Question 1

(a) $V = g(X, Y)$, $W = h(X, Y)$, $X = g^{-1}(v, w)$, $Y = h^{-1}(v, w)$.
$a_1 = a_2 = \cos(\theta)$, $a_3 = a_4 = \sin(\theta)$.

$$\begin{aligned} V &= g(X, Y) = X\cos(\theta) + Y\sin(\theta) \\ W &= h(X, Y) = X\sin(\theta) + Y\cos(\theta) \end{aligned}$$

$a_1 = a_2 = a_3 = 1$, $a_4 = -1$.

$$\begin{pmatrix} V \\ W \end{pmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{aligned} V &= g(X, Y) = X - Y \\ W &= h(X, Y) = X + Y \end{aligned}$$

V subtracts two Gaussian noise signals, W sums two Gaussian noise signals.

(b) Solving the system, we get:

$$\begin{aligned} x_1 = \phi_1(V, W) &= \frac{V\cos(\theta) - W\sin(\theta)}{\cos^2(\theta) - \sin^2(\theta)} \\ y_1 = \psi_1(V, W) &= \frac{W\cos(\theta) - V\sin(\theta)}{\cos^2(\theta) - \sin^2(\theta)} \end{aligned}$$

$$\begin{aligned} \left|\tilde{\mathbf{J}}_1\right| &= \left|\frac{\partial \phi_1}{\partial v}\frac{\partial \psi_1}{\partial w} - \frac{\partial \phi_1}{\partial w}\frac{\partial \psi_1}{\partial v}\right| \\ &= \Big|\frac{\partial}{\partial v}\left(\frac{V\cos(\theta) - W\sin(\theta)}{\cos^2(\theta) - \sin^2(\theta)}\right)\frac{\partial}{\partial w}\left(\frac{W\cos(\theta) - V\sin(\theta)}{\cos^2(\theta) - \sin^2(\theta)}\right) \\ &\quad - \frac{\partial}{\partial w}\left(\frac{V\cos(\theta) - W\sin(\theta)}{\cos^2(\theta) - \sin^2(\theta)}\right)\frac{\partial}{\partial v}\left(\frac{W\cos(\theta) - V\sin(\theta)}{\cos^2(\theta) - \sin^2(\theta)}\right)\Big| \\ &= \Big|\left(\frac{\cos(\theta)}{\cos^2(\theta) - \sin^2(\theta)}\right)\left(\frac{\cos(\theta)}{\cos^2(\theta) - \sin^2(\theta)}\right) \\ &\quad - \left(\frac{\sin(\theta)}{\cos^2(\theta) - \sin^2(\theta)}\right)\left(\frac{\sin(\theta)}{\cos^2(\theta) - \sin^2(\theta)}\right)\Big| \\ &= \frac{1}{\cos^2(\theta) - \sin^2(\theta)} \\ &= \frac{1}{\cos(2\theta)} \end{aligned}$$

Using multivariate transform technique, the joint pdf of $f_{VW}(v,w)$ is:

$$\begin{aligned}
f_{VM}(v,w) &= \sum_{i=1}^{n} f_{XY}(x_i, y_i)\left|\tilde{\mathbf{J}}_i\right| \\
&= f_{XY}\big(g^{-1}(v,w), h^{-1}(v,w)\big) \times \left|\tilde{\mathbf{J}}\right| \\
&= \boxed{\frac{1}{\cos(2\theta)} f_{XY}\left(\frac{V\cos(\theta) - W\sin(\theta)}{\cos^2(\theta) - \sin^2(\theta)}, \frac{W\cos(\theta) - V\sin(\theta)}{\cos^2(\theta) - \sin^2(\theta)}\right)}
\end{aligned}$$

(c) The marginal pdf of V is:

$$\begin{aligned}
f_V(v) &= \int_{-\infty}^{\infty} f_{VW}(v,w) dw \\
&= \int_{-\infty}^{\infty} \frac{1}{\cos(2\theta)} f_{XY}\left(\frac{V\cos(\theta) - W\sin(\theta)}{\cos^2(\theta) - \sin^2(\theta)}, \frac{W\cos(\theta) - V\sin(\theta)}{\cos^2(\theta) - \sin^2(\theta)}\right) dw
\end{aligned}$$

Question 2

Joint pdf: $f_{XY}(x+y) = A(x+y), \quad 0 \le x \le 1, \quad 0 \le y \le 1$

(a) Since the integral over the pdf is one:

$$\begin{aligned}
1 &= A\int_0^1 \int_0^1 (x+y)\ dxdy \\
&= A\int_0^1 \left(\frac{1}{2}x^2 + yx\right)\Big|_0^1 dy \\
&= A\int_0^1 \left(\frac{1}{2} + y\right) dy \\
&= A\left(\frac{y+y^2}{2}\right)\Big|_0^1 \\
&= \boxed{A = 1}
\end{aligned}$$

(b) Joint cdf:

$$\begin{aligned}
F_{XY}(x,y) &= \int_0^y \int_0^x f(u,v)\ dudv \\
&= \int_0^y \int_0^x u + v\ dudv \\
&= \int_0^y \frac{x^2 + 2xv}{2}\ dv \\
&= \boxed{\frac{y^2x + yx^2}{2}}
\end{aligned}$$

(c) Marginal cdfs $F_X(x)$ and $F_Y(y)$:

$$F_X(x) = F_{XY}(x, \infty) = \lim_{y \to \infty} \frac{y^2 x + y x^2}{2}$$
$$F_Y(y) = F_{XY}(\infty, y) = \lim_{x \to \infty} \frac{y^2 x + y x^2}{2}$$

Marginal pdfs $f_X(x)$ and $f_Y(y)$:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y) dy = \int_0^1 x + y \; dy = \boxed{\frac{1 + 2x}{2}}$$
$$f_Y(y) = \int_0^1 x + y \; dx = \boxed{\frac{1 + 2y}{2}}$$