Ch. 4 - Set proofs

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4.4 PROOFS INVOLVING SETS

Intersection $A \cap B = \{x : x \in A \text{ and } x \in B\}$ 

Union $A \cup B = \{x : x \in A \text{ or } x \in B\}$ 

Difference $A - B = \{x : x \in A \text{ and } x \notin B\}$ 

Ex) Prove for every 2 sets A & B, $\boxed{A - B = A \cap \overline{B}}$

For $A - B = A \cap \overline{B}$, we must show (i) $A - B \subseteq A \cap \overline{B}$ and (ii) $A \cap \overline{B} \subseteq A - B$.

Proof of i)

Let $x \in A - B$. Then $x \in A$ and $x \notin B$.
So $x \in A$ and $x \in \overline{B}$, thus $x \in A \cap \overline{B}$.

Hence $A - B \subseteq A \cap \overline{B}$

Proof of ii)

Let $x \in A \cap \overline{B}$. Then $x \in A$ and $x \in \overline{B}$.
So $x \in A$ and $x \notin B$, thus $x \in A - B$

Hence $A \cap \overline{B} \subseteq A - B$.

Thus by (i) and (ii), $A - B = A \cap \overline{B}$

Ex) Prove for every 2 sets A & B $\boxed{(A \cup B) - (A \cap B) = (A - B) \cup (B - A)}$

We must prove

i) $(A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A)$

Let $x \in (A \cup B) - (A \cap B)$, then $x \in (A \cup B)$ and $x \notin (A \cap B)$.
So $x \in A$ or $x \in B$.

Case 1: assume $x \in A$

Since we know $x \notin (A \cap B)$, then
$x \notin A$ or $x \notin B$. But by our assumption,
it must be $x \notin B$. So $x \in A$ and $x \notin B$,
thus $x \in (A - B)$.

case 2. assume $x \in B$.

WLOG from case 1, $x \in (B-A)$

Hence $x \in (A-B)$ or $x \in (B-A)$. So $x \in (A-B) \cup (B-A)$.

Finally, we know $(A \cup B) - (A \cap B) \subseteq (A-B) \cup (B-A)$

ii) $(A-B) \cup (B-A) \subseteq (A \cup B) - (A \cap B)$

Let $x \in (A-B) \cup (B-A)$. So $x \in A-B$ or $x \in B-A$.

Case 1: $x \in A-B$

So $x \in A$ and $x \notin B$. Thus we can say
$x \in (A \cup B)$ and $x \notin (A \cap B)$. So
$x \in (A \cup B) - (A \cap B)$.

Case 2: $x \in B-A$.

WLOG from case 1, again we see
$x \in (A \cup B) - (A \cap B)$.

Hence $(A-B) \cup (B-A) \subseteq (A \cup B) - (A \cap B)$

Ex) Prove $\boxed{A \cup B = A \quad \text{iff} \quad B \subseteq A}$

i) if $A \cup B = A$, then $B \subseteq A$

Proof by contrapositive "if $B \nsubseteq A$, then $A \cup B \neq A$"

Assume $B \nsubseteq A$, then there is an element x
such that $x \in B$ and $x \notin A$. Since $x \in A$,
$x \in A \cup B$. However, $x \notin A$. So $A \cup B \neq A$.

ii) if $B \subseteq A$, then $A \cup B = A$

Assume $B \subseteq A$, then for any $x \in B$ also $x \in A$.

4.5 FUNDAMENTAL PROPERTIES OF SET OPERATIONS

Commutative	$A \cup B = B \cup A$
Associative	$A \cup (B \cup C) = (A \cup B) \cup C$
Distributive	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
DeMorgans	$\overline{A \cup B} = \bar{A} \cap \bar{B}$

Ex) Prove the version of DeMorgans law above

We will prove

i) $\overline{A\cup B} \subseteq \bar{A}\cap\bar{B}$

Let $x \in \overline{A\cup B}$, then $x \notin A\cup B$. Then $x \notin A$ and $x \notin B$. So $x \in \bar{A}$ and $x \in \bar{B}$, Hence $x \in \bar{A}\cap\bar{B}$, as required.

ii) $\bar{A}\cap\bar{B} \subseteq \overline{A\cup B}$

Let $x \in \bar{A}\cap\bar{B}$, then $x \in \bar{A}$ and $x \in \bar{B}$. So $x \notin A$ and $x \notin B$, so $x \notin A\cup B$. Hence $x \in \overline{A\cup B}$, as required.

4.6 PROOFS INVOLVING CARTESIAN PRODUCTS OF SETS

Cartesian product $\quad A\times B = \{(a,b) : a\in A \text{ and } b \in B\}$

if $A \neq \emptyset$ and $B \neq \emptyset$, then $A\times B = \emptyset$

$$\boxed{A\times(B\cup C) = (A\times B)\cup(A\times C)}$$

Ex) Prove $\boxed{\text{if } A\subseteq C \text{ and } B\subseteq D, \text{ then } A\times B \subseteq C\times D}$

Assume $A\subseteq C$ and $B \subseteq D$. Let $(x,y) = A\times B$. Then $x\in A$ and $y \in B$. And since $A\subseteq C$ and $B\subseteq D$, then $x \in C$ and $y \in D$. So $(x,y)\in C\times D$.

Thus $A\times B \subseteq C\times D$.

Ex) Prove $\boxed{A\times(B-C) = (A\times B)-(A\times C)}$

i) $A\times(B-C) \subseteq (A\times B)-(A\times C)$

Let $(x,y)\in A\times(B-C)$. Then $x\in A$ and $y \in B-C$.

Then $y\in B$ and $y \notin C$. Since $x \in A$ and $y\in B$, then $(x,y)\in A\times B$. Since $y\notin C$, then $(x,y)\notin A\times C$.

Thus $(x,y)\in(A\times B)-(A\times C)$, as required.

ii) $(A\times B)-(A\times C) \subseteq A\times(B-C)$

Let $(x,y) \in (A \times B) - (A \times C)$. Then $(x,y) \in A \times B$ and $(x,y) \notin A \times C$. So $x \in A$ and $y \in B$, but $y \notin C$.

Since $y \in B$ and $y \notin C$, $y \in B - C$. Thus because $x \in A$, $(x,y) \in A \times (B - C)$.

$=$

4.1 PROOFS INVOLVING DIVISIBILITY OF INTEGERS

a divides b, $a|b$ if $c = \frac{b}{a}$ for some $c \in \mathbb{Z}$

Ex) Prove $\boxed{\text{if } a|c \text{ and } b|d \text{ then } ab|cd}$

$=$

Assume $a|c$ and $b|d$. Then there exist integers n and m, s.t. $n = \frac{c}{a}$ and $m = \frac{d}{b}$

Then $nm = \frac{cd}{ab}$ Since $mn \in \mathbb{Z}$, $ab|cd$.

$=$

Ex) Prove $\boxed{\text{if } a|b \text{ and } a|c, \text{ then } a|(bn+cm) \quad n,m \in \mathbb{Z}}$

$=$

$p = \frac{b}{a}$, $q = \frac{c}{a}$ for some $p, q \in \mathbb{Z}$

Then $bn + cm = apn + aqm = a(pn + qm)$

$pn + qm = \frac{bn+cm}{a}$, $pn + qm \in \mathbb{Z}$, so $a|(bn+cm)$

$=$

Ex) Prove if $2|(x^2+1)$, then $4|(x^2-1)$ $x \in \mathbb{Z}$

$=$

Assume $2|(x^2+1)$, then $2k = x^2 + 1$ for some $k \in \mathbb{Z}$

$x^2 = 2k - 1 = 2(k-1) + 1$, since $k - 1 \in \mathbb{Z}$ x^2 is odd.

By theorem 3.12, if x^2 is odd, then x is odd too.

So $x = 2n + 1$, $n \in \mathbb{Z}$ thus $x^2 = (4n^2 + 4n + 1)$.

So $x^2 - 1 = 4n^2 + 4n$, $n^2 + n = \frac{x^2-1}{4}$. Since $n^2 + n \in \mathbb{Z}$ $4|(x^2-1)$

$=$

Ex) Prove if $3 \nmid (x^2 - 1)$, then $3|x$

$=$

Assume to the contrapositive that $3 \nmid x$.
Then $3n \neq x$ for some $n \in \mathbb{Z}$. Then
$3n = x + 1$ or $3n = x + 2$.

Case 1) $3n = x + 1$

So $x = 3n - 1$

$x^2 = 9n^2 - 6n + 1$

$x^2 - 1 = 9n^2 - 6n = 3(3n^2 - 6n)$

Since $3n^2 - 6n \in \mathbb{Z}$, $3 \mid (x^2 - 1)$ as required

Case 2) $3n = x + 2$

So $x = 3n - 2$

$x^2 = 9n^2 - 12n + 4$

$x^2 - 1 = 9n^2 - 12n + 3 = 3(3n^2 - 4n + 1)$

$3n^2 - 4n + 1 \in \mathbb{Z}$, $3 \mid (x^2 - 1)$ as required.

$=$

4.2 PROOFS INVOLVING CONGRUENCE OF INTEGERS

For integers a, b, and $n \geq 2$. We say a is congruent to b modulo n, $a \equiv b \pmod{n}$, if $n \mid (a - b)$.

$a \equiv b \pmod{n} \longrightarrow k = \frac{a - b}{n}, \ k \in \mathbb{Z}$

Ex) Let $a, b, k, n \in \mathbb{Z}$ where $n \geq 2$. Prove if $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $a + c \equiv (b + d) \pmod{n}$.

$=$

Assume $p = \frac{a-b}{n}$ and $q = \frac{c-d}{n}$, $p, q \in \mathbb{Z}$

Then $p + q = \frac{a + c - b - d}{n} = \frac{(a+c) - (b+d)}{n}$

And since $p + q \in \mathbb{Z}$, $a + c \equiv (b + d) \pmod{n}$

$=$

4.3 PROOFS INVOLVING REAL NUMBERS

$a^2 \geq 0$ for all $a \in \mathbb{R}$

Ex) Prove $\boxed{\text{if } xy = 0, \text{ then } x = 0 \text{ or } y = 0}$

$=$

Assume $xy = 0$. Then consider cases

case 1: $x = 0$, as required

case 2: $x \neq 0$, then $y = 0$, as required.

$=$

Ex) Prove if $x^3 - 5x^2 + 3x = 15$, then $x = 5$ where $x \in \mathbb{R}$

$=$

$$x^3 - 5x^2 + 3x - 15 = 0$$
$$(x-5)(x^2+3) = 0$$

Since $x \in \mathbb{R}$, $x^2 \geq 0$, so $x^2 + 3 > 0$

Thus $(x-5) = 0$, so $x = 5$ as required

$=$

Triangle Inequality $\boxed{|x+y| \leq |x| + |y|}$ $x, y \in \mathbb{R}$