

9. Give an example of a family of intervals  $A_n, n = 1, 2, \dots$ , such that  $A_{n+1} \subset A_n$  and  $\bigcap_{n=1}^{\infty} A_n = \emptyset$ .

Consider the set of intervals  $A_n$ , where each interval is given by  $(0, \frac{n}{n^2}]$ .

$A_{n+1} \subset A_n$  for all  $n$ .

Proof:

Suppose  $A_{n+1} \not\subset A_n$ .

Then  $\exists x > 0 \in A_n, x \notin A_{n+1}$ .

$x \in A_n$  implies  $x \leq \frac{n}{n^2}$  (since  $x$  is in the interval  $(0, \frac{n}{n^2}]$ ).

$x \notin A_{n+1}$  implies  $x > \frac{n+1}{(n+1)^2}$  (since  $x > 0$ , it must be a positive upper bound of the interval  $(0, \frac{n+1}{(n+1)^2}]$ ).

But  $\frac{n+1}{(n+1)^2} < \frac{n}{n^2}$ .

So, according to the above logic,  $\frac{n+1}{(n+1)^2} < x < \frac{n+1}{(n+1)^2} \leq \frac{n}{n^2}$ , which is a contradiction.

Therefore,  $A_{n+1} \subset A_n$  for all  $n$ .