Ch. 8/9 - Relations and Functions

Saturday, July 16, 2016 10:44 AM

8.1 RELATIONS

A relation, R, from A to B is a subset of $A \times B$

For example, $A = \{x, y, z\}$ $B = \{1, 2\}$

$R = \{(x,2)(y,1)(y,2)\}$

$R \subseteq A \times B$, so it is a relation

↳ maps $a \in A$ to $b \in B$

Domain $R = \{a \in A : (a,b) \in R$ for some $b \in B\}$

↳ the first coordinates of the relation elements

$\text{dom}(R) = \{x, y\}$

Range $R = \{b \in B : (a,b) \in R$ for some $a \in A\}$

↳ the second coordinates of the relation elements

$\text{range}(R) = \{1, 2\}$

Inverse relation, $R^{-1} = \{(b,a) : (a,b) \in R\}$

↳ swap the coordinates of the relation

$R^{-1} = \{(2,x)(1,y)(2,y)\}$

9.1 FUNCTIONS

$f : A \to B$

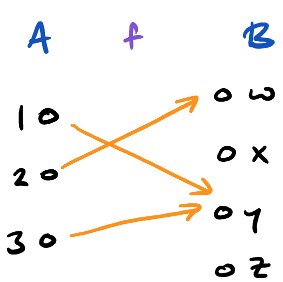
↳ a relation such that each element of A is mapped to <u>exactly one</u> element of B

$f : \{(1,y)(1,x)(2,y)\}$ ✗

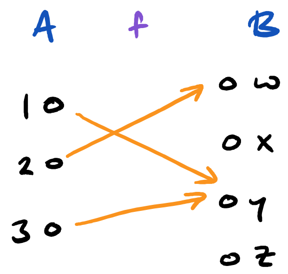
1 cannot go to both y and x

$$\boxed{\text{if } f(a) = b \text{ and } f(c) = b, \text{ then } a = c}$$

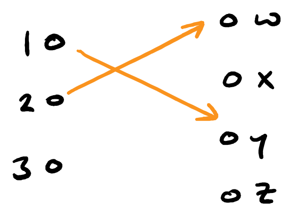
Codomain → subset of the range, the set B



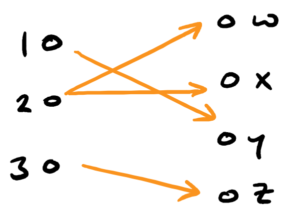
✓ well-defined



✓ well-defined



✗ not all domain mapped
(not function)



✗ one item of A mapped
to two items of B

↳ doesn't pass vertical line test

9.2 SET OF ALL FUNCTIONS FROM A to B

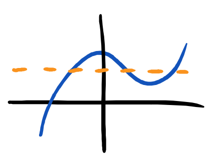
$B^A = \{ f : f : A \to B \}$

$$\boxed{|B^A| = |B|^{|A|}}$$

9.3 ONE-TO-ONE AND ONTO FUNCTIONS

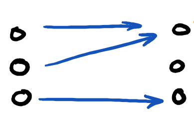
One-to-One (Injective) $\boxed{|A| \leq |B|}$ $f : A \to B$

↳ if it passes horizontal line test



✗ not injective

two elements in domain to one in codomain

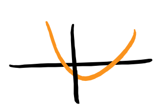


✗ not injective

Prove: If $f(a) = f(b)$, then $a = b$

Ex) Determine if $f(x) = x^2 - 3x - 2$ is bijective

$f(0) = -2$, $f(3) = -2$



$f(0) = f(3)$, where $0 \neq 3$ so not injective

$f(0) = -2, \quad f(3) = -2$

$f(0) = f(3)$, where $0 \neq 3$ so not injective

Onto (surjective) $\boxed{|A| \geq |B|}$ $\quad f: A \to B$

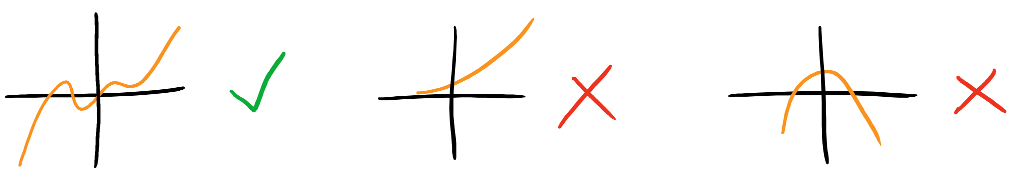
↳ if every element of the codomain is mapped to by some element of the domain

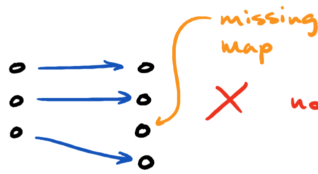
$f: A \to B \qquad A = \{1, 2, 3\} \quad B = \{x, y, z, w\}$

$f = \{(1, y)(2, w)(3, y)\}$

✗ not onto b/c z and x are not defined in the function

↳ numerically, view as function spanning entire vertical axis



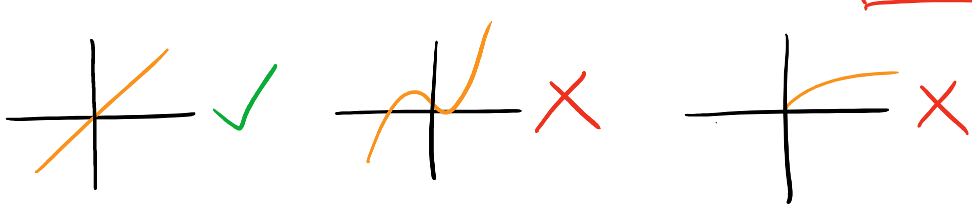


not surjective

<u>Prove:</u> Let $f: A \to B$. We say f is surjective when for every b in B, there is an a in A such that $f(a) = b$

$\forall b \in B, \exists a \in A$ s.t. $f(a) = b$

Bijective (both surjective and injective) $\boxed{|A| = |B|}$



Ex) Prove $f: \mathbb{R} \to \mathbb{R}$ $f(x) = 7x - 2$ is bijective

We must prove

1) f is injective

Consider $f(a) = f(b)$, then $7a - 2 = 7b - 2$

So $7a = 7b$. Thus $a = b$, hence f is injective.

2) f is surjective

For every $a \in \mathbb{R}$ we must show there exists $x \in \mathbb{R}$ such that $f(x) = a$.

Consider $x = \frac{a+2}{7}$, then $f(x) = 7\left(\frac{a-2}{7}\right) + 2 = a$

Hence f is surjective.

Since (1) and (2) are true, f is bijective.

9.5 COMPOSITION OF FUNCTIONS

Composition: Let $f: A \to B$ and $g: B \to C$ be two functions. The composition is a new function $g \circ f: A \to C$

$$(g \circ f)(a) = g(f(a))$$

Notes: if f & g are both injective, $g \circ f$ is injective
if f & g are both surjective, $g \circ f$ is surjective

$$(f + g)(x) = f(x) + g(x)$$

$$(fg)(x) = f(x) \cdot g(x)$$

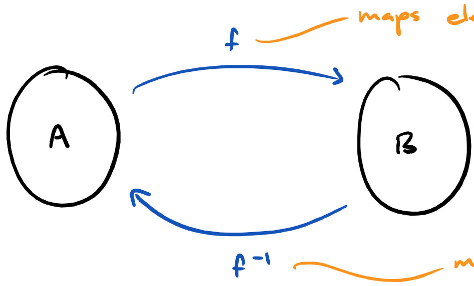
$$f \circ g \neq g \circ f$$

$$(h \circ g) \circ f = h \circ (g \circ f)$$

9.6 INVERSE FUNCTIONS

Inverse Relation, $R^{-1} = \{(b,a) : (a,b) \in R\}$

maps elements from A to B



maps elements back from B to A

THEOREM 9.15: f^{-1} exists if and only if f is bijective

↳ furthermore f^{-1} is also bijective

An inverse function is always bijective.

THEOREM 9.16: Let $f: A \to B$ and $f^{-1}: B \to A$.
$f \circ f^{-1} = i_B$ and $f^{-1} \circ f = i_A$

Ex1 $f: \mathbb{R} - \{2\} \to \mathbb{R} - \{3\}$ defined by $f(x) = \frac{3x}{}$

$f \circ f^{-1} = i_B$ and $f^{-1} \circ f = i_A$

Ex) $f : \mathbb{R} - \{2\} \longrightarrow \mathbb{R} - \{3\}$ defined by $f(x) = \dfrac{3x}{x-2}$

is known to be bijective. Determine f^{-1}

$$x = \frac{3f^{-1}}{f^{-1} - 2}$$

$$x(f^{-1} - 2) = 3f$$

$$xf^{-1} - 3f^{-1} = 2x$$

$$f^{-1}(x) = \frac{2x}{x-3}$$