

Continued Fraction for e

Show that $e = [2; 1, 2, 1, 1, 4, \dots]$. Amazingly, this continues in a regular fashion as $e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, \dots]$, and is sometimes called Euler's continued fraction.

Solution

We have

$$\begin{aligned}
 e &= 2 + 0.718281 \dots \\
 &= 2 + \frac{1}{1.392211 \dots} \\
 &= 2 + \frac{1}{1 + \frac{1}{2.549646 \dots}} \\
 &= 2 + \frac{1}{\frac{1}{1 + \frac{1}{2 + \frac{1}{1.819350}}}} \\
 &= 2 + \frac{1}{\frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1.220479}}}}} \\
 &= 2 + \frac{1}{\frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{4.535573 \dots}}}}}}
 \end{aligned}$$

This gives the beginning of the expression $e = [2; 1, 2, 1, 1, 4, \dots]$.