Midterm 2 Review Problems

Monday, July 18, 2016 10:52 AM

1. (4.46) Let A and B be sets. Prove that $A \cup B = A \cap B$ if and only if $A = B$.

Prove

1) if $A \cup B = A \cap B$, then $A = B$

Assume $A \cup B = A \cap B$. Let $x \in A$. Then $x \in A \cup B$.

Then $x \in A \cap B$. So $x \in B$. Thus $A \subseteq B$. *

Let $x \in B$. Then $x \in A \cup B$. Then $x \in A \cap B$.

So $x \in A$. Thus $B \subseteq A$. Hence $A = B$, as required.

2) if $A = B$, then $A \cup B = A \cap B$.

Assume $A = B$. Then $A \cup B = B \cup B = B$. And $A \cap B = B \cap B = B$.

So $A \cup B = A \cap B$, as required.

2. (4.54) Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$ for every two sets A and B (Theorem 4.22(4b)).

i) Let $x \in \overline{A \cap B}$. Then $x \notin A \cap B$. So $x \notin A$ or $x \notin B$.

So $x \in \overline{A}$ or $x \in \overline{B}$. Thus $x \in \overline{A} \cup \overline{B}$.

Hence $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$, as required.

ii) Let $x \in \overline{A} \cup \overline{B}$. Then $x \in \overline{A}$ or $x \in \overline{B}$. So $x \notin A$ or $x \notin B$.
Then $x \notin A \cap B$. So $x \in \overline{A \cup B}$.

Hence $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$, as required.

$x \notin A$.

3. (4.56) Let A, B, and C be sets. Prove that $(A - B) \cup (A - C) = A - (B \cap C)$.

i) Let $x \in (A-B) \cup (A-C)$. Then $x \in (A-B)$ or $x \in (A-C)$.

Then $x \in A$ and $x \notin B$ or $x \in A$ and $x \notin C$
So $x \in A$ and $x \notin B$ or $x \notin C$. So $x \in A$ and $x \notin B \cap C$.

Thus $x \in A - (B \cap C)$, hence $(A-B) \cup (A-C) \subseteq A - (B \cap C)$, as required.

ii) Let $x \in A - (B \cap C)$. Then $x \in A$ and $x \notin B \cap C$.

So $x \notin B$ or $x \notin C$. Assume $x \notin B$. Then $x \in A$ and $x \notin B$.
So $x \in A - B$. WLOG, assume $x \notin A$, then $x \in B - A$.

Thus $x \in (A-B) \cup (B-A)$, hence $A - (B \cap C) \subseteq (A-B) \cup (B-A)$, as required.

4. (4.64) For sets A and B, find a necessary and sufficient condition for $(A \times B) \cap (B \times A) = \emptyset$.
Verify that this condition *is* necessary and sufficient.

Necessary and sufficient implies if and only if.

This is true if and only if A and B share no elements. That is, $A \cap B = \emptyset$

i) if $A \cap B = \emptyset$, then $(A \times B) \cap (B \times A) = \emptyset$

Take the contrapositive

if $(A \times B) \cap (B \times A) \neq \emptyset$, then $A \cap B \neq \emptyset$.

Assume $(A \times B) \cap (B \times A) \neq \emptyset$. Let $x \in (A \times B) \cap (B \times A)$
Then $x \in (A \times B)$ and $x \in (B \times A)$. Let $x = (a, b)$
$a \in A$ and $b \in B$, and $a \in B$ and $b \in A$.

So $a \in A \cap B$ and $b \in B \cap A$. So $A \cap B \neq \emptyset$, as required.

ii) if $(A \times B) \cap (B \times A) = \emptyset$, then $A \cap B = \emptyset$

Take the contrapositive, Assume $A \cap B \neq \emptyset$.

Let $x \in A \cap B$. Then $a \in A$ and $a \in B$.
So $(a, a) = A \times B$. And $(a, a) = B \times A$.

Then $A \times B \cap B \times A \neq \emptyset$, as required.

5. (4.60) For $A = \{x, y\}$, determine $A \times \mathcal{P}(A)$

$P(A) = \{\{x, y\} \{x\}, \{y\}, \emptyset\}$

$A \times P(A) = \{ (x, \{x, y\})\ (x, \{x\})\ (x, \{y\})\ (x, \emptyset)$
$(y, \{x, y\})\ (y, \{x\})\ (y, \{y\})\ (y, \emptyset) \}$

6. (4.62) Let A and B be sets. Prove that $A \times B = \emptyset$ if and only if $A = \emptyset$ or $B = \emptyset$.

i) If $A = \emptyset$ or $B = \emptyset$, then $A \times B = \emptyset$

Assume $A = \emptyset$. Then $A \times B = \emptyset$, as required.

ii) If $A \times B = \emptyset$, then $A = \emptyset$ or $B = \emptyset$.

Assume the contrapositive $A \neq \emptyset$ and $B \neq \emptyset$.

Then $a \in A$ and $b \in B$. So $(a, b) \in A \times B$.

So $A \times B \neq \emptyset$, as required.

7. (4.66) Result 4.23 states that if A, B, C, and D are sets such that $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$.

(a) Show that the converse of Result 4.23 is false.

Converse: if $A \times B \subseteq C \times D$ then $A \subseteq C$ and $B \subseteq D$

Consider $A = \{a\}$ $B = \emptyset$ $C = \emptyset$ $D = \emptyset$

$A \times B = \emptyset$ and $C \times D = \emptyset$, so $A \times B \subseteq C \times D$.

However, $A \nsubseteq C$, so the converse is false.

(b) Under what added hypothesis is the converse true? Prove your assertion.

If $A \neq \emptyset$ and $B \neq \emptyset$.

If $A \times B \subseteq C \times D$ and $A \neq \emptyset$ and $B \neq \emptyset$, then $A \subseteq C$ and $B \subseteq D$.

Assume $A \times B \subseteq C \times D$ and $C \neq \emptyset$ and $D \neq \emptyset$.

Let $(a,b) \in A \times B$. Then $a \in A$ and $b \in B$.
and because $A \times B \subseteq C \times D$, then $a \in C$ and $b \in D$.

Thus $A \subseteq C$ and $B \subseteq D$, as required.

8. (4.70) Let A and B be sets. Show, in general, that $\overline{A \times B} \neq \overline{A} \times \overline{B}$.

Consider a counterexample.

Let $a \in A$ and $b \in B$. And we define the universal set $U = \{a, b\}$.

can define U if having troubles w/ U

The universal set for $A \times B$ is $U \times U = \{(a,a)(a,b)(b,a)(b,b)\}$

Then $\{(a,b)\} \in A \times B$. So $\overline{A \times B}$ is $U \times U - A \times B = \{(a,a)(b,a)(b,b)\}$

However $\overline{A} = U - a = \{b\}$ and $\overline{B} = U - b = \{a\}$, so $\overline{A} \times \overline{B} = \{(b,a)\}$

Hence $\overline{A \times B} \neq \overline{A} \times \overline{B}$

does not divides break into cases

9. (4.4) Let $x, y \in \mathbb{Z}$. Prove that if $3 \nmid x$ and $3 \nmid y$, then $3 | (x^2 - y^2)$.

Assume $3 \nmid x$ and $3 \nmid y$. Then $x = 3n+1$ or $x = 3n+2$
and $y = 3m+1$ or $y = 3m+2$ $n, m \in \mathbb{Z}$.

case 1: $x = 3n+1$, $y = 3m+1$

$$x^2 - y^2 = 9n^2 + 6n + 1 - 9m^2 - 6m - 1$$
$$= 3(3n^2 + 2n - 3m^2 - 2m)$$

Since $3n^2 + 2n - 3m^2 - 2m \in \mathbb{Z}$, $3 | x^2 - y^2$

case 2: $x = 3n+1$, $y = 3m+2$

$$x^2 - y^2 = 9n^2 + 6n + 1 - 9m^2 - 12m - 4$$
$$= 3(3n^2 + 2n - 3m^2 - 4m - 1)$$

Since $3n^2 + 2n - 3m^2 - 4m - 1 \in \mathbb{Z}$, $3 | x^2 - y^2$

case 3: $x = 3n+2$, $y = 3m+1$

$$x^2 - y^2 = 9n^2 + 6n + 4 - 9m^2 - 12m - 1$$
$$= 3(3n^2 + 2n - 3m^2 - 4m + 1)$$

Since $3n^2 + 2n - 3m^2 - 4m + 1 \in \mathbb{Z}$, $3 | x^2 - y^2$

case 4: $x = 3n+2$, $y = 3m+2$

$$x^2 - y^2 = 9n^2 + 12n + 4 - 9m^2 - 12m - 4$$

$= 3(3n^2+4n+3m^2-4m)$

Since $3n^2+4n+3m^2-4m \in \mathbb{Z}$, $3 \mid x^2-y^2$

10. (4.10) Let $n \in \mathbb{Z}$. Prove that $2|(n^4-3)$ if and only if $4|(n^2+3)$.

i) if $2 \mid n^4-3$, then $4 \mid n^2+3$

Assume $k = \dfrac{n^4-3}{2}$, $n^4 = 2k+3 = 2(k+1)+1$

$k+1 \in \mathbb{Z}$ so n^4 is odd. So n^2 is odd, so $n^2 = 2m+1$ $m \in \mathbb{Z}$

So $n^2+3 = 2m+1+3 = 4m+4 = 4(m+1)$

Since $m+1 \in \mathbb{Z}$, $4 \mid n^2+3$.

ii) if $4|(n^2+3)$, then $2 \mid n^4-3$

Assume $4|(n^2+3)$. Then $m = \dfrac{n^2+3}{4}$, $m \in \mathbb{Z}$

So $n^2+3 = 4m = 2(2m)$. Since $2m \in \mathbb{Z}$, $2 \mid n^2+3$.

11. (4.16) Let $a, b \in \mathbb{Z}$. Prove that if $a^2+2b^2 \equiv 0 \pmod 3$, then either a and b are both congruent to 0 modulo 3 or neither is congruent to 0 modulo 3.

$Q(x) =$ (a is cong. and b is cong.) or (a is not and b is not)

$\sim Q(x) =$ (a is not cong or b is not cong) and (a is cong or b is cong)
= only one of a or b is congruent

Assume the contrapositive if one of $a \equiv 0 \pmod 3$ or $b \equiv 0 \pmod 3$, then a^2+2

Case 1) $a \equiv 0 \pmod 3$ and $b \not\equiv 0 \pmod 3$

$\boxed{\text{Since } a \equiv 0 \pmod 3, \text{ then } a^2 \equiv 0 \pmod 3}$. So $a^2 = 3m$ $m \in \mathbb{Z}$

Then i) $b = 3k+1$, $k \in \mathbb{Z}$ ← note, useful theorem.

$$a^2+2b^2 = 3m + 2(3k+1)^2 = 3m + 2(9k^2+6k+1)$$
$$= 3m + 18k^2 + 12k + 2$$
$$= 3(m+6k^2+4k)+2$$

Since $m+6k^2+4k \in \mathbb{Z}$, $a^2+2b^2 \not\equiv 0 \pmod 3$

ii) $b = 3k+2$

$$a^2+2b^2 = 3m + 2(3k+2)^2 = 3m + 18k^2 + 24k + 8$$
$$= 3(m+6k^2+8k+2)+2$$

So $a^2+2b^2 \not\equiv 0 \pmod 3$

Case 2) $a \not\equiv 0 \pmod 3$ and $b \equiv 0 \pmod 3$

Since $b \equiv 0 \pmod 3$, then $b^2 \equiv 0 \pmod 3$. So $b^2 = 3m$ $m \in \mathbb{Z}$

Then i) $a = 3k+1$, $k \in \mathbb{Z}$

$$a^2+2b^2 = 9k^2+6k+1+2(3m)$$

$b^2 \not\equiv 0 \pmod{3}$

$= 9k^2 + 6k + 6m + 1$
$= 3(3k^2 + 2k + 2m) + 1$

$3k^2 + 2m + 2k \in \mathbb{Z}$, $\quad a^2 + 2b^2 \not\equiv 0 \pmod 3$

ii) $a = 3k + 2$, $\quad k \in \mathbb{Z}$

$a^2 + 2b^2 = 9k^2 + 12k + 4 + 6m$
$= 3(3k^2 + 4k + 2m + 1) + 1$

So $\quad a^2 + 2b^2 \not\equiv 0 \pmod 3$.

12. (4.24) Let x and y be even integers. Prove that $x^2 \equiv y^2 \pmod{16}$ if and only if either (1) $x \equiv 0 \pmod 4$ and $y \equiv 0 \pmod 4$ or (2) $x \equiv 2 \pmod 4$ and $y \equiv 2 \pmod 4$.

1) if $\quad x^2 \equiv y^2 \pmod{16}$, then $(x \equiv 0 \pmod 4$ and $y \equiv 0 \pmod 4))$ or $(x \equiv 2 \pmod 4)$ and y

1. (4.28)

(a) Prove that if r is a real number such that $0 < r < 1$, then

$$\frac{1}{r(1-r)} \geq 4.$$

Assume $\quad 0 < r < 1$. $\quad$ Since $\quad 2r - 1 \in \mathbb{R}$. $\quad$ Then $\quad (2r-1)^2 \geq 0$

$-4r + 4r^2 + 1 \geq 0$
$-4r(1-r) + 1 \geq 0$

$\equiv 2 (\mod 4))$

$\frac{1}{r(1-r)} \geq 4$ $r \neq 0$ and $1-r \neq 0$

As required.

(b) If the real number r in part (a) is an integer, is the implication true in this case? Explain.

No, since $0 < r < 1$, r cannot be an integer.

2. (4.34) Prove for every three real numbers x, y and z that $|x - z| \leq |x - y| + |y - z|$.

Consider $a = x - y$ and $b = y - z$, then $a + b = x - z$

Then by the triangle inequality $|a+b| \leq |a| + |b|$

So $|x - y + y - z| = |x - z| \leq |x - y| + |y - z|$

3. (4.36) Prove for every positive real number x that $1 + \frac{1}{x^4} \geq \frac{1}{x} + \frac{1}{x^3}$.

$x^4 + 1 \geq x + x^3$
$x^4 - x^3 - x + 1 \geq 0$

4. (5.2) Disprove the statement: If $n \in \{0, 1, 2, 3, 4\}$, then $2^n + 3^n + n(n-1)(n-2)$ is prime.

Case 1: $n = 0$, $2^n + 3^n + n(n-1)(n-2) = 1 + 1 = 2$

case 2: $n = 1$, $2 + 3 = 5$

case 3: $n = 2$, $4 + 9 = 13$

case 4: $n = 3$, $8 + 27 + 3(2)(1) = 14 + 27 = 41$

case 5: $n = 4$ $16 + 81 + 4(3)(2) = 97 + 24 = 121$

$121 = 11^2$ so it is not prime.

5. (5.4) Disprove the statement: Let $n \in \mathbb{N}$. If $\frac{n(n+1)}{2}$ is odd, then $\frac{(n+1)(n+2)}{2}$ is odd.

Consider $n = 2$ $\frac{n(n+1)}{2} = \frac{2 \cdot (2+1)}{2} = 3$

But $\frac{(n+1)(n+2)}{2} = \frac{3 \cdot 4}{2} = \frac{12}{2} = 6$ is even.

6. (5.6) Let $a, b \in \mathbb{Z}$. Disprove the statement: If ab and $(a+b)^2$ are of opposite parity, then a^2b^2 and $a + ab + b$ are of opposite parity.

Consider $a = 1$ and $b = 1$

$ab=1,\quad (a+b)^2=4\quad$ opposite parity

$a^2b^2=1,\quad 1+1+1=3\quad$ same parity

Thus proven.

7. (5.12) Prove that 200 cannot be written as the sum of an odd integer and two even integers.

Assume to the contrary that it can be.

$200=(2k+1)+2n+2m \qquad k,n,m\in\mathbb{Z}$

$=2(k+n+m)+1$

However 200 is even, so contradiction

8. (5.18) Let a be an irrational number and r a nonzero rational number. Prove that if s is a real number, then either $ar+s$ or $ar-s$ is irrational.

Suppose to the contrary that if $s\in\mathbb{R}$, then $ar+s\in\mathbb{Q}$ and $ar-s\in\mathbb{Q}$.

Then The sum of two rationals is rational

$ar+s+ar-s=\frac{p}{q} \qquad p,q\in\mathbb{Z}$

Since $r\in\mathbb{Z}$, then $r=\frac{n}{m} \qquad n,m\in\mathbb{Z}$

$a\frac{n}{m}+a\frac{n}{m}=\frac{p}{q}\,. \qquad 2a\frac{n}{m}=\frac{p}{q}$

$a=\frac{pm}{2qn}$

Since $pm\in\mathbb{Z}$ and $2qn\in\mathbb{Z}$, $a\in\mathbb{Q}$.

which is a contradiction.

9. (5.28) Prove that there do not exist positive integers m and n such that $m^2-n^2=1$.

Assume to the contrary they do exist

then $m^2-n^2=1$

$(m+n)(m-n)=1$

Since $m+n\in\mathbb{Z}$ and $m-n\in\mathbb{Z}$, then

case 1: $m+n=1$ and $m-n=1$

So $m=1-n \qquad m=1+n$

$1+n=1-n$

$n=-n$, which since $n>0$, is a contradiction

case 2: $m+n=-1$ and $m-n=-1$

so $m+n=m-n$
$n=-n$

10. (5.32) Prove that there exist no positive integers m and n for which $m^2+m+1=n^2$.

Assume they do $m^2+m+1=n^2$

$m^2+m+1 < m^2+2m+1$
$n^2 < (m+1)^2$
$n < m+1$

$m^2+m+1=n^2$
so $m^2 < n^2$
$m<n$

So $m<n<m+1$. But since $m,n \in \mathbb{Z}$, n cannot be between m and $m+1$. So there is a contradiction.

11. (5.36) Let $a,b \in \mathbb{R}$. Prove that if $ab \neq 0$, then $a \neq 0$ by using as many of the three proof techniques as possible.

Proof by contrapositive.

If $a=0$, then $ab=0$.

Assume $a=0$, then $ab=0$, as required.

12. (5.38) Prove the following statement using more than one method of proof.
Let $a,b \in \mathbb{Z}$. If a is odd and $a+b$ is even, then b is odd and ab is odd.

Assume a is odd and $a+b$ is even.

$a=2m+1$ $\quad a+b=2n$ $\quad m,n \in \mathbb{Z}$

Then $b=2n-2m-1=2(n-m-1)+1$

Since $n-m-1 \in \mathbb{Z}$ b is odd, as required.

And $ab=(2m+1)(2(n-m-1)+1) = 4(mn-m^2-m)+2m+2(n-m-1)+1$
$= 2(2(mn-m^2-m)+m+(n-m-1))+1$

ab is odd

1. (5.42) Show that there exist two distinct irrational numbers a and b such that a^b is rational.

$a=\sqrt{2}$ $\quad b=2\sqrt{2}$

$a^b = \sqrt{2}^{2\sqrt{2}}$

Assume $\sqrt{2}^{2\sqrt{2}}$ is rational Then we are done.

Assume $\sqrt{2}^{2\sqrt{2}}$ is irrational. Then

Assume $\sqrt{2}^{\sqrt{2}}$ is irrational. Then

$$\sqrt{2}^{\sqrt{2}^{2\sqrt{2}}} = \sqrt{2}^{2\sqrt{2}\sqrt{2}} = \sqrt{2}^{2\cdot 2} = 2^2 = 4,$$ which is rational

2. (5.46)

(a) Prove that there exist four distinct positive integers such that each integer divides the sum of the remaining integers.

Consider $a=1$, $a=2$, $a=3$, $a=6$

$\frac{1+2+3}{6}$ $\frac{2+3+6}{1}$ $\frac{1+2+6}{3}$ $\frac{1+3+6}{2}$

$\frac{6}{6}$ $\frac{11}{1}$ $\frac{9}{3}$ $\frac{10}{2}$

3. (5.50) Disprove the statement: There is a real number x such that $x^6 + x^4 + 1 = 2x^2$.

$\sim(\exists x \in \mathbb{R}$ s.t $x^6 + x^4 + 1 = 2x^2)$

$\forall x \in \mathbb{R},\ x^6 + x^4 + 1 \neq 2x^2$

$x^6 + x^4 - 2x^2 + 1 \neq 0$

$x^6 + (x^2 - 1)^2 \neq 0$

$x^6 = 0$ when $x = 0$. $(0-1)^2 = 1$. so both $x^6 = 0$ and $(x^2-1)^2 = 0$ cannot be true, so $\neq 0$

4. (5.52) The integers 1, 2, 3 have the property that each divides the sum of the other two. Indeed, for each positive integers a, the integers a, $2a$, $3a$ have the property tht each divides the sum of the other two. Show that the following statement is false.
There exists an example of three distinct positive integers different from a, $2a$, $3a$ for some $a \in \mathbb{N}$ have the property that each divides the sum of the other two.

7. (7.22)

(a) Express the following quantified statement in symbols:

There exist two integers a and b such that for every positive integer n, $a < \frac{1}{n} < b$.

$\exists a, b \in \mathbb{Z}$ s.t. $\forall n \in \mathbb{Z}^+,\ a < \frac{1}{n} < b$

(b) Prove that the statement in (a) is true.

Consider $a = 0$, $b = 2$

$\forall n \in \mathbb{Z}^+,\ 0 < \frac{1}{n} < 2$

$n \in [1, \infty)$

$n \in (0, 1]$

so $0 < \frac{1}{n} < 2$

8. (7.26) Prove the following statement. For every positive real number a and positive rational number b, there exist a real number c and irrational number d such that $ac + bd = 1$.

$\forall a, b \in \mathbb{Q}^+, \ \exists c \in \mathbb{R} \ d \in \mathbb{I}$ s.t $ac + bd = 1$

Choose $c = \frac{1 - \sqrt{2}}{a}$ $\quad d = \frac{\sqrt{2}}{b}$

13. (8.4) Let $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$. Then $R_1 = \{(a, 2), (a, 3), (b, 1), (b, 3), (c, 4)\}$ is a relation from A to B, which $R_2 = \{(1, b), (1, c), (2, a), (2, b), (3, c), (4, a), (4, c)\}$ is a relation from B to A. A relation R is defined on A by xRy if there exists $z \in B$ such the xR_1z and zR_2y. Express R by listing its elements.

$R_1 : A \to B$

$R_2 : B \to A$

$R = \{ (x, y) : \exists z \in B \text{ s.t. } (x, z) \in R_1 \text{ and } (z, y) \in R_2 \}$

$z=1$	$\exists\, (b,1)\ (1,b)$	(b,b)
	$(b,1)\ (1,c)$	$(b,c$
$z=2$	$\exists\, (a,2)\ (2,a)$	(a,a)
	$\exists\, (a,2)\ (2,b)$	(a,b)
$z=3$	$\exists\, (a,3)\ (3,c)$	(a,c)
	$\exists\, (b,3)\ (3,c)$	(b,c)
$z=4$	$\exists\, (c,4)\ (4,a)$	(c,a)
	$\exists\, (c,4)\ (4,a)$	(c,c)

$R = \{(a,a)(a,b)(a,c)(b,b)(b,c)(c,a)(c,c)\}$

1. (9.2) Let $A = \{1, 2, 3\}$ and $B = \{a, b, c, d\}$. Give an example of a relation from A to B containing exactly three elements such that R is not a function from A to B. Explain why R is not a function.

$R = \{(1,a)(2,c)(2,d)\}$

not a f b/c 3 not defined
and b/c 2 mapped to 2 items

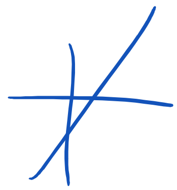
2. (9.4) For the given subset A_i of $\mathbb{R}$ and the relation R_i $(1 \le i \le 3)$ from A_i to $\mathbb{R}$, determine whether R_i is a function from A_i to $\mathbb{R}$.

(a) $A_1 = \mathbb{R}, R_1 = \{(x, y) : x \in A_1, y = 4x - 3\}$

$A_1 = \mathbb{R}$

$R_1 = \{(x,y) : x \in \mathbb{R}, y = 4x - 3\}$

yes, is a function



(b) $A_2 = [0, \infty)$, $R_2 = \{(x, y) : x \in A_2, (y + 2)^2 = x\}$

$R_2 = \{(x,y) : x \in [0,8), (y+2)^2 = x\}$

$\pm(y+2) = \sqrt{x}$

$y = \sqrt{x} - 2$

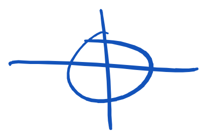
no, not a function.

(c) $A_3 = \mathbb{R}$, $R_3 = \{(x, y) : x \in A_3, (x + y)^2 = 4\}$

$R_3 = \{(x,y) : x \in \mathbb{R}, (x+y)^2 = 4$

$x^2 + 2xy + y^2 = 4$

not a function



3. (9.8) Let $A = \{5, 6\}$, $B = \{5, 7, 8\}$ and $S = \{n : n \ge 3$ is an odd integer$\}$. A relation R from $A \times B$ to S is defined as $(a, b)\ R\ s$ if $s|(a + b)$. Is R a function from $A \times B$ to S?

$R : A \times B \to S$

$A \times B = \{(5,5)(5,7)(5,8)(6,5)(6,7)(6,8)\}$

$S = \{3, 5, 7, 9, 11, 13, 15, 17, 19 \ldots\}$

$R = \{((a,b), s) : s | a+b\}$

$R = \{((5,5),5)$
$((5,7),3)$
$((5,8),13)$
$((6,5),11)$
$((6,7),13)$
$((6,8),7)\}$

$\frac{5+5}{5}$ $\frac{5+7}{3}$ $\frac{5+8}{13}$ $\frac{6+5}{11}$ $\frac{6+7}{13}$ $\frac{6+8}{7}$

4. (9.12, (a) and (b) only) For a function $f : A \to B$ and subsets C and D of A, and E and F of B, prove the following.

(a) $f(C \cup D) = f(C) \cup f(D)$

$C \subseteq A$, $D \subseteq A$, $E \subseteq B$, $F \subseteq B$.

$f(C \cup D) \subseteq f(C) \cup f(D)$

Let $y \in f(C \cup D)$ $\therefore$ $x \in C \cup D$ s.t $(x, y) \in f$

So $x \in C$ or $x \in D$.
So $y \in f(C)$ or $y \in f(D)$.

Thus $y \in f(C) \cup f(D)$

$f(C) \cup f(D) \subseteq f(C \cup D)$

Let $y \in f(C) \cup f(D)$. $\therefore$ $x \in C$ or $x \in D$ s.t

(1 pt) In order to disprove the statement
$\forall x \in S, P(x)$
one needs to prove which of the following?

- **A.** $\forall x \in S, \sim P(x)$
- **B.** $\forall x \in S, P(x)$
- **C.** $\exists x \in S$ s.t. $P(x)$
- **D.** $\exists x \in S$ s.t. $\sim P(x)$
- **E.** None of the above

$\sim(\forall x \in S, Px)$

$\exists x \in S, \sim P(x)$

(1 pt)

Mark each of the following true T or false F.

1. The numbers 24 and 24 provide a counterexample to the statement
"The sum of a multiple of 4 and a multiple of 8 must be a multiple of 8"

2. The numbers 7 and 4 provide a counterexample to the statement
"If the product of two integers is even, then the two integers are even"

3. The numbers 45 and 3 provide a counterexample to the statement
"The sum of a multiple of 5 and a multiple of 2 must be a multiple of 10"

4. The number 2 is a counterexample to the statement
"All prime numbers are odd numbers"

$4|24 \quad 8|24 \quad 8 \nmid 48$, F

$7 \cdot 4 = 28 \quad 2|28, \quad 2 \nmid 7. \quad$ T

$5|45 \quad 2 \nmid 3 \quad$ F

T.

(1 pt) Consider the proof provided below. Determine if the proof given is valid, and if so, which method of proof is being employed.

- **A.** The proof is valid and it is a direct proof.
- **B.** The proof is valid and it is a proof by contrapositive.
- **C.** The proof is valid and it is a proof by contradiction.
- **D.** This is not a valid proof.
- **E.** The proof is valid, but it uses a method different from those listed above.

Statement: Let $x, y \in \mathbb{R}$. Then $|xy| = |x||y|$

Proof:
Assuming $|xy| = |x||y|$, square both sides to get $(xy)^2 = x^2y^2$.

Next, take the square root of both sides to get $\sqrt{(xy)^2} = \sqrt{x^2}\sqrt{y^2}$.

Therefore, $xy = xy$, which is true for all real numbers x and y.
Thus, the statement is proved.

Note: For the purposes of this question, the proof deliberately omits an explicit statement of what method is being used. In general, unless it is a direct proof, your proofs SHOULD tell the reader what method you are using.

Not valid. starts by assuming given.

(1 pt) Consider the proof provided below. Determine if the proof given is valid, and if so, which method of proof is being employed.

- **A.** The proof is valid and it is a direct proof.
- **B.** The proof is valid and it is a proof by contrapositive.
- **C.** The proof is valid and it is a proof by contradiction.
- **D.** This is not a valid proof.
- **E.** The proof is valid, but it uses a method different from those listed above.

Statement: If x is an irrational number, then $x^{1/3}$ is also irrational.

Proof:
Let x be an irrational number, and suppose that $x^{1/3}$ is rational. Then there exist integers a and b, with $b \neq 0$, such that $x^{1/3} = \frac{a}{b}$. Then,

$$x = \left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}.$$

Hence, since a^3 and b^3 are integers, x is rational. This disagrees with our assumption that x is irrational. Thus, it must be that $x^{1/3}$ is also irrational.

Note: For the purposes of this question, the proof deliberately omits an explicit statement of what method is being used. In general, unless it is a direct proof, your proofs SHOULD tell the reader what method you are using.

---

valid. contradiction.

(1 pt) Consider the proof provided below. Determine if the proof given is valid, and if so, which method of proof is being employed.

- **A.** The proof is valid and it is a direct proof.
- **B.** The proof is valid and it is a proof by contrapositive.
- **C.** The proof is valid and it is a proof by contradiction.
- **D.** This is not a valid proof.
- **E.** The proof is valid, but it uses a method different from those listed above.

Statement: If x is an irrational number, then $x^{1/3}$ is also irrational.

Proof:
Suppose that $x^{1/3}$ is rational. Then there exist integers a and b, with $b \neq 0$, such that $x^{1/3} = \frac{a}{b}$. Then,

$$x = \left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}.$$

Hence, since a^3 and b^3 are integers, x is rational.
Thus, the statement is proved.

Note: For the purposes of this question, the proof deliberately omits an explicit statement of what method is being used. In general, unless it is a direct proof, your proofs SHOULD tell the reader what method you are using.

---

valid. contrapositive

(1 pt)

Consider a statement of the form:

$$\exists x \in S, \text{ s.t. } P(x)$$

True or False: Are the following valid ways to prove the above statement? Enter T if it is a valid approach, F if it is not valid.

1. Find specific values of x for which $\sim P(x)$ is false.

2. Find specific values of x for which $P(x)$ is true.

3. Prove that the statement $\forall x \in S, \sim P(x)$ is false.

4. Show that one can construct a value of $x \in S$ with the desired property $P(x)$

5. Show that if there does not exist an $x \in S$ with the desired property, then there is a contradiction to a previously established result or theorem

T

T

T

T

T

(1 pt)

Each of the statements below can be disproved using one of the approaches (a)-(d) listed below. Put the appropriate letter next to the corresponding statement.

a) Find an $x \in S$ such that ~$P(x)$ is true.

b) Find an $x \in S$ such that ~$P(x)$ and ~$Q(x)$ are both true.

c) Show that for all $x \in S$, ~$P(x)$ is true.

d) Show that for all $x \in S$, ~$P(x)$ and ~$Q(x)$ are both true.

1. $\forall x \in S, (P(x) \wedge Q(x))$

2. $\forall x \in S, (P(x) \vee Q(x))$

3. $\exists x \in S, P(x)$

4. $\forall x \in S, P(x)$

5. $\exists x \in S, (P(x) \vee Q(x))$

6. $\exists x \in S, (P(x) \wedge Q(x))$

$\sim = \exists x \in S$ s.t $\sim P(x) \vee \sim Q(x)$

a

$\sim = \exists x \in S$ s.t. $\sim P(x) \wedge \sim Q(x)$

b

$\sim = \forall x \in S, \sim P(x)$

c

$\sim = \exists x \in S,$ s.t. $\sim P(x)$

a

$\sim = \forall x \in S, \sim P(x) \wedge \sim Q(x)$

d

$\sim = \forall x \in S, \sim P(x) \vee \sim Q(x)$

c

(1 pt)

Let $I(x)$ be the statement "x has an Internet connection", let $C(x, y)$ be the statement "x and y have chatted over the internet". Let S be the set of students in your Math 220 class.

Express each of the following statements in terms of $I(x)$ and $C(x, y)$, quantifiers, and logical connectives. Put the appropriate letter next to the corresponding symbolic form.

1. $\exists x \in S \text{ s.t. } \exists y \in S \text{ s.t. } (y \neq x \wedge {}^{\sim} C(x, y))$
2. $\exists x \in S \text{ s.t } \exists y \in S \text{ s.t. } (y \neq x \wedge \forall z \in S, {}^{\sim}(C(x, z) \wedge C(y, z)))$
3. $\exists x \in S \text{ s.t. } {}^{\sim} I(x)$
4. $\forall x \in S, I(x) \Rightarrow \exists y \in S \text{ s.t. } (x \neq y \wedge C(x, y))$

a) Someone in your class does not have an internet connection.

b) There are two students in your class who have not chatted with each other over the internet.

c) Everyone in your class with an internet connection has chatted over the internet with at least one other student in your class.

d) There are at least two students in your class who have not chatted with the same person in your class.

1. there are two stuents that are different that have not chatted. B

2. there are two seperate students that have not both chatted to every other student ?

3. there is a student without internet (A)

4. for all student w/ internet, there is a student that they have chatted with. (E)

(1 pt) Determine whether the given proposition is true or false. Enter T for true and F for false.

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \text{ s.t. } x^2 + y = 0$$

Answer:

$$\exists x \in \mathbb{R} \text{ s.t. } \forall y \in \mathbb{R}, x^2 + y = 0$$

Answer:

$$\exists x \in \mathbb{R} \text{ s.t. } \exists y \in \mathbb{R} \text{ s.t. } x^2 + y = 0$$

Answer:

$$\forall y \in \mathbb{R}, \exists x \in \mathbb{R} \text{ s.t. } y = x^2$$

Answer:

$$\forall y \in \mathbb{R}, y \geq 0 \implies (\exists x \in \mathbb{R} \text{ s.t. } y = x^2)$$

Answer:

$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ s.t $x^2 + y = 0$

$\sim\rightarrow$ $\exists x \in \mathbb{R}$ s.t $\forall y \in \mathbb{R}$, $x^2 + y \neq 0$

negation is false so T

$\exists x \in \mathbb{R}$ s.t $\forall y \in \mathbb{R}$, $x^2 + y = 0$

F

T

$\exists y \in \mathbb{R}$ s.t $\forall x \in \mathbb{R}$, $y \neq x^2$

negation is true. $y = -1$

$\therefore$ (F)

$y \geq 0 \quad \exists x \in \mathbb{R} \quad s.t. \quad y = x^2$

(T)

(1 pt) Which of the relations below is NOT a function?

- **A.** {(0,-2), (1,0), (-1,-3), (0,-1)}
- **B.** {(2,5), (3,6), (4,7), (5,8)}
- **C.** {(6,-2), (-4,6), (-2,4), (1,0)}
- **D.** {(-1,5), (-2,5), (-3,5), (-4,5)}
- **E.** None of the above.

A $\quad 0 \to -1, \; 0 \to -2$

(1 pt) For the congruences below, enter the **lowest possible non-negative** integer that is congruent to the value given.

$28 \equiv$ ____ (mod 10)

$-21 \equiv$ ____ (mod 6)

$24 \equiv$ ____ (mod 9)

$-20 \equiv$ ____ (mod 6)

$25 \equiv$ ____ (mod 4)

$-28 \equiv$ ____ (mod 8)

$523913060 \equiv$ ____ (mod 101)

$551153244 \equiv$ ____ (mod 101)

$$\frac{28 - 8}{10} = 2$$

$$\frac{-21 - 3}{6} = -4$$

$$\frac{24 - 6}{9} = 2$$

$$\frac{-20-4}{6}=-4$$

$$\frac{25-1}{4}=6$$

$$\frac{-28-4}{8}=4$$

(1 pt) For n a nonnegative integer, either $n \equiv 0 \pmod 3$ or $n \equiv 1 \pmod 3$ or $n \equiv 2 \pmod 3$. In each case, fill out the following table with the canonical representatives (that is, the **lowest possible non-negative integer**) modulo 3 of the expressions given:

$n \pmod 3$	$n^3 \pmod 3$	$2n \pmod 3$	$n^3+2n \pmod 3$
0			
1			
2			

From this, we can conclude:

- **A.** Since $n^3+2n \not\equiv 0 \bmod 3$ for all n, we conclude that 3 does not necessarily divide n^3+2n for all nonnegative integers n.
- **B.** Since $n^3+2n \equiv 0 \bmod 3$ for all n, we conclude that 3 divides n^3+2n for any nonnegative integer n.

$0 \equiv n \pmod 3$

$\frac{0-n}{3}=k$ $n=3$

$x \equiv a \pmod 3$

$\frac{x-a}{3}$ 0

$\frac{1-n}{3}=k$ $n=4$

$\frac{x-4^3}{3}$ 1 $\therefore \frac{63}{3}$

$16\cdot 4=32\cdot 2=64$

$\frac{2-n}{3}=k$ $n=5$

$\frac{x-125}{3}$ $x=2$ $\therefore \frac{123}{3}$

(1 pt) Let x and y be real numbers. Which of the following below illustrates a correct application of the triangle inequality?

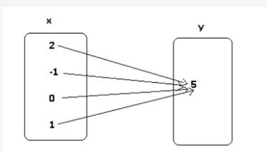
- **A.** $|x-y| < |x| - |y|$
- **B.** $|x-y| \leq |x| + |y|$
- **C.** $|x-y| < |x| + |y|$
- **D.** $|x-y| \leq |x| - |y|$
- **E.** None of the above

triangle $|a+b| \leq |a|+|b|$

(1 pt) Which of the relations below is NOT a function?
(Note: You can click to enlarge the images)

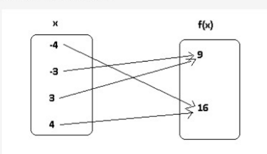
- **A.** Relation 1
- **B.** Relation 2
- **C.** Relation 3
- **D.** Relation 4
- **E.** None of the above.

Relation 1



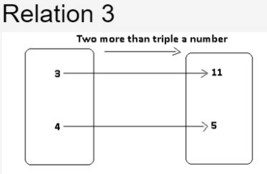
function

Relation 2



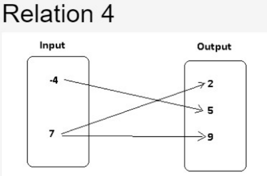
function

Relation 3



function

Relation 4



not function

(1 pt) Enter Y (for yes) or N (for no) in each answer space below to indicate whether the corresponding equation defines y as a function of x.

1. $x^2 + 3y = 7$
2. $1x = y^2$
3. $7 + x = y^3$
4. $x + 1 = y^2$

$y = \frac{7 - x^2}{3}$ y

$y = \pm\sqrt{x}$ n

$y = (x - 7)^{\frac{1}{3}}$ y

$y = \pm\sqrt{x+1}$ n

(1 pt)

Let $S = \{-9, -7, -5, -3, -1, 1, 3, 5, 7, 9\}$ and define $f : S \to \mathbb{N}$ by $f(x) = |x|$.
Answer the following questions by putting next to them the appropriate letter for the corresponding set below.

1. What is the range of the function f? S
2. What is the image of -3? 3
3. What is the codomain of the function f? $\mathbb{N}$
4. What is the domain of the function f? $\{1, 3, 5, 7, 9\}$

a) S

b) $\mathbb{N}$

c) $\{1, 3, 5, 7, 9\}$

d) $\{3\}$

a
d
b
c