More on Relations

Example 1

$R_0 \subseteq \mathbb{R} \times \mathbb{R}$ with $(x, y) \in R_0 \text{ iff } x \cdot y \geq 1$.

- $(4, 1.3) \in R_0$ because $4 \cdot 1.3 \geq 1$
- $(0.2, 0.4) \notin R_0$ because $0.2 \cdot 0.4 < 1$

Describing the Relation R_0

- Reflexive? **NO!** $(0, 0) \notin R_0$ because $0 \cdot 0 \ngeq 1$.

- Symmetric? **YES!** Suppose we have a pair $(a, b) \in R_0$. $a \cdot b \geq 1$, which means $b \cdot a \geq 1$. Therefore, $(b, a) \in R_0$ as well.

- Antisymmetric? **NO!** $(4, 1.3) \in R_0$ and $(1.3, 4) \in R_0$, but $4 \neq 1.3$

- Transitive? **NO!** $(0.5, 4) \in R_0$. $(4, 1) \in R_0$. If the relation is transitive, then $(0.5, 1) \in R_0$. But $0.5 \cdot 1 < 1$, so it's not in the relation.

> "Symmetric" is NOT the opposite of "antisymmetric."

Define W to be the set of nonempty subsets of $\mathbb{N}$. That is, $W = \mathcal{P}(\mathbb{N}) \setminus \{\emptyset\}$. For example, $\{3, 5, 6\} \in W$.

Define $T \subseteq W \times W$ as:

$$(X, Y) \in T \text{ iff } X \cap Y \neq \emptyset$$

In other words, W contains (X, Y) only if X and Y have something in common.

- Reflexive?
  - Suppose $A \in W$. Then A is a nonempty subset of $\mathbb{N}$, which means $A \cap A \neq \emptyset$. Thus, $(A, A) \in T$.
  - **YES**

- Symmetric?
  - Suppose $(X, Y) \in T$. Then, $X \cap Y \neq \emptyset$. That means $Y \cap X \neq \emptyset$. Therefore, $(Y, X) \in T$.
  - **YES**

- Antisymmetric?
    - Suppose $(X, Y) \in T$ and $(Y, X) \in T$. Thus, $X \cap Y \neq \emptyset$ and $Y \cap X \neq \emptyset$. Hmm... it seems like we're stuck. Let's try hunting for counterexamples.
    - Counterexample: $(\{1, 6\}, \{1, 3, 5\}), \{1, 3, 5\}), (\{1, 6\}$ are in T, but $\{1, 6\} \neq \{1, 3, 5\}$

Inverse of a Relation

$$R^{-1} = \{(y, x) \mid (x, y) \in R\}$$

New Material

Claim: Let R,S be relations on set U. If R and S are transitive, then $R \cap S^{-1}$ is transitive.

Proof: (direct)

Suppose R and S are transitive. [NTS: $R \cap S^{-1}$ is transitive, which means that for all choices of $x, y, z \in U$, if $(x, y) \in R \cap S^{-1}$ and $(y, z) \in R \cap S^{-1}$, then $(x, z) \in R \cap S^{-1}$]

Consider arbitrary $a, b, c \in U$ such that $(a, b) \in R \cap S^{-1}$ and $(b, c) \in R \cap S^{-1}$.

By definition of $\cap$,

- $(a, b) \in R$
- $(a, b) \in S^{-1}$
- $(b, c) \in R$
- $(b, c) \in S^{-1}$.

Because R is transitive, we know $(a, c) \in R$.

By definition of S^{-1}, $(ba) \in S$ and $(c, b) \in S$. Because S is transitive, $(c, a) \in S$. Therefore, $(a, c) \in S^{-1}$. So $(a, c) \in R$ and $(a, c) \in S^{-1}$. Therefore, $(a, c) \in R \cap S^{-1}$.

Because a,b,c were arbitrary, $R \cap S^{-1}$ is transitive.

An **equivalence relation** is a relation that is reflexive, symmetric, and transitive.

Example: $=$ as defined over $\mathbb{R}$.

- $\forall x \in \mathbb{R},\ x = x$
- $\forall x, y \in \mathbb{R},\ (x = y \rightarrow y = x)$
- $\forall x, y, z \in \mathbb{R},\ ((x = y \wedge y = z) \rightarrow (x = z))$

Let R be an equivalence relation on set A, and let $w \in A$. The **equivalence class** of w (under R) is defined as:

$$[w]_R = \{b \in A \mid (w, b) \in R\}$$

(i.e., set of all elements of A that R relates to b)

$$\begin{aligned} R_M &= \{(A, A). . (E, E), (A, E), (E, A), (B, D), (D, B)\} \\ [A]_{R_M} &= \{A, E\} \\ [B]_{R_M} &= \{B, D\} \\ [C]_{R_M} &= \{C\} \end{aligned}$$