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Pîrlea Silvia-Cristina
grupa 215

Examen sisteme dinamice

Exercițiul 1

$$\begin{cases} x'(t) = r x \\ x(0) = x_0 \end{cases}$$

$x' = r x$ $\qquad x' = \frac{dx}{dt}$

$\frac{dx}{dt} = r x$

$\frac{dx}{x} = r\,dt \Rightarrow \int \frac{dx}{x} = \int r\,dt$

$\ln x = r \cdot t + \ln c$

$x = c \cdot e^{rt}$ – soluția generală $\quad c \in \mathbb{R}$

$x(0) = x_0$

$\Rightarrow c = x_0 \Rightarrow x = x_0 e^{rt}$ – soluție modelului lui Malthus

$x_0 = 1000$

$x(t_1) = x_1 \qquad t_1 = 10 \qquad x_1 = 50000$

$\Rightarrow x_1 = x_0 e^{rt_1}$

$50000 = 1000 e^{10r} \quad |:1000$

$e^{10r} = 50 \quad | \ln$

$10r = \ln 50$

$r = \frac{1}{10} \ln 50$ $\qquad S = \{\frac{1}{10} \ln 50\}$

Exercițiul 2:

a) $x^2 y' = xy + y^2$

$y' = \frac{xy + y^2}{x^2}$

$y' = \frac{y}{x} + \frac{y^2}{x^2}$

notăm $z = \frac{y}{x} \Rightarrow y = zx$

$y' = z'x + z$

$(=) \quad z'x + z = z + z^2 \quad |-z$

$z'x = z^2$

$z' = \frac{z^2}{x} \qquad z' = \frac{dz}{dx}$

$\frac{dz}{dx} = \frac{z^2}{x}$

$\frac{dz}{z^2} = \frac{1}{x}dx$

$\int \frac{dz}{z^2} = \int \frac{dx}{x}$

$-\frac{1}{z} = \ln x + \ln c$

$-\frac{1}{z} = \ln cx \qquad c \in \mathbb{R}$

$z = -\frac{1}{\ln cx}$

$y = zx \Rightarrow y(x) = -\frac{x}{\ln cx}$ soluția generală a ecuației

b) $y'' - 4y' + 8y = (25x - 5)e^x$

I Ecuația omogenă:

$y'' - 4y' + 8y = 0$

ecuația caracteristică asociată este

$r^2 - 4r + 8 = 0 \qquad \Delta = 16 - 32 = -16 = 16i^2$

$r_{1,2} = \frac{4 \pm 4i}{2} = 2 \pm 2i \quad < \begin{matrix} y_1 = e^{2x}\cos 2x \\ y_2 = e^{2x}\sin 2x \end{matrix}$

$y_0 = c_1 y_1 + c_2 y_2$

$y_0 = c_1 e^{2x}\cos 2x + c_2 e^{2x}\sin 2x$

II Sol. particulară

$y'' - 4y' + 8y = (25x - 5)e^x$

$f(x) = (25x - 5)e^x \Rightarrow$ căutăm y_p de forma $y_p = Q_1(x)e^x$

$y_p = (ax + b)e^x$

$y_p' = (ax + b)e^x + ae^x = e^x(ax + a + b)$

$y_p'' = (ax + b)e^x + ae^x + ae^x = e^x(ax + 2a + b)$

înlocuim în ecuație și avem

$e^x(ax+2a+b) - 4e^x(ax+a+b) + 8e^x(ax+b) = (25x-5)e^x$

$e^x(5ax - 2a + 5b) = (25x-5)e^x$

$\Leftrightarrow \begin{cases} 5a = 25 \\ -2a+5b = -5 \end{cases} \Leftrightarrow \begin{cases} a = 5 \\ -10+5b = -5 \end{cases} \Leftrightarrow \begin{cases} a=5 \\ b=1 \end{cases}$

$\Rightarrow y_p = (5x+1)e^x$

$y = y_0 + y_p$

$y(x) = c_1 e^{2x}\cos 2x + c_2 e^{2x}\sin 2x + (5x+1)e^x$ soluția generală a ecuației

Exercițiul 3

$$\begin{cases} xy'' + y' = 4x \\ y(1) = 1 \\ y'(1) = 4 \end{cases}$$

$xy'' + y' = 4x$

notăm $z = y' \Rightarrow z' = y''$

Avem:

$xz' + z = 4x$ – ecuație dif. liniară de ordinul I

I Ecuația omogenă:

$xz' + z = 0$

$z' = -\frac{z}{x}$ $\quad z' = \frac{dz}{dx}$

$\frac{dz}{dx} = -\frac{z}{x}$

$\int \frac{dz}{z} = \int -\frac{dx}{x}$

$\ln z = -\ln x + \ln c$

$\ln z = \ln \frac{c}{x}$

$z_0 = \frac{c}{x}$ $\quad c \in \mathbb{R}$

II Sol particulară (utilizăm metoda variației constantelor)

$z_p = \frac{f}{x}$

$z_p' = \frac{f'x - f}{x^2} = \frac{f'}{x} - \frac{f}{x^2}$

înlocuim în ecuație și avem:

$$x z_p' + z_p = 4x$$

$$x\left(\frac{f'}{x} - \frac{f}{x^2}\right) + \frac{f}{x} = 4x$$

$$f' - \cancel{\frac{f}{x}} + \cancel{\frac{f}{x}} = 4x$$

$$f' = 4x$$

$$f = \int 4x\,dx$$

$$f = 2x^2$$

$\Rightarrow z_p = \frac{2x^2}{x}$ $z_p = 2x$

$z = z_0 + z_p$

$z(x) = \frac{c}{x} + 2x$

$y'(x) = z(x) \Rightarrow y(x) = \int z(x)\,dx = \int \frac{c}{x} + 2x\,dx$ $c = c_1$

$\boxed{y(x) = c_1 \ln x + x^2 + c_2}$ soluția generală a ecuației dif.

$y(x) = c_1 \ln x + x^2 + c_2$ $c_1, c_2 \in \mathbb{R}$

$y'(x) = \frac{c_1}{x} + 2x$

$\begin{array}{l} y(1) = 1 \\ y'(1) = 4 \end{array} \Leftrightarrow \begin{cases} 1 + c_2 = 1 \\ \frac{c_1}{1} + 2 = 4 \end{cases} \begin{cases} 1 + c_2 = 1 \\ c_1 + 2 = 4 \end{cases} \Leftrightarrow \begin{cases} c_2 = 0 \\ c_1 = 2 \end{cases}$

$\Rightarrow y(x) = 2\ln x + x^2$ – soluția problemei Cauchy

Exercițiul 4

$$\begin{cases} x'(t) = -x + e^x y \\ y'(t) = -e^x x - y \end{cases}$$

$x^*(0,0)$ punct de echilibru

$V(x,y) = x^2 + y^2$

$V(0,0) = 0$

$V(x,y) = x^2 + y^2 > 0 \ \forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ $\Big| \Rightarrow$ funcția V e o funcție de tip Lyapunov

$f_1(x,y) = -x + e^x y$

$f_2(x,y) = -e^x x - y$

$$\dot{V}(x,y) = \frac{\partial V}{\partial x}(x,y) f_1(x,y) + \frac{\partial V}{\partial y}(x,y) f_2(x,y)$$

$$\dot{V}(x,y) = 2x(-x + e^x y) + 2y(-e^x x - y)$$

$\dot{V}(x,y) = -2x^2 + \cancel{2xye^x} - \cancel{2xye^x} - 2y^2$

$\dot{V}(x,y) = -2x^2 - 2y^2 < 0 \quad \forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$\Rightarrow$ $X^*(0,0)$ punct de echilibru local ~~stabil~~ asimptotic stabil

Exercițiul 6:

$$\begin{cases} y' = 3x^2 + xy^2 \\ y(0) = 1 \end{cases}$$

Ecuația integrală Volterra:

$$y(x) = y^0 + \int_{x_0}^{x} f(s, y(s))\,ds$$

$x_0 = 0 \quad y^0 = 1$

$$\Rightarrow \quad y(x) = 1 + \int_0^x f(s, y(s))\,ds$$

$$y(x) = 1 + \int_0^x 3s^2 + sy^2(s) = 1 + \int_0^x 3s^2\,ds + \int_0^x sy^2(s)\,ds$$

$$y(x) = 1 + s^3\Big|_0^x + \int_0^x sy^2(s)\,ds$$

$$y(x) = 1 + x^3 + \int_0^x sy^2(s)\,ds$$

Șirul aproximațiilor succesive:

$$y_{n+1}(x) = y^0 + \int_{x_0}^{x} f(s, y_n(s))\,ds$$

$$y_{n+1}(x) = 1 + x^3 + \int_0^x s y_n^2(s)\,ds$$

$y_0(x) \equiv 1$

$$\Rightarrow \quad y_1(x) = 1 + x^3 + \int_0^x s y_0^2(s)\,ds$$

$$y_1(x) = 1 + x^3 + \int_0^x s\,ds = 1 + x^3 + \frac{s^2}{2}\Big|_0^x$$

$$y_1(x) = x^3 + \frac{x^2}{2} + 1$$

$$y_2(x) = 1 + x^3 + \int_0^x s y_1^2(s)\,ds$$

$$y_2(x) = 1 + x^3 + \int_0^x s\left(s^3 + \frac{s^2}{2} + 1\right)^2 ds = 1 + x^3 + \int_0^x s\left(s^6 + \frac{s^4}{4} + 1 + s^5 + 2s^3 + s^2\right) ds$$

$$y_2(x) = 1 + x^3 + \int_0^x \left(s^7 + \frac{s^5}{4} + s + s^6 + 2s^4 + s^3\right) ds$$

$$y_2(x) = 1 + x^3 + \frac{s^8}{8} + \frac{s^6}{24} + \frac{s^2}{2} + \frac{s^7}{7} + \frac{2s^5}{5} + \frac{s^4}{4}\Big|_0^x$$

$$y_2(x) = 1 + x^3 + \frac{x^8}{8} + \frac{x^6}{24} + \frac{x^2}{2} + \frac{x^7}{7} + \frac{2x^5}{5} + \frac{x^4}{4}$$

$$y_2(x) = \frac{x^8}{8} + \frac{x^7}{7} + \frac{x^6}{24} + \frac{2x^5}{5} + \frac{x^4}{4} + x^3 + \frac{x^2}{2} + 1$$

Exercițiul 7:

$$\begin{cases} x'(t) = -4y(t) \\ y'(t) = x(t) \end{cases}$$

a) $$\begin{cases} x' = -4y \\ y' = x \\ x(0) = \eta_1 \\ y(0) = \eta_2 \end{cases}$$

derivăm prima ecuație și avem:

$$x'' = -4y'$$

$x'' = -4x$ - ec. dif de ord. II liniară omogenă cu coef. const.

ecuația caracteristică asociată este:

$$r^2 = -4 = 4i^2 \Rightarrow r_{1,2} = \pm 2i \quad \begin{matrix} x_1 = \cos 2t \\ x_2 = \sin 2t \end{matrix}$$

$$\Rightarrow x = c_1 \cos 2t + c_2 \sin 2t$$

$$x' = -2c_1 \sin 2t + 2c_2 \cos 2t$$

$$y = \frac{x'}{-4} \Rightarrow y = \frac{-2c_1 \sin 2t + 2c_2 \cos 2t}{-4} = -\frac{c_2}{2} \cos 2t + \frac{c_1}{2} \sin 2t$$

$$\begin{cases} x(t) = c_1 \cos 2t + c_2 \sin 2t \\ y(t) = -\frac{c_2}{2} \cos 2t + \frac{c_1}{2} \sin 2t \end{cases}$$ soluția generală a sistemului liniar

$$\begin{matrix} x(0) = \eta_1 \Leftrightarrow c_1 = \eta_1 \\ y(0) = \eta_2 \Leftrightarrow -\frac{c_2}{2} = \eta_2 \end{matrix} \Leftrightarrow \begin{cases} c_1 = \eta_1 \\ c_2 = -2\eta_2 \end{cases}$$

$I_\eta = \mathbb{R}$

$\Rightarrow$ ~~$x(t, \eta_1, \eta_2) = \eta_1 \cos 2t + 2\eta_2$~~

$$x(t, \eta_1, \eta_2) = \eta_1 \cos 2t - 2\eta_2 \sin 2t$$

$$y(t, \eta_1, \eta_2) = \eta_2 \cos 2t + \frac{\eta_1}{2} \sin 2t$$

$f: \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2 \quad f(t, \eta_1, \eta_2) = (x(t, \eta_1, \eta_2), y(t, \eta_1, \eta_2))$

$$\Rightarrow f(t, \eta_1, \eta_2) = \left(\eta_1 \cos 2t - 2\eta_2 \sin 2t, \; \eta_2 \cos 2t + \frac{\eta_1}{2} \sin 2t\right)$$

fluxul generat de sistem

b) $\begin{cases} x' = -4y \\ y' = x \end{cases}$

ecuația diferențială a orbitelor: $\dfrac{dx}{dy} = \dfrac{-4y}{x}$

$$-4y\,dy = x\,dx$$

$$\int 4y\,dy = \int -x\,dx$$

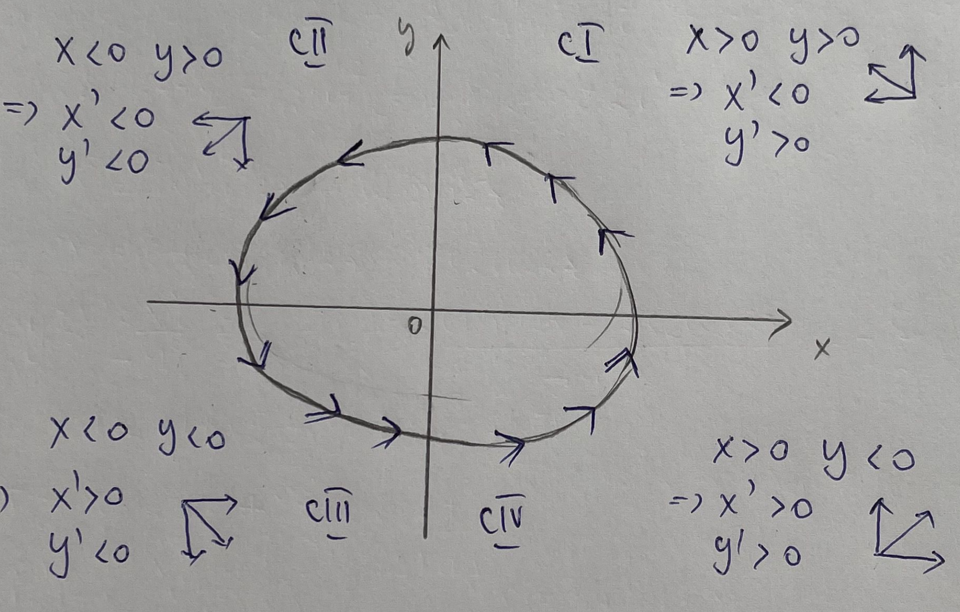
$$2y^2 = -\frac{x^2}{2} + c$$

$$2y^2 + \frac{x^2}{2} = c \quad \text{- ecuația orbitelor}$$

Se observă că ecuația orbitelor reprezintă ecuația unei elipse

$\Rightarrow$ Orbitele au forma unor elipse

Portretul fazic:



$X^*(0,0)$ punct de echilibru

$$A = \begin{pmatrix} 0 & -4 \\ 1 & 0 \end{pmatrix}$$

$$\det(\lambda I_2 - A) = 0$$

$$\begin{vmatrix} \lambda & 4 \\ -1 & \lambda \end{vmatrix} = 0$$

$$\lambda^2 + 4 = 0 \Rightarrow \lambda_{1,2} = \pm 2i$$

$$Re(\lambda_1) = Re(\lambda_2) = 0 \leq 0$$

$\Rightarrow X^*(0,0)$ punct de echilibru local stabil de tip centru

Exercițiul 5

$$x' = ax^2 - x^3 - a + x$$

$f(x) = ax^2 - x^3 - a + x$

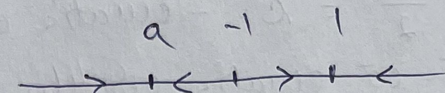
$f(x) = 0 \iff ax^2 - x^3 - a + x = 0$

$x^2(a-x) - (a-x) = 0$

$(a-x)(x^2-1) = 0 \Rightarrow x_1 = a,\ x_2 = 1,\ x_3 = -1$ puncte de echilibru

c I $a < -1$

x	$-\infty$		a		-1		1		$+\infty$
$f(x)$	+	+	0	−	0	+	0	−	−
		$\to$		$\leftarrow$		$\to$		$\leftarrow$	



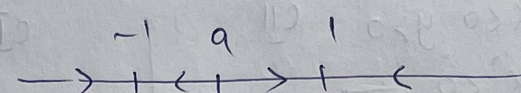
$x_1 = a$ – local as stabil

$x_2 = 1$ – local as stabil

$x_3 = -1$ – instabil

c II $a \in (-1, 1)$

x	$-\infty$		-1		a		1		$+\infty$
$f(x)$	+	+	0	−	0	+	0	−	
		$\to$		$\leftarrow$		$\to$		$\leftarrow$	



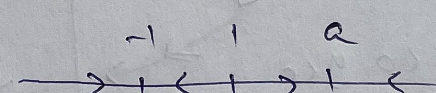
$x_1 = a$ – instabil

$x_2 = 1$ – local as stabil

$x_3 = -1$ – local as stabil

c III $a \in (1, \infty)$

x	$-\infty$		-1		1		a		
$f(x)$	+	+	0	−	0	+	0	−	−
		$\to$		$\leftarrow$		$\to$		$\leftarrow$	



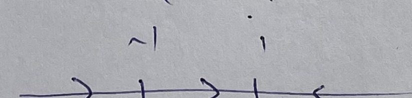
$x_1 = a$ local as stabil

$x_2 = 1$ instabil

$x_3 = -1$ local as stabil

c IV $a = -1 \Rightarrow f(x) = (-1-x)(x-1)(x+1) = -(x+1)^2(x-1)$

x	$-\infty$		-1		1		$+\infty$
$f(x)$	+	+	0	+ +	0	−	− −
		$\to$		$\to$		$\leftarrow$	

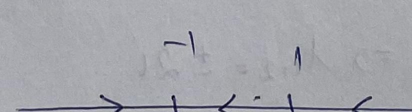


$x_1 = -1$ – instabil

$x_2 = 1$ – local as stabil

c V $a = 1 \Rightarrow f(x) = (1-x)(x-1)(x+1) = -(x-1)^2(x+1)$

x	$-\infty$		-1		1		$+\infty$
$f(x)$	+	+	0	− −	0	−	− −
		$\to$		$\leftarrow$		$\leftrightarrow$	



$x_1 = -1$ local as stabil

$x_2 = 1$ instabil