Orthogonality

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Outline

- Reference: Chapter 7.1

Norm & Distance

模 $\|v\|_p = \left(\sum_{i=1}^{n} |v_i|^p\right)^{\frac{1}{p}}$

- **Norm**: Norm of vector v is the length of v
  - Denoted $\|v\|$

$$\|v\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

- **Distance**: The distance between two vectors u and v is defined by $\|v - u\|$

$$\|v\| = \sqrt{1^2 + 2^2 + 3^2}$$
$$= \sqrt{14}$$

$$v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad u = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix} \quad v - u = \begin{bmatrix} -1 \\ 5 \\ 3 \end{bmatrix}$$

$$\|v - u\| = \sqrt{(-1)^2 + 5^2 + 3^2}$$
$$= \sqrt{35}$$

Orthogonal

https://www.youtube.com/watch?v=43BfcSkctYA

https://www.youtube.com/watch?v=EktZVposDMU

Dot Product & Orthogonal

① 计算 $\cos\theta$

$u \cdot v = \|u\| \cdot \|v\| \cdot \cos\theta$

② 计算投影

$u_v = \|u\| \cdot \cos\theta \cdot \frac{v}{\|v\|} = \frac{u \cdot v}{\|v\|} \cdot \frac{v}{\|v\|} = \frac{u \cdot v}{\|v\|^2} \cdot v$

- **Dot product**: dot product of u and v is

inner product

$$u \cdot v = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

$$= \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \boxed{= u^T v}$$

- **Orthogonal**: u and v are orthogonal if $u \cdot v = 0$

Orthogonal is actually "perpendicular" 垂直

Zero vector is orthogonal to every vector

More about Dot Product

- Let u and v be vectors, A be a matrix, and c be a scalar
- $u \cdot u = \|u\|^2$ Connect norm and dot product
- $u \cdot u = 0$ if and only if $u = 0$
- $u \cdot v = v \cdot u$
- $u \cdot (v + w) = u \cdot v + u \cdot w$
- $(v + w) \cdot u = v \cdot u + w \cdot u$
- $(cu) \cdot v = c(u \cdot v) = u \cdot (cv)$
- $\|cu\| = |c|\|u\|$
- $Au \cdot v = (Au)^T v$

Example $||2\mathbf{u} + 3\mathbf{v}||^2 =$ …… $= 4||\mathbf{u}||^2 + 12(\mathbf{v} \cdot \mathbf{u}) + 9||\mathbf{v}||^2.$

Example $||2\mathbf{u}+3\mathbf{v}||^2 =$ ......

$= (2u+3v)\cdot(2u+3v)$

$= 4u\cdot u + 12u\cdot v + 9v\cdot v$

$= 4||u||^2 + 12u\cdot v + 9||v||^2$

$$= 4||\mathbf{u}||^2 + 12(\mathbf{v}\cdot\mathbf{u}) + 9||\mathbf{v}||^2.$$

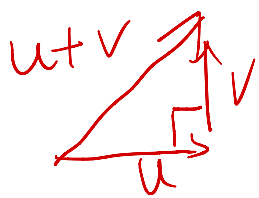
Pythagorean Theorem

毕达哥拉斯 hhh 也是勾股定理

$\mathbf{u}$ and $\mathbf{v}$ are orthogonal $\Longleftrightarrow$ $||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$

Proof: $||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + \underline{2\mathbf{u}\cdot\mathbf{v}} + ||\mathbf{v}||^2$

=0 if and only if u and v are orthogonal



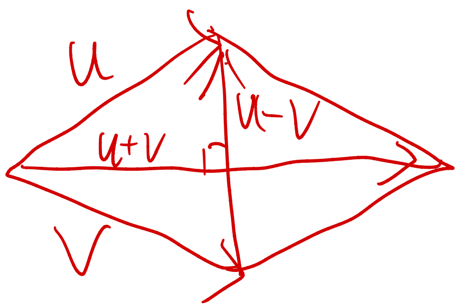
Pythagorean Theorem

The diagonals of a parallelogram are orthogonal. ⟺ The parallelogram is a rhombus. (菱形)

Proof: $(u+v)\cdot(u-v)=0$

$$= \|u\|^2 - \|v\|^2$$

$\|\mathbf{u}\| = \|\mathbf{v}\|$



Triangle Inequality 三角不等式

- For any vectors u and v,

$$||\mathbf{u}+\mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$$

Proof:

$$\|u+v\|^2 = \|u\|^2 + 2u \cdot v + \|v\|^2$$

$$\leq \|u\|^2 + 2|u \cdot v| + \|v\|^2$$

Cauchy-Schwarz Inequality $\leq \|u\|^2 + 2\|u\| \cdot \|v\| + \|v\|^2$

$$\leq (\|u\| + \|v\|)^2$$

证明柯西不等式

令 $\|u\|^2 = \|v\|^2 = 1$ $\because |u_i v_i| \le \frac{1}{2}(u_i^2 + v_i^2)$

$\therefore \left|\sum_{i=1}^{n} u_i v_i\right| \le \frac{1}{2}\sum_{i=1}^{n} u_i^2 + \frac{1}{2}\sum_{i=1}^{n} v_i^2 = 1$

对于任意 u, v

令 $\hat{u} = \frac{u}{\|u\|}$, $\hat{v} = \frac{v}{\|v\|}$

那么 $\left|\sum_{i=1}^{n} \hat{u}_i \hat{v}_i\right| = \left|\sum_{i=1}^{n} \frac{u_i}{\|u\|} \cdot \frac{v_i}{\|v\|}\right| \le 1$

$\Rightarrow \left|\sum_{i=1}^{n} u_i v_i\right| \le \|u\| \cdot \|v\|$

$\Rightarrow |u \cdot v| \le \|u\| \cdot \|v\|$