Assignment 2: Due Friday Mar 17 at 1:00 pm in class

Please include your name and student number when submitting the assignments for grading. Additionally, please note that you are to solve the problems below on your own. Collaborative work will be considered as cheating and UBC plagiarism penalties will apply. This assignment is worth 2.5% of your final grade. If you hand in the assignment X days late, then you will be penalized as follows: the max mark you can get is $\max\{5 - X, 0\}$.

The core concepts this assignment is designed to asses are:

- **Distributions:** cumulative distribution function (cdf) $F_X(x)$, probability density function (pdf) $f_X(x)$, and probability mass function (pmf).
- **Transformations:** Probability integral transform, Univariate transform, Multivariate transform.
- **Generating Samples:** Inverse transform, Polar, Composition, and Acceptance-Rejection.

Questions

1. θ**-Rotational Transformer**: The circuit illustrated in Fig.1 is common in communication systems (e.g. stereo baseband systems). The parameters $\{a_i\}$ are gains. If the parameters $a_1 = a_2 = \cos(\theta)$ and $a_3 = a_4 = \sin(\theta)$ then the circuit in Fig.1 is known as a θ-Rotational Transformer. Let us assume that only Guassian noise is present at the inputs X and Y, the goal is compute the pdf of the outputs W and Y. Note that here we assume that all amplifiers are noiseless linear amplifiers.

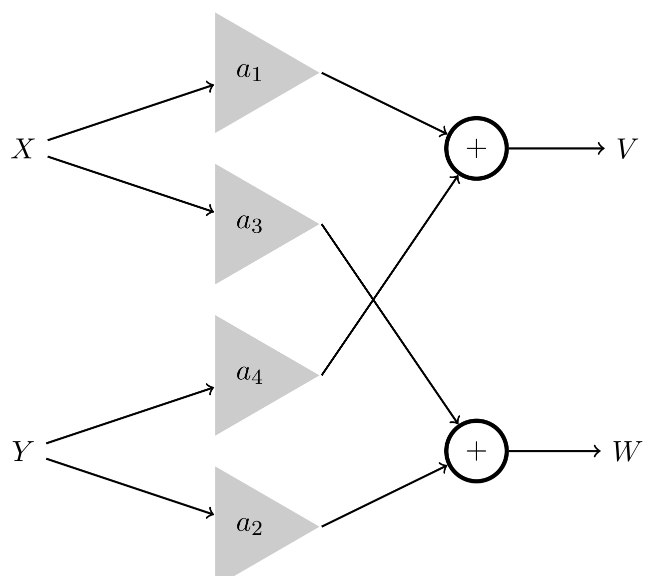

Figure 1: A two-variable-to-two-variable matrixer.

   (a) Construct the expressions $V = g(X, Y)$ and $W = h(X, Y)$. What is the physical interpretation of the output signals V and W if $a_1, a_3, a_2 = 1$ and $a_4 = -1$?

   (b) Using the multivariate transform technique, construct the joint pdf of $f_{VW}(v, w)$ using the joint pdf of $f_{XY}(x, y)$ and the expressions $g(X, Y)$ and $h(X, Y)$.

   (c) Construct the marginal pdf of V (i.e. $f_V(v)$).

2. Consider a two dimensional random variable with joint pdf given by:

$$f_{XY}(x, y) = A(x + y), \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1$$

for some constant A.

(a) Compute the normalization constant A.

(b) Compute the joint cumulative distribution function $F_{XY}(x, y)$.

(c) Compute the marginal cumulative distribution functions $F_X(x), F_Y(y)$ and marginal probability density functions $f_X(x), f_Y(y)$ of both X and Y.

3. Use the inverse transform method to generate the following random variables. In each case verify your answer by plotting a histogram constructed from 10,000 samples. Additionally plot the actual density to check how close this empirical density is to the true density.

(a) Random variable with density:

$$f_X(x) = \frac{e^x}{e - 1}, \quad 0 \leq x \leq 1.$$

(b) Random variable with distribution:

$$F_X(x) = \frac{x^2 + x}{2}, \quad 0 \leq x \leq 1.$$

4. Use the compositional method to generate the following random variables in either R or MATLAB. In each case verify your answer by plotting a histogram constructed from 10,000 samples. Additionally plot the actual density to check how close this empirical density is to the true density.

(a) Random variable with distribution:

$$F_X(x) = \frac{x + x^3 + x^5}{3}, \quad 0 \leq x \leq 1.$$

(b) Random variable with distribution: (Hint: the sum of a Gaussian and Logistic distribution)

$$F_X(x) = \frac{1}{4}\big[2 + \tanh(\frac{x-1}{2}) + \text{erf}(\frac{x-1}{\sqrt{2}})\big], \quad -\infty < x < \infty.$$

5. Generate samples from the following distribution

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-(x-1)^2/2}, \quad -\infty \leq x \leq \infty$$

in either R or MATLAB using the methods described below. Verify your answer by plotting a histogram constructed from 10,000 samples. Additionally plot the actual density to check how close this empirical density is to the true density.

(a) Polar method.

(b) Acceptance-Rejection method.

(c) Which method should be used to generate the samples from $f_X(x)$ and why?