Linear Function in Coordinate System

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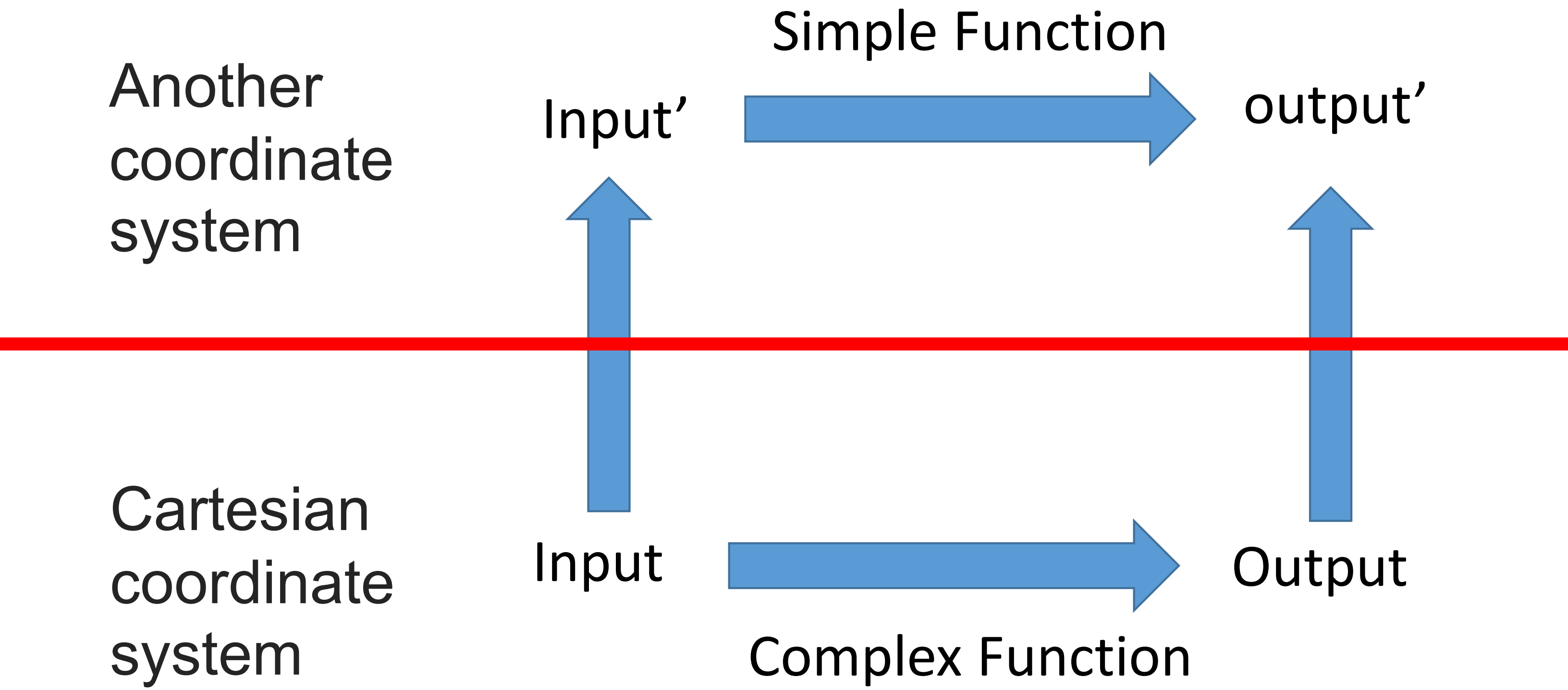
Outline

- Describing a function in a coordinate system
  - A complex function in one coordinate system can be simple in other systems.
- Reference: Textbook Chapter 4.5

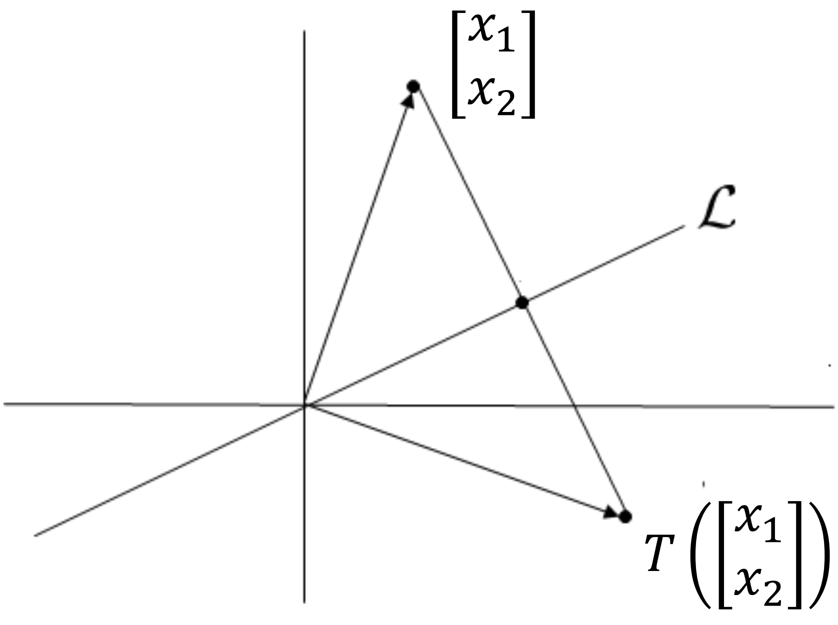
Basic Idea



Another coordinate system
Simple Function
Input’
output’
Cartesian coordinate system
Input
Output
Complex Function

Sometimes a function can be complex ……

- Example: reflection about a line $\mathcal{L}$ through the origin in $\mathcal{R}^2$



$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = ?$$

$$[\mathrm{T}] = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix}$$

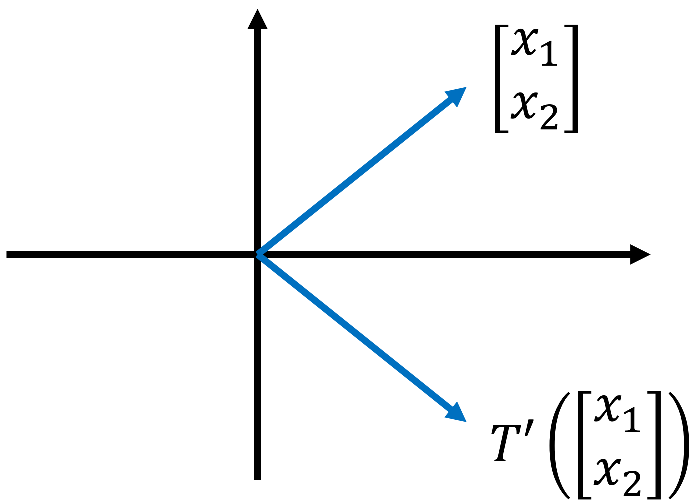
$$= ?$$

这个很聪明

Sometimes a function can be complex ……

- Example: reflection about a line $\mathcal{L}$ through the origin in $\mathcal{R}^2$

special case: $\mathcal{L}$ is the *horizontal axis*



$$T'\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) =? \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$

$$[T'] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

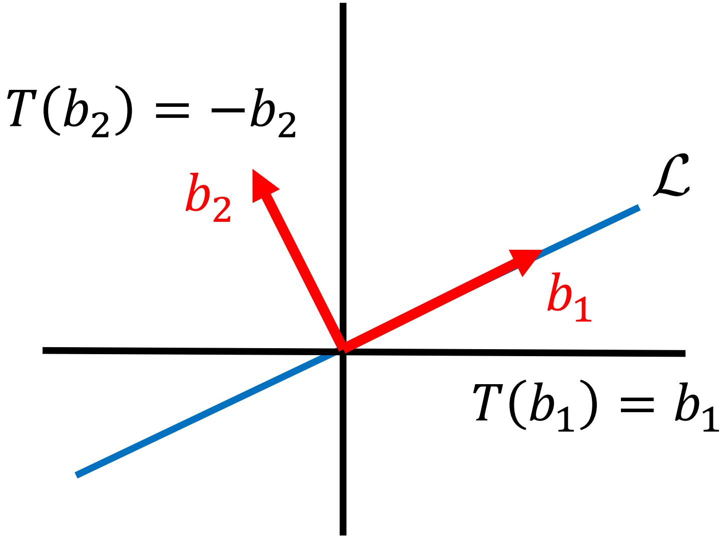
$T'(e_1)$
$= e_1$

$T'(e_2)$
$= -e_2$

Describing the function in another coordinate system

- Example: reflection about a line $\mathcal{L}$ through the origin in $\mathcal{R}^2$

In another coordinate system $\mathcal{B}$ ....



$\mathcal{B} = \{b_1, b_2\}$

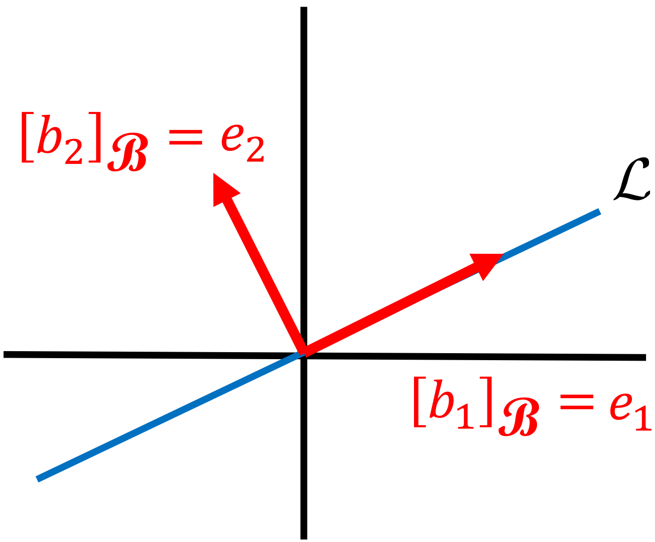
Describing the function in another coordinate system

- Example: reflection about a line $\mathcal{L}$ through the origin in $\mathcal{R}^2$

In another coordinate system $\mathcal{B}$ ....

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Input and output are both in $\mathcal{B}$



$[T]b_1 = b_1$

⇒ $[T]_{\mathcal{B}}([b_1]_{\mathcal{B}}) = [b_1]_{\mathcal{B}}$

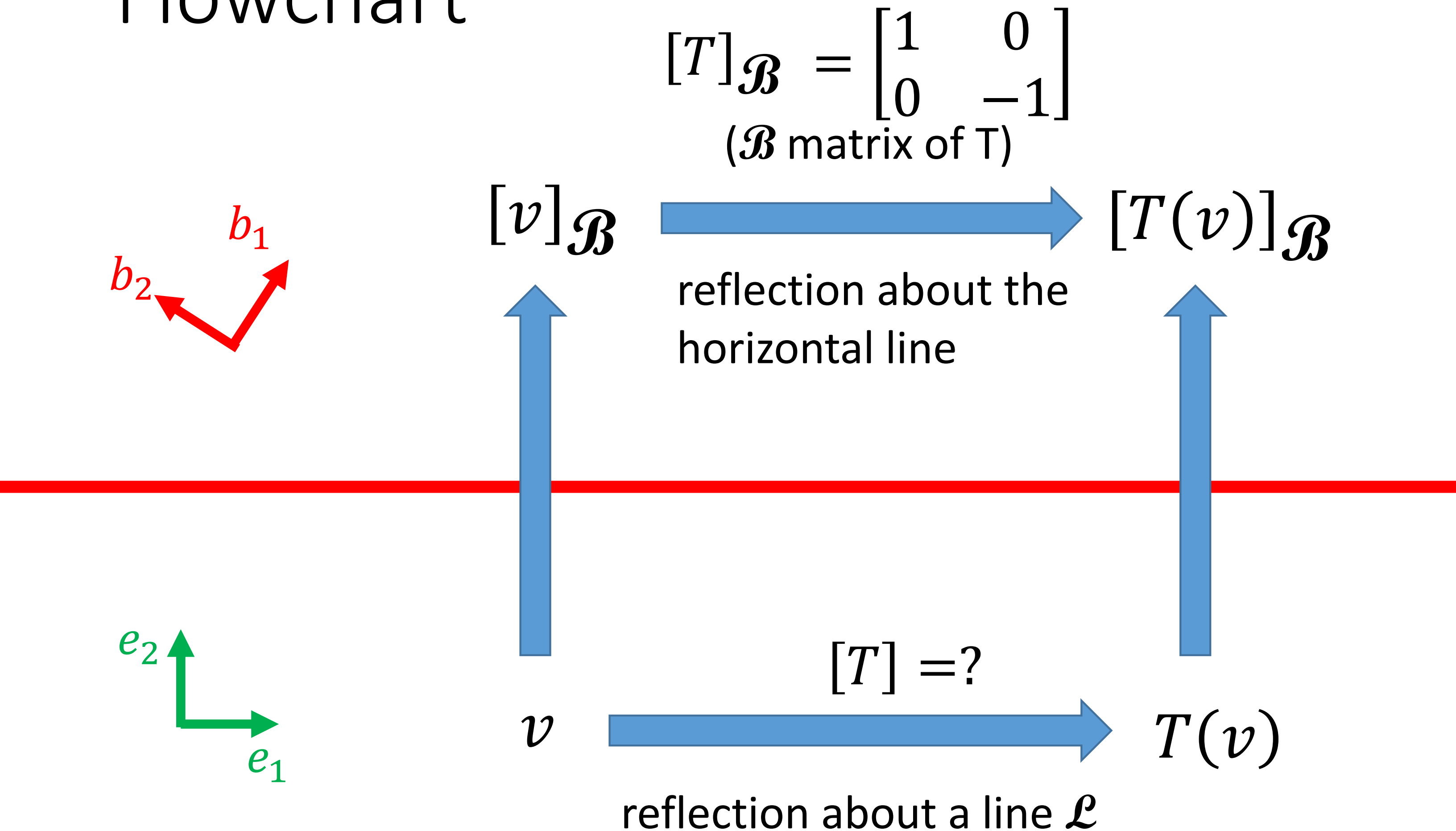
⇒ $[T]_{\mathcal{B}}(e_1) = e_1$

$[T]b_2 = -b_2$

⇒ $[T]_{\mathcal{B}}([b_2]_{\mathcal{B}}) = [-b_2]_{\mathcal{B}}$

⇒ $[T]_{\mathcal{B}}(e_2) = -e_2$

Flowchart



$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
($\mathcal{B}$ matrix of T)
b_1
b_2
$[v]_{\mathcal{B}}$
$[T(v)]_{\mathcal{B}}$
reflection about the horizontal line
e_2
e_1
$[T] = ?$
v
$T(v)$
reflection about a line $\mathcal{L}$

Flowchart

$\mathcal{B}$ coordinate system

Cartesian coordinate system

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
{[v]_{\mathcal{B}}} \arrow[r, "{[T]_{\mathcal{B}}}"] & {[T(v)]_{\mathcal{B}}} \arrow[d, "B"] \\
v \arrow[u, "B^{-1}"] \arrow[r, "{[T] =?}"'] & T(v)
\end{tikzcd}
$$

物理意义

$$A = B[T]_{\mathcal{B}}B^{-1}$$

similar

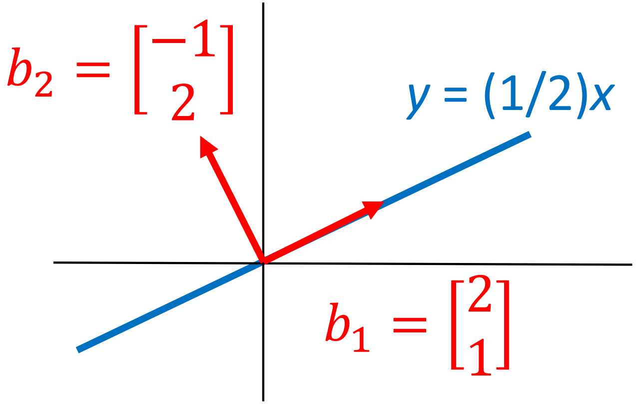
$$[T]_{\mathcal{B}} = B^{-1}AB$$

similar

相似阵

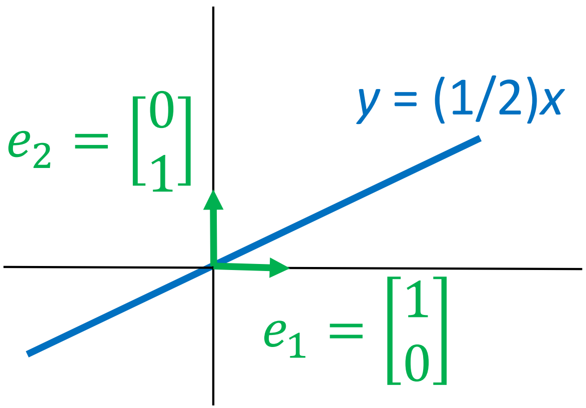
真的牛逼

- Example: reflection operator *T* about the line *y* = (1/2)*x*



$$B^{-1} = \begin{bmatrix} 0.4 & 0.2 \\ -0.2 & 0.4 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

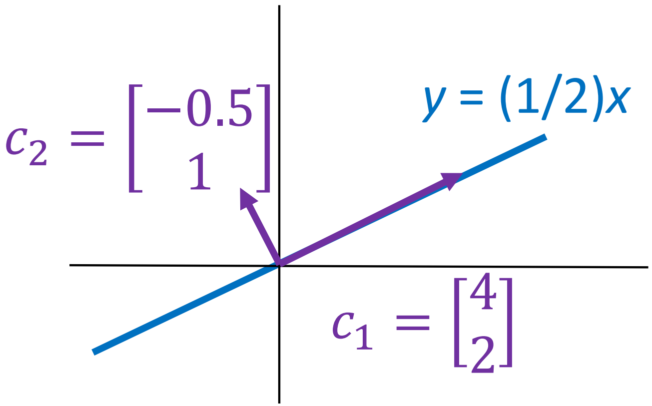
$$\begin{tikzcd}[column sep=huge, row sep=huge]
{[v]_{\mathcal{B}}} \arrow[r, "{[T]_{\mathcal{B}}}", "{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}"'] & {[T(v)]_{\mathcal{B}}} \arrow[d, "B"] \\
v \arrow[u, "B^{-1}"'] \arrow[r, "A = ?"] & T(v)
\end{tikzcd}$$



$$A = B[T]_{\mathcal{B}}B^{-1}$$

- Example: reflection operator T about the line $y = (1/2)x$

$$A = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}$$



$$A = C[T]_C C^{-1}$$

$$= \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 0.4 & 0.2 \\ -0.2 & 0.4 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$\begin{tikzcd}
{[v]_{\mathcal{B}}} \arrow[r, "{[T]_{\mathcal{B}}}"] & {[T(v)]_{\mathcal{B}}} \arrow[d, "B"] \\
v \arrow[u, "B^{-1}"] \arrow[r, "A\,=?"] & T(v)
\end{tikzcd}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A = B[T]_{\mathcal{B}} B^{-1}$$

Example 2 (P279)

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + x_3 \\ x_1 + x_2 \\ -x_1 - x_2 + 3x_3 \end{bmatrix} \qquad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 1 & 0 \\ -1 & -1 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 3 & 1 \end{bmatrix}$$

$$[T]_{\mathcal{B}} = \begin{bmatrix} 3 & -9 & 8 \\ -1 & 3 & -3 \\ 1 & 6 & 1 \end{bmatrix}$$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
{[v]_{\mathcal{B}}} \arrow[r, "{[T]_{\mathcal{B}} = ?}", "{[T]_{\mathcal{B}} = B^{-1}AB}"'] & {[T(v)]_{\mathcal{B}}} \arrow[d, "B"] \\
v \arrow[u, "B^{-1}"] \arrow[r, "A \text{ is known}", "{A = B\,[T]_{\mathcal{B}}B^{-1}}"'] & T(v)
\end{tikzcd}$$

Example 3 (P279) Determine T

$$T\left(\underset{\mathbf{b}_1}{\begin{bmatrix}1\\1\\0\end{bmatrix}}\right)=\underset{\mathbf{c}_1}{\begin{bmatrix}1\\2\\1\end{bmatrix}}\qquad T\left(\underset{\mathbf{b}_2}{\begin{bmatrix}1\\0\\1\end{bmatrix}}\right)=\underset{\mathbf{c}_2}{\begin{bmatrix}3\\-1\\1\end{bmatrix}}\qquad T\left(\underset{\mathbf{b}_3}{\begin{bmatrix}0\\1\\1\end{bmatrix}}\right)=\underset{\mathbf{c}_3}{\begin{bmatrix}2\\0\\1\end{bmatrix}}$$

$$\begin{tikzcd}
\mathbf{e}_1\ \mathbf{e}_2\ \mathbf{e}_3 & {[v]_{\mathcal{B}}} \arrow[r, "{[T]_{\mathcal{B}}}"'] & {[T(v)]_{\mathcal{B}}} & \mathbf{B}^{-1}\mathbf{c}_1\ \mathbf{B}^{-1}\mathbf{c}_2\ \mathbf{B}^{-1}\mathbf{c}_3 \\
\mathbf{b}_1\ \mathbf{b}_2\ \mathbf{b}_3 & v \arrow[u, "B^{-1}"'] \arrow[r, "A"] & T(v) \arrow[u, "B^{-1}"] & \mathbf{c}_1\ \mathbf{c}_2\ \mathbf{c}_3
\end{tikzcd}$$

$\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ as a coordinate system

{$\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$} is a basis of R^3

Example 3 (P279) Determine T

$$[T]_{\mathcal{B}} = [B^{-1}c_1 \quad B^{-1}c_2 \quad B^{-1}c_3] = B^{-1}C$$

$$A = B[T]_{\mathcal{B}}B^{-1} = BB^{-1}CB^{-1} = CB^{-1}$$

$$\begin{tikzcd}
\mathbf{e}_1 \ \mathbf{e}_2 \ \mathbf{e}_3 \quad [v]_{\mathcal{B}} \arrow[rr, "{[T]_{\mathcal{B}}}"'] & & {[T(v)]_{\mathcal{B}}} \quad \mathbf{B^{-1}c_1} \ \mathbf{B^{-1}c_2} \ \mathbf{B^{-1}c_3} \arrow[d, shift left, "B"] \\
\mathbf{b}_1 \ \mathbf{b}_2 \ \mathbf{b}_3 \quad v \arrow[u, "B^{-1}"'] \arrow[rr, "A"] & & T(v) \quad \mathbf{c}_1 \ \mathbf{c}_2 \ \mathbf{c}_3 \arrow[u, shift left, "B^{-1}"]
\end{tikzcd}$$

$\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ as a coordinate system

{$\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$} is a basis of R^3

Inception

小開的父親說：

"I'm disappointed that you're trying so hard to be me."

小開有了不要繼承父業的念頭

$\mathcal{B}$ coordinate system

夢境

$$\begin{tikzcd}[column sep=huge, row sep=huge]
{[v]_{\mathcal{B}}} \arrow[r, "{[T]_{\mathcal{B}}}"] & {[T(v)]_{\mathcal{B}}} \arrow[d, "B"] \\
v \arrow[u, "B^{-1}"] \arrow[r, "{[T]}"] & T(v)
\end{tikzcd}$$

$[T]_{\mathcal{B}} = B^{-1}AB$

清醒

Cartesian coordinate system

做夢

現實

$[T] = B\,[T]_{\mathcal{B}}B^{-1}$

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