Catalan's Identity

Using the Fibonacci addition formula

$$F_{n+m} = F_{n-1}F_m + F_nF_{m+1}$$

and Cassini's identity,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n.$$

prove Catalan's identity

$$F_n^2 - F_{n-r}F_{n+r} = (-1)^{n-r}F_r^2.$$

Solution

Let $x = n - r$ and $y = r$. This means $n = x + y$, so Catalan's identity becomes

$$F_{x+y}^2 - F_xF_{x+2y} = (-1)^xF_y^2.$$

Using the Fibonacci addition formula we can write

$$\begin{aligned}
F_{x+y}^2 - F_xF_{x+2y} = & \left(F_{x-1}F_y + F_xF_{y+1}\right)^2 - \left(F_{x-1}F_{2y} + F_xF_{2y+1}\right)F_x \\
= & F_{x-1}^2F_y^2 + 2\,F_{x-1}F_xF_yF_{y+1} + F_x^2F_{y+1}^2 - F_{x-1}F_x\left(F_{y-1}F_y + F_yF_{y+1}\right) - F_x^2\left(F_y^2 + F_{y+1}^2\right) \\
= & F_{x-1}F_xF_y\left(F_{y+1} - F_{y-1}\right) + F_y^2\left(F_{x-1}^2 - F_x^2\right) \\
= & F_y^2\left(F_{x-1}\left(F_{x-1} + F_x\right) - F_x^2\right) \\
= & F_y^2\left(F_{x-1}F_{x+1} - F_x^2\right) \\
= & (-1)^xF_y^2
\end{aligned}$$

Which is the same as Catalan's identity above.