Diagnosed Particle Disaggregation

Jacob A. Cram

December 2, 2020

1 Definitions and Units

$$m = C_m r^{\alpha} \tag{1}$$

As in DeVries et al. [2014] particle mass m is a function of radius r and scales with a fractal dimension . C_m is a constant.

$$w = C_w r^{\gamma} \tag{2}$$

Sinking speed also scales with mass to another constant . According to Guidi et al. [2008] $\gamma = \alpha - 1$, but we'll keep things in terms of going forward.

$$F = nmw = nC_m C_w r^{\alpha+\gamma} \tag{3}$$

Flux F is a function of particle numbers, mass, and sinking speed.

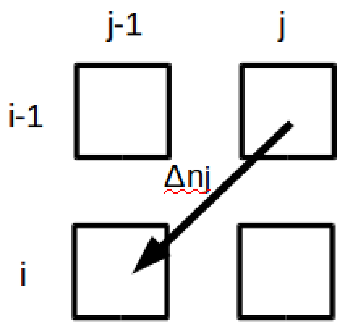

Figure 1: Some number of particles Δn_j of size "j" remineralize to size "j-1" as they sink from depth "i-1" to depth "i".

Going forward we are going to determine the calculations for how many particles of size j in shallow depth i-1 remineralize into particles of size j-1 in deeper depth i. We will call this term Δn_j

2 Conservation of particle number flux

In the absence of disaggregation, the number of particles leaving a box of water is equal to the number of particles going into that box from above. Thus the number of particles in the box is a function of the number of particles going into that box, and the difference in velocities between when the particle enters and when that particle leaves.

$$n_{i-1,j-1}\frac{w_{j-1}}{w_j} + n_{i,j-1} = n_{i,j-1}\frac{w_{j-1}}{w_j} + n_{i,j} \tag{4}$$

Where $n_{i-1,j}$ is the number of particles of size j (the bigger size) at depth i-1 (the shallower depth). The subscripts correspond to locations in Figure 1.

We can re-arrange equation 4

$$n_{i-1,j-1}w_{j-1} + n_{i-1,j}w_j = n_{i,j-1}w_{j-1} + n_{i,j}w_j \tag{5}$$

Substitue in equation 2 into equation 5.

$$n_{i-1,j-1}r_{j-1}^{\gamma} + n_{i-1,j}r_j^{\gamma} = n_{i,j-1}r_{j-1}^{\gamma} + n_{i,j}r_j^{\gamma} \tag{6}$$

Rearrange equation 6

$$r_{j-1}^{\gamma}(n_{i-1,j-1} - n_{i,j-1}) = r_j^{\gamma}(n_{i,j} - n_{i-1,j}) = \Phi \tag{7}$$

Where Φ is a placeholder standing for either side of equation 7, which I will subsequently substitute into things.

Solve for Δn_j

$$\Delta n_j = n_{i,j} - n_{i-1,j} = \frac{r_{j-1}^{\gamma}}{r_j^{\gamma}}(n_{i-1,j-1} - n_{i,j-1}) \tag{8}$$

3 Conservation of Mass Flux

Total flux defined is the sum of flux in each (observed) particle size bin. Particles not in an observed bin don't count towards total flux.

$$\Delta F = \sum \Delta f_j + \Delta f_0 \tag{9}$$

Here Δf_j is the mass loss from bin of size j and $Delta f_0$ is the loss that comes from particles in bin 1 becoming small enough that you can no longer see them with the UVP.

$$\Delta f_j = \frac{\partial f}{\partial z} \Delta z n_{i-1,j} \tag{10}$$

$$\frac{\partial f}{\partial z} = \frac{\partial m}{\partial z}\frac{\partial f}{\partial m} = \frac{\partial m}{\partial t}\frac{\partial t}{\partial z}\frac{\partial f}{\partial m} \tag{11}$$

In PRiSM, fractional mass loss as a function of time is the same for all particles of all sizes.

Now we are going to come up with the values for each of these terms.

Particle remineralization.

$$\frac{\partial m}{\partial t} = C_r * m = C_r C_m r^{\alpha} \tag{12}$$

Sinking speed definition, substituting from equation 2

$$\frac{\partial t}{\partial z} = \frac{1}{w} = \frac{1}{C_w r^{\gamma}} \tag{13}$$

Flux for a given size class, substituting eqation 1, and finally putting everything in terms of mass (rather than mass and radius, since the two are related)

$$f = mw = m * C_w r^{\gamma} = m * C_w (\frac{m}{C_m})^{\frac{\gamma}{\alpha}} \tag{14}$$

Derriving equation 14 with respect to mass, and substituting equation 1

$$\frac{\partial f}{\partial m} = Cw(1 + \frac{\gamma}{\alpha})(\frac{m}{C_m})^{\frac{\gamma}{\alpha}} = C_w(1 + \frac{\gamma}{\alpha})r^{\gamma} \tag{15}$$

Finally, we can construct our equation for flux attenuation by substituting equations 12, 13 and 15 into equation 11

$$\frac{\partial f}{\partial z} = C_r C_m r^{\alpha}(1 + \frac{\gamma}{\alpha}) \tag{16}$$

And now we can solve for equation 17.

$$\Delta f_j = C_r C_m r^{\alpha}(1 + \frac{\gamma}{\alpha})\Delta z * n_{i-1,j} \tag{17}$$

We also need to solve for Δf_0 the flux "attenuation" that actually comes from particles leaving the smallest bin and escaping from what the UVP sees.

$$\Delta f_0 = \Delta n_1 m_1 w_1 = \Delta n_1 C_m C_w r_1^{\gamma} \tag{18}$$

Here, Δn_1 is the number of particles leaving bin j = 1, but we haven't solved for that yet.

4 Solving for Δn_j

Recall that Δn_j is the number of particles that migrate between bin "j" and bin "j-1" as the particles sink from depth "i-1" to depth "i".

The flux at the shallower depth is equal to the flux at the deeper depth, plus the flux that attenuated between those two depths. Since $f = nmw$ and we know m and w

$$n_{i-1,j-1}C_mC_wr_{j-1}^{\alpha+\gamma}+n_{i-1,j}C_mC_wr_j^{\alpha+\gamma} = n_{i,j-1}C_mC_wr_{j-1}^{\alpha+\gamma}+n_{i,j}C_mC_wr_j^{\alpha+\gamma}+\Delta f_j \tag{19}$$

This equation can be re-arranged, and we can substitute in equation 17 for Δf_j.

The C_m cancel out.

$$C_wr_{j-1}^{\alpha+\gamma}(n_{i-1,j-1}-n_{i,j-1}) = C_wr_j^{\alpha+\gamma}(n_{i,j}-n_{i-1,j})+C_r(1+\frac{\gamma}{\alpha})\Delta z n_{i-1,j}r^\alpha \tag{20}$$

We can then substitute in Φ from equation 7.

$$C_wr_{j-1}^\alpha\Phi = C_wr_j^\alpha\Phi + C_r(1+\frac{\gamma}{\alpha})\Delta z n_{i-1,j}r^\alpha \tag{21}$$

Rearrange

$$C_w\Phi(r_{j-1}^\alpha - r_j^\alpha) = Cr(1+\frac{\gamma}{\alpha})\Delta z r^\alpha n_{i-1,j} \tag{22}$$

solve for Φ

$$\Phi = \frac{\frac{C_r}{C_w}\Delta z r^\alpha n_{i-1,j}(1+\frac{\gamma}{\alpha})}{r_{j-1}^\alpha - r_j^\alpha} \tag{23}$$

$$\Delta n_j = \frac{\Phi}{r_j^\gamma} = \frac{\frac{C_r}{C_w}\Delta z r^\alpha n_{i-1,j}(1+\frac{\gamma}{\alpha})}{r_j^\gamma(r_{j-1}^\alpha - r_j^\alpha)} \tag{24}$$

$$\Delta n_{j-1} = \frac{\Phi}{r_{j-1}^\gamma} = \frac{\Delta n_j r_j^\gamma}{r_{j-1}^\gamma} \tag{25}$$

At this point, the only unsolved variable is C_r, which we can now calculate.

5 Solving for C_r

We can calculate ΔF, the attenuation of flux and can impose the size spectrum and all of the other constants. Here we find the C_r that gives us the correct ΔF

First, to solve equation 9 by substituting in equaitons 17 and 18

$$\Delta F = \sum_j \Delta f_j + \Delta f_0 = \sum_{j=1}^{n} \left\{ C_r C_m r_j^{\alpha} (1 + \frac{\gamma}{\alpha}) \Delta z n_{i-1,j} \right\} + \Delta n_1 C_m C_w r_1^{\gamma} \quad (26)$$

Substitute equation 24 for Δn_j when j = 1 for Δn_1

$$\Delta F = \sum_j \Delta f_j + \Delta f_0 = \sum_{j=1}^{n} \left\{ C_r C_m r_j^{\alpha} (1 + \frac{\gamma}{\alpha}) \Delta z n_{i-1,j} \right\} + \frac{\frac{C_r}{C_w} \Delta z r^{\alpha} n_{i-1,1} (1 + \frac{\gamma}{\alpha})}{r_1^{\gamma} (r_0^{\alpha} - r_1^{\alpha})} C_m C_w r_1^{\gamma} \quad (27)$$

Pull what I can out of the sum operation

$$\Delta F = C_r C_m \Delta z (1 + \frac{\gamma}{\alpha}) \sum_{j=1}^{n} \left\{ r_j^{\alpha} ni - 1, j \right\} + \frac{\frac{C_r}{C_w} \Delta z r^{\alpha} n_{i-1,1} (1 + \frac{\gamma}{\alpha})}{r_1^{\gamma} (r_0^{\alpha} - r_1^{\alpha})} C_m C_w r_1^{\gamma} \quad (28)$$

Now we can solve for C_r

$$C_r = \frac{\Delta F}{C_m \Delta z \left[(1 + \frac{\gamma}{\alpha}) \sum_{j=1}^{n} \left\{ r_j^{\alpha} n_{i-1,j} \right\} + \frac{r_1^{2\alpha} n_{i-1,1}}{r_0^{\alpha} - r_1^{\alpha}} \right]} \quad (29)$$

Thus for a pair of profiles, we can estemate the flux attenuation, calculate Cr from that, and then plug Cr (and the profile) into the equation 24 for Δn_j. We can thus compute Δn_j for each size class to see how many particles from that bin move to the next bin smaller.