Relations

> Definition: Let A, B be sets. A **binary relation** from A to B is simply a subset of $A \times B$.

If $R \subseteq A \times B$ is a binary relation,

- $a \, R \, b$ denotes $(a, b) \in R$, which reads "a is related to b by R"
- $a \, R \, b$ denotes $(a, b) \notin R$

Example: $\leq \subseteq \mathbb{N} \times \mathbb{N}$
$2 \leq 5$ means $(2, 5) \in \leq$

Example:

- Let Courses be set of all SU courses
- Let Semesters be set of all possible semesters
- Your transcript is a relation from Semesters to Courses

So

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(Fall 2013, CIS 275)
(Fall 2013, Mat 397)
(Fall 2014, CIS 351)
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Example:

- Let $X = \{1, 2\}$
- Let $Y = \{a, b\}$
- Following are relations form X to Y:
  - $\{1, a), (2, b), (2, a)\}$
  - $\{(3, a)\}$

Question: Suppose $|A| = m$,
$|B| = n$

How many binary relations are there from A to B. In other words,

$$|\mathcal{P}(A \times B)| = 2^{|A \times B|} = 2^{mn}$$

> Definition: a **relation on the set** A is a binary relation from A to A.

Examples: Following are relations on set ${2, 4, 8}:

- $\{(2,2),(4,8),(2,8)\}$
- $\{(4,8),(8,4)\}$
- $\emptyset$

> Definition: Let $R \subseteq X \times Y$ be a relation. The **inverse of** R is
>
> $$R^{-1} \subseteq Y \times X$$
>
> where
>
> $$R^{-1} = \{(y,x) | (x,y) \in R\}$$

Examples

1. Suppose $T = \{(a,1),(b,2),(c,2),(c,8)\}$. Then $T^{-1} = \{(1,a),(2,b),(2,c),(8,c)\}$
2. For $\leq \subseteq \mathbb{N} \times \mathbb{N}$:

$$\leq^{-1} = \geq$$

3. Define P as the relation on the set of people such that xPy iff x is a parent of y. Then

$$P^{-1} = \text{"is a child of"}$$
$$(a,b) \in P^{-1} \text{ iff } (b,a) \in P$$

Representing Relations Graphically

Example: T can be represented as $T \subseteq \{a,b,c,d\} \times \{1,2,4,8,10\}$

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a.        1.
b.        2.
c.        4.
d.        8.
         10.
```

Draw arrows from left column entries to right column entries.

R is reflexive iff $\forall x \in A, (x, x) \in R$

R is symmetric iff $\forall x, y \in A, ((x, y) \in A \rightarrow (y, x) \in A)$

R is antisymmetric iff $\forall x, y \in A, ((x, y) \in R \wedge (y, x) \in R \rightarrow x = y)$

R is transitive iff $\forall x, y, z \in A, (((x, y) \in R \wedge (y, z) \in R) \rightarrow (x, z) \in R)$