Theorem 88 The Dot Product and Angles

Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$. Then

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta,$$

where θ, $0 \leq \theta \leq \pi$, is the angle between $\vec{u}$ and $\vec{v}$.

e^(x)+e^(-x)+2

factors as (e^(x)+e^(-x))^2

for all even x

Key Idea 53 Distances to Lines

1. Let P be a point on a line ℓ that is parallel to $\vec{d}$. The distance h from a point Q to the line ℓ is:

$$h = \frac{\left\|\vec{PQ} \times \vec{d}\right\|}{\left\|\vec{d}\right\|}.$$

2. Let P_1 be a point on line ℓ_1 that is parallel to $\vec{d_1}$, and let P_2 be a point on line ℓ_2 parallel to $\vec{d_2}$, and let $\vec{c} = \vec{d_1} \times \vec{d_2}$, where lines ℓ_1 and ℓ_2 are not parallel. The distance h between the two lines is:

$$h = \frac{|\vec{P_1P_2} \cdot \vec{c}|}{\|\vec{c}\|}.$$

Definition 66 Equations of a Plane in Standard and General Forms

The plane passing through the point $P = (x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c \rangle$ can be described by an equation with standard form

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0;$$

the equation's general form is

$$ax + by + cz = d;$$

the equation's vector form is

$$\langle x, y, z \rangle \cdot \vec{n} = \langle x_0, y_0, z_0 \rangle \cdot \vec{n} = d.$$

the same line	they share all points;
intersecting lines	share only 1 point;
parallel lines	$\vec{d}_1 \parallel \vec{d}_2$, no points in common; or
skew lines	$\vec{d}_1 \nparallel \vec{d}_2$, no points in common.

1. Are unit vectors equal?
2. Do they share any points?
3. If yes & ||, same line
4. If yes & !||, Intersect
5. If No, Skew

Theorem 90 The Cross Product and Angles
Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^3$. Then

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \, \|\vec{v}\| \sin\theta,$$

where θ, $0 \le \theta \le \pi$, is the angle between $\vec{u}$ and $\vec{v}$.

Definition 65 Equations of Lines in Space
Consider the line in space that passes through $\vec{p} = \langle x_0, y_0, z_0 \rangle$ in the direction of $\vec{d} = \langle a, b, c \rangle$.

1. The vector equation of the line is

$$\vec{\ell}(t) = \vec{p} + t\vec{d}.$$

2. The parametric equations of the line are

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$

3. The symmetric equations of the line are

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Key Idea 54 Distance from a Point to a Plane
Let a plane with normal vector $\vec{n}$ be given, and let Q be a point. The distance h from Q to the plane is

$$h = \frac{|\vec{n} \cdot \vec{PQ}|}{\|\vec{n}\|},$$

where P is any point in the plane.

Key Idea 55 Cylindrical coordinates (r, θ, z)

$$\begin{aligned} x &= r\cos\theta & r &= \sqrt{x^2+y^2} \\ y &= r\sin\theta & \theta &= \tan^{-1}\left(\tfrac{y}{x}\right) \\ z &= z & z &= z \end{aligned}$$

where $0 \leq \theta \leq \pi$ if $y \geq 0$ and $\pi < \theta < 2\pi$ if $y < 0$.

Theorem 98 Distance Traveled

Let $\vec{v}(t)$ be a velocity function for a moving object. The distance traveled by the object on $[a, b]$ is:

$$\text{distance traveled} = \int_a^b \|\vec{v}(t)\|\ dt.$$

Key Idea 58 Average Speed, Average Velocity

Let $\vec{r}(t)$ be a continuous position function on an open interval I containing $a < b$.

The average speed is:

$$\frac{\text{distance traveled}}{\text{travel time}} = \frac{\int_a^b \|\vec{v}(t)\|\ dt}{b-a} = \frac{1}{b-a}\int_a^b \|\vec{v}(t)\|\ dt.$$

The average velocity is:

$$\frac{\text{displacement}}{\text{travel time}} = \frac{\int_a^b \vec{r}\,'(t)\,dt}{b-a} = \frac{1}{b-a}\int_a^b \vec{r}\,'(t)\,dt.$$

Theorem 102 Formulas for Curvature

Let C be a smooth curve on an open interval I in the plane or in space.

1. If C is defined by $y = f(x)$, then

$$\kappa = \frac{|f''(x)|}{\left(1+\left(f'(x)\right)^2\right)^{3/2}}.$$

2. If C is defined as a vector–valued function in the plane, $\vec{r}(t) = \langle x(t), y(t)\rangle$, then

$$\kappa = \frac{|x'y''-x''y'|}{\left((x')^2+(y')^2\right)^{3/2}}.$$

3. If C is defined in space by a vector–valued function $\vec{r}(t)$, then

$$\kappa = \frac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|} = \frac{\|\vec{r}\,'(t)\times\vec{r}\,''(t)\|}{\|\vec{r}\,'(t)\|^3} = \frac{\vec{a}(t)\cdot\vec{N}(t)}{\|\vec{v}(t)\|^2}.$$

Theorem 100 Acceleration in the Plane Defined by $\vec{T}$ and $\vec{N}$

Let $\vec{r}(t)$ be a position function with acceleration $\vec{a}(t)$ and unit tangent and normal vectors $\vec{T}(t)$ and $\vec{N}(t)$. Then $\vec{a}(t)$ lies in the plane defined by $\vec{T}(t)$ and $\vec{N}(t)$; that is, there exists scalars a_T and a_N such that

$$\vec{a}(t) = a_T\vec{T}(t) + a_N\vec{N}(t).$$

Moreover,

$$a_T = \vec{a}(t)\cdot\vec{T}(t) = \frac{d}{dt}\Big(\|\vec{v}(t)\|\Big)$$

$$a_N = \vec{a}(t)\cdot\vec{N}(t) = \sqrt{\|\vec{a}(t)\|^2 - a_T^2} = \frac{\|\vec{a}(t)\times\vec{v}(t)\|}{\|\vec{v}(t)\|} = \|\vec{v}(t)\|\ \|\vec{T}\,'(t)\|$$

Key Idea 56 Spherical coordinates (ρ, θ, ϕ)

$$x = \rho\sin\phi\,\cos\theta \qquad \rho = \sqrt{x^2+y^2+z^2}$$

$$y = \rho\sin\phi\,\sin\theta \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$z = \rho\cos\phi \qquad \phi = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

where $0 \le \theta \le \pi$ if $y \ge 0$ and $\pi < \theta < 2\pi$ if $y < 0$.

$$\text{arc length} = \int_0^t \|\vec{r}\,'(u)\|\ du.$$