The Golden Angle and the Ratio of Fibonacci Numbers

(a) Define x_n to be the nth rational approximation to x obtained from its continued fraction, where, for example, $x_0 = [a_0;\ ]$, $x_1 = [a_0; a_1]$, and $x_2 = [a_0; a_1, a_2]$. Using $g/2\pi = [0; 2, \overline{1}]$, determine $g_0/2\pi$, $g_1/2\pi$,
$g_2/2\pi$, and $g_3/2\pi$.

(b) Prove by induction that $g_n/2\pi = F_n/F_{n+2}$.

Solution

(a)

We have

$$g_0/2\pi = [0;\ ] = 0,$$

$$g_1/2\pi = [0; 2] = \frac{1}{2} = \frac{F_1}{F_3},$$

$$g_2/2\pi = [0; 2, 1] = 1 + \cfrac{1}{2 + \cfrac{1}{1}} = \frac{1}{3} = \frac{F_2}{F_4},$$

$$g_3/2\pi = [0; 2, 1, 1] = 1 + \cfrac{1}{2 + \cfrac{1}{\cfrac{1}{1 + \cfrac{1}{1}}}} = \frac{2}{5} = \frac{F_3}{F_5}.$$

(b)

Base Case: (a) shows the equality is true for $n = 0, 1, 2,$ and 3.

Induction Step: Suppose that the equality is true for positive integers $n = k$. then

$$\frac{F_{k+1}}{F_{k+3}} = \frac{F_{k+1}}{F_{k+1} + F_{k+2}}$$

$$= \cfrac{1}{1 + \cfrac{F_{k+1}}{F_{k+2}}}$$

$$= \frac{1}{1 + \Phi_k}$$

$$= \frac{g_{k+1}}{2\pi}$$

So $g_n/2\pi = F_n/F_{n+2}$ is true for $n = k+1$ and, by induction, is true for all positive integers.