Some Algebra Practice

The golden ratio ϕ and the golden ratio conjugate φ are defined as

$$\phi = \frac{\sqrt{5}+1}{2}, \qquad \varphi = \frac{\sqrt{5}-1}{2}$$

Prove the following identities by direct calculation:

(a) $\varphi = \phi - 1$

$$\begin{aligned}
\phi - 1 &= \frac{\sqrt{5}+1}{2} - 1 \\
&= \frac{1}{2}\left(\sqrt{5} + 1 - 2\right) \\
&= \frac{\sqrt{5}-1}{2} \\
&= \varphi
\end{aligned}$$

(b) $\varphi = \frac{1}{\phi}$

$$\begin{aligned}
\frac{1}{\phi} &= \frac{2}{\sqrt{5}+1} \times \frac{1-\sqrt{5}}{1-\sqrt{5}} \\
&= \frac{2\left(1-\sqrt{5}\right)}{-4} \\
&= \frac{\sqrt{5}-1}{2} = \varphi
\end{aligned}$$

(c) $\phi^2 = \phi + 1$

$$\begin{aligned}
\phi^2 &= \left(\frac{\sqrt{5}+1}{2}\right)^2 \\
&= \frac{5 + 2\sqrt{5} + 1}{4} \\
&= \frac{\sqrt{5}+3}{2} = \phi + 1
\end{aligned}$$

(d) $\varphi^2 = 1 - \varphi$

$$\varphi^2 = \left(\frac{\sqrt{5} - 1}{2}\right)^2$$

$$= \frac{5 - 2\sqrt{5} + 1}{4}$$

$$= \frac{-\sqrt{5} + 3}{2} =$$

$$-\varphi + 1$$

(1)