Final Worksheet Problems

Sunday, August 14, 2016 7:19 PM

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1. Prove that the function $f : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$f(x) = e^{-x^2}$$

is injective.

Take $f(a) = f(b)$. Then $e^{-a^2} = e^{-b^2}$

$\ln(e^{-2a}) = \ln(e^{-2b})$

$-2a \cancel{\ln e} = -2b \cancel{\ln e}$

$-2a = -2b$

$a = b$

$\therefore$ f is injective

2. Prove that the function $f : \mathbb{Z} - \{0\} \to \mathbb{N}$ defined by

$$f(n) = |n|$$

is surjective.

Take $a \in \mathbb{Z} - \{0\}$ For each $b \in \mathbb{N}$, there exists some a such that

$f(a) = b$.

consider $a = -b$ And since $b \in \mathbb{N}$, $a \in \mathbb{Z} - \{0\}$

$f(a) = |a| = |-b| = b$

Thus f is surjective

3. Write four different bijections $f : \mathbb{N} \to \mathbb{N}$.

$f(x) = x$

$f(x) = \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n-1 & \text{if } n \text{ is even} \end{cases}$

$f(x) = x + 2$

$f(x) = -x$

4. (a) Prove or disprove: For all non-empty sets A, there exists an injective function $f : A \to \mathcal{P}(A)$. (Note: this includes infinite sets too!)

$|A| \leq |P(A)|$ so there exists an injection from A to B

<u>Proof</u> Consider $f : A \to P(A)$, $f(a) = \{a\}$ for some $a \in A$. Thus $\{a\} \in P(A)$.

Then the function is well-defined.

Now consider $f(a_1) = f(a_2)$. Then $\{a_1\} = \{a_2\}$

So $\{a_1\} \subseteq \{a_2\}$ and $\{a_2\} \subseteq \{a_1\}$

Thus if $\{a_1\} \subseteq \{a_2\}$ so $a_1 \in \{a_2\}$ so $a_1 = a_2$ $\therefore$ injective

(b) Prove that for all non-empty sets A, there exists no surjective function $f : A \to \mathcal{P}(A)$.

$|A| \leq |P(A)|$ so not every element of $P(A)$ could be mapped to by an element of A.

<u>Proof</u> suppose to the contrary there exists some surjection $f : A \to P(A)$.

Proof suppose to the contrary there exists some surjection $f: A \to P(A)$.

Then for each $b \in P(A)$, there exists some $a \in A$ such that $f(a) = b$

assume $b = \{a \in A : a \notin f(a)\}$

these are elements of A for which there is something in P(A) being mapped to

5. (a) For the set $A = \{1, 2\}$ and the function $f : A \to \mathcal{P}(A)$ defined by $f(1) = \emptyset$, $f(2) = \{1\}$, determine the set B in the proof that there exists no surjective function $f : A \to \mathcal{P}(A)$.

$A = \{1,2\}$ $P(A) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

$f = \{(1, \emptyset), (2, \{1\})\}$

$\{2\}$ and $\{1,2\}$ are not mapped to

$B = \{a \in A : a \notin f(a)\} = \{$ 1 : $1 \notin \emptyset$ ✓, 2 : $2 \notin \{1\}$ ✓ $\}$

$= \{1,2\}$

(b) Let A be any non-empty set and define $f : A \to \mathcal{P}(A)$ by $f(a) = \{a\}$. Determine the set B in the proof that there exists no surjective function $f : A \to \mathcal{P}(A)$.

$f(a) = \{a\}$

$B = \{a \in A : a \notin f(a)\}$

a : $a \notin \{a\}$ so nothing

$B = \emptyset$

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1. (a) For the set $A = \{1, 2\}$ and the function $f : A \to \mathcal{P}(A)$ defined by $f(1) = \emptyset$, $f(2) = \{1\}$, determine the set $B = \{a \in A : a \notin f(a)\}$

$A = \{1,2\}$ $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

$f = \{(1, \emptyset)\ (2, \{1\})\}$

$B = \{1,2\}$ $B \in P(A)$

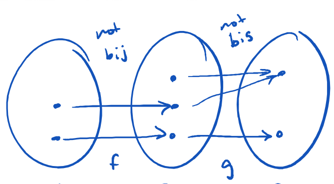
(b) Let A be any non-empty set and define $f : A \to \mathcal{P}(A)$ by $f(a) = \{a\}$. Determine the set $B = \{a \in A : a \notin f(a)\}$

$A = \{a, b$

$B = \emptyset$

2. Prove that for all non-empty sets A, there exists no surjective function $f : A \to \mathcal{P}(A)$.

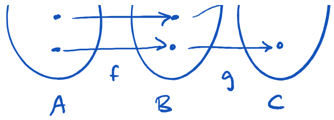
3. Give an example of two functions $f : A \to B$ and $g : B \to C$ so that the composition $g \circ f : A \to C$ is a bijection, but f and g are not bijections. (Note: you must specify the sets A, B and C too)



$A = \{1,2\}$ $f = \{(1,b), (2,c)\}$

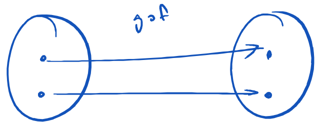
$B = \{a,b,c\}$ $g = \{(a,\theta)\ (b,\theta), (c,\phi)\}$

$C = \{\theta, \phi\}$ $g \circ f = \{(1,\theta), (2,\phi)\}$



$B = \{a, b, c\}$ $\quad g = \{(a,\theta)\ (b,\theta), (c,\phi)\}$

$C = \{\theta, \phi\}$ $\quad g\circ f = \{(1,\theta), (2,\phi)\}$



$g\circ f$ bijective

↳ ∴ f injective (not necessarily surjective)

g surjective (not necessarily injective)

4. Let $f : A \to B$ and $g : B \to C$ be functions such that $g \circ f$ is bijective.
   (a) Prove that f is injective.

Assume $g\circ f$ is bijective. so $g\circ f$ is injective.

Take $f(a) = f(b)$. Then apply g to both sides so

$g(f(a)) = g(f(b))$

so $(g\circ f)(a) = (g\circ f)(b)$

Since $g\circ f$ is injective, then $a = b$. so f is injective.

   (b) Prove that g is surjective.

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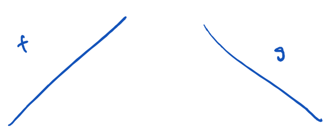
$\forall c \in C$, $\exists\, b \in B$ s.t. $g(b) = c$.

Since $g\circ f$ is surjective, then $\forall c \in C$, $\exists a \in A$ s.t. $g(f(a)) = c$

$f(a) \in B$. so $f(a) = b$, thus $g(f(a)) = g(b) = c$.

5. Let $f : A \to A$ and $g : A \to A$ be bijections, where A is a subset of $\mathbb{R}$. Show that it is not necessarily true that $f + g$ is a bijection on A.

$A \subseteq \mathbb{R}$. Consider $f(x) = x$
$g(x) = -x$



bijection

$f+g$

$(f+g)(x) = x - x = 0$

6. Prove part (b) of Theorem 10.1: Let A, B, and C be sets. If $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

↳ $f : A \to B$ bijection

$|A| = |B|$ $|B| = |C|$, so bijections $f : A \to B$, $g : B \to C$

so $f\circ g : A \to C$ is a bijection, so $|A| = |C|$

1. Let A and B be countable sets. Prove that $A \cup B$ is countable.

case 1: A and B are finite

$A\cup B$ are finite, so $A \cup B$ is countable

case 2: A is finite and B is denumerable (WLOG for vice versa)

$|A| = m$ for some $m \in \mathbb{N} \cup \{0\}$

case 2: A is finite and B is denumerable (WLOG for vice versa)

$|A| = m$ for some $m \in \mathbb{N} \cup \{0\}$

$|B| =$

2. Prove that the set of rational numbers is denumerable.

3. Prove that the set $S = \{x \in \mathbb{R} : x^2 \in \mathbb{N}\}$ is countable.

$S = \underbrace{\{\sqrt{1}, \sqrt{2}, \sqrt{3} \ldots\}}_{S_1} \cup \underbrace{\{-\sqrt{1}, -\sqrt{2}, -\sqrt{3} \ldots\}}_{S_2}$

define $f: \mathbb{N} \to S_1$, $f(n) = \sqrt{n}$

to show $|\mathbb{N}| = |S|$, we will show f is bijective

$f(a) = f(b)$ $\sqrt{a} = \sqrt{b}$ $a = b$ $\therefore$ injective

$\forall b \in S$, $\exists a \in \mathbb{N}$ s.t. $f(a) = b$

consider $a = b^2$ $f(a) = \sqrt{a} = \sqrt{b^2} = b$ $\therefore$ surjective

so $|\mathbb{N}| = |S_1| = |S_1 \cup S_2|$ so bijective

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1. Prove Theorem 10.10. (Hint: use a proof by contrapositive)

Theorem 10.10: Let A and B be sets such that $A \subseteq B$. If A is uncountable, then B is uncountable.

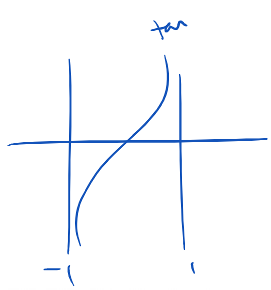
Assume B is countable.

case 1: B is finite. then A is finite. $\to$ A is countable.

case 2: B is infinite. if A is finite, then it countable.
if A is infinite, then an infinite subset of a countable set is countable, so A is countable.

2. Prove Theorem 10.14: $|(-1, 1)| = |\mathbb{R}|$

(a) Draw a graph of a function $f : (-1, 1) \to \mathbb{R}$ that is bijective. What basic properties should it have?



(b) Come up with an equation for the function you drew in part (a).

$f(x) = \tan\left(\frac{\pi}{2} x\right)$ $f(x) = \dfrac{x}{1 - |x|}$

(c) Prove that the function you created in (b) is bijective.
Hint: The proof of injective is usually easier, so you may want to start there.

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1. Let $A = \{0, 1, 4, 9, 16, ...\}$ and $B = \{0, 1, 8, 27, 64, ...\}$. Prove that $|A| = |B|$ using the Schroder-Bernstein theorem.

There exists an injection $f: A \to B$, $f(a) = a^{3/2}$ so $|A| \leq |B|$
$f: B \to A$, $f(a) = a^{2/3}$ so $|A| \geq |B|$

Thus $|A| = |B|$

2. Prove that $2^n > n$ for any natural number n.

↑ induction

Proof by induction

Base case: ~~$n=0$~~ $n=1$ $2^1 > 1$
$n=2$ $2^2 > 2$

Assume $2^k > k$ for some $k \in \mathbb{N}$

$2^{k+1} = 2^k 2 > 2k = k+k$ since $k > 2$, then $2k > k+1$
$> k+1$

so $2^{k+1} > k+1$

thus by induction...

3. Prove the following:

$$\forall n \in \mathbb{N},\ 3 | (4^n - 1)$$

Base case: $n=1$ $3 | (4^1 - 1)$ $3|3$ ✓
$3 | (4^2 - 1)$ $3 | (16-1)$
$3 | 15$ ✓

Assume $3 | (4^k - 1)$ for some $k \in \mathbb{N}$

so $3m = 4^k - 1$ for some $m \in \mathbb{N}$

Consider $3 | (4^{k+1} - 1)$
$3 | (4^k 4 - 1)$
$3 | (4(4k - 1) + 3)$
$3 | (4(3m) + 3$
$3 | 3(4m+1)$

since $4m+1 \in \mathbb{Z}$, $3 | (4^{k+1} - 1)$
so by induction its true

✓

4. Prove that $\sum_{k=1}^{n}(2k-1) = n^2$.

$\sum_{k=1}^{n} (2k-1) = n^2$

base step $n=1$ $(2 \cdot 1 - 1) = 1^2$
$1 = 1$

$n=2$ $(2 \cdot 1 - 1) + (2 \cdot 2 - 1) + (2 \cdot 3 - 1) + (2 \cdot 4 - 1)$

Inductive step: $\sum_{m=1}^{k} (2m-1) = k^2$ for some $k \in \mathbb{Z}$

$\sum_{m=1}^{k+1} (2m-1) = \underbrace{\sum_{m=1}^{k} (2m-1)} + (2(k+1) - 1)$

$= k^2 + (2k + 2 - 1)$
$= k^2 + 2k + 1$
$= (k+1)^2$

$\sum_{k=1}^{\wedge} (2(k+1) - 1) = (k+1)^2$

SO BY INDUCTION!

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$$= k^2 + (2k+2-1)$$
$$= k^2 + 2k + 1$$
$$= (k+1)^2$$

$k=1$

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SO BY INDUCTION!

1. For what integers is the inequality $2^n > 2n+1$ true? Prove your answer.

$2^n > 2n+1$

$n=1$: $2 > 3$ ✗
$n=2$: $4 > 4+1$ ✗
$n=3$: $8 > 7$ ✓
$n=4$: $16 > 9$ ✓

$S = \{n \in \mathbb{Z} : n \geq 3\}$

$2(k+1)+1$
$2k+2+1$
$2k+3$

Base case: $n=3$ $\quad 2^3 > 2(3)+1$
$8 > 7$ ✓

Inductive step: assume $2^k > 2k+1$ for all $k \in \mathbb{Z}$, $k \geq 3$

hypothesis: $2^{(k+1)} = 2^k 2 = 2^k + 2^k$ $\quad$ since $k \geq 3$, $2^k \geq 8$
$> 2^k + 8$
$> 2^k + 3 = 2k+2+1 = 2(k+1)+1$

so $2^{k+1} > 2(k+1)+1$, so by induction statement is true

3. Prove that $\{1+1/n\}$ converges to 1.

$\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$ for all $n > N$.

$\left|1 + \frac{1}{n} - 1\right| < \varepsilon$

$\left|\frac{1}{n}\right| < \varepsilon$

$N = \left\lceil \frac{1}{\varepsilon} \right\rceil$

So $n > N \geq \frac{1}{\varepsilon}$ so $\frac{1}{\varepsilon} < n$

$\frac{1}{n} < \varepsilon$

$\left|\frac{1}{n}\right| < \varepsilon$

$\frac{1}{n} < \varepsilon$

$\frac{1}{\varepsilon} < n$

As required, so $\{1 + \frac{1}{n}\}$ does converge to 1

4. Find the limit of the sequence $\left\{\frac{n}{2n+1}\right\}$ and prove your answer.

$\varepsilon > 0$. There exists some N for which $n > N$ is $|a_n - L| < \varepsilon$.

$L = \frac{1}{2 + \frac{1}{n}} = \frac{1}{2}$

so $\frac{n}{2n+1} - \frac{1}{2} = \frac{n}{2n+1} - \frac{1\frac{1}{2}(2n+1)}{2\frac{1}{2}(2n+1)} = \frac{n - \frac{1}{2}(2n+1)}{2n+1}$

$N = \left\lceil \frac{\frac{1}{\varepsilon} - 2}{4} \right\rceil \qquad n > N \geq \frac{\frac{1}{\varepsilon} - 2}{4}$

$= \frac{n - n - \frac{1}{2}}{2n+1} = \frac{-1}{4n+2} < \varepsilon$

$|a_n - L| = \left|\frac{n}{2n+1} - \frac{1}{2}\right| = \left|\frac{-1}{4n+2}\right| = \frac{1}{4n+2} < \frac{1}{4\left(\frac{\frac{1}{\varepsilon}-2}{4}\right)+2} = \varepsilon$

b/c $n > \frac{\frac{1}{\varepsilon} - 2}{4}$

$\frac{-1}{4n+2} < \varepsilon$

so $\left|\frac{n}{2n+1} - \frac{1}{2}\right| < \varepsilon$ $\therefore$ converges to $\frac{1}{2}$

$\frac{-1}{\varepsilon} < 4n+2$

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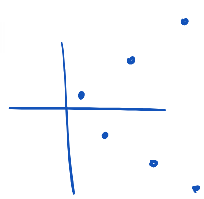
$\frac{-1}{\varepsilon} - 2 < 4n$

$\frac{\frac{-1}{\varepsilon} - 2}{4} < n$

1. Prove that the sequence $\{(-1)^n n\}$ diverges.



$\{(-1)^n n\} = \{-1, 2, -3, 4 \ldots\}$

Assume to the contrary that $\{(-1)^n n\}$ converges to some value L.

$\varepsilon > 0$. There exists some $N < n$ such that $|(-1)^n n - L| < \varepsilon$.

Consider $\varepsilon = \frac{1}{2}$, and $n = N+1$ as well as $n = N+2$

so $|(-1)^{N+1}(N+1) - L| < \frac{1}{2}$

$|(-1)^{N+2}(N+2) - L| < \frac{1}{2}$

$$|(-1)^{N+2}(N+2)-L|<\frac{1}{2}$$

case 1: N is even

$|-N-1-L|<\frac{1}{2}$ which indicates $L<0$

$|N+2-L|<\frac{1}{2}$ which indicates $L>0$

contradiction

same as when N is odd

2. Prove that the sequence $\{n^2-n\}$ diverges to infinity.

Let $M\in\mathbb{R}^+$. Consider $N=\lceil M\rceil+1$ and $n>N$

Then $n>N\geq 2$

$a_n=n^2-n=n(n-1)$ and since $n\geq 2$, $(n-1)\geq 1$

so $n(n-1)\geq n(1)$

So $a_n\geq n>N>M$ so $a_n>M$ for all $n>N$

3. Determine if the following series converges or diverges. Prove your answer.

$$\sum_{k=1}^{\infty}\frac{2}{\cancel{k}^2+4\cancel{k}+3}$$

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Hint: Consider a partial fraction decomposition.

$$\frac{2}{(n+1)(n+3)}=\frac{A}{(n+1)}+\frac{B}{(n+3)}$$

$$2=(n+3)A+(n+1)B$$

$n=-3\rightarrow 2=-2B\rightarrow B=-1$

$n=-1\rightarrow 2=2A\rightarrow A=1$

$$\sum_{n=1}^{\infty}\frac{1}{n+1}+\frac{-1}{n+3}$$

$n=1$ $\quad\frac{1}{2}-\frac{1}{4}$

$n=2$ $\quad\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{5}$

$n=3$ $\quad\frac{1}{2}-\cancel{\frac{1}{4}}+\frac{1}{3}-\frac{1}{5}+\cancel{\frac{1}{4}}-\frac{1}{6}$

$n=4$ $\quad\frac{1}{2}-\cancel{\frac{1}{4}}+\frac{1}{3}-\cancel{\frac{1}{5}}+\cancel{\frac{1}{4}}-\frac{1}{6}+\cancel{\frac{1}{5}}-\frac{1}{7}$ $\quad+\frac{1}{6}-\frac{1}{6}$

$$\boxed{S_n=\frac{1}{2}+\frac{1}{3}-\frac{1}{n+2}-\frac{1}{n+3}}$$

$\lim_{n\to\infty}S_n=\frac{5}{6}$ $\quad L=\frac{1}{2}+\frac{1}{3}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$

converges

proof

$$\left\{\frac{2}{n^2+4n+3}\right\}$$

$\varepsilon>0$. There exists some N where for $n>N$.

$$|S_n-L|=\left|\frac{1}{2}+\frac{1}{3}-\frac{1}{n+2}-\frac{1}{n+3}-\frac{5}{6}\right|$$

$$\left|\frac{-1}{n+2}-\frac{1}{n+3}\right|<\varepsilon$$

$$\frac{1}{n+2}+\frac{1}{n+3}<\frac{1}{n+2}+\frac{1}{n+2}=\frac{2}{n+2}$$

$$<\frac{2}{\frac{2}{\varepsilon}-2+2}=\varepsilon$$

$\frac{2}{n+2}<\varepsilon$

$\frac{2}{\varepsilon}-2<n$

so take $N=\lceil\frac{2}{\varepsilon}-2\rceil$

$$< \frac{2}{\frac{2}{\varepsilon}-2+2} = \varepsilon$$

So $|s_n - L| < \varepsilon$ $\therefore$ converges

so take $N = \lceil \frac{2}{\varepsilon} - 2 \rceil$

4. Prove Result 12.10: $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

Suppose to the contrary $\sum_{k=1}^{\infty} \frac{1}{k}$ converges.

Then $\varepsilon > 0$. There exists some $N < n$ such that

Consider $\varepsilon = 1$. Then

$|s_n - L| < 1$ so $-1 < s_n - L < 1$

Example Problems

Problem 1: Prove that $\left\{\frac{2n^2+1}{3n-1}\right\}$ diverges to infinity

$\forall M \in \mathbb{R}^+ \ \exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow a_n > M$

Consider $N = \lceil \frac{3M}{2} \rceil$

since $n > N$

$$\frac{2n^2+1}{3n-1} > \frac{2n^2}{3n} = \frac{2n}{3} > \frac{2\frac{3}{2}M}{3} = M$$

so $\frac{2n^2+1}{3n-1} > M$, as required.

$\frac{2n^2+1}{3n-1} > M$

$\frac{2n^2+1}{3n-1} > M$

$\frac{2n^2+1}{3n-1} > \frac{2n^2}{3n} = \frac{2n}{3} > M$

$n > \frac{3M}{2}$

Problem 2: Find the limit of the sequence $\left\{\frac{n^2+1}{n^2+100n}\right\}$ and prove your answer.

$$\frac{n^2+1}{n^2+100n} = \frac{\frac{n^2}{n^2} + \frac{1}{n^2}}{\frac{n^2}{n^2} + \frac{100n}{n^2}} = \frac{1+\frac{1}{n^2}}{1+\frac{100}{n}} \to L = 1$$

$\varepsilon > 0$. There exists some $N \in \mathbb{N}$, such that if $n > N$

$$\left|\frac{n^2+1}{n^2+100n} - 1\right| < \varepsilon$$

Consider $N = \lceil \frac{100}{\varepsilon} \rceil$ $n > N$

$$\left|\frac{n^2+1}{n^2+100n} - 1\right| = \left|\frac{n^2+1-n^2-100n}{n^2+100n}\right| = \left|\frac{1-100n}{n^2+100n}\right|$$

since $n \geq 1$, then $1 - 100n < 0$

so $|1-100n| = 100n - 1$

$$= \frac{100n-1}{n^2+100n} < \frac{100n}{n^2+100n} = \underbrace{\frac{100}{n}}_{} < \varepsilon$$

$$= \frac{100n-1}{n^2+100n} < \frac{100n}{n^2+100n} = \underbrace{\frac{100}{n} < \varepsilon}$$

So choose $N = \left\lceil \frac{100}{\varepsilon} \right\rceil$

Problem 3:

(a) Prove that $\sum_{k=2}^{n} \frac{2}{k^2-1} = 1+\frac{1}{2}-\frac{1}{n}-\frac{1}{n+1}$

(b) The series converges.

Proof by induction: Base case $n=2$: $\frac{2}{4-1} = 1+\cancel{\frac{1}{2}}-\cancel{\frac{1}{2}}-\frac{1}{3}$

$$\frac{2}{3} = \frac{3}{3}-\frac{1}{3} = \frac{2}{3} \checkmark$$

Assume $\sum_{m=2}^{k} \frac{2}{m^2-1} = 1+\frac{1}{2}-\frac{1}{k}-\frac{1}{k+1}$

IH $\sum_{m=2}^{k+1} \frac{2}{m^2-1} = \sum_{m=2}^{k} \frac{2}{m^2-1} + \frac{2}{(k+1)^2-1}$ $\qquad 1+\frac{1}{2}-\frac{1}{k+1}-\frac{1}{k+1+1}$

$$= 1+\frac{1}{2}-\frac{1}{k}-\frac{1}{k+1}+\frac{2}{(k+1)^2-1}$$

$$= 1+\frac{1}{2}-\frac{1}{k+1}+\underbrace{\frac{2}{(k+1)^2-1}-\frac{1}{k}}$$

$$\frac{2}{k^2+2k+1-1}-\frac{1}{k} \quad \frac{k\,2}{k(k^2+2k)}-\frac{1}{k}\frac{(k^2+2k)}{(k^2+2k)} \quad \frac{2k-k^2-2k}{k(k^2+2k)} \quad \frac{-k^2}{k^3+2k^2} \quad \frac{-1}{k+2}$$

$$= 1+\frac{1}{2}-\frac{1}{k+1}-\frac{1}{(k+1)+1}$$

Thus by induction its true.

As $n\to\infty$, $\sum_{k=2}^{n} \frac{2}{k^2-1} = 1+\frac{1}{2} = \frac{3}{2}$

$\varepsilon > 0$. There exists some N for which all $n > N$

$|S_n - L| < \varepsilon$

Consider $N = \left\lceil \frac{2}{\varepsilon} \right\rceil$

$$\left|1+\frac{1}{2}-\frac{1}{n}-\frac{1}{n+1}-\frac{3}{2}\right|$$

$$= \left|-\frac{1}{n}-\frac{1}{n+1}\right|$$

$$= \frac{1}{n}+\frac{1}{n+1} \qquad \frac{1}{n+1} < \frac{1}{n} \text{ since } n>0$$

$$< \frac{1}{n}+\frac{1}{n} = \frac{2}{n} \le \frac{2}{\frac{2}{\varepsilon}} = \varepsilon \quad \text{as required.}$$

Problem 4: Prove that if $\{a_n\}$ converges, then for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $m > N$ and $n > N$ then $|a_m - a_n| < \varepsilon$.

$|a_n - L| < \frac{\varepsilon}{2}$ $\qquad |a_m - L| < \frac{\varepsilon}{2}$

$$|a_n - a_m| = |a_n - a_m + L - L|$$

$$= |(a_n - L) - (a_m - L)|$$

$$\le |a_n - L| + |-(a_m - L)| \quad \text{by triangle inequality}$$

$$= |a_n - L| + |a_m - L|$$

$\leq |a_n - L| + |-(a_m - L)|$ by triangle inequality

$= |a_n - L| + |a_m - L|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\therefore \quad |a_n - a_m| < \varepsilon$

Problem 5: Prove or disprove: "If $f: A \to B$ is a function and X and Y are non-empty subsets of A, then $f(X) \cap f(Y) \subseteq f(X \cap Y)$."

Disproof: Consider counterexample

$A = \{1, 2, 3\}$ $\quad X = \{1, 2\}$ $\quad Y = \{2, 3\}$

$B = \{a, b\}$

$f = \{(1, a)\ (2, b)\ (3, a)\}$

$f(x) = \{a, b\}$ $\quad f(Y) = \{b, a\}$ $\quad f(X \cap Y) = \{b\}$

$f(X) \cap f(Y) = \{a, b\}$

$\{a, b\} \nsubseteq \{b\}$

Problem 6: Let $f: A \to B$ and let X, Y be subsets of A. Prove or disprove

$$X \subseteq Y \Rightarrow f(X) \subseteq f(Y)$$

$X \subseteq A$, $Y \subseteq A$ $\quad A = \{1, 2, 3\}$ $\quad B = \{a, b\}$ $\quad f = \{(1, a)\ (2, a)\ (3, a)\}$

$X = \{1, 2\}$ $\quad Y = \{1, 2, 3\}$

$f(x) = \{a\}$ $\quad f(Y) = \{a\}$

Prove

Assume $X \subseteq Y$. Then

let $a \in f(X)$. Then $b \in X$ s.t. $a = f(b)$

Since $X \subseteq Y$, then $b \in Y$. so $f(b) \in f(Y)$

since $f(b) = a$, $a \in$

Since $a \in f(x)$, then

Problem 7: Use the Schroder-Bernstein Theorem to prove that $|\mathbb{Q}| = |\mathbb{N}|$

Problem 8

(a) Consider the set $A = \mathbb{Z} - \{220\}$. Show that $|A| = |\mathbb{Z}|$

(b) Consider the set $B = \mathbb{R} - \mathbb{Z}$. Prove that B is uncountable

a) consider the function

$f : A \to \mathbb{Z}$

$f(x) = \{$

b) Assume to the contrary that B is countable.

Case 1: B is finite

However, B is infinite so this is a contradiction

case 2: B is denumerable

note: $\mathbb{Z} \cup B = \mathbb{R}$

we know $\mathbb{Z}$ is denumerable and B is denumerable.

we know the union of two denumerable sets is denumerable.

But we know $\mathbb{R}$ is not denumerable, so there's a contradiction.

Thus B is uncountable.