1. Prove or disprove the claim that there are integers m,n such that $m^2 + mn + n^2$ is a perfect square.

   Zero is an integer. So if $m = 0$, then $m^2 + mn + n^2 = n^2$ which is a perfect square.

2. Prove or disprove the claim that for any positive integer m there is a positive integer n such that $mn + 1$ is a perfect square.

   For a positive integer p, given m, take $n = p + 1$. Then $mn + 1 = m(m + 2 + 1) = (m + 1)^2$.

3. Prove that there is a quadratic $f(n) = n^2 + bn + c$ with positive integer coefficients b, c, such that $f(n)$ is composite (i.e. not prime) for all positive integers n, or else prove that the statement is false.
   $f(n) = (n + 1)(m + 2) = n^2 + 3n + 2$

4. Prove that if every even natural number greater than 2 is a sum of two primes (the Goldbach Conjecture), then every odd natural number greater than 5 is a sum of three primes.

   Assume $2n = p + q$ where p and q are primes and n is any integer such that $n > 1$. If $m = 2n$, then every odd natural number greater than 5 is given my $m + 3$. So $m + 3 = p + q + 3$. But 3 is a prime number, so $m + 3$ is the sum of three primes.

5. Use the method of induction to prove that the sum of the first n odd numbers is equal to n^2.

   $P(1)$ is true since $1 = 1^2$
   Assuming $P(k)$ is true, we show that $P(k + 1)$ is true.
   $P(k) = 1 + 3 + 5... + (2k - 1) = k^2$
   $P(k + 1) = 1 + 3 + 5 + ... + (2k - 1) + (2k + 1)$
   $P(k + 1) = P(k) + (2k + 1) = k^2 + (2k + 1) = (k + 1)^2$
   So $P(k) \to P(k + 1)$ and the result follows by induction.

6. Prove by induction that $\forall \in \mathbb{N} : \sum_{r=1}^{n} r^2 = \frac{1}{6}n(n + 1)(2n + 1)$.
   $P(1)$ is true since $\frac{1}{6}n(n + 1)(2n + 1) = 1$
   Assuming $P(k)$ is true, we show that $P(k + 1)$ is true.
   $\sum_{r=1}^{k+1} r^2 = \frac{1}{6}(k + 1)(k + 2)(2(k + 1) + 1)$
   $= \frac{1}{6}(k + 1)(k + 2)(2(k + 1) + 1)$
   $= \frac{1}{6}(k + 1)(k + 2)(2k + 3)$
   $= \frac{1}{6}(k + 1)(k + 2)(2k + 3)$
   $= \frac{2k^3+9k^2+13k+6}{6}$
   If $P(k)$ is true, then $P(k + 1) = P(k) + (k + 1)^2$.
   $P(k) + (k + 1)^2 = \frac{1}{6}k(k + 1)(2k + 1) + (k + 1)^2$.
   $P(k) + (k + 1)^2 = \frac{1}{6}k(k + 1)(2k + 1) + (k^2 + 2k + 1)$.
   Simplifying gives $P(k) + (k + 1)^2 = \frac{2k^3+9k^2+13k+6}{6}$ which is the same as the result above.
   So $P(k) \to P(k + 1)$ and the result follows by induction.