1. Let $A = r \in \mathbb{Q} | r > 0 \wedge r^2 > 3$. Show that A has a lower bound in Q but no greatest lower bound in $\mathbb{Q}$. Give all details of the proof along the lines of the proof given in the lecture that the rationals are not complete.

   Let $x \in \mathbb{Q}$ be any lower bound of A and show there's a larger one in $\mathbb{Q}$.

   Let $x = \frac{p}{q}$ for integers p, q. Now, suppose $x^2 > 3$. This means $p^2 - 3q^2 > 0$.

   Consider the fact that as n gets larger $\frac{n^2}{2n+1}$ increases without bound.

   So there is $n \in \mathbb{N}$ such that $\frac{n^2}{2n+1} > \frac{p^2}{3q^2-p^2}$.

   Let $y = (\frac{n+1}{n})\frac{p}{q}$. y is a rational and $y^2 < 3$. Since $\frac{n+1}{n} > 1$, $y > x$. But for any $a \in A$, $y^2 < 3 < a^2$, so $a > y$. Therefore, y is a lower bound of a greater than x. This proves that for any lower bound in $\mathbb{Q}$ there is a larger one in $\mathbb{Q}$.

2. In addition to the completeness property, the *Archimedean property* is an important fundamental property of $\mathbb{Q}$. Use the Archimedean property to show that if $r, s \in \mathbb{R}$ and $r < s$, there is a $q \in \mathbb{Q}$ such that $r < q < s$.
   Let $\frac{1}{s-r} \in \mathbb{R}.\exists n \in \mathbb{N}$ such that $n > \frac{1}{s-r}$. Let m be the smallest natural number such that $\frac{m}{n} > r$. This means $m - 1 \leq rn$. Since $n > \frac{1}{s-r}$, it follows that $\frac{1}{n} < s - r$. So $m - 1 \leq rn \rightarrow m \leq rn + 1 \rightarrow \frac{m}{n} \leq r + \frac{1}{n} < r + (s - r) = s$. Therefore, $r < \frac{m}{n} < s$ which is equivalent to the original statement to be proven: $r < q < s$.

3. Formulate both in symbols and in words what it means to say that $a_n \not\rightarrow a$ as $n \rightarrow \infty$.

   $(\forall \epsilon > 0)(\exists n \in \mathbb{N})(\forall m \geq n)(|a_n - a| > \epsilon)$. This means $|a_n - a|$ does not become arbitrarily close to 0.

4. Prove that $(n/(n + 1))^2 \rightarrow 1$ as $n \rightarrow \infty$.

   Show that $\forall n \geq N$, $|(n/(n + 1))^2 - 1| < \epsilon$. $|\frac{n^2-n^2-2n-1}{(n+1)^2}| < \epsilon$. $\frac{2n+1}{(n+1)^2} < \epsilon$. Pick N such that $\frac{(N+1)^2}{2N+1} > \frac{1}{\epsilon}$. $\forall n \geq N$ $\frac{2n+1}{(n+1)^2} \leq \frac{2N+1}{(N+1)^2} < \epsilon$.

5. Prove that $1/n^2 \rightarrow 0$ as $n \rightarrow \infty$.

   Show that $\forall n \geq N$, $|1/n^2 - 0| < \epsilon$. $|1/n^2| < \epsilon$. $|1/n^2| < 1/n(n - 1)$. Pick N such that $N(N - 1) > 1/\epsilon$. $\forall n \geq N, 1/n^2 < 1/n(n - 1) \leq 1/N(N - 1) < \epsilon$.

6. Prove that $1/2^n \rightarrow 0$ as $n \rightarrow \infty$.

   Show that $\forall n \geq N$, $|1/2^n - 0| < \epsilon$. $1/2^n < \epsilon$. $n \ln(1/2) < \ln \epsilon$. $n < \ln(1/2)\epsilon$. $n > \ln(\epsilon)/\ln(1/2)$. $ln(\epsilon)/\ln(1/2) > 0$. Pick N such that $N > ln(\epsilon)/\ln(1/2)$. $\forall n \geq N, (1/2)^n < \epsilon \rightarrow 1/2^n < \epsilon$.

7. We say a sequence $\{a_n\}_{n=1}^{\infty}$ *tends to infinity* if, as n increases, a_n increases without bound. For instance, the sequence $\{2^n\}_{n=1}^{\infty}$. Formulate a precise definition of this notion, and prove that both of these examples fulfill the definition.

   $\forall x \in \mathbb{R}$, $\exists n \in \mathbb{N}, a_n \geq x$. $a_n \geq x$ is satisfied if $n \geq x$, so choose $N \geq x$ and the condition is satisfied. Similarly, $2^n \geq x$ is satisfied if $n \ln 2 \geq \ln x$, $n \geq \ln x - \ln 2$. So set $N = \ln x - \ln 2$ and the condition is satisfied.

8. Let $\{a_n\}_{n=1}^{\infty}$ *tends to infinity* be an increasing sequence. Suppose $a_n \rightarrow a$ as $n \rightarrow \infty$. Prove that $a =$lub$\{a_n | n \in \mathbb{N}\}$.

   $\forall n \geq N$, we have $a_n < a + \epsilon$ and $a_n > a - \epsilon$ with $\epsilon > 0$. The first inequality shows that a is an upper bound. If there were an upper bound less than a, then $a \in A$, where A is the set of all a_n. But this would imply $|a_n - a| = 0$ for some n. This is a contradiction on the definition of limit.

9. Prove that if $\{a_n\}_{n=1}^{\infty}$ is increasing and bounded above, then it tends to a limit.

   a_n has a lub c. $\forall \epsilon > 0, \exists N, a_N > c - \epsilon$. $\forall n > N, |c - a_n| \leq |c - a_N| < \epsilon$. So the limit is the least upper bound.