Orthogonal Matrices & Symmetric Matrices

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Outline

Orthogonal Matrices

- Reference: Chapter 7.5

Symmetric Matrices

- Reference: Chapter 7.6

Norm-preserving 保持

- A linear operator is norm-preserving if

$$\|T(u)\| = \|u\| \qquad \text{For all u}$$

Example: linear operator T on $\mathcal{R}^2$ that rotates a vector by θ.

$\Rightarrow$ Is T norm-preserving?

yes

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Example: linear operator T is refection

$\Rightarrow$ Is T norm-preserving?

yes

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Norm-preserving

- A linear operator is norm-preserving if

$$\|T(u)\| = \|u\| \qquad \text{For all u}$$

Example: linear operator T is projection
$\Rightarrow$ Is T norm-preserving?

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

no

Example: linear operator U on $\mathcal{R}^n$ that has an eigenvalue $\lambda \neq \pm 1$.

$$\|Ax\| = \|\lambda x\| = |\lambda| \|x\| \quad \Rightarrow \lambda = \pm 1$$

no

Orthogonal Matrix

- An nxn matrix Q is called an orthogonal matrix (or simply orthogonal) if the columns of Q form an orthonormal basis for R^n
- Orthogonal operator: standard matrix is an orthogonal matrix.

$$A_\theta = \begin{array}{c} \begin{array}{cc} \text{unit} & \text{unit} \end{array} \\ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \\ \text{orthogonal} \end{array}$$ is an orthogonal matrix.

Norm-preserving

- Necessary conditions:

Norm-preserving $\rightarrow$ Orthogonal Matrix

Norm-preserving $\leftarrow$ Orthogonal Matrix ???

Linear operator Q is norm-preserving

$\Rightarrow$ $\|\mathbf{q}_j\| = 1$ $\qquad \|\mathbf{q}_j\| = \|Q\mathbf{e}_j\| = \|\mathbf{e}_j\|$

$\Rightarrow$ $\mathbf{q}_i$ and $\mathbf{q}_j$ are orthogonal 畢式定理

$$\|\mathbf{q}_i + \mathbf{q}_j\|^2 = \|Q\mathbf{e}_i + Q\mathbf{e}_j\|^2 = \|Q(\mathbf{e}_i + \mathbf{e}_j)\|^2 = \|\mathbf{e}_i + \mathbf{e}_j\|^2 = 2 = \|\mathbf{q}_i\|^2 + \|\mathbf{q}_j\|^2$$

Orthogonal Matrix

Those properties are used to check orthogonal matrix.

- Q is an orthogonal matrix
- $QQ^T = I_n$ $= Q^TQ$
- Q is invertible, and $Q^{-1} = Q^T$

Simple inverse

- $Qu \cdot Qv = u \cdot v$ for any u and v

Q preserves dot projects

- $\|Qu\| = \|u\|$ for any u

Q preserves norms

$Qu \cdot Qv = (Qu)^T Qv = u^T Q^T Q v = u^T v = u \cdot v$

Norm-preserving $\rightleftarrows$ Orthogonal Matrix

$Qu \cdot Qu = u \cdot u \Rightarrow \|Qu\|^2 = \|u\|^2 \Rightarrow \|Qu\| = \|u\|$

Orthogonal Matrix

- Let P and Q be n x n orthogonal matrices
  - $detQ = \pm 1$
  - PQ is an orthogonal matrix
  - Q^{-1} is an orthogonal matrix
  - Q^T is an orthogonal matrix

Check by $(PQ)^{-1} = (PQ)^T$

Check by $(Q^{-1})^{-1} = (Q^{-1})^T$

Proof ① $QQ^T = I \Rightarrow det(I) = det(QQ^T) = det(Q) \cdot det(Q^T) = det(Q)^2$

$\Rightarrow det(Q) = \pm 1$

② $(PQ)^{-1} = Q^{-1}P^{-1} = Q^T P^T = (PQ)^T$

③ $(Q^{-1})^{-1} = Q = (Q^T)^T = (Q^{-1})^T$

④ $Q^{-1} = Q^T$

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}$$

Rows and columns

Orthogonal Operator

- Applying the properties of orthogonal matrices on orthogonal operators
- T is an orthogonal operator
  - $T(u) \cdot T(v) = u \cdot v$ for all u and v
  - $\|T(u)\| = \|u\|$ for all u

Preserves dot product

Preserves norms

- T and U are orthogonal operators, then TU and T^{-1} are orthogonal operators.

Example: Find an orthogonal operator T on $\mathcal{R}^3$ such that

$$T\left(\begin{bmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{bmatrix}\right)=\begin{bmatrix}0\\1\\0\end{bmatrix}$$

Norm-preserving

$$v=\begin{bmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{bmatrix} \qquad Av=e_2 \qquad v=A^{-1}e_2$$

Find A^{-1} first

Because $A^{-1}=A^T$

$$A^{-1}=\begin{bmatrix}* & 1/\sqrt{2} & *\\ * & 0 & *\\ * & 1/\sqrt{2} & *\end{bmatrix}$$

Also orthogonal

$$\begin{bmatrix}1/\sqrt{2}\\0\\-1/\sqrt{2}\end{bmatrix} \qquad \begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$A^{-1}=\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2} & 0\\ 0 & 0 & 1\\ -1/\sqrt{2} & 1/\sqrt{2} & 0\end{bmatrix}$$

$$A=(A^{-1})^T$$

Conclusion

- Orthogonal Matrix (Operator)
  - Columns and rows are orthogonal unit vectors
  - Preserving norms, dot products
  - Its inverse is equal its transpose

Outline

Orthogonal Matrices

- Reference: Chapter 7.5

Symmetric Matrices

- Reference: Chapter 7.6

Eigenvalues are real

- The eigenvalues for symmetric matrices are always real. 只有實數特征值

How about more general cases?

實係數多項式虛根共軛

若有 $\lambda = a + bi$

$\Downarrow$

則有 $\bar{\lambda} = a - bi$

Consider 2 x 2 symmetric matrices

$$A = A^T = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \in R^{2\times 2}$$

$$det(A - tI_2) = t^2 - (a+c)t + ac - b^2$$

$$\text{Since } (a+c)^2 - 4(ac - b^2) = (a-c)^2 + 4b^2 \geq 0$$

The symmetric matrices always have real eigenvalues.

假设A存在特征值λ，$Av=\lambda v$，则有$Aw=\bar{\lambda} w$

$$
w^TAv \begin{cases} = w^T\lambda v = \lambda w^T v = \lambda\, w\cdot v \\ = w^TA^Tv = (Aw)^Tv = \bar{\lambda} w^T v = \bar{\lambda}\, w\cdot v \end{cases}
$$

$$\Longrightarrow \lambda\, w\cdot v = \bar{\lambda}\, w\cdot v$$

$\because \overline{AB} = \bar{A}\bar{B}$ $\therefore \overline{Av} = \bar{A}\bar{v} = A\bar{v} = \overline{\lambda v} = \bar{\lambda}\bar{v}$ $\Longrightarrow w = \bar{v}$

$\therefore w\cdot v = \bar{v}\cdot v > 0$

$\Longrightarrow \lambda = \bar{\lambda}$

$\because$ 只有实数才会和自己共轭

$\therefore \lambda$为实数

Orthogonal Eigenvectors

A is symmetric

$det(A - tI_n)$ Factorization

$$= (t-\lambda_1)^{\underline{m_1}}(t-\lambda_2)^{\underline{m_2}} \ldots (t-\lambda_k)^{\underline{m_k}}(\ldots\ldots)$$

Eigenvalue:	λ_1	λ_2	……	λ_k
Eigenspace: (dimension)	$d_1 \leq m_1$	$d_2 \leq m_2$	……	$d_k \leq m_k$

Independent → orthogonal

Orthogonal Eigenvectors

- A is symmetric.
- If u and v are eigenvectors corresponding to eigenvalues λ and μ $(\lambda \neq \mu)$

$\Rightarrow$ u and v are orthogonal.

$$A\mathbf{u} \cdot \mathbf{v} = \lambda \mathbf{u} \cdot \mathbf{v} = \lambda(u \cdot v)$$

$$= (Au)^T v$$

$$= \mathbf{u} \cdot A^T \mathbf{v} = u \cdot Av$$

$$= \mathbf{u} \cdot \mu \mathbf{v} = \mu(u \cdot v)$$

$\because \lambda \neq \mu$

$\therefore u \cdot v = 0$

Diagonalization

A is symmetric

P560 ?

$A = PDP^T$

$P^T A P = D$

P is an orthogonal matrix

D is a diagonal matrix

: simple

$P^T A P = D$

$P^{-1} A P = D$

$A = PDP^{-1}$

Diagonalization

$A = PDP^T$

P consists of eigenvectors , D are eigenvalues

Diagonalization

- Example

$$A = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$$

$\text{A} = P\text{DP}^{-1}$ ⇒ $\text{A} = P\text{DP}^{T}$, $\text{P}^{T}\text{A}P = \text{D}$

A has eigenvalues $\lambda_1 = 6$ and $\lambda_2 = 1$,

with corresponding eigenspaces $\mathcal{E}_1 = \text{Span}\{[\,-1\ \ 2\,]^T\}$ and $\mathcal{E}_2 = \text{Span}\{[\,2\ \ 1\,]^T\}$

orthogonal

$\Rightarrow \mathcal{B}_1 = \{[\,-1\ \ 2\,]^T/\sqrt{5}\}$ and $\mathcal{B}_2 = \{[\,2\ \ 1\,]^T/\sqrt{5}\}$

$$P = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}.$$

Example of Diagonalization of Symmetric Matrix

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

$A = PDP^{-1}$ → $A = PDP^{T}$

P is an orthogonal matrix

$\lambda_1 = 2$

Intendent

independent

Eigenspace: $Span\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$ $\xrightarrow{\text{Gram-Schmidt}}_{\text{normalization}}$ $Span\left\{\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}\right\}$

Not orthogonal

$\lambda_2 = 8$

Eigenspace: $Span\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}$ $\xrightarrow[\text{normalization}]{}$ $Span\left\{\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}\right\}$

$$P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

Diagonalization

P is an orthogonal matrix

A is symmetric $\rightleftarrows$ $P^{T}AP = D$ / $A = PDP^{T}$

P consists of eigenvectors , D are eigenvalues

Finding an orthonormal basis consisting of eigenvectors of *A*

(1) 计算特征值

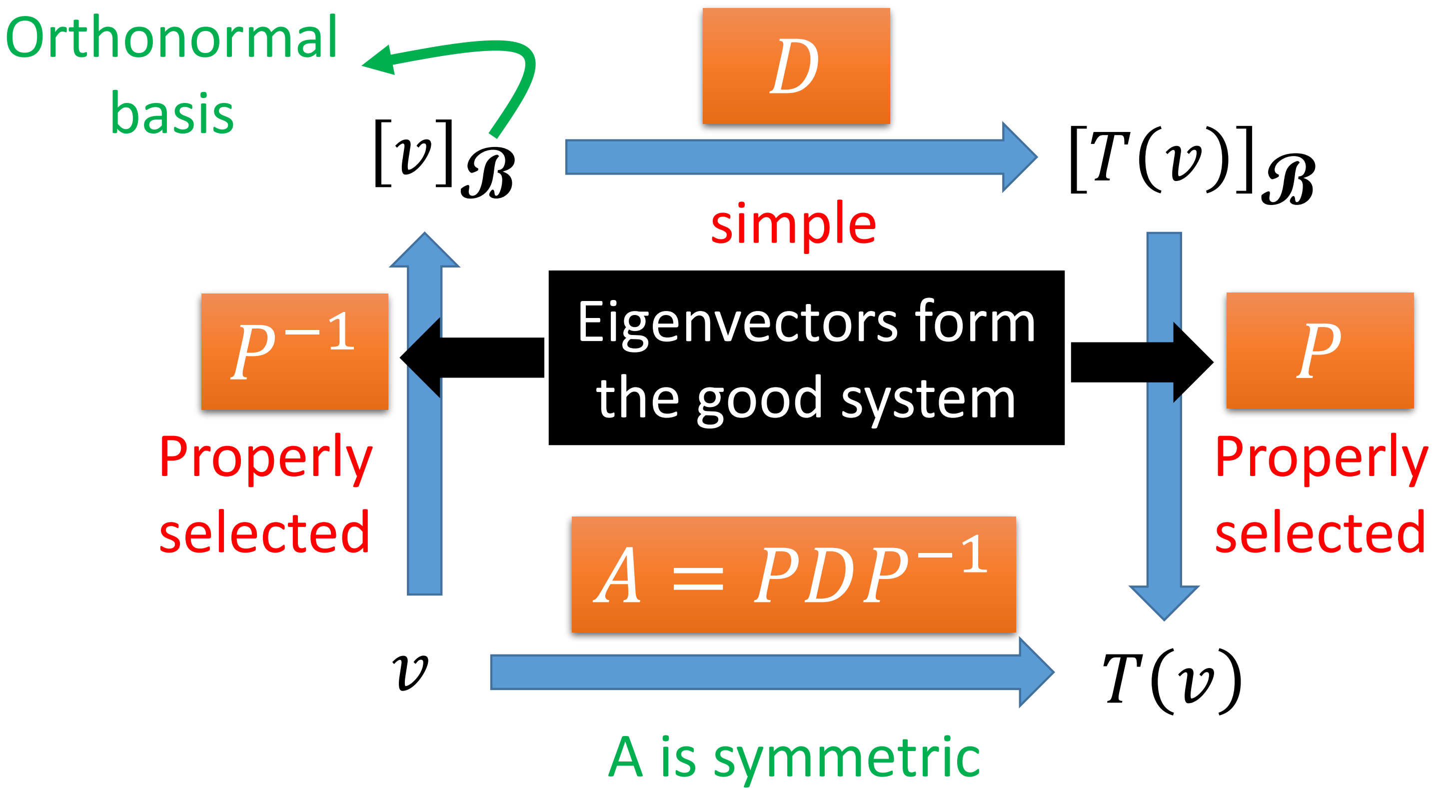
(2) 计算对应特征向量

(3) 把向量标准正交化

Diagonalization of Symmetric Matrix

$$u = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k$$

$$u \cdot v_1 \qquad u \cdot v_2 \qquad u \cdot v_k$$



Spectral Decomposition

谱分解

Orthonormal basis

$A = PDP^T$ Let $P = [\ \mathbf{u}_1\ \ \mathbf{u}_2\ \ \cdots\ \ \mathbf{u}_n\ ]$ and $D = \text{diag}[\ \lambda_1\ \ \lambda_2\ \ \cdots\ \ \lambda_n\ ]$.

$$= P[\ \lambda_1\mathbf{e}_1\ \ \lambda_2\mathbf{e}_2\ \ \cdots\ \ \lambda_n\mathbf{e}_n\ ]P^T$$

$$= [\ \lambda_1 P\mathbf{e}_1\ \ \lambda_2 P\mathbf{e}_2\ \ \cdots\ \ \lambda_n P\mathbf{e}_n\ ]P^T$$

$$= [\ \lambda_1\mathbf{u}_1\ \ \lambda_2\mathbf{u}_2\ \ \cdots\ \ \lambda_n\mathbf{u}_n\ ]\begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \vdots \\ \mathbf{u}_n^T \end{bmatrix}$$

P_1 P_2 P_n

$$= \lambda_1 \mathrm{P}_1 + \lambda_2 \mathrm{P}_2 + \cdots + \lambda_n \mathrm{P}_n$$

P_i are symmetric

Rank=1

Spectral Decomposition

Orthonormal basis

$A = PDP^T$ Let $P = [\,\mathbf{u}_1 \;\; \mathbf{u}_2 \;\; \cdots \;\; \mathbf{u}_n\,]$ and $D = \mathrm{diag}[\,\lambda_1 \;\; \lambda_2 \;\; \cdots \;\; \lambda_n\,]$.

$$= \lambda_1 \mathrm{P}_1 + \lambda_2 \mathrm{P}_2 + \cdots + \lambda_n \mathrm{P}_n$$

$$\mathrm{rank}\, P_i = \mathrm{rank}\, \mathbf{u}_i \mathbf{u}_i^T = 1.$$

$$P_i P_i = \mathbf{u}_i \underline{\mathbf{u}_i^T \mathbf{u}_i} \mathbf{u}_i^T = \mathbf{u}_i \mathbf{u}_i^T = P_i$$

$$P_i P_j = \mathbf{u}_i \underline{\mathbf{u}_i^T \mathbf{u}_j} \mathbf{u}_j^T = O$$

$$P_i \mathbf{u}_i = u_i$$

$$P_i \mathbf{u}_j = 0$$

Spectral Decomposition

- Example

$$A = \begin{bmatrix} 3 & -4 \\ -4 & -3 \end{bmatrix}$$ Find spectrum decomposition.

Eigenvalues $\lambda_1 = 5$ and $\lambda_2 = -5$.

$$P_1 = u_1 u_1^T$$

An orthonormal basis consisting of eigenvectors of *A* is

$$B = \left\{ \underset{u_1}{\begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}}, \underset{u_2}{\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}} \right\}$$

$$P_2 = u_2 u_2^T$$

$$A = \lambda_1 P_1 + \lambda_2 P_2$$

Conclusion

- Any symmetric matrix
  - has only real eigenvalues
  - has orthogonal eigenvectors.
  - is always diagonalizable

A is symmetric $\Leftrightarrow$ $P^TAP = D$ $\quad$ $A = PDP^T$

P is an orthogonal matrix

Appendix

Diagonalization

- By induction on n.
- $n = 1$ is obvious.
- Assume it holds for $n \geq 1$, and consider $A \in \mathcal{R}^{(n+1)\times(n+1)}$.
- A has an eigenvector $\mathbf{b}_1 \in \mathcal{R}^{n+1}$ corresponding to a real eigenvalue λ, so $\exists$ an orthonormal basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{n+1}\}$
  - by the **Extension Theorem** and Gram-Schmidt Process.

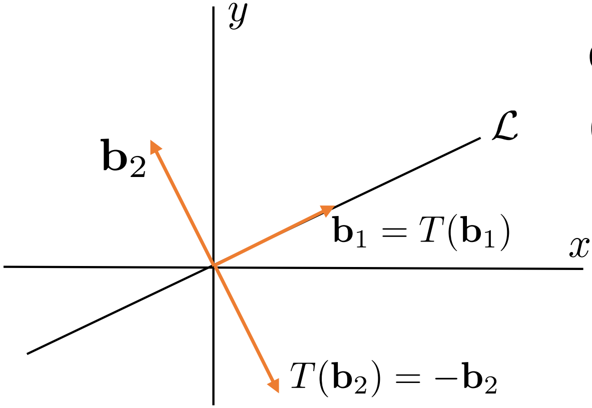
$$B^T AB = \begin{bmatrix} \mathbf{b}_1^T \\ \mathbf{b}_2^T \\ \vdots \\ \mathbf{b}_{n+1}^T \end{bmatrix} \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_{n+1} \end{bmatrix} = \left[\begin{array}{c|ccc} \mathbf{b}_1^T A\mathbf{b}_1 & \mathbf{b}_1^T A\mathbf{b}_2 & \cdots & \mathbf{b}_1^T A\mathbf{b}_{n+1} \\ \hline \mathbf{b}_2^T A\mathbf{b}_1 & \mathbf{b}_2^T A\mathbf{b}_2 & \cdots & \mathbf{b}_2^T A\mathbf{b}_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{b}_{n+1}^T A\mathbf{b}_1 & \mathbf{b}_{n+1}^T A\mathbf{b}_2 & \cdots & \mathbf{b}_{n+1}^T A\mathbf{b}_{n+1} \end{array}\right]$$

$$= \left[\begin{array}{c|c} \lambda & \mathbf{0}^T \\ \hline \mathbf{0} & S \end{array}\right], \text{since } \mathbf{b}_1^T A\mathbf{b}_1 = \lambda \mathbf{b}_1^T \mathbf{b}_1 = \lambda \text{ and } \mathbf{b}_j^T A\mathbf{b}_1 = \mathbf{b}_1^T A\mathbf{b}_j = 0 \ \forall j \neq 1.$$

$S = S^T \in \mathscr{R}^{n \times n} \Rightarrow \exists$ an orthogonal $C \in \mathscr{R}^{n \times n}$ and a diagonal $L \in \mathscr{R}^{n \times n}$ such that $C^T SC = L$ by the induction hypothesis.

$$\Rightarrow \underbrace{\begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & C^T \end{bmatrix}}_{\text{orthogonal } P^T} B^T AB \underbrace{\begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & C \end{bmatrix}}_{\text{orthogonal } P} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & C^T \end{bmatrix} \begin{bmatrix} \lambda & \mathbf{0}^T \\ \mathbf{0} & S \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & C \end{bmatrix} = \begin{bmatrix} \lambda & \mathbf{0}^T \\ \mathbf{0} & C^T SC \end{bmatrix} = \underbrace{\begin{bmatrix} \lambda & \mathbf{0}^T \\ \mathbf{0} & L \end{bmatrix}}_{\text{diagonal } D}$$

Example: reflection operator T about a line $\mathcal{L}$ passing the origin.



Question: Is T an orthogonal operator?

(An easier) Question:
Is T orthogonal if $\mathcal{L}$ is the x-axis?

$\mathbf{b}_1$ is a unit vector along $\mathcal{L}$.
$\mathbf{b}_2$ is a unit vector perpendicular to $\mathcal{L}$.
$P = [\ \mathbf{b}_1\ \ \mathbf{b}_2\ ]$ is an orthogonal matrix.
$\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ is an orthonormal basis of $\mathcal{R}^2$.
$[T]_{\mathcal{B}} = \text{diag}[1\ -1]$ is an orthogonal matrix.

Let the standard matrix of T be Q. Then $[T]_{\mathcal{B}} = P^{-1}QP$, or $Q = P[T]_{\mathcal{B}}P^{-1} \Rightarrow Q$ is an orthogonal matrix. $\Rightarrow T$ is an orthogonal operator.