Ch. 6 - Mathematical Induction

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6.1 THE PRINCIPLE OF MATHEMATICAL INDUCTION

Some sets have a minimum element, while others do not.

→ $\mathbb{N}$ and $[0,1]$ have a minimum element

→ $\mathbb{Z}$ and $(0,1]$ do not have one

↳ if the set does have a minimum element, it is unique

PRINCIPLE OF MATHEMATICAL INDUCTION:

For $n>0$ and some statement $P(n)$.

If $P(1)$ is true and the implication "If $P(k)$, then $P(k+1)$" is true for all $k \in \mathbb{N}$, then $P(n)$ for all $n \in \mathbb{N}$

↳ proving the statement $\forall n \in \mathbb{N},\ P(n)$ with this is called a proof by induction.

STEPS

Base case: prove $P(1)$

Induction hypothesis: $\forall k \in \mathbb{N},\ P(k) \Rightarrow P(k+1)$

Inductive step: prove hypothesis (usually through direct proof)

Ex) Prove $P(n): 1+2+3+\ldots+n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$

Proof by induction:

Base case $P(1): 1 = \frac{1(1+1)}{2} = 1$, which is true

Induction hypothesis: If $P(k)$, then $P(k+1)$ for all $k \in \mathbb{N}$

Assume $P(k): \frac{k(k+1)}{2} = 1+2+3+\cdots+k$

Then $P(k+1): 1+2+3+\cdots+k+(k+1)$

$$= \frac{k(k+1)}{2} + (k+1)$$

$$= \frac{k(k+1)+2(k+1)}{2}$$

$$= \frac{k^2+3k+2}{2}$$

$$= \frac{(k+1)(k+1+1)}{2}$$

$$\overline{}$$
$$2$$

which is our original equation with $k+1$ instead of k

Thus by induction, the statement is true

Induction can be more generally described with the base step being $P(m)$ for some integer m

Ex) Prove $2^n > n$ for all $n \in \mathbb{N}$

Base case: $n=1$, $2^1 > 1$ which is true

Also, note $n=2$ $\quad 2^2 > 2$, $4 > 2$ which is true

So we can assume $k \geq 2$

Hypothesis: Assume $2^k > k$ for all $k \in \mathbb{N}$

Then $2^{k+1} = 2^k 2 > 2k = k + k > k+1$ b/c $k \geq 2$

Thus $2^{k+1} > k+1$, so by induction it is proved.

Ex) Prove $\forall n \in \mathbb{N}$, $3 \mid (2^{2n} - 1)$

Base case: $n=1$, $3 \mid (2^2 - 1)$, $3 \mid (4-1)$, $3 \mid 3$ which is true

Hypothesis: $3 \mid (2^{2k} - 1)$ for all $k \in \mathbb{N}$

So $3m = 2^{2k} - 1$ for some $m \in \mathbb{Z}$

$$\text{So } 2^{2(k+1)} - 1 = 2^{2k}2^2 - 1 = 4(2^{2k}) - 1 = 4(2^{2k}-1) + 3$$
$$= 4(3m) + 3$$
$$= 3(4m+1)$$

Since $4m+1 \in \mathbb{Z}$, $3 \mid (2^{2(k+1)} - 1)$ as required.

So by induction the statement is proved.

Ex) Prove $\overline{A_1 \cup A_2 \cup A_3 \cdots \cup A_n} = \bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cdots \cap \bar{A}_n$