Area of the Spiraling Squares

Prove that

$$\sum_{i=0}^{\infty} \phi^{2i} = \Phi$$

Solution

Let $x = \sum_{i=0}^{\infty} \phi^{2i}$. Then

$$\begin{aligned} x &= 1 + \phi^2 + \phi^4 + \phi^6 + \ldots \\ \phi &= \phi^2 + \phi^4 + \phi^6 + \ldots \end{aligned}$$

Subtracting equations, we get $\left(1 - \phi^2\right)x = 1$, or

$$\begin{aligned} x &= \frac{1}{1 - \phi^2} \\ &= \frac{\Phi^2}{\Phi^2 - 1} \qquad (\text{from } \phi = 1/\Phi) \\ &= \frac{\Phi^2}{\Phi} \qquad (\text{from } \Phi^2 - \Phi - 1 = 0) \\ &= \Phi \end{aligned}$$