> Notes for 10/8/13

Proof Techniques

$$(H_1 \wedge H_2 \wedge \ldots \wedge H_k) \Rightarrow C$$

Direct Proof

- What to assume: $H_1 \wedge H_2 \wedge \ldots \wedge H_k$
- What to show: C

Proof by Contraposition

- What to assume: $\neg C$
- What to show: at least one hypothesis is false
  - $(\neg H_1 \vee \neg H_2 \vee \ldots \vee \neg H_k)$

Proof by Contradiction

- What to assume: $(H_1 \wedge H_2 \wedge \ldots \wedge H_k) \wedge \neg C$
- What to show: a contradiction

Example Proof

Claim: Let n be an integer. If n is even, then is not odd.

Proof (contradiction):

- Assumptions
  - n is even
  - it is not the case that n is not odd (i.e., n **is odd**)

[NTS: a contradiction]

Since n is even, there exists an integer k such that $n = 2k$.

Because n is odd, there exists an integer l such that $n = 2l + 1$

Therefore,

$$
\begin{aligned}
2k &= 2l + 1 \\
k &= \frac{2l+1}{2} \\
k &= l + \frac{1}{2}
\end{aligned}
$$

Since $k - l = \frac{1}{2}$, at least one of k, l, is not an integer, contradicting claims that k and l were both integers.

Since negating the desired conclusion led to a contradiction, the claim itself is true.

Definitions

- A number is **rational** IFF it can be expressed in the form p/q where p and q are both integers and $q \neq 0$.
- A real number is **irrational** IFF it is not rational.

Claim: $\sqrt{2}$ is irrational.

> In a direct proof, we would have to show there is no choice of p and q that would have the property that $\sqrt{2} = p/q$. This is impossible, so we need to choose a different proof method.

Proof (contradiction):

Suppose $\sqrt{2}$ is not irrational. This means $\sqrt{2}$ is rational.

[NTS: a contradiction]

Because $\sqrt{2}$ is rational, there exist integers p and q such that $p/q = \sqrt{2}$ and $q \neq 0$.

Furthermore, p and q can be chosen such that $\gcd(p, q) = 1$. Thus, by algebra,

$$
\begin{aligned}
2 &= \frac{p^2}{q^2} \\
p^2 &= 2q^2
\end{aligned}
$$

Since q (and thus q^2) are integers, p^2 is even.

> Fact ★: if n^2 is even, then n is even.

By fact ★, p is even, which means there exists an integer k such that $p = 2k$. Therefore,

$$(2k) = 2q^2$$
$$q^2 = \frac{(2k)^2}{2} = \frac{4k^2}{2} = 2k^2$$

and hence q^2 is even.

By fact ★, q is also even. Because p and q are both even, their $\gcd \neq 1$. This contradicts an earlier statement.

Thus, negating desired conclusion led to contradiction, so the desired claim is true.
