligned} &= (F_{k-1}p + F_k q) \\ &\quad + (F_k p \\ &\quad + F_{k+1} q) \end{aligned}$$

$$\begin{aligned} &= (F_{k-1} + F_k)p \\ &\quad + (F_{k+1} + F_k)q \end{aligned}$$

$$= F_{k+1}p + F_{k+2}q$$

So $P(k+1)$ and the results follows by the Principle of Induction.

(b) Form a table containing the first 12 Fibonacci and Lucas numbers that can be referred to in the next two problems.

n	F_n	L_n
0	0	0
1	1	1
2	1	3
3	2	4
4	3	7
5	5	11
6	8	18
7	13	29
8	21	47
9	34	76
10	55	123
11	89	199
12	144	322

(c) Prove that $L_n = F_{n-1} + F_{n+1}$.

From the result in (a):

$$
\begin{aligned}
L &= F_{n-1}(L_1) \\
&\quad + F_n(L_2) \\
&= F_{n-1}(1) \\
&\quad + F_n(3) \\
&= 2\,F_{n-1} \\
&\quad + \left(F_{n-1}\right. \\
&\quad \left. + F_{n-2}\right) \\
&= F_{n-1} + \left(F_{n-1}\right. \\
&\quad \left. + F_n\right) \\
&= F_{n-1} + F_{n+1}
\end{aligned}
$$

(d) Prove that $\boldsymbol{F_n = \frac{1}{5}\left(L_{n-1} + L_{n+1}\right)}$.

$$
\begin{aligned}
\frac{1}{5}\left(L_{n-1}\right. \\
\left. + L_{(n}\right. &= \frac{1}{5}\left(\left(F_{n-2} + F_n\right)\right. \\
&\quad + \left(F_n\right. \\
&\quad \left.\left. + F_{n+2}\right)\right) \\
&= \frac{1}{5}\left(F_{n-2} + 2\,F_n\right. \\
&\quad + + F_n \\
&\quad \left. + F_{n+1}\right) \\
&= \frac{1}{5}\left(F_{n-2} + 3\,F_n\right. \\
&\quad + + F_n \\
&\quad \left. + F_{n+1}\right) \\
&= F_n
\end{aligned}
$$