Math 415. Final Exam. December 16, 2017

Full Name: ____________________

Net ID: ________________________

Discussion Section: ___________

- There are 30 problems worth 5 points each.
- You must not communicate with other students.
- No books, notes, calculators, or electronic devices allowed.
- This is a 180 minute exam.
- Do not turn this page until instructed to.
- Fill in the answers on the scantron form provided. Also circle your answers on the exam itself.
- Hand in both the exam and the scantron.
- On the scantron make sure you bubble in **your name, your UIN and your NetID**.
- There are several different versions of this exam.
- Please double check that you correctly bubbled in your answer on the scantron. It is the answer on the scantron that counts!
- Good luck! Happy holidays!

Fill in the following information on the scantron form:

On the first page of the scantron bubble in **your name, your UIN and your NetID**!
On the back of the scantron bubble in the following:

95. D
96. C

1. (5 points) Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 4 \\ 0 & 1 & 1 \end{bmatrix}$. Which one of the following sets is a basis for $\mathrm{Nul}(A)$?

(A) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$

(B) ★ $\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}$

(C) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$

(D) None of the answers.

(E) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \right\}$

Solution. First, we find the reduced echelon form of A, which is

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus x_3 is the only independent variable so the null space is one dimensional. Since $x_1 = -2x_3$ and $x_2 = -x_3$ it follows that $\mathrm{Nul}\, A = \mathrm{Span} \left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} \right\}$.

---

2. (5 points) Suppose A is a 3×6 matrix

$$A = \begin{bmatrix} 1 & * & * & 0 & * & * \\ 0 & * & * & 1 & * & * \\ 1 & * & * & -2 & * & * \end{bmatrix}$$

whose column space is

$$\mathrm{Col}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \right\}.$$

Which of the following is a basis for $\mathrm{Nul}(A^T)$?

(A) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

(B) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}$

(C) No vectors, since $\dim \mathrm{Nul}(A^T) = 0$.

(D) ★ $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$

(E) There is not enough information to determine the answer.

Solution. By the FTLA, $\mathrm{Nul}(A^T)$ is the orthogonal complement of $\mathrm{Col}(A)$, thus $\mathrm{Nul}(A^T)$ is one dimensional and spanned by the vector $\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ since the orthogonality conditions give the equations $x_1 = -x_3$ and $x_2 = 2x_3$.

---

3. (5 points) Let $\mathbb{P}^2$ be the vector space of polynomials of degree at most 2. Which of the following subsets of $\mathbb{P}^2$ is linearly ***dependent***?

(A) $\{1-t^2, 1+t^2, t\}$

(B) ★ $\{t^2-1, 1-t, t^2-t\}$

(C) $\{1, t, t^2\}$

(D) $\{t^2+t, 1+t, 1\}$

Solution. Since $(t^2-1)+(1-t)-(t^2-t)=0$, this set is linearly dependent.

---

4. (5 points) Suppose A is a 5×4 matrix with rank 4. Which of the following statements is FALSE?

(A) ★ $A\mathbf{x} = \mathbf{b}$ has exactly one solution for each $\mathbf{b} \in \mathbb{R}^5$

(B) $\text{Nul}(A) = \{\mathbf{0}\}$

(C) A has four pivots.

(D) The columns of A are linearly independent

Solution. If A has no free variables, then the only solution to $A\mathbf{x} = \mathbf{0}$ is the trivial solution. This implies that the columns of A are linearly independent and $\text{Nul}(A) = \{\mathbf{0}\}$. Furthermore, A has 4 pivots, so $\text{rank}(A) = 4$. Because A only has 3 columns, $\text{Col}(A)$ cannot be $\mathbb{R}^5$. This means there exists some $\mathbf{b} \in \mathbb{R}^5$ such that $A\mathbf{x} = \mathbf{b}$ has no solutions.

---

5. (5 points) Consider the two matrices

$$A = \begin{bmatrix} 0 & 3 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Which of the following statements is correct?

(A) $\mathrm{Col}(A) = \mathrm{Col}(B)$ and $\mathrm{Nul}(A) \neq \mathrm{Nul}(B)$

(B) ★ $\mathrm{Col}(A) = \mathrm{Col}(B)$ and $\mathrm{Nul}(A) = \mathrm{Nul}(B)$

(C) $\mathrm{Col}(A) = \mathrm{Col}(B)$ and $\mathrm{Nul}(A) \neq \mathrm{Nul}(B)$

(D) $\mathrm{Col}(A) \neq \mathrm{Col}(B)$ and $\mathrm{Nul}(A) = \mathrm{Nul}(B)$

Solution. Since $\mathrm{Col}(A)$ and $\mathrm{Col}(B)$ are both equal to $\mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$, we have $\mathrm{Col}(A) = \mathrm{Col}(B)$. Since A and B are equivalent, $\mathrm{Nul}(A) = \mathrm{Nul}(B)$.

---

6. (5 points) Suppose A is a 2×2 matrix with eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with eigenvalue 2 and eigenvector $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ with eigenvalue 1. If $\mathbf{w} = \mathbf{v}_1 + 2\mathbf{v}_2$, what is $A^3\mathbf{w}$?

(A) ★ $\begin{bmatrix} 6 \\ 10 \end{bmatrix}$

(B) Not enough information to tell

(C) $\begin{bmatrix} 8 \\ 2 \end{bmatrix}$

(D) $\begin{bmatrix} -8 \\ 3 \end{bmatrix}$

(E) $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$

Solution. $A^3\mathbf{w} = (\lambda_1)^3\mathbf{v}_1 + 2(\lambda_2)^3\mathbf{v}_2 = 8\mathbf{v}_1 + 2\mathbf{v}_2 = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$.

---

7. (5 points) Let A be a 3×3 matrix with rows $\mathbf{R}_1, \mathbf{R}_2$ and $\mathbf{R}_3$. Suppose the following row operations bring A to the identity matrix:

$$A \xrightarrow{R_1 \leftrightarrow R_3} A_1 \xrightarrow{R_1 \to \frac{1}{2} R_1} A_2 \xrightarrow{R_3 \to R_3 + 4R_1} I_3$$

where I_3 is the 3×3 identity matrix. Then,

(A) ★ $\det A = -2$

(B) $\det A = -5$

(C) $\det A = \frac{1}{2}$

(D) $\det A = -\frac{1}{2}$

(E) $\det A = 2$

Solution. Of course, $\det(I_3) = 1$. Then also $\det(A_2) = 1$, since this row operation does not change determinants. Then $\det(A_1) = 2$, (row 1 in A_1 is 2 times row 1 in A_2) and finally $\det(A) = -2$ because of the row swap.

---

8. (5 points) The 3×3 matrix A is reduced to the echelon matrix U using the following row operations (in the given order):

1. $R_2 \to R_2 + R_1$;

2. $R_3 \to R_3 - 2R_1$;

3. $R_3 \to R_3 - R_2$.

What is the matrix L in the decomposition $A = LU$?

(A) ★ $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$

(B) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

(C) None of the other answers.

(D) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$

(E) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

Solution. Let E_1, E_2, E_3 denote the elementary matrices corresponding the the first, second and third operations above, when multiplying a matrix from the left. Explicitly, $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$, and $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$. Then $E_3E_2E_1A = U$, an upper triangular matrix. Therefore, $A = E_1^{-1}E_2^{-1}E_3^{-1}U$, multiplying both sides by the inverse matrices.

---

9. (5 points) Consider the following subspace V of $\mathbb{R}^4$:

$$\operatorname{Span}\left\{ \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\0\\1\end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix}0\\1\\1\\1\end{bmatrix} \right\}.$$

What is the projection matrix for orthogonal projection onto V?

(A) $\begin{bmatrix}1 & -1 & 0 & 1\\ -1 & 2 & 1 & 0\\ 0 & 1 & 1 & 1\\ 1 & 0 & 1 & 2\end{bmatrix}$

(B) $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

(C) $\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$

(D) $\frac{1}{3}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

(E) ★ $\frac{1}{3}\begin{bmatrix}1 & -1 & 0 & 1\\ -1 & 2 & 1 & 0\\ 0 & 1 & 1 & 1\\ 1 & 0 & 1 & 2\end{bmatrix}$

Solution. The projection matrix is $QQ^T = \frac{1}{3}\begin{bmatrix}1 & -1 & 0 & 1\\ -1 & 2 & 1 & 0\\ 0 & 1 & 1 & 1\\ 1 & 0 & 1 & 2\end{bmatrix}$.

---

10. (5 points) The matrix $A = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$ is reduced to the identity matrix $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ using the following row operations (in the given order):

(1) $R_2 \to R_2 + R_1$,

(2) $R_2 \leftrightarrow R_4$.

(3) $R_3 \to R_3 - R_4$

Which of the following matrices is A^{-1}?

(A) None of the other answers.

(B) ★ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(D) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(E) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Solution. Let $E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ be the elementary matrix corresponding to $R2 \to$

$R2 + R1$, and $E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ be the elementary matrix corresponding to $R2 \leftrightarrow R4$, and let $E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ be the elementary matrix corresponding to $R3 \to R3 - R4$. Then $E_3E_2E_1A = I$. Thus $E_3E_2E_1$ is the inverse of A.

---

11. (5 points) Consider the vector space $\mathbb{P}^3$ of polynomials of degree at most 3. Let V be the set of polynomials $p(t)$ of degree at most 3 such that $p(0) = p(1) = 0$. This set V is a subspace of $\mathbb{P}^3$. What is the dimension of V?

(A) 4

(B) ★ 2

(C) 0

(D) 1

(E) 3

Solution. Let $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$. Then $p(0) = a_0$ and $p(1) = a_0 + a_1 + a_2 + a_3$. Therefore, if $p(1) = p(0) = 0$, then $a_0 = 0$ and $a_1 + a_2 + a_3 = 0$. Therefore in this situation, $a_0 = 0$ and $a_3 = -a_1 - a_2$. Thus if $p(t)$ is in V, then

$$p(t) = a_1(t - t^3) + a_2(t^2 - t^3).$$

Thus $V = \text{Span}(t - t^3, t^2 - t^3)$. Thus dimension of V is 2.

---

12. (5 points) Let $\mathcal{B} := (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ be an orthonormal basis of $\mathbb{R}^3$ such that $\mathbf{b}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Let $\mathbf{v} = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$ and let c_1, c_2, c_3 be scalars such that $\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3$. What is c_2?

(A) $\frac{5}{\sqrt{2}}$

(B) ★ $\frac{3}{\sqrt{2}}$

(C) There is not enough information to determine c_2

(D) -$\frac{3}{\sqrt{2}}$

(E) -3

Solution. From page 4, Lecture 18, since the basis is orthonormal we obtain that $c_2 = \mathbf{v} \cdot \mathbf{b_2} = \frac{3}{\sqrt{2}}$.

---

13. (5 points) Let A be an **invertible** $n \times n$ matrix. Consider the following four statements:

I. A has an LU decomposition.

II. $\det(A) \neq 0$.

III. The reduced row echelon form of A is the identity matrix.

IV. $\mathrm{Nul}(A) = \{\mathbf{0}\}$.

Which of the above statements are always true?

(A) None of the statements are true.

(B) ★ II, III and IV only.

(C) All statements are true.

(D) II and III only

(E) I, II, and III only.

Solution. The matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is invertible, but does not have an LU-decomposition. If A is invertible, the rank is n, and there are no free variables. If the rank is n then the dimension of $\mathrm{Nul}(A)$ is zero. Thus $\mathrm{Nul}(A)$ must be $\{\mathbf{0}\}$. Also, if the rank is n then the RREF of A must be the identity. The determinant of matrix is non-zero if and only if the matrix is invertible.

---

14. (5 points) For which values of h is $\begin{bmatrix} 1 \\ 0 \\ h \end{bmatrix}$ in the column space of

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 0 & 5 \end{bmatrix}?$$

(A) Only for $h = 2$

(B) ★ Only for $h = 1$

(C) Only for $h = 0$

(D) For no values of h

(E) For all values of h

Solution. To answer this question, we have to check for which values of h the system $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ h \end{bmatrix}$ is consistent. So we form an augmented matrix and row reduce:

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ -1 & 3 & 0 \\ 0 & 5 & h \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 0 & 5 & h \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & h-1 \end{array}\right]$$

This system is only consistent if $h - 1 = 0$, i.e. if $h = 1$.

---

15. (5 points) What is the determinant of the matrix $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 3 \\ 0 & 0 & 0 & 2 \\ 1 & 0 & 1 & 3 \end{bmatrix}$?

(A) None of the other answers.

(B) -2

(C) 4

(D) -4

(E) ★ 2

Solution.

$$\det(A) = (-1) \cdot 2 \begin{vmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = -2 \left(\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} \right) = (-2)\,(1-2) = 2.$$

---

16. (5 points) Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$. Suppose that Q is a 3×2-matrix with orthonormal columns and R is an invertible upper triangular matrix such that $A = QR$. Which of the following matrices can be R?

(A) $\begin{bmatrix} \sqrt{2} & 3/\sqrt{2} \\ 0 & 3/\sqrt{2} \end{bmatrix}$.

(B) $\begin{bmatrix} \sqrt{2} & 3/\sqrt{2} \\ 0 & 2/\sqrt{6} \end{bmatrix}$.

(C) ★ $\begin{bmatrix} \sqrt{2} & 3/\sqrt{2} \\ 0 & 3/\sqrt{6} \end{bmatrix}$.

(D) $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{6} \\ 0 & 2/\sqrt{6} \end{bmatrix}$

(E) None of the other answers.

Solution. From Gram-Schmidt, we have $\boldsymbol{q_1} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$ and $\boldsymbol{q_2} = \frac{\boldsymbol{a_2} - (\boldsymbol{a_2} \cdot \boldsymbol{q_1})\boldsymbol{q_1}}{\|\boldsymbol{a_2} - (\boldsymbol{a_2} \cdot \boldsymbol{q_1})\boldsymbol{q_1}\|} = \begin{bmatrix} 1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$. And $R = Q^T A = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 3/\sqrt{2} \\ 0 & 3/\sqrt{6} \end{bmatrix}$

---

17. (5 points) Let A be an $n \times n$ matrix. Consider the following statements:

(T1) If A is not the zero matrix, then A^2 is also not the zero matrix.

(T2) If A is invertible, then A^2 is also invertible.

Which of the statements are ALWAYS TRUE?

(A) Neither Statement (T1) nor Statement (T2).

(B) Both Statement (T1) and Statement (T2).

(C) ★ Only Statement (T2).

(D) Only Statement (T1).

Solution. (T2) is true because if A is invertible, then $\det(A) \neq 0$. Therefore $\det(A^2) = \det(A)^2 \neq 0$.

(T1) is false. Take $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $A^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

---

18. (5 points) Let A be a diagonalizable 3×3 matrix with only two distinct eigenvalues. Which of the following statements is FALSE?

(A) The matrix $5A$ is diagonalizable.

(B) The matrix A has an eigenbasis.

(C) ★ There are no more than two linearly independent eigenvectors of A.

(D) There is an eigenvalue of A for which the corresponding eigenspace is spanned by two linearly independent eigenvectors.

Solution. Since A is diagonalizable, it has an eigenbasis. Since we know that the matrix has an eigenbasis but only two distinct eigenvalues, it follows that one of those eigenvalues is a repeated eigenvalue and its corresponding eigenspace must be spanned by two eigenvectors. The vectors in an eigenbasis are always linearly independent eigenvectors. Lastly, since an eigenbasis exists, the matrix A is diagonalizable and so $5A$ should be diagonalizable too.

---

19. (5 points) Let A be a 4×4 matrix with eigenvectors

$$\begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\-1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$$

corresponding to the eigenvalues $\lambda = -2, 0, 1, 3$. What is A?

(A) $\begin{bmatrix}-2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 3\end{bmatrix}$

(B) ★ $\begin{bmatrix}1 & 1 & 0 & 0\\ 1 & -1 & 0 & 0\\ 1 & 0 & -1 & 1\\ 0 & 0 & 1 & 1\end{bmatrix} \begin{bmatrix}-2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 3\end{bmatrix} \begin{bmatrix}1 & 1 & 0 & 0\\ 1 & -1 & 0 & 0\\ 1 & 0 & -1 & 1\\ 0 & 0 & 1 & 1\end{bmatrix}^{-1}$

(C) $\begin{bmatrix}1 & 1 & 0 & 0\\ 1 & -1 & 0 & 0\\ 1 & 0 & -1 & 1\\ 0 & 0 & 1 & 1\end{bmatrix}^{-1} \begin{bmatrix}-2 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 3\end{bmatrix} \begin{bmatrix}1 & 1 & 0 & 0\\ 1 & -1 & 0 & 0\\ 1 & 0 & -1 & 1\\ 0 & 0 & 1 & 1\end{bmatrix}$

(D) Not enough information to determine A

(E) $\begin{bmatrix}1 & 1 & 0 & 0\\ 1 & -1 & 0 & 0\\ 1 & 0 & -1 & 1\\ 0 & 0 & 1 & 1\end{bmatrix}$

Solution. If P is the eigenvector matrix and D is the eigenvalues diagonal matrix, then $A = PDP^{-1}$.

---

20. (5 points) Suppose there are two fictional cities Champaign and Urbana. Every year, 40% of the residents in Champaign move to Urbana (and the other 60% remains in Champaign), and 50% of the residents in Urbana move to Champaign (and the other 50% remains in Urbana). What is the steady state distribution of the population in Champaign and Urbana?

(A) 4/10 of the population lives in Champaign and 6/10 of the population lives in Urbana.

(B) None of the other answers.

(C) 4/9 of the population lives in Champaign and 5/9 of the population lives in Urbana.

(D) 6/10 of the population lives in Champaign and 4/10 of the population lives in Urbana.

(E) ★ 5/9 of the population lives in Champaign and 4/9 of the population lives in Urbana.

Solution. This system has the corresponding Markov matrix $A = \begin{bmatrix} .6 & .5 \\ .4 & .5 \end{bmatrix}$ where the first row represents Champaign and the second Urbana. To find the steady state solution, we look for the probability eigenvector corresponding to eigenvalue $\lambda = 1$.

$A - 1I = \begin{bmatrix} -.4 & .5 \\ .4 & -.5 \end{bmatrix}$ so the appropriate vector is $\begin{bmatrix} 5/9 \\ 4/9 \end{bmatrix}$.

---

21. (5 points) Let $\mathbb{P}^2$ be the vector space of polynomials of degree at most 2. Let T be a linear transformation given by

$$\begin{aligned} T : \mathbb{P}^2 &\rightarrow \mathbb{P}^2 \\ T(p(t)) &= \frac{dp(t)}{dt} + 2p(t) \end{aligned}$$

Consider the standard basis $\mathcal{E} = \{1, t, t^2\}$. What is $T_{\mathcal{E}\mathcal{E}}$?

(A) ★ $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

(B) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

(D) None of the other answers

(E) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

Solution. It follows that $T_{\mathcal{E},\mathcal{E}} = [T(1)_{\mathcal{E}}, T(t)_{\mathcal{E}}, T(t^2)_{\mathcal{E}}] = [(2)_{\mathcal{E}}, (1+2t)_{\mathcal{E}}, (2t+2t^2)_{\mathcal{E}}] = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

---

22. (5 points) Let A be the following 3×3-matrix

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

What are the eigenvalues of A?

(A) ★ $\lambda = 1, i, -i$

(B) $\lambda = 1, -1$

(C) None of the other answers

(D) $\lambda = 0, i, -i$

(E) $\lambda = 0, 1, -1$

Solution.

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 0 & -1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = (1-\lambda) \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = (1-\lambda)(\lambda^2 + 1).$$

Thus the eigenvalues of A are $1, i, -i$.

---

23. (5 points) Let A be a **symmetric** $n \times n$ matrix, that is $A^T = A$. Let $\mathbf{v}_1, \mathbf{v}_2$ in $\mathbb{R}^n$ be such that $\mathbf{v}_1$ is an eigenvector of A to eigenvalue λ_1 and $\mathbf{v}_2$ is an eigenvector of A to eigenvalue λ_2. Consider the following two statements:

(T1) If $\lambda_1 \neq \lambda_2$, then $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent.

(T2) If $\lambda_1 \neq \lambda_2$, then $\mathbf{v}_1, \mathbf{v}_2$ are orthogonal to each other.

Are these statements always correct?

(A) ★ Statement T1 and Statement T2 are correct.

(B) Neither Statement T1 nor Statement T2 is correct.

(C) Only Statement T1 is correct.

(D) Only Statement T2 is correct.

Solution. (T1) is true, see Lecture 30 Theorem 1. (T2) fails for arbitrary matrix, but is true for symmetric matrices by Lecture 34.

---

24. (5 points) Consider the following statements:

(T1) If a consistent system of linear equations has no free variables, then it has a unique solution.

(T2) If the augmented matrix of a linear system has two identical rows, the linear system has infinitely many solutions.

Which of the statements are TRUE?

(A) ★ Only (T1) is correct.

(B) Both (T1) and (T2) are correct.

(C) Neither (T1) nor (T2) is correct.

(D) Only (T2) is correct.

Solution. If a system of linear equations is consistent and has no free variables, then it has a unique solution. If the augmented matrix of a linear system has two identical rows, the linear system could have 0, 1 or infinitely many solutions. For example, the system

$$\begin{aligned} x_1 - x_2 &= 0 \\ x_2 &= 1 \\ x_2 &= 1 \end{aligned}$$

has only one solution.

---

25. (5 points) Consider the following subset of the vector space of 2×2 matrices $M_{2\times 2}$:

$$D = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \ : \ 0 \text{ is an eigenvalue of } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\}$$

It can be shown that D is NOT a subspace of $M_{2\times 2}$. Which of the following tests does D **fail to satisfy**? (Select all that apply.)

I. contains the zero matrix

II. closed under addition

III. closed under scalar multiplication

(A) II. and III. only

(B) ★ II. only

(C) III. only

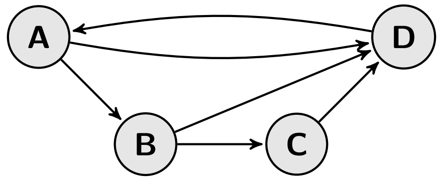
(D) I., II., and III.

(E) I. and III. only

Solution. The zero matrix has eigenvalue 0. Therefore I. holds. For example both $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ have 0 as an eigenvalue, but their sum $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ does not. Thus II. fails. Let c be a scalar and A be a 2×2-matrix with eigenvalue 0. Then cA is also a matrix with eigenvalue 0. Thus III. holds.

---

26. (5 points) Which matrix is the PageRank matrix (that is the Markov matrix that is used when calculating the Google PageRank) for the following system of webpages?



(A) None of the other answers.

(B) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \end{bmatrix}$

(C) ★ $\begin{bmatrix} 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 & 0 \end{bmatrix}$

(D) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 & 0 \end{bmatrix}$

(E) $\begin{bmatrix} -1 & 0 & 0 & 1 \\ \frac{1}{2} & -1 & 0 & 0 \\ 0 & \frac{1}{2} & -1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 & -1 \end{bmatrix}$

Solution. The pagerank matrix records for each column for site X links to sites Y in the corresponding rows.

27. (5 points) Let A be a $m \times n$ matrix, where $m < n$. Which one of the statements is FALSE?

(A) $\dim \mathrm{Col}(A^T) + \dim \mathrm{Nul}(A) = n$.

(B) $\dim \mathrm{Col}(A) = \dim \mathrm{Col}(A^T)$.

(C) $\dim \mathrm{Col}(A) + \dim \mathrm{Nul}(A^T) = m$.

(D) ★ $\dim \mathrm{Nul}(A) = \dim \mathrm{Nul}(A^T)$.

Solution. Let r be the rank of matrix A. By Fundamental theorem of linear algebra, $\dim \mathrm{Col}(A) = \dim \mathrm{Col}(A^T) = r$, $\dim \mathrm{Nul}(A) = n - r$ and $\dim \mathrm{Nul}(A^T) = m - r$.

---

28. (5 points) Let $V = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ and let $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. What is the orthogonal projection of $\mathbf{b}$ onto $V^{\perp}$?

(A) $\frac{1}{3} \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$

(B) None of the other answers.

(C) ★ $\frac{1}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

(D) $\frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

(E) $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Solution. From FTLA, we know that $V^{\perp} = \text{Nul}(A^T)$, where $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$. Thus we have $V^{\perp} = span\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$. So $Proj_{V^{\perp}} b = \dfrac{b \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

---

29. (5 points) Consider the matrix

$$A = \begin{bmatrix} 0 & 3 & -1 & -1 & 0 \\ 0 & 0 & 4 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

What are the dimensions of the null space and the left null space of A?

(A) ★ $\dim \mathrm{Nul}(A) = 3, \dim \mathrm{Nul}(A^T) = 1$

(B) $\dim \mathrm{Nul}(A) = 2, \dim \mathrm{Nul}(A^T) = 2$

(C) $\dim \mathrm{Nul}(A) = 3, \dim \mathrm{Nul}(A^T) = 2$

(D) $\dim \mathrm{Nul}(A) = 1, \dim \mathrm{Nul}(A^T) = 3$

(E) $\dim \mathrm{Nul}(A) = 2, \dim \mathrm{Nul}(A^T) = 1$

Solution. The dimension of a null space is the number of free variables. The matrix A has rank 2 and 5 columns, so 3 free variables. The matrix A^T has also rank 2 but only 3 columns, so a single free variable.

---

30. (5 points) The least squares solution of

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

is

(A) $\begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$.

(B) ★ $\begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix}$.

(C) $\begin{bmatrix} \frac{1}{3} \\ \frac{-2}{3} \\ \frac{1}{3} \end{bmatrix}$.

(D) none of the other answers.

(E) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Solution. We solve for the equation

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}^T \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} \hat{\boldsymbol{x}} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}^T \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

Multiplying out we get

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \hat{\boldsymbol{x}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$
