Machine Learning Principles

Class8 : Sept 28

Linear Classification II: Logistic Regression

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Today's Lecture

1. Modeling of Logistic Regression :

   - $P[C_k|x]$: sigmoid logistic function embedding "a linear regression function"

   - decision rule/ boundary

   - orientation and steepness of a sigmoid function

2. Training:

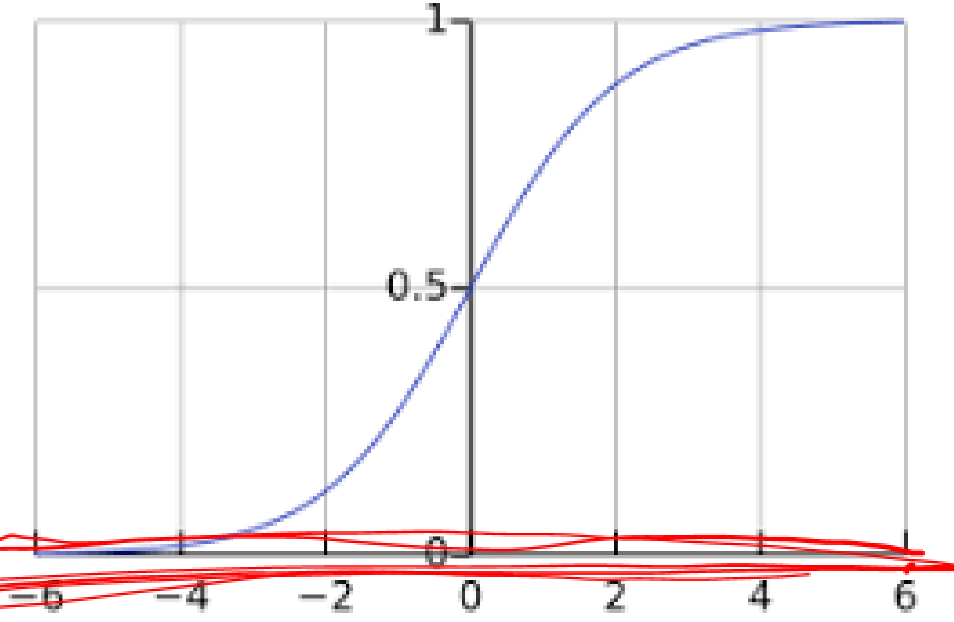
   - defining objective Function (MLE) and its optimization

   - overfitting and regularization

3. Multinomial (Multiclass) Logistic Regression

[1] Logistic Regression (sigmoid function)

[logistic sigmoid function]

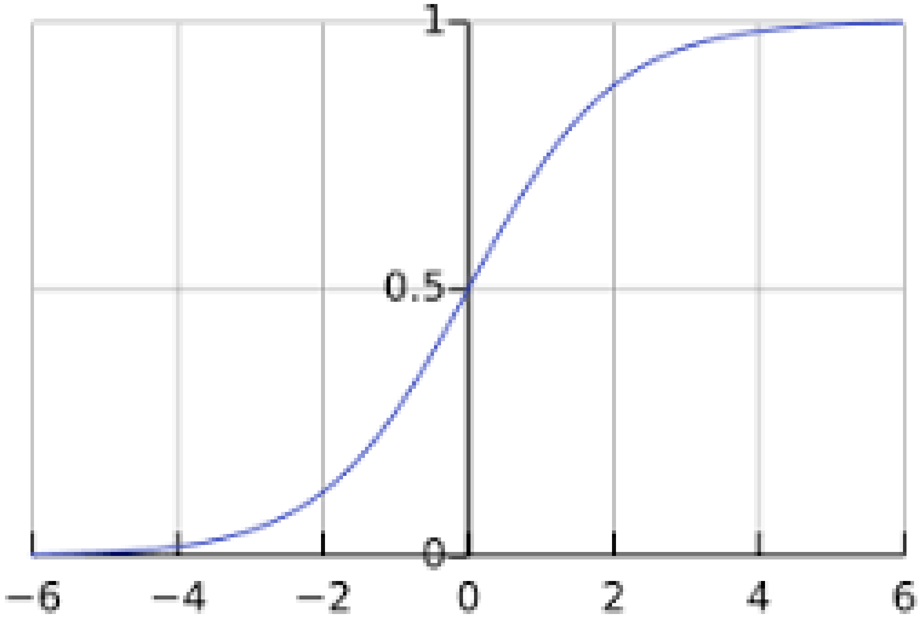


$$\sigma(x) = \frac{1}{1 + exp^{-x}}$$

$w^T . x + b$

[2] Logistic Regression (sigmoid function + linear function)

[logistic sigmoid function]



$$P[C_1|\vec{x}] = \frac{1}{1+\exp\left(-\vec{w}^t\vec{x} - b\right)}$$

$$P[C_0|\vec{x}] = \frac{\exp\left(-\vec{w}^t\vec{x} - b\right)}{1+\exp\left(-\vec{w}^t\vec{x} - b\right)}$$

$w^T \cdot x + b = 0$

"decision boundary"

$$\sigma(x) = \frac{1}{1+exp^{-x}}$$

[decision rule]

$$\begin{cases} w^t x + b \geq 0 \;\; then \;\; P[C_1|x] \geq P[C_0|x] \rightarrow x \in C_1 \\ w^t x + b < 0 \;\; then \;\; P[C_1|x] < P[C_0|x] \rightarrow x \in C_0 \end{cases}$$

[3] Logistic Regression (example)

ex] suppose we learned logistic regression on the 2-D data space like below.
what are the decision rule/boundary/ region?

$$P[C_1|x] = \sigma\left(\begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}x \\ y\end{bmatrix}\right) = \frac{1}{1 + \exp - \begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}x \\ y\end{bmatrix}}$$

$1 \cdot x + 2y \geq 0$ $\quad (x, y) \in C_1$

$x + 2y < 0$ $\quad (x, y) \in C_0$

[4] Logistic Regression (example)

ex] suppose we learned logistic regression on the 2-D data space like below.
what are the decision rule/boundary/ region?

$$P[C_1|x] = \sigma\left(\begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}x \\ y\end{bmatrix}\right) = \frac{1}{1 + \exp - \begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}x \\ y\end{bmatrix}}$$

- Decision Boundary (linear regression function/ hyperplane)
- The direction of the normal vector of the hyperplane
- The magnitude of the normal vector of the hyperplane

[5] Logistic Regression (normal vector's direction and magnitude)

Q: $w^t x + \mathrm{b} = 0$ defines a decision boundary. do they define the same boundaries ?

$$(1) \quad \sigma([1 \quad 2] \begin{bmatrix} x \\ y \end{bmatrix}) = \frac{1}{1 + \exp - [1 \quad 2] \begin{bmatrix} x \\ y \end{bmatrix}}$$

$$(2) \quad \sigma([2 \quad 4] \begin{bmatrix} x \\ y \end{bmatrix}) = \frac{1}{1 + \exp - [2 \quad 4] \begin{bmatrix} x \\ y \end{bmatrix}}$$

$$(3) \quad \sigma([3 \quad 6] \begin{bmatrix} x \\ y \end{bmatrix}) = \frac{1}{1 + \exp - [3 \quad 6] \begin{bmatrix} x \\ y \end{bmatrix}}$$

- same decision

$1x + 2y = 0$

$||w|| \uparrow$

$$||w||(w'^t x + b') = 0$$

$$\text{where} \quad || \begin{bmatrix} w' \\ b' \end{bmatrix} || = 1$$

There could be infinitely many solutions defining the same decision boundary for different $||w||$, but they differ in the steepness of their logistic sigmoid functions. The larger $||w||$ gives the steeper sigmoid.

[6] Logistic Regression (properties)

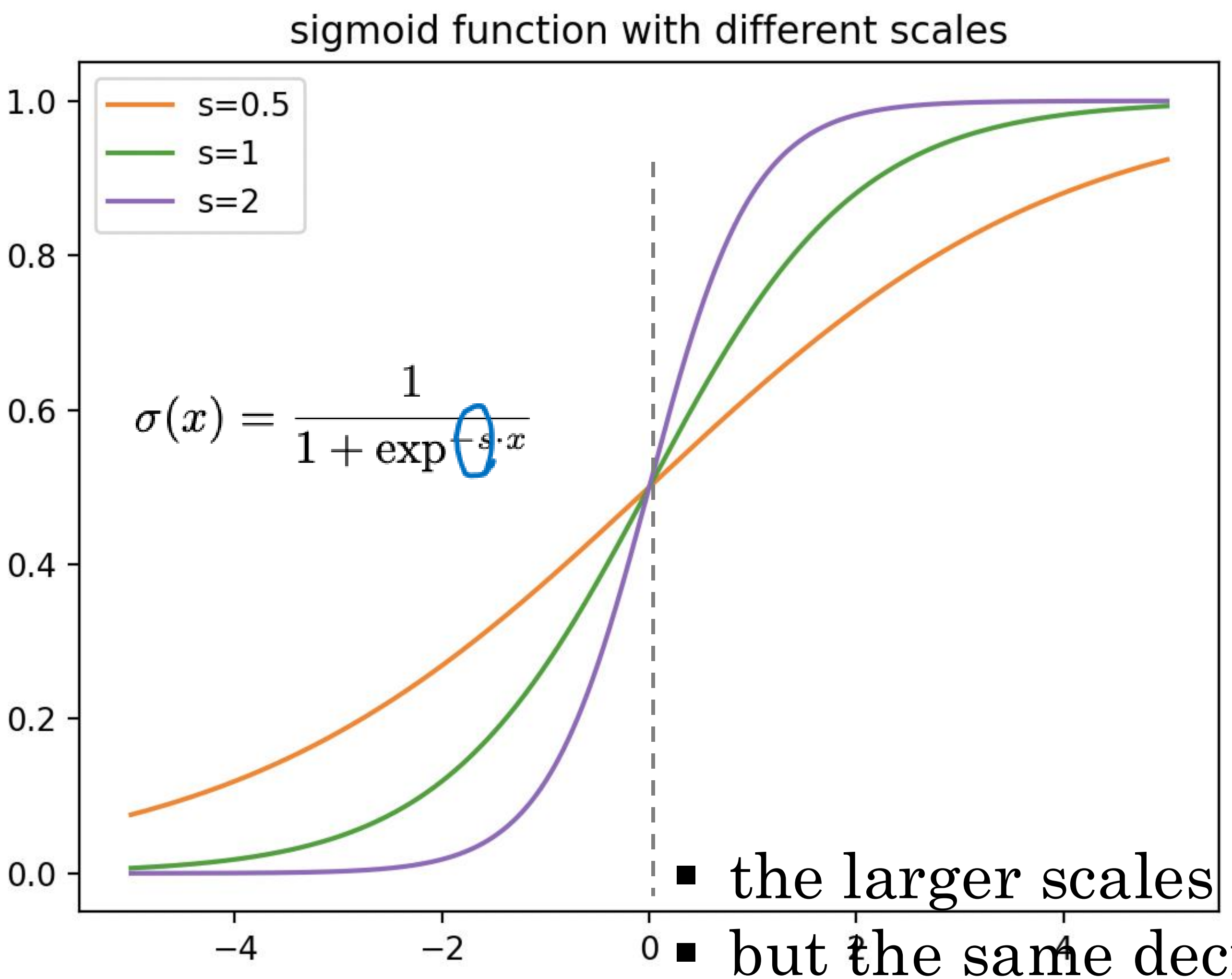
- The direction & magnitude of the normal vector of the hyperplane determines orientation & steepness of the sigmoid functions.

  ✓ $\vec{w}x + b = 0$
  ✓ $\vec{w}$: normal vector (direction to be the class 1)

$$P[C_1|\vec{x}] = \frac{1}{1 + \exp\left(-\vec{w}^t\vec{x} - b\right)}$$

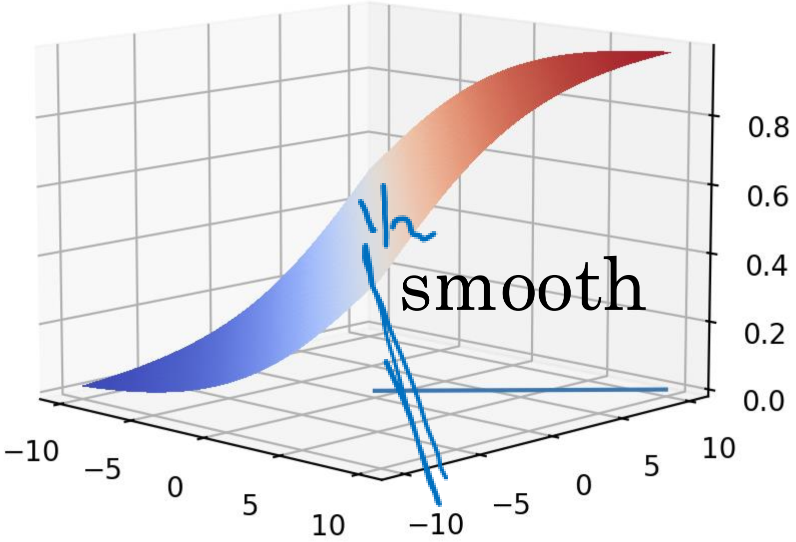
$$P[C_0|\vec{x}] = \frac{\exp\left(-\vec{w}^t\vec{x} - b\right)}{1 + \exp\left(-\vec{w}^t\vec{x} - b\right)}$$

**example 1D: sigmoid functions with different scales

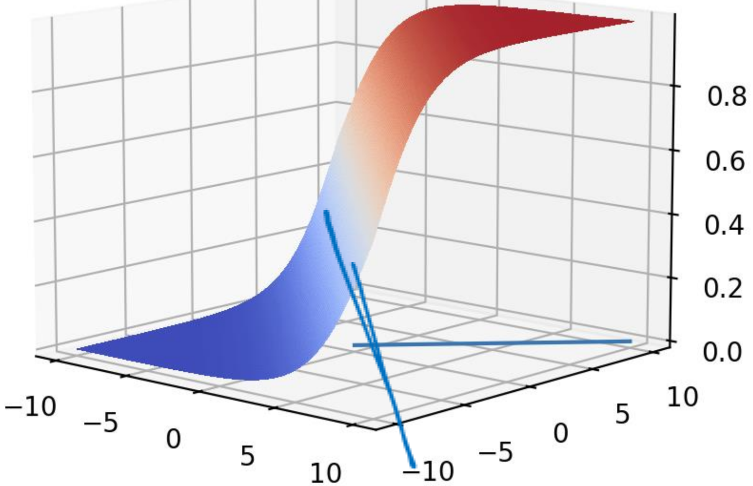


- the larger scales makes the sigmoid steeper.
- but the same decision boundary $x = 0$

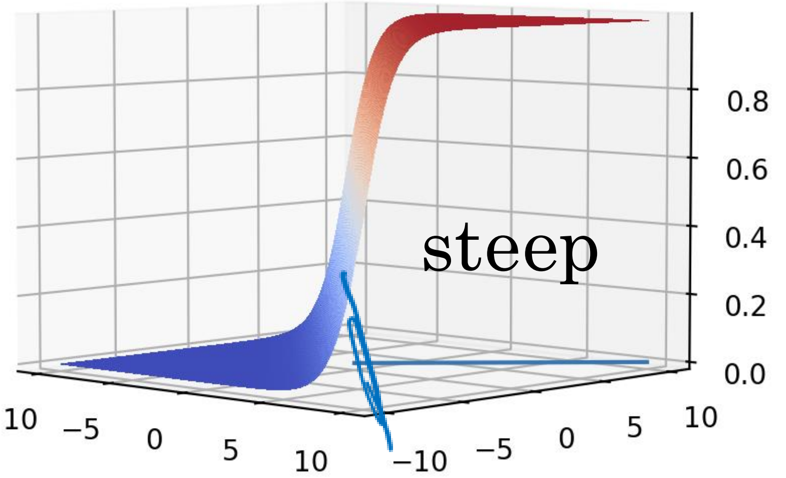
**example 2D: same decision region but different sigmoid functions



$\vec{w} = (0.2,0.2)$



$\vec{w} = (0.5,0.5)$



$\vec{w} = (1,1)$

- Their decision boundaries are the same as $X_1 + X_2 = 0$
  but their steepness varies according to the magnitude of $||w||$

$$P(C_1 | x) = \frac{1}{1 + \exp^{-w^T \cdot x - b}}$$

- Training Logistic Regression

  how do we learn the parameters $\vec{w}$ and b?

[1] Training (Maximum Likelihood Estimator: MLE)

$$P(w|D) = \frac{p(w,D)}{P(D)} = \frac{p(D|w)p(w)}{p(D)}$$

$w *= argmax\ \ P(D|w)$: **M**aximum **L**liklihood **E**stimation (MLE)

[2] Training (MLE)

- suppose we have a data set $\{\overrightarrow{x_n}, t_n\}$ $where\ t_n \in \{0,1\}\ \&\ n = \{1,2,\ldots,N\}$
- like 1,0, 0,…,1,0,1…
- $P(t_1, t_1, t_3, \ldots t_N | \vec{w}, b, \overrightarrow{x_n})$

$$= \prod_{n=1}^{N} P(t_n | \vec{w}, b, \overrightarrow{x_n}) = \mathrm{P}(t_n = 1 | \vec{w}, b, \overrightarrow{x_n})^{t_n}\ \mathrm{P}(t_n = 0 | \vec{w}, b, \overrightarrow{x_n})^{1-t_n}$$

Q: how we modeled $\mathrm{P}(t_n = 1 | \vec{w}, b)$ in logistic regression?

[3] Training (cross entropy loss)

$$P(t_1, ..., t_N|w) = \prod_{n=1}^{N} \sigma(w^t x_n)^{t_n} (1 - \sigma(w^t x_n))^{1-t_n}$$

$$J(\vec{w}) = -\ln P(t_1, ..., t_N|w) = \boxed{\sum_{n=1}^{N} -t_n \ln \sigma(w^t x_n) - (1 - t_n) \ln (1 - \sigma(w^t x_n))}$$

$$w* = \arg\min_{w} J(\vec{w})$$

[cross entropy loss]

** Entropy / Cross Entropy / KL Divergence (Kullback-Leibler)

$$H(X) = E[\log \frac{1}{P(X)}] = \sum_x p(x) \log \frac{1}{p(x)}$$

- **[Entropy]**:
  the smallest number bits to encode R.V X

$$D(p||q) = \sum_x p(x) \log \frac{p(x)}{q(x)}$$
$$= \sum_x p(x) \log \frac{1}{q(x)} - \sum_x p(x) \log \frac{1}{p(x)}$$

- **[KL divergence]**:
  how many bits more needed when using $q(x)$ instead of the original density $p(x)$?
- this measures
  the difference/distance
  between the two densities: $p(x)$ and $q(x)$

** Entropy / Cross Entropy / KL Divergence (Kullback-Leibler)

- KL divergence is always non-negative!

$$D(p||q) = \sum_x p(x) \log \frac{p(x)}{q(x)} \geq 0$$

$$-D(p||q) = \sum_x p(x) \log \frac{q(x)}{p(x)}$$

[by Jensen's Inequality]

$$\leq \log \sum_x \frac{p(x)q(x)}{p(x)} = \log 1 = 0$$

** Entropy / Cross Entropy / KL Divergence (Kullback-Leibler)

$$
\begin{aligned}
H(p,q) &= \sum_x p(x) \log \frac{1}{q(x)} \\
&= \sum_x p(x) \log \frac{p(x)}{q(x)p(x)} = \sum_x p(x) \log \frac{p(x)}{q(x)} + \sum_x p(x) \log \frac{1}{p(x)} \\
&= D(p||q) + H(X)
\end{aligned}
$$

- **[Cross Entropy]**
  minimizing cross entropy directly minimizes KL divergence.

[4] Training (cross entropy)

$$P(t_1, ..., t_N|w) = \prod_{n=1}^{N} \sigma(w^t x_n)^{t_n} (1 - \sigma(w^t x_n))^{1-t_n}$$

$$J(\vec{w}) = -\ln P(t_1, ..., t_N|w) = \boxed{\sum_{n=1}^{N} -t_n \ln \sigma(w^t x_n) - (1 - t_n) \ln (1 - \sigma(w^t x_n))}$$

$$w* = \arg\min_w J(\vec{w})$$

[cross entropy loss]

- as taking the ground truth (1 / 0) as $p(t)$
  minimizing the cross entropy minimizes the distance
  between $p(t)$ and $q(t) = \sigma(w^t x)$

[5] Training (computing Gradient)

$$\nabla_w J(\vec{w}) = \sum_{n=1}^{N} -t_n \frac{\sigma(w^t x_n)(1-\sigma(w^t x_n))}{\sigma(w^t x_n)} x_n - (1-t_n)\frac{-\sigma(w^t x_n)(1-\sigma(w^t x_n))}{1-\sigma(w^t x_n)} x_n$$

$$= \sum_{n=1}^{N} \{-t_n(1-\sigma(w^t x_n)) - (1-t_n)(-\sigma(w^t x_n)\} x_n$$

$$\boxed{= \sum_{n=1}^{N} (\sigma(w^t x_n) - t_n) x_n}$$

[5] Training (computing Gradient)

**useful properties of logistic sigmoid function

- symmetric & complement probability

$$\sigma(-x) = 1 - \sigma(x)$$
$$\sigma(x) = 1 - \sigma(-x)$$

- derivative

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\sigma(x) &= \frac{\exp^{-x}}{(1+\exp^{-x})^2} \\
&= \frac{1}{(1+\exp^{-x})} \cdot \frac{\exp^{-x}}{(1+\exp^{-x})} \\
&= \frac{1}{(1+\exp^{-x})} \cdot \frac{1+\exp^{-x} - 1}{(1+\exp^{-x})} \\
&= \sigma(x) \cdot (1 - \sigma(x))
\end{aligned}$$

[6] Training (computing Hessian)

$$\nabla^2_w J(\vec{w}) = \sum_{n=1}^{N} \sigma(w^t x_n) \cdot (1 - \sigma(w^t x_n)) x_n x_n^t \succeq 0$$

- non-negative definite

[6] Training ($\nabla^2 J(w) = 0$)

$$\nabla_w^2 J(\vec{w}) = \sum_{n=1}^{N} \sigma(w^t x_n) \cdot (1 - \sigma(w^t x_n)) x_n x_n^t \succeq 0$$

non-negative definite

- The two possible cases of $\nabla^2 J(w) = 0$

- no minimum solution $\overrightarrow{w*}$ when data is linearly separable.

- infinitely many minimum points: covariance matrix $X^t X$ is singular, collinearity exist.

[7] Training ($\nabla^2 J(w) > 0$)

$$\nabla_w^2 J(\vec{w}) = \sum_{n=1}^{N} \sigma(w^t x_n) \cdot (1 - \sigma(w^t x_n)) x_n x_n^t \succeq 0$$

non-negative definite

- unique solution: $\nabla^2 J(w) > 0$

- when data is not linearly separable and $X^t X$ is full-rank matrix.
  then $\nabla^2 J(w) \succ 0 \;\rightarrow \nabla J(w*) = 0$ becomes a sufficient condition for $w*$

[8] Training (summary)

$\nabla^2 J(w) = 0$		$\nabla^2 J(w) > 0$
data is linearly separable	collinearity $X^t X$ is singular	data is not linearly separable & $X^t X$ is a full rank matrix
no minimum exist. no w $*$ $\nabla J(w*) = 0$ $\|\|w\|\|$ going to ∞	multiple w is possible $s.t\ \nabla J(w*) = 0$ infinitely many solutions, "**global**" minimum	unique solution exist $s.t\ \nabla J(w*) = 0$ unique, "**global**" minimum

- depending on the structure of data, the shape of $J(w)$ varies.

[9] Training (no closed solution $\nabla J(\vec{w}) = 0$)

Let's try finding the optimal point $\overrightarrow{w*}$ by $\nabla J(\vec{w}) = 0$!

$$\nabla_w J(\vec{w}) = \sum_{n=1}^{N} -t_n \frac{\sigma(w^t x_n)(1-\sigma(w^t x_n))}{\sigma(w^t x_n)} x_n - (1-t_n)\frac{-\sigma(w^t x_n)(1-\sigma(w^t x_n))}{1-\sigma(w^t x_n)} x_n$$

$$= \sum_{n=1}^{N} \{-t_n(1-\sigma(w^t x_n)) - (1-t_n)(-\sigma(w^t x_n)\} x_n$$

$$\boxed{= \sum_{n=1}^{N} (\sigma(w^t x_n) - t_n) x_n = 0}$$

no closed solution for $\nabla_w J(w) = 0$

- Steepest Gradient Descent Algorithm
  (an iterative optimization method to minimize $J(\vec{w})$)

[1] Steepest Gradient Descent Algorithm

- it finds iteratively a local/ global minimum of $J(\vec{w})$
- it computes the steepest descent direction at the current and take a step to that direction as amount of predefined and compute the next direction again

$$w_{i+1} = w_i - \eta \nabla J(w)$$ (gradient gives the steepest & increasing direction)

[2] Steepest Gradient Descent Algorithm (direction by gradient)

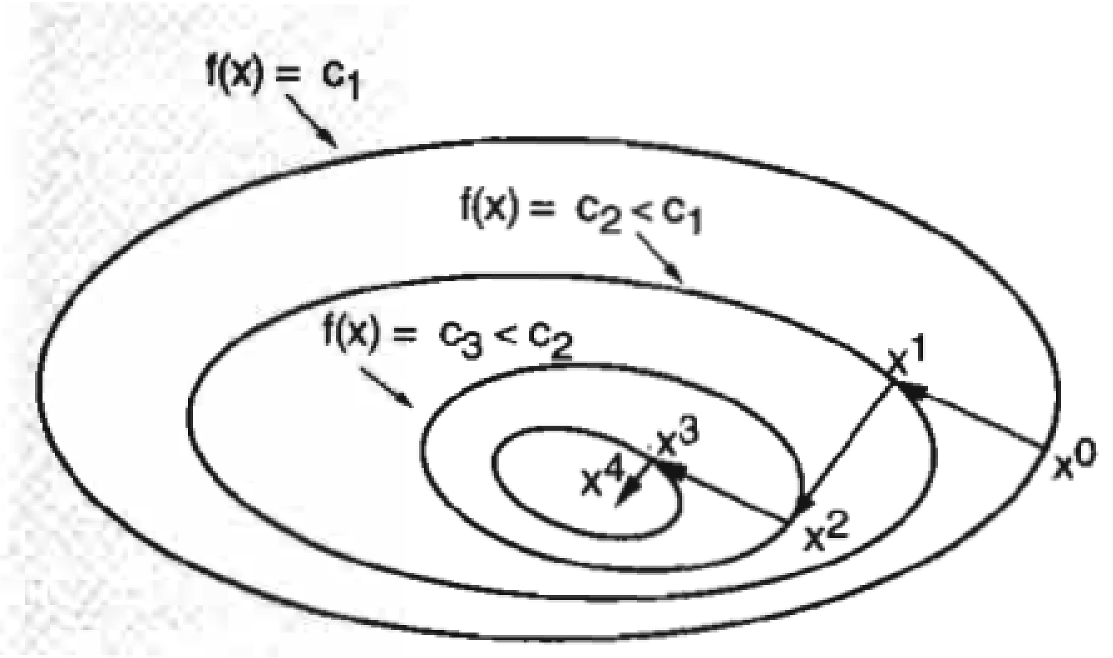


Figure 1.2.1. Iterative descent for minimizing a function f. Each vector in the generated sequence has a lower cost than its predecessor.

[Nonlinear Programming, Dimitri P. Bertsekas]

- gradient direction is orthogonal to contours.

[3] Steepest Gradient Descent Algorithm

Too **Large** step size η (overshoot)	Too **Small** step size η (undershoot)
▪ slow convergence ▪ instead of smoothly moving toward the minimum, it keeps bouncing back and forth (wasted movements)	▪ slow convergence
▪ possibility of divergence	▪ possibility stuck in local minimum

❖ The step size should not be too large / too small.

[4] Steepest Gradient Descent Algorithm

- do we need to worry about step size near optimal value $w * s.t \nabla J(w *) = 0$?

- Training Logistic Regression using Gradient Steepest Descent

[1] Training (gradient/steepest decent algorithm)

$$w_{i+1} = w_i - \eta \nabla J(w)$$

(gradient gives the steepest ascent direction)

$$\nabla_w J(\vec{w}) = \sum_{n=1}^{N} -t_n \frac{\sigma(w^t x_n)(1-\sigma(w^t x_n))}{\sigma(w^t x_n)} x_n - (1-t_n) \frac{-\sigma(w^t x_n)(1-\sigma(w^t x_n))}{1-\sigma(w^t x_n)} x_n$$

$$= \sum_{n=1}^{N} \{-t_n(1-\sigma(w^t x_n)) - (1-t_n)(-\sigma(w^t x_n)\} x_n$$

$$\boxed{= \sum_{n=1}^{N} (\sigma(w^t x_n) - t_n) x_n}$$

[2] Training (convergence?)

$\nabla^2 J(w) = 0$		$\nabla^2 J(w) > 0$
data is linearly separable	collinearity $X^t X$ is singular	data is not linearly separable & $X^t X$ is a full rank matrix
no minimum exist. $s.t\ \nabla J(w*) = 0$ $\|\|w\|\|$ goes to ∞	multiple w is possible $s.t\ \nabla J(w*) = 0$ global minimum	unique solution exist $s.t\ \nabla J(w*) = 0$ global minimum

- does gradient descent algorithm converge?

(if tolerance set, improvement $< \varepsilon$ & if we set the proper step size.)

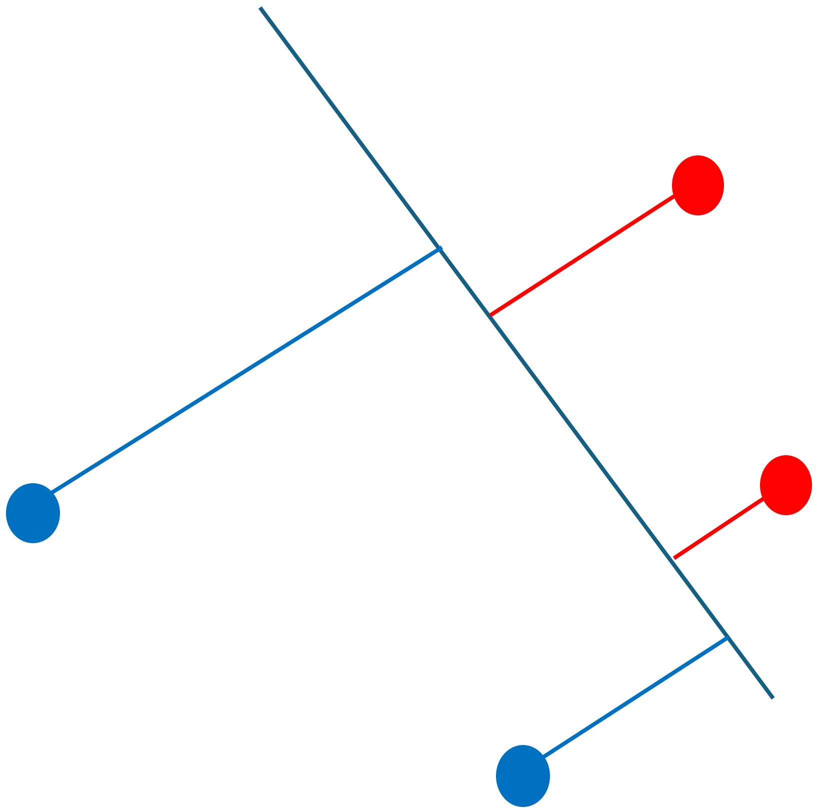
- Overfitting

  MLE is sensitive to data
  (data structure determines the direction & steepness $\vec{w}$)

[**] logistic regression encourages the larger margins.

**concept of margin

margin: the smallest distance
between the decision boundary and any of the samples



[1] Overfitting Scenario (limited data size)

Like regression model,
logistic regression classifier overfits to train data
when too many features relative to # training samples.

Q: even worse, when the training dataset is linearly separable, the overfitted model tends to have excessively high confidence in its predictions.

[2] Overfitting Scenario (steep logistic sigmoid)

the steep sigmoid function would be appropriate

- if we have a well-designed feature map making data linearly separable.
- if we we have a sufficient number of data points.
- however, hard to achieve the condition in practice.

- Regularization (MAP rule)

$$w* = argmax\, p(w|D) \propto p(D|w)p(w)$$

[1] Regularization (ridge logistic regression)

$p(w)$: $\vec{w} \sim N(0, \sigma^2 I)$

prior

$$P(t_1, ...t_N|w)P(w) = \prod_{n=1} \sigma(w^t x_n)^{t_n}(1 - \sigma(w^t x_n))^{1-t_n} \cdot \boxed{\frac{1}{\sqrt{2\pi\sigma^{2M}}} \cdot \exp^{-\frac{1}{2\sigma^2}||W||^2}}$$

$$-\ln P(t_1, ...t_N|w)P(w) = \sum_{n=1}^{N} -t_n \ln \sigma(w^t x_n) - (1 - t_n) \ln (1 - \sigma(w^t x_n)) - \ln \frac{1}{\sqrt{2\pi\sigma^{2M}}} + \frac{1}{2\sigma^2}||W||^2$$

$$J(w) = \sum_{n=1}^{N} -t_n \ln \sigma(w^t x_n) - (1 - t_n) \ln (1 - \sigma(w^t x_n)) + \frac{1}{2\sigma^2}||W||^2$$

$$J(w) = \sum_{n=1}^{N} -t_n \ln \sigma(w^t x_n) - (1 - t_n) \ln (1 - \sigma(w^t x_n)) \boxed{+ \lambda \cdot} ||W||^2$$

regularization parameters /
set using cross validation.

[2] Regularization (gradient)

- $\nabla J(\overrightarrow{W})$

$$\nabla_w J(\vec{w}) = \sum_{n=1}^{N} -t_n \frac{\sigma(w^t x_n)(1-\sigma(w^t x_n))}{\sigma(w^t x_n)} x_n - (1-t_n)\frac{-\sigma(w^t x_n)(1-\sigma(w^t x_n))}{1-\sigma(w^t x_n)} x_n + 2\lambda\vec{w}$$

$$= \sum_{n=1}^{N} \{-t_n(1-\sigma(w^t x_n)) - (1-t_n)(-\sigma(w^t x_n)\} x_n + 2\lambda\vec{w}$$

$$= \sum_{n=1}^{N} (\sigma(w^t x_n) - t_n) x_n + 2\lambda\vec{w}$$

- gradient descent

$$w_{i+1} = w_i - \eta \nabla J(w)$$

- Multinomial Logistic Regression

  (Learning K posterior $P[C_k|x]$)

[1] Multinomial Logistic Regression (posterior using linear functions)

$$P[C_0|x] = \frac{P[x|C_0]P[C_0]}{P[x|C_0]P[C_0] + P[x|C_1]P[C_1] + P[x|C_2]P[C_2]}$$

$$P[C_0|x] = \frac{\exp[\ln P[x|C_0]P[C_0]]}{\exp\boxed{[\ln P[x|C_0]P[C_0]]} + \exp\boxed{[\ln P[x|C_1]P[C_1]]} + \exp\boxed{[\ln P[x|C_2]P[C_2]]}}$$

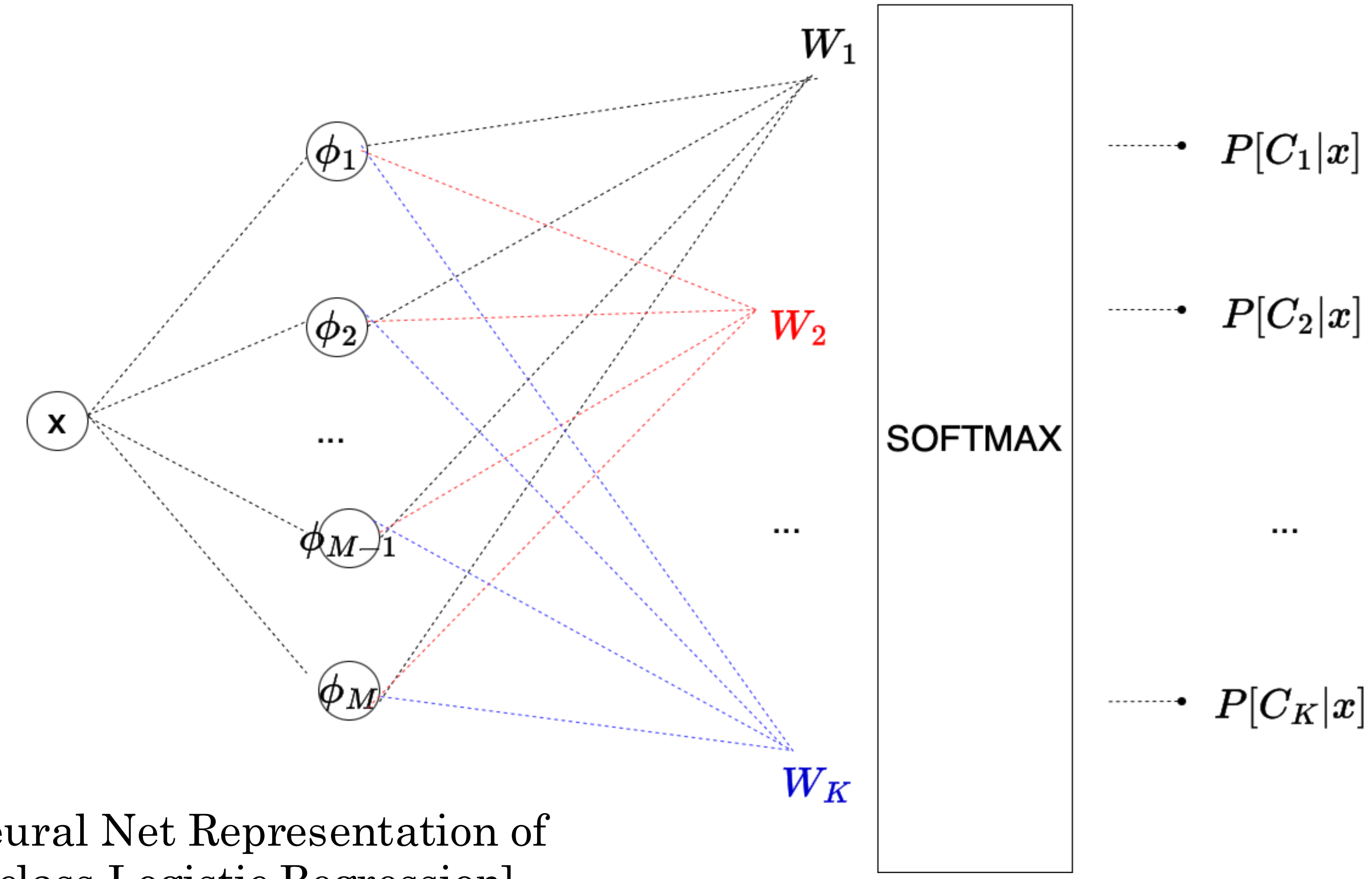
when Gaussain assumed, these log elements are quadratic.
But each exponent becomes a linear function: $w^t x + b$; the second order terms are cancelled out when covariances are equal.

[2] Multinomial Logistic Regression (softmax function)

$$P[C_k|x] = \frac{\exp w_k^t \phi(x)}{\sum_{k=1}^{K} \exp w_k^t \phi(x)}$$

- using normalized exponential (softmax) and K linear functions, the K posteriors are represented.

[3] Multinomial Logistic Regression (softmax function)



[the Neural Net Representation of Multiclass Logistic Regression]

[4] Multinomial Logistic Regression

Multiclass logistic regression learns K discriminant linear functions
the functions are translated into posterior by softmax
(deep neural net classifier has the same structure in the last layer)
Q: how can we learn the discriminant linear functions?

[6] Multinomial Logistic Regression (training)

$$P[C_k|x] = \frac{\exp w_k^t \phi(x)}{\sum_{k=1}^{K} \exp w_k^t \phi(x)}$$

- then how to learn the parameters $\vec{w}$?
- MLE!

  we plug in the posteriors in k-categorical density and estimate the parameter using MLE as we did in binary logistic regression.

[7] Multinomial Logistic Regression (cross entropy based on MLE)

- likelihood

$$P[T|W,x] = \prod_{n=1}^{N}\prod_{k=1}^{K} P[C_k|W,x_n]^{t_{nk}}$$

- negative-log (cross entropy)

$$J(W) = -\ln P[T|W,x] = -\sum_{n=1}^{N}\sum_{k=1}^{K} t_{nk}\ln P[C_k|W,x_n]$$

[8] Multinomial Logistic Regression (softmax gradient)

- $\nabla P[C_j|x]$ $r.t$ w_i $(i{=}j)$

$$P[C_j|x] = \frac{\exp w_j^t \phi(x)}{\sum_{k=1}^{K} \exp w_k^t \phi(x)}$$

$$\nabla_{W_j} P[C_j|x] = \{\frac{(\sum_{k=1}^{K} \exp w_k^t \phi(x))(\exp w_j^t \phi(x)) - (\exp w_j^t \phi(x))^2}{(\sum_{k=1}^{K} \exp w_k^t \phi(x))^2}\}\phi(x)$$

$$\nabla_{W_j} P[C_j|x] = \{\frac{\exp w_j^t \phi(x)}{\sum_{k=1}^{K} \exp w_k^t \phi(x)} \cdot (1 - \frac{\exp w_j^t \phi(x)}{\sum_{k=1}^{K} \exp w_k^t \phi(x)})\}\phi(x)$$

- ∇ $P[C_j|x]$ $r.t$ w_i $(i \neq j)$

$$P[C_j|x] = \frac{\exp w_j^t \phi(x)}{\sum_{k=1}^{K} \exp w_k^t \phi(x)}$$

$$\nabla_{W_i} P[C_j|x] = \{\frac{-(\exp w_j^t \phi(x))(\exp w_i^t \phi(x))}{(\sum_{k=1}^{K} \exp w_k^t \phi(x))^2}\}\phi(x)$$

$$\nabla_{W_i} P[C_j|x] = \{\frac{\exp w_j^t \phi(x)}{\sum_{k=1}^{K} \exp w_k^t \phi(x)} \cdot (-\frac{\exp w_i^t \phi(x)}{\sum_{k=1}^{K} \exp w_k^t \phi(x)})\}\phi(x)$$

[9] Multinomial Logistic Regression (softmax gradient for gradient descent)

$$\nabla_{W_j} P[C_j|x] = P[C_j|x] \cdot (1 - P[C_j|x])\phi(x) \qquad (i=j)$$
$$\nabla_{W_i} P[C_j|x] = P[C_j|x] \cdot (-P[C_i|x])\phi(x) \qquad (i \neq j)$$

Q: can we compute $P[C_j|x]$?
yes, during training, we can compute $P[C_j|x]$ using the current parameter W!

[10] Multinomial Logistic Regression (cross entropy gradient)

$$
\begin{aligned}
\nabla_{W_j} J(W_1, W_2, \ldots W_K) &= -\sum_{n=1}^{N}\sum_{k=1}^{K} t_{nk} \nabla_{W_j} \ln P[C_k|\phi(x)] \\
&= \sum_{n=1}^{N} t_{n1} P[C_j|\phi(x)] + \ldots + t_{nj}(P[C_j|\phi(x)] - 1) + \ldots + t_{nK} P[C_j|\phi(x)] \\
&= \sum_{n=1}^{N} \{P[C_j|\phi(x)]\underbrace{(t_{n1} + t_{n2} + \ldots t_{nK})}_{\text{sum} = 1} - t_{nj}\}\phi(x) \\
&= \sum_{n=1}^{N} \{P[C_j|\phi(x)] - t_{nj}\}\phi(x)
\end{aligned}
$$

[11] Multinomial Logistic Regression (gradient descent)

$$w_{i+1} = w_i - \eta \nabla J(w)$$

$$\nabla J(W) = \begin{bmatrix} \nabla_{W_1} J(W) \\ \nabla_{W_2} J(W) \\ \cdots \\ \cdots \\ \nabla_{W_{K-1}} J(W) \\ \nabla_{W_K} J(W) \end{bmatrix}$$