

Matrix-Vector Product

李宏毅

Hung-yi Lee



$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$



Matrix-vector product: $A\mathbf{x} = \mathbf{b}$

Row Aspect

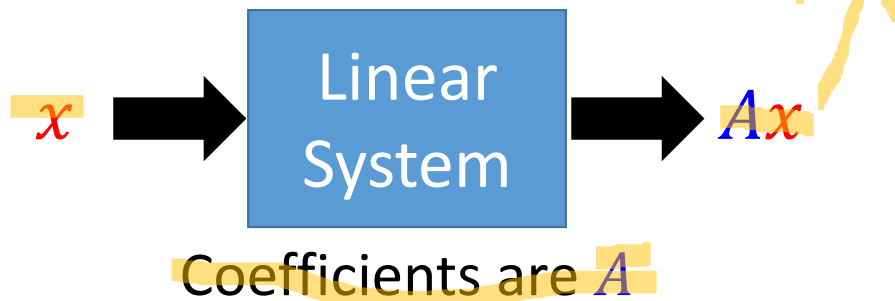
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$
$$Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad Ax = \begin{bmatrix} \\ \end{bmatrix}$$

Matrix-Vector Product

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{array} = \begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_m \end{array}$$

$Ax = b$



Column Aspect

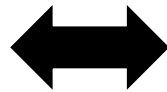
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$Ax =$$

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

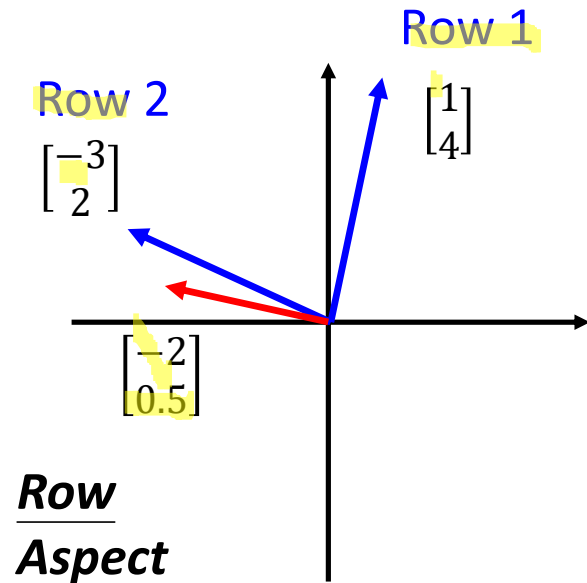
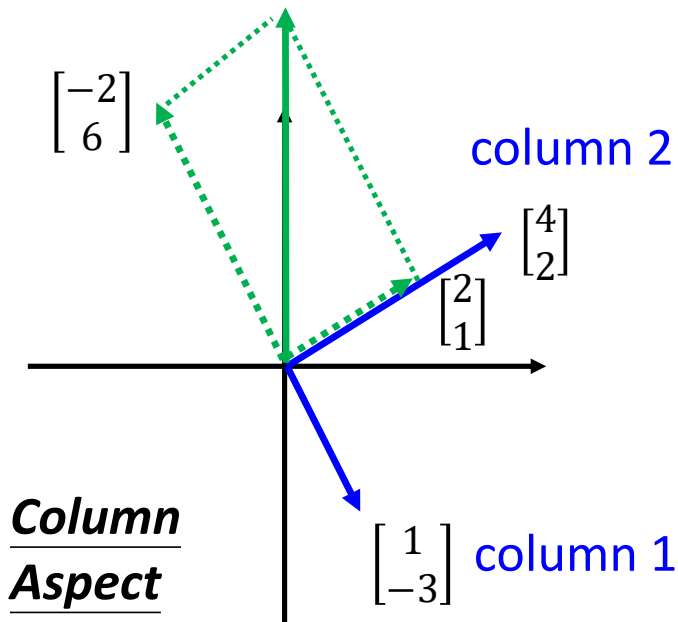
Example

$$\begin{aligned} x_1 + 4x_2 &= b_1 \\ -3x_1 + 2x_2 &= b_2 \end{aligned}$$



$$\begin{bmatrix} -2 & x_1 \\ 0.5 & x_2 \end{bmatrix} \rightarrow \boxed{A} \rightarrow \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = Ax$$

$$A = \begin{bmatrix} 1 & 4 \\ -3 & 2 \end{bmatrix}$$



Matrix-vector Product

- The size of matrix and vector should be matched.

Diagram illustrating matrix-vector multiplication with dimension mismatches and corrections:

Matrix $A = \begin{bmatrix} 2 & 3 & 5 \\ 3 & 1 & -1 \\ -2 & 1 & 1 \end{bmatrix}$ (3x3) and vector $x = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ (3x1) are shown. A large red 'X' is drawn over the product Ax , indicating it is invalid due to dimension mismatch.

Below, two corrected matrices are shown:

Matrix $A' = \begin{bmatrix} 1 & -1 \\ 2 & 3 \\ 1 & 4 \end{bmatrix}$ (3x2) and Matrix $A'' = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 0 & -1 \\ 1 & -3 \end{bmatrix}$ (4x2).

Red arrows indicate the correction: one arrow points from the second column of A to the first column of A' , and another arrow points from the second column of A to the second column of A'' . The second column of A is $\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, which matches the dimensions of both A' and A'' .

Properties of Matrix-vector Product

- A and B are $m \times n$ matrices, \mathbf{u} and \mathbf{v} are vectors in \mathcal{R}^n , and c is a scalar.
- $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$
- $A(c\mathbf{u}) = c(A\mathbf{u}) = (cA)\mathbf{u}$
- $(A + B)\mathbf{u} = A\mathbf{u} + B\mathbf{u}$
- $A\mathbf{0}^k$ is the $m \times 1$ zero vector
- $\mathbf{0}\mathbf{v}$ is also the $m \times 1$ zero vector
- $I_n \mathbf{v} = \mathbf{v}$



Properties of Matrix-vector Product

$$j \leftarrow \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

- A and B are $m \times n$ matrices. If $A\mathbf{w} = B\mathbf{w}$ for all \mathbf{w} in \mathcal{R}^n . Is it true that $A = B$?

$A\mathbf{e}_j = \mathbf{a}_j$ for $j = 1, 2, \dots, n$, where \mathbf{e}_j is the j -th standard vector in \mathcal{R}^n

\mathbf{e}_1 = $\begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix}$ $A\mathbf{e}_1 = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix} = 1 \cdot a_1 + 0 \cdot a_2 + \cdots + 0 \cdot a_n = a_1$

Column Aspect

$$A\mathbf{e}_1 = B\mathbf{e}_1 \quad A\mathbf{e}_2 = B\mathbf{e}_2$$



$$a_1 = b_1$$



$$a_2 = b_2$$

.....

$$A\mathbf{e}_n = B\mathbf{e}_n$$



$$a_n = b_n$$

只需要
 $\Rightarrow A = B$
 便

Concluding Remarks

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

Row
Aspect

$$= x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Column
Aspect

Concluding Remarks

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

Row Aspect

$$= x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Column Aspect

Concluding Remarks

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

Row Aspect

$$= x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Column Aspect