Journal Pre-proof

Work fluctuations in a generalized Gaussian active bath

Koushik Goswami

PII: S0378-4371(20)30907-9
DOI: https://doi.org/10.1016/j.physa.2020.125609
Reference: PHYSA 125609

To appear in: *Physica A*

Received date : 4 January 2020
Revised date : 26 November 2020

Please cite this article as: K. Goswami, Work fluctuations in a generalized Gaussian active bath, *Physica A* (2020), doi: https://doi.org/10.1016/j.physa.2020.125609.

Work fluctuations in a generalized Gaussian active bath

Koushik Goswami

Department of Inorganic and Physical Chemistry, Indian Institute of Science, Bangalore 560012, India

Abstract

We theoretically investigate the dynamics and work distribution of a Brownian particle in a Gaussian active bath. By modeling the active noise as a generalized form of Ornstein-Uhlenbeck process (OUP), we show that the dynamics approaches asymptotically to a superdiffusive regime. Two protocols are considered to perform work on the system, and exact expressions for the probability distribution function (PDF) of work are obtained. Further, we show, by employing the large deviation principle (LDP), that the PDF follows an anomalous scaling with time, in contrast to the normal LDP. Then, fluctuation relations (FR) of work are studied to find that the transient FR does not exist, but a non-conventional FR emerges in the long-time limit. Also, the known results for the usual OUP bath are recovered.

1. Introduction

When a passive Brownian particle diffuses in a bath containing active particles (active bath), it exhibits some interesting characteristics which cannot be fully explained by Einstein's theory [1]. In an active bath, the Brownian particle is subjected to two random forces - one is the thermal noise which arises due to collisions with solvent molecules, and the other is an athermal noise exerted by the active particles [2, 3, 4, 5, 6]. The second noise usually called active noise does not follow fluctuation-dissipation theorem (FDT) and drives the system away from equilibrium [7, 8]. One of the most common and simple model of active noise is the Ornstein-Uhlenbeck process (OUP) where the dynamical persistence is incorporated through an exponential correlation function [2, 9, 4, 10]. It is worth mentioning here that the OUP model has been widely used for the dynamical description of active systems, *e.g.*, see

Refs. [11, 12, 13, 14]. In the OUP bath, the distribution of particle's displacement is always Gaussian with an enhanced diffusivity. But the dynamics is non-Fickian at short times, and it approaches to a Fickian regime in the long-time limit, *i.e.*, its mean square displacement (MSD) grows asymptotically with time T as: $< x^2(T) >\sim T$. Such traits have been observed experimentally, for example, in Ref. [2] where the dynamics of colloidal silica beads in a low-dense bacterial bath was studied. However, the OUP model fails to describe a system which displays a superdiffusive behavior at long timescales. In the case of superdiffusion, the MSD behaves like: $< x^2(T) >\sim T^\mu$, where $\mu > 1$.

An anomalous dynamical behavior is often observed in biological systems, such as diffusing vesicles or organelles in an intracellular environment [15, 16], motile amoeboid [17], migrating biological cell [18, 19], etc [20, 21]. The dynamics is anomalous in the sense that $< x^2(T) >\propto T^\mu$, for $\mu \neq 1$, and it is usually modeled by the generalized Langevin equation which includes a retarded dissipative term and an (active) external noise [22, 23, 24]. At initial timescales, the motion has a subdiffusive regime ($0 < \mu < 1$) due to the memory effects of the frictional kernel, and it transits to a superdiffusive regime ($\mu > 1$) at later times. However, for the case of a memory-less dissipation, the subdiffusive regime does not exist [25], and thermal fluctuations are treated as the delta-correlated Gaussian noise. The origin of superdiffusion is attributed to the external noise which has a long-range temporal correlation such as a power-law decay. In a recent experiment [26], similar model has been employed to explain the superdiffusive behavior of colloidal particles dispersed in a nematic liquid crystal. In general, one may use such stochastic model for the description of superdiffusive feature of a tracer particle in an active bath [25]. However, it cannot be mapped into the OUP model, and thus, in practice, its applicability in modeling active bath seems limited. In this paper, we characterize the bath by an active noise which is considered here as a generalized O-U process (GOUP), or in other words, the noise is taken as the position coordinate of a fictitious Brownian particle trapped in a harmonic potential and driven by the fractional Gaussian noise in the overdamped limit. With this model, one can show that superdiffusive nature can be obtained in specific parametric regions. Moreover, it is of theoretical importance to generalize our previous results of Ref. [27] in a unified framework. In a recent theoretical study [28], a generalization of the OUP model has been proposed to describe the persistence of velocity fluctuations in the motion of self-propelled particles.

An important relation to characterizing nonequilibrium fluctuations is the large deviation principle (LDP) which suggests that the probability distribution function (PDF) of a scaled time-integrated observable (*e.g.*, work) decays exponentially with time T in the asymptotic limit [29, 30]. Mathematically speaking,

$$P(\omega) \approx e^{-T\,I(\omega)}, \tag{1}$$

where $P(\omega)$ is the PDF at a time T, of a dynamical observable Ω, defined by $\Omega[y(t)] = \frac{1}{T}\int_0^T dt\, y^\theta(t)$, $\theta = 1, 2, 3, \cdots$. Sometimes, T is called the speed, and $I(\omega)$ represents the rate function, given by $I(\omega) = \lim_{T\to\infty} -\frac{1}{T}\ln P(\omega)$. For $\theta = 1$, Ω can be related to the intensive work (*i.e.* work per unit time) done on a diffusive particle dragged by an external force, and it follows a normal LDP (1) [29]. However, for Ω with $\theta > 2$, an anomalous scaling arises at large T, *viz.* [31, 32]

$$P(\omega) \approx e^{-T^\rho\, I(\omega)}, \tag{2}$$

where $\rho < 1$, and the rate function is redefined as $I(\omega) = \lim_{T\to\infty} -\frac{1}{T^\rho}\ln P(\omega)$, with a reduced speed T^ρ. Recently, similar scaling behavior was observed in "Run-and-Tumble particles" model involving large fluctuations [33]. Usually, a system having long-range memory is associated with an LDP with a reduced speed [34]. Since the long-order persistence is included in our system through an active noise, we envisage an occurrence of the LDP of anomalous type, as discussed in Sec. 3.

From the energetic perspective, it is of great interest to understand the thermodynamic properties of a dynamical system, in particular, for the one driven out of equilibrium. On the single trajectory level, thermodynamic variables, such as work, heat or entropy production, are characterized by *conventional* fluctuation relations (FR) which relate probabilities in the forward and backward trajectories to an exponential function involving the ambient temperature [35, 36, 37]. But a system subjected to an external noise often violates the conventional FR [38]. Recently, the diffusion of a passive particle in an active bath has been studied experimentally and theoretically [9, 27, 3, 39], and it has been found that the conventional FR does not exist at transient times. But at the steady state, an (extended) FR emerges (see Eq. (30)) if the ambient temperature occurring in the exponent is replaced by a new, stationary quantity usually referred to as "effective temperature ".

Note that in all those studies, the dynamics is Fickian in the long time limit, and thus irrespective of the nature of active noises, the effective temperature is a unique, time-independent quantity which is given by the sum of two diffusivities in the thermal and active noise [3, 27, 10]. However, the system which executes an anomalous dynamics does not follow the conventional FR even with the modification just mentioned (*i.e.*, the extended version of FR). For example, a particle that is driven externally by a power-law correlated noise obeys an FR where the ratio of probabilities of a thermodynamic variable (*e.g.*, work) in the forward and reverse protocols is given by a stretched exponential [38, 40]. To differentiate such FR from a conventional one, it is referred to as *non-conventional* fluctuation relation. As announced earlier, here our system of interest is a passive Brownian particle diffusing in a GOUP bath, and we study work fluctuations of this system. For that, we consider two different protocols under which the work is performed on the particle. In Sec. 4, we investigate the work fluctuation relation applicable to our system. All the results are summarized in Sec. 5.

2. Dynamics

Here we consider that a passive Brownian particle is diffusing in an active bath at the ambient temperature $\mathbb{T}$. Apart from the thermal noise $\eta_T(t)$, it is subjected to an active noise $\chi_A(t)$. So its motion in the overdamped limit can be described by the Langevin equation,

$$\dot{x}(t) = \eta_T(t) + \chi_A(t). \tag{3}$$

Here $\eta_T(t)$ is the Gaussian white noise and it follows the fluctuation-dissipation theorem (FDT), *i.e.*, $\langle \eta_T(t)\eta_T(t')\rangle = 2D_T\delta(t-t')$. D_T denotes the thermal diffusivity and is related to the temperature $\mathbb{T}$ through the Einstein relation, $D_T = k_B\mathbb{T}/\zeta$. The drag coefficient ζ is included in the noise terms of Eq. (3), and k_B is taken as unity. So the energy and temperature are expressed in the unit of D_T. $\chi_A(t)$ is usually modeled as the Ornstein-Uhlenbeck process (OUP). Note that $\langle \eta_T(t_i)\chi_A(t_j)\rangle = 0$, $\forall\, i,\, j$, as both noises have different origins. Here we take $\chi_A(t)$ to be more general, namely, the generalized Ornstein-Uhlenbeck process (GOUP). It may be described by the following stochastic equation:

$$\int_{-\infty}^{t} dt'\, \mathcal{K}(t-t')\, \dot{\chi}_A(t') = -\frac{1}{\tau_A^\alpha}\chi_A(t) + \frac{1}{\tau_A^\alpha}\,\xi(t), \tag{4}$$

where the friction kernel is given by [41]

$$\mathcal{K}(t-t') = \frac{\Gamma(3-\alpha)}{\Gamma(1-\alpha)}\,|t-t'|^{-\alpha},$$

for $0 < \alpha \leq 1$. Here $\xi(t)$ is the zero-centered fractional Gaussian noise with power-law correlation function of the form

$$< \xi(t)\xi(t') >= 2D_A\mathcal{K}(t-t').$$

D_A represents the strength of correlation, and τ_A is the correlation time. Using Eq. (4) one can find the autocorrelation function of $\chi_A(t)$, and it is given by [42]

$$\langle \chi_A(t)\chi_A(t')\rangle = \frac{D_A}{\tau_A^\alpha}\,E_{\alpha,1}\left(-\frac{1}{\Gamma(3-\alpha)}\left(\frac{|t-t'|}{\tau_A}\right)^\alpha\right), \tag{5}$$

where $E_{\alpha,\beta}$ is the Mittag-Leffler function (MLF), defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}. \tag{6}$$

Notice that for $\alpha = 1$, $\langle \chi_A(t)\chi_A(t')\rangle = (D_A/\tau_A)\,e^{-|t-t'|/\tau_A}$, which corresponds to the usual OUP. For $0 < \alpha < 1$, the correlation function given by MLF (5) can be understood as the generalization of the exponential function, and hence, it corresponds to the generalized OUP. It is worth noting here that similar model has been extensively used in different contexts, in particular, to capture the subdiffusive nature of many biological processes, *e.g.*, see Refs. [42, 43, 44].

We compute the mean square displacement (MSD) using Eq. (3) (see AppendixA for details), and it reads

$$\left\langle x^2(T)\right\rangle = 2D_T T + \frac{2D_A}{\tau_A^\alpha}T^2\,E_{\alpha,3}\left(-\frac{1}{\Gamma(3-\alpha)}\left(\frac{T}{\tau_A}\right)^\alpha\right). \tag{7}$$

Applying the following relation for $z \to 0$,

$$E_{\alpha,\beta}(-z) \approx \frac{1}{\Gamma(\beta)} - \frac{z}{\Gamma(\alpha+\beta)} + \frac{z^2}{\Gamma(2\alpha+\beta)} + \cdots,$$

from Eq. (7) we get in the limit $T \ll \tau_A$, $\langle x^2(T)\rangle \approx 2D_T T + (D_A/\tau_A^\alpha)T^2$. For $D_T\,\tau_A^{\alpha-1} \approx D_A$, the motion is diffusive at short times. However, it suggests a ballistic behavior in the case $D_T \to 0$. At large z, the MLF asymptotically approaches to

$$E_{\alpha,\beta}(-z) \approx \frac{1}{z\Gamma(\beta-\alpha)}, \quad z \to \infty. \tag{8}$$

Using the above in Eq. (7), in the limit $T \gg \tau_A$, one obtains

$$\left\langle x^2(T)\right\rangle \approx 2D_T T + 2D_A T^{2-\alpha}. \tag{9}$$

For $0 < \alpha < 1$, the dynamics is non-Fickian, specifically superdiffusive at the long timescale. However, for $\alpha = 1$, the motion is Fickian.

Since the motion is subjected to two Gaussian noises, the PDF of displacement is also Gaussian over the entire time, and it can be written as

$$P(x,T) = \sqrt{\frac{1}{2\pi\,\langle x^2(T)\rangle}}\, e^{-\frac{x^2}{2\langle x^2(T)\rangle}}. \tag{10}$$

At large times, the distribution spreads linearly in time with a total diffusivity $D_T + D_A$ for $\alpha = 1$. But for $\alpha < 1$, it grows in time with a fixed power scaling, and the growth is higher for the smaller α, as can be seen in Fig. 1.

3. Expression for work distribution

Here we discuss work distributions of a particle diffusing in the GOUP bath, subjected to two different kinds of protocol.

3.1. Model I

Let us consider that the passive particle is pulled by a constant force $\mathcal{F}$, and so the potential experienced over a time period T is $U(x) = -\mathcal{F}x$. Its governing equation can be written as

$$\dot{x}(t) = -U'(x(t)) + \eta_T(t) + \chi_A(t). \tag{11}$$

Here one can define the mechanical work performed on the particle as [45]: $W_M[x(t)] = \int_{x(0)=x_0}^{x(T)=x_f} \mathcal{F}\,dx = \mathcal{F}x_f$, where, without loss of generality, the

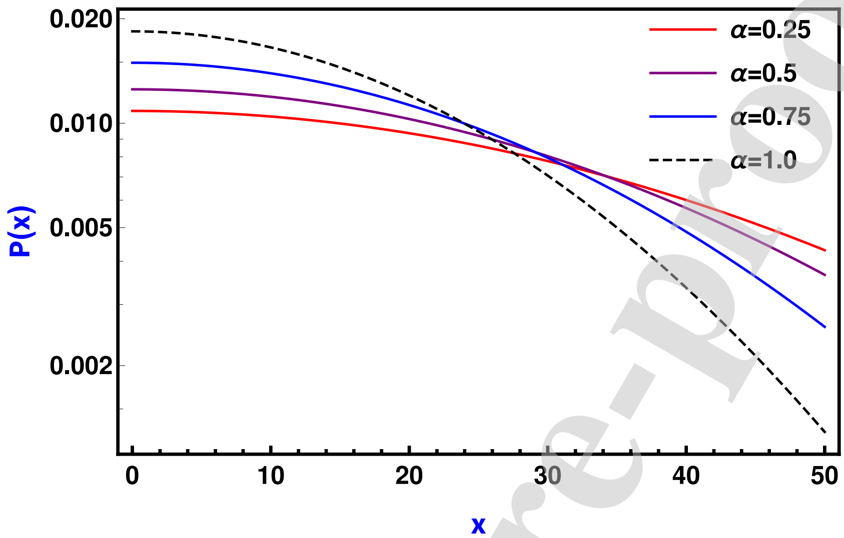

Figure 1: Logarithmic plot of the PDF as a function of displacement x at time $T = 10$ for different values of α. The parameters used here are (in their respective units): $D_T = 1$, $D_A = 25$, $\tau_A = 1$.

initial position is taken to be $x_0 = 0$. Now we can express the distribution of work using the definition of delta functional, as

$$\begin{aligned} P(W_M) &= \langle \delta\left(W_M - W_M[x(t)]\right)\rangle \\ &= \frac{1}{2\pi}\int d\psi\, e^{i\psi W_M}\left\langle e^{-i\psi\, W_M[x(t)]}\right\rangle. \end{aligned} \tag{12}$$

Here the ensemble-averaged quantity, $\left\langle e^{-i\psi\, W_M[x(t)]}\right\rangle$ corresponds to the characteristic function of work. By virtue of Eq. (11), it can be written as

$$\left\langle e^{-i\psi\, W_M[x(t)]}\right\rangle = \left\langle e^{-i\psi\,\mathcal{F}x_f}\right\rangle = e^{-i\psi\,\mathcal{F}^2 T}\,\left\langle e^{-i\psi\,\mathcal{F}\int_0^T dt\,\eta_T(t)}\right\rangle_{\eta_T(t)}\,\left\langle e^{-i\psi\,\mathcal{F}\int_0^T dt\,\chi_A(t)}\right\rangle_{\chi_A(t)}. \tag{13}$$

Here $\langle\cdots\rangle_\chi$ represents the ensemble average over all possible realizations of a noise χ. As $\eta_T(t)$ and $\chi_A(t)$ are both Gaussian in nature, their characteristic functionals can be easily calculated using the formula [46],

$$\left\langle e^{i\int_0^T dt\,q(t)\chi(t)}\right\rangle_{\chi(t)} = \exp\left[-\frac{1}{2}\int_0^T\int_0^T dt\,dt'\,q(t)\,\langle\chi(t)\chi(t')\rangle\,q(t')\right], \tag{14}$$

where $q(t)$ denotes the conjugate variable to a Gaussian noise $\chi(t)$. Thus, using Eq. (14) in Eq. (13), one gets

$$\left\langle e^{-i\psi\,\mathcal{F}x_f}\right\rangle = e^{-i\psi\,\mathcal{F}^2T}\;e^{-\mathcal{F}^2\psi^2\,D_T T}\,\exp\left[-\mathcal{F}^2\psi^2\,\frac{D_A}{\tau_A^\alpha}T^2\,E_{\alpha,3}\left(-\frac{1}{\Gamma(3-\alpha)}\left(\frac{T}{\tau_A}\right)^\alpha\right)\right]. \tag{15}$$

Substituting Eq. (15) in Eq. (12), we finally obtain the distribution of work, and it reads

$$P(W_M) = \sqrt{\frac{1}{2\pi\sigma^2_{W_M}}}\,e^{-\frac{(W_M-\overline{W}_M)^2}{2\sigma^2_{W_M}}}. \tag{16}$$

Here $\overline{W}_M$ is the average work given by $\overline{W}_M = \mathcal{F}^2T$, and the variance of work is

$$\sigma^2_{W_M} = 2D_T\,\mathcal{F}^2T + \frac{2D_A}{\tau_A^\alpha}\,\mathcal{F}^2T^2\,E_{\alpha,3}\left(-\frac{1}{\Gamma(3-\alpha)}\left(\frac{T}{\tau_A}\right)^\alpha\right). \tag{17}$$

Notice that the PDF (16) is equivalent to the scaled distribution of x, and thus it shares similar characteristics as the one discussed in Sec. 2.

A convenient way to rewrite Eq. (16) is by defining a non-dimensional work $\omega = W/\overline{W}$. At long timescales, the PDF of work can be approximated using Eq. (8), and it takes the form of

$$P(\omega) \approx e^{-T^\alpha\,\frac{\mathcal{F}^2}{4D_A}(\omega-1)^2}. \tag{18}$$

For $\alpha < 1$, it conforms to Eq. (2), indicating a long-tail behavior of the distribution. For $\alpha = 1$, $P(\omega)$ follows the linear scaling with time as given in Eq. (1).

3.2. Model II

Here we consider that the particle is confined in a harmonic trap of stiffness λ. Also it is assumed that the particle is initially in a thermal equilibrium with the bath at the temperature $\mathbb{T}$, and therefore, the initial distribution is given by

$$P(x_0) = \sqrt{\frac{\lambda}{2\pi D_T}}e^{-\frac{\lambda x_0^2}{2D_T}}. \tag{19}$$

To perform work on the particle, the center of the trap is dragged with a constant velocity v over a time period T. So the potential acting upon the particle is

$$U(x(t),t) = \frac{\lambda}{2}(x(t) - v\,t)^2, \tag{20}$$

and the performed (thermodynamic) work can be written as [45]

$$\begin{aligned} W[x(t)] &= \int_0^T \frac{\partial U(x(t),t)}{\partial t} dt = -\lambda v \int_0^T [x(t) - v\,t]\, dt \\ &= -\lambda v \int_0^T dt\, x(t) + \frac{\lambda}{2} v^2 T^2. \end{aligned} \tag{21}$$

With the help of Eqs. (11) and (20), the displacement after time T can be expressed as

$$x(T) = x_0\, e^{-\lambda T} + \lambda \int_0^T dt\, vt\, e^{-\lambda(T-t)} + \int_0^T dt\, \eta_T(t)\, e^{-\lambda(T-t)} + \int_0^T dt\, \chi_A(t)\, e^{-\lambda(T-t)}. \tag{22}$$

Now by virtue of Eqs. (21)-(22), we can write down the PDF of work as

$$\begin{aligned} P(W) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\psi\, e^{i\psi W} \left\langle e^{-i\psi W[x(t)]} \right\rangle \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\psi\, e^{i\psi W}\, e^{-i\psi \frac{\lambda}{2} v^2 T^2} \left\langle e^{i\psi\lambda v \int_0^T dt\, x(t)} \right\rangle \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\psi\, e^{i\psi W}\, e^{-i\psi \frac{v^2}{\lambda}(T\lambda - 1 + e^{-\lambda T})} \left\langle e^{i\psi v x_0 (1-e^{-\lambda T})} \right\rangle_{x_0} \\ &\quad \left\langle e^{iv\psi \int_0^T dt\, (1-e^{-\lambda(T-t)})\eta_T(t)} \right\rangle_{\eta_T(t)} \left\langle e^{iv\psi \int_0^T dt\, (1-e^{-\lambda(T-t)})\chi_A(t)} \right\rangle_{\chi_A(t)}. \end{aligned} \tag{23}$$

As usual, the bracket $\langle \cdots \rangle$ carries the similar meaning of averaging over all histories of the variables mentioned in the subscript. Using the initial distribution (19), the average over x_0 is calculated, and it reads

$$\left\langle e^{i\psi v x_0 (1-e^{-\lambda T})} \right\rangle_{x_0} = \int dx_0\, P(x_0) e^{i\psi v x_0 (1-e^{-\lambda T})} = e^{-\frac{D_T}{2\lambda}\psi^2 v^2 (1 - 2e^{-\lambda T} + e^{-2\lambda T})}. \tag{24}$$

With the help of Eq. (14), one can compute the ensemble averages over the Gaussian noises, and the results are

$$\left\langle e^{iv\psi\int_0^T dt\,(1-e^{-\lambda(T-t)})\eta_T(t)}\right\rangle_{\eta_T(t)} = e^{-D_T v^2\psi^2\int_0^T dt\,(1-e^{-\lambda(T-t)})^2}$$
$$= e^{-\frac{D_t}{2\lambda}v^2\psi^2(2\lambda T-3+4e^{-\lambda T}-e^{-2\lambda T})}, \qquad (25)$$

and

$$\left\langle e^{iv\psi\int_0^T dt\,(1-e^{-\lambda(T-t)})\chi_A(t)}\right\rangle_{\chi_A(t)}$$
$$= e^{-\frac{v^2\psi^2}{2}\int_0^T dt_1\int_0^T dt_2(1-e^{-\lambda(T-t_1)})\langle\chi(t_1)\chi(t_2)\rangle(1-e^{-\lambda(T-t_2)})}. \qquad (26)$$

Therefore by virtue of Eqs. (24)-(26), the PDF of work (23) can be rewritten as

$$P(W) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\psi\, e^{i\psi W}\, e^{-i\psi\frac{v^2}{\lambda}(T\lambda-1+e^{-\lambda T})-D_T\psi^2\frac{v^2}{\lambda}(T\lambda-1+e^{-\lambda T})}$$
$$\times\, e^{-\frac{v^2\psi^2}{2}\int_0^T dt_1\int_0^T dt_2(1-e^{-\lambda(T-t_1)})\langle\chi(t_1)\chi(t_2)\rangle(1-e^{-\lambda(T-t_2)})}. \qquad (27)$$

From Eq. (21), one can compute the average work which is given by

$$\overline{W} = \langle W[x(t)]\rangle = \frac{v^2}{\lambda}(T\lambda - 1 + e^{-\lambda T}). \qquad (28)$$

Now defining

$$\sigma_W^2 = 2D_T\overline{W}+v^2\int_0^T dt_1\int_0^T dt_2(1-e^{-\lambda(T-t_1)}) < \chi(t_1)\chi(t_2) > (1-e^{-\lambda(T-t_2)}),$$

Eq. (27) can be recast as

$$P(W) = \sqrt{\frac{1}{2\pi\sigma_W^2}}\, e^{-\frac{(W-\overline{W})^2}{2\sigma_W^2}}. \qquad (29)$$

So the distribution of work is Gaussian with a time-dependent variance whose behavior at different times are discussed in AppendixB. Like Eq. (18), the rescaled PDF follows an anomalous LDP for $0 < \alpha < 1$. For both models, the work distributions in the GOUP bath that we have found are exact and are one of the main results of this paper.

4. Fluctuation relation

For a diffusive system, the change in free energy associated with model I and II is zero. Therefore the conventional fluctuation relation (FR) exists if the distributions of positive and negative work are related via [35]

$$\frac{P(W)}{P(-W)} = e^{\frac{W}{\mathbb{T}_r}}. \tag{30}$$

In a thermal bath, $\mathbb{T}_r = \mathbb{T} \equiv D_T$, for any timescale. In our previous study [27], we have shown that for processes occurring in an active bath, an (extended) FR (30) holds only at the steady state with $\mathbb{T}_r = D_T + D_A$, and the dynamics is Fickian at the stationary limit, in contrast to the present case. Here we analyze FRs for the two models.

From Eq. (16), we can write

$$\ln\left[\frac{P(W_M)}{P(-W_M)}\right] = \frac{2\overline{W}_M}{\sigma^2_{W_M}} W_M = \frac{W_M}{D_r(T)}, \tag{31}$$

for model I. Here a nonequilibrium temperature at time T, denoted by $D_r(T)$, is defined as, $D_r(T) = D_T + \frac{D_A}{\tau_A^\alpha}\, T\, E_{\alpha,3}\left(-\frac{1}{\Gamma(3-\alpha)}\left(\frac{T}{\tau_A}\right)^\alpha\right)$. Similarly, for model II, the probability ratio of work can be expressed using Eq. (29), as

$$\ln\left[\frac{P(W)}{P(-W)}\right] = \frac{2\overline{W}}{\sigma^2_W} W = \frac{W}{D_r(T)}, \tag{32}$$

where $D_r(T)$ can be written explicitly as

$$D_r(T) = D_T + \frac{v^2\, D_A}{2\overline{W}\, \tau_A^\alpha}\int_0^T dt_1 \int_0^T dt_2 (1 - e^{-\lambda(T-t_1)})\,(1 - e^{-\lambda(T-t_2)}) \\ E_{\alpha,1}\left(-\frac{1}{\Gamma(3-\alpha)}\left(\frac{|t-t'|}{\tau_A}\right)^\alpha\right). \tag{33}$$

From Eqs. (31)-(32), it is evident that the system does not follow the conventional FR (30) at any transient time. Moreover, $D_r(T)$ does not have any definite scaling in T for any arbitrary time. It suggests that any kind of transient FR does not exist.

Here we consider the limit $T \gg \tau_A$, for the model I. Using the relation (8), $D_r(T)$ can be asymptotically written as

$$D_r(T) \approx D_T + D_A\, T^{1-\alpha}. \tag{34}$$

Notice that $D_r(T)$ has a definite power-law scaling with time T, and so one can define a renormalized temperature as: $\mathbb{T}_r = \lim_{T\gg\tau_A} D_r(T) = D_T + D_A\, T^{1-\alpha}$. Thus, it may be argued that the probability ratio at large T can be related to a stretched exponential through an FR of the following form:

$$\lim_{T\gg\tau_A} \ln\left[\frac{P(W_M)}{P(-W_M)}\right] = \frac{W_M}{D_T + D_A\, T^{1-\alpha}} \approx \frac{W_M}{D_A}\, T^{\alpha-1}. \tag{35}$$

Note that the FR (35) has been reported before for systems driven by long-correlated Gaussian noise [38], and it is usually termed as the non-conventional fluctuation relation.

For model II, the average work in the long-time limit (*i.e.*, $T \gg \tau_A$ and $T \gg \tau_R$) can be computed using Eq. (28), and it is given by $\overline{W} = v^2 T$. Now by virtue of Eq. (33) and (B.9), $D_r(T)$ can be approximated for $T \gg \tau_A$, as

$$\begin{aligned} &D_r(T) \approx \\ &D_T + D_A\, T^{1-\alpha} + \frac{D_A}{2T\,\tau_A^{\alpha}}\left[-\frac{2T^{1-\alpha}}{\lambda\,\gamma\,\Gamma(2-\alpha)} - \frac{2\lambda^{\alpha-2}}{\gamma+\lambda^{\alpha}} + \frac{\lambda^{\alpha-2}}{\gamma+\lambda^{\alpha}} - \frac{1}{\lambda^2}\,\frac{e^{-\lambda T}}{\gamma\, T^{\alpha}\Gamma(1-\alpha)}\right] \\ &\approx D_T + D_A\, T^{1-\alpha} + \mathcal{O}(T^{-\alpha}), \end{aligned} \tag{36}$$

which is the same as model I. Therefore, in general, one can conclude that for the non-Fickian dynamics (*i.e.*, $\langle x^2(T)\rangle \sim T^{2-\alpha}$), a non-conventional (extended) FR holds in the long-time limit with a renormalized temperature $(D + D_A\, T^{1-\alpha})$, where $0 < \alpha < 1$. This is one of the important results of this work.

For $\alpha = 1$, $\mathbb{T}_r$ can be conceptualized in terms of an effective temperature denoted as $\mathbb{T}_{\text{eff}}$. By relating fluctuations and dissipation in the nonequilibrium regime, one can usually define a frequency-dependent effective temperature. Its large-T behavior can be obtained in the low frequency limit. For more details on this topic, the reader is referred to Refs. [47, 48, 49]. From the equipartition theorem, one can also define the effective temperature for large T, as [2]

$$\mathbb{T}_{\text{eff}} = \lambda\,\left\langle x^2\right\rangle_{\text{har}}, \tag{37}$$

where $\langle x^2\rangle_{\text{har}}$ is the long-time MSD of a Brownian particle trapped in a harmonic potential of the form: $U(x) = \lambda x^2/2$. By taking $v = 0$ in Eq. (22), and then using the correlation of $\eta_T(t)$ and $\chi_A(t)$, one can compute the MSD.

In the limit $T \to \infty$, it is given by

$$\left\langle x^2 \right\rangle_{\text{har}} = \frac{D_T}{\lambda} + \frac{D_A}{\tau_A^\alpha} \frac{\lambda^{\alpha-2}}{\gamma + \lambda^\alpha}.$$

The last term in the right-hand side (RHS) is obtained by approximating Eq. (B.4) to Eq. (B.8). Using the above equation in Eq. (37) and putting the value of γ (Eq. (A.6)), one can find the effective temperature to be

$$\mathbb{T}_{\text{eff}} = D_T + D_A \frac{\lambda^{\alpha-1}}{\frac{1}{\Gamma(3-\alpha)} + \lambda^\alpha \tau_A^\alpha}. \tag{38}$$

For $\alpha = 1$, $\mathbb{T}_{\text{eff}} = D_T + \frac{D_A}{1+\lambda\tau_A}$, as reported earlier in Ref. [39].

Now consider the limit where the correlation timescale of the active noise is very small, *i.e.*, the limit: $\tau_A \to 0$. So using Eq. (38), the effective temperature can be computed as

$$\mathbb{T}_{\text{eff}} = D_T + D_A \Gamma(3-\alpha) \lambda^{\alpha-1}. \tag{39}$$

For $0 < \alpha < 1$, $\mathbb{T}_{\text{eff}}$ has a similar scaling form as the one obtained in Ref. [25]. However, it cannot be realized as $\mathbb{T}_r$, and consequently, any conventional FR does not hold. Notice that for $\alpha = 1$, $\mathbb{T}_{\text{eff}} = D_T + D_A$, which can be viewed as $\mathbb{T}_r$. Note that the same expression of $\mathbb{T}_{\text{eff}}$ for the OUP model was reported in some recent studies [3, 27, 10]. Therefore, a (extended) conventional FR is satisfied with a constant renormalized temperature for the case $\alpha = 1$, in agreement with our previous result [27].

5. Summary

In this paper, we have found the superdiffusive behavior of a passive particle diffusing in an active bath characterized by a generalized Ornstein-Uhlenbeck process (GOUP). The distribution of displacement as well as performed work is Gaussian at all times, and at long times, its width spreads with time following a power-law. In what follows, the large deviation principle does not hold in its normal form, rather the distribution decays in time with a reduced speed as given by the stretched exponential. For such models, any kinds of conventional fluctuation relations (FR) do not exist, even by defining an effective temperature. But in the long-time limit, it satisfies a non-conventional FR of a stretched exponential form. This work generalizes the most common model for an active bath - the Ornstein-Uhlenbeck process, thus claiming its potential applicability in the future.

6. Acknowledgements

The author thanks the anonymous referees for useful comments and pointing out some relevant references. The article is dedicated to Prof. K. L. Sebastian for his seminal contribution in theoretical chemistry.

AppendixA. Computation of Eq. (7)

Here we compute the mean square displacement (MSD) of a free particle in a GOUP bath. From Eq. (3), the MSD can be expressed as ($x_0 = 0$)

$$< x^2(T) >= \int_0^T dt_1 \int_0^T dt_2 \, \langle \eta(t_1)\eta(t_2)\rangle + \int_0^T dt_1 \int_0^T dt_2 \, \langle \chi(t_1)\chi(t_2)\rangle \, . \tag{A.1}$$

Using the correlation function of white noise $\eta_T(t)$, it can be easily shown that

$$\int_0^T dt_1 \int_0^T dt_2 \, \langle \eta(t_1)\eta(t_2)\rangle = 2D_T T. \tag{A.2}$$

Using Eq. (5), the second term in the RHS of Eq. (A.1) can be rewritten as

$$\int_0^T dt_1 \int_0^T dt_2 \, \langle \chi(t_1)\chi(t_2)\rangle = \frac{D_A}{\tau_A^\alpha} \int_0^T dt_1 \int_0^T dt_2 \, E_{\alpha,1}\left(-\frac{1}{\Gamma(3-\alpha)} \left(\frac{|t_1 - t_2|}{\tau_A} \right)^\alpha \right). \tag{A.3}$$

The Laplace transform of Mittag-Leffler function from t to s domain is given by

$$\int_0^\infty dt \, e^{-st} \, t^{\beta-1} E_{\alpha,\beta}\left(\pm\gamma t^\alpha\right) \equiv \mathcal{L}\left[t^{\beta-1} E_{\alpha,\beta}\left(\pm\gamma t^\alpha\right)\right] = \frac{s^{\alpha-\beta}}{s^\alpha \mp \gamma}. \tag{A.4}$$

Applying the above equation in Eq. (A.3), one can obtain

$$\mathcal{L}\left[\int_0^T dt_1 \int_0^T dt_2 \, E_{\alpha,1}\left(-\frac{1}{\Gamma(3-\alpha)} \left(\frac{|t_1 - t_2|}{\tau_A} \right)^\alpha \right) \right] = 2\frac{s^{\alpha-3}}{s^\alpha + \gamma}. \tag{A.5}$$

Here $\mathcal{L}[f(x)]$ represents the Laplace transform of a function $f(x)$, *viz.* $\mathcal{L}[f(x)] = \int_0^\infty dx \, e^{-s\,x} f(x)$, and γ is defined as

$$\gamma = \frac{\tau_A^{-\alpha}}{\Gamma(3-\alpha)}. \tag{A.6}$$

On performing the inverse Laplace transform of Eq. (A.5), we get

$$\int_0^T dt_1 \int_0^T dt_2\, E_{\alpha,1}\left(-\frac{1}{\Gamma(3-\alpha)}\left(\frac{|t_1-t_2|}{\tau_A}\right)^\alpha\right) = 2T^2 E_{\alpha,3}(-\gamma\, T^\alpha). \quad \text{(A.7)}$$

Therefore, substituting the results of Eqs. (A.2) and (A.7) in Eq. (A.1), we obtain the MSD as given in Eq. (7).

AppendixB. Finding σ_W^2 in Eq. (29)

Here, σ_W^2 in Eq. (29) is computed by finding its different terms. First, we perform the Laplace transform of one term as follows:

$$\begin{aligned}
&\mathcal{L}\left[\int_0^T dt_1 \int_0^T dt_2\, e^{-\lambda(T-t_1)}\, E_{\alpha,1}\left(-\frac{1}{\Gamma(3-\alpha)}\left(\frac{|t_1-t_2|}{\tau_A}\right)^\alpha\right)\right] \\
&= \mathcal{L}\Big[\int_0^T dt_1 \int_0^{t_1} dt_2\, e^{-\lambda(T-t_1)}\, E_{\alpha,1}\left(-\gamma(t_1-t_2)^\alpha\right) \\
&\quad + \int_0^T dt_2 \int_0^{t_2} dt_1\, e^{-\lambda(T-t_1)}\, E_{\alpha,1}\left(-\gamma(t_2-t_1)^\alpha\right)\Big] \\
&= \frac{1}{s+\lambda}\,\frac{1}{s}\,\frac{s^{\alpha-1}}{s^\alpha+\gamma} + \frac{1}{s+\lambda}\,\frac{1}{s}\,\frac{(s+\lambda)^{\alpha-1}}{(s+\lambda)^\alpha+\gamma} \\
&= \sum_{k=0}^{\infty}(-1)^k\gamma^k\,\frac{s^{-2-\alpha\,k}}{s+\lambda} + \sum_{k=0}^{\infty}(-1)^k\gamma^k\,\frac{(s+\lambda)^{-2-\alpha\,k}}{s+\lambda-\lambda}.
\end{aligned} \quad \text{(B.1)}$$

In the final step of the above, it is written as a series expansion, and thence the inverse Laplace transform is performed to obtain

$$\begin{aligned}
&\int_0^T dt_1 \int_0^T dt_2\, e^{-\lambda(T-t_1)}\, E_{\alpha,1}\left(-\frac{1}{\Gamma(3-\alpha)}\left(\frac{|t_1-t_2|}{\tau_A}\right)^\alpha\right) \\
&= \sum_{k=0}^{\infty}(-1)^k\gamma^k\, T^{2+\alpha k}\, E_{1,3+\alpha k}(-\lambda T) + \sum_{k=0}^{\infty}(-1)^k\gamma^k\, e^{-\lambda T}\, T^{2+\alpha k}\, E_{1,3+\alpha k}(\lambda T).
\end{aligned} \quad \text{(B.2)}$$

The Laplace transform of another term in σ_W^2 can be written as

$$\begin{aligned}
&\mathcal{L}\left[\int_0^T dt_1 \int_0^T dt_2\, e^{-\lambda(T-t_1)-\lambda(T-t_2)}\, E_{\alpha,1}\left(-\gamma(t_1-t_2)^\alpha\right)\right] \\
&= \frac{2}{s(s+2\lambda)}\,\frac{(s+\lambda)^{\alpha-1}}{(s+\lambda)^\alpha+\gamma} \\
&= \frac{1}{\lambda}\,\frac{1}{s}\,\frac{(s+\lambda)^{\alpha-1}}{(s+\lambda)^\alpha+\gamma} - \frac{1}{\lambda}\,\frac{1}{s+2\lambda}\,\frac{(s+\lambda)^{\alpha-1}}{(s+\lambda)^\alpha+\gamma} \\
&= \frac{1}{\lambda}\sum_{k=0}^{\infty}(-1)^k\gamma^k\,\frac{(s+\lambda)^{-1-\alpha k}}{s+\lambda-\lambda} - \frac{1}{\lambda}\sum_{k=0}^{\infty}(-1)^k\gamma^k\,\frac{(s+\lambda)^{-1-\alpha k}}{s+\lambda+\lambda}.
\end{aligned} \tag{B.3}$$

Like the previous case, the above equation, after doing the inverse Laplace transform, can be written as a sum of Mittag-Leffler functions as given below.

$$\begin{aligned}
&\int_0^T dt_1 \int_0^T dt_2\, e^{-\lambda(T-t_1)-\lambda(T-t_2)}\, E_{\alpha,1}\left(-\gamma(t_1-t_2)^\alpha\right) \\
&= \frac{1}{\lambda}\sum_{k=0}^{\infty}(-1)^k\gamma^k\, e^{-\lambda T}\, T^{1+\alpha k}\, E_{1,2+\alpha k}(\lambda T) - \frac{1}{\lambda}\sum_{k=0}^{\infty}(-1)^k\gamma^k\, e^{-\lambda T}\, T^{1+\alpha k}\, E_{1,2+\alpha k}(-\lambda T).
\end{aligned} \tag{B.4}$$

So using Eqs. (A.7), (B.2) and (B.4), $\sigma_W^2(T)$ can be rewritten as

$$\begin{aligned}
&\sigma_W^2(T) = 2D_T\overline{W} + v^2\int_0^T dt_1\int_0^T dt_2 (1-e^{-\lambda(T-t_1)}) < \chi(t_1)\chi(t_2) > (1-e^{-\lambda(T-t_2)}) \\
&= 2D_T\frac{v^2}{\lambda}(T\lambda - 1 + e^{-\lambda T}) + \frac{D_A}{\tau_A^\alpha}v^2\Big[2T^2E_{\alpha,3}(-\gamma\, T^\alpha) - 2\sum_{k=0}^{\infty}(-1)^k\gamma^k\, T^{2+\alpha k}\, E_{1,3+\alpha k}(-\lambda T) \\
&- 2\sum_{k=0}^{\infty}(-1)^k\gamma^k\, e^{-\lambda T}\, T^{2+\alpha k}\, E_{1,3+\alpha k}(\lambda T) + \frac{1}{\lambda}\sum_{k=0}^{\infty}(-1)^k\gamma^k\, e^{-\lambda T}\, T^{1+\alpha k}\, E_{1,2+\alpha k}(\lambda T) \\
&- \frac{1}{\lambda}\sum_{k=0}^{\infty}(-1)^k\gamma^k\, e^{-\lambda T}\, T^{1+\alpha k}\, E_{1,2+\alpha k}(-\lambda T)\Big].
\end{aligned} \tag{B.5}$$

In the following section, we analyze two limiting cases of $\sigma_W^2(T)$.

Let us consider the limit $T \ll \tau_R < \tau_A$, where the relaxation time of the particle is $\tau_R = 1/\lambda$. Now using $E_{\alpha,\beta}(z) \approx 1/\Gamma(\beta)$ in this limit, σ_W^2 can be approximated up to second-order as

$$\sigma_W^2(T) \approx D_T\,\frac{v^2\,T^2}{\lambda} + D_A\frac{v^2\,T^2}{\tau_A^\alpha},$$

which suggests a ballistic behavior.

Now we consider the limit $T \gg \tau_A$. In the limit $z \to \infty$, the Mittag-Leffler function behaves asymptotically as [50]

$$E_{\alpha,\beta}(z) \sim \frac{z^{\frac{1-\beta}{\alpha}}}{\alpha}\, e^{z^{1/\alpha}} - \frac{1}{z\Gamma(\beta-\alpha)}. \tag{B.6}$$

Using the above relation, Eq. (B.2) can be approximated in the limit $T \gg \tau_A$ as follows:

$$\begin{aligned}
&\sum_{k=0}^{\infty}(-1)^k\gamma^k\, T^{2+\alpha k}\, E_{1,3+\alpha k}(-\lambda T) + \sum_{k=0}^{\infty}(-1)^k\gamma^k\, e^{-\lambda T}\, T^{2+\alpha k}\, E_{1,3+\alpha k}(\lambda T) \\
&\approx \frac{1}{\lambda}\sum_{k=0}^{\infty}(-1)^k\gamma^k\, \frac{T^{1+\alpha k}}{\Gamma(2+\alpha k)} + \frac{1}{\lambda^2}\sum_{k=0}^{\infty}(-1)^k\frac{\gamma^k}{\lambda^{\alpha k}} \\
&\approx \frac{T}{\lambda}\, E_{\alpha,2}(-\gamma T^\alpha) + \frac{\lambda^{\alpha-2}}{\gamma+\lambda^\alpha} \\
&\approx \frac{T^{1-\alpha}}{\lambda\,\gamma\,\Gamma(2-\alpha)} + \frac{\lambda^{\alpha-2}}{\gamma+\lambda^\alpha}.
\end{aligned} \tag{B.7}$$

Similarly, Eq. (B.4) can be approximated as

$$\begin{aligned}
&\frac{1}{\lambda}\sum_{k=0}^{\infty}(-1)^k\gamma^k\, e^{-\lambda T}\, T^{1+\alpha k}\, E_{1,2+\alpha k}(\lambda T) - \frac{1}{\lambda}\sum_{k=0}^{\infty}(-1)^k\gamma^k\, e^{-\lambda T}\, T^{1+\alpha k}\, E_{1,2+\alpha k}(-\lambda T) \\
&\approx \frac{\lambda^{\alpha-2}}{\gamma+\lambda^\alpha} - \frac{1}{\lambda^2}\,\frac{e^{-\lambda T}}{\gamma\, T^\alpha\Gamma(1-\alpha)}.
\end{aligned} \tag{B.8}$$

Therefore, with the help of Eq. (9) and Eqs. (B.7)-(B.8), $\sigma_W^2(T)$ in Eq. (B.5) can be written in the long-time limit, as

$$\begin{aligned}
\sigma_W^2(T) &\approx 2D_T\, v^2 T + 2D_A\, v^2 T^{2-\alpha} \\
&\quad + \frac{D_A}{\tau_A^\alpha}\, v^2\left[-\frac{2T^{1-\alpha}}{\lambda\,\gamma\,\Gamma(2-\alpha)} - \frac{2\lambda^{\alpha-2}}{\gamma+\lambda^\alpha} + \frac{\lambda^{\alpha-2}}{\gamma+\lambda^\alpha} - \frac{1}{\lambda^2}\,\frac{e^{-\lambda T}}{\gamma\, T^\alpha\Gamma(1-\alpha)}\right] \\
&\approx 2D_T\, v^2 T + 2D_A\, v^2 T^{2-\alpha} + \mathcal{O}(T^{1-\alpha}).
\end{aligned} \tag{B.9}$$

[1] A. Einstein, Investigations on the Theory of the Brownian Movement (Courier Corporation, 1956).

[2] C. Maggi, M. Paoluzzi, N. Pellicciotta, A. Lepore, L. Angelani, and R. Di Leonardo, Phys. Rev. Lett. **113**, 238303 (2014).

[3] S. Chaki and R. Chakrabarti, Physica A **511**, 302 (2018).

[4] K. Goswami and K. L. Sebastian, J. Stat. Mech.: Theory Exp **2019**, 083501 (2019).

[5] L. Dabelow, S. Bo, and R. Eichhorn, Phys. Rev. X **9**, 021009 (2019), URL `https://link.aps.org/doi/10.1103/PhysRevX.9.021009`.

[6] S. Ye, P. Liu, F. Ye, K. Chen, and M. Yang, Soft Matter **16**, 4655 (2020).

[7] D. Mizuno, C. Tardin, C. F. Schmidt, and F. C. MacKintosh, Science **315**, 370 (2007).

[8] C. Maggi, M. Paoluzzi, L. Angelani, and R. Di Leonardo, Sci. Rep **7**, 17588 (2017).

[9] A. Argun, A.-R. Moradi, E. Pinçe, G. B. Bagci, A. Imparato, and G. Volpe, Phys. Rev. E **94**, 062150 (2016).

[10] S. Chaki and R. Chakrabarti, Soft Matter **16**, 7103 (2020), URL `http://dx.doi.org/10.1039/D0SM00711K`.

[11] D. Mandal, K. Klymko, and M. R. DeWeese, Phys. Rev. Lett **119**, 258001 (2017).

[12] É. Fodor, C. Nardini, M. E. Cates, J. Tailleur, P. Visco, and F. van Wijland, Phys. Rev. Lett **117**, 038103 (2016).

[13] S. Das, G. Gompper, and R. G. Winkler, New J. Phys **20**, 015001 (2018).

[14] A. Kumari, P. S. Pal, A. Saha, and S. Lahiri, Phys. Rev. E **101**, 032109 (2020), URL `https://link.aps.org/doi/10.1103/PhysRevE.101.032109`.

[15] P. Dieterich, R. Klages, R. Preuss, and A. Schwab, Proc. Natl. Acad. Sci. U.S.A. **105**, 459 (2008), ISSN 0027-8424, `https://www.pnas.org/content/105/2/459.full.pdf`, URL `https://www.pnas.org/content/105/2/459`.

[16] E. Hatta, J. Phys. Chem. B **112**, 8571 (2008), pMID: 18582101, https://doi.org/10.1021/jp801764z, URL https://doi.org/10.1021/jp801764z.

[17] S. S. Rogers, T. A. Waigh, and J. R. Lu, Biophys. J. **94**, 3313 (2008), ISSN 0006-3495, URL http://www.sciencedirect.com/science/article/pii/S0006349508704879.

[18] L. Bruno, V. Levi, M. Brunstein, and M. A. Desposito, Phys. Rev. E **80**, 011912 (2009).

[19] J. F. Reverey, J.-H. Jeon, H. Bao, M. Leippe, R. Metzler, and C. Selhuber-Unkel, Scientific reports **5**, 11690 (2015).

[20] S. Khan, A. M. Reynolds, I. E. G. Morrison, and R. J. Cherry, Phys. Rev. E **71**, 041915 (2005), URL https://link.aps.org/doi/10.1103/PhysRevE.71.041915.

[21] X. Trepat, G. Lenormand, and J. J. Fredberg, Soft Matter **4**, 1750 (2008), URL http://dx.doi.org/10.1039/B804866E.

[22] A. D. Viñales and M. A. Despósito, Phys. Rev. E **75**, 042102 (2007), URL https://link.aps.org/doi/10.1103/PhysRevE.75.042102.

[23] M. A. Despósito and A. D. Viñales, Phys. Rev. E **77**, 031123 (2008), URL https://link.aps.org/doi/10.1103/PhysRevE.77.031123.

[24] M. A. Despósito, C. Pallavicini, V. Levi, and L. Bruno, Physica A **390**, 1026 (2011).

[25] M. A. Despósito, Phys. Rev. E **84**, 061114 (2011), URL https://link.aps.org/doi/10.1103/PhysRevE.84.061114.

[26] J. M. Pagès, J. Ignés-Mullol, and F. Sagués, Phys. Rev. Lett **122**, 198001 (2019).

[27] K. Goswami, Physica A **525**, 223 (2019).

[28] F. J. Sevilla, R. F. Rodríguez, and J. R. Gomez-Solano, Phys. Rev. E **100**, 032123 (2019), URL https://link.aps.org/doi/10.1103/PhysRevE.100.032123.

[29] H. Touchette, Phys. Rep. **478**, 1 (2009), ISSN 0370-1573, URL `http://www.sciencedirect.com/science/article/pii/S0370157309001410`.

[30] H. Touchette, Physica A **504**, 5 (2018).

[31] D. Nickelsen and H. Touchette, Phys. Rev. Lett. **121**, 090602 (2018), URL `https://link.aps.org/doi/10.1103/PhysRevLett.121.090602`.

[32] B. Meerson, Phys. Rev. E **100**, 042135 (2019), URL `https://link.aps.org/doi/10.1103/PhysRevE.100.042135`.

[33] G. Gradenigo and S. N. Majumdar, J. Stat. Mech.: Theory Exp **2019**, 053206 (2019).

[34] R. L. Jack and R. J. Harris, Phys. Rev. E **102**, 012154 (2020), URL `https://link.aps.org/doi/10.1103/PhysRevE.102.012154`.

[35] G. E. Crooks, Phys. Rev. E **60**, 2721 (1999), URL `https://link.aps.org/doi/10.1103/PhysRevE.60.2721`.

[36] C. Jarzynski, Phys. Rev. Lett. **78**, 2690 (1997), URL `https://link.aps.org/doi/10.1103/PhysRevLett.78.2690`.

[37] C. Jarzynski, Phys. Rev. E **56**, 5018 (1997), URL `https://link.aps.org/doi/10.1103/PhysRevE.56.5018`.

[38] A. V. Chechkin and R. Klages, J. Stat. Mech. **2009**, L03002 (2009), URL `https://doi.org/10.1088%2F1742-5468%2F2009%2F03%2Fl03002`.

[39] K. Goswami, Phys.Rev.E **99**, 012112 (2019).

[40] A. V. Chechkin, F. Lenz, and R. Klages, Journal of Statistical Mechanics: Theory and Experiment **2012**, L11001 (2012), URL `https://doi.org/10.1088%2F1742-5468%2F2012%2F11%2Fl11001`.

[41] H. Qian, in Processes with Long-Range Correlations (Springer, 2003), pp. 22–33.

[42] S. Kou and X. S. Xie, Phys. Rev. Lett. **93**, 180603 (2004).

[43] I. Goychuk, Adv. Chem. Phys **150**, 187 (2012).

[44] K. L. Sebastian, J. Chem. Phys **151**, 025101 (2019).

[45] U. Seifert, Rep. Prog. Phys **75**, 126001 (2012), URL `http://stacks.iop.org/0034-4885/75/i=12/a=126001`.

[46] N. G. Van Kampen, Stochastic processes in physics and chemistry, vol. 1 (Elsevier, 1992).

[47] S. K. Nandi and N. S. Gov, Soft Matter **13**, 7609 (2017), URL `http://dx.doi.org/10.1039/C7SM01648D`.

[48] R. R. Netz, J. Chem. Phys **148**, 185101 (2018), `https://doi.org/10.1063/1.5020654`, URL `https://doi.org/10.1063/1.5020654`.

[49] S. Chaki and R. Chakrabarti, Physica A **530**, 121574 (2019), ISSN 0378-4371, URL `http://www.sciencedirect.com/science/article/pii/S0378437119309306`.

[50] H. J. Haubold, A. M. Mathai, and R. K. Saxena, J. Appl. Math. **2011** (2011).

Highlights

Re: PHYSA-2046

- Superdiffusion of a Brownian particle in an active bath.
- Non-conventional work fluctuation relation
- Anomalous large deviation

CRediT author statement

The corresponding author is responsible for ensuring that the descriptions are accurate and agreed by all authors.

Declaration of interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.