ELEC 321 / STAT 321

HOMEWORK 1

Problems marked with (*) have a numerical component. For these problems, computations can be done using R or Matlab. Please, submit a copy of your computer script and display your results using tables, pictures, etc. when convenient.

Problem 1: Let A, B and C be three events. Find expressions for the events so that of A, B and C :
(a) only A occurs;
(b) both A and B but not C occurs;
(c) at least one of the events occurs;
(d) at least two of the events occurs;
(e) all three occur;
(f) none of the events occurs;
(g) at most one of them occurs;
(h) at most two of them occurs;
(i) exactly two of them occur;
(j) at most three of them occur.

Problem 2:
(a) Show that if $P(A) = 0.9$ and $P(B) = 0.8$ then $P(A \cap B) \geq 0.7$.
(b) Generally, prove Bonferroni's inequality, namely $P(A \cap B) \geq P(A) + P(B) - 1$.
(c) Even more generally, $P(A_1 \cap A_2 \cap \cdots \cap A_n) \geq P(A_1) + P(A_2) + \cdots + P(A_n) - (n-1)$. **Hint:** use induction.
(d) Prove the following inequality:

$$P\left(\cup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i) \tag{1}$$

Hint: use induction.

Problem 3 (Total Probability): Suppose that $A_1, A_2, ..., A_n$ are a partition of the sample space. That is $A_i \cap A_j = \phi$ for $i \neq j$ and $A_1 \cup A_2 \cup \cdots \cup A_n = \Omega$. Show that, for all B,

$$P(B) = \sum_{i=1}^{n} P(B|A_i)\, P(A_i)$$

Problem 4: Consider a system of n antennae arranged in a linear order. Communication flows through the system provided no two consecutive antennae are down.

(a) Suppose that $m < n$ antennae are down and the remaining $n - m$ are functional. How many linear orderings are there in which no pair of consecutive antennae are down?

(b) Suppose that there are $n = 10$ antennae and the probability that m of them are down are as in the following table:

m	0	1	2	3	4	5	6
Probability	0.11	0.27	0.30	0.20	0.09	0.02	0.01

Calculate the probability that communication flows through this system.

[**Hint:** use the result of Problem 3]

Problem 5: Suppose that the sample space has a countably infinite number of points.

(a) Show that not all points can be equally likely.

(b) Can all the points have positive probability of occurring? Why?

Problem 6: (*) Let f_n denote the number of ways of tossing a coin n times such that successive heads never appear.

(a) Argue that

$$f_n = f_{n-1} + f_{n-2}, \qquad n \geq 2 \text{ where } f_0 = 1, f_2 = 2$$

Hint: how many outcomes are there that start with a head, and how many start with a tail?

(b) Suppose that all possible outcomes of n tosses are equally likely. If P_n denotes the probability that successive heads never appear when a coin is tossed n times, find P_n (in terms of f_n) and compute.

(c) Complete the following table.

n	P_n
0	1
1	1
2	
$\vdots$	
25	

Problem 7: Suppose that an urn has 20 balls, 6 green, 6 red and 8 blue. Three balls are selected sequentially and without replacement.

(a) Calculate the probability for the following events: (i) they are all red, (ii) they are all of the same color (iii) they are all of different colors.

(b) Let

$$A_i = \{\text{There are exactly i red balls }\}$$

for $i = 0, 1, 2, 3$. Calculate $P(A_i)$ for $i = 0, 1, 2, 3$.

Problem 8: Suppose that A and B are independent events. (a) Show that A^c and B are also independent. (b) Show that A^c and B^c are also independent.

Problem 9: Total Probability: Suppose that $A_1, A_2, ..., A_n$ are a partition of the sample space. That is $A_i \cap A_j = \phi$ for $i \neq j$ and $A_1 \cup A_2 \cup \cdots \cup A_n = \Omega$. Show that, for all B,

$$P(B) = \sum_{i=1}^{n} P(B|A_i) P(A_i)$$

Problem 10: Two independent remote sensing devices, A and B, mounted on an airplane are used to detect and locate dead trees in a large area of forest land. The detectability of device A is 0.8 (that is, the probability that a dead tree will be detected by device A is 0.8), whereas the detectability of device B is 0.9. However, when a dead tree has been detected its location may not be pinpointed accurately by either device. Based on a detection from device A alone, the location of the dead tree can be accurately determined with probability 0.7, whereas the corresponding probability with device B alone is only 0.4. If a dead tree is detected by both devices, its location can be pinpointed with certainty. Determine the following.

(a) The probability that a forest fire will be detected.

(b) The probability that a forest fire will be detected by only one device

(c) The probability of accurately locating a forest fire.

Problem 11:

(a) A sequence of binary digits is transmitted in a certain communication system. Any given digit is received erroneously with probability p and received correctly with probability $1-p$. Errors occur independently from digit to digit. Out of a sequence of n digits transmitted, what is the probability than j digits are received erroneously?

(b) A certain error-correcting code is applied to the communication system described in Part (a). The code can correct a single error in any digit, but it cannot correct any pattern of two or more errors. What is the probability that the code can correct the error pattern that occurs when *n-digits* are transmitted?

Problem 12 (*): It is known that 15% of the parts coming off the production line in a certain company are defective. Parts are submitted to 10 cheap and fast non-destructive tests before they are shipped to customers. The conditional probabilities of testing positive for each of the 10 tests are given below:

Test	Given the part is defective	Given the part is non defective
T_1	0.80	0.20
T_2	0.64	0.16
T_3	0.86	0.07
T_4	0.80	0.09
T_5	0.74	0.18
T_6	0.76	0.13
T_7	0.77	0.17
T_8	0.84	0.14
T_9	0.85	0.21
T_{10}	0.75	0.08

Let

$$D = \{\text{Part is defective}\}$$

and let B_i represent the result of the i^{th} test (that is $B_i = +$ means the i^{th} test resulted positive and $B_i = -$ means the i^{th} test resulted negative). Let $I_k = \cap_{i=1}^{k} B_i$.

Suppose that the tests results are conditionally independent given that the part is defective. Suppose also that the tests results are conditionally independent given that the part is non defective.

(a) Calculate $P(D|I_k)$, $k = 1, 2, ..., 10$ for a part with the following test results

(i) (+ - + - + + + - - +)

(ii) (+ + + - - + - - - -)

(iii) (- - - + + - + + + +)

(b) Suppose that parts with $P(D|I_{10}) > \alpha$ are not shipped to customers. Let E be the event that we make an error following this strategy (that is, we ship a faulty part or reject a good part). Explain why

$$P(E) = P(E|\text{Part is non-defective}) \times 0.85 + P(E|\text{Part is defective}) \times 0.15 \quad (2)$$

The following numerical procedure can be used to estimate $P(E|\text{Part is non-defective})$ and $P(E|\text{Part is defective})$

For a given value of α, (take $\alpha = .50$, to fix ideas):

1) Generate $N = 50000$ sequences of test results assuming that the tested parts are defective. For each sequence calculate and save $P(D|I_{10})$.

2) Use the results of 1) to calculate the fraction of wrongly approved parts which would accrue for the given cutoff value α. Call this fraction f_d. Notice that f_d estimates $P(E|\text{Part is defective})$.

3) Generate $N = 50000$ sequences of test results assuming that the tested boards are non defective. For each sequence calculate and save $P(D|I_{10})$.

4) Use the results of 3) to calculate the fraction of wrongly rejected parts which would accrue for the given cutoff α. Call this fraction f_{nd}. Notice that f_{nd} estimates $P(E|\text{Part is non defective})$

5) Use equation (2) and the results of 2) and 4) to estimate the probability of error for the given α. Call this estimate $J(\alpha)$.

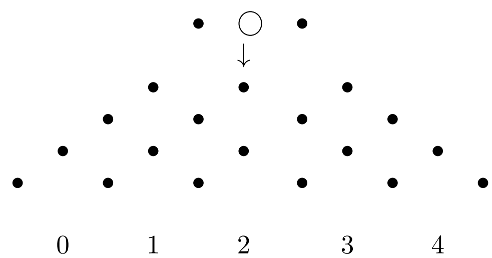
(c) Calculate $J(\alpha)$ for $\alpha = 0.01, 0.02, 0.03, ..., 0.99$ and recommend the optimal cutoff value α for rejecting parts.

Problem 13 (*): (a) An urn contains $n = 100$ chips numbered 1 through 100. A chip is randomly drawn and its number is recorded. Let Y be a random variable representing the number in the drawn chip. What are the probability mass function $f(y)$ and distribution function $F(y)$ for Y? Calculate the mean and variance for Y.

(b) Five chips are sequentially randomly drawn from the urn to have their number recorded. The drawn chips are put back into the urn before the next chip is selected. Let X represent the smallest of the 5 recorded numbers. Derive the distribution function and probability mass function for X. Calculate the mean and variance of X. **Hint:** Use R or Matlab to calculate the mean and variance of X.

Problem 14 : An airline books a daily shuttle service from Vancouver to Boonies. They offer two round-trip flights, one on a two-engine plane, the other on a four-engine plane. Suppose that each engine on each airplane will fail independently with the same probability p and that each plane will arrive safely at its destination if at least half its engine remain working. For what values of p would you prefer to fly on the two-engine plane?

Problem 15 (*): In his 1889 publication *Natural Inheritance*, the renewed British scientist Sir Francis Galton described a pinball-type board that he called a *quincunx*.



As pictured, the quincunx has five rows of pegs, the pegs in each row being the same distance apart. At the bottom of the board are five cells, numbered 0 through 4. A ball is introduced at the top of the board between the pegs in the first row. After wedging past those two pegs, it will hit the middle peg second row and veer either to the right or to the left, then strike a peg in the third row, again veering to the right or to the left, and so on.

(a) If the ball has a 50-50 chance of going either direction each time it hits a peg, what is the probability it ends in cell 0? What about for the other cells?

(b) Write a code that simulates this experiment and run it 1000 times, recording the number of times the ball lands on each cell. Compare the frequencies with the theoretical probabilities derived in Part (a).

(c) Suppose now that the quincunx is enlarged to have 100 rows. Derive the probabilities of landing on cells 0, 1, 2, ...,100?

(d) Update your code to run this experiment and run it 1000 times. Compare the frequencies with the theoretical probabilities derived in Part (c).

(e) Summarize your learning from Parts (a) through (d).