Midterm 2 Review Worksheets

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1. Prove part (4) of the "sets lemma." Let A and B be sets.

"If $x \notin A \cap B$, then $x \notin A$ or $x \notin B$."

Assume $x \notin A \cap B$, then $x \in \overline{A \cap B}$.

By DeMorgan's, then $x \in \bar{A} \cup \bar{B}$, so $x \in \bar{A}$ or $x \in \bar{B}$

Finally $x \notin A$ or $x \notin B$.

2. Let A and B be sets. Prove that

$$(A \cap B = A) \Leftrightarrow (A \subseteq B)$$

i) if $A \cap B = A$, then $A \subseteq B$

Assume $A \cap B = A$. Then if $x \in A$, $x \in A \cap B$.
So $x \in B$. Thus $A \subseteq B$ as required.

ii) if $A \subseteq B$, then $A \cap B = A$

Assume $A \subseteq B$, then $x \in A$ and $x \in B$.
Thus $x \in A \cap B$ and $x \in A$.

So $A \cap B \subseteq A$

Let $y \in A$. Since $A \subseteq B$, $y \in B$ too.
Thus $y \in A \cap B$.

So $A \subseteq A \cap B$

Hence $A \cap B = A$

3. Let A and B be sets. Prove that

$$\underbrace{(A \times B) \cap (B \times A)}_{LHS} = \underbrace{(A \cap B) \times (B \cap A)}_{RHS}$$

i) $(A \times B) \cap (B \times A) \subseteq (A \cap B) \times (B \cap A)$

$(x,y) \in (A \times B) \cap (B \times A)$. Then $(x,y) \in A \times B$ and $(x,y) \in B \times A$

So $x \in A$ and $x \in B$ and $y \in A$ and $y \in B$.

So $x \in (A \cap B)$ and $y \in (B \cap A)$

So $(x,y) \in (A \cap B) \times (B \cap A)$, as required

ii) $(A \cap B) \times (B \cap A) \subseteq (A \times B) \cap (B \times A)$

$(x,y) \in (A \cap B) \times (B \cap A)$

So $x \in (A \cap B)$ and $y \in (B \cap A)$

So $x \in A$ and $x \in B$ and $y \in B$ and $y \in A$.
So $(x,y) \in A \times B$ and $(x,y) \in (B \times A)$

So $(x,y) \in (A \times B) \cap (B \times A)$ as required.

4. Write the definition for "a divides b" symbolically

$\forall a, b \in \mathbb{Z}$, $\exists k \in \mathbb{Z}$ s.t. $k = \frac{b}{a}$

5. Let $a, b, c, d \in \mathbb{Z}$. Prove that if $a|b$ and $c|d$, then $ac|bd$.

Assume $a|b$ and $c|d$. Then $n = \frac{b}{a}$ and $m = \frac{d}{c}$ $n, m \in \mathbb{Z}$

So $nm = \frac{bd}{ac}$, since $nm \in \mathbb{Z}$, $ac | bd$ as required.

Example: Let $a, b \in \mathbb{Z}$ with $a \neq 0, b \neq 0$.
Prove that if $a|b$ and $b|a$ then
$a = b$ or $a = -b$.

$n = \frac{b}{a}$ and $m = \frac{a}{b}$ $n, m \in \mathbb{Z}$, thus $a = \frac{b}{n} = bm = amn$

So $mn = 1$. Thus there are two cases

case 1: $m = 1$ and $n = 1$

So $a = bm = b$

case 2: $m = -1$ and $n = -1$

So $a = bm = -b$

Hence $a = \pm b$, as required.

2. "The square of any odd integer gives a remainder of 1 when divided by 4".

(a) Write the above statement in "if-then" form.

if n is odd, then $n^2 \equiv 1 \pmod 4$ so $\frac{n^2 - 1}{4} = m$ $m \in \mathbb{Z}$

(b) Prove the statement

Assume n is odd, then $n = 2k + 1$ $k \in \mathbb{Z}$

$$\frac{n^2 - 1}{4} = \frac{4k^2 - 4k + 1 - 1}{4} = \frac{4(k^2 - 4k)}{4} = k^2 - 4k$$

Since $k^2 - 4k \in \mathbb{Z}$, $4 \mid (n^2 - 1)$ so $n^2 \equiv 1 \pmod 4$

1. Prove that for any real numbers a,b, the following inequality holds

$$|a| - |b| \leq |a + b|$$

Recall the triangle inequality $|a+b| \leq |a| + |b|$

$|a+b| - |b| \leq |a|$

Let $a+b=c$ $\quad |c| - |b| \leq |c-b|$

Let $-b=d$ $\quad |c| - |-d| \leq |c+d|$

$|c| - |d| \leq |c+d|$

Thus the inequality holds.

2. Let a, b be positive real numbers such that $0 < a < b$.

(a) Prove that $a < \frac{a+b}{2} < b$

Assume $0<a<b$, then $a+a<a+b$ and $b+b>a+b$

So $a+a<a+b<b+b$, thus $\frac{a+a}{2} < \frac{a+b}{2} < \frac{b+b}{2}$.

Hence $a< \frac{a+b}{2} < b$, as required.

(b) Prove that $a < \sqrt{ab} < b$

$a \cdot a < ab < bb$, so $\sqrt{a^2} < \sqrt{ab} < \sqrt{b^2}$

Hence $a < \sqrt{ab} < b$, as required.

4. Let $a, b, c, d \in \mathbb{R}$. Is the conjecture

$$\text{if } 0 < \frac{a}{b} < \frac{c}{d}, \text{ then } \frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

true or false? Prove your answer.

It is false. For a counterexample consider $a=1$ $b=2$ $c=-3$ $d=-4$

Then $0< \frac{1}{2} < \frac{-3}{-4}$, $0< 0.5 < 0.75$, as required

However $\frac{1}{2} < \frac{1-3}{2-4} < \frac{-3}{-4}$, $0.5 < 1 < 0.75$, which is false. Thus the statement is false.

alternatively, if we choose such that $b+d=0$, $\frac{a+c}{b+d}$ is undefined. Which would make the statement false.

5. Use proof by contradiction to prove the following statement:

"The negative of an irrational number is irrational"

Assume to the contrary that the negative of an irrational number is rational.

Then $-i=q$ $\quad i \in \mathbb{I}, q \in \mathbb{Q}$ and $q = \frac{n}{m}$ $\quad n,m \in \mathbb{Z}$

So $i = \frac{-n}{m}$ however an irrational number i cannot be represented as the quotient of two integers, so this is a contradiction

2. Is the number $\sqrt{2} + 8$ irrational? Why or why not? Prove your answer.

Irrational. Assume to the contrary that $\sqrt{2}+8$ is rational, then $\sqrt{2} + 8 = \frac{p}{q}$ $p, q \in \mathbb{Z}$

so $\sqrt{2} = \frac{p}{q} - 8 = \frac{p-8q}{q}$ Since $p-8q, q \in \mathbb{Z}$

then $\sqrt{2}$ is rational, but this is a contradiction.

Thus $\sqrt{2}+8$ is rational.

3. Prove that there do not exist positive integers a and n such that $a^2 + 3 = 3^n$.

$\sim(\exists a, n \in \mathbb{Z}^+ \text{ s.t. } a^2+3=3^n) = \forall a, n \in \mathbb{Z}^+, a^2+3 \neq 3^n$

Assume to the contrary that $\exists a, n \in \mathbb{Z}^+$ st. $a^2+3=3^n$

Consider $n=1$, then $a^2+3=3$ which is impossible as $a^2>0$
So we can see $n \geq 2$

Take $a^2+3=3^n$ so $a^2 = 3^n - 3 = 3(3^{n-1}-1)$

$$\frac{a^2}{3} = 3^{n-1} - 1$$

by lemma from class

So $3 | a^2$, so $3 | a$, thus $\frac{a}{3} = k$ $k \in \mathbb{Z}$ $k>0$

Thus $a^2+3 = 9k^2+3 = 3^n$

$$3 = 3^n - 9k^2 = 9(3^{n-2} - k^2)$$

$$\frac{3}{9} = 3^{n-2} - k^2$$

since $n \geq 2$, $3^{n-2} - k^2 \in \mathbb{Z}$, thus $9 | 3$ which is a contradiction.

5. Let x be a positive real number. Prove

$$x - \frac{2}{x} > 1 \Rightarrow x > 2$$

by

1. a direct proof:

Assume $x - \frac{2}{x} > 1$, then $x^2 - x - 2 > 0$
$(x-2)(x+1) > 0$

So $x+1>0$ and $x-2>0$. Since $x>0$, $x+1>0$ already.

For $x-2>0$, then $x>2$ as required.

2. a proof-by-contrapositive:

Assume $x \le 2$, and since $x>0$

$1 \le \frac{2}{x}$ $\quad -1 \ge -\frac{2}{x}$ $\quad x-1 \ge x-\frac{2}{x}$

Since $2 \ge x$, $1 \ge x-1$ so $1 \ge x-\frac{2}{x}$ as required.

3. a proof-by-contradiction:

Assume to the contrary that $x-\frac{2}{x}>1$ and $x \le 2$

$$x^2-x-2=(x-2)(x+1)$$

So $x+1>0$ and $x-2>0$. Since $x>0$, $x+1>0$ already.

For $x-2>0$, then $x>2$, which contradicts our assumption $x \le 2$.

1. Let

$$S=\{p+q\sqrt{2} : p,q \in \mathbb{Q}\}, \qquad T=\{a+b\sqrt{3} : a,b \in \mathbb{Q}\}$$

Prove that $S \cap T = \mathbb{Q}$.

Hint: You may use the fact that $\sqrt{6}$ is irrational without proving it.

We must prove

i) $S \cap T \subseteq \mathbb{Q}$

Assume to the contrary $S \cap T \not\subseteq \mathbb{Q}$

Let $x \in S \cap T$, then $x \in S$ and $x \in T$.

So $x=p+q\sqrt{2}$ and $x=a+b\sqrt{3}$ for $p,q,a,b \in \mathbb{Q}$

so $p+q\sqrt{2}=a+b\sqrt{3}$

$$p-a=b\sqrt{3}-q\sqrt{2}$$

$$(p-a)^2=3b^2-2bq\sqrt{2}\sqrt{3}+2q^2$$

$$p^2-2pa+a^2=3b^2-2bq\sqrt{6}+2q^2$$

$$\sqrt{6}=\frac{p^2-2pa+a^2-3b^2-2q^2}{2bq}=\frac{k}{j} \qquad k,j \in \mathbb{Z}$$

However, since $k,j \in \mathbb{Z}$, $\sqrt{6} \in \mathbb{Q}$, which is a contradiction.

ii) $\mathbb{Q} \subseteq S \cap T$

Let $x \in \mathbb{Q}$, then $x=p+q\sqrt{2}$ $\quad p,q \in \mathbb{Q}$ where $q=0$
and $x=a+b\sqrt{3}$ $\quad a,b \in \mathbb{Q}$ where $b=0$

So $x \in S$ and $x \in T$, so $x \in S \cap T$ as required.

3. A "pythagorean triple" is a triple of positive integers (a, b, c) that satisfies $a^2 + b^2 = c^2$. Prove that there exists a pythagorean triple (a, b, c) such that $a < b < c$.

Consider $a=3$, $b=4$, and $c=5$.

$a^2 + b^2 = 3^2 + 4^2 = 9 + 16 = 25 = c^2$

and $a<b<c$, and $a>0, b>0, c>0$ as required

4. Prove that the polynomial $p(x) = x^3 + x - 1$ has at least one real root.

$p(1) = 1 + 1 - 1 = 1$

$p(-1) = -1 + -1 - 1 = -3$

By intermediate value theorem, since $p(x)$ is continuous and $p(1) = 1 > 0$ and $p(-1) = -3 < 0$, then there exists a real value a between 1 and -1 such that $p(a) = 0$

5. Prove that there is a unique set A such that for every set B, $A \cup B = B$.

Consider $A = \emptyset$. Let B be any set, then $A \cup B = \emptyset \cup B = B$, as required.

To prove the uniqueness, we must prove

That if $A \cup B = B$ and $C \cup B = B$, then $A = C$.

Suppose $C \cup B = B$ and $C \neq \emptyset$. Then

Consider $B = \emptyset$. Then $C \cup \emptyset = \emptyset$, so $C = \emptyset$

However, this contradicts our assumption $C \neq \emptyset$.

So the solution is unique.

1. Disprove the statement

"There exists an integer x such that $x^2 - 2x + 7 = 0$."

$\exists x \in \mathbb{Z}$ s.t. $x^2 - 2x + 7 = 0$

we will instead prove the negation $\forall x \in \mathbb{Z}$, $x^2 - 2x + 7 \neq 0$

$x^2 - 2x + 1 + 6 \neq 0$
$(x-1)^2 + 6 \neq 0$

Since $(x-1)^2 \geq 0$ and $6 > 0$, then $(x-1)^2 + 6 > 0$,
so $(x-1)^2 + 6 \neq 0$, as required.

2. Disprove the statement

"There exists an integer k such that $4k + 3$ is a perfect square"

We will prove the negation $\forall k \in \mathbb{Z}$, $4k+3 \neq n^2$ $n \in \mathbb{Z}$

Suppose to the contrary $\exists k \in \mathbb{Z}$ s.t $4k+3 = n^2$ $n \in \mathbb{Z}$

$4k+3 = 2(2k+1)+1$, since $2k+1 \in \mathbb{Z}$, $4k+3$ is odd.

So n^2 is odd. Thus n is odd. So $n = 2m+1$ $m \in \mathbb{Z}$

Thus $4k+3 = 4m^2+4m+1 = 2(2m^2+2m)+1$

$$3-1 = 2(2m^2+2m) - 4k$$

$$2 = 2(2m^2+2m-2k)$$

$$1 = 2(m^2+m-k)$$

However, since $m^2+m-k \in \mathbb{Z}$, 1 is even which is a contradiction.

3. Determine if each of the following quantified statements is true or false and explain why.

(a) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ s.t. $x+y=1$

Negation $\exists x \in \mathbb{R}$ s.t $\forall y \in \mathbb{R}$, $x+y \neq 1$

The negation is false, so (a) is True.

Consider $y = 1-x$, $y \in \mathbb{R}$ and $x+y = x+1-x = 1-1$

So for each x, there does exist a y.

(b) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ s.t. $(x+y=2) \wedge (2x-y=1)$

Negation: $\exists x \in \mathbb{R}$ s.t. $\forall y \in \mathbb{R}$, $(x+y \neq 2) \vee (2x-y \neq 1)$

Consider $x=2$ and $y=0$.

Then $x+y = 2 \neq 2$, and $2x-y = 4 \neq 1$ as required.

Thus the negation is true

has nothing to do w/ $\forall$

just an example...

(c) $\exists x \in \mathbb{R}$ s.t. $\forall y \in \mathbb{R}, xy = 0$

Consider $x=0$, then $\forall y \in \mathbb{R}$ $xy=0$ which is true.

(d) $\exists x \in \mathbb{R}$ s.t. $\forall y \in \mathbb{R}, y \neq 0 \Rightarrow xy = 1$

Negation: $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ s.t. $y \neq 0$ and $xy \neq 1$

Negation is true, so original is false.

Let $x \in \mathbb{R}$. Consider $y = \frac{2}{x}$. then $y \neq 0$ and $xy = x\frac{2}{x} = 2 \neq 1$

Let $x \in \mathbb{R}$. consider $y = \frac{2}{x}$. then $y \neq 0$ and $xy = x\frac{2}{x} = 2 \neq 1$

4. For each of the following quantified statements, determine the negation and decide which statement (the original or the negation) is true. Prove your answer

   (a) $\forall x \in \mathbb{Z}$, if $x > 0$ then $\exists y \in \mathbb{Q}$ s.t. $x^2 > y$

$\sim(P \to Q) \equiv P \wedge \sim Q$

Negation: $\exists x \in \mathbb{Z}$ s.t. $x > 0$ and $\forall y \in \mathbb{Q}$, $x^2 \leq y$

Negation is false, so original is true.

Let $x \in \mathbb{Z}$ and $x > 0$. Then $x > 0$. so $x^2 > 0$.

Consider $y = 0$. Then since $x^2 > 0$ $x^2 > y$ as required.

   (b) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}$, if $x < y$ then $\exists q \in \mathbb{R}$ s.t. $x + q < y$

Negation: $\exists x, y \in \mathbb{R}$ s.t. $x < y$ and $\forall q \in \mathbb{R}$, $x + q \geq y$

Negation is false. Original is false.

Take $x \in \mathbb{R}$ and $y \in \mathbb{R}$. If $x < y$, consider $q = 0$.

Thus $x + q = x < y$ as required.

5. Consider the following two statements:

   (a) For all $w \in \mathbb{R}$, there exists $x \in \mathbb{R}$ such that $w < x$.

   (b) There exists $y \in \mathbb{R}$ such that for all $z \in \mathbb{R}$, $y < z$.

   One of these is true, the other is false. Determine which is which and prove your answers.

(a) $\forall w \in \mathbb{R}$, $\exists x \in \mathbb{R}$ s.t $w < x$ True

Let $w \in \mathbb{R}$. consider $x = w + 1$.

Thus $w < w + 1$ which is true.

(b) $\exists y \in \mathbb{R}$ s.t. $\forall z \in \mathbb{R}$, $y < z$ False

easier $\exists, \forall$

let $e \in E$. consider $a \in A$.

Consider negation $\forall y \in \mathbb{R}$, $\exists z \in \mathbb{R}$ s.t $y \geq z$

Let $y \in \mathbb{R}$. Consider $w = y$.

Thus $y \geq w = y$, so $y = y$ which is True.

Hence (b) is false.

1. Let $A = \mathbb{N}$ and $B = \mathbb{Q}$

   (a) Give an example of a relation R from A to B that is a function.

$R = \{(a, 1) : a \in \mathbb{N}\}$

must not have double mapped x coordinate.
must have all x coordinates defined.

(b) Give an example of a relation R from A to B that is *not* a function.

$R = \{(1,1)(1,2)(2,2)\}$

$R = \{(1,b) : b \in \mathbb{Q}\}$

2. Determine the range of the functions $f : \mathbb{R} \to \mathbb{R}$ defined below. Prove your answers.

(a) $f(x) = \dfrac{x^2}{1+x^2}$

$x \in \mathbb{R}$ so $x^2 \geq 0$, so $f(x) \geq 0$.

Also, $1 + x^2 > x^2$, so $\dfrac{x^2}{1+x^2} < 1$, so $f(x) < 1$

ran $f = [0, 1)$

2. Determine the range of the functions $f : \mathbb{R} \to \mathbb{R}$ defined below. Prove your answers.

$\cancel{(a)\ f(x) = \frac{x^2}{1+x^2}}$

(b) $f(x) = \dfrac{x}{1+|x|}$

ran $f = (-1, 1)$

$1 + |x| > |x|$, so $\dfrac{|x|}{1+|x|} < 1$

where $x > 0$

$$\frac{|x|}{1+|x|} = \frac{x}{1+x} < 1$$

where $x = 0$

$$\frac{x}{1+|x|} = 0$$

where $x < 0$

$$\frac{x}{1+|x|} = \frac{-|x|}{1+|x|} > -1$$

so $-1 < \dfrac{x}{1+|x|} < 1$, so ran $f = (-1, 1)$

3. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be defined by

$$f(a,b) = \frac{(a+1)(a+2b)}{2}$$

Prove that the image of $\mathbb{N} \times \mathbb{N}$ under f is a subset of $\mathbb{N}$.

Must prove $\frac{(a+1)(a+2b)}{2} \in \mathbb{N}$ $\qquad a,b \in \mathbb{N}$

So must show i) $(a+1)(a+2b) > 0$
and ii) $(a+1)(a+2b) = 2k$ $\in k$.

Proof of i) since $a>0$, $a+1>0$. Since $b>0$, $a+2b>0$

Thus $(a+1)(a+2b)>0$, as required.

Proof of ii)

case 1: $a=2n$, $b=2m$ $\quad n, \quad \mathbb{Z}$

$$\begin{aligned}(2n+1)(2n+2b) &= 4n^2+2n+4nb+2n+2b \\ &= 4n^2+4n+4nb+2b \\ &= 2(n^2+2n+2nb+b)\end{aligned}$$

case 2: $a=2n+1$, $b=2m$

$$\begin{aligned}(2n+1+1)(2n+1+4m) &= 4n^2+2n+8mn+4n+2+8m \\ &= 2(n^2+n+4mn+2n+4m+1)\end{aligned}$$

case 3: $a=2n$, $b=2m+1$

$$\begin{aligned}(2n+1)(2n+4m+2) &= 4n^2+8mn+6n+4m+2 \\ &= 2(2n^2+4mn+3n+2m+1)\end{aligned}$$

case 4: $a=2n+1$, $b=2m+1$

$(2n+2)(2n+4m+2) =$

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4. Let $f : A \to B$ be a function and let X and Y be subsets of B.

(a) Determine whether $f^{-1}(X \cup Y)$ must equal $f^{-1}(X) \cup f^{-1}(Y)$. Prove your answer.

$$\begin{aligned}f^{-1}(X\cup Y) &= \{a\in A : f(a)\in X\cup Y\} \\ &= \{a\in A : f(a)\in X \text{ or } f(a)\in Y\} \\ &= \{a\in A : f(a)\in X\} \cup \{a\in A : f(a)\in Y\} \\ &= f^{-1}(X)\cup f^{-1}(Y)\end{aligned}$$

So true $\qquad$ Note: $f^{-1}(X) = \{a\in A : f(a)\in X\}$

(b) Determine whether $f^{-1}(X \cap Y)$ must equal $f^{-1}(X) \cap f^{-1}(Y)$. Prove your answer.

$$\begin{aligned}f^{-1}(X\cap Y) &= \{a\in A : f(a)\in X\cap Y\} \\ &= \{a\in A : f(a)\in X \text{ and } f(a)\in Y\} \\ &= \{a\in A : f(a)\in X\} \text{ and } \{a\in A : f(a)\in Y\} \\ &= f^{-1}(X)\cap f^{-1}(Y)\end{aligned}$$

so true

5. Let $f : A \to B$ be a function and let $C \subseteq A$. Give an example for which $f^{-1}(f(C)) \not\subseteq C$.

Consider $A = \{1, 2, 3\}$ $B = \{a, b, c\}$ $C = \{1, 2\}$ so $C \subseteq A$

$f = \{(1, a)(2, b)(3, b)\}$

$f(C) = f(\{1,2\}) = \{a, b\}$

$f^{-1}(f(C)) = f^{-1}(\{a, b\}) = \{1, 2, 3\}$

$\therefore$ $f^{-1}(f(C)) \not\subseteq C$

note for an injective function this would be true.