Matrix Multiplication

Hung-yi Lee

Reference

- Textbook: Chapter 2.1

Matrix Multiplication

- Given two matrices A and B, the (i, j)-entry of AB is the inner product of row i of A and column j of B

column j of B

row i of A

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & \cdots & b_{2j} & \cdots & b_{2p} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix}$$

A B

$C = AB$

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

Matrix Multiplication

- Given two matrices A and B, the (i, j)-entry of AB is the inner product of row i of A and column j of B

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix}$$

$$\boxed{C = AB} = \begin{bmatrix} (-1) \times 1 + 3 \times 2 & 1 \times 1 + 2 \times 2 \\ (-1) \times 3 + 3 \times 4 & 1 \times 3 + 2 \times 4 \\ (-1) \times 5 + 3 \times 6 & 1 \times 5 + 2 \times 6 \end{bmatrix}$$

Matrix Multiplication – 4 ways

- Way 1: inner product

Given two matrices A and B, the (*i*, *j*)-entry of AB is the inner product of row i of A and column j of B

B $\begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & \cdots & b_{2j} & \cdots & b_{2p} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix}$

A $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

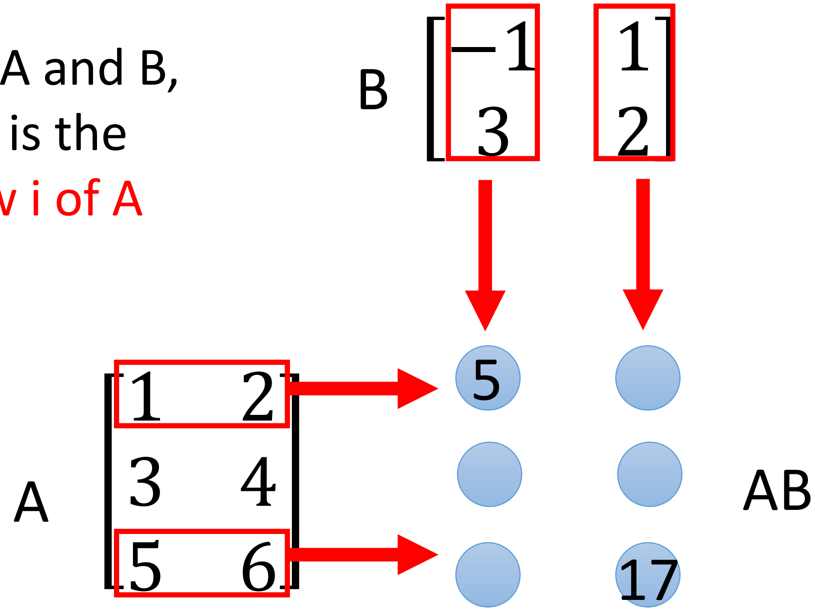
$\begin{bmatrix} & & \\ & \text{(i,j)-entry} & \\ & & \end{bmatrix}$

AB

Matrix Multiplication – 4 ways

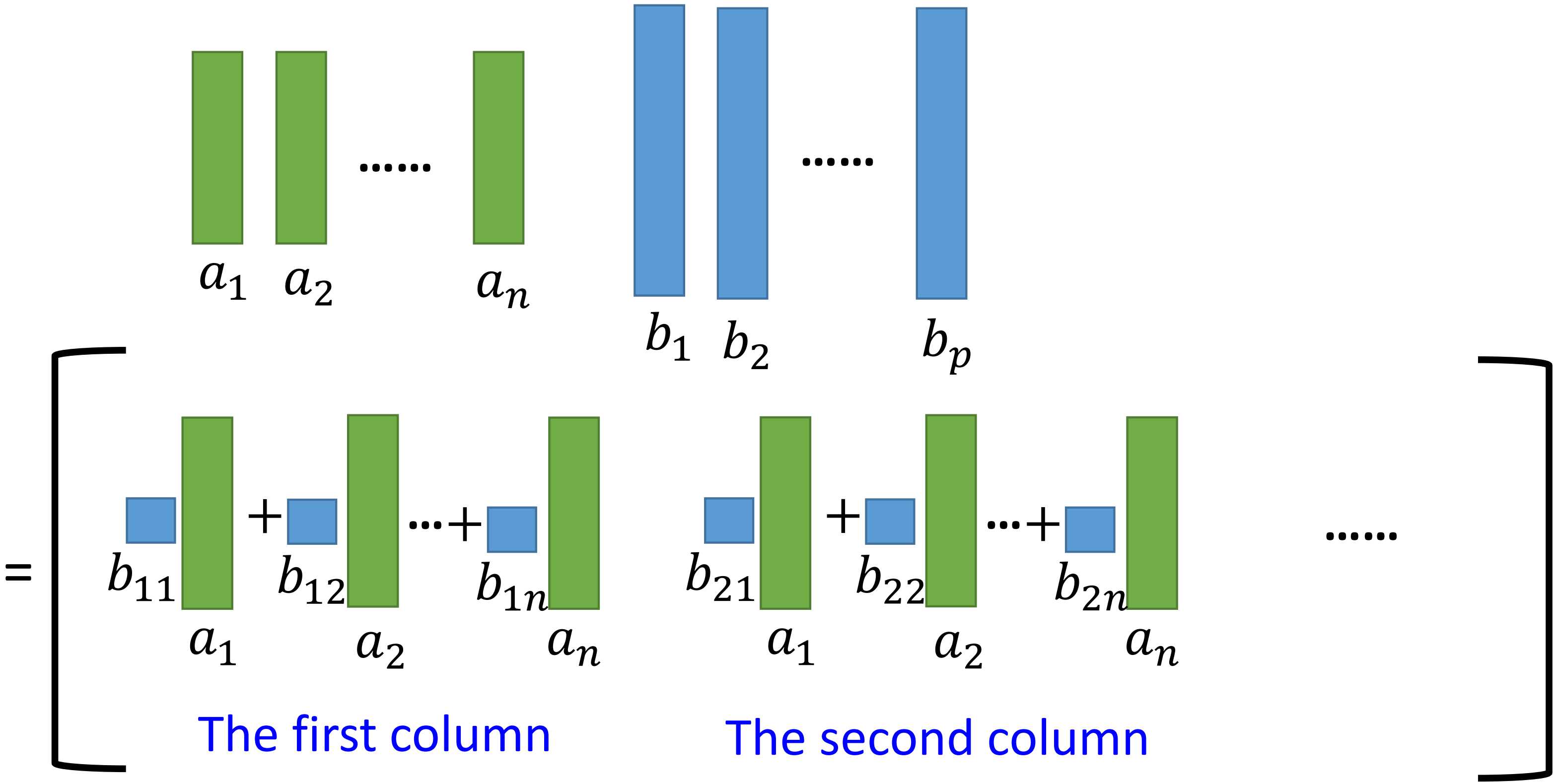
- Way 1: inner product

Given two matrices A and B, the (i, j)-entry of AB is the inner product of row i of A and column j of B



Matrix Multiplication – 4 ways

- Way 2: Linear combination of columns



Matrix Multiplication – 4 ways

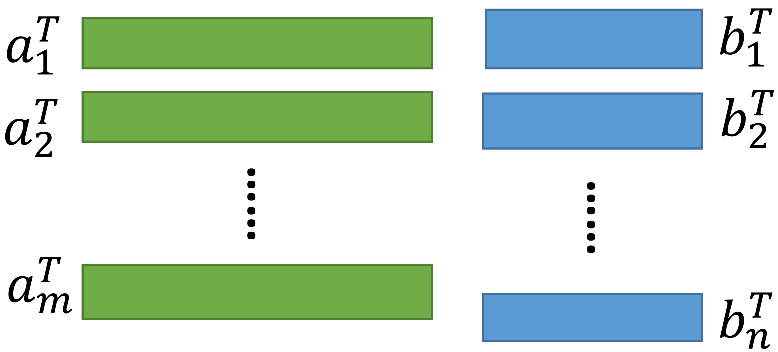
- Way 2: Linear combination of columns

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix}$$

$$= \left[\begin{array}{cc} \underbrace{-1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}}_{\text{The first column}} & \underbrace{1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}}_{\text{The second column}} \end{array} \right]$$

Matrix Multiplication – 4 ways

- Way 3: Linear combination of rows



$$= \begin{bmatrix} a_{11}b_1^T + a_{12}b_2^T \cdots + a_{1n}b_n^T \\ a_{21}b_1^T + a_{22}b_2^T \cdots + a_{2n}b_n^T \\ \vdots \\ a_{m1}b_1^T + a_{m2}b_2^T \cdots + a_{mn}b_n^T \end{bmatrix}$$

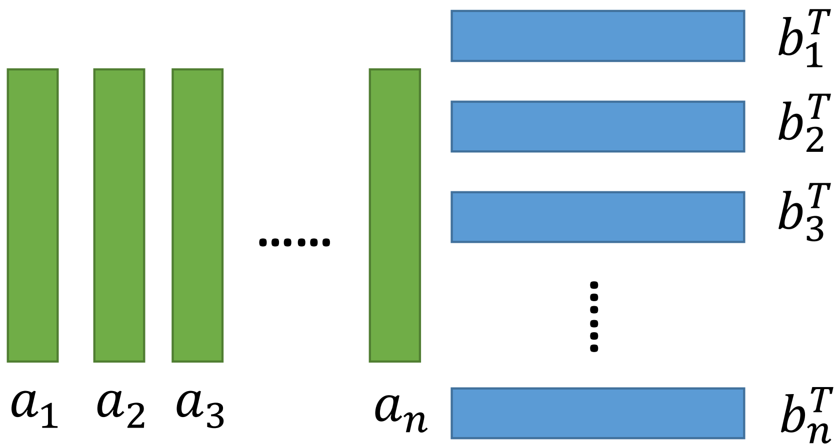
Matrix Multiplication – 4 ways

- Way 3: Linear combination of rows

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1[-1 \quad 1] + 2[3 \quad 2] \\ \text{The first row} \\ 3[-1 \quad 1] + 4[3 \quad 2] \\ \text{The second row} \\ 5[-1 \quad 1] + 6[3 \quad 2] \\ \text{The third row} \end{bmatrix}$$

Matrix Multiplication – 4 ways

- Way 4: summation of matrices



$$= a_1 b_1^T + a_2 b_2^T + \cdots + a_n b_n^T$$

matrices

Matrix Multiplication – 4 ways

- Way 4: summation of matrices

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 3 & 2 \end{bmatrix} = \left[\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \begin{bmatrix} -1 & 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \begin{bmatrix} 3 & 2 \end{bmatrix} \right]$$

"1 x 2" "2 x 1"

"1 x 1"

$$= \left[\begin{bmatrix} -1 & 1 \\ -3 & 3 \\ -5 & 5 \end{bmatrix} + \begin{bmatrix} 6 & 4 \\ 12 & 8 \\ 18 & 12 \end{bmatrix} \right]$$

Rank = ? Rank = ?

Augmentation and Partition

- Augment: the augment of A and B is [A B]
- Partition:

$$A = \left[\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 4 & 3 & 2 & 1 \end{array}\right] \qquad A = \left[\begin{array}{cc|cc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \hline 4 & 3 & 2 & 1 \end{array}\right]$$

$$A = \left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \hline 4 & 3 & 2 & 1 \end{array}\right] \qquad A = \left[\begin{array}{c|cc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 4 & 3 & 2 & 1 \end{array}\right]$$

Block Multiplication

$$A = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 0 & 5 & -1 & 6 \\ 1 & 0 & 3 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 0 \\ 2 & -1 & 2 \\ 0 & 3 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

Multiply as the small matrices are scalar

$$= \begin{bmatrix} & \\ & \end{bmatrix}$$

Don’t switch the order

Block Multiplication

$$A = \left[\begin{array}{cc|cc} 1 & 3 & 4 & 2 \\ 0 & 5 & -1 & 6 \\ \hline 1 & 0 & 3 & -1 \end{array}\right]$$

"2 x 2"

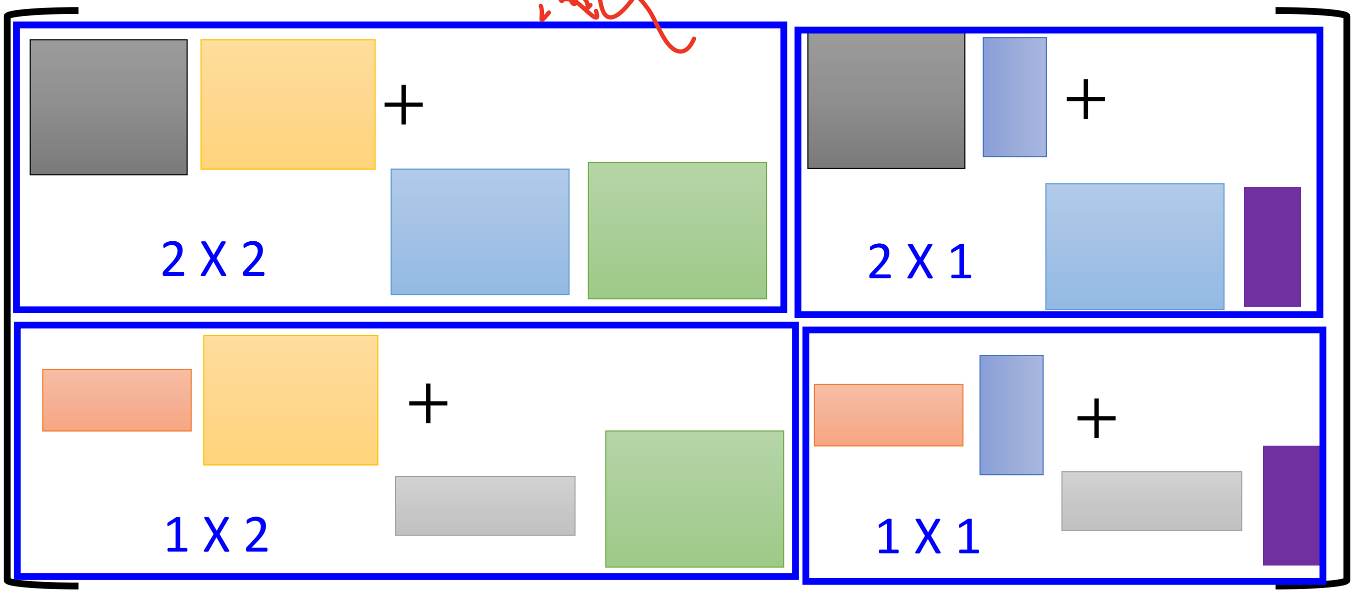
不是所有partition相乘都有意义

$$B = \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 1 & 2 & 0 \\ \hline 2 & -1 & 2 \\ 0 & 3 & 1 \end{array}\right]$$

"2 x 2"

$AB =$

"2 x 2"



Block Multiplication

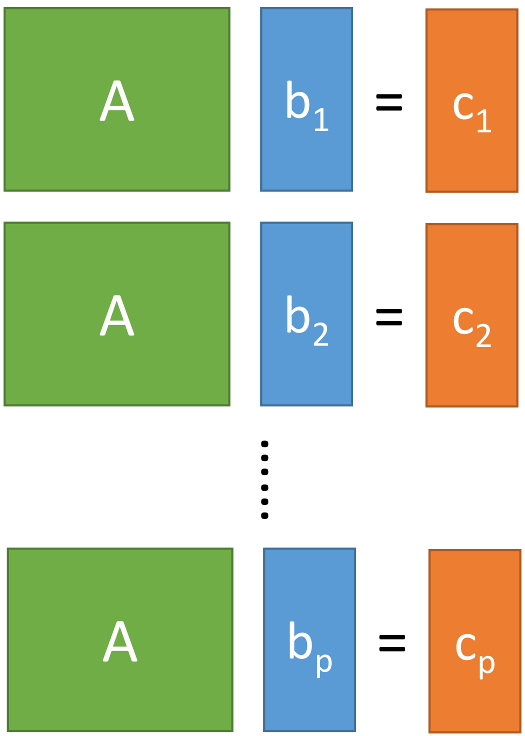
$$A = \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 6 & 8 & 5 & 0 \\ -7 & 9 & 0 & 5 \end{array}\right] \qquad A = \begin{bmatrix} I_2 & O \\ B & 5I_2 \end{bmatrix} \qquad B = \begin{bmatrix} 6 & 8 \\ -7 & 9 \end{bmatrix}$$

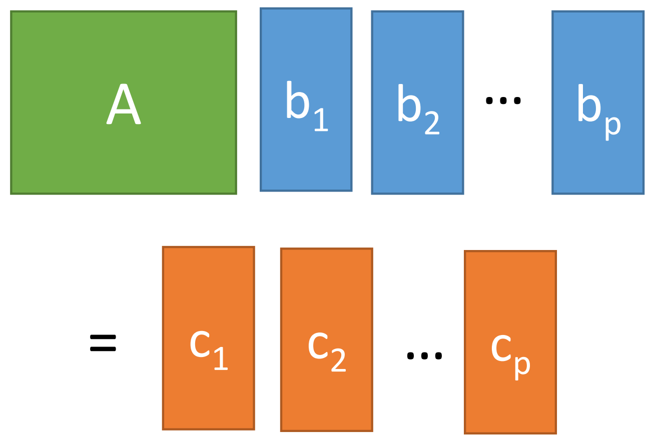
$$A^2 = \begin{bmatrix} I_2 & O \\ B & 5I_2 \end{bmatrix}\begin{bmatrix} I_2 & O \\ B & 5I_2 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

$$A^3 = AA^2 = \begin{bmatrix} I_2 & O \\ B & 5I_2 \end{bmatrix}\begin{bmatrix} I_2 & O \\ 6B & 25I_2 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

Matrix Multiplication - Meaning

- ***Multiple Input*** $C = AB$



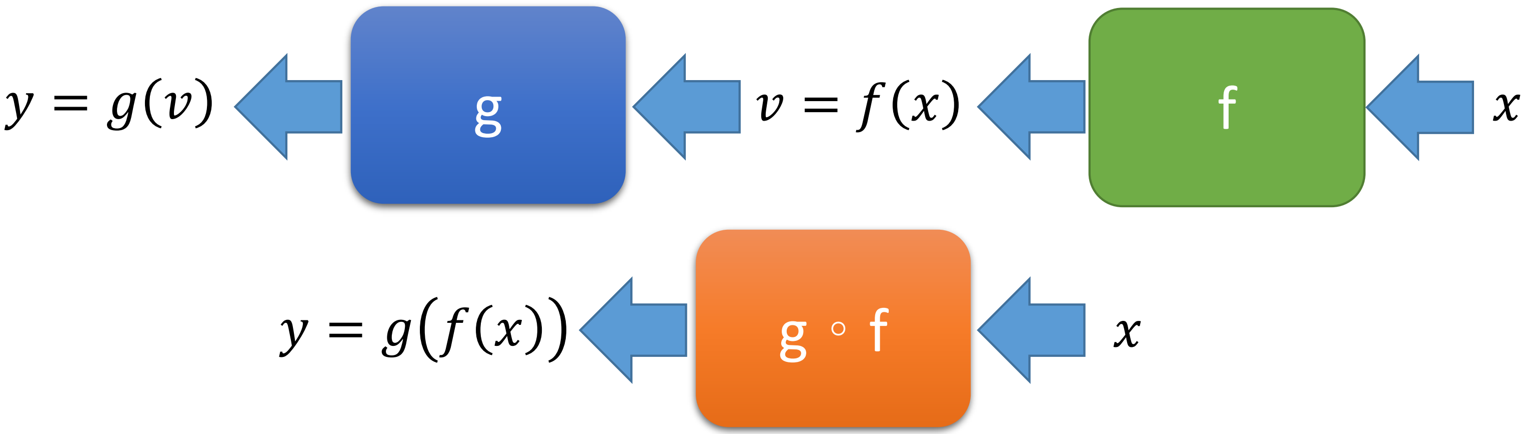


$$AB = A[b_1 \quad b_2 \quad \cdots \quad b_p]$$

$$= [Ab_1 \quad Ab_2 \quad \cdots \quad Ab_p]$$

Matrix Multiplication - Meaning

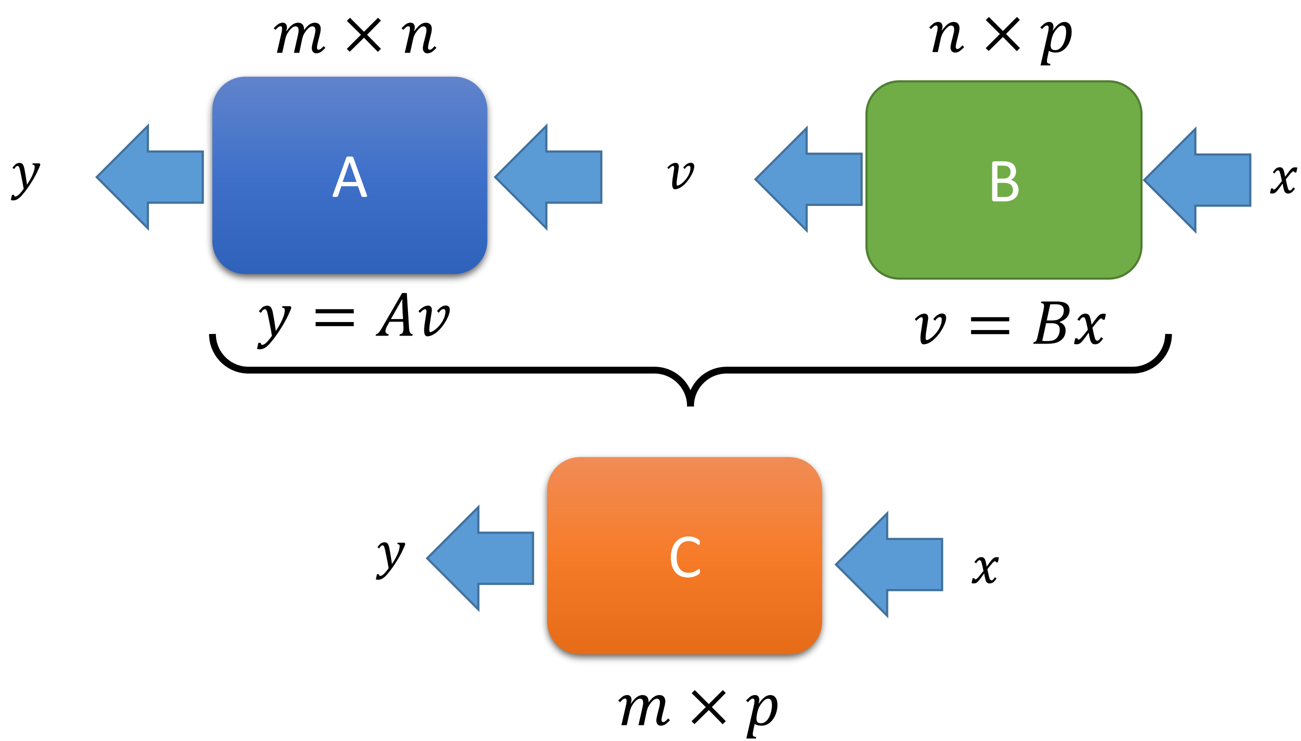
- ***<u>Composition</u>***
  - Given two function f and g, the function $g(f(.))$ is the composition g°f.



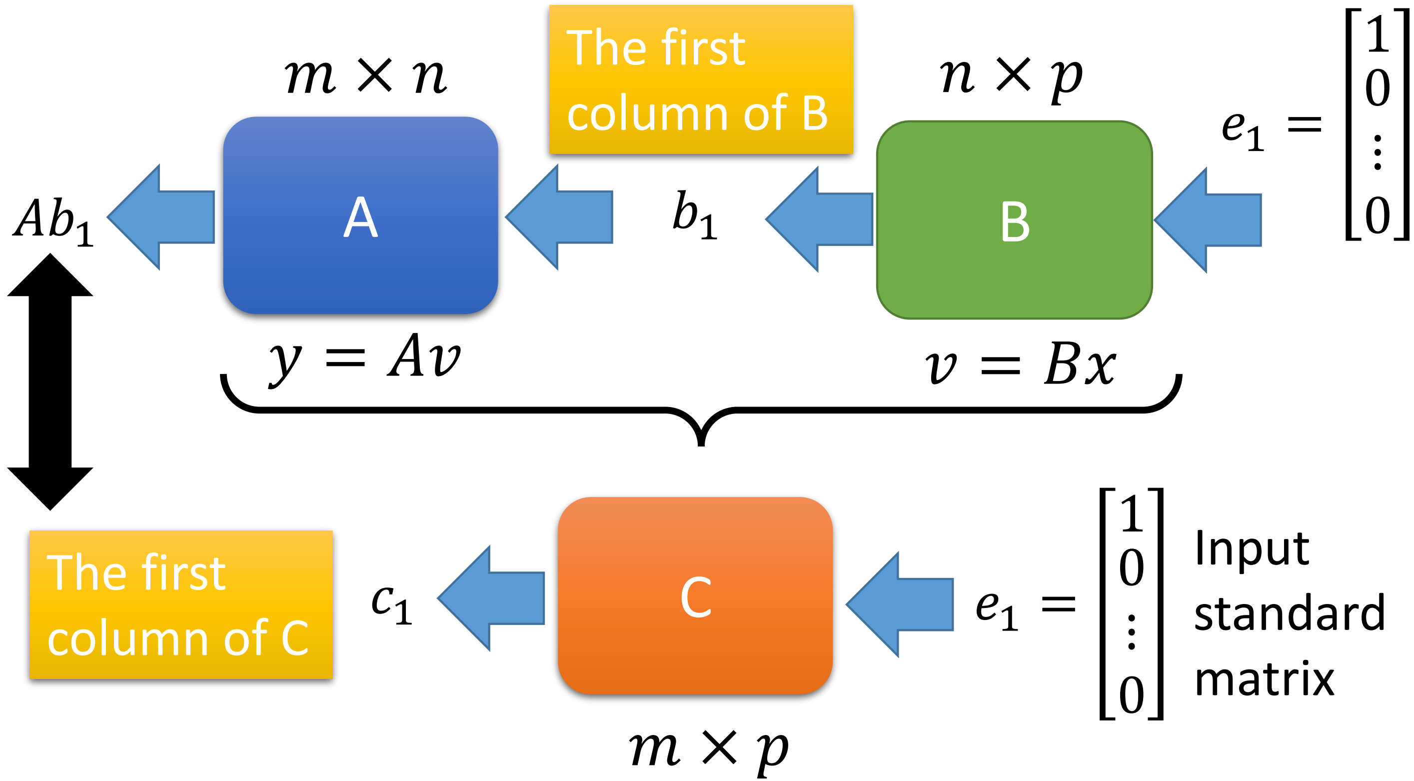
Matrix multiplication is the composition of two linear functions.

Matrix Multiplication - Meaning

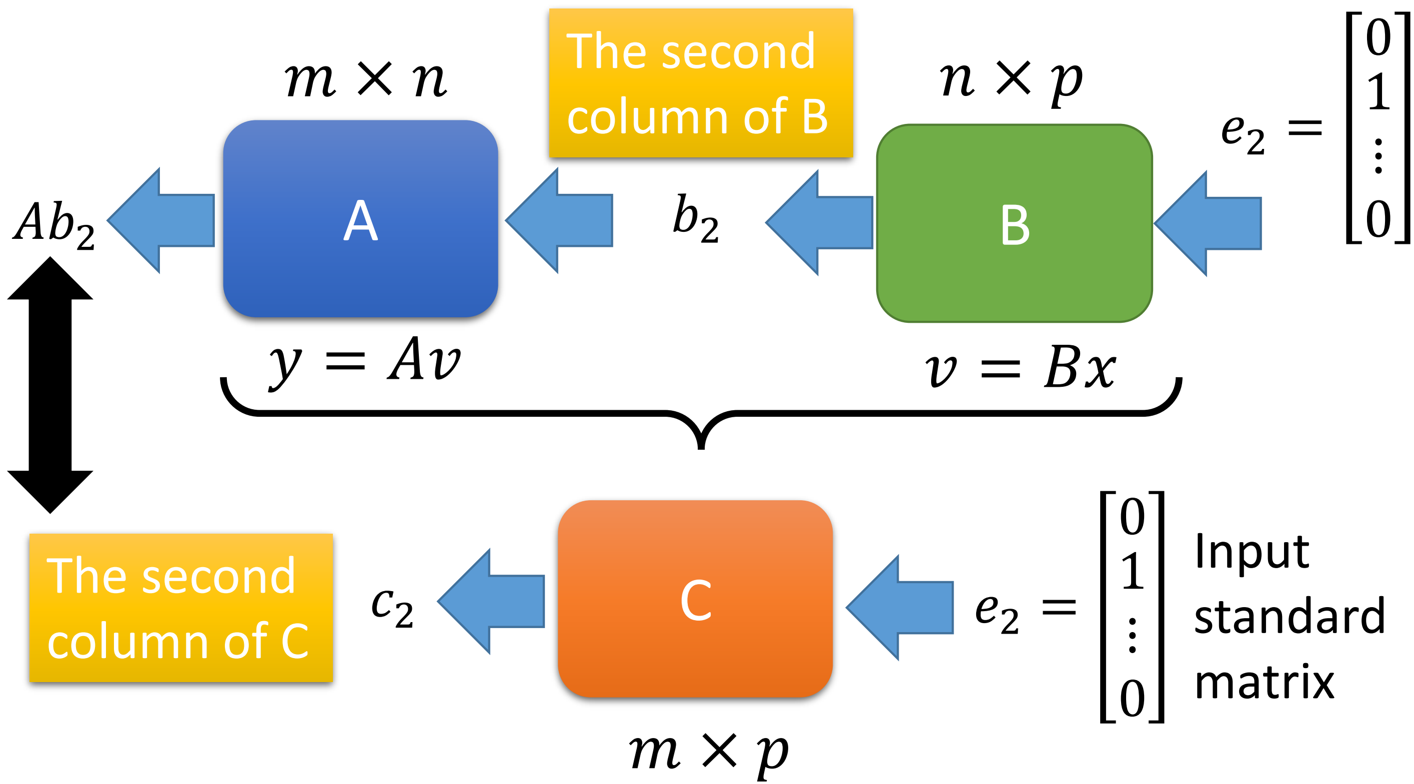
- ***Composition***

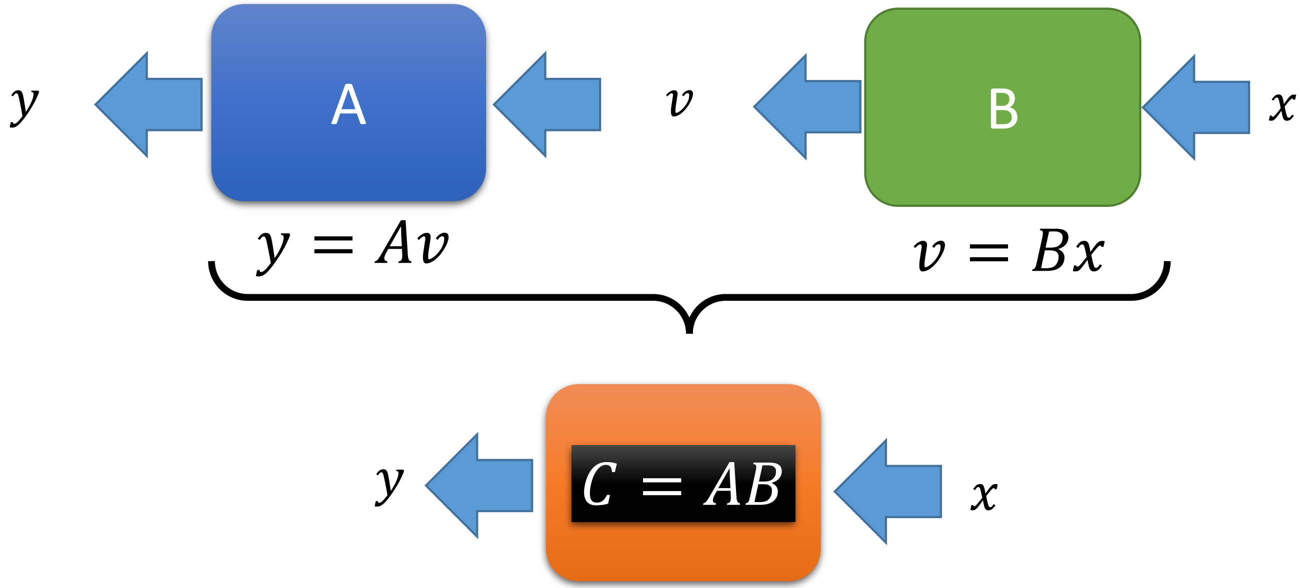


Matrix Multiplication - Meaning



Matrix Multiplication - Meaning





The composition of A and B is

$$C = [Ab_1 \quad Ab_2 \quad \cdots \quad Ab_p\,]$$

Matrix Multiplication

Example

$$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

reflection about the x-axis

rotation by 180°

R^2 y $\longleftarrow \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} \longleftarrow$ R^2 v $\longleftarrow \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix} \longleftarrow$ R^2 x

$$\underbrace{y = Av \qquad v = Bx}$$

$y \longleftarrow \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix} \longleftarrow x$

reflection about the y-axis

Not Communicative

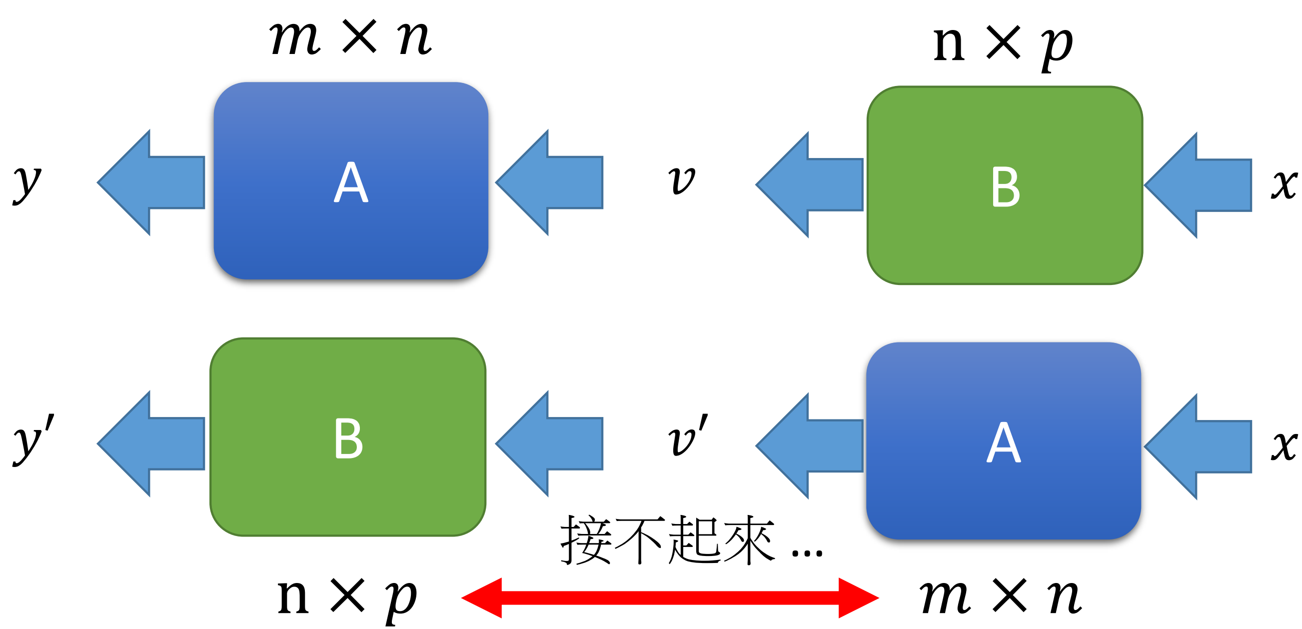
- $AB \neq BA$

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$$

$$\neq BA = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$

Not Communicative



If A and B are matrices, then both AB and BA are defined if and only if A and B are square matrices?

$m \times n$ $n \times m$

A B

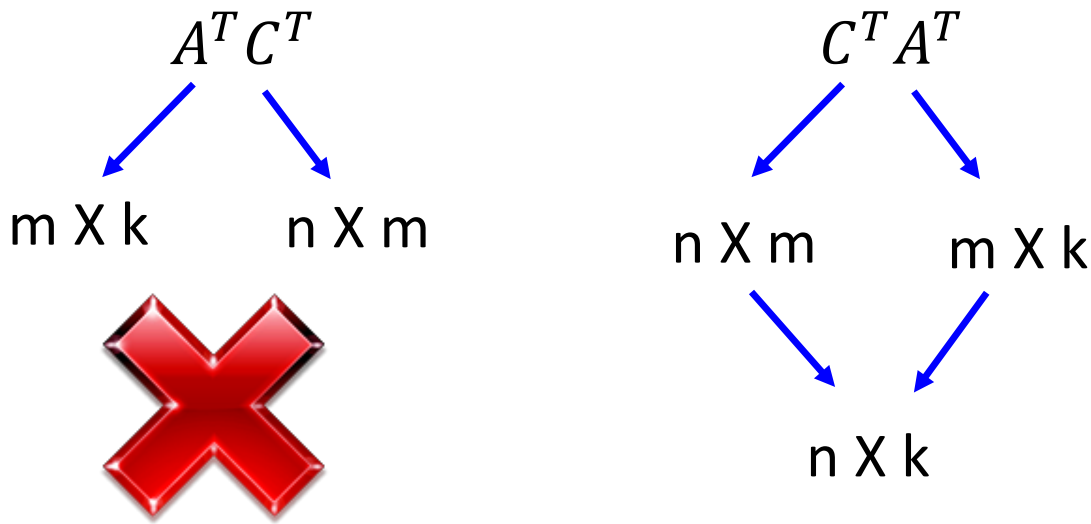
Properties

- Let A and B be k x m matrices, C be an m x n matrix, and P and Q be n x p matrices
  - For any scalar s, s(AC) = (sA)C = A(sC)
  - (A + B)C = AC + BC
  - C(P+Q)=CP+CQ
  - $I_k A = A = A I_m$
  - The product of any matrix and a zero matrix is a zero matrix
- Power of square matrices: $A \in \mathcal{M}_{n\times n}$, $A^k = A\,A \cdots A$ (k times), and by convention, $A^1 = A$, $A^0 = I_n$.

Properties

AC: k X n $\quad (AC)^T$: n X k

- Let A be kxm matrices, C be an mxn matrix,
  - $(AC)^T = ? \quad C^T A^T$



Special Matrix

对角上对应元素、相乘即可

- Diagonal Matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \underline{AB} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

- Symmetric Matrix $\quad A^T = A$

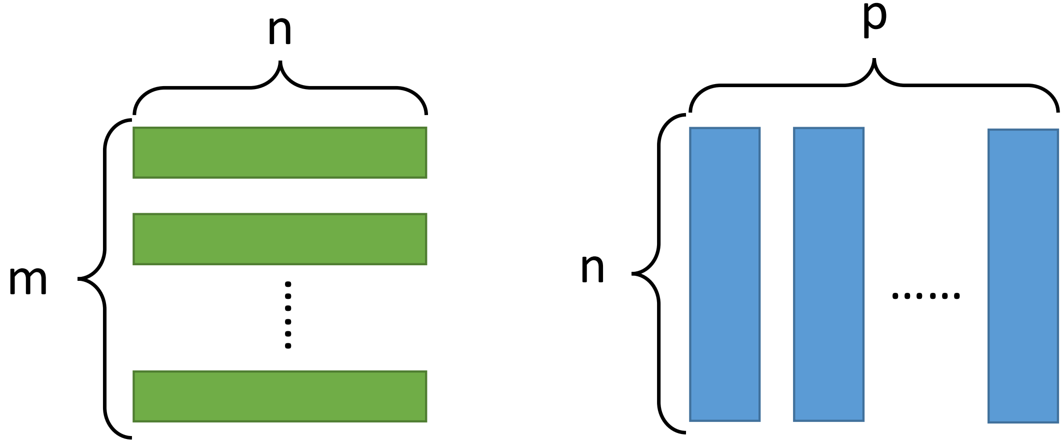
$$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & -1 \\ 4 & -1 & 5 \end{bmatrix} = A^T \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \neq B^T$$

AA^T and A^TA are square and symmetric

$$(AA^T)^T = A^{TT}A^T = AA^T \qquad (A^TA)^T = A^TA^{TT} = A^TA$$

Practical Issue

- Let A and B be k x m matrices, C be an m x n matrix, and P and Q be n x p matrices
  - A(CP) = (AC)P



Multiplication count: m X n X p

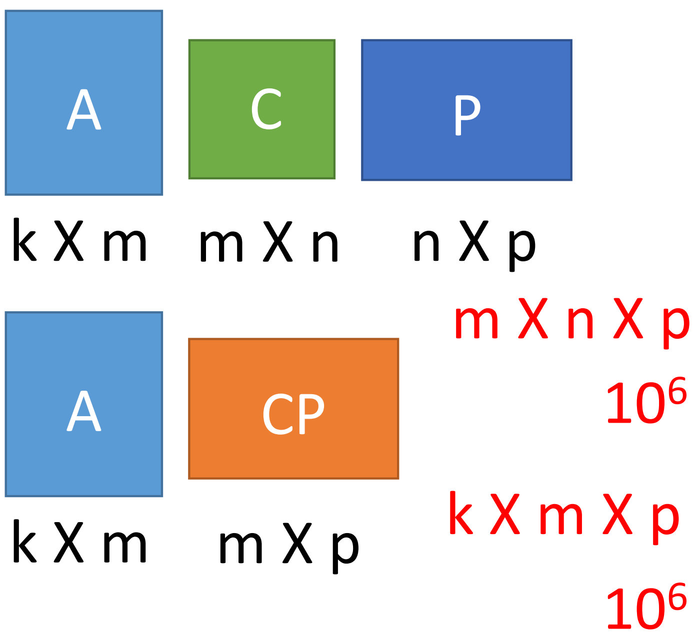
Practical Issue

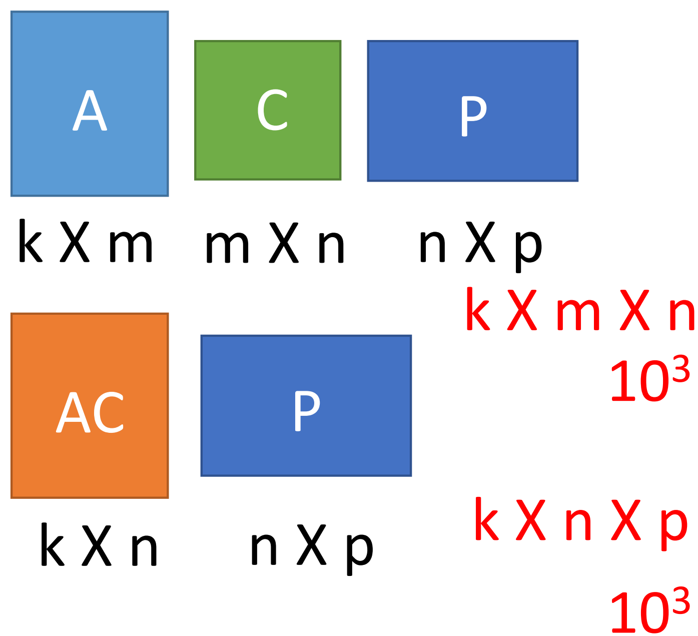
k=1 m=1000

n=1 p=1000

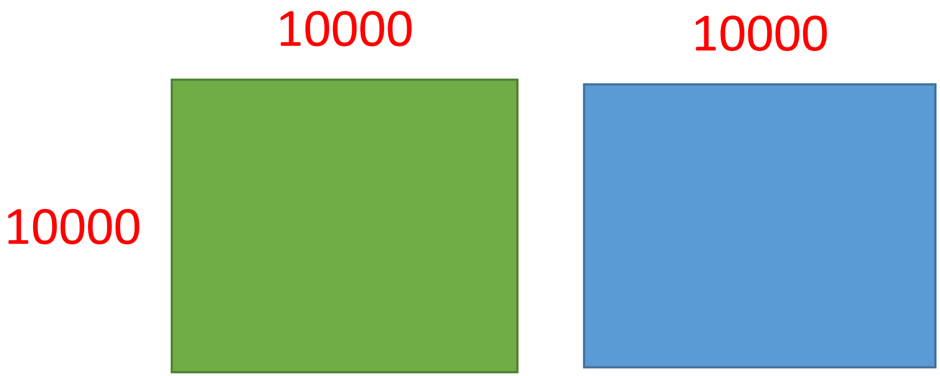
- Let A and B be k x m matrices, C be an m x n matrix, and P and Q be n x p matrices
  - A(CP) = (AC)P

经典动态规化





Practical Issue - GPU



Multiplying two 10000 X 10000 matrices

CPU
```
It cost 21.249996 sec
```

GPU
```
It cost 0.843893 sec
```

(GTX 980 Ti)

More than 20 times faster