How many solutions?

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Reference

- Textbook: Chapter 1.7

Review

Given a system of linear equations with m equations and n variables

$A\mathbf{x} = \mathbf{b}$ $\quad A: m \times n \quad x \in R^n \quad b \in R^m$

Is b a linear combination of columns of A?

Is b in the span of the columns of A?

NO

No solution

YES

Have solution

How many solutions?

Today

Given a system of linear equations with m equations and n variables

$A\mathbf{x} = \mathbf{b}$ $\quad A: m \times n \quad x \in R^n \quad b \in R^m$

Is b a linear combination of columns of A?

Is b in the span of the columns of A?

NO

No solution

Other cases?

YES

The columns of A are ***independent***.

Rank A = n

Nullity A = 0

Unique solution

The columns of A are ***dependent***.

Rank A < n

Nullity A > 0

Infinite solution

$$x \rightarrow \boxed{A} \rightarrow b$$

$$x' \rightarrow \boxed{A} \rightarrow b$$

$$0.8x + 0.2x' \rightarrow \boxed{A} \rightarrow b$$

(依賴的、不獨立的)

Dependent and Independent

(獨立的、自主的)

Definition

- A set of n vectors $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n\}$ is linear dependent

  Find one → Obtain many

  - If there exist scalars $\underline{x_1, x_2, \cdots, x_n}$, **not all zero**, such that

$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = \boldsymbol{0}$$

线性相关

- A set of n vectors $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n\}$ is linear independent

$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_n\boldsymbol{a}_n = \boldsymbol{0}$$

Only if $x_1 = x_2 = \cdots = x_n = 0$ unique

Dependent and Independent

Linear Dependent

Given a vector set, $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, if there exists any $\mathbf{a}_i$ that is a linear combination of other vectors

$$\left\{\begin{bmatrix}-4\\12\\6\end{bmatrix}, \begin{bmatrix}-10\\30\\15\end{bmatrix}\right\} \times \frac{5}{2}$$

Dependent or Independent? ✓

$$\left\{\begin{bmatrix}6\\3\\3\end{bmatrix} + \begin{bmatrix}1\\8\\3\end{bmatrix} = \begin{bmatrix}7\\11\\6\end{bmatrix}\right\}$$

Dependent or Independent? ✓

Dependent and Independent

Linear Dependent

Given a vector set, $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, if there exists any $\mathbf{a}_i$ that is a linear combination of other vectors

$$0 \times \left\{ \begin{bmatrix} 3 \\ -1 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, 0 \times \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix} \right\}$$

Dependent or Independent?

Zero vector is the linear combination of any other vectors

Any set contains zero vector would be linear dependent

How about a set with only one vector?

zero vector 相关

Dependent and Independent

Linear Dependent

Given a vector set, $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, if there exists any $\mathbf{a}_i$ that is a linear combination of other vectors

$$2\boldsymbol{a}_i + \boldsymbol{a}_j + 3\boldsymbol{a}_k = \mathbf{0}$$

$$2\boldsymbol{a}_i + \boldsymbol{a}_j = -3\boldsymbol{a}_k$$

$$\left(-\frac{2}{3}\right)\boldsymbol{a}_i + \left(-\frac{1}{3}\right)\boldsymbol{a}_j = \boldsymbol{a}_k$$

$\Updownarrow$

$$\boldsymbol{a}_{i'} = 3\boldsymbol{a}_{j'} + 4\boldsymbol{a}_{k'}$$

$$\boldsymbol{a}_{i'} - 3\boldsymbol{a}_{j'} - 4\boldsymbol{a}_{k'} = \mathbf{0}$$

Given a vector set, $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, there exists scalars $x_1, x_2, \ldots, x_n$, that are **not all zero**, such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$.

Intuition

Dependent:
Once we have solution, we have infinite.

- Intuitive link between dependence and the number of solutions

$$\begin{bmatrix} 6 & 1 & 7 \\ 3 & 8 & 11 \\ 3 & 3 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 22 \\ 12 \end{bmatrix} \qquad 1 \cdot \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix}$$

$$1 \cdot \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix} + 1 \cdot \begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix} + 1 \cdot \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix} = \begin{bmatrix} 14 \\ 22 \\ 12 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$2 \cdot \begin{bmatrix} 6 \\ 3 \\ 3 \end{bmatrix} + 2 \cdot \begin{bmatrix} 1 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 14 \\ 22 \\ 12 \end{bmatrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

Infinite Solution

Proof

- Columns of A are **dependent** → If $Ax=b$ have solution, it will have Infinite Solutions

- If $Ax=b$ have Infinite solutions → Columns of A are dependent

Proof

齐次方程

Homogeneous linear equations

$$\color{blue}{A}\color{red}{x}\color{black}{} = \mathbf{0} \qquad \color{blue}{A}\color{black}{} = [a_1 \quad a_2 \quad \cdots \quad a_n] \qquad \color{red}{x}\color{black}{} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

(always having $\mathbf{0}$ as solution)

Based on the definition

A set of n vectors $\{\boldsymbol{a_1}, \boldsymbol{a_2}, \cdots, \boldsymbol{a_n}\}$ is linear dependent $\Leftrightarrow$ $Ax = \mathbf{0}$ have non-zero solution

infinite

A set of n vectors $\{\boldsymbol{a_1}, \boldsymbol{a_2}, \cdots, \boldsymbol{a_n}\}$ is linear independent $\Leftrightarrow$ $Ax = \mathbf{0}$ only have zero solution

Proof

对 太深刻了

反证

- Columns of A are **dependent** → If Ax=b have solution, it will have Infinite solutions

indepent = 唯一解

We can find non-zero solution u such that $Au = \mathbf{0}$

There exists v such that $Av = \mathrm{b}$

$A(u+v) = \mathrm{b}$

$u+v$ is another solution different to v

- If Ax=b have Infinite solutions → Columns of A are dependent

$u \neq v$

$Au = \mathrm{b}$

$Av = \mathrm{b}$

$\underbrace{A(u-v)}_{\text{Non-zero}} = \mathbf{0}$

Rank and Nullity

Rank and Nullity

- The **rank** of a matrix is defined as the maximum number of *linearly independent columns* in the matrix.
- **Nullity** = Number of columns - **rank**

$$\begin{bmatrix} -3 & 2 & -1 \\ 7 & 9 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 3 & 10 \\ 2 & 6 & 20 \\ 3 & 9 & 30 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$R = 3$ $\qquad R = 1$ $\qquad R = 0$

$N = 0$ $\qquad N = 2$ $\qquad N = 3$

Rank and Nullity

- The **rank** of a matrix is defined as the maximum number of *linearly independent columns* in the matrix.
- **Nullity** = Number of columns - **rank**

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 8 \end{bmatrix} \qquad \begin{bmatrix} 0 & 3 \\ 0 & 5 \end{bmatrix} \qquad \begin{bmatrix} 5 \\ 2 \end{bmatrix} \qquad [6]$$

R=1	R=1	R=1	R=1
N=2	N=1	N=0	N=0

Rank and Nullity

- The **rank** of a matrix is defined as the maximum number of *linearly independent columns* in the the matrix.
- **Nullity** = Number of columns - **rank**

If A is a mxn matrix:

Rank A = n

Nullity A = 0

$\Longleftrightarrow$

Columns of A are independent

Conclusion

$$A\mathbf{x} = \mathbf{b}$$

$A: m \times n \quad x \in R^n \quad b \in R^m$

Is b a linear combination of columns of A?

Is b in the span of the columns of A?

NO

No solution

YES

The columns of A are ***independent***.

Rank $A = n$

Nullity $A = 0$

Unique solution

The columns of A are ***dependent***.

Rank $A < n$

Nullity $A > 0$

Infinite solution

Conclusion

The columns of A are ***independent***.

Rank A = n

Nullity A = 0

$A: m \times n$

$x \in R^n \quad b \in R^m$

NO

Is b a linear combination of columns of A?

Is b in the span of the columns of A?

NO

No solution

YES

Infinite solution

YES

Is b a linear combination of columns of A?

Is b in the span of the columns of A?

NO

No solution

YES

Unique solution

Question

- True or False
  - If the columns of A are linear independent, then $Ax=b$ has unique solution. ✗
  - If the columns of A are linear independent, then $Ax=b$ has at most one solution. ✓
  - If the columns of A are linear dependent, then $Ax=b$ has infinite solution. ✗
  - If the columns of A are linear independent, then $Ax=0$ (homogeneous equation) has unique solution. ✓
  - If the columns of A are linear dependent, then $Ax=0$ (homogeneous equation) has infinite solution. ✓