

Subspace

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In Chapter 4

The same vector or operation is represented differently when they are in different coordinate systems.

民濕寢則腰疾偏死，鰭然乎哉？木處則惴慄恂懼，
猿猴然乎哉？三者孰知正處？民食芻豢，麋鹿食
薦，螂蛆甘帶，鴟鴞嗜鼠，四者孰知正味？猿獼
狙以為雌，麋與鹿交，鰭與魚游。毛嬙、西施，
人之所美也；魚見之深入，鳥見之高飛，麋鹿見
之決驟，四者孰知天下之正色哉？《莊子·齊物論》

Subspace

向量子空间

Reference

- Textbook: chapter 4.1

Subspace

- A vector set V is called a subspace if it has the following three properties:
- 1. The zero vector $\mathbf{0}$ belongs to V
- 2. If \mathbf{u} and \mathbf{w} belong to V , then $\mathbf{u} + \mathbf{w}$ belongs to V

零向量

Closed under (vector) addition

- 3. If \mathbf{u} belongs to V , and c is a scalar, then $c\mathbf{u}$ belongs to V

Closed under scalar multiplication

2+3 is linear combination

Examples

$$W = \left\{ \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \in \mathcal{R}^3 : 6w_1 - 5w_2 + 4w_3 = 0 \right\} \quad \text{Subspace?}$$

Property 1. $\mathbf{0} \in W \longrightarrow 6(0) - 5(0) + 4(0) = 0$



Property 2. $\mathbf{u}, \mathbf{v} \in W \Rightarrow \mathbf{u} + \mathbf{v} \in W$

$$\mathbf{u} = [u_1 \ u_2 \ u_3]^T, \mathbf{v} = [v_1 \ v_2 \ v_3]^T \quad \mathbf{u} + \mathbf{v} = [u_1 + v_1 \ u_2 + v_2 \ u_3 + v_3]^T$$

$$6(u_1 + v_1) - 5(u_2 + v_2) + 4(u_3 + v_3)$$

$$= (6u_1 - 5u_2 + 4u_3) + (6v_1 - 5v_2 + 4v_3) = 0 + 0 = 0$$

Property 3. $\mathbf{u} \in W \Rightarrow c\mathbf{u} \in W$

$$6(cu_1) - 5(cu_2) + 4(cu_3) = c(6u_1 - 5u_2 + 4u_3) = c0 = 0$$

Examples

$$V = \{c\mathbf{w} \mid c \in \mathcal{R}\}$$

Subspace?



$$\mathcal{S}_1 = \left\{ \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathcal{R}^2 : w_1 \geq 0 \text{ and } w_2 \geq 0 \right\}$$



Subspace?

$\mathbf{u} \in \mathcal{S}_1, \mathbf{u} \neq \mathbf{0} \Rightarrow -\mathbf{u} \notin \mathcal{S}_1$

$$\mathcal{S}_2 = \left\{ \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathcal{R}^2 : w_1^2 = w_2^2 \right\}$$



Subspace?

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \in \mathcal{S}_2 \text{ but } \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \notin \mathcal{S}_2$$

\mathcal{R}^n

Subspace?



$\{\mathbf{0}\}$

Subspace?



zero subspace

Subspace v.s. Span 张成空间

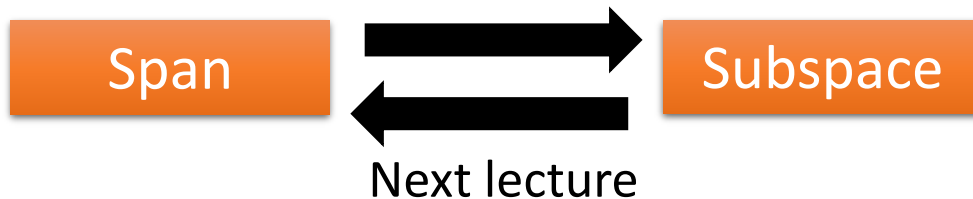
- The span of a vector set is a subspace

Let $S = \{w_1, w_2, \dots, w_k\}$ $V = \text{Span } S$

Property 1. $\mathbf{0} \in V$

Property 2. $\mathbf{u}, \mathbf{v} \in V, \mathbf{u} + \mathbf{v} \in V$

Property 3. $\mathbf{u} \in V, c\mathbf{u} \in V$



Null Space

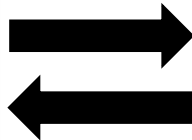
- The null space of a matrix A is the solution set of $A\mathbf{x}=\mathbf{0}$. It is denoted as $\text{Null } A$.

$$\text{Null } A = \{ \mathbf{v} \in \mathcal{R}^n : A\mathbf{v} = \mathbf{0} \}$$

The solution set of the homogeneous linear equations $A\mathbf{v} = \mathbf{0}$.

- $\text{Null } A$ is a subspace

A linear function is
one-to-one



Null space only
contain $\mathbf{0}$

Null Space - Example

$$T : \mathcal{R}^3 \rightarrow \mathcal{R}^2 \text{ with } T \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} x_1 - x_2 + 2x_3 \\ -x_1 + x_2 - 3x_3 \end{bmatrix}$$

Find a generating set for the null space of T .

The null space of T is the set of solutions to $A\mathbf{x} = \mathbf{0}$

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -3 \end{bmatrix} \xrightarrow{\text{blue arrow}} R = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{ccc} x_1 = x_2 & & \\ \text{---} x_1 & \text{---} x_2 & \text{---} 0 \\ & x_3 & = 0 \end{array} \xrightarrow{\text{blue arrow}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

a generating set for the null space

Column Space and Row Space

- Column space of a matrix A is the span of its columns. It is denoted as $\text{Col } A$.

$$A \in \mathcal{R}^{m \times n} \Rightarrow \text{Col } A = \{A\mathbf{v} : \mathbf{v} \in \mathcal{R}^n\}$$

If matrix A represents a function

$\text{Col } A$ is the range of the function

- Row space of a matrix A is the span of its rows. It is denoted as $\text{Row } A$.

$$\text{Row } A = \text{Col } A^T$$

Column Space = Range

- The range of a linear transformation is the same as the column space of its matrix.

Linear Transformation

$$T \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \right) = \begin{bmatrix} x_1 + 2x_2 + x_3 - x_4 \\ 2x_1 + 4x_2 - 8x_4 \\ 2x_3 + 6x_4 \end{bmatrix}$$

Standard matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 4 & 0 & -8 \\ 0 & 0 & 2 & 6 \end{bmatrix} \Rightarrow \text{Range of } T = \text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -8 \\ 6 \end{bmatrix} \right\}$$

RREF

- Original Matrix A v.s. its RREF R
 - Columns:
 - The relations between the columns are the same.
 - The span of the columns are different.
- Rows:
 - The relations between the rows are changed.
 - The span of the rows are the same.

$$\text{Col } A \neq \text{Col } R$$

$$\text{Row } A = \text{Row } R$$

Consistent

$Ax = b$ have solution (consistent)

b is the linear combination of columns of A

b is in the span of the columns of A

b is in $\text{Col } A$

$$A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 4 & 0 & -8 \\ 0 & 0 & 2 & 6 \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \in \text{Col } A? \quad \mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \in \text{Col } A?$$

Solving $Ax = u$

$$\text{RREF}([A \ u]) = \begin{bmatrix} 1 & 2 & 0 & -4 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & \textcircled{1} \end{bmatrix}$$

Solving $Ax = v$

$$\text{RREF}([A \ v]) = \begin{bmatrix} 1 & 2 & 0 & -4 & 0.5 \\ 0 & 0 & 1 & 3 & 1.5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Conclusion: Subspace is Closed under addition and multiplication

