Diagonalization

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Review

- If $Av = \lambda v$ (v is a vector, λ is a scalar)
  - v is an eigenvector of A excluding zero vector
  - λ is an eigenvalue of A that corresponds to v
- Eigenvectors corresponding to λ are nonzero solution of $(A - \lambda I_n)\mathbf{v} = \mathbf{0}$

Eigenvectors corresponding to λ

$$= \underbrace{Null(A - \lambda I_n)}_{\text{eigenspace}} - \{\mathbf{0}\}$$

Eigenspace of λ:

Eigenvectors corresponding to λ + $\{\mathbf{0}\}$

- A scalar t is an eigenvalue of A

$$\Leftrightarrow \boxed{det(A - tI_n) = 0}$$

Review

- Characteristic polynomial of A is

$det(A - tI_n)$ Factorization

multiplicity

$$= (t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2}\ldots(t-\lambda_k)^{m_k}\underline{(\ldots\ldots)}$$

Eigenvalue:	λ_1	λ_2	λ_k
Eigenspace:	d_1	d_2	d_k
(dimension)	$\leq m_1$	$\leq m_2$	$\leq m_k$

Outline

- An nxn matrix A is called diagonalizable if $A = PDP^{-1}$
  - D: nxn diagonal matrix
  - P: nxn invertible matrix
- Is a matrix A diagonalizable?
  - If yes, find D and P
- Reference: Textbook 5.3

Diagonalizable

- An nxn matrix A is called diagonalizable if $A = PDP^{-1}$
  - D: nxn diagonal matrix
  - P: nxn invertible matrix
- Not all matrices are diagonalizable

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $A^2 = 0$ (?)

If $A = PDP^{-1}$ for some invertible P and diagonal D

$\Rightarrow$ $A^2 = PD^2P^{-1} = 0$ $\Rightarrow$ $D^2 = 0$ $\Rightarrow$ $D = 0$

D is diagonal

$\Rightarrow$ $A = 0$

Diagonalizable

$$P = [p_1 \quad \cdots \quad p_n]$$

$$D = \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}$$

- If A is diagonalizable

$$A = PDP^{-1} \Rightarrow AP = PD$$

$$\Rightarrow AP = [\underline{Ap_1} \quad \cdots \quad \underline{Ap_n}]$$

$$\Rightarrow PD = P[d_1e_1 \quad \cdots \quad d_ne_n]$$

$$= [Pd_1e_1 \quad \cdots \quad Pd_ne_n]$$

$$= [d_1Pe_1 \quad \cdots \quad d_nPe_n]$$

$$= [\underline{d_1p_1} \quad \cdots \quad \underline{d_np_n}] \Rightarrow Ap_i = d_ip_i$$

p_i is an eigenvector of A corresponding to eigenvalue d_i

Diagonalizable

$$P = \begin{bmatrix} p_1 & \cdots & p_n \end{bmatrix}$$ 线性

$$D = \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}$$ 无关

- If A is diagonalizable

$$A = PDP^{-1}$$

p_i is an eigenvector of A corresponding to eigenvalue d_i

$\Vert$

There are n eigenvectors that form an invertible matrix

$\Vert$

There are n independent eigenvectors

$\Vert$

The eigenvectors of A can form a basis for R^n.

Diagonalizable

$$P = [p_1 \quad \cdots \quad p_n]$$

$$D = \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}$$

- If A is diagonalizable

$$A = PDP^{-1}$$

p_i is an eigenvector of A corresponding to eigenvalue d_i

How to diagonalize a matrix A?

Step 1: Find *n* L.I. eigenvectors corresponding if possible, and form an invertible *P*

Step 2: The eigenvalues corresponding to the eigenvectors in P form the diagonal matrix *D.*

Diagonalizable

A set of eigenvectors that correspond to distinct eigenvalues is linear independent.

$det(A - tI_n)$ Factorization

$$= (t-\lambda_1)^{\underline{m_1}}(t-\lambda_2)^{\underline{m_2}} \ldots (t-\lambda_k)^{\underline{m_k}}(\ldots\ldots)$$

Eigenvalue: λ_1 λ_2 …… λ_k

Eigenspace: $d_1 \leq m_1$ $d_2 \leq m_2$ …… $d_k \leq m_k$

(dimension)

Independent

Diagonalizable

A set of eigenvectors that correspond to distinct eigenvalues is linear independent.

Eigenvalue: λ_1 λ_2 ...... λ_m

Assume dependent

Eigenvector: v_1 v_2 ...... v_m

➡ a contradiction

$$\mathbf{v}_k = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_{k-1}\mathbf{v}_{k-1}$$

$$A\mathbf{v}_k = c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 + \cdots + c_{k-1}A\mathbf{v}_{k-1}$$

$$\lambda_k\mathbf{v}_k = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + \cdots + c_{k-1}\lambda_{k-1}\mathbf{v}_{k-1}$$

(λ_k)

$$- \quad \lambda_k\mathbf{v}_k = c_1\lambda_k\mathbf{v}_1 + c_2\lambda_k\mathbf{v}_2 + \cdots + c_{k-1}\lambda_k\mathbf{v}_{k-1}$$

$$\mathbf{0} = c_1(\lambda_1 - \lambda_k)\,\mathbf{v}_1 + c_2(\lambda_2 - \lambda_k)\mathbf{v}_2 + \cdots + c_{k-1}(\lambda_{k-1} - \lambda_k)\mathbf{v}_{k-1}$$

Not $c_1 = c_2 = \cdots = c_{k-1} = 0$ ➡ *Same eigenvalue* ➡ a contradiction

证明

给定一个 n 维矩阵 A，其具有 n 个不等的特征值，分别为 $\lambda_1, ..., \lambda_n$，而 $x_1, ..., x_2$ 为分别对应 n 个不等特征值的特征向量。我们需要证明这些特征向量线性无关。

先假设这些特征向量线性相关，则存在 n 个不全为零的常数（c_i）使得如下式子成立：

$$c_1x_1 + c_2x_2 + ... + c_nx_n = 0 \tag{1}$$

用矩阵 A 左乘式 (1)，根据 $Ax_i = \lambda_i x_i$ 得：

$$c_1\lambda_1x_1 + c_2\lambda_2x_2 + ... + c_n\lambda_nx_n = 0 \tag{2}$$

再用式 (2) 减去 $\lambda_n * (1)$，得：

$$c_1(\lambda_1 - \lambda_n)x_1 + c_2(\lambda_2 - \lambda_n)x_2 + ... + c_{n-1}(\lambda_{n-1} - \lambda_n)x_{n-1} = 0 \tag{3}$$

接下来，可将 x_i 前面的系数 $c_i(\lambda_i - \lambda_n)$ 用常数 d_i 代替，则式 (3) 可写成：

$$d_1x_1 + d_2x_2 + ... + d_{n-1}x_{n-1} = 0 \tag{4}$$

式 (4) 是不是与式 (1) 形式一样？只是少了一个 x_n。那么对式 (4) 也进行类似式 (1) 的处理，可得：

$$d_1(\lambda_1 - \lambda_{n-1})x_1 + d_2(\lambda_2 - \lambda_{n-1})x_2 + ... + d_{n-2}(\lambda_{n-2} - \lambda_{n-1})x_{n-2} = 0 \tag{5}$$

若是按照前面的步骤（式 (1) 至式 (3)）重复进行 $n-2$ 次（每次都用一个不同的单个字符代替 x_i 前面的系数）后，可得：

$$m_1(\lambda_1 - \lambda_3)x_1 + m_2(\lambda_2 - \lambda_3)x_2 = 0 \tag{6}$$

用 n_i 代替式 (6) 中 x_i 的系数，即令 $n_1 = m_1(\lambda_1 - \lambda_3)$，$n_2 = m_2(\lambda_2 - \lambda_3)$。

再按照前面的步骤（式 (1) 至式 (3)）进行一次处理，可得 $n_1(\lambda_1 - \lambda_2)x_1 = 0$（$n_1$ 为常数），**由于特征向量不为零且各特征值都不相等**，所以只能是 $n_1 = 0$，又因为 $n_1 = m_1(\lambda_1 - \lambda_3)$，所以 $m_1 = 0$，带入到式 (6) 中可得 $m_2 = 0$，如此往后迭代最终可得：

$$c_i = 0 \quad \text{for i } = 1, 2, ..., n$$

则说明前面的假设（n 个特征向量 $\lambda_1, ..., \lambda_n$ 是线性相关）是错误的，故 **矩阵不同特征值对应的特征向量线性无关** 得证。

Diagonalizable

$$P = [p_1 \quad \cdots \quad p_n]$$

$$D = \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}$$

- If A is diagonalizable

$$A = PDP^{-1}$$

p_i is an eigenvector of A corresponding to eigenvalue d_i

$$det(A - tI_n)$$
$$= (t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2} \ldots (t-\lambda_k)^{m_k}(\ldots\ldots)$$

Eigenvalue: λ_1 λ_2 …… λ_k

Eigenspace: ……

$\{$ Basis for λ_1 Basis for λ_2 Basis for λ_3 $\}$

Independent Eigenvectors

You can't find more!

Diagonalizable - Example

- Diagonalize a given matrix $A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$

characteristic polynomial is $-(t+1)^2(t-3)$ ⇒ eigenvalues: 3, –1

eigenvalue 3

$$\mathbb{B}_1 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

eigenvalue –1

$$\mathbb{B}_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

$A = PDP^{-1}$, where

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

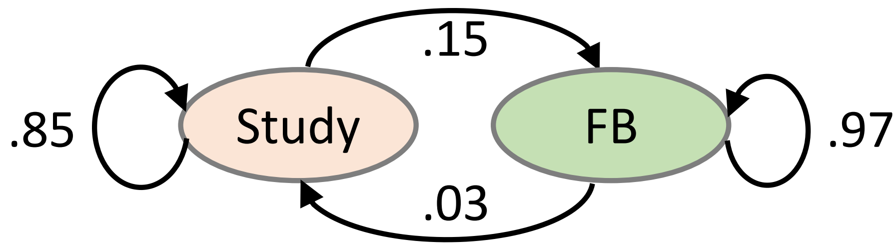
$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

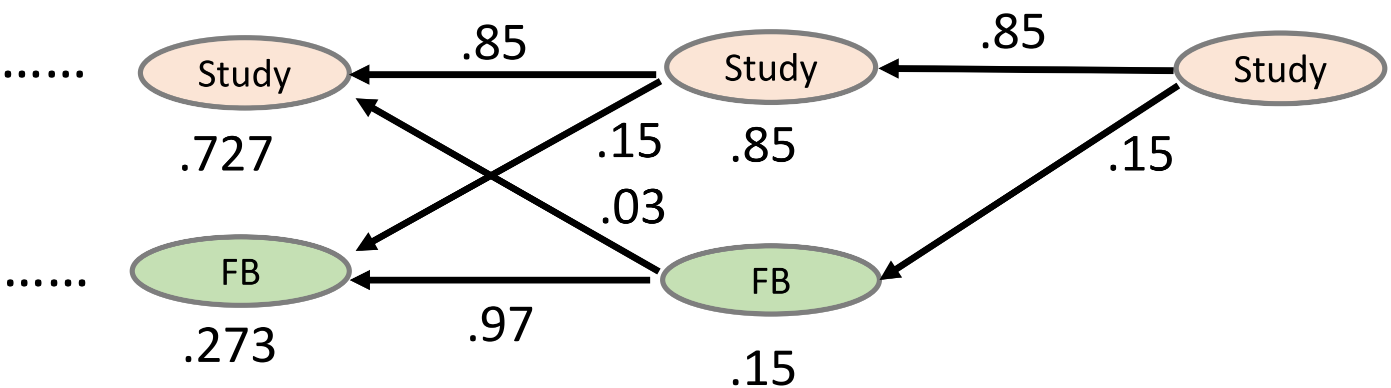
Application of Diagonalization

- If A is diagonalizable,

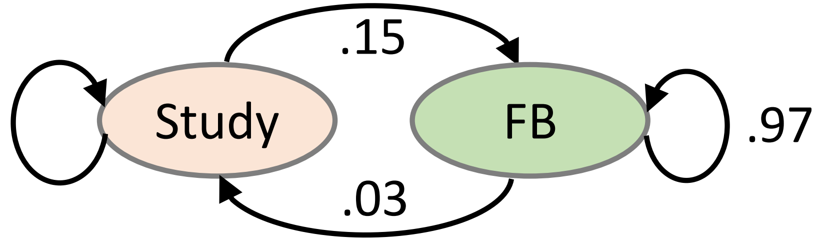
$$A = PDP^{-1} \Longrightarrow A^m = PD^mP^{-1}$$

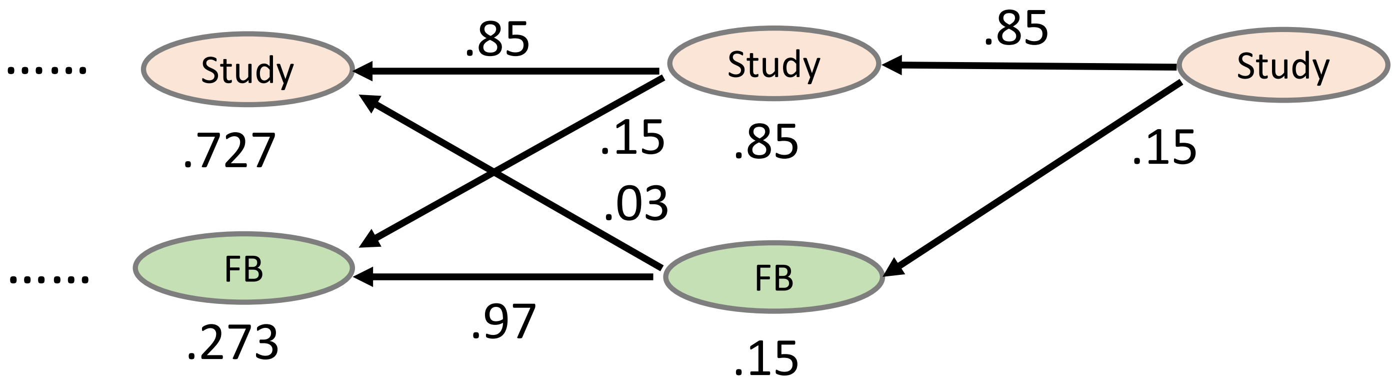
- Example:





$$\begin{array}{ccc} & & \text{From} & \\ & & \text{Study} & \text{FB} \\ \text{To} & \text{Study} & \multicolumn{2}{c}{\multirow{2}{*}{$\begin{bmatrix} .85 & .03 \\ .15 & .97 \end{bmatrix}$}} \\ & \text{FB} & & \end{array} = A$$





$$\begin{bmatrix} .727 \\ .273 \end{bmatrix} \quad \begin{bmatrix} .85 & .03 \\ .15 & .97 \end{bmatrix} \quad \begin{bmatrix} .85 \\ .15 \end{bmatrix} \quad \begin{bmatrix} .85 & .03 \\ .15 & .97 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A = PDP^{-1} \Longrightarrow A^m = PD^mP^{-1}$$

Diagonalizable

- Diagonalize a given matrix $A = \begin{bmatrix} .85 & .03 \\ .15 & .97 \end{bmatrix}$

$$\det\ (A - tI_2)$$

$$A - .82I_2 \xrightarrow[\text{RREF}]{} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathbf{p}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$A - I_2 \xrightarrow[\text{RREF}]{} \begin{bmatrix} 1 & -.2 \\ 0 & 0 \end{bmatrix} \Rightarrow \mathbf{p}_2 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

$$P = \begin{bmatrix} -1 & 1 \\ 1 & 5 \end{bmatrix}$$

(invertible)

$$D = \begin{bmatrix} .82 & 0 \\ 0 & 1 \end{bmatrix}$$

Application of Diagonalization

$$A = PDP^{-1} \text{ where } P = \begin{bmatrix} -1 & 1 \\ 1 & 5 \end{bmatrix}, D = \begin{bmatrix} .82 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} A^m &= PD^mP^{-1} \\ &= \begin{bmatrix} -1 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} (.82)^m & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 5 \end{bmatrix}^{-1} \\ &= \frac{1}{6} \begin{bmatrix} 1+5(.82)^m & 1-(.82)^m \\ 5-5(.82)^m & 5+(.82)^m \end{bmatrix} \end{aligned}$$

When $m \to \infty$,

$$A^m = \begin{bmatrix} 1/6 & 1/6 \\ 5/6 & 5/6 \end{bmatrix}$$

The beginning condition does not influence.

Test for a Diagonalizable Matrix

- An n x n matrix A is diagonalizable if and only if both the following conditions are met.

The characteristic polynomial of A factors into a product of linear factors.

$$det(A - tI_n) \quad \boxed{\text{Factorization}}$$

$$= (t - \lambda_1)^{m_1}(t - \lambda_2)^{m_2} \dots (t - \lambda_k)^{m_k}\cancel{(\dots\dots)}$$

For each eigenvalue λ of A, the multiplicity of λ equals the dimension of the corresponding eigenspace.

Independent Eigenvectors

An n x n matrix A is diagonalizable

=

The eigenvectors of A can form a basis for R^n.

=

$$det(A - tI_n)$$

$$= (t-\lambda_1)^{\underline{m_1}}(t-\lambda_2)^{\underline{m_2}} \ldots (t-\lambda_k)^{\underline{m_k}}\cancel{(\ldots\ldots)}$$

Eigenvalue:	λ_1	λ_2	……	λ_k
Eigenspace: (dimension)	$d_1 = m_1$	$d_2 = m_2$	……	$d_k = m_k$

怎么证明?

This lecture

- Reference: Chapter 5.4

$$
\begin{tikzcd}
{[v]_{\mathcal{B}}} \arrow[r, "{[T]_{\mathcal{B}}}", "\text{simple}"'] & {[T(v)]_{\mathcal{B}}} \arrow[d, "B", "\text{Properly selected}"'] \\
v \arrow[u, "B^{-1}", "\text{Properly selected}"'] \arrow[r, "T"] & T(v)
\end{tikzcd}
$$

Review

$$P = [p_1 \quad \cdots \quad p_n]$$

eigenvector

$$D = \begin{bmatrix} d_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_n \end{bmatrix}$$

eigenvalue

An n x n matrix A is diagonalizable ($A = PDP^{-1}$)

=

The eigenvectors of A can form a basis for R^n.

$$det(A - tI_n)$$
$$= (t-\lambda_1)^{m_1}(t-\lambda_2)^{m_2} \ldots (t-\lambda_k)^{m_k}(\ldots\ldots)$$

Eigenvalue: λ_1 λ_2 …… λ_k

Eigenspace: ……

{ Basis for λ_1 Basis for λ_2 Basis for λ_3 }

Independent Eigenvectors

Diagonalization of Linear Operator

- Example 1: $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 8x_1 + 9x_2 \\ -6x_1 - 7x_2 \\ 3x_1 + 3x_2 - x_3 \end{bmatrix}$

det

The standard matrix is $A = \begin{bmatrix} 8 \;\boxed{-t} & 9 & 0 \\ -6 & -7 \;\boxed{-t} & 0 \\ 3 & 3 & -1 \;\boxed{-t} \end{bmatrix}$

$\Rightarrow$ the characteristic polynomial is $-(t + 1)^2(t - 2)$

Eigenvalue -1:

$\mathcal{B}_1 = \left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$

Eigenvalue 2:

$\mathcal{B}_2 = \left\{\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}\right\}$

$\Rightarrow \mathcal{B}_1 \cup \mathcal{B}_2$ is a basis of $\mathcal{R}^3$

Diagonalization of Linear Operator

- Example 2: $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -x_1 + x_2 + 2x_3 \\ x_1 - x_2 \\ 0 \end{bmatrix}$

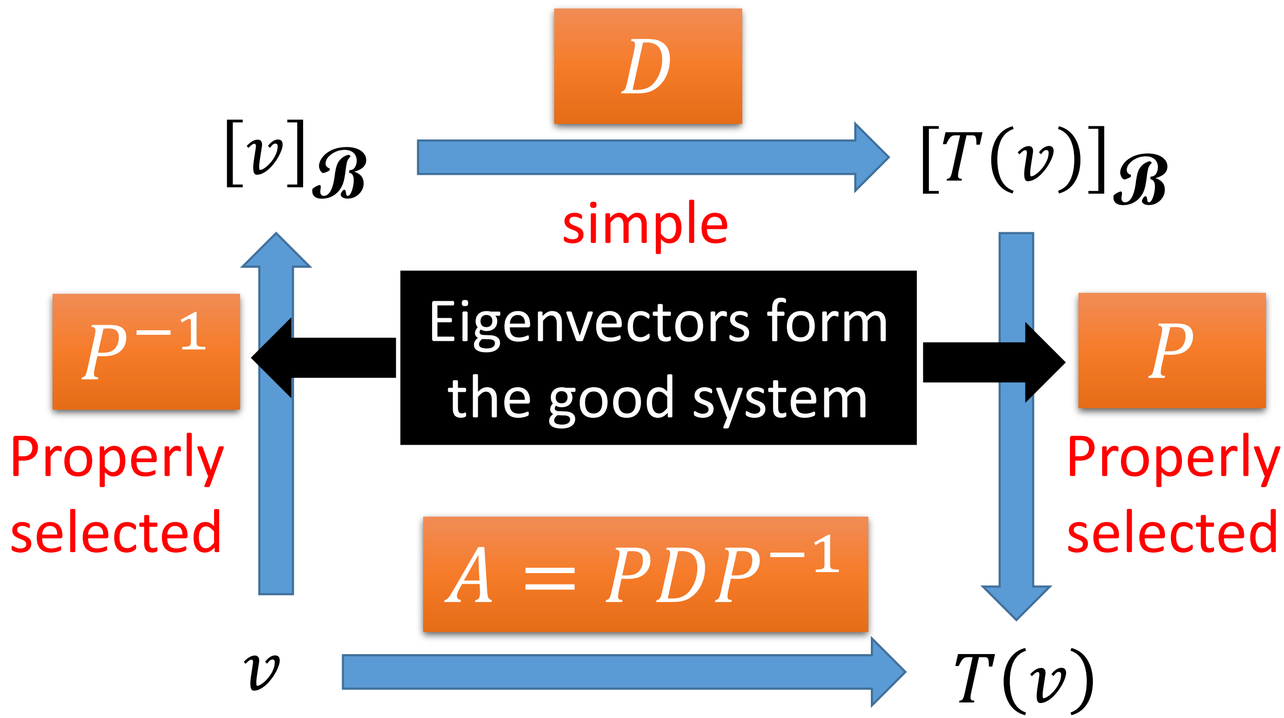
The standard matrix is $A = \begin{bmatrix} -1 & 1 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

det -t -t -t

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Diagonalization of Linear Operator

- If a linear operator T is diagonalizable



Diagonalization of Linear Operator

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 8x_1 + 9x_2 \\ -6x_1 - 7x_2 \\ 3x_1 + 3x_2 - x_3 \end{bmatrix}$$

-1:

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

2:

$$\mathcal{B}_2 = \left\{ \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \right\}$$

$$\begin{tikzcd}
{[v]_{\mathcal{B}}} \arrow[r, "{[T]_{\mathcal{B}}}", "{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}"'] & {[T(v)]_{\mathcal{B}}} \arrow[d, "{\begin{bmatrix} -1 & 0 & 3 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}}"] \\
v \arrow[u, "{\begin{bmatrix} -1 & 0 & 3 \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}^{-1}}"] \arrow[r, "{\begin{bmatrix} 8 & 9 & 0 \\ -6 & -7 & 0 \\ 3 & 3 & -1 \end{bmatrix}}"] & T(v)
\end{tikzcd}$$