# Final Review Problems

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9.18. Let $A = \{w, x, y, z\}$ and $B = \{r, s, t\}$. Give an example of a function $f : A \to B$ that is neither one-to-one nor onto. Explain why $f$ fails to have these properties.

$f = \{(w,r), (x,r), (y,s), (z,s)\}$

↳ noninjective b/c w and r both go to r

↳ nonsurjective b/c not mapped to t

9.26. Give an example of a function $f : \mathbf{N} \to \mathbf{N}$ that is
(a) one-to-one and onto (b) one-to-one but not onto
(c) onto but not one-to-one (d) neither one-to-one nor onto.

$f(x) = x$ $\qquad f(x) = 2x$

$f(x) = \begin{cases} 1 & x=1 \\ 2 & x=2 \\ x-1 & x \geq 2 \end{cases}$ $\qquad f(x) = x^2 + 5$

9.28. Let $A = \{2, 4, 6\}$ and $B = \{1, 3, 4, 7, 9\}$. A relation $f$ is defined from $A$ to $B$ by $a\ f\ b$ if 5 divides $ab + 1$. Is $f$ a one-to-one function?

$f : A \to B$ $\qquad (a,b) \in f$ if $5 \mid (ab+1)$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 2 1 | $5 \mid (2+1)$ | × | 4 1 | $5 \mid (4+1)$ | ✓ | 6 1 | $5 \mid (7)$ | × |
| 2 3 | $5 \mid (6+1)$ | × | 4 3 | $5 \mid (13)$ | × | 6 3 | $5 \mid (18+1)$ | × |
| 2 4 | $5 \mid (8+1)$ | × | 4 4 | $5 \mid (17)$ | × | 6 4 | $5 \mid (25)$ | ✓ |
| 2 7 | $5 \mid (14+1)$ | ✓ | 4 7 | $5 \mid (29)$ | × | 6 7 | $5 \mid (43)$ | × |
| 2 9 | $5 \mid (18+1)$ | | 4 9 | $5 \mid (37)$ | × | 6 9 | $5 \mid (55)$ | ✓ |

$f = \{(2,7), (4,1), (6,4), (6,9)\}$

f is not well-defined, so no its not

9.30. Prove that the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 7x - 2$ is bijective.

$f(a) = f(b)$
$7a - 2 = 7b - 2$ $\qquad$ injective
$a = b$

For all $b \in \mathbb{R}$, $\exists a \in \mathbb{R}$ s.t. $f(a) = b$

Consider $a = \frac{b+2}{7}$

$f(a) = 7\left(\frac{b+2}{7}\right) - 2 = b + 2 - 2 = b$ $\qquad$ surjective

9.32. Prove that the function $f : \mathbf{R} - \{2\} \to \mathbf{R} - \{5\}$ defined by $f(x) = \frac{5x+1}{x-2}$ is bijective.

$\frac{5a+1}{a-2} = \frac{5b+1}{b-2} \rightarrow a = b$

$b(a-2) = 5a+1$
$ab - 2b = 5a+1$
$a(b-5) = 1+2b$

For all $b \in \mathbb{R} - \{5\}$, $\exists a \in \mathbb{R} - \{2\}$ s.t. $f(a) = b$

Consider $b = \frac{5a+1}{a-2}$

$a = \frac{1+2b}{b-5}$ $\qquad f(a) = \dfrac{5\frac{1+2b}{b-5} + 1}{\frac{1+2b}{b-5} - 2} = b$

9.36. Let $A = \{a, b, c, d, e, f\}$ and $B = \{u, v, w, x, y, z\}$. With each element $r \in A$, there is associated a list or subset $L(r) \subseteq B$. The goal is to define a "list function" $\phi : A \to B$ with the property that $\phi(r) \in L(r)$ for each $r \in A$.

(a) For $L(a) = \{w, x, y\}$, $L(b) = \{u, z\}$, $L(c) = \{u, v\}$, $L(d) = \{u, w\}$, $L(e) = \{u, x, y\}$, $L(f) = \{v, y\}$, does there exist a bijective list function $\phi : A \to B$ for these lists?

(b) For $L(a) = \{u, v, x, y\}$, $L(b) = \{v, w, y\}$, $L(c) = \{v, y\}$, $L(d) = \{u, w, x, z\}$, $L(e) = \{v, w\}$, $L(f) = \{w, y\}$, does there exist a bijective list function $\phi : A \to B$ for these lists?

9.40. Let $A$ and $B$ be nonempty sets. Prove that if $f : A \to B$, then $f \circ i_A = f$ and $i_B \circ f = f$.

Let $b \in B$. Then for some $a \in A$ $f(a) = b$

$i_A(a) = a$, so $(f \circ i_A)(a) = f(i_A(a)) = f(a)$

so $f \circ i_A = f$

$i_B(a) = b$, so $(i_B \circ f)(a) = i_B(f(a)) = f(a)$

so $i_B \circ f = f$

9.42. Prove or disprove the following:

(a) If two functions $f : A \to B$ and $g : B \to C$ are both bijective, then $g \circ f : A \to C$ is bijective.
(b) Let $f : A \to B$ and $g : B \to C$ be two functions. If $g$ is onto, then $g \circ f : A \to C$ is onto.
(c) Let $f : A \to B$ and $g : B \to C$ be two functions. If $g$ is one-to-one, then $g \circ f : A \to C$ is one-to-one.
(d) There exist functions $f : A \to B$ and $g : B \to C$ such that $f$ is not onto and $g \circ f : A \to C$ is onto.
(e) There exist functions $f : A \to B$ and $g : B \to C$ such that $f$ is not one-to-one and $g \circ f : A \to C$ is one-to-one.

a) $f: A \to B$ and $g: B \to C$

$(g \circ f)(a_1) = (g \circ f)(a_2)$

$g(f(a_1)) = g(f(a_2))$, since $g$ is injective, then $f(a_1) = f(a_2)$
since $f$ is injective, then $a_1 = a_2$

since $g$ is surjective, then for all $c \in C$ $\exists b \in B$ s.t $g(b) = c$

So $(g \circ f)(a) = g(f(a)) = g(b) = c$.

So surjective

b) consider $A = \{1, 2, 3\}$ $B = \{a, b, c\}$ $C = \{x, y, z\}$

$f = \{(1, a), (2, a), (3, b)\}$

$g = \{(a, x), (b, y), (c, z)\}$

$g \circ f = \{(1, x), (2, x), (3, y)\}$ not surjective, DISPROVED

c) consider $A = \{1, 2, 3\}$ $B = \{a, b, c\}$ $C = \{x, y, z\}$

$g \circ f = \{(1,x),(2,x),(3,y)\}$ not surjective, DISPROVED

c) consider $A=\{1,2,3\}$ $B=\{a,b,c\}$ $C=\{x,y,z\}$

$f=\{(1,a),(2,a),(3,a)\}$

$g=\{(a,x),(b,y),(c,z)\}$ g is one to one

$g \circ f=\{(1,x),(2,x),(3,x)\}$ g is not

d)

9.46. Let $A$ be the set of odd integers and $B$ the set of even integers. A function $f : A \times B \to A \times A$ is defined by $f(a, b) = (3a - b, a + b)$ and a function $g : A \times A \to B \times A$ is defined by $g(c, d) = (c - d, 2c + d)$.

(a) Determine $(g \circ f)(3, 8)$.
(b) Determine whether the function $g \circ f : A \times B \to B \times A$ is one-to-one.
(c) Determine whether $g \circ f$ is onto.

a) $g(f(3,8))$

$f(3,8)=(3\cdot 3-8,\ 3+8)=(1,11)$

$g(1,11)=(1-11,\ 2\cdot 1+11)=(-10,13)$

b) $g(f(a,b)) = g(3a-b,\ a+b) = ((3a-b)-(a+b),\ 2(3a-b)+a+b)$
$= (3a-b-a-b,\ 6a-2b+a+b)$
$= (2a-2b,\ 7a-b)$

$g(f(a_1,b_1)) = g(f(a_2,b_2))$

$$\left.\begin{aligned} 2a_1-2b_1 &= 2a_2-2b_2 \\ 7a_1-b_1 &= 7a_2-b_2 \end{aligned}\right\} \quad a_1=a_2 \quad b_1=b_2 \quad \therefore \text{ injective}$$

9.54. Let the functions $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 2x + 3$ and $g(x) = -3x + 5$.

(a) Show that $f$ is one-to-one and onto.
(b) Show that $g$ is one-to-one and onto.
(c) Determine the composition function $g \circ f$.
(d) Determine the inverse functions $f^{-1}$ and $g^{-1}$.
(e) Determine the inverse function $(g \circ f)^{-1}$ of $g \circ f$ and the composition $f^{-1} \circ g^{-1}$.

$(g\circ f)(x) = g(f(x)) = -3f(x)+5 = -3(2x+3)+5$

$x = 2f+3 \longrightarrow f^{-1} = \frac{x-3}{2}$

$x=-3g+5 \longrightarrow g^{-1} = \frac{x-5}{-3}$

$x = -3(2(gf)+3)+5$

$x = -6gf-9+5 = -6gf-4$

$g\circ f^{-1} = -4-x$ $-x-4$

$x = -6gf - 9 + 5 = -6gf - 4$

$g \circ f^{-1} = \frac{-4-x}{6} = \frac{-x-4}{6}$

$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

$f^{-1} \circ g^{-1} = f^{-1}(g^{-1}(x)) = \dfrac{\left(\frac{x-5}{-3}\right) - 3}{2}$

$= \dfrac{\frac{x-5}{-3} - \frac{3\cdot(-3)}{-3}}{2}$

$= \frac{x-5+9}{-6} = \frac{x+4}{-6} = \frac{-x-4}{6}$

9.58. Suppose, for a function $f : A \to B$, that there is a function $g : B \to A$ such that $f \circ g = i_B$. Prove that if $g$ is surjective, then $g \circ f = i_A$.

Assume $g$ is surjective. Then for each $a \in A$, there exists a $b \in B$ such that $g(b) = a$.

$f \circ g = i_B$, so $(f \circ g)(b) = b = f(g(b))$

so $g(b) = g(f(\underbrace{g(b)}_{a})) = g(f(a)) = a$

$f(g(b))$

so $(g \circ f)(a) = a$

so $g \circ f = i_A$

10.4. Let $\mathbf{R}^+$ denote the set of positive real numbers and let $A$ and $B$ be denumerable subsets of $\mathbf{R}^+$. Define $C = \{x \in \mathbf{R} : -x \in B\}$. Show that $A \cup C$ is denumerable.

$A \subseteq \mathbb{R}^+ \quad |A| = |\mathbb{N}|$

$B \subseteq \mathbb{R}^+ \quad |B| = |\mathbb{N}|$

and $|C| = |B|$

so $A$ is denumerable and $C$ is denumerable.

The union of two denumerable sets is denumerable.

10.6. (a) Prove that the function $f : \mathbf{R} - \{1\} \to \mathbf{R} - \{2\}$ defined by $f(x) = \frac{2x}{x-1}$ is bijective.
(b) Explain why $|\mathbf{R} - \{1\}| = |\mathbf{R} - \{2\}|$.

$f(a) = f(b) \qquad \frac{2a}{a-1} = \frac{2b}{b-1} \qquad a = b$

$\forall b \in \mathbb{R} - \{2\}, \ \exists a \in \mathbb{R} - \{1\}$ s.t. $f(a) = b$

consider $a = \frac{b}{b-2}$

$f(a) = \dfrac{\frac{b}{b-2} \cdot 2}{\frac{b}{b-2} - \frac{b-2}{b-2}} = \dfrac{\frac{2b}{b-2}}{\frac{2}{b-2}} = b$

$b = \frac{2a}{a-1}$

$ab - b = 2a$

$ab - 2a = b$

$a(b-2) = b$

an uncountable set minus a finite set is still uncountable

10.12. Prove that the set of all 2-element subsets of $\mathbf{N}$ is denumerable.

10.12. Prove that the set of all 2-element subsets of **N** is denumerable.

10.16. Let $A_1, A_2, A_3, \ldots$ be pairwise disjoint denumerable sets. Prove that $\cup_{i=1}^{\infty} A_i$ is denumerable.

10.20. Prove that the set of irrational numbers is uncountable.

Assume to the contrary that $\mathbb{I}$ is countable.

case 1: $\mathbb{I}$ is finite, then since $\mathbb{Q}$ is denumerable and so $\mathbb{I} \cup \mathbb{Q}$ is denumerable.
Then $\mathbb{R} = \mathbb{I} \cup \mathbb{Q}$ is denumerable
But we know $\mathbb{R}$ is uncountable so this is a contradiction

case 2: $\mathbb{I}$ is denumerable.

Then $\mathbb{I} \cup \mathbb{Q}$ is denumerable b/c $\mathbb{Q}$ is denumerable and the union of denumerable sets is denumerable.
However $\mathbb{I} \cup \mathbb{Q} = \mathbb{R}$, and $\mathbb{R}$ is uncountable. So this is a contradiction.

10.24. Prove that **R** and $\mathbf{R}^+$ are numerically equivalent.

There exists an injective function from $\mathbb{R}$ to $\mathbb{R}^+$

Consider $f(x) = e^x$, then $f(a) = f(b) \to a = b$
so $f$ is injective.

Since, $\mathbb{R}^+$ is an infinite subset of $\mathbb{R}$ and there is an injection $f: \mathbb{R} \to \mathbb{R}^+$

Then there exists a bijection $f: \mathbb{R} \to \mathbb{R}^+$

10.26. Prove or disprove the following:

(a) If $A$ is an uncountable set, then $|A| = |\mathbf{R}|$.
(b) There exists a bijective function $f : \mathbf{Q} \to \mathbf{R}$.
(c) If $A$, $B$ and $C$ are sets such that $A \subseteq B \subseteq C$ and $A$ and $C$ are denumerable, then $B$ is denumerable.
(d) The set $S = \left\{ \frac{\sqrt{2}}{n} : n \in \mathbf{N} \right\}$ is denumerable.
(e) There exists a denumerable subset of the set of irrational numbers.
(f) Every infinite set is a subset of some denumerable set.
(g) If $A$ and $B$ are sets with the property that there exists an injective function $f : A \to B$, then $|A| = |B|$.

a) $P(A)$ is uncountable, and $|P(A)| > |\mathbb{R}|$ so FALSE

b) $|\mathbb{Q}| < |\mathbb{R}|$ so no bijection FALSE

c) TRUE

d) $f(x) = \frac{\sqrt{2}}{x}$ and is bijective, so TRUE

e) TRUE

f) FALSE $\mathbb{R}$ is infinite but not denumerable

e) TRUE

f) FALSE $\mathbb{R}$ is infinite but not denumerable

g) FALSE

10.28. Prove or disprove: If $A$ and $B$ are two sets such that $A$ is countable and $|A| < |B|$, then $B$ is uncountable.

Disprove Consider $A = \{1, 2\}$ and $B = \{1, 2, 3\}$.

Then $|A| < |B|$, but $B$ is countable.

10.32. Prove that if $A$, $B$ and $C$ are nonempty sets such that $A \subseteq B \subseteq C$ and $|A| = |C|$, then $|A| = |B|$.

Assume $A \subseteq B \subseteq C$ and $|A| = |C|$.

Since $A \subseteq B$, then $|A| \leq |B|$

Since $B \subseteq C$, then $|B| \leq |C|$

and since $|A| = |C|$, then

$|B| \leq |A|$.

So By Shroder Bernstein

$|A| = |B|$

10.34. Prove that $|\mathbf{Q} - \{q\}| = \aleph_0$ for every rational number $q$ and $|\mathbf{R} - \{r\}| = c$ for every real number $r$.

$\mathbb{Q} - \{q\}$ is an infinite subset of a denumerable set $\mathbb{Q}$, so its denumerable
so $|\mathbb{Q} - \{q\}| = \aleph_0$

$\mathbb{R} - \{r\}$ is

10.42. Let $S$ and $T$ be two sets. Prove that if $|S - T| = |T - S|$, then $|S| = |T|$.

6.8. Find a formula for $1 + 4 + 7 + \cdots + (3n - 2)$ for positive integers $n$, and then verify your formula by mathematical induction.

$$
\begin{array}{rl}
 & S = 1 + 4 + 7 + \dots + (3n-5) + (3n-2) \\
+ & S = (3n-2) + (3n-5) + \dots + 7 + 4 + 1 \\
\hline
 & 2S = (3n-1) + (3n-1) + \dots
\end{array}
$$

$2S = n(3n-1) \longrightarrow S = \frac{n(3n-1)}{2}$

Base case : $S_1 = 1 = \frac{1(3\cdot 1-1)}{2} = 1$

Assume: $S = \frac{k(3k-1)}{2}$

$$\frac{(k+1)[3(k+1)-1]}{2}$$

Inductive step: $\underbrace{1+4+7+\dots+(3k-5)+(3k-2)}+(3(k+1)-2)$

$$\frac{k(3k-1)}{2}+3k+1 = \frac{1}{2}k(3k-1)+3k+1 = \frac{1}{2}(3k^2-k)+3k+1 =$$

$$\frac{3}{2}k^2-\frac{1}{2}k+3k+1 = \frac{1}{2}[3k^2-k+6k+2]$$

$$= \frac{1}{2}[3k^2+5k+2]$$

$$= \frac{1}{2}(3k+2)(k+1)$$

$$= \frac{(k+1)[3(k+1)-1]}{2}$$

6.10. Let $r \neq 1$ be a real number. Use induction to prove that $a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$ for every positive integer $n$.

Base case: $n=1$ $\quad a = \frac{a(1-r^1)}{1-r} = a$ ✓

There exists some $k \in \mathbb{N}$ s.t.

$$a + ar + ar^2 + \dots + ar^{k-1} = \frac{a(1-r^k)}{1-r} \qquad \frac{a(1-r^{k+1})}{1-r}$$

IH: $a + ar + ar^2 + \dots + ar^{k-1} + ar^k$

$$= a\frac{(1-r^k)}{1-r} + ar^k\,\frac{1-r}{1-r}$$

$$= \frac{a(1-r^k)+ar^k(1-r)}{1-r} = \frac{a \cancel{-ar^k} + \cancel{ar^k} - rar^k}{1-r}$$

$$= \frac{a[1-rr^k]}{1-r} = \frac{a(1-r^{k+1})}{1-r}$$

6.12. Consider the open sentence $P(n)$: $9 + 13 + \cdots + (4n+5) = \frac{4n^2+14n+1}{2}$, where $n \in \mathbf{N}$.

(a) Verify the implication $P(k) \Rightarrow P(k+1)$ for an arbitrary positive integer $k$.
(b) Is $\forall n \in \mathbf{N}$, $P(n)$ true?

$$9+13+\dots+(4n+5) = \frac{4n^2+14n+1}{2} \qquad \frac{4(k+1)^2+14(k+1)+1}{2}$$

$$9+13+\dots+(4k+5)+4(k+1)+5$$

$$\frac{4k^2+14k+1}{2} + \frac{2(4(k+1)+5)}{2} \qquad 4k^2+8k+4$$

$$\frac{4k^2+14k+1+8k+8+10}{2}$$

$$\frac{4k^2+22k+19}{2} = \frac{(4k^2+8k+4)+(14k+14)+1}{2} = \frac{4(k+1)^2+14(k+1)+1}{2}$$

$$9 = \frac{4+14+1}{2} = \frac{19}{2} = 8.5 \quad \times$$

6.20. (a) Use mathematical induction to prove that every finite nonempty set of real numbers has a largest element.
(b) Use (a) to prove that every finite nonempty set of real numbers has a smallest element.

Base case. A set with one element has a largest element that is the one element.

Inductive step: Assume a set with k elements has a largest element.

Then a set with k+1 elements has a largest element in the first k elements, and if that is larger than the k+1th element, that's the largest. If k+1th element . . . .

6.22. Prove that $3^n > n^2$ for every positive integer $n$.

Base case: $n=1$ $3>1$ so true
$n=2$ $9>4$
$n=3$ $27>9$

$(k+1)^2$
$k^2+2k+1$

Inductive step: $3^k > k^2$ for some $k \in \mathbb{N}$

IH: $3^{k+1} = 3 \cdot 3^k > 3k^2 = k^2 + 2k^2 > k^2 + 2 \cdot k \cdot 2 > k^2 + 2k + 1 = (k+1)^2$

since $k > 2$

$\blacksquare$

6.24. Prove Bernoulli's Identity: For every real number $x > -1$ and every positive integer $n$,

$$(1+x)^n \geq 1 + nx.$$

6.26. Prove that $81 \mid (10^{n+1} - 9n - 10)$ for every nonnegative integer $n$.

6.30. Recall for integers $n \geq 2$, $a, b, c, d$, that if $a \equiv b \ (mod\ n)$ and $c \equiv d \ (mod\ n)$, then both $a + c \equiv b + d \ (mod\ n)$ and $ac \equiv bd \ (mod\ n)$. Use these results and mathematical induction to prove the following: For any $2m$ integers $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_m$ for which $a_i \equiv b_i \ (mod\ n)$ for $1 \leq i \leq m$,

(a) $a_1 + a_2 + \cdots + a_m \equiv b_1 + b_2 + \cdots + b_m \ (mod\ n)$ and
(b) $a_1 a_2 \cdots a_m \equiv b_1 b_2 \cdots b_m \ (mod\ n)$.

12.4. Prove that the sequence $\left\{\frac{1}{n^2+1}\right\}$ converges to 0.

$\varepsilon > 0$. There exists $N \in \mathbb{N}$ s.t $n > N$

$|a_n - L| = \left|\frac{1}{n^2+1} - 0\right| < \varepsilon$

Consider $N = \left\lceil \sqrt{\frac{1}{\varepsilon} - 1} \right\rceil$

$\frac{1}{n^2+1} < \varepsilon$

Consider $N = \left\lceil \sqrt{\frac{1}{\varepsilon} - 1} \right\rceil$

Then $\left| \frac{1}{n^2+1} \right| = \frac{1}{n^2+1} < \frac{1}{\left(\sqrt{\frac{1}{\varepsilon}-1}\right)^2+1}$

$= \frac{1}{\frac{1}{\varepsilon}-1+1}$

$= \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$

$\frac{1}{n^2+1} < \varepsilon$

$\frac{1}{\varepsilon} < n^2+1$

$\sqrt{\frac{1}{\varepsilon}-1} < n$

12.8. Show that the sequence $\{n^4\}$ diverges to infinity.

$\forall M > 0$, $\exists N \in \mathbb{N}$ s.t. $n > M \Rightarrow n^4 > M$

Consider $N = \lceil M^{1/4} \rceil$. Then if $n > M$, $n > N \geq M^{1/4}$

so $n^4 > M$

1. (12.12) Prove that the series $\Sigma_{k=1}^{\infty} \frac{1}{(3k-2)(3k+1)}$ converges and determine its sum by

(a) computing the first few terms of the sequence $\{s_n\}$ of partial sums and conjecturing a formula for $s_n$.

$$\frac{1}{(3k-2)(3k+1)} = \frac{A}{(3k-2)} + \frac{B}{(3k+1)} = \frac{1}{3(3k-2)} - \frac{1}{3(3k+1)}$$

$$1 = (3k+1)A + (3k-2)B$$

$k = -\frac{1}{3}$ $\quad 1 = (-\frac{1}{3}3 - 2)B$ $\quad -\frac{1}{3} = B$

$k = \frac{2}{3}$ $\quad 1 = (3\frac{2}{3}+1)A$ $\quad \frac{1}{3} = A$

$s_1 = \frac{1}{3} - \frac{1}{12} = \frac{4}{12} - \frac{1}{12} = \frac{3}{12} = \frac{1}{4}$

$s_2 = \frac{1}{3} - \cancel{\frac{1}{12}} + \cancel{\frac{1}{12}} - \frac{1}{21} = \frac{2}{7}$

$s_3 = \frac{1}{3} - \cancel{\frac{1}{12}} + \cancel{\frac{1}{12}} - \cancel{\frac{1}{21}} + \cancel{\frac{1}{21}} - \frac{1}{30} = \frac{3}{10}$

$L = 1/3$ $\quad \boxed{s_n = \frac{n}{3n+1}}$

(b) using mathematical induction to verify that your conjecture in (a) is correct

$s_n = \frac{n}{3n+1}$ $\quad L = 1/3$

$\varepsilon > 0$. $\exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow |s_n - L| < \varepsilon$

Consider $N = \left\lceil \frac{\frac{1}{\varepsilon} - 3}{9} \right\rceil$

$\left| \frac{n}{3n+1} - \frac{1}{3} \right| = \left| \frac{3n}{3(3n+1)} - \frac{(3n+1)}{3(3n+1)} \right|$

$= \left| \frac{3n - 3n - 1}{} \right|$

$$\left| \frac{\cdots}{3n+1} - \frac{\cdots}{3} \right| = \left| \frac{\cdots}{3(3n+1)} - \frac{\cdots}{3(3n+1)} \right|$$

$$= \left| \frac{3n - 3n - 1}{9n+3} \right|$$

$$= \left| \frac{-1}{9n+3} \right|$$

$$= \frac{1}{9n+3} \qquad\qquad \frac{1}{9n+3} < \varepsilon$$

$$1 < \varepsilon(9n+3)$$

$$< \frac{1}{9\left(\frac{\frac{1}{\varepsilon}-3}{9}\right)+3} = \varepsilon \qquad\qquad \frac{\frac{1}{\varepsilon}-3}{9} < n$$

$$\therefore \quad \left| \frac{n}{3n+1} - \frac{1}{3} \right| < \varepsilon$$

2. (12.16)

(a) Prove that if $\Sigma_{k=1}^{\infty} a_k$ is a convergent series, then $\lim_{n\to\infty} a_n = 0$.

3. (12.46) Prove that $\lim_{n\to\infty} \dfrac{2n^2}{4n^2+1} = \dfrac{1}{2}$

$\varepsilon > 0$. $\exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow \left| \frac{2n^2}{4n^2+1} - \frac{1}{2} \right| < \varepsilon$

Consider $N = \left\lceil \sqrt{\frac{1/\varepsilon - 2}{8}} \right\rceil$

$$\left| \frac{2(2n^2)}{2(4n^2+1)} - \frac{(4n^2+1)}{2(4n^2+1)} \right| = \left| \frac{4n^2 - 4n^2 - 1}{2(4n^2+1)} \right| = \left| \frac{-1}{2(4n^2+1)} \right| = \frac{1}{8n^2+2}$$

$$< \frac{1}{8\,\frac{\frac{1}{\varepsilon}-2}{8} + 2} = \frac{1}{\frac{1}{\varepsilon} - 2 + 2} = \varepsilon$$

4. (12.47) Prove that the sequence $\{1 + (-2)^n\}$ diverges.

Assume to contrary that $\{1 + (-2)^n\}$ converges.

$\varepsilon > 0$. $\exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow |1 + (-2)^n - L| < \varepsilon$ for some $L \in \mathbb{R}$

Consider $\varepsilon = 1$. So $|1 + (-2)^n - L| < 1$

Case 1: $n$ is odd, so $n = 2k+1$ $k \in \mathbb{Z}$

$$|1 + (-2)^n - L| = |1 - (2)^n - L| < 1$$

So $-1 < 1 - 2^n - L < 1$

$$-2 < -2^n - L < 0$$

$\hookrightarrow$ $L - 2 < -2^n$

$L < -2^n + 2$

since $n > 1$, then $-2^n > 2$ so $L < 0$

case 2: n is even, so $n=2k$ $k\in\mathbb{Z}$

$|1+2^n-L|<\varepsilon$

so $-1 < 1+2^n-L < 1$

$-2 < 2^n-L < 0$

$2^n-L<0 \longrightarrow L>0$

However, since $L<0$ and $L>0$, this is a contradiction

5. (12.48) Prove that $\lim_{n\to\infty}(\sqrt{n^2+1}-n)=0$.

$\varepsilon>0$. $\exists N\in\mathbb{N}$ s.t. $n>N \Rightarrow |\sqrt{n^2+1}-n-0|<\varepsilon$

Consider $N=\left\lceil \frac{\varepsilon^2-1}{2\varepsilon} \right\rceil$

$\sqrt{n^2+1}-n<\varepsilon$

$\cancel{n^2}+1<\varepsilon^2-2\varepsilon n+\cancel{n^2}$

$|\sqrt{n^2+1}-n| = \sqrt{n^2+1}-n$

$2\varepsilon n < \varepsilon^2-1$

$< \varepsilon$

$n < \frac{\varepsilon^2-1}{2\varepsilon}$