CSC 226 - ASSIGNMENT 3 - SOLUTIONS

1. Suppose G is a connected graph with distinct, positive integer edge weights. Prove that G has <u>exactly</u> one MST T.

<u>Proof</u>: Assume for purposes of contradiction that there exists two MSTs for G, call them T and T'.

→ Since T + T' are distinct MSTs each contains at least one edge that the other does not.

→ Let $e \in T$ + $e' \in T'$ be the edges with minimum weight that do not exist in the other tree, respectively.

→ i.e. $e \in T$ but $e \notin T'$ + $e' \in T'$ but $e' \notin T$,

→ Since all the edge weights are distinct, assume without loss of generality that $w(e) < w(e')$.

→ Now, place edge e in tree T' creating cycle C. Furthermore, cycle C must contain at least one edge $f \in T'$ not in T otherwise C would be a cycle in T + T would not be a tree.

→ $w(f)$ must be greater than $w(e)$ since $w(e) < w(e') \leq w(f)$ by assumption above.

→ Remove f from T' and we have a new spanning tree T^* with $w(T^*) < w(T)$.

→ This is a contradiction since T' was a minimum spanning tree.

→ Therefore, G has exactly one MST.

2. Let V' be any subset of V in $G = (V, E)$ and let e be a minimum weight edge that has exactly one endpoint in V'. Then, there exists a minimum spanning tree T for G that contains e.

Proof: By contradiction, assume that no MST exists that contains e as described above.

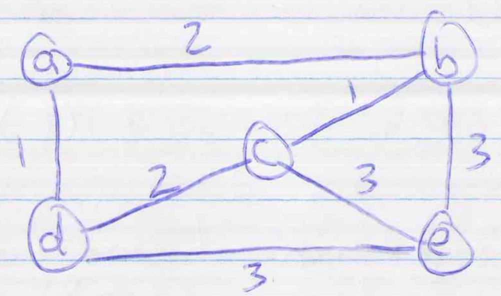
→ Let T be a MST of G and add e to T, creating a cycle C.

→ Since e connects V' + $V-V'$ then there must exist some other edge f that connects V' and $V-V'$ or e would have had to be part of T, such that $f \in C$ too.

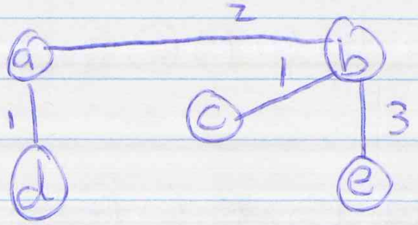
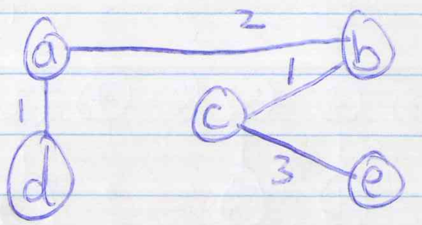
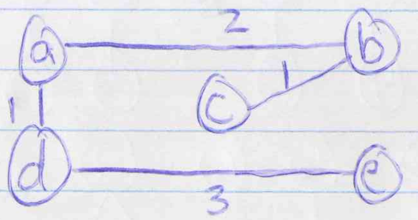
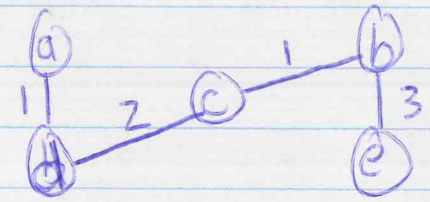
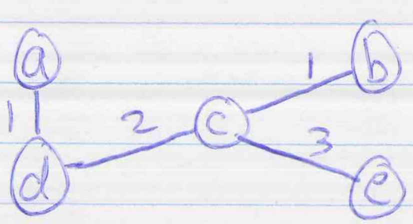
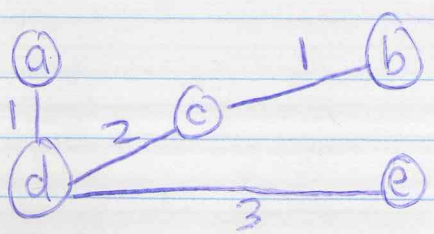
→ $w(f) \geq w(e)$ by definition of e above and thus, if we remove f from C we get a new spanning tree T' with weight $w(T') \leq w(T)$, implying that T' is a MST containing e, a contradiction.

→ Therefore, there must exist at least one MST of G containing e.

3. $G =$



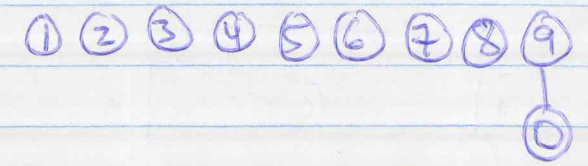
Conjecture: All MSTs T of a graph G have the same profile.

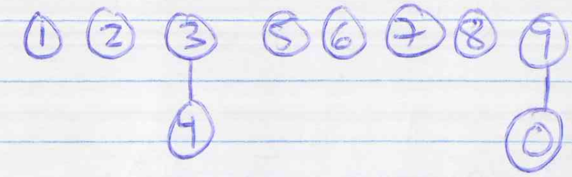
MST		edges	weight	profile
(1.)		(a,b) (b,c) (b,e) (d,a)	7	1,1,2,3
(2.)		(a,b) (b,c) (c,e) (d,a)	7	1,1,2,3
(3.)		(a,b) (b,c) (d,a) (d,e)	7	1,1,2,3
(4.)		(b,c) (b,e) (c,d) (d,a)	7	1,1,2,3
(5.)		(b,c) (c,d) (c,e) (d,a)	7	1,1,2,3
(6.)		(b,c) (c,d) (d,a) (d,e)	7	1,1,2,3

4.

	i	0	1	2	3	4	5	6	7	8	9
(1)	id[i]	0	1	2	3	4	5	6	7	8	9
(2)	9-0	$\boxed{9}$	1	2	3	4	5	6	7	8	9
(3)	3-4	9	1	2	3	$\boxed{3}$	5	6	7	8	9
(4)	5-8	9	1	2	3	3	5	6	7	$\boxed{5}$	9
(5)	7-2	9	1	$\boxed{7}$	3	3	5	6	7	5	9
(6)	2-1	9	$\boxed{7}$	7	3	3	5	6	7	5	9
(7)	5-7	9	7	7	3	3	7	6	7	5	9
(8)	0-3	9	7	7	$\boxed{9}$	3	7	6	7	5	9
(9)	4-2	9	7	7	9	3	7	6	7	5	$\boxed{7}$

(1)


(2)


(3)


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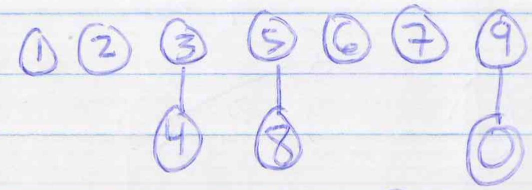
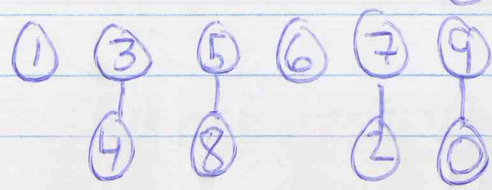
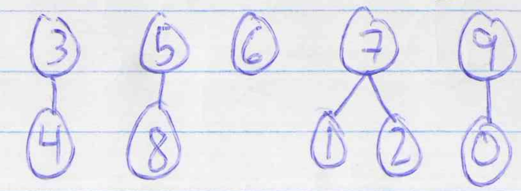
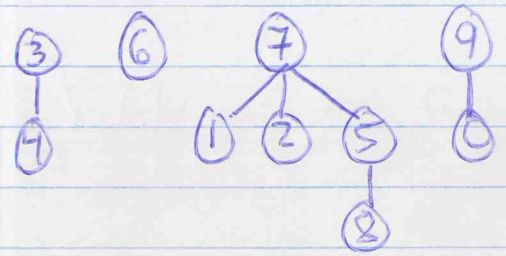
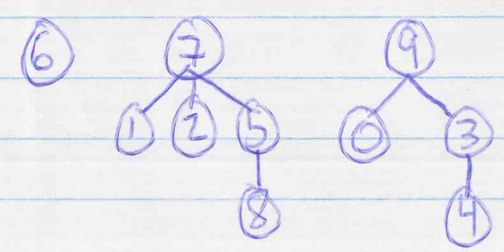
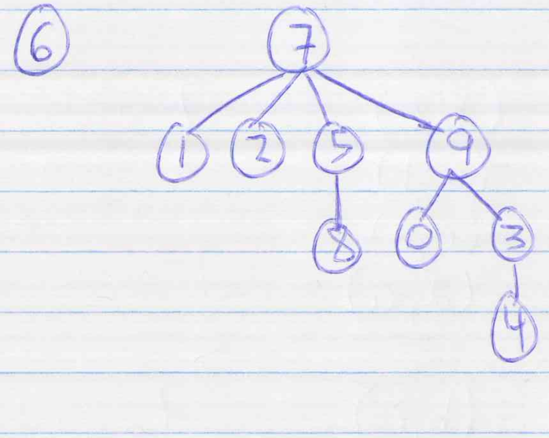


(5)


(6)


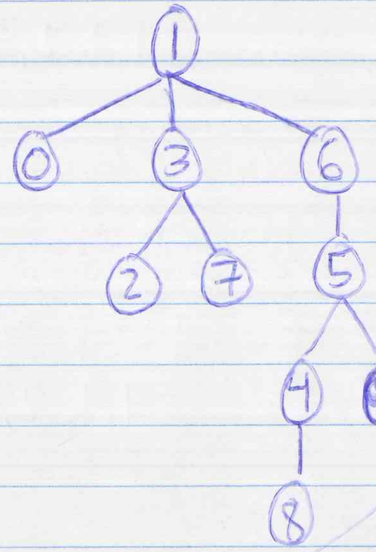
(7)


(8)


(9)


5.

i	0	1	2	3	4	5	6	7	8	9
id[i]	1	1	3	1	5	6	1	3	4	5



→ At some point node 5 was merged to node 6.

→ At that time size of 5 ~~was~~ had to be less than or equal to size of 6.

→ subsequently, any more unions with this tree would append the other tree directly to 6 as a child.

→ this could only happen when 6 + 5 were singletons.

Notes:

1. The proofs in question 1 + 2 will not necessarily be the same as mine. Try to follow their logic and mark accordingly.
2. Question 4: when the tree's are the same size they can arbitrarily add one to the other. So, their final forest may look slightly different

→ also, they may union-by-height instead of size (number of nodes) and that is okay too.