Midterm 1 Review Problems

Monday, June 6, 2016 3:43 PM

1.6. The set $E = \{2x : x \in \mathbf{Z}\}$ can be described by listing its elements, namely $E = \{\dots, -4, -2, 0, 2, 4, \dots\}$. List the elements of the following sets in a similar manner.

(a) $A = \{2x + 1 : x \in \mathbf{Z}\}$
(b) $B = \{4n : n \in \mathbf{Z}\}$
(c) $C = \{3q + 1 : q \in \mathbf{Z}\}$

a) $A = \{\dots, -3, -1, 1, 3 \dots\}$

b) $B = \{\dots, -8, -4, 0, 4, 8 \dots\}$

c) $C = \{\dots -5, -2, 1, 4, 7\}$

1.16. Find $\mathcal{P}(\mathcal{P}(\{1\}))$ and its cardinality.

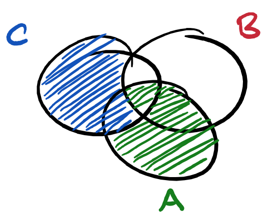
$\mathcal{P}(\{1\}) = \{\{1\}, \emptyset\}$

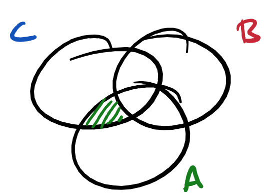
$P(\{1\}, \emptyset) = \boxed{\{\{\{1\}\}, \{\emptyset\}, \{\{1\}, \emptyset\}, \emptyset\}}$

$|P(P(\{1\}))| = \boxed{4}$

1.28. Let A, B and C be nonempty subsets of a universal set U. Draw a Venn diagram for each of the following set operations.

(a) $(C - B) \cup A$
(b) $C \cap (A - B)$.





1.36. For a real number r, define S_r to be the interval $[r - 1, r + 2]$. Let $A = \{1, 3, 4\}$. Determine $\bigcup_{\alpha \in A} S_\alpha$ and $\bigcap_{\alpha \in A} S_\alpha$.

$$\bigcup_{\alpha \in A} S_\alpha = S_1 \cup S_3 \cup S_4 = [0,3] \cup [2,5] \cup [3,6]$$
$$= \boxed{[0,6]}$$

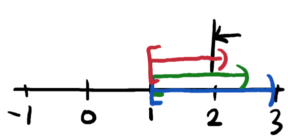
$$\bigcap_{\alpha \in A} S_\alpha = S_1 \cap S_2 \cap S_4 = [0,3] \cap [2,5] \cap [3,6]$$
$$= \boxed{\{3\}}$$

1.42. For each of the following collections of sets, define a set A_n for each $n \in \mathbf{N}$ such that the indexed collection $\{A_n\}_{n \in \mathbf{N}}$ is precisely the given collection of sets. Then find both the union and intersection of the indexed collection of sets.

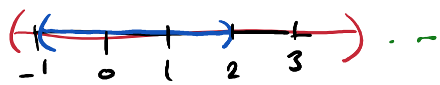
(a) $\{[1, 2 + 1), [1, 2 + 1/2), [1, 2 + 1/3), \dots\}$
(b) $\{(-1, 2), (-3/2, 4), (-5/3, 6), (-7/4, 8), \dots\}$.

a) $A_n = \{n \in \mathbb{N} : [1, 2 + \frac{1}{n})\}$

$\bigcup_{n \in \mathbb{N}} A_n = \boxed{[1,3)}$ $\bigcap_{n \in \mathbb{N}} A_n = \boxed{[1,2]}$



b) $A_n = \{n \in \mathbb{N} : (-\frac{2n-1}{n}, 2n)\}$



$\bigcup_{n\in\mathbb{N}} A_n = \boxed{(-2, \infty)}$ $\bigcap_{n\in\mathbb{N}} A_n = \boxed{(-1, 2)}$

1.46. Which of the following are partitions of $A = \{a, b, c, d, e, f, g\}$? For each collection of subsets that is not a partition of A, explain your answer.
(a) $S_1 = \{\{a, c, e, g\}, \{b, f\}, \{d\}\}$ (b) $S_2 = \{\{a, b, c, d\}, \{e, f\}\}$
(c) $S_3 = \{A\}$ (d) $S_4 = \{\{a\}, \emptyset, \{b, c, d\}, \{e, f, g\}\}$
(e) $S_5 = \{\{a, c, d\}, \{b, g\}, \{e\}, \{b, f\}\}$.

partitions: union is entire original set
intersect is nothing
no empty sets

a) S_1 ✓ b) ✗ missing g

c) ✓, $A \subseteq A$ d) ✗ includes $\emptyset$

e) ✗ double b

1.60. For $A = \{\emptyset, \{\emptyset\}\}$, determine $A \times \mathcal{P}(A)$.

$P(A) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

$A \times P(A) = \{(\emptyset, \emptyset), (\emptyset, \{\emptyset\}), (\emptyset, \{\{\emptyset\}\}), (\emptyset, \{\emptyset, \{\emptyset\}\}),$
$(\{\emptyset\}, \emptyset), (\{\emptyset\}, \{\emptyset\}), (\{\emptyset\}, \{\{\emptyset\}\}), (\{\emptyset\}, \{\emptyset, \{\emptyset\}\})\}$

2.2. Consider the sets A, B, C and D below. Which of the following statements are true? Give an explanation for each false statement.

$A = \{1, 4, 7, 10, 13, 16, \ldots\}$ $C = \{x \in \mathbf{Z} : x \text{ is prime and } x \neq 2\}$
$B = \{x \in \mathbf{Z} : x \text{ is odd}\}$ $D = \{1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots\}$

(a) $25 \in A$ (b) $33 \in D$ (c) $22 \notin A \cup D$ (d) $C \subseteq B$ (e) $\emptyset \in B \cap D$ (f) $53 \notin C$.

a) … 19, 22, 25 ✓

b) ✗

c) $22 \in A$ ✗

d) all primes except 2 are odd ✓

e) $\emptyset$ is a subset, not an element ✗

f) 53 is prime. ✗

2.6. For the open sentence $P(A) : A \subseteq \{1, 2, 3\}$ over the domain $S = \mathcal{P}(\{1, 2, 4\})$, determine:
(a) all $A \in S$ for which $P(A)$ is true.
(b) all $A \in S$ for which $P(A)$ is false.
(c) all $A \in S$ for which $A \cap \{1, 2, 3\} = \emptyset$.

$S = \{\{1\}, \{2\}, \{4\}, \{1,2\}, \{1,4\}, \{2,4\}, \{1,2,4\}, \emptyset\}$ $2^3 = 8$ ✓

a) $\{1\}, \{2\}, \{1,2\}, \emptyset$

b) $\{4\}, \{1,4\}, \{2,4\}, \{1,2,4\}$

c) $\{4\}, \emptyset$

2.16. For the sets $A = \{1, 2, \cdots, 10\}$ and $B = \{2, 4, 6, 9, 12, 25\}$, consider the statements

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$$P\colon\ A \subseteq B. \qquad Q\colon\ |A - B| = 6.$$

Determine which of the following statements are true.
(a) $P \vee Q$ (b) $P \vee (\sim Q)$ (c) $P \wedge Q$
(d) $(\sim P) \wedge Q$ (e) $(\sim P) \vee (\sim Q)$.

$P \equiv F$

$A - B = \{1, \cancel{2}, 3, \cancel{4}, 5, \cancel{6}, 7, 8, \cancel{9}, 10\}$
$- \{2, 4, 6, 9, 12, 25\}$
$= \{1, 3, 5, 7, 8, 10, 12, 25\}$

$|A - B| = 8 \quad \therefore \quad Q \equiv F$

a) $F \vee F \equiv F$

b) $F \vee T \equiv T$

c) F

d) F

e) T

2.20. For statements P and Q, construct a truth table for $(P \Rightarrow Q) \Rightarrow (\sim P)$.

$(P \to Q) \to (\sim P)$

P	Q	$P \to Q$	$\sim P$	$(P \to Q) \to \sim P$
T	T	T	F	F
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

2.22. Consider the statements:

$$P\colon \sqrt{2} \text{ is rational.} \qquad Q\colon \tfrac{2}{3} \text{ is rational.} \qquad R\colon \sqrt{3} \text{ is rational.}$$

Write each of the following statements in words and indicate whether the statement is true or false.
(a) $(P \wedge Q) \Rightarrow R$ (b) $(P \wedge Q) \Rightarrow (\sim R)$
(c) $((\sim P) \wedge Q) \Rightarrow R$ (d) $(P \vee Q) \Rightarrow (\sim R)$.

P is F, Q is T, R is F

a) if $\sqrt{2}$ is rational and $\frac{2}{3}$ is rational, then $\sqrt{3}$ is rational

$(F \wedge T) \Rightarrow F \equiv F \to F \equiv \boxed{T}$

b) if $\sqrt{2}$ is rational and $2/3$ is rational, then $\sqrt{3}$ is irrational

$F \to T \equiv \boxed{T}$

c) if $\sqrt{2}$ is irrational and $\frac{2}{3}$ is rational, then $\sqrt{3}$ is rational

$T \wedge T \to F \equiv \boxed{F}$

d) if $\sqrt{2}$ is rational or $\frac{2}{3}$ is rational, then $\sqrt{3}$ is irrational

$T \vee F \to T \equiv \boxed{T}$

2.32. In each of the following, two open sentences $P(x)$ and $Q(x)$ over a domain S are given. Determine all $x \in S$ for which $P(x) \Rightarrow Q(x)$ is a true statement.

(a) $P(x): x - 3 = 4;\ Q(x): x \geq 8;\ S = \mathbf{R}$.
(b) $P(x): x^2 \geq 1;\ Q(x): x \geq 1;\ S = \mathbf{R}$.
(c) $P(x): x^2 \geq 1;\ Q(x): x \geq 1;\ S = \mathbf{N}$.
(d) $P(x): x \in [-1, 2];\ Q(x): x^2 \leq 2;\ S = [-1, 1]$.

$P(x) \to Q(x)$ requires that x s.t. $P(x) \equiv F$ or
$P(x) \equiv T$ and $Q(x) \equiv T$

a) $(x \neq 7) \vee (\underbrace{x = 7 \wedge x \geq 8}_{\text{contradiction}})$ $\therefore$ $\boxed{x \neq 7}$

b) $(x < 1) \vee (x \geq 1)$ $\therefore$ $\boxed{x \in \mathbb{R}}$

c) $(x < 1) \vee (x \geq 1)$ but $x \not< 1$ b/c $x \in \mathbb{N}$ $\therefore$ $\boxed{x \in \mathbb{N}}$

d) $(-1 > x > 2) \vee (x \in [-1, 2] \wedge x^2 \leq 2)$

all values of $x \in S$ make $P(x)$ true, so only most make $Q(x)$ true.

$x^2 \leq 2$ if $x \leq \pm\sqrt{2}$

$\therefore$ $\boxed{x \in [-1, \sqrt{2}]}$

2.34. Each of the following describes an implication. Write the implication in the form "if, then."

(a) Any point on the straight line with equation $2y + x - 3 = 0$ whose x-coordinate is an integer also has an integer for its y-coordinate.
(b) The square of every odd integer is odd.
(c) Let $n \in \mathbf{Z}$. Whenever $3n + 7$ is even, n is odd.
(d) The derivative of the function $f(x) = \cos x$ is $f'(x) = -\sin x$.
(e) Let C be a circle of circumference 4π. Then the area of C is also 4π.
(f) The integer n^3 is even only if n is even.

$P \to Q$
P suff. for Q
Q nec. for P
if P then Q

a) $2y + x - 3 = 0$ $\underbrace{\text{x is an integer}}_{P}$ is suff. for $\underbrace{\text{y is integer}}_{Q}$

$\boxed{\text{if x is an integer, then y is an integer}}$

b) the square of every odd integer is odd
being odd is sufficient for square being odd

$\boxed{\text{if odd, then square is odd}}$

c) whenever $3n+7$ is even, n is odd

$\underbrace{3n+7 \text{ is even}}_{P}$ is suff. for $\underbrace{n \text{ is odd}}_{Q}$

$\boxed{\text{if } 3n+7 \text{ is even, then } n \text{ is odd}}$

d) the derivative of $\cos x$ is $-\sin x$

$\underbrace{f(x) \text{ being } \cos x}_{P}$ is sufficient for $\underbrace{f'(x) = -\sin x}_{Q}$

$\boxed{\text{if } f(x) = \cos(x), \text{ then } f'(x) = -\sin x}$

e) $\boxed{\text{if the circumference is } 4\pi, \text{ then the area is } 4\pi}$

e) $\boxed{\text{if the circumference is } 4\pi, \text{ then the area is } 4\pi}$

f) $\underbrace{n \text{ is even}}_{Q}$ is necessary for $\underbrace{n^3 \text{ is even}}_{P}$

$\boxed{\text{if } n^3 \text{ is even, then } n \text{ is even}}$

2.42. Determine all values of n in the domain $S = \{2, 3, 4\}$ for which the following is a true statement: The integer $\frac{n(n-1)}{2}$ is odd if and only if $\frac{n(n+1)}{2}$ is even.

if and only if $\therefore$ biconditional

$\frac{n(n-1)}{2}$ is odd $\rightarrow$ $\frac{n(n+1)}{2}$ is even

$\wedge$

$\frac{n(n+1)}{2}$ is even $\rightarrow$ $\frac{n(n-1)}{2}$ is odd

$n=2$ $\quad$ -1 is odd $\rightarrow$ 3 is even

$T \rightarrow F \equiv F$

$\therefore$ @ $n=2$, F

$n=3$ $\quad$ $\frac{3}{\cancel{2}}(\cancel{2})$ is odd $\rightarrow$ $\frac{3}{\cancel{2}}(\cancel{4})\,2$ 6 is even $\equiv T$

$\wedge$ $\quad$ 6 is even $\rightarrow$ 3 is od $\equiv T$

$n=4$ $\quad$ $2\,\frac{\cancel{4}(4-1)}{\cancel{2}}$ is odd $\rightarrow$ $\frac{\cancel{4}(4+1)}{2\,\cancel{2}}$ is even

6 is odd $\rightarrow$ 10 is even $\quad$ T

$\wedge$

10 is even $\rightarrow$ 6 is odd $\quad$ F

@ $n=4$, F

$\boxed{n=3}$

(2.48) For statements P and Q, show that $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$ is a tautology. Then state $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$ is words. (This is an important logical form, called **modus ponens**.)

We will show this is a tautology with a truth table.

P	Q	$P \Rightarrow Q$	$P \wedge (P \Rightarrow Q)$	$(P \wedge (P \Rightarrow Q)) \Rightarrow Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

(2.52) Let P and Q be statements.

(a) Is $\sim (P \vee Q)$ logically equivalent to $(\sim P) \vee (\sim Q)$? Explain.

By DeMorgans law $\sim(P \vee Q) \equiv \sim P \wedge \sim Q$ which is not equivalent to $\sim P \vee \sim Q$.

using a truth table to prove it.

P	Q	$\sim P$	$\sim Q$	$P \vee Q$	$\sim(P \vee Q)$	$\sim P \vee \sim Q$
T	T	F	F	T	F	F
T	F	F	T	T	F	T
F	T	T	F	T	F	T
F	F	T	T	F	T	T

diffrent

(b) What can you say about the biconditional $\sim (P \vee Q) \Leftrightarrow ((\sim P) \vee (\sim Q))$?

$\sim(P \vee Q)$	$\Rightarrow$	$(\sim P \vee \sim Q)$
F	T	F
F	T	T
F	T	T
T	T	T

$(\sim P \Rightarrow \sim Q)$	$\Rightarrow$	$\sim(P \vee Q)$
F	T	F
T	F	F
T	F	F
T	T	T

the biconditional is false because only one implication is true.

3. (2.54) For statements P and Q, show that $(\sim Q) \Rightarrow (P \wedge (\sim P))$ and Q are logically equivalent.

$$\sim Q \Rightarrow (P \wedge \sim P) \equiv \sim Q \Rightarrow F \quad \text{by contradiction}$$
$$\equiv (\sim\sim Q \vee F) \quad \text{by theorem 2.17}$$
$$\equiv (Q \vee F) \quad \text{by negation}$$
$$\equiv Q \quad \text{by identity law}$$

4. (2.60) Consider the implication: If x and y are even, then xy is even.

(a) State the implication using "only if"

if P, then $Q \equiv Q$ if $P \equiv P$ only if Q

x and y are even only if xy is even

(b) State the converse of the implication

$P \Rightarrow Q$ is converse of $Q \Rightarrow P$

if xy is even, x and y are even

(c) State the implication as a disjunction (see Theorem 2.17).

$P \Rightarrow Q \equiv \sim P \vee Q$

x and y are odd or xy is even (assuming $x, y \in \mathbb{Z}$)

(d) State the negation of the implication as a conjunction (see Theorem 2.21(a)).

$\sim(P \Rightarrow Q) \equiv P \wedge \sim Q$

x and y are even and xy is odd

5. (2.64) For which biconditional is its negation the following?

"n^3 and $7n+2$ are odd or n^3 and $7n+2$ are even."

~~(n^3 and $7n+2$ are odd) $\wedge$ (n^3 and $7n+2$ are even)~~

~~$P \wedge \sim Q \equiv \sim(P \Rightarrow Q)$~~

~~$\sim$(if n^3 and $7n+2$ are odd, then n^3 and $7n+2$ are odd)~~

~~n^3 and $7n+2$ are odd if and only if n^3 and $7n+2$ are odd~~

the statement could be said as

"n^3 and $7n+2$ have the same parity"

the negation could be stated as

"n^3 and $7n+2$ have opposite parity."

which could be stated as the biconditional

n^3 is even if and only if $7n+2$ is odd

2.68. State the negations of the following quantified statements:

(a) For every rational number r, the number $1/r$ is rational.

(b) There exists a rational number r such that $r^2 = 2$.

a) $\sim(\forall r \in \mathbb{Q}, \frac{1}{r}$ is rational$)$

$\exists r \in \mathbb{Q}$ s.t. $\frac{1}{r}$ is irrational

b) $\sim(\exists r \in \mathbb{Q}$ s.t. $r^2 = 2)$

$\forall r \in \mathbb{Q}, r^2 \neq 2$

2.74. Consider the open sentence

$$P(x, y, z) : \ (x-1)^2 + (y-2)^2 + (z-2)^2 > 0.$$

where the domain of each of the variables x, y and z is $\mathbf{R}$.

(a) Express the quantified statement $\forall x \in \mathbf{R}, \forall y \in \mathbf{R}, \forall z \in \mathbf{R}, P(x, y, z)$ in words.
(b) Is the quantified statement in (a) true or false? Explain.
(c) Express the negation of the quantified statement in (a) in symbols.
(d) Express the negation of the quantified statement in (a) in words.
(e) Is the negation of the quantified statement in (a) true or false? Explain.

a) for all real numbers x, y, z, $P(x)$

b) false, if $x=1$, $y=2$, $z=2$ then $0>0$ is false

c) $\exists x, y, z \in \mathbb{R}$ s.t. $\sim P(x)$

d) there exist real numbers x, y, z such that $\sim P(x)$

e) true, if $x=1$, $y=2$, $z=2$

8. (3.14) Let $S = \{1, 5, 9\}$. Prove that if $n \in S$ and $\frac{n^2+n-6}{2}$ is odd, then $\frac{2n^3+3n^2+n}{6}$ is even.

We will solve this using the trivial proof, showing for all $n \in S$, the hypothesis $\frac{n^2+n-6}{2}$ is odd is false, as such the implication is true.

case 1. $n=1$, $\frac{n^2+n-6}{2} = \frac{1+1-6}{2} = -2$

-2 is even, so the hypothesis is false

case 2. $n=5$, $\frac{n^2+n-6}{2} = \frac{25+5-6}{2} = 12$

12 is even, so hypothesis is false.

case 3. $n=9$, $\frac{n^2+n-6}{2} = \frac{81+9-6}{2} = \frac{90-6}{2} = \frac{84}{2} = 42$

42 is even so hypothesis is false.

In all case, the hypothesis is false, so the implication is true for all cases

3.16. Let $x \in \mathbf{Z}$. Prove that if $7x + 5$ is odd, then x is even.

We will prove this by proving the contrapositive "if x is odd, then $7x+5$ is even".

Assume x is odd, then its of the form

$x = 2k+1$ for some integer k. Thus

$$7x+5 = 7(2k+1)+5 = 14k+7+5 = 14k+12$$
$$= 2(7k+6)$$

$7k+6$ is an integer, thus $7x+5$ is of the form $2m$ where m is an integer, so $7x+5$ is even.

Thus by proving the contrapositive, we have proved the original statement.

3.18. Let $x \in \mathbf{Z}$. Prove that $5x - 11$ is even if and only if x is odd.

"If and only if" means the statement is a biconditional. So we must prove statements 1) if $5x-11$ is even, then x is odd. And 2) if x is odd, then $5x-11$ is even.

Proof of 1)

We will prove this implication by the contrapositive "if x is odd, then $5x-11$ is even"

Assume x is odd, then it is of the form $2k+1$ for some integer k. Thus.

$$5x-11 = 5(2k+1)-11 = 10k+5-11 = 10k-6 = 2(5k-3)$$

Since $5k-3$ is an integer, then $5x-11$ is even, and so the contrapositive is proved.

By proving the contrapositive, we have proved the original statement.

Proof of 2)

WLOG from our proof by contrapositive of 1) we have also proved the second implication

By proving both implications, we have proved the biconditional.

3.20. Let $x \in \mathbf{Z}$. Prove that $3x + 1$ is even if and only if $5x - 2$ is odd.

To prove this, we will first prove the following lemma:

$3x+1$ is even if and only if $5x-$

"If $3x+1$ is even if and only if x is odd"

first prove lemma "$3x+1$ is even

1) if $3x+1$ is even, x is odd

contra: x is even $\therefore$ $x = 2m$

2 is odd.

if and only if x is odd"

$6m+1 = 2(3m)+1 \quad \therefore \quad$ odd

2) if x is odd, then $3x+1$ is even

$$x = 2k+1 \quad \therefore \quad 3x+1 = 6k+3+1 = 6k+4 = 2(3k+2)$$

$\therefore$ $3x+1$ is even.

Having proved the lemma. We can now prove the biconditional.

1) if $3x+1$ is even, $5x-2$ is odd.

Assume $3x+1$ is even, we know from the above lemma that

x is odd, $x=2k+1$

$$\therefore \quad 5x-2 = 10k+5-2 = 10k+3 = 2(5k+1)+1$$

$\therefore$ $5x-2$ is odd.

2) if $5x-2$ is odd, $3x+1$ is even

contra: $3x+1$ is odd, $5x-2$ is even.

if $3x+1$ is odd, then from lemma x is even.

so $x=2m$, so $5x-2 = 10m-2 = 2(5m-1)$

$\therefore$ x is even

so contrapositive is true.

So everything is true.

3.28. Let $x, y \in \mathbf{Z}$. Prove that if xy is odd, then x and y are odd.

proof by contrapositive "if x and y are even, then xy is even"

assume $x=2m$, $y=2n$ $\quad m,n \in \mathbb{Z}$

then $xy = 2m \cdot 2n = 4(mn) = 2\underbrace{(2mn)}_{\text{integer}}$ $\quad \therefore$ xy is even

proved by contrapositive.

CASES

1) x is even, y is odd, $x=2n+1$, $y=2m$ $\quad \therefore \quad xy = 4nm+2m = 2(2nm+m)$

xy is even $\therefore$ implication true b/c hypothesis

2) x and y are even $x=2n$, $y=2m$ $\therefore$ $xy = 4mn = 2(2mn)$ $\therefore$ implication true b/c false.

3) x and y are odd $\therefore$ $x=2n+1$, $y=2m+1$ $\therefore$ $xy = 4mn + 2m + 2n + 1$
$= 2(2mn+m+n)+1$ $\therefore$ x

4) x is odd, y is even WLOG, 1).

So, in all cases, the implication is true.

3.32. (a) Let x and y be integers. Prove that $(x + y)^2$ is even if and only if x and y are of the same parity.
(b) Restate the result in (a) in terms of odd integers.

1) if $(x+y)^2$ is even, then x and y same parity

2) if x and y same parity, then $(x+y)^2$ is even

cases. 1) x even, y even $x=2m$, $y=2n$

$(2m+2n)(2m+2n) = 4m^2 + 8mn + 4n^2$
$= 2(2m^2+2n^2+4mn)$

$\therefore (x+y)^2$ is even

WLOG

case 2) x even, y odd $x=2m$, $y=2n+1$

$(2m+2n+1)(2m+2n+1)$

$= 4m^2 + 4mn + 2m + 4mn + 4n^2 + 2n$
$+ 2m + 2n + 1$

$= 4m^2 + 8mn + 4m + 4n + 1$

$= 2(2m^2 + 4mn + 2m + 2n) + 1$

$\therefore (x+y)^2$ is odd

false

hypothesis

y is odd

implication true

case 3) x odd, y odd $\quad x = 2m+1,\ y = 2n+1$

$$(2m+2n+2)(2m+2n+2)$$
$$= 4m^2 + \underline{4mn} + 4m + \underline{4mn} + 4n^2 + 4n + 4m + 4n + 4$$
$$= 4m^2 + 8mn + 8m + 8n + 4$$
$$= 2(2m^2 + 4mn + 4m + 4n + 2)$$

$\underline{(x+y)^2 \text{ is even}}$

thus for all cases, when x and y are same polarity $(x+z)^2$ is even

3.11 - 3.35

3.11. Let $n \in \mathbf{Z}$. Prove that if $1 - n^2 > 0$, then $3n - 2$ is an even integer.

Assume $1-n^2 > 0$. Then $n^2 < 1$, since $n \in \mathbb{Z}$, this can only be true for $n=0$. Thus, $n=0$. So

$$3n-2 = 3\cdot 0 - 2 = -2$$

-2 is an even integer, so the statement is proved.

3.13. Let $S = \{0, 1, 2\}$ and let $n \in S$. Prove that if $(n+1)^2(n+2)^2/4$ is even, then $(n+2)^2(n+3)^2/4$ is even.

we will do a proof by cases for all possible case of $n \in S$.

$n=0 \quad \frac{1^2 3^2}{4} = \frac{9}{4}$ not even $\therefore$ hyp. false $\therefore$ true

$n=1 \quad \frac{\cancel{2}^2 3^2}{\cancel{4}} = 9$ not even $\therefore$ hyp false $\therefore$ true

$n=2 \quad \frac{3^2 \cancel{4^2}}{\cancel{4}} = 9 \cdot 4 = 18 \cdot 2 = 36$ hyp. even $\quad \frac{\cancel{4^2} 5^2}{\cancel{4}} = 2(2 \cdot 5^2) \therefore$ concl. even.

$T \Rightarrow T \therefore$ true

3.15. Let $A = \{n \in \mathbf{Z} : n > 2$ and n is odd$\}$ and $B = \{n \in \mathbf{Z} : n < 11\}$. Prove that if $n \in A \cap B$, then $n^2 - 2$ is prime.

Assume $n \in A \cap B$. then $n \in \{3, 5, 7, 9\}$

$3^2 - 2 = 9 - 2 = 7$

$5^2 - 2 = 25 - 2 = 23$

$7^2 - 2 = 47$

$9^2 - 2 = 79$

All prime.

3.17. Let $n \in \mathbf{Z}$. Prove that if $15n$ is even, then $9n$ is even.

$15n = 2k \quad \therefore \quad 9n = 18k = 2(9k)$

3.18. Let $x \in \mathbf{Z}$. Prove that $5x - 11$ is even if and only if x is odd.

1) $5x - 11$ is even $\rightarrow$ x is odd

P.B.C. x is even $\rightarrow$ $5x - 11$ is odd

$x = 2n$, $5x - 11 = 10n - 11 = 10n - 12 + 1$
$= 2(5n - 6) + 1$

2) x is odd $\Rightarrow$ $5x - 11$ is even

$x = 2m + 1$, $5x - 11 = 5(2m+1) - 11 = 10m + 5 - 11$
$= 10m - 6$
$= 2(5m - 3)$

3.20. Let $x \in \mathbf{Z}$. Prove that $3x + 1$ is even if and only if $5x - 2$ is odd.

first prove lemma L1 $3x+1$ is even if & only if x is odd

i) x is odd $\Rightarrow$ $3x + 1$ is even

$x = 2k + 1$, $3x + 1 = 6k + 3 + 1 = 2(3k + 2) \therefore$ even

ii) P.B.C. x is even $\Rightarrow$ $3x + 1$ is odd

$x = 2k$, $3x + 1 = 6k + 1 = 2(3k) + 1 \therefore$ odd

with that

if $3x+1$ is even, then $5x-2$ is odd

if $3x+1$ is even, then by lemma L1
x is odd, so $x=2k+1$

$5x-2 = 10k+5-2 = 10k+3 = 2(5k+1)+1 \therefore$ odd

if $5x-2$ is odd, then $3x+1$ is even

PBC. if $3x+1$ is odd, then $5x-2$ is even
$3x+1$ is odd, by L1 x is even $\therefore x=2k$
$5x-2 = 10k-2 = 2(5k-1) \therefore$ even

3.21. Let $n \in \mathbf{Z}$. Prove that $(n+1)^2 - 1$ is even if and only if n is even.

n is even $\rightarrow (n+1)^2-1$ is even

$n=2k$, $k \in \mathbb{Z}$ $\therefore (2k+1)^2-1$

$= (2k+1)(2k+1) - 1$
$= 4k^2+4k+1-1 = 2(2k^2+2k)$
$\therefore$ even

$(n+1)^2-1$ is even $\rightarrow n$ is even

P.B.C. n is odd $\rightarrow (n+1)^2-1$ is odd

$n=2k+1 \therefore (2k+2)(2k+2)-1$

$= 4k^2+8k+4-1$
$= 2(2k^2+8k+1)+1$
$\therefore$ odd

3.23. Let $A = \{0, 1, 2\}$ and $B = \{4, 5, 6\}$ be subsets of $S = \{0, 1, \ldots, 6\}$. Let $n \in S$. Prove that if $\frac{n(n-1)(n-2)}{6}$ is even, then $n \in A \cup B$.

$n \in S$

if $\frac{n(n-1)(n-2)}{6}$ is even $\rightarrow n \in \{0,1,2,4,5,6\}$

PBC

if $n \notin \{0,1,2,4,5,6\}$ $\rightarrow$ $\frac{n(n-1)(n-2)}{6}$ is odd

$n \in \{0,1,2,3,4,5,6\}$

PBC

if $n \notin \{0,1,2,4,5,6\} \rightarrow \frac{n(n-1)(n-2)}{6}$ is odd

$n \in \{0,1,2,3,4,5,6\}$

then $n = 3$

$\therefore \frac{n(n-1)(n-2)}{6} = \frac{3 \cdot 2 \cdot 1}{6} = 1$ is odd

$\therefore$ by PBC, prove original statement.

3.25. Let $\{A, B\}$ be a partition of the set of $S = \{1, 2, \ldots, 7\}$, where $A = \{1, 4, 5\}$ and $B = \{2, 3, 6, 7\}$. Let $n \in S$. Prove that if $\frac{n^2+3n-4}{2}$ is even, then $n \in A$.

PBC.

if $n \notin A$ then $\frac{n^2+3n-4}{2}$ is odd

$n \in S-A$

$n \in B$

$n \in \{2,3,6,7\}$

case 1: $n=2$ $\quad \frac{\cancel{4}+6-\cancel{4}}{2} = 3$ is odd ✓

case 2: $n=3$ $\quad \frac{9+9-4}{2} = \frac{18-4}{2} = 7$ is odd ✓

case 3 $n=6$ $\quad \frac{36+18-4}{2} = \frac{\cancel{2}}{\cancel{2}}(18+9-2) = 25$ is odd ✓

case 4 $n=7$ $\quad \frac{49+21-4}{2} = \frac{70-4}{2} = \frac{66}{2} = 33$ is odd ✓

3.27. Prove that if $n \in \mathbf{Z}$, then $n^3 - n$ is even.

assume $n \in \mathbb{Z}$

case 1: n is odd, $n = 2k+1$, $k \in \mathbb{Z}$

$$
\begin{aligned}
n^3 - n &= (2k+1)^3 - n \\
&= (2k+1)(2k+1)(2k+1) - n \\
&= (4k^2+4k+1)(2k+1) - n \\
&= 8k^3 + 8k + 2k + 4k^2 + 4k + 1 - (2k+1) \\
&= 8k^3 + 4k^2 \cancel{+2k} \cancel{+1} + \cancel{-2k} \cancel{-1} \\
&= 8k^3 + 4k^2 = 2(4k^3 + 2k^2) \quad \therefore \text{ even}
\end{aligned}
$$

case 2: n is even, $n = 2k$ $k \in \mathbb{Z}$

$n^3 - n = 8k^3 - 2k = 2(2k^3 - k)$ $\therefore$ even

3.29. Let $a, b \in \mathbf{Z}$. Prove that if ab is odd, then $a^2 + b^2$ is even.

case 1: a & b are even

$a=2n \quad b=2m$

$ab = 2(2mn)$ is even. hyp $\equiv F \therefore$ imp $\equiv T$

case 2: a & b are odd

$a = 2n+1, \quad b=2m+1$

$ab = 4mn + 2n + 2m + 1$
$= 2(2mn + n + m) + 1$ odd

$a^2 + b^2 = 4m^2 + 4n^2 = 2(2m^2 + 2n^2)$ is even

$\therefore T \Rightarrow T$ is true

case 3. a is even, b is odd

$(2m+1)(2n) = 4mn + 2n = 2(2mn+n)$ is even

hyp. is false $\therefore$ imp. T

case 4. WLOG from case 3

3.31. Let $a, b \in \mathbf{Z}$. Prove that if $a + b$ and ab are of the same parity, then a and b are even.

PBC if a or b are odd then

a+b and ab are of difs. parity

case 1: $a = 2m+1, \quad b = 2n$

$a+b = 2m + 2n + 1$ is odd ✓ diff par
$ab = 2mn + 2n$ is even

case 2: WLOG ✓

case 3: $a = 2m+1, \quad b = 2n+1$

$a+b = 2m + 2n + 2$ is even
$ab = 2mn + 2n + 2n + 1$ is odd ✓ diff par

case 4: a is even and b is even

hyp. false $\therefore$ implication T ✓

By proving contrapos.

3.33. Let $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$ be subsets of $S = \{1, 2, 3, 4\}$. Let $n \in S$. Prove that $2n^2 - 5n$ is either (a) positive and even or (b) negative and odd if and only if $n \notin A \cap B$.

1) $n \notin A \cap B \rightarrow (2n^2 - 5 > 0 \wedge \text{even}) \vee (2n^2 - 5 < 0 \wedge \text{odd})$

$A \cap B = \{2, 3\} \quad \therefore \quad n \in (S - (A \cap B)) = \{1, 4\}$

case 1: $n = 1 \quad 2 - 5 = -3$ is < 0 and odd ✓

case 2: $n = 4 \quad 2 \cdot 16 = 32$ is > 0 and even ✓

2) by contra

$n \in A \cap B \rightarrow \sim(2n^2 - 5 > 0 \wedge \text{even}) \wedge \sim(2n^2 - 5 < 0 \wedge \text{odd})$

$n \in \{2, 3\} \quad (2n^2 - 5 \leq 0 \vee \text{odd}) \wedge (2n^2 - 5 \geq 0 \vee \text{even})$

case $n = 2 \quad 2 \cdot 2^2 - 5 = 2 \cdot 4 - 5 = 8 - 5 = 3$ odd & > 0 ✓

case $n = 3 \quad 2 \cdot 3^2 - 5 = 18 - 5 = 13$ odd & > 0 ✓

3.35. Prove for every nonnegative integer n that $2^n + 6^n$ is an even integer.

if $n \geq 0$, then $2^n + 6^n$ is even

case 1. $n = 0$

$2^0 + 6^0 = 2$ is even

case 2. $n > 0$. if $n > 0$ & is integer, then $n \geq 1$, then $n - 1$ is ≥ 0

$2^n + 6^n = 2^n + 2 \cdot 3^n = 2^n + 2^n 3^n = 2(\underbrace{2^{n-1} + 2^{n-1} \cdot 3^n}_{\text{integer}})$ $\therefore$ even

much to clear for me

(1 pt)

What is the converse of the following: "If this triangle has two 45 degree angles then it is a right triangle."

- **A.** If this triangle does not have two 45 degree angles then it is not a right triangle.
- **B.** If this triangle has two 45 degree angles then it is a right triangle.
- **C.** If it is a right triangle then this triangle has two 45 degree angles.
- **D.** If it is not a right triangle then this triangle does not have two 45 degree angles.
- **E.** If it is a right triangle then this triangle does not have two 45 degree angles.
- **F.** If this triangle has two 45 degree angles then it is not a right triangle.

Converse $P \to Q$, $Q \to P$ ANS: C

(1 pt)

Rewrite the following as an equivalent if then statement: "'n is divisible by both 2 and 3' is a necessary condition for 'n is divisible by 6'."

- **A.** If n is not divisible by 6 then n is divisible by both 2 and 3.
- **B.** If n is divisible by both 2 and 3 then n is divisible by 6.
- **C.** If n is not divisible by both 2 and 3 then n is divisible by 6.
- **D.** If n is divisible by 6 then n is divisible by both 2 and 3.
- **E.** If n is not divisible by both 2 and 3 then n is not divisible by 6.
- **F.** If n is not divisible by 6 then n is not divisible by both 2 and 3.
- **G.** If n is divisible by 6 then n is not divisible by both 2 and 3.
- **H.** If n is divisible by both 2 and 3 then n is not divisible by 6.

Rewrite the following as an equivalent if then statement: "'n is divisible by both 2 and 3' is a sufficient condition for 'n is divisible by 6'."

- **A.** If n is not divisible by both 2 and 3 then n is divisible by 6.
- **B.** If n is divisible by 6 then n is not divisible by both 2 and 3.
- **C.** If n is divisible by 6 then n is divisible by both 2 and 3.
- **D.** If n is not divisible by 6 then n is divisible by both 2 and 3.
- **E.** If n is divisible by both 2 and 3 then n is divisible by 6.
- **F.** If n is divisible by both 2 and 3 then n is not divisible by 6.
- **G.** If n is not divisible by 6 then n is not divisible by both 2 and 3.
- **H.** If n is not divisible by both 2 and 3 then n is not divisible by 6.

P is suff. for Q

Q is nec. for P

ANS: D , ANS: E

(1 pt) Complete the following truth table by filling in the blanks with T or F as appropriate.

P	Q	$\sim P$	$P \vee Q$	$\sim P \wedge (P \vee Q)$	$[\sim P \wedge (P \vee Q)] \Rightarrow Q$
T	T				
T	F				
F	T				
F	F				

The statement in the final column is

- **A.** a contradiction
- **B.** neither a tautology nor a contradiction
- **C.** a tautology

F	T	F	T
F	T	F	T
T	T	T	T
T	F	F	T

ANS: C

(1 pt) Complete the following truth table by filling in the blanks with T or F as appropriate.

P	Q	$P \wedge Q$	$\sim P \wedge \sim Q$	$(P \wedge Q) \vee (\sim P \wedge \sim Q)$	$P \Leftrightarrow Q$
T	T				
T	F				
F	T				
F	F				

The statements in the last two columns are

- **A.** not logically comparable
- **B.** not logically equivalent
- **C.** logically equivalent

			$P \to Q$	$Q \to P$	$P \leftrightarrow Q$
T	F	T	T	T	T
F	F	F	F	T	F
F	F	F	T	F	F
F	T	T	T	T	T

ANS: C

(1 pt) For the following proof (of equivalence of 2 statements) provide the justifications at each step, using the following equivalences. Use the following key:

a	Double Negation
b	De Morgan's Law
c	Commutative Properties
d	Associative Properties
e	Distributive Properties
f	Equivalence of Contrapositive
g	Definition of Implication
h	Definition of Equivalence
i	Identity Laws ($P \vee F \equiv P \wedge T \equiv P$)
j	Tautology ($P \vee {\sim}P \equiv T$)
k	Contradiction ($P \wedge {\sim}P \equiv F$)

$(P \wedge {\sim}Q) \vee (P \wedge Q) \equiv P \wedge ({\sim}Q \vee Q)$ by [ ] dist.

$\equiv P \wedge (Q \vee {\sim}Q)$ by [ ] comm.

$\equiv P \wedge T$ by [ ] tautology.

$\equiv P$ by [ ] identity law

e c j i

(1 pt)

Let $C(x)$ be the statement "x has a cat", let $D(x)$ be the statement "x has a dog" and let $F(x)$ be the statement "x has a ferret". Let S be the set of all students in your class. Express each of the following statements in terms of $C(x)$, $D(x)$, and $F(x)$, quantifiers, and logical connectives. Put the appropriate letter next to the corresponding symbolic form.

a) A student in your class has a cat, a dog, and a ferret.

b) All students in your class have a cat, a dog, or a ferret.

c) Some student in your class has a cat and a ferret but not a dog.

d) No student in this class has a cat, a dog, and a ferret.

e) For each of the three animals, cats, dogs, and ferrets, there is a student in your class who has one of these animals.

a) $\exists x \in S$ s.t. $C(x) \wedge D(x) \wedge F(x)$

b) $\forall x \in S,\ C(x) \vee D(x) \vee F(x)$

c) $\exists x \in S,\ C(x) \wedge D(x) \wedge {\sim}F(x)$

d) ${\sim}(\exists x \in S$ st $C(x) \wedge D(x) \wedge F(x))$

$\forall x \in S,\ {\sim}(C(x) \wedge D(x) \wedge F(x))$

e) $\forall x \in S,\ C(x) \vee D(x) \vee F(x)$

[ ] **1.** $\exists x \in S, (C(x) \wedge D(x) \wedge F(x))$ A

[ ] **2.** $\exists x \in S, (C(x) \wedge F(x) \wedge {\sim}D(x))$ C

[ ] **3.** ${\sim}(\exists x \in S, (C(x) \wedge D(x) \wedge F(x)))$ D

[ ] **4.** $\forall x \in S, (C(x) \vee D(x) \vee F(x))$ B

[ ] **5.** $(\exists x \in S, C(x)) \wedge (\exists x \in S, D(x)) \wedge (\exists x \in S, F(x))$ E

(1 pt)

Which of the following most accurately describes the process of a direct proof of the statement $\forall x \in S, P(x) \Rightarrow Q(x)$?"

- **A.** Show that $P(x)$ and $Q(x)$ are both true for some $x \in S$.
- **B.** Show that $P(x)$ and $Q(x)$ are both true for all $x \in S$
- **C.** Take an arbitrary $x \in S$ that makes $Q(x)$ true and show that $P(x)$ is true.
- **D.** Show that $Q(x)$ is true for all $x \in S$
- **E.** Take an arbitrary $x \in S$ that makes $P(x)$ true and show that $Q(x)$ is true.

E

(1 pt) Let S be the statement: *All natural numbers are real numbers.*
This statement can be put as an implication: *if x is a natural number, then x is a real number.* Complete the sentences below, filling in A-G from the following list, and T or F for true or false, as appropriate.

A. Numbers aren't natural.
B. All real numbers are natural.
C. Some natural numbers aren't real numbers.
D. Some real numbers aren't natural numbers.
E. No real number is natural.
F. A number can't be natural if it isn't real.
G. A number can't be real if it isn't natural.

S is ____ (Enter T for true or F for false).
The converse of S is ____ (enter a letter from A-G) and that statement is ____ (true or false).
The contrapositive of S is ____ and that statement is ____ . (true or false)
The negation of S is ____ and that statement is ____ (true or false).

S is T

the converse: if x is real then x is natral. F

real is suff. for natural, B

the contrapositive: if x isn't real, then x isn't natural. T

x isn't real is suff for x isn't natural F
x isn't natural is nec. for x isn't real

the negation: x is natural or x is not real F

C

(1 pt) Complete the following truth table by filling in the blanks with T or F as appropriate.

P	Q	$P \Rightarrow Q$	$\sim P$	$\sim Q$	$\sim Q \Rightarrow \sim P$
T	T				
T	F				
F	T				
F	F				

$P \Rightarrow Q$ and $\sim Q \Rightarrow \sim P$ are

- **A.** logically equivalent
- **B.** not logically equivalent
- **C.** not logically comparable

T	F	F	T	A
F	F	T	F	
T	T	F	T	
T	T	T	T	

(1 pt)

Which of the following is equivalent to the statement $(P \vee Q) \Rightarrow R$?"

- **A.** $(P \Rightarrow R) \vee (Q \Rightarrow R)$
- **B.** $(P \wedge Q) \Rightarrow R$
- **C.** $(P \Rightarrow R) \wedge (Q \Rightarrow R)$

What do you think this equivalency has to do with "proof by cases"?
(Note: This part of the problem is for marks, but needs to be manually graded. Any answer that looks to me like a reasonable attempt will get full credit)

P	Q	R	P v Q	→ R
T	T	T	T	–
T	T	F	T	
T	F	T	T	
T	F	F	T	
F	T	T	T	

T	T	T	T	
F	T	F	T	
F	F	T	T	
F	F	F	F	T

$$P \vee Q \to R \equiv (P \to R) \wedge (Q \to R)$$

(1 pt)

For each of the pairs of integers listed below, enter "S" if the two integers have the SAME parity. Enter "O" if the two integers have the OPPOSITE parity.

1. (-8,-4)

2. (-1024,1024)

3. (7,7)

4. (3,0)

5. (2,-1)

S
S
S
O
O

(1 pt) Consider the following proof of the statement "If x and y are integers of the same parity, then $x - y$ is even."

Proof: Let x and y be two integers of the same parity. Without loss of generality, assume that x and y are both even. Hence, there exist integers m and n such that $x = 2m$ and $y = 2n$. We then have
$x - y = 2m - 2n = 2(m - n)$
and so $x - y$ is even.

Is this an appropriate use of the phrase "without loss of generality"? Enter "Y" for yes, and "N" for no.

Try to explain to yourself why or why not.

$x = 2m+1, \quad y = 2n+1$

N

(1 pt)

Match each statement with an equivalent symbolic sentence.

1. $A \cup B \neq \emptyset$ C
2. $A \not\subseteq B$ B
3. $B \subseteq A$ A

A. $\forall x, \ (x \notin A \implies x \notin B)$
B. $\exists x \text{ s.t. } (x \in A \ \wedge \ x \notin B)$
C. $\exists x \text{ s.t. } (x \in A \ \vee x \in B)$

$x \notin A \rightarrow x \notin B$

$\forall x, \ x \notin A \ \vee \ x \in B$

$B \subseteq A$ $\therefore$ all elements of B are also in A

$\forall x \in B, \ x \in A$

but there may exist an element of A not in B

$A \not\subseteq B$ $\therefore$ A has some element not in B

$\exists x \in A$ s.t. $x \notin B$ B

$A \cup B \neq \emptyset$ $\therefore$ A is not empty and B is not empty

(1 pt)

Let A be the following set. A = $\{\emptyset, 1, \{1,2\}\}$. Mark each of the following true T or false F.

1. $\{1,2\} \subseteq A$ F
2. $\{\emptyset, 1\} \in A \times A$
3. $\{\emptyset, 1\} \in P(A)$
4. $(1,2) \in A \times A$
5. $\{\emptyset\} \in P(A)$