Solving System of Linear Equations

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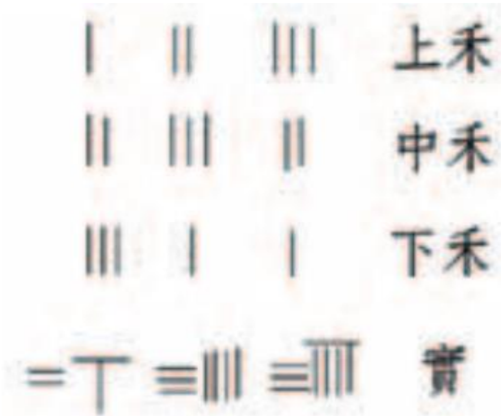
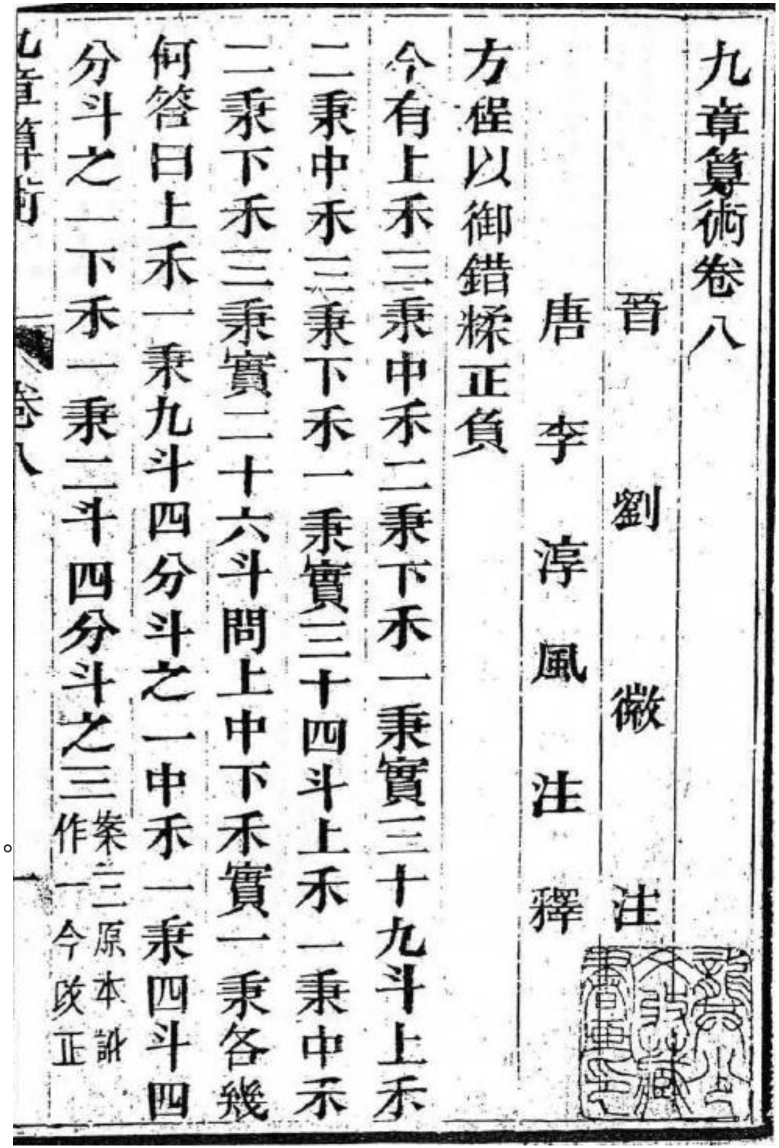
Reference

- Textbook: Chapter 1.3, 1.4

九章算術



置上禾三秉，中禾二秉，下禾一秉，實三十九斗，於右方。中、左禾列如右方。以右行上禾遍乘中行而以直除。又乘其次，亦以直除。然以中行中禾不盡者遍乘左行而以直除。左方下禾不盡者，上為法，下為實。實即下禾之實。求中禾，以法乘中行下實，而除下禾之實。餘如中禾秉數而一，即中禾之實。求上禾亦以法乘右行下實，而除下禾、中禾之實。餘如上禾秉數而一，即上禾之實。實皆如法，各得一斗。



九章算術卷八

晉 劉徽 注

唐 李淳風 注釋

方程以御錯糅正負

今有上禾三秉中禾二秉下禾一秉實三十九斗上禾二秉中禾三秉下禾一秉實三十四斗上禾一秉中禾二秉下禾三秉實二十六斗問上中下禾實一秉各幾何答曰上禾一秉九斗四分斗之一中禾一秉四斗四分斗之一下禾一秉二斗四分斗之三案三原本訛作一今改正

算法統宗

四色方程法可誇須存末位作根芽諸行乘減同前倒偶與奇行認莫差若遇奇行須減價偶行之價要相加加減作實須加法減法亦須減法佳隨問搜多繁雜色憑斯推廣更無他

今有瓜二箇梨四箇共價四分梨二箇桃七箇共價四分桃四箇橘七箇共價三分瓜一箇橘八箇共價二分四釐問各該價若干

荅曰瓜八釐　梨六釐　桃四釐　橘二釐

法曰列所問數　以一行三行爲奇二行四行爲偶

一	瓜二	梨四	空	空	價四分
二	空	梨二	桃七	空	價四分
三	空	空	桃四	橘七	價三分
四	瓜一	空負四	空	橘八得一十六	價二分四釐　得四分八釐

先以一行瓜二爲法遍乘四行梨空負四桃空橘八得一十六價二分四釐得四分八釐却以四行瓜一遍乘一行梨四得四第四行梨空無減桃空價四分得四分與四行四分八釐對減餘八釐次以二行梨二遍乘四行梨負四得八桃空橘十六得三十二價八釐得一分六釐却以四行梨負四遍乘二行梨二得八與二行梨八對減盡桃七得二十八橘空價四分得一錢六分加四行一分六釐共一錢七分六釐又以三行桃四遍乘四行桃負二十八得一百一十二橘三十二得一百二十八價一錢七分六釐得七錢零四釐却以四行桃負二十八遍乘三行桃四得一百一十二與四行桃減盡橘七得一百九十六減四行橘一百二十八餘六十八爲法價三分得八錢四分減四行價七錢零四釐餘一錢三分六釐爲實

Equivalent

- Two systems of linear equations are **equivalent** if they have exactly **the same solution set**.

$$\begin{cases} 3x_1 & +x_2 & = 10 \\ x_1 & -3x_2 & = 0 \end{cases}$$

Solution set: $\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$

$\Longleftrightarrow$ equivalent

$$\begin{cases} x_1 & & = 3 \\ & x_2 & = 1 \end{cases}$$

Solution set: $\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$

Equivalent

- Applying the following three operations on a system of linear equations will produce an equivalent one.
- 1. Interchange

$$\begin{cases} 3x_1 & +x_2 & = 10 \\ x_1 & -3x_2 & = 0 \end{cases} \Rightarrow \begin{cases} x_1 & -3x_2 & = 0 \\ 3x_1 & +x_2 & = 10 \end{cases}$$

- 2. Scaling

$$\begin{cases} 3x_1 & +x_2 & = 10 \\ x_1 & -3x_2 & = 0 \quad \times(-3) \end{cases} \Rightarrow \begin{cases} 3x_1 & +x_2 & = 10 \\ -3x_1 & +9x_2 & = 0 \end{cases}$$

- 3. Row Addition

$$\begin{cases} 3x_1 & +x_2 & = 10 \\ x_1 & -3x_2 & = 0 \quad \times(-3) \end{cases} \Rightarrow \begin{cases} & 10x_2 & = 10 \\ x_1 & -3x_2 & = 0 \end{cases}$$

Solving system of linear equation

- Two systems of linear equations are **equivalent** if they have exactly **the same solution set**.
- Strategy:

$$\begin{cases} x_1 & -3x_2 & = 0 \\ 3x_1 & +x_2 & = 10 \end{cases} \Rightarrow \begin{cases} x_1 & -3x_2 & = 0 \\ & 10x_2 & = 10 \end{cases} \Rightarrow \begin{cases} x_1 & -3x_2 & = 0 \\ & x_2 & = 1 \end{cases} \Rightarrow \begin{cases} x_1 & & = 3 \\ & x_2 & = 1 \end{cases}$$

We know how to transform the given system of linear equations into another equivalent system of linear equations.

We do it again and again until the system of linear equation is so simple that we know its answer at a glance.

Augmented Matrix

- a system of linear equation

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
\quad \Rightarrow \quad A\mathbf{x} = \mathbf{b}
$$

$$
\underset{m \times n}{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}
$$

coefficient matrix

Augmented Matrix

- a system of linear equation

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
\quad \Rightarrow \quad A\mathbf{x} = \mathbf{b}
$$

m x n   m x 1

m x (n+1)

$$
\left[\ A \mid \mathbf{b}\ \right] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}
$$

augmented matrix

Solving system of linear equation

- Two systems of linear equations are **equivalent** if they have exactly **the same solution set**.
- Strategy of solving:

$$\begin{cases} x_1 & -3x_2 & = 0 \\ 3x_1 & +x_2 & = 10 \end{cases} \Rightarrow \begin{cases} x_1 & -3x_2 & = 0 \\ & 10x_2 & = 10 \end{cases} \Rightarrow \begin{cases} x_1 & -3x_2 & = 0 \\ & x_2 & = 1 \end{cases} \Rightarrow \begin{cases} x_1 & & = 3 \\ & x_2 & = 1 \end{cases}$$

$$\begin{bmatrix} 1 & -3 & 0 \\ 3 & 1 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 0 \\ 0 & 10 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

1. Interchange any two rows of the matrix
2. Multiply every entry of some row by the same nonzero scalar
3. Add a multiple of one row of the matrix to another row

Solving system of linear equation

A ***complex*** system of linear equations

A ***simple*** system of linear equations

$A\mathbf{x} = \mathbf{b}$ $\longleftrightarrow$ $R'\mathbf{x} = \mathbf{b}'$

equivalent

A'=[A $\mathbf{b}$] $\Rightarrow$ A'' $\Rightarrow$ A''' $\Rightarrow$ …… $\Rightarrow$ R=[R' $\mathbf{b}'$]

reduced row echelon form

1. Interchange any two rows of the matrix
2. Multiply every entry of some row by the same nonzero scalar
3. Add a multiple of one row of the matrix to another row

elementary row operations

行初等变换

Reduced Row Echelon Form

- A system of linear equations is easily solvable if its augmented matrix is in ***reduced row echelon form***
- ***Row Echelon Form***

1. Each nonzero row lies above every zero row

2. The leading entries are in echelon form

$$\begin{bmatrix} 1 & 7 & 2 & -3 & 9 & 4 \\ 0 & 0 & 1 & 4 & 6 & 8 \\ 0 & 0 & 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced Row Echelon Form

- A system of linear equations is easily solvable if its augmented matrix is in ***reduced row echelon form***
- ***Row Echelon Form***

1. Each nonzero row lies above every zero row

2. The leading entries are in echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 6 & 3 & 0 \\ 0 & 0 & 1 & 5 & 7 & 0 \\ 0 & 1 & 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

No zero rows

Reduced Row Echelon Form

- A system of linear equations is easily solvable if its augmented matrix is in ***reduced row echelon form***
- ***Reduced Row Echelon Form***

1-2 The matrix is in row echelon form

3. The columns containing the leading entries are standard vectors.

$$\begin{bmatrix} 1 & 7 & 2 & -3 & 9 & 4 \\ 0 & 0 & 1 & 4 & 6 & 8 \\ 0 & 0 & 0 & 2 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

只有某一个是1，其它是0

Reduced Row Echelon Form

- A system of linear equations is easily solvable if its augmented matrix is in ***reduced row echelon form***
- ***Reduced Row Echelon Form***

1-2 The matrix is in row echelon form

3. The columns containing the leading entries are standard vectors.

$$\begin{bmatrix} 1 & -3 & 0 & 2 & 0 & 7 \\ 0 & 0 & 1 & 6 & 0 & 9 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

RREF is unique

- A matrix can be transformed into multiple REF by row operation, but only one RREF

REF

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 3 & 3 & -6 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$\times 3$

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 3 & -6 & -5 & 3 \\ 2 & -1 & 1 & 0 \end{bmatrix}$$

REF

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 3 & 3 & -6 \\ 0 & 0 & 3 & 9 \end{bmatrix}$$

REF

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 3 & 0 & -15 \\ 0 & 0 & 3 & 9 \end{bmatrix}$$

RREF

$$\begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Pivot 中樞

Reduced Row Echelon Form

A

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ -1 & -2 & 1 & 2 & 3 & 6 \\ 2 & 4 & -3 & 2 & 0 & 3 \\ -3 & -6 & 2 & 0 & 3 & 9 \end{bmatrix} \Rightarrow$$

R

$$\begin{bmatrix} 1 & 2 & 0 & 0 & -1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Leading Entry

The pivot positions of A are (1,1), (2,3) and (3,4).

The pivot columns of A are 1st, 3rd and 4th columns.

Reduced Row Echelon Form

- A system of linear equations is easily solvable if its augmented matrix is in ***reduced row echelon form***

Example 1. Unique Solution

$$\begin{array}{cccc} x_1 & x_2 & x_3 & b \\ \end{array}$$
$$\left[\begin{array}{cccc} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right] \Rightarrow \begin{aligned} x_1 \qquad\qquad &= -4 \\ x_2 \qquad &= -5 \\ x_3 &= \;\; 3 \end{aligned}$$

If RREF looks like [I $\mathbf{b}'$]

unique solution

Example 2. Infinite Solution

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & b \\ \end{array}$$
$$\left[\begin{array}{cccccc} 1 & -3 & 0 & 2 & 0 & 7 \\ 0 & 0 & 1 & 6 & 0 & 9 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{aligned} x_1 - 3x_2 \quad + 2x_4 \quad &= 7 \\ x_3 + 6x_4 \quad &= 9 \\ x_5 &= 2 \\ \cancel{0 = 0} \end{aligned}$$

Free variables

Basic variables

x_1

x_2

x_3

x_4

x_5

With free variables, there are infinitely many solutions.

Parametric Representation:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 7 + 3x_2 - 2x_4 \\ \\ 9 - 6x_4 \\ \\ 2 \end{bmatrix}$$

Reduced Row Echelon Form

- ***Example 3. No Solution***

$$\begin{array}{cccc} x_1 & x_2 & x_3 & b \\ \end{array}$$

$$\left[\begin{array}{cccc} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \Leftrightarrow \begin{aligned} x_1 \qquad\quad -3x_3 &= 0 \\ x_2 + 2x_3 &= 0 \\ 0x_1 + 0x_2 + 0x_3 &= 1 \\ 0x_1 + 0x_2 + 0x_3 &= 0 \end{aligned}$$

inconsistent

When an augmented matrix contains a row in which **the only nonzero entry lies in the last column**

The corresponding system of linear equations has **no solution** (**inconsistent**).

http://www.ams.org/notices/201106/rtx110600782p.pdf
http://www.dougbabcock.com/matrix.php

Reduced Row Echelon Form

- RREF of a matrix is unique.
- Gaussian elimination: an algorithm for finding the **reduced** **row echelon form** of a matrix.

Original augmented matrix | **A row echelon form** | **The reduced row echelon form**

$$\begin{bmatrix} 1 & -2 & -1 & 3 \\ 3 & -6 & -5 & 3 \\ 2 & -1 & 1 & 0 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & -2 & -1 & 3 \\ 0 & 3 & 3 & -6 \\ 0 & 0 & 1 & 3 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Elementary row operations | Elementary row operations

Please refer to the steps of Gaussian Elimination in the textbook by yourself.

Example 1

$$
\begin{aligned}
x_1 \;\; &+2x_2 - \;\; x_3 + 2x_4 + \;\; x_5 = 2\\
-x_1 \;\; &-2x_2 + \;\; x_3 + 2x_4 + 3x_5 = 6\\
2x_1 \;\; &+4x_2 - 3x_3 + 2x_4 \qquad\quad\;\; = 3\\
-3x_1 &-6x_2 + 2x_3 \qquad\qquad\;\; + 3x_5 = 9
\end{aligned}
$$

$$
\begin{array}{c}
\\
1\\
-2\\
3
\end{array}
\begin{bmatrix}
1 & 2 & -1 & 2 & 1 & 2\\
-1 & -2 & 1 & 2 & 3 & 6\\
2 & 4 & -3 & 2 & 0 & 3\\
-3 & -6 & 2 & 0 & 3 & 9
\end{bmatrix}
$$

Example 1

$$\begin{aligned}
x_1 &+2x_2 - x_3 + 2x_4 + x_5 = 2\\
-x_1 &-2x_2 + x_3 + 2x_4 + 3x_5 = 6\\
2x_1 &+4x_2 - 3x_3 + 2x_4 = 3\\
-3x_1 &-6x_2 + 2x_3 + 3x_5 = 9
\end{aligned}$$

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 4 & 4 & 8 \\ 0 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & -1 & 6 & 6 & 15 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 2 & -1 & 2 & 1 & 2 \\ 0 & 0 & -1 & -2 & -2 & -1 \\ 0 & 0 & 0 & 4 & 4 & 8 \\ 0 & 0 & -1 & 6 & 6 & 15 \end{bmatrix}$$

Example 1

$$
\begin{aligned}
x_1 + 2x_2 - x_3 + 2x_4 + x_5 &= 2\\
-x_1 - 2x_2 + x_3 + 2x_4 + 3x_5 &= 6\\
2x_1 + 4x_2 - 3x_3 + 2x_4 &= 3\\
-3x_1 - 6x_2 + 2x_3 + 3x_5 &= 9
\end{aligned}
$$

$$
\begin{array}{c c}
 & \\
 & \\
-1 & \left[\begin{array}{cccccc}
1 & 2 & -1 & 2 & 1 & 2\\
0 & 0 & -1 & -2 & -2 & -1\\
0 & 0 & 0 & 4 & 4 & 8\\
0 & 0 & -1 & 6 & 6 & 15
\end{array}\right]
\end{array}
$$

Example 1

$$
\begin{aligned}
x_1 + 2x_2 - x_3 + 2x_4 + x_5 &= 2 \\
-x_1 - 2x_2 + x_3 + 2x_4 + 3x_5 &= 6 \\
2x_1 + 4x_2 - 3x_3 + 2x_4 &= 3 \\
-3x_1 - 6x_2 + 2x_3 + 3x_5 &= 9
\end{aligned}
$$

$$
\begin{array}{c c}
 & \\
 & \\
-2 & \\
 &
\end{array}
\begin{bmatrix}
1 & 2 & -1 & 2 & 1 & 2 \\
0 & 0 & -1 & -2 & -2 & -1 \\
0 & 0 & 0 & 4 & 4 & 8 \\
0 & 0 & 0 & 8 & 8 & 16
\end{bmatrix}
$$

Example 1

$$
\begin{array}{rrrrrl}
x_1 & +2x_2 & -\ x_3 & +2x_4 & +\ x_5 & =2 \\
-x_1 & -2x_2 & +\ x_3 & +2x_4 & +3x_5 & =6 \\
2x_1 & +4x_2 & -3x_3 & +2x_4 & & =3 \\
-3x_1 & -6x_2 & +2x_3 & & +3x_5 & =9
\end{array}
$$

$$
\begin{bmatrix}
1 & 2 & -1 & 2 & 1 & 2 \\
0 & 0 & -1 & -2 & -2 & -1 \\
0 & 0 & 0 & 4 & 4 & 8 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Example 1

$$\begin{aligned}
x_1 + 2x_2 - x_3 + 2x_4 + x_5 &= 2\\
-x_1 - 2x_2 + x_3 + 2x_4 + 3x_5 &= 6\\
2x_1 + 4x_2 - 3x_3 + 2x_4 &= 3\\
-3x_1 - 6x_2 + 2x_3 + 3x_5 &= 9
\end{aligned}$$

$$\begin{bmatrix} 1 & 2 & -1 & 0 & -1 & -2 \\ 0 & 0 & -1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Example 1

$$
\begin{array}{cccccc}
\color{red}{x_1} & \color{red}{x_2} & \color{red}{x_3} & \color{red}{x_4} & \color{red}{x_5} & \color{green}{b} \\
\end{array}
$$

$$
\left[\begin{array}{cccccc}
1 & 2 & 0 & 0 & -1 & -5 \\
0 & 0 & 1 & 0 & 0 & -3 \\
0 & 0 & 0 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

$$
\begin{aligned}
x_1 + 2x_2 \qquad + \qquad - x_5 &= -5 \\
x_3 \qquad\qquad &= -3 \\
x_4 + x_5 &= \ \ 2
\end{aligned}
$$

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}
=
\begin{bmatrix} -5 - 2x_2 + x_5 \\ x_2 \\ -3 \\ 2 - x_5 \\ x_5 \end{bmatrix}
= x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}
+ x_5 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}
+ \begin{bmatrix} -5 \\ 0 \\ -3 \\ 2 \\ 0 \end{bmatrix}
$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -5 - 2x_2 + x_5 \\ x_2 \quad 1 \\ -3 \\ 2 - x_5 \quad 3 \\ x_5 \quad -1 \end{bmatrix} -8 = \overset{1}{x_2} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \overset{-1}{x_5} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} -5 \\ 0 \\ -3 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_2 \quad + \quad - x_5 &= -5 \\ x_3 \qquad &= -3 \\ x_4 + x_5 &= 2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} & x_1 \ free \\ & x_2 = -\frac{5}{2} - \frac{1}{2}x_1 + \frac{1}{2}x_5 \\ & \cancel{x_1 = -5 - 2x_2 + x_5} \\ & \cancel{x_2 \quad \text{free}} \\ & x_3 = -3 \\ & x_4 = 2 - x_5 \\ & x_5 \quad \text{free} \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} x_1 \quad -8 \\ -5/2 - 1/2x_1 + 1/2\overset{1}{x_5} \\ -3 \\ 2 - x_5 \quad 3 \\ x_5 \quad -1 \end{bmatrix} = \overset{-8}{x_1} \begin{bmatrix} 1 \\ -1/2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \overset{-1}{x_5} \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -5/2 \\ -3 \\ 2 \\ 0 \end{bmatrix}$$

Example 2

- Find the RREF of

$$R = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & -6 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\times(-2) \quad \begin{pmatrix} R \\ 2R \end{pmatrix} \Rightarrow \begin{pmatrix} R \\ \boldsymbol{O} \end{pmatrix}$$

Example 3

- Find the RREF of $\begin{bmatrix} R & 2R \\ R & -R \end{bmatrix}$ $\quad R = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} R & 2R \\ R & -R \end{bmatrix} \rightarrow \begin{bmatrix} R & 2R \\ \boldsymbol{O} & -3R \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} R & \boldsymbol{O} \\ \boldsymbol{O} & -3R \end{bmatrix} \rightarrow \begin{bmatrix} R & \boldsymbol{O} \\ \boldsymbol{O} & R \end{bmatrix}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & -3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Checking Independence

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$$

Linear independent or not?

A set of n vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ is linear dependent

Given a vector set, $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, if there exists any $\mathbf{a}_i$ that is a linear combination of other vectors

matrix A

Given a vector set, $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$, there exists scalars $x_1, x_2, \ldots, x_n$, that are **not all zero**, such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$.

vector x

$Ax = \mathbf{0}$ have non-zero solution

Checking Independence

$$\underbrace{\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}}_{\mathbf{A}}$$

Linear independent or not?

$Ax = \mathbf{0}$ have non-zero solution or not

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 0 & 4 & 2 \\ 1 & 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 2 & 0 & 4 & 2 & 0 \\ 1 & 1 & 1 & 3 & 0 \end{bmatrix} \end{array} \xrightarrow{\text{RREF}} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \end{array} \\ \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Checking Independence

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$
$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 2 & 0 & 4 & 2 & 0 \\ 1 & 1 & 1 & 3 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \\ \end{array} \begin{bmatrix} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 + 2x_3 &= 0 \\ x_2 - x_3 &= 0 \\ x_4 &= 0 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} x_1 &= -2x_3 \\ x_2 &= x_3 \\ x_3 &\ is\ free \\ x_4 &= 0 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

setting $x_3 = 1$ $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$