Sum of Even and Odd Fibonacci Numbers

Show that the sums over the first n even or odd Fibonacci numbers are given by

$$\sum_{i=1}^{n} F_{2i} = F_{2n+1} - 1, \qquad \sum_{i=1}^{n} F_{2i-1} = F_{2n}$$

Solution

The relation $F_{n+1} = F_{n+2} - F_n$ is used to form a list:

$$\begin{aligned}
F_{2n_} &= F_{2n} - F_{2n-2} \\
F_{2n_} &= F_{2n-2} - F_{2n-4} \\
F_{2n_} &= F_{2n-4} - F_{2n-6} \\
. \quad & \quad . \\
. \quad & \quad . \\
. \quad & \quad . \\
F_2 &= F_4 - F_2 \\
F_1 &= F_2 - F_0
\end{aligned}$$

By adding the right side we obtain $\sum_{i=1}^{n} F_{2i-1} = F_{2n} - F_0$. Substituting $F_0 = 0$ gives F_{2n}.

Next we list the identities

$$\begin{aligned}
F_{2n} &= F_{2n+1} - F_{2n-1} \\
F_{2n_} &= F_{2n-1} - F_{2n-3} \\
F_{2n_} &= F_{2n-3} - F_{2n-5} \\
. \quad & \quad . \\
. \quad & \quad . \\
. \quad & \quad . \\
F_4 &= F_5 - F_3 \\
F_2 &= F_3 - F_1
\end{aligned}$$

By adding the right side we obtain $\sum_{i=1}^{n} F_{2i} = F_{2n+1} - F_1$. Substituting $F_1 = 1$ gives $F_{2n+1} - 1$.