Having Solution or Not

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Learning Target

Review

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

System of Linear Equations

Matrix-vector product: $A\mathbf{x} = \mathbf{b}$

- Given A and **b**, sometimes **x** exists (having solution), and sometimes doesn't (no solution)
- New terms: "*linear combination*" and "*span*"

Solution

- Given a system of linear equation

$$\begin{array}{rcl} 2x_1 - 3x_2 + x_3 & = & -10 \\ x_1 \qquad\qquad + x_3 & = & 3 \end{array} \quad \vdots$$

$\begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$ is a solution

$\begin{bmatrix} -4 \\ 3 \\ 7 \end{bmatrix}$ is also a solution

There are other possible solutions ……

The **set** of **all solutions** of a **system of linear equations** is called the **solution set**.

Solution

- A system of linear equations is called consistent if it has one or more solutions.
- A system of linear equations is called inconsistent if its solution set is empty.

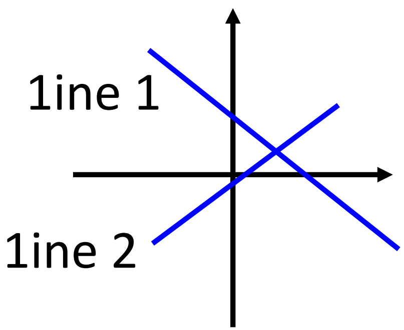
	$\begin{aligned} 3x_1 + x_2 &= 10 \\ x_1 - 3x_2 &= 0 \end{aligned}$	$\begin{aligned} 3x_1 + x_2 &= 10 \\ 6x_1 + 2x_2 &= 20 \end{aligned}$	$\begin{aligned} 3x_1 + x_2 &= 10 \\ 6x_1 + 2x_2 &= 0 \end{aligned}$
Solution set	$\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$	$\left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} + t \begin{bmatrix} -1 \\ 3 \end{bmatrix} : \forall t \in \mathcal{R} \right\}$	$\{\}$, or ϕ
Consistent or Inconsistent?	Consistent	Consistent	Inconsistent

Solution (High School)

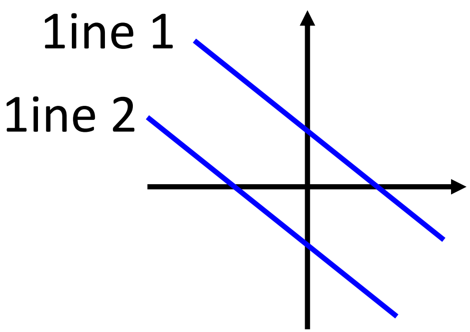
More Variables?

- Considering any system of linear equations with 2 variables and 2 equations

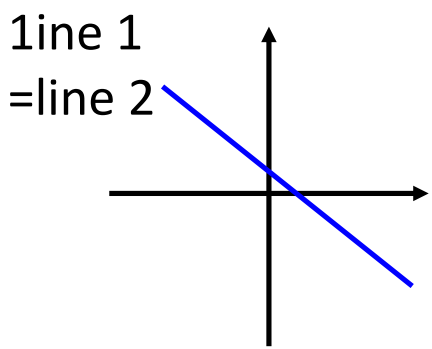
$$a_{11}x_1 + a_{12}x_2 = b_1 \quad \text{...... line 1}$$
$$a_{21}x_1 + a_{22}x_2 = b_2 \quad \text{...... line 2}$$



unique solution



no solution



infinitely many solution

Does a system of linear equations have solutions?

Linear Combination

Linear Combination 线性组合

- Given a vector set $\{u_1, u_2, \cdots, u_k\}$
- The linear combination of the vectors in the set:
  - $v = c_1 u_1 + c_2 u_2 + \cdots + c_k u_k$
  - $c_1, c_2, \cdots, c_k$ are scalars (Coefficients of linear combination)

vector set: $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

coefficients: $\{-3,4,1\}$

What is the result of linear combination?

Column Aspect

$$
\begin{array}{ccccccccc}
\boxed{a_{11}}x_1 & + & \boxed{a_{12}}x_2 & + & \cdots & + & \boxed{a_{1n}}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
 & & & & \vdots & & & & \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \\
a_1 & & a_2 & & & & a_n & &
\end{array}
$$

$$A = [a_1 \quad a_2 \quad \cdots \quad a_n]$$

Vector set

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

coefficients

$$\mathbf{Ax} = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

Linear Combination

System of Linear Equations v.s. Linear Combination

$$A\mathbf{x} = \mathbf{b}$$

(A system of linear equations)

Non empty solution set?

Has solution or not?

Consistent?

The Same question

Column Aspect

$$A\mathbf{x}$$

$$= x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = b$$

the linear combination of columns of A

Is b the linear combination of columns of A?

Example 1

$$3x_1 + 6x_2 = 3$$
$$2x_1 + 4x_2 = 4$$

Has solution or not?

$$\mathbf{Ax} = \mathbf{b}$$

$$A = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad b = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Is b the linear combination of columns of A?

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} \qquad \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \end{bmatrix} \right\}$$

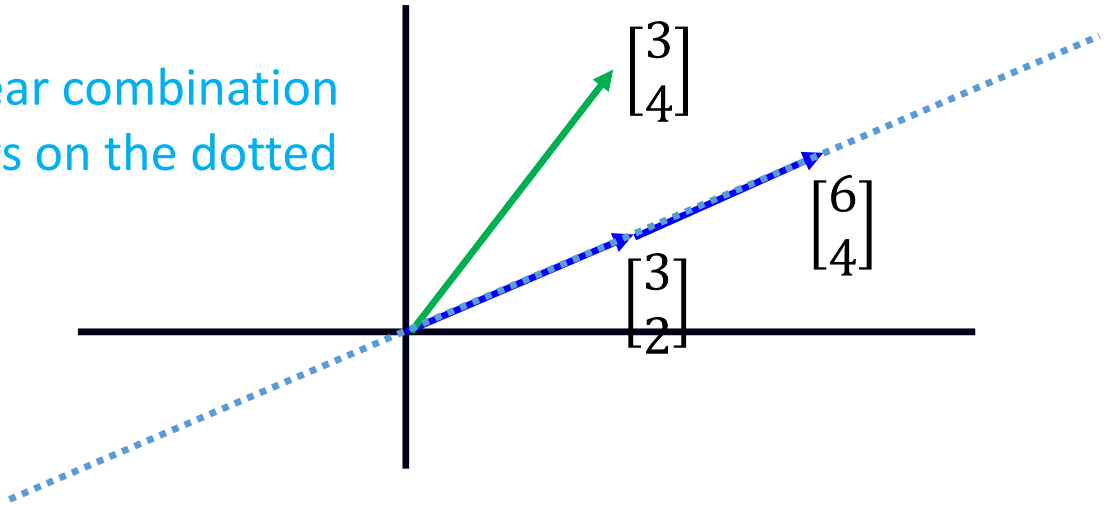
Example 1

$$\begin{aligned} 3x_1 + 6x_2 &= 3 \\ 2x_1 + 4x_2 &= 4 \end{aligned}$$

Has solution or not?

- Vector set: $\left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \end{bmatrix} \right\}$
- Is $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ a linear combination of $\left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \end{bmatrix} \right\}$? No

The linear combination is always on the dotted line.



Example 2

$$2x_1 + 3x_2 = 4$$
$$3x_1 + 1x_2 = -1$$

Has solution or not?

$$A\mathbf{x} = \mathbf{b}$$

$A = \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}$ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $b = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

Is b the linear combination of columns of A?

$\begin{bmatrix} 4 \\ -1 \end{bmatrix}$ $\left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$

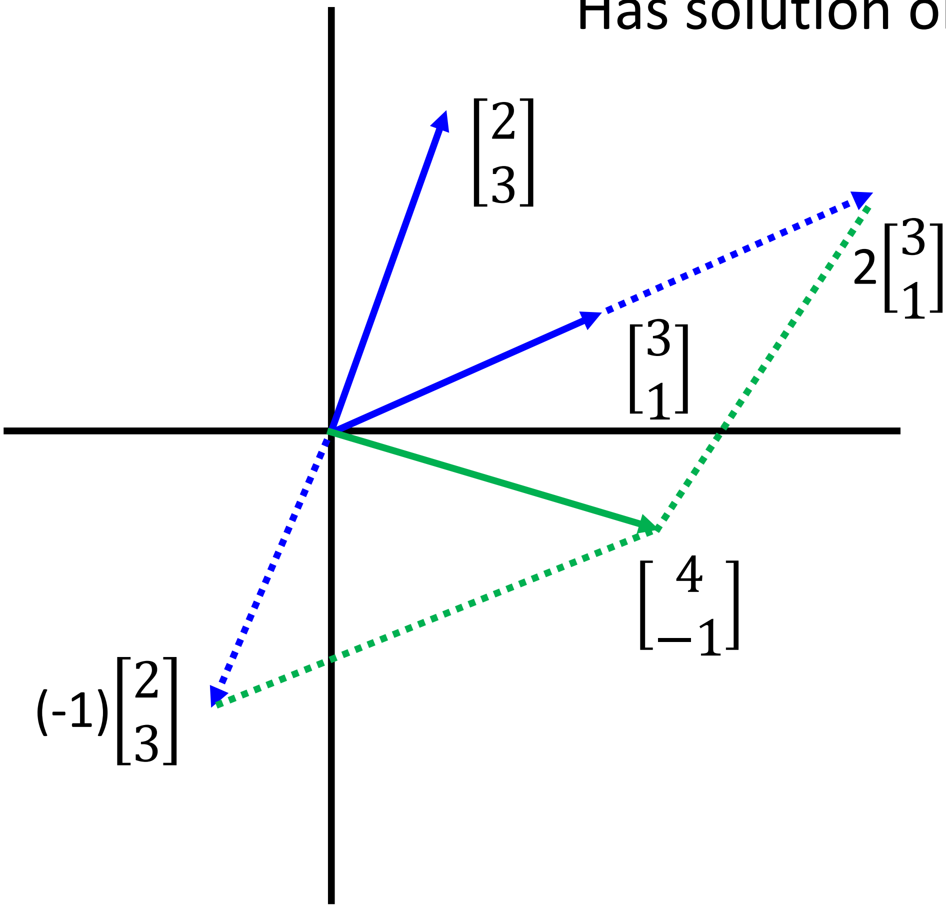
Example 2

$$\begin{aligned} 2x_1 + 3x_2 &= 4 \\ 3x_1 + 1x_2 &= -1 \end{aligned}$$

Has solution or not?

$\left\{ \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$

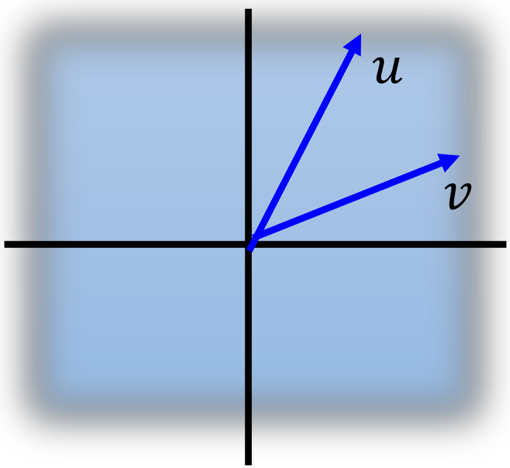
$\begin{bmatrix} 4 \\ -1 \end{bmatrix}$



Example 2

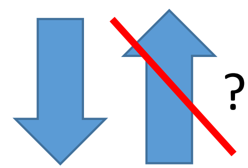
How about in $\mathcal{R}^4$?

- If **u** and **v** are any nonparallel vectors in $\mathcal{R}^2$, then every vector in $\mathcal{R}^2$ is a linear combination of **u** and **v**
  - Nonparallel: **u** and **v** are nonzero vectors, and **u** $\neq c$**v**.



$$\begin{aligned} u_1 x_1 \quad &+ v_1 x_2 \quad = b_1 \\ u_2 x_1 \quad &+ v_2 x_2 \quad = b_2 \end{aligned}$$

u and **v** are not parallel



Has solution

- If **u, v** and **w** are any nonparallel vectors in $\mathcal{R}^3$, then every vector in $\mathcal{R}^3$ is a linear combination of **u, v** and **w**? 不行

比如在xy平面中的
这个的平行 Vector
线性的相关才行

Example 3

$$2x_1 + 6x_2 = -4$$
$$1x_1 + 3x_2 = -2$$

Has solution or not?

$$A\mathbf{x} = \mathbf{b}$$

$$A = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad b = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$$

Is b the linear combination of columns of A?

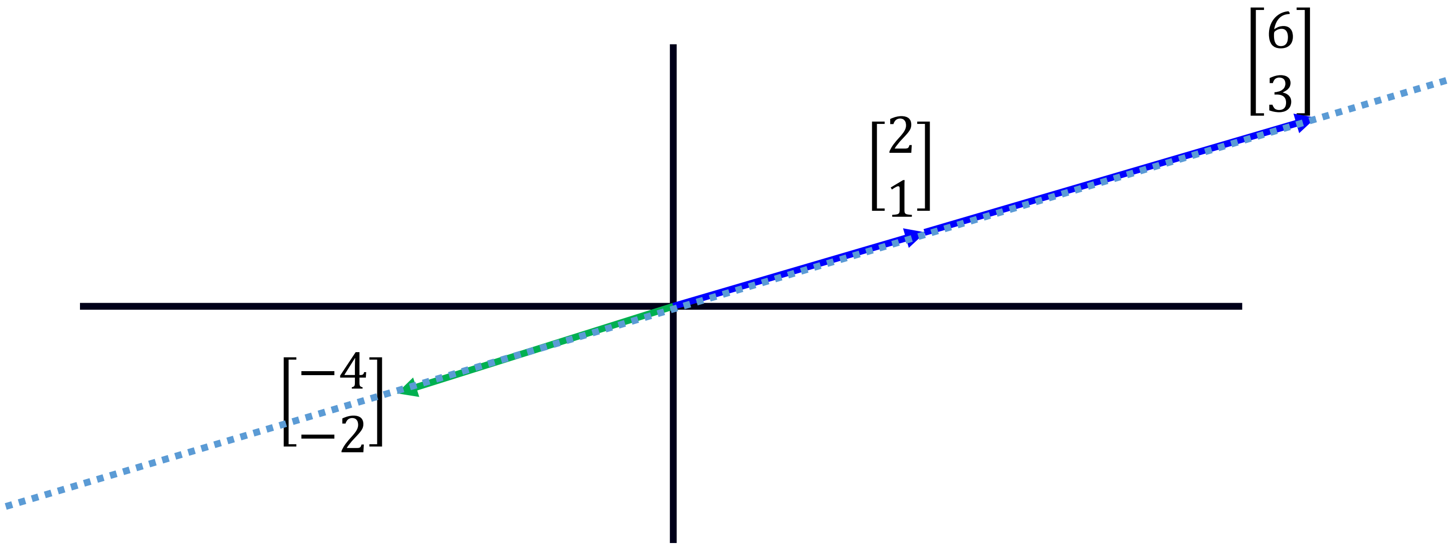
$$\begin{bmatrix} -4 \\ -2 \end{bmatrix} \qquad \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \end{bmatrix} \right\}$$

Example 3

$$2x_1 + 6x_2 = -4$$
$$1x_1 + 3x_2 = -2$$

Has solution or not?

- Vector set: $\left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}6\\3\end{bmatrix}\right\}$
- Is $\begin{bmatrix}-4\\-2\end{bmatrix}$ a linear combination of $\left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}6\\3\end{bmatrix}\right\}$? Yes



Does a system of linear equations have solutions?

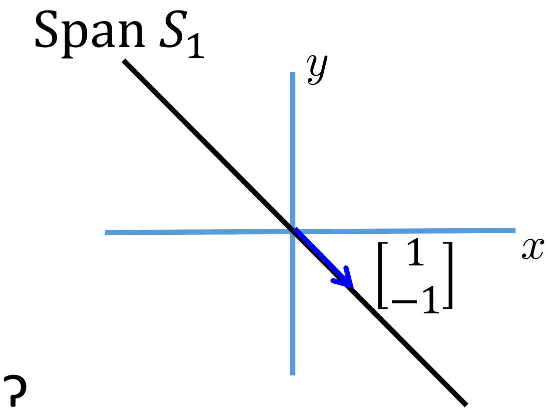
Span

Span

- A vector set $S = \{u_1, u_2, \cdots, u_k\}$
- Span of S is the vector set of all linear combinations of $u_1, u_2, \cdots, u_k$
  - Denoted by $Span\ \{u_1, u_2, \cdots, u_k\}$ or $Span\ S$
  - $Span\ S = \{c_1 u_1 + c_2 u_2 + \cdots + c_k u_k \mid for\ all\ c_1, c_2, \cdots, c_k\}$
- Vector set $V = Span\ S$
  - "S is a generating set for V" or "S generates V"
  - A vector set generated by another vector set is called ***Space***

向量空间

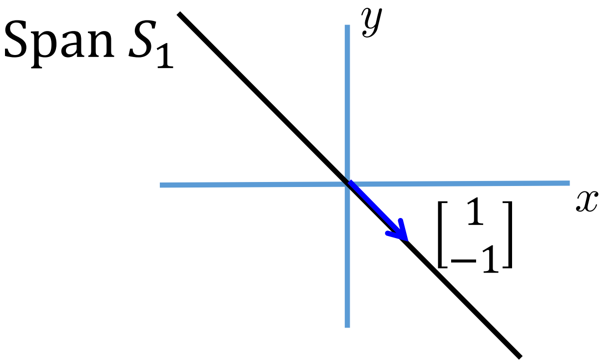
Span

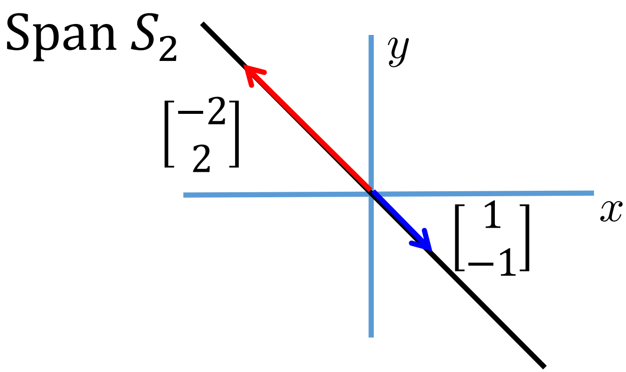


- Let $S_0 = \{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\}$, what is Span S_0?
  - Ans: $\{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\}$ (only one member)
- Let $S_1 = \{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\}$, what is Span S_1?
  - If S contains a non zero vector, then Span S has infinitely many vectors

Span

- Let $S_1 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$, what is Span S_1?
- Let $S_2 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right\}$, what is Span S_2?





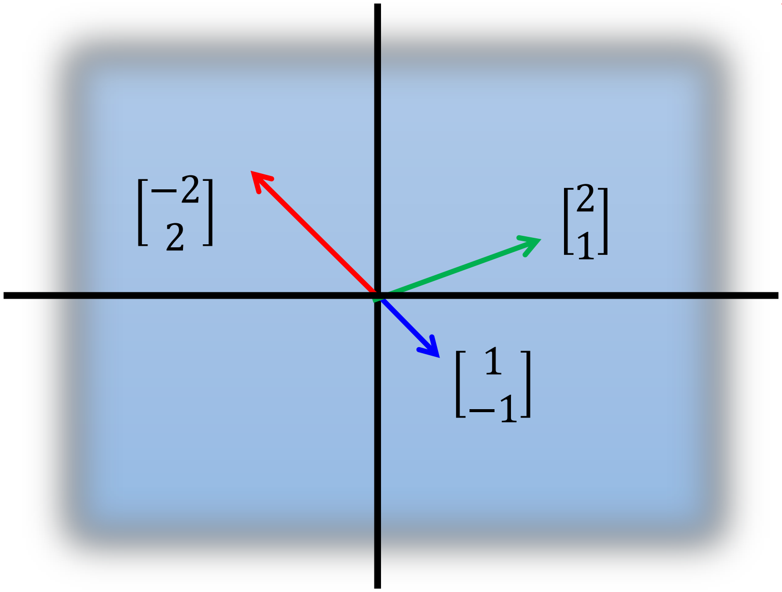
$$\text{Span } S_1 = \text{Span } S_2$$

(Different number of vectors can generate the same space.)

Span

- Let $S_3 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$, what is Span S_3?

nonparallel vectors

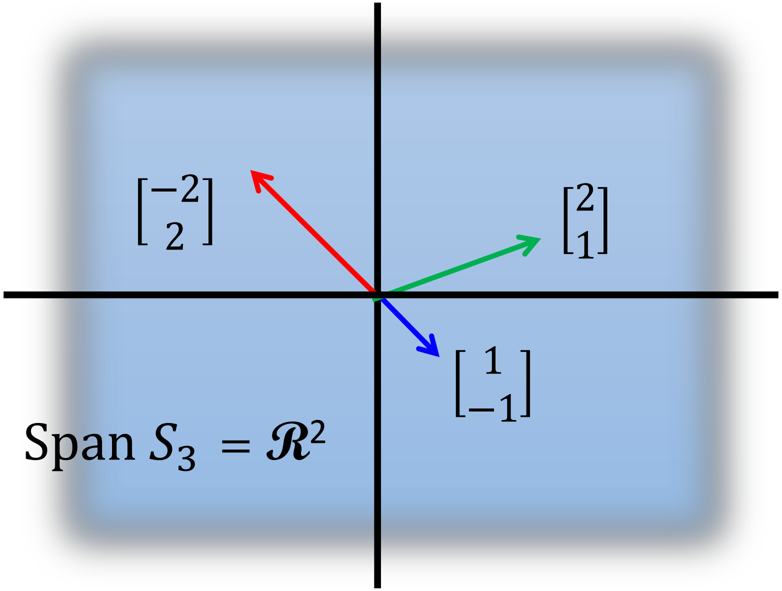


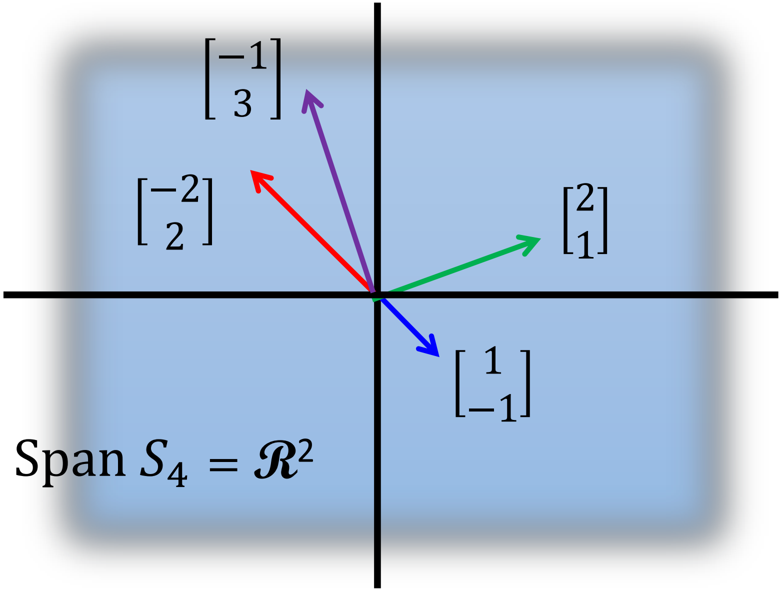
Every vector in $\mathcal{R}^2$ is their linear combination

$$\text{Span } S_3 = \mathcal{R}^2$$

Span

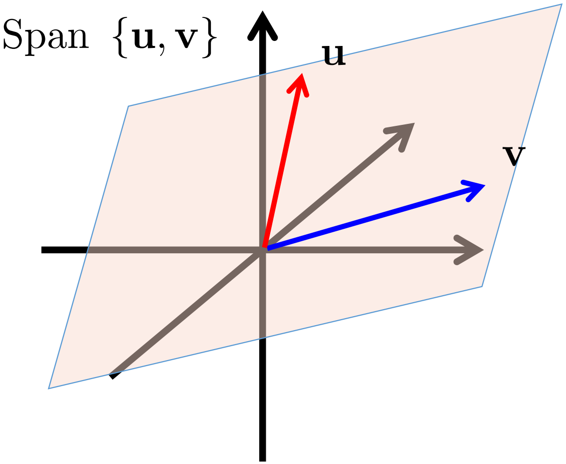
- Let $S_3 = \{\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}-2\\2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}\}$, what is Span S_3?
- Let $S_4 = \{\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}-2\\2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}-1\\3\end{bmatrix}\}$, what is Span S_4?



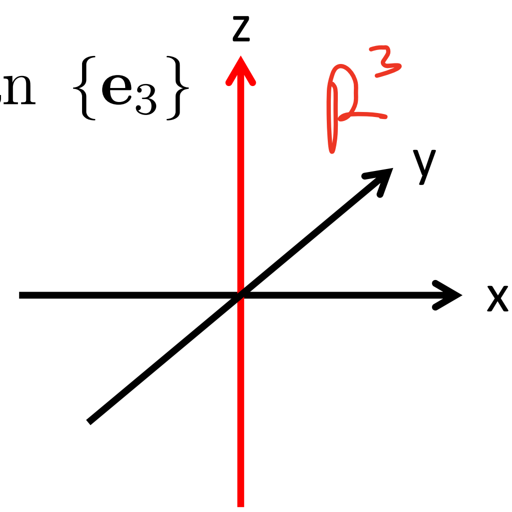


Span

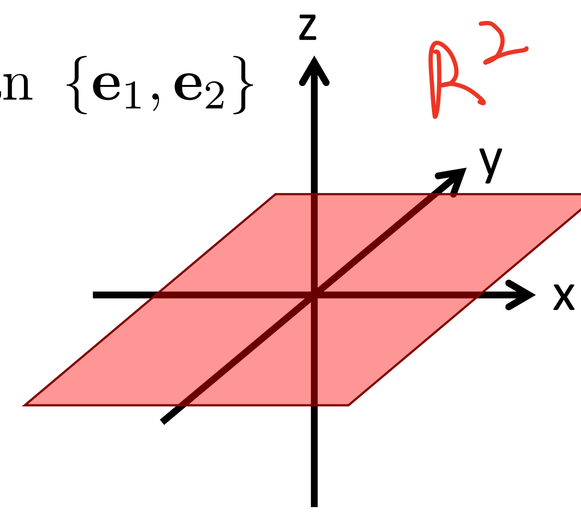
$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



Span $\{\mathbf{e}_3\}$



Span $\{\mathbf{e}_1, \mathbf{e}_2\}$



$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

$$A\mathbf{x} = \mathbf{b}$$

Has solution or not?

The same question

$$x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

Is b the linear combination of columns of A?

The same question

$$\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \in Span \left\{ \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} \cdots \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} \right\}$$

Is b in the span of the columns of A?

Summary

$$A\mathbf{x} = \mathbf{b}$$

Does a system of linear equations have solution?

Is b a linear combination of columns of A?

Is b in the span of the columns of A?

YES → Have solution

NO → No solution