Math 415. Final Exam. May 6, 2019

Full Name: ____________________

Net ID: ______________________

Discussion Section: ___________

- Do not turn this page until instructed to.
- There are 33 problems worth 3 points each. You get one point for free. Therefore the total number of points is 100.
- Each question has only one correct answer. You can choose up to two answers. If you choose just one answer, then you will get 3 points if the answer is correct, and 0 points otherwise. However, if you choose two answers, you will get 1.5 points if one of the answers is correct, and 0 points otherwise.
- You must not communicate with other students.
- No books, notes, calculators, or electronic devices allowed.
- This is a 180 minute exam. There are several different versions of this exam.
- Fill in the answers on the scantron form provided, **and** circle your answers on the exam itself. Hand in both the exam and the scantron.
- Please double check that you correctly bubbled in your answer on the scantron. It is the answer on the scantron that counts!
- If you have to erase something on the scantron, please make sure to do so thoroughly.
- Good luck!

Fill in the following information on the scantron form:

On the first page of the scantron bubble in **your name, your UIN and your NetID**! On the back of the scantron bubble in the following:

95. D
96. C

1. (3 points) The matrix

$$A = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 2 & 4 & -6 & 8 \\ 10 & -4 & 6 & 10 \end{bmatrix}$$

has echelon form

$$E = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 24 & -36 & 30 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Which of the following sets are a basis for $\mathrm{Col}(A^T)$?

I. $\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -6 \\ 8 \end{bmatrix} \right\}$

II. $\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 24 \\ -36 \\ 30 \end{bmatrix} \right\}$

III. $\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -6 \\ 8 \end{bmatrix}, \begin{bmatrix} 10 \\ -4 \\ 6 \\ 10 \end{bmatrix} \right\}$

(A) ★ II only

(B) I and III only

(C) II and III only

(D) I only

(E) I and II only

Solution. Observe that A has 2 pivot positions. Thus $\dim \mathrm{Col}(A^T) = 2$. So III can not be a basis of $\mathrm{Col}(A^T)$. Note that the vectors in I are linear dependent. Hence I is not basis. Finally observe that II is a basis, since it contains the non-zero rows of an echelon form of A. By a theorem from class, these rows form a basis of $\mathrm{Col}(A^T)$.

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2. (3 points) Consider the bases $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2\}$ of $\mathbb{R}^2$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\}$ of $\mathbb{R}^4$. Let $T : \mathbb{R}^2 \to \mathbb{R}^4$ given by

$$T(\mathbf{a}_1) = \mathbf{b}_3, \qquad T(\mathbf{a}_2) = -8\mathbf{b}_3 + 6\mathbf{b}_4.$$

Which of the following matrices is $T_{\mathcal{B}\mathcal{A}}$?

(A) $\begin{bmatrix} 1 & 0 \\ -8 & 6 \end{bmatrix}$

(B) None of the other answers.

(C) $\begin{bmatrix} 1 & -8 \\ 0 & 6 \end{bmatrix}$

(D) ★ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -8 \\ 0 & 6 \end{bmatrix}$

(E) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -8 & 6 \end{bmatrix}$

Solution. The columns of $T_{\mathcal{B}\mathcal{A}}$ are given by $\begin{bmatrix} T(\mathbf{a}_1)_{\mathcal{B}} & T(\mathbf{a}_2)_{\mathcal{B}} \end{bmatrix}$. $T(\mathbf{a}_1)_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $T(\mathbf{a}_2)_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ -8 \\ 6 \end{bmatrix}$.

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3. (3 points) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be non-zero vectors in $\mathbb{R}^3$ such that

$$2\mathbf{a} - \mathbf{b} = 0$$

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$$

Which of the following describes the set $\text{span}(\mathbf{a}, \mathbf{b}, \mathbf{c})$?

(A) It is empty.

(B) ★ It is a line in $\mathbb{R}^3$.

(C) It is a plane in $\mathbb{R}^3$.

(D) None of the other answers.

(E) It is $\mathbb{R}^3$.

Solution. The given relations let us rewrite any vector in the span

$$\mathbf{v} = d_1\mathbf{a} + d_2\mathbf{b} + d_3\mathbf{c}$$

as

$$\begin{aligned} \mathbf{v} &= d_1\mathbf{a} + d_2(2\mathbf{a}) + d_3(-\mathbf{a} - \mathbf{b}) \\ &= d_1\mathbf{a} + d_2(2\mathbf{a}) + d_3(-3\mathbf{a}) \\ &= (d_1 + 3d_2 - 3d_3)\mathbf{a} \end{aligned}$$

Hence the span is the set of all scalar multiples of the nonzero vector $\mathbf{a}$, i.e. a line.

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4. (3 points) Let V be a vector space of dimension at least 3. Let $\mathcal{B} = \big(\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\big)$, $\mathcal{C} = \big(\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\big)$ be two different ordered bases of V. Which of the following statements are always true?

(I) If the second entry of $\mathbf{v}_{\mathcal{B}}$ is zero, then at least one entry of $\mathbf{v}_{\mathcal{C}}$ is zero.

(II) If $\mathbf{v}_{\mathcal{B}}$ is not the zero vector, then $\mathbf{v}_{\mathcal{C}}$ is not the zero vector.

(III) If $\mathbf{c}_1 = \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$, then $\{\mathbf{c_1}, \mathbf{b_2}, \mathbf{b_3}, \ldots, \mathbf{b_n}\}$ is a basis for V. That is, if $\mathbf{c}_1 = \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$, then substituting $\mathbf{c_1}$ for $\mathbf{b_1}$ in $\mathcal{B}$ creates a basis for V.

(A) Only statements (I) and (II).

(B) Only statements (I) and (III).

(C) All three statements.

(D) None of the other answers.

(E) ★ Only statements (II) and (III).

Solution.

- Consider $\mathbf{v} = \begin{bmatrix}1\\0\\1\end{bmatrix}$, a vector in $\mathbb{R}^3$ written in terms of the standard basis. Consider the basis $\mathcal{C} = \left(\begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix} \begin{bmatrix}0\\-2\\0\end{bmatrix} \right)$. Then $\mathbf{v}_{\mathcal{C}} = \begin{bmatrix}1\\1\\1\end{bmatrix}$. The second component of $\mathbf{v}$ is zero, but every component of $\mathbf{v}_{\mathcal{C}}$ is nonzero.

- The coordinate vector for $\mathbf{v}$ with respect to a basis $\mathcal{B}$ is given by $\mathbf{v}_{\mathcal{C}} = \begin{bmatrix}c_1\\c_2\\\vdots\\c_n\end{bmatrix}$ where $\mathbf{v} = k_1\mathbf{c_1} + k_2\mathbf{c_2} + \cdots + k_n\mathbf{c_n}$. So $\mathbf{v}_{\mathcal{B}} = \mathbf{0}$ if and only if $\mathbf{v}_{\mathcal{C}} = \mathbf{0}$.

- If $\mathbf{c_1} = \mathbf{b_1} + \mathbf{b_2} + \mathbf{b_3}$, then $\mathcal{B}' = \{\mathbf{c_1}, \mathbf{b_2}, \mathbf{b_3} \ldots \mathbf{b_n}\}$ is a basis for V.

  Any vector v that can be written as a linear combination of vectors in $\mathcal{B}$ can be rewritten in terms of a linear combination of vectors in $\mathcal{B}'$.

  The vectors in $\mathcal{B}'$ must be linearly independent: If $k_1\mathbf{c_1} + k_2\mathbf{b_2} + \cdots + k_n\mathbf{b_n} = 0$, then $k_1(\mathbf{b_1} + \mathbf{b_2} + \mathbf{b_3}) + k_2\mathbf{b_2} + \cdots + k_n\mathbf{b_n} = 0$. By linear independence of $\mathcal{B}$, we know that all k_i are zero, and so $\mathcal{B}'$ is also linearly independent.

  So $\mathcal{B}'$ is a basis for V.

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5. (3 points) Let A be a real 5×5 matrix, and suppose that there exists an orthonormal basis of $\mathbb{R}^5$ consisting of eigenvectors for A. Which of the following statements must be TRUE?

(A) All of the eigenvalues of A are real and nonnegative.

(B) The matrix A is invertible.

(C) ★ The matrix A is symmetric.

(D) None of the other answers.

(E) The matrix A is orthogonal.

Solution. If A has an orthonormal eigenbasis, then $A = QDQ^T$, where Q is orthogonal (with columns given by the eigenbasis) and D is diagonal. Thus, $A^T = (QDQ^T)^T = QDQ^T = A$, i.e. A is symmetric.

The zero matrix is a counterexample to the other statements, which are all false.

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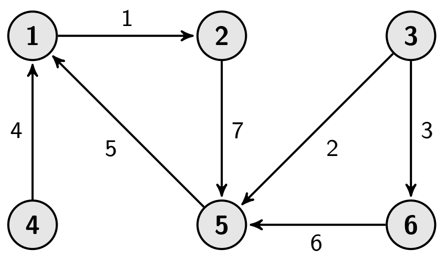
6. (3 points) Let A be the edge-node incidence matrix of the directed graph below.



What is the dimension of $\mathrm{Nul}(A)$ and $\mathrm{Nul}(A^T)$?

(A) None of the other answers.

(B) $\dim \mathrm{Nul}(A) = 1$, $\dim(\mathrm{Nul}(A^T)) = 1$.

(C) ★ $\dim \mathrm{Nul}(A) = 1$, $\dim(\mathrm{Nul}(A^T)) = 2$.

(D) $\dim \mathrm{Nul}(A) = 2$, $\dim(\mathrm{Nul}(A^T)) = 1$.

(E) $\dim \mathrm{Nul}(A) = 2$, $\dim(\mathrm{Nul}(A^T)) = 2$.

Solution. The graph has one connected component and two independent loops.

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7. (3 points) Suppose that A is a 3×3 matrix that satisfies

$$\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -5 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 4\end{bmatrix} A = I.$$

Which of the following matrices is A^{-1}?

(A) ★ $\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -5 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 4\end{bmatrix}$

(B) None of the other answers.

(C) $\begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 5 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1/4\end{bmatrix}$

(D) $\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1/4\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 5 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

(E) $\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 4\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -5 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

Solution. The inverse of A is any matrix A^{-1} that satisfies $A^{-1}A = I$.
Since $\left(\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -5 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 4\end{bmatrix}\right) A = I$, $\left(\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -5 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 4\end{bmatrix}\right) = A^{-1}$.

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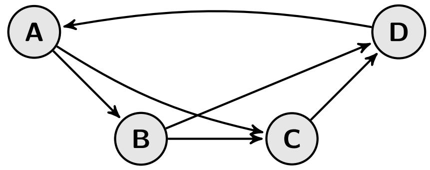
8. (3 points) Which matrix is the PageRank matrix for the following system of webpages?



(A) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 & 0 \end{bmatrix}$

(B) $\begin{bmatrix} -1 & 0 & 0 & 1 \\ \frac{1}{2} & -1 & 0 & 0 \\ 0 & \frac{1}{2} & -1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 & -1 \end{bmatrix}$

(C) None of the other answers.

(D) ★ $\begin{bmatrix} 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \end{bmatrix}$

(E) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 & 0 \end{bmatrix}$

Solution. The pagerank matrix records for each column for site X links to sites Y in the corresponding rows.

9. (3 points) Let $W := \operatorname{span}\left(\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right)$. What is the projection matrix for the orthogonal projection onto W with respect to the standard basis?

(A) $\begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}$

(B) None of the other answers.

(C) $\begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0\\\frac{1}{2} & \frac{3}{2} & 0\\0 & 0 & 0\end{bmatrix}$

(D) ★ $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$

(E) $\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\\frac{1}{2} & \frac{3}{2}\end{bmatrix}$

Solution. Let π_W be the projection onto W and P be the corresponding projection matrix. Then

$$\pi_W(\begin{bmatrix}1\\0\\0\end{bmatrix}) = \begin{bmatrix}1\\0\\0\end{bmatrix},$$

since $\begin{bmatrix}1\\0\\0\end{bmatrix}$ is in W. Furthermore,

$$\pi_W(\begin{bmatrix}0\\1\\0\end{bmatrix}) = \begin{bmatrix}0\\1\\0\end{bmatrix},$$

becuause $\begin{bmatrix}0\\1\\0\end{bmatrix}$ is also in W. Since $\begin{bmatrix}0\\0\\1\end{bmatrix}$ is orthogonal to W, we get that

$$\pi_W(\begin{bmatrix}0\\0\\1\end{bmatrix}) = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Thus

$$P = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}.$$

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10. (3 points) Let

$$\mathbf{u} = \begin{bmatrix} 2 \\ -6 \\ 7 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 6 \\ 0 \\ 3 \end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix} 2k \\ 3k \\ 2 \end{bmatrix}.$$

For which value of k are $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ linearly dependent?

(A) There is no value of k for which $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are linearly dependent.

(B) ★ $k = -1$.

(C) $k = 2$.

(D) None of the other answers.

(E) $k = 1$.

Solution. $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent if and only if the matrix $\begin{bmatrix} 2 & 6 & 2k \\ -6 & 0 & 3k \\ 7 & 3 & 2 \end{bmatrix}$ has rank less than 3. Row reduction gives $\begin{bmatrix} 1 & 0 & -k/2 \\ 0 & 1 & k/2 \\ 0 & 0 & 2k+2 \end{bmatrix}$ and so $0 = 2k + 2$. Solve for k to get $k = -1$.

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11. (3 points) Calculate the singular values of $A = \begin{bmatrix} 0 & -1 \\ 0 & -3 \end{bmatrix}$.

(A) $0, \sqrt{3}$

(B) ★ $0, \sqrt{10}$

(C) $0, 10$

(D) None of the other answers.

(E) $0, -3$

Solution. The singular values of A are the positive square roots of the eigenvalues of $A^T A$. $A^T A = \begin{bmatrix} 0 & 0 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 10 \end{bmatrix}$ which has eigenvalues 0 and 10.

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12. (3 points) Recall that $M_{2\times 2}$ is the vector space of 2×2-matrices. Consider the following subspaces of $M_{2\times 2}$:

$$V = \{A \in M_{2\times 2} \mid A^T = -A\} \qquad W = \{B \in M_{2\times 2} \mid B \text{ is diagonal}\}.$$

What are the dimensions of V and W?

(A) $\dim V = 4$ and $\dim W = 4$

(B) ★ $\dim V = 1$ and $\dim W = 2$

(C) $\dim V = 3$ and $\dim W = 2$

(D) None of the other answers

(E) $\dim V = 2$ and $\dim W = 2$

Solution. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is in V, then $a = -a$, $c = -b$, and $d = -d$, so $A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} = b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and so $V = \operatorname{span}\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$ is 1-dimensional.

A general element of W looks like

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

so $W = \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

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13. (3 points) Let A be an $\ell \times m$ matrix such that for every $\boldsymbol{b}$ in $\mathbb{R}^{\ell}$, the equation $A\boldsymbol{x} = \boldsymbol{b}$ has at most one solution. What does this statement imply about the relative size of ℓ and m?

(A) $\ell \leq m$

(B) None of the other answers.

(C) ★ $\ell \geq m$

(D) $\ell = m$

(E) nothing (ℓ and m can be any positive integers).

Solution. If for every $\boldsymbol{b}$ in $\mathbb{R}^{\ell}$ the equation $A\boldsymbol{x} = \boldsymbol{b}$ has at most one solution, there are no free variables and so the columns of A must be independent. Hence there must be at most ℓ columns.

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14. (3 points) Let $\mathbb{P}_n$ be the vector space of all polynomials of degree at most n. Let $T : \mathbb{P}_2 \to \mathbb{P}_1$ be defined by

$$T(p(t)) = 3\frac{d}{dt}p(t)$$

and let $\mathcal{A} = (1+t, t^2, 1-t)$ and $\mathcal{B} = (1, t)$ be bases for $\mathbb{P}_2$ and $\mathbb{P}_1$ respectively. Which one of the following matrices is $T_{\mathcal{B}\mathcal{A}}$?

(A) $\begin{bmatrix} 3 & 0 \\ 0 & 6 \\ -3 & 0 \end{bmatrix}$

(B) ★ $\begin{bmatrix} 3 & 0 & -3 \\ 0 & 6 & 0 \end{bmatrix}$

(C) $\begin{bmatrix} 3 & 0 \\ -3 & 0 \\ 0 & 6 \end{bmatrix}$

(D) $\begin{bmatrix} 3 & -3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

(E) None of the other answers.

Solution. From Sections 3 and 4, Lecture 17, we obtain that

$$T_{\mathcal{B}\mathcal{A}} = \left[T(1+t)_{\mathcal{B}}\ T(t^2)_{\mathcal{B}}\ T(1-t)_{\mathcal{B}}\right] = [3_{\mathcal{B}}\ (6t)_{\mathcal{B}}\ (-3)_{\mathcal{B}}] = \begin{bmatrix} 3 & 0 & -3 \\ 0 & 6 & 0 \end{bmatrix}.$$

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15. (3 points) Let A be a 3×3 matrix. Consider the following linear system:

$$A\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Which of the following are always true?

I. The system always has a solution.

II. If A is invertible, the system has exactly one solution.

III. If two rows of A are equal, then the system has no solution.

IV. If there is a zero row in the row echelon form of A, then the system has infinitely many solutions.

(A) Only I and II.

(B) ★ Only I, II, and IV.

(C) None of the other answers.

(D) Only I and IV.

(E) I, II, III and IV.

Solution. $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is always a solution, so I is true. Invertibility always implies existence and uniqueness of a solution, so II is true. III is false as it's in opposition with I. IV is true, since it implies singularity of the matrix A, and we already know there is at least *one* solution for $\mathbf{x}$.

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16. (3 points) Suppose

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

has QR-decomposition

$$Q = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}, R = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

Which of the following is a least-squares solution to $A\mathbf{x} = \mathbf{b}$ for $\mathbf{b} = \begin{bmatrix} 0 \\ 2 \\ -2 \\ 0 \end{bmatrix}$?

(A) $\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$.

(B) ★ $\begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$.

(C) $\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$.

(D) $\begin{bmatrix} -3 \\ 1 \\ -1 \end{bmatrix}$.

(E) None of the other answers.

Solution. To make this feasible to do on a test, we need to use that finding the least-squares solution is equivalent to solving the system

$$R\hat{\mathbf{x}} = Q^T\mathbf{b}$$

We have $Q^T\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$, so we end up row reducing the augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 4 & 2 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 2 & -2 \end{array}\right] \xrightarrow[\text{divide by 2}]{} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow[R_1 \to R_1 - 2R_2]{} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow[R_1 \to R_1 - 3R_3, R_2 \to R_2 + R_3]{} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

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17. (3 points) Let W be a subspace of $\mathbb{R}^n$ with $\dim W < n$. Let P be the projection matrix of the orthogonal projection onto W with respect to the standard basis. Select the statement about P which is FALSE, or if all statements about P are correct, select "None of the other options".

(A) The nullspace of P is equal to $W^\perp$.

(B) None of the other options.

(C) ★ The columns of P form a basis of W.

(D) The rank of P is equal to the dimension of W.

(E) All eigenvalues of P are either 0 or 1.

Solution. The columns of P span W, but may not be linearly independent. For example, take the orthogonal projection onto $W := \operatorname{span}(\begin{bmatrix}1\\0\end{bmatrix})$. Then $P := \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$. The columns of P are not linearly independent, but span W.

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18. (3 points) For the ordered basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}$$

which of the following is the ordered basis obtained by applying the Gram-Schmidt process to $\mathcal{B}$?

(A) $\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \right\}.$

(B) $\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$

(C) $\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$

(D) ★ $\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \right\}.$

(E) None of the other answers.

Solution. For Gram-Schmidt, it's important that we start with the first vector and proceed down the order of the basis. First, we renormalize $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ down to $\mathbf{q_1} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$. Moving onto $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, we can either apply the algorithm OR eyeball that its projection onto $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$ is $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. We get $\mathbf{b_2} = \mathbf{q_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Finally, the third vector is already orthogonal to $\mathbf{q_1}$ and $\mathbf{q_2}$, so we just need to renormalize it down to $\mathbf{q_3} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$.

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19. (3 points) Consider the following two statements:

(S1) If $\mathcal{B} = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is an orthogonal basis of $\mathbb{R}^n$ and $\mathbf{w}$ is in $\mathbb{R}^n$, then the projections of $\mathbf{w}$ onto each of the vectors $\mathbf{v_1}, \ldots, \mathbf{v_n}$ must sum back to $\mathbf{w}$.

(S2) There exists a subspace U of $\mathbb{R}^5$ such that $\dim U = \dim U^{\perp}$.

Which of the above two statements is always true?

(A) Only Statement S2 is correct.

(B) ★ Only Statement S1 is correct.

(C) Statement S1 and Statement S2 are correct.

(D) Neither Statement S1 nor Statement S2 is correct.

Solution. Statement (S2) is false, because $\dim U + \dim U^{\perp} = 5$, and there is no integer x such that $x + x = 5$. Statement (1) is correct by Theorem 1 of Lecture notes 21.

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20. (3 points) Let A be a 5×2 matrix and B be a 3×5 matrix. Which of the following statements is correct?

(A) None of the other answers.

(B) AB^T is a 5×5 matrix.

(C) ★ AB^T is not defined.

(D) AB^T is a 2×3 matrix.

(E) AB^T is a 5×3 matrix.

Solution. B^T is a 5×3 matrix, and the number of columns of A is not equal to the number of rows of B^T.

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21. (3 points) Let $A = \begin{bmatrix} 1 & 2 & 0 & 3 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 \\ 2 & 3 & 4 & -5 & 2 \\ 0 & 1 & 3 & 5 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 2 & 0 & 3 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 \\ 2 & 3 & 4 & -5 & 2 \\ 0 & 1 & 3 & 5 & 7 \end{bmatrix}$. Which of the following is true of the determinants of A and B?

(A) None of the other answers.

(B) $\det(A) = -\det(B)$

(C) $\det(A) = 0$

(D) ★ $\det(A) = \det(B)$

(E) $\det(B) = 0$

Solution. B is obtained from A via the single row operations $R1 \to R1 + R2$. Such row operations do not change the determinant, so $\det(A) = \det(B)$. Furthermore, $\det(A) = 11$, so none of the other choices are correct.

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22. (3 points) Consider

$$A = \begin{bmatrix} 9 & 3 & 1 \\ 0 & 5 & 1 \\ 0 & 1 & 5 \end{bmatrix}.$$

What are the eigenvalues of A?

(A) $9, 4$

(B) None of the other answers.

(C) ★ $9, 4, 6$

(D) $4, 6$

(E) $9, 5, 1$

Solution. $\det(A - \lambda I) = \det(\begin{bmatrix} 9-\lambda & 3 & 1 \\ 0 & 5-\lambda & 1 \\ 0 & 1 & 5-\lambda \end{bmatrix}) = (9-\lambda)[(5-\lambda)^2 - 1] + 0 + 0 = (9-\lambda)[\lambda^2 - 10\lambda + 24] = (9-\lambda)(\lambda-4)(\lambda-6)$.

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23. (3 points) Let $\mathcal{B} = (\mathbf{b}_1, \mathbf{b}_2)$ be a basis of $\mathbb{R}^2$, let A be a 2×2-matrix and $\mathbf{v}$ be a vector in $\mathbb{R}^2$. Suppose that $\mathbf{v}_\mathcal{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and that A can be decomposed as follows:

$$A = I_{\mathcal{E},\mathcal{B}} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} I_{\mathcal{B},\mathcal{E}}.$$

Then what is $A\mathbf{v}$?

(A) $A \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

(B) ★ $2\mathbf{b}_1 + 3\mathbf{b}_2$

(C) $3\mathbf{b}_1 + 2\mathbf{b}_2$

(D) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

(E) None of the other answers.

Solution. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation such that $T(\mathbf{v}) = A\mathbf{v}$. Then

$$T_{\mathcal{B}\mathcal{B}} = I_{\mathcal{B},\mathcal{E}} A I_{\mathcal{E},\mathcal{B}} = I_{\mathcal{B},\mathcal{E}} I_{\mathcal{E},\mathcal{B}} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} I_{\mathcal{B},\mathcal{E}} I_{\mathcal{E},\mathcal{B}} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

Thus

$$(A\mathbf{v})_\mathcal{B} = T_{\mathcal{B}\mathcal{B}} \mathbf{v}_\mathcal{B} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Thus $A\mathbf{v} = 2\mathbf{b}_1 + 3\mathbf{b}_2$.

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24. (3 points) Let A be a 3×3 matrix with eigenvalues $0, 1, 2$. Which of the following statements about A must be true?

I. A is invertible

II. A is diagonalizable

III. A has an LU-decomposition

(A) I., II., and III.

(B) ★ II. only

(C) None of the other answers

(D) III. only

(E) II. and III. only

Solution. Since $det(A) = 0 \times 1 \times 2 = 0$, A is not invertible. Because A has distinct eigenvalues, A is necessarily diagonalizable. $A = \begin{bmatrix} 0 & -1 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ has no LU-decomposition and has eigenvalues 0,1,2.

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25. (3 points) Consider the following subset of the vector space of 2×2 matrices $M_{2\times 2}$:

$$D = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;:\; \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = 0 \right\}$$

It can be shown that D is NOT a subspace of $M_{2\times 2}$. Which of the following tests does D **fail to satisfy**? (Select all that apply.)

I. contains the zero matrix

II. closed under addition

III. closed under scalar multiplication

(A) III. only

(B) ★ II. only

(C) II. and III. only

(D) I. and III. only

(E) I., II., and III.

Solution. The zero matrix has determinant 0. Therefore I. holds. For example both $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ have 0 as determinant, but their sum $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ does not. Thus II. fails. Let c be a scalar and A be a 2×2-matrix with determinant 0. Then cA is also a matrix with determinant 0. Thus III. holds.

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26. (3 points) Let $\mathcal{B} := (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ be an orthonormal basis of $\mathbb{R}^3$ such that $\mathbf{b}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. Let $\mathbf{v} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$ and let c_1, c_2, c_3 be scalars such that $\mathbf{v} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3$. What is c_2?

(A) There is not enough information to determine c_2

(B) -3

(C) $\frac{5}{\sqrt{2}}$

(D) $\frac{3}{\sqrt{2}}$

(E) ★ $\frac{-3}{\sqrt{2}}$

Solution. From page 4, Lecture 18, since the basis is orthonormal we obtain that $c_2 = \mathbf{v} \cdot \mathbf{b_2} = \frac{-3}{\sqrt{2}}$.

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27. (3 points) Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} -4 & 0 \\ 1 & 2 \\ 0 & 2 \\ 2 & 2 \end{bmatrix}$$

Which of the following statements is true about $\mathrm{Nul}(A)$ and $\mathrm{Col}(B)$?

(A) $\mathrm{Col}(B)$ is contained in $\mathrm{Nul}(A)$ but they are not equal.

(B) $\mathrm{Nul}(A)$ is the orthogonal complement of $\mathrm{Col}(B)$.

(C) $\mathrm{Nul}(A)$ is contained in $\mathrm{Col}(B)$ but they are not equal.

(D) ★ $\mathrm{Nul}(A) = \mathrm{Col}(B)$.

(E) None of the other answers.

Solution. Since $AB = 0$, it follows that $\mathrm{Col}(B)$ is contained in $\mathrm{Nul}(A)$. A is in echelon form with 2 free variables, so $\mathrm{Nul}(A)$ is 2 dimensional. Since the two columns of B are clearly linearly independent, $\mathrm{Col}(B)$ is also 2 dimensional. Hence we must have $\mathrm{Nul}(A) = \mathrm{Col}(B)$.

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28. (3 points) Let B be a 4×3 matrix. Let $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ be the columns of B and suppose that $\{\mathbf{b}_1, \mathbf{b}_2\}$ is linearly independent and $\mathbf{b}_3 = \mathbf{b}_1 + 7\mathbf{b}_2$. Which of the following is a basis of $\mathrm{Nul}(B)$?

(A) None of the other answers

(B) ★ $\left\{ \begin{bmatrix} -1 \\ -7 \\ 1 \end{bmatrix} \right\}$

(C) $\{\mathbf{b}_3\}$

(D) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \end{bmatrix} \right\}$

(E) $\{\mathbf{b}_1, \mathbf{b}_2\}$

Solution. B has rank 2, three columns, so $\dim \mathrm{Nul}(B) = 1$. Now $-1\mathbf{b}_1 - 7\mathbf{b}_2 + \mathbf{b}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = B \begin{bmatrix} -1 \\ -7 \\ 1 \end{bmatrix}$, so that $\begin{bmatrix} -1 \\ -7 \\ 1 \end{bmatrix}$ is a basis for $\mathrm{Nul}(B)$.

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29. (3 points) The student population in a small town has three preferences for dinner: a Chinese place, a Mexican place or having dinner at home. Everyone in town eats dinner in one of these places or has dinner at home. Assume that one half of those who eat in Chinese restaurant return to the restaurant next time and one half decides to go to the Mexican next time. From those who eat in Mexican restaurant, one quarter come back, one half go to the Chinese place next time, and one quarter stay home next time. Form those who stay home, one half decides to go to the Mexican and one half decides to go to the Chinese. In the steady state, what is the percentage of students who decide to stay home?

(A) 100%

(B) None of the other answers

(C) ★ 20%

(D) 25%

Solution. The transition matrix is $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$. The vector $x = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to eigenvalue 1, so the steady state is $\frac{1}{2+2+1}x$, and so 20% of the students decide to stay home.

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30. (3 points) The invertible matrix $A = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$ is reduced to the echelon matrix $U = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix}$ using the following row operations (in the given order):

A. $R_2 \to R_2 + 2R_1$,

B. $R_3 \to R_3 - R_1$,

C. $R_3 \to R_3 + R_2$,

D. $R_4 \to R_4 + R_2$.

Which of the following matrices is the matrix L in the LU-decomposition $A = LU$?

(A) ★ $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$

(B) None of the other answers.

(C) $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$

(D) $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

(E) $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Solution. Using the algorithm to get the L matrix, using the opposite sign for the replacements and the given order of operations we obtain the lower triangular matrix $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$.

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31. (3 points) Observe that 1 is an eigenvalue of the matrix $A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 2 \end{bmatrix}$. What is the dimension of the corresponding eigenspace?

(A) 0

(B) 2

(C) None of the other answers.

(D) ★ 1

(E) 3

Solution. The corresponding eigenspace is $\mathrm{Nul}(A + (-1)I) = \mathrm{Nul} \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. To find its dimension, we bring the matrix to an echelon form and count the number of free variables.

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

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32. (3 points) Let A be an $m \times n$-matrix and $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be in $\mathbb{R}^n$. Consider the following statements:

I. If $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$, then $A\mathbf{w}$ is a linear combination of $A\mathbf{u}$ and $A\mathbf{v}$.

II. If $A\mathbf{w}$ is a linear combination of $A\mathbf{u}$ and $A\mathbf{v}$, then $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

Which of the two statements is always true?

(A) ★ Only I is correct.

(B) Neither I nor II are correct.

(C) Both I and II are correct.

(D) Only II is correct.

Solution. For I., suppose that $\mathbf{w} = c\mathbf{u} + d\mathbf{v}$. Then

$$A(\mathbf{w}) = A(c\mathbf{u} + d\mathbf{v}) = cA(\mathbf{u}) + dA(\mathbf{v}).$$

For II., let A be the zero matrix. Then $A\mathbf{w}$ is always the zero vector and hence a linear combination of $A\mathbf{u}$ and $A\mathbf{v}$. However, $\mathbf{w}$ does not have to be a linear combination of $\mathbf{u}$ and $\mathbf{v}$.

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33. (3 points) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Let A be the matrix representing T with respect to the standard basis of $\mathbb{R}^n$ (that is, $A = T_{\mathcal{E}\mathcal{E}}$). Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ be such that

$$\mathbf{x} \in \mathrm{Nul}(A), \text{ and } \quad \mathbf{y} \in \mathrm{Col}(A).$$

Consider the following statements:

I. $T(\mathbf{x}) = \mathbf{0}$.

II. For every vector $\mathbf{v} \in \mathbb{R}^n$, $A\mathbf{v} = T(\mathbf{v})$.

III. There exists a vector $\mathbf{z} \in \mathbb{R}^n$ such that $T(\mathbf{z}) = \mathbf{y}$.

Which of the statements are ALWAYS TRUE?

(A) II. only

(B) I. and II. only

(C) I. and III. only

(D) I. only

(E) ★ I., II., and III.

Solution. Because A is the matrix of T with respect to the standard basis, for any $\mathbf{v} \in \mathbb{R}^n$, $T(\mathbf{v}) = A\mathbf{v}$. In particular, if $\mathbf{x} \in \mathrm{Nul}(A)$, then $\mathbf{0} = A\mathbf{x} = T(\mathbf{x})$. Also if $\mathbf{y} \in \mathrm{Col}(A)$, then there exists some vector $\mathbf{z} \in \mathbb{R}^n$ such that $A\mathbf{z} = \mathbf{y}$. But that means $T(\mathbf{z}) = \mathbf{y}$.
