Math 415. Final Exam. December 16, 2017

Full Name: ____________________

Net ID: ________________________

Discussion Section: ____________

- There are 31 problems worth 5 points each.
- You must not communicate with other students.
- No books, notes, calculators, or electronic devices allowed.
- This is a 180 minutes exam.
- Do not turn this page until instructed to.
- Fill in the answers on the scantron form provided. Also circle your answers on the exam itself.
- Hand in both the exam and the scantron.
- On the scantron make sure you bubble in **your name, your UIN and your NetID**.
- There are several different versions of this exam.
- Please double check that you correctly bubbled in your answer on the scantron. It is the answer on the scantron that counts!
- Good luck!

Fill in the following information on the scantron form:

On the first page of the scantron bubble in **your name, your UIN and your NetID**!
On the back of the scantron bubble in the following:

95. D
96. C

1. (5 points) Let A be a 3×5-matrix such that

$$\mathrm{Col}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}.$$

Then the dimension of $\mathrm{Col}(A^T)$ is

(A) 1

(B) 3

(C) 4

(D) 0

(E) ★ 2

Solution. The rank of A is the same as the rank of A^T, so also dimension of the row space is 2.

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2. (5 points) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 8 \end{bmatrix}$. Let L be a lower triangular 3×3-matrix and U be an upper triangular matrix such that $A = LU$. Which of the following matrices can be L?

(A) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

(B) None of the other answers

(C) $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$

(D) ★ $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$

(E) $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

Solution.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 4 \\ 1 & 4 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{bmatrix}$$

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3. (5 points) Which of the following is a basis of Span $\left(\begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 4 \end{bmatrix}\right)$?

(A) ★ $\begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$

(B) $\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$

(C) $\begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 4 \end{bmatrix}$

(D) $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$

Solution. $(-1)\begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \\ 4 \end{bmatrix}$, but the other two vectors are linearly independent.

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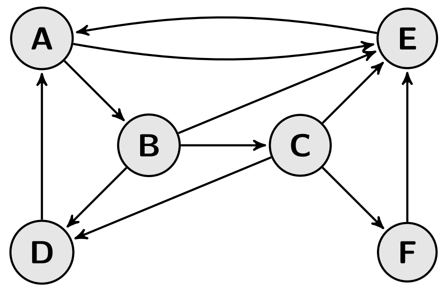
4. (5 points) Which matrix is the PageRank matrix for the following system of webpages?



(A) ★ $\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 \end{bmatrix}$

(B) $\begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

(C) $\begin{bmatrix} -1 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & -1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & -1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$

(D) $\begin{bmatrix} -1 & 0 & 0 & 1 & 1 & 0 \\ \frac{1}{2} & -1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & -1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & -1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{3} & -1 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & -1 \end{bmatrix}$

(E) None of the other answers.

Solution. The pagerank matrix records for each column for site X links to sites Y in the corresponding rows.

5. (5 points) Let A be an $\ell \times m$ matrix such that for every $\boldsymbol{b}$ in $\mathbb{R}^\ell$, the equation $A\boldsymbol{x} = \boldsymbol{b}$ has a solution. What does this statement imply about the relative size of ℓ and m?

(A) $\ell = m$

(B) nothing (ℓ and m can be any positive integers)

(C) $\ell \geq m$

(D) ★ $\ell \leq m$

Solution. If for every $\boldsymbol{b}$ in $\mathbb{R}^\ell$ the equation $A\boldsymbol{x} = \boldsymbol{b}$ has a solution, then $\mathrm{Col}(A) = \mathbb{R}^\ell$. Therefore the columns of A span $\mathbb{R}^\ell$. Hence there must be at least ℓ columns.

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6. (5 points) Consider the following two statements:

(T1) Every linearly independent set of vectors in a vector space V is a basis of V or can be extended to a basis of V.

(T2) If $\boldsymbol{v_1}, \ldots, \boldsymbol{v_n}$ are linearly independent vectors in a vector space V, then $\dim(V) \geq n$.

Then:

(A) Only Statement T2 is correct.

(B) Neither Statement T1 nor Statement T2 is correct.

(C) Only Statement T1 is correct.

(D) ★ Statement T1 and Statement T2 are correct.

Solution. (T2) is correct, because every set of linearly independent vectors can be extended to basis of V. For (T1), see Theorem 1 of Lecture 13.

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7. (5 points) Which of the following subsets of $\mathbb{R}^2$ is a subspace?

$$U = \{ \begin{bmatrix} a \\ b \end{bmatrix} \ : \ ab \geq 0\}$$
$$V = \{ \begin{bmatrix} 3a \\ -b \end{bmatrix} \ : \ a - b = 0\}$$
$$W = \{ \begin{bmatrix} a \\ 2b+1 \end{bmatrix} \ : \ a + b = 0\}$$

(A) V and W, but not U

(B) ★ V only

(C) U, V and W

(D) W only

(E) None of them

Solution. U is not closed under addition, W does not contain the zero vector, but $V = \text{Span}\left(\begin{bmatrix} 3 \\ -1 \end{bmatrix}\right)$.

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8. (5 points) Let A be an $n \times n$ matrix, let D be a diagonal matrix and let P be an invertible matrix such that $A = PDP^{-1}$. Which of the following statements is always true?

(A) $\mathrm{Nul}(A) = \mathrm{Nul}(D)$

(B) $\dim \mathrm{Nul}(A) \neq \dim \mathrm{Nul}(D)$

(C) None of the other answers

(D) ★ $\dim \mathrm{Nul}(A) = \dim \mathrm{Nul}(D)$

Solution. We check that $\dim \mathrm{Nul}(A) = \dim \mathrm{Nul}(D)$. Let $\mathbf{v}$ in $\mathbb{R}^n$. Since P is invertible, $A\mathbf{v} = 0$ if and only if $DP^{-1}\mathbf{v} = 0$. Thus $\mathbf{v}$ is in $\mathrm{Nul}(A)$ if and only if $P^{-1}\mathbf{v}$ is in $\mathrm{Nul}(D)$.

Let $\mathbf{v}_1, \ldots, \mathbf{v}_d$ be a basis of $\mathrm{Nul}(A)$. It is left to check that $P^{-1}\mathbf{v}_1, \ldots, P^{-1}\mathbf{v}_d$ is a basis of $\mathrm{Nul}(D)$.

First observe that $v_i = PP^{-1}\mathbf{v}_i$. Thus $P^{-1}\mathbf{v}_1, \ldots, P^{-1}\mathbf{v}_d$ are linearly independent by Worksheet 7 Problem 7(3). Thus it is left to check that $P^{-1}\mathbf{v}_1, \ldots, P^{-1}\mathbf{v}_d$ span $\mathrm{Nul}(D)$.

Let $\mathbf{z}$ in $\mathrm{Nul}(D)$. Then $A(P\mathbf{z}) = PDP^{-1}P\mathbf{z} = PD\mathbf{z} = P\mathbf{0} = \mathbf{0}$. Thus $P\mathbf{z}$ is in $\mathrm{Nul}(A)$. Therefore there are scalars $c_1, \ldots, c_d$ such that

$$P\mathbf{z} = c_1\mathbf{v}_1 + \cdots + c_d\mathbf{v}_d.$$

Then

$$\mathbf{z} = P^{-1}P\mathbf{z} = P^{-1}(c_1\mathbf{v}_1 + \cdots + c_d\mathbf{v}_d) = c_1P^{-1}\mathbf{v}_1 + \cdots + c_dP^{-1}\mathbf{v}_d.$$

Thus $P^{-1}\mathbf{v}_1, \ldots, P^{-1}\mathbf{v}_d$ span $\mathrm{Nul}(D)$.

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9. (5 points) Consider the following two statements:

(T1) If a vector space V has dimension d, then every set of d linearly independent vectors $\boldsymbol{v_1}, \ldots, \boldsymbol{v_d}$ in V forms a basis of V.

(T2) Every linearly independent set of vectors in a vector space V is a basis of V or can be extended to a basis of V.

Then:

(A) Only Statement T2 is correct.

(B) ★ Statement T1 and Statement T2 are correct.

(C) Neither Statement T1 nor Statement T2 is correct.

(D) Only Statement T1 is correct.

Solution. For (T1), see Theorem 1 of Lecture 13. For (T2), see Section 3 of Lecture 13.

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10. (5 points) Let V be a subspace of $\mathbb{R}^n$, $n > 0$ and let P be the projection matrix of the projection onto V. Which of the following statements is not always true?

(A) $\mathrm{Nul}(P) = V^{\perp}$.

(B) $\mathrm{rank}(P) = \dim V$.

(C) $\mathrm{Col}(P) = V$.

(D) ★ $\mathrm{Col}(P^T) = V^{\perp}$.

Solution. For any projection matrix P, $P^T = P$ (for instance P has an orthonormal eigenbasis). So $\mathrm{Col}(P^T) = \mathrm{Col}(P) = V$ and this is never equal to $V^{\perp}$.

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11. (5 points) Let A be a 5×4 matrix. Let a_1, a_2, a_3, a_4 be the columns of A and suppose that a_1, a_2, a_3 are linearly independent and $a_4 = a_1 - a_2 + a_3$. Which of the following is a basis of $\mathrm{Nul}(A)$?

(A) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

(B) None of the other answers

(C) a_1, a_2, a_3

(D) a_4

(E) ★ $\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

Solution. A has rank 3, four columns, so $\dim \mathrm{Nul}(A) = 1$. Now $1a_1 - 1a_2 + 1a_3 - 1a_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = A \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$, so that $\begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$ is a basis for $\mathrm{Nul}(B)$.

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12. (5 points) Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ for which

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} \frac{x+y}{2} \\ \frac{x+y}{2} \\ z \end{bmatrix}.$$

What is the matrix $T_{\mathcal{B}\mathcal{B}}$ that represents T with respect to the basis

$$\mathcal{B} := \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

(A) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(B) ★ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(C) $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(D) None of the other answers.

(E) $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Solution. Note that $Tb_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = b_1$, $Tb_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0$, $Tb_3 = b_3$. Thus $T_{\mathcal{B}\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

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13. (5 points) Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ be 4 row vectors of length 4 (so that the transpose $\mathbf{a}_i^T$ belongs to $\mathbb{R}^4$). Let

$$A = \begin{bmatrix} - & \mathbf{a}_1 & - \\ - & \mathbf{a}_2 & - \\ - & \mathbf{a}_3 & - \\ - & \mathbf{a}_4 & - \end{bmatrix}$$

be the 4×4 matrix with these vectors as rows. Assume $\det(A) = 3$. What is the determinant of the matrix

$$B = \begin{bmatrix} -\mathbf{a}_1 \\ 3\mathbf{a}_2 - 2\mathbf{a}_3 \\ 3\mathbf{a}_3 + \mathbf{a}_4 \\ \mathbf{a}_3 \end{bmatrix}$$

(A) None of the other answers is correct.

(B) ★ $\det(B) = 9$.

(C) $\det(B) = 27$.

(D) $\det(B) = -9$.

(E) $\det(B) = -27$.

Solution.

$$\det(B) = \det \begin{bmatrix} -\mathbf{a}_1 \\ 3\mathbf{a}_2 \\ \mathbf{a}_4 \\ \mathbf{a}_3 \end{bmatrix} = -3\det \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_4 \\ \mathbf{a}_3 \end{bmatrix} = 3\det(A) = 9$$

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14. (5 points) Let A be an $n \times n$ matrix, and let $\boldsymbol{v}_1, \boldsymbol{v}_2$ in $\mathbb{R}^n$ be such that $\boldsymbol{v}_1$ is an eigenvector of A to eigenvalue λ_1 and $\boldsymbol{v}_2$ is an eigenvector of A to eigenvalue λ_2. Consider the following two statements:

(T1) If $\lambda_1 \neq \lambda_2$, then $\boldsymbol{v}_1, \boldsymbol{v}_2$ are linearly independent.

(T2) If $\lambda_1 \neq \lambda_2$, then $\boldsymbol{v}_1, \boldsymbol{v}_2$ are orthogonal to each other.

Are these statements always correct?

(A) Neither Statement T1 nor Statement T2 is correct.

(B) Only Statement T2 is correct.

(C) Statement T1 and Statement T2 are correct.

(D) ★ Only Statement T1 is correct.

Solution. (T1) is true, see Lecture 30 Theorem 1. (T2) fails. For example, look at Example 7 in Lecture 30. The eigenvectors in that example are not orthogonal.

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15. (5 points) Consider the following bases:

$$\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix} \text{ of } \mathbb{R}^2, \text{ and } \quad \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \text{ of } \mathbb{R}^3.$$

With respect to these bases the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ is represented by the matrix

$$\begin{bmatrix}1 & 1\\0 & 1\\0 & 1\end{bmatrix}.$$

What is $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$?

(A) ★ $\begin{bmatrix}0\\-1\\1\end{bmatrix}$

(B) $\begin{bmatrix}1\\0\\0\end{bmatrix}$

(C) none of the other answers

(D) $\begin{bmatrix}1\\1\\1\end{bmatrix}$

(E) $\begin{bmatrix}0\\1\\1\end{bmatrix}$

Solution. $T\begin{bmatrix}1\\0\end{bmatrix} = T(-\begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix}) = -\begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\1\\1\end{bmatrix} = -b_2 + b_3$

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16. (5 points) Let $\mathbb{P}_2$ be the vector space of polynomials of degree at most 2. Consider the following sets of polynomials in $\mathbb{P}_2$.

$$\begin{aligned}\mathcal{A} &= \{1, t, t+1\} \\ \mathcal{B} &= \{1, t, t^2\} \\ \mathcal{C} &= \{1, 2t, t^2, t^2 - t\}.\end{aligned}$$

Which of these sets are linearly independent?

(A) $\mathcal{A}$ only.

(B) ★ $\mathcal{B}$ only.

(C) $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$.

(D) $\mathcal{B}$ and $\mathcal{C}$ only.

(E) None of them.

Solution. $\mathcal{A}$ is obviously dependent. $\mathcal{B}$ is the standard basis of $\mathbb{P}_2$, hence linearly independent. $\mathcal{C}$ is linearly dependent, since $t^2 - t = t^2 + \frac{1}{2}2t$.

17. (5 points) Let A be 2×2-matrix with eigenvalues $\frac{1}{2}$ and $\frac{1}{3}$. What can you say about $\lim_{k\to\infty} A^k$?

(A) $\lim_{k\to\infty} A^k$ does not exist.

(B) Not enough information to say anything

(C) $\lim_{k\to\infty} A^k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(D) ★ $\lim_{k\to\infty} A^k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(E) $\lim_{k\to\infty} A^k = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$

Solution.

$$A^k = V \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}^k V^{-1} \to \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

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18. (5 points) Let A be a 3×5-matrix such that

$$\mathrm{Col}(A) = \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}.$$

Then the dimension of $\mathrm{Nul}(A)$ is

(A) 4

(B) ★ 3

(C) 2

(D) 1

(E) 0

Solution. The rank of A is 2, so the dimension of the null space is # of columns minus the rank, is $5 - 2 = 3$.

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19. (5 points) Consider the following two subsets of the vector space of 2×2 matrices.

$$V_0 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;:\; a = 0 \right\}, \quad V_1 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;:\; 0 \text{ is an eigenvalue of } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\}$$

Then:

(A) V_0 and V_1 are subspaces.

(B) ★ Only V_0 is a subspace.

(C) Neither V_0 nor V_1 is a subspace.

(D) Only V_1 is a subspace.

Solution. V_0 is a subspace. It is easy to check that V_0 contains the zero vector and is closed under scalar multiplication and addition. However, V_1 is not a subspace. For example both $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ have 0 as an eigenvalue, but their sum $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ does not. Thus V_1 is not closed under addition.

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20. (5 points) Consider the following basis $\mathcal{B}$ of $\mathbb{R}^3$:

$$\left(\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}\right).$$

Let $x = \begin{bmatrix}1\\1\\1\end{bmatrix}$. What is the coordinate vector $x_{\mathcal{B}}$ of x with respect to the basis $\mathcal{B}$?

(A) ★ $\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\0 & 0 & 1\\\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}$

(B) None of the other answers

(C) $\begin{bmatrix}1\\1\\1\end{bmatrix}$

(D) $\begin{bmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\\0 & 1 & 0\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}$

(E) $\begin{bmatrix}2\\1\\0\end{bmatrix}$

Solution. The basis is orthonormal, so the matrix Q with these columns has inverse Q^T. Now $Q = I_{\mathcal{E}\mathcal{B}}$ and $Q^T = I_{\mathcal{B}\mathcal{E}}$, so $x_{\mathcal{B}} = I_{\mathcal{B}\mathcal{E}}x_E = Q^Tx$.

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21. (5 points) Let $\mathbb{P}^2$ be the vector space of all polynomials of degree at most 2. Let L be the linear transformation from $\mathbb{P}^2$ to $\mathbb{P}^2$ given by

$$L(p(t)) = 3p'(t) + 2p(t).$$

Then the matrix that represents L with respect to the bases $\{1, t, t^2\}$ and $\{1, t, t^2\}$ is

(A) $\begin{bmatrix} 3 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 4 & 3 \end{bmatrix}$

(B) $\begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 0 & 6 & 2 \end{bmatrix}$.

(C) $\begin{bmatrix} 3 & 2 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 3 \end{bmatrix}$

(D) ★ $\begin{bmatrix} 2 & 3 & 0 \\ 0 & 2 & 6 \\ 0 & 0 & 2 \end{bmatrix}$

Solution. $L(1) = 2$, $L(t) = 3 + 2t$, $L(t^2) = 6t + 2t^2$.

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22. (5 points) For which value of t is the vector $\begin{bmatrix} 3 \\ 1+t \end{bmatrix}$ in the span of the vectors $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$?

(A) Only for $t = -3$.

(B) For no t.

(C) Only for $t = 2$.

(D) For all t.

(E) ★ Only for $t = -4$.

Solution.

$$\left[\begin{array}{cc|c} 1 & -2 & 3 \\ -1 & 2 & 1+t \end{array}\right] \simeq \left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 0 & 4+t \end{array}\right]$$

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23. (5 points) Let V be the subset of all 2×2-matrices A such that $A^T = -A$. This is a subspace of the vector space of all 2×2-matrices. What is the dimension of V?

(A) ★ 1

(B) 0

(C) 2

(D) 4

(E) 3

Solution. $V = \{\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}\}$, so that $\dim(V) = 1$.

24. (5 points) A financial company has assets in countries A, B and C. Each year half of the money invested in country A stays in country A, and a quarter of the money invested in country A goes to country B and C each. For country B and C, one half of the money stays in each country and the other half is invested in country A. In the steady state, what is the percentage of the assets of the company that are invested in country A?

(A) None of the other answers

(B) 25%

(C) ★ 50%

(D) 100%

Solution. The transition matrix is $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} \end{bmatrix}$. The vector $x = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector for A, so the steady state is $\frac{1}{4}x$, and so 50% of the assets invested in country A.

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25. (5 points) Let A be a 4×3-matrix and $\boldsymbol{b}$ in $\mathbb{R}^4$. Consider the following two statements:

(S1) The equation $A\boldsymbol{x} = \boldsymbol{b}$ is consistent.

(S2) If $A\boldsymbol{x} = \boldsymbol{b}$ is consistent, the solution to $A\boldsymbol{x} = \boldsymbol{b}$ is unique.

Which of the two statements is correct for all such A and $\boldsymbol{b}$?

(A) ★ Neither Statement S1 nor Statement S2.

(B) Only Statement S2.

(C) Both Statement S1 and Statement S2.

(D) Only Statement S1.

Solution. For statement (S1), note that $Col(A) \neq \mathbb{R}^4$. For statement (S2), it can be that $Nul(A) \neq \{\mathbf{0}\}$.

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26. (5 points) Consider the following two statements:

(T1) If $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ are three orthonormal vectors, then the projection of $\boldsymbol{v_3}$ onto the span of $\boldsymbol{v_1}, \boldsymbol{v_2}$ is $\boldsymbol{v_3}$.

(T2) The Gram–Schmidt process produces from a linearly independent set $\{\boldsymbol{v_1}, \ldots, \boldsymbol{v_n}\}$ an orthonormal set $\{\boldsymbol{q_1}, \ldots, \boldsymbol{q_n}\}$ with the property that for each $k \leq n$ the vectors $\{\boldsymbol{q_1}, \ldots, \boldsymbol{q_k}\}$ span the same subspace as $\{\boldsymbol{v_1}, \ldots, \boldsymbol{v_k}\}$.

Then:

(A) Neither Statement T1 nor Statement T2 is correct.

(B) Only Statement T1 is correct.

(C) ★ Only Statement T2 is correct.

(D) Statement T1 and Statement T2 are correct.

Solution. Since $\boldsymbol{v_3}$ is orthogonal to both $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$, the projection of $\boldsymbol{v_3}$ onto the span of $\boldsymbol{v_1}, \boldsymbol{v_2}$ is $\mathbf{0}$. Thus (T1) is false.

Statement (T2) is correct, as can be observed from the definition of the Gram-Schmidt process.

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27. (5 points) Let W be the Span$\{\begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\1\end{bmatrix}\}$, and let $y = \begin{bmatrix}2\\4\\2\\1\end{bmatrix}$. Let y_W be in W and $y_{W^\perp}$ in $\mathbb{R}^4$ be orthogonal to W such that $y = y_W + y_{W^\perp}$. Then

(A) $y_W = \begin{bmatrix}0\\2\\0\\1\end{bmatrix}$ and $y_{W^\perp} = \begin{bmatrix}2\\2\\2\\0\end{bmatrix}$

(B) ★ $y_W = \begin{bmatrix}0\\2.5\\2\\2.5\end{bmatrix}$ and $y_{W^\perp} = \begin{bmatrix}2\\1.5\\0\\-1.5\end{bmatrix}$

(C) None of the other answers

(D) $y_W = \begin{bmatrix}0\\2\\2\\2\end{bmatrix}$ and $y_{W^\perp} = \begin{bmatrix}2\\2\\0\\-2\end{bmatrix}$

Solution. If $y_w = \frac{1}{2}\begin{bmatrix}0\\1\\0\\1\end{bmatrix} + 2\begin{bmatrix}0\\1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\2\frac{1}{2}\\2\\2\frac{1}{2}\end{bmatrix}$, and $y_{W^\perp} = \begin{bmatrix}2\\1\frac{1}{2}\\0\\-1\frac{1}{2}\end{bmatrix}$, then $y_W \in W$ and $y_W \cdot y_{W^\perp} = 0$ and $y = y_W + y_{W^\perp}$ as required.

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28. (5 points) Let $A = \begin{bmatrix} 0 & 2 & -1 \\ -2 & 8 & -5 \\ -3 & 10 & -7 \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$. Then:

(A) $\boldsymbol{w}$ is an eigenvector of A for the eigenvalue -2,

(B) $\boldsymbol{v}$ is an eigenvector of A for the eigenvalue 2,

(C) ★ $\boldsymbol{v}$ is an eigenvector of A for the eigenvalue -2,

(D) None of the other answers.

(E) $\boldsymbol{w}$ is an eigenvector of A for the eigenvalue 2,

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Solution.

$$\begin{bmatrix} 0 & 2 & -1 \\ -2 & 8 & -5 \\ -3 & 10 & -7 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ -4 \end{bmatrix} = -2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

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29. (5 points) Let A be an $m \times n$-matrix with echelon form U. Which of the following statement is true for all such A?

(T1) $\mathrm{Nul}(A) = \mathrm{Nul}(U)$.

(T2) $\mathrm{Col}(A^T) = \mathrm{Col}(U^T)$.

(A) Only (T2).

(B) ★ Both (T1) and (T2)

(C) Neither (T1) nor (T2).

(D) Only (T1).

Solution. The solutions of $Ax = 0$ and $Ux = 0$ are the same, so (T1) holds. The row space is preserved by elementary row operations so also (T2) holds.

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30. (5 points) Consider the following two statements:

(S1) There exists a subspace V of $\mathbb{R}^7$ such that $\dim V = \dim V^\perp$.

(S2) If V is a subspace of $\mathbb{R}^7$, then the zero vector is the only vector which is in V as well as in $V^\perp$.

Then:

(A) ★ Only Statement S2 is correct.

(B) Statement S1 and Statement S2 are correct.

(C) Neither Statement S1 nor Statement S2 is correct.

(D) Only Statement S1 is correct.

Solution. Statement (S1) is false, because $\dim V + \dim V^\perp = 7$, and there is no integer x such that $x + x = 7$. Statement (2) is correct, since the zero vector is the only vector orthogonal to itself.

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31. (5 points) Let $B = \begin{bmatrix} 0 & 1 \\ 3 & 1 \\ 0 & 1 \end{bmatrix}$. Let Q be a 3×2-matrix with orthonormal columns and R be an upper triangular matrix such that $B = QR$. Which of the following matrices can be R?

(A) ★ $\begin{bmatrix} 3 & 1 \\ 0 & \sqrt{2} \end{bmatrix}$

(B) $\begin{bmatrix} 1 & \sqrt{3} \\ 0 & 2 \end{bmatrix}$

(C) $\begin{bmatrix} 3 & 2 \\ 0 & \sqrt{2} \end{bmatrix}$

(D) $\begin{bmatrix} \sqrt{3} & 2 \\ 0 & 2 \end{bmatrix}$

(E) None of the other answers

Solution.

$$\begin{bmatrix} 0 & 1 \\ 3 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \\ 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & \sqrt{2} \end{bmatrix}$$
