Question 1

$E_1 = \{\text{1st die turns up odd}\}$, $E_2 = \{\text{2nd die turns up odd}\}$, $E_3 = \{\text{total number of spots is odd}\}$

Die 1	**Die 2**	**Spots**
E_1	E_2	$\overline{E_3}$
E_1	$\overline{E_2}$	E_3
$\overline{E_1}$	E_2	E_3
$\overline{E_1}$	$\overline{E_2}$	$\overline{E_3}$

(a)

$$\begin{aligned} P(E_1 \cap E_2) &= 1/4 & P(E_1)P(E_2) &= (1/2)(1/2) = 1/4 \\ P(E_1 \cap E_3) &= 1/4 & P(E_1)P(E_3) &= (1/2)(1/2) = 1/4 \\ P(E_2 \cap E_3) &= 1/4 & P(E_2)P(E_3) &= (1/2)(1/2) = 1/4 \end{aligned}$$

Since $P(E_1 \cap E_2) = P(E_1)P(E_2)$, $P(E_1 \cap E_3) = P(E_1)P(E_3)$, $P(E_2 \cap E_3) = P(E_2)P(E_3)$, so yes, the events $\boxed{\textbf{are independent}}$.

(b) $E_1 \cap E_2 \cap E_3$ is never true simultaneously, so the events $\boxed{\textbf{are mutually exclusive}}$.

Question 2

There are four possible routes, each with a 1/4 probability of being taken. Of those routes, E_1 has a 1/3 probability of taking the route to A, E_2 has a 1/2 probability of taking the route to A, E_3 has a 1 probability of taking the route to A, E_4 has a 1/4 probability of taking the route to A. So:

$$P = (1/4)(1/3) + (1/4)(1/2) + (1/4)(1) + (1/4)(1/4) \approx \boxed{0.52083}$$

Just for fun, I confirmed this for myself using the Python code in appendix A to simulate the situation 1000000 times, of which the robot reached A 52.1% of the time.

```
import random
n = 10000000
A = "Arrived"

E1 = [None, None, A]
E2 = [None, A]
E3 = [A]
E4 = [None, None, None, A]
O  = [E1, E2, E3, E4]

reached_a = 0
for i in range(n):
    c1 = random.choice(O)
    c2 = random.choice(c1)
    if c2 == A: reached_a += 1

print(reached_a/n) # approximately 0.5210729
```

Question 3

Use the inclusion-exclusion principle to determine the probability that there is a functioning path from n_1 to n_4, where:

(a) The probability that link l_i is functioning is p_i.

$$\bigcup_{i=1}^{n} A_i = \sum_{J\subseteq\{1,2,\ldots,n\},J\neq\emptyset} (-1)^{|J|-1} \bigcap_{j\in J} A_j$$
$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{J\subseteq\{1,2,\ldots,n\},J\neq\emptyset} (-1)^{|J|-1} P\left(\bigcap_{j\in J} A_j\right)$$

There are four[1] possible paths to get from n_1 to n_4:

- $l_1 \to l_5$
- $l_1 \to l_3 \to l_6$
- $l_2 \to l_6$
- $l_2 \to l_4 \to l_5$

So these are our events, $A_1 = l_1 \cap l_5$, $A_2 = l_1 \cap l_3 \cap l_6$, $A_3 = l_1 \cap l_6$, and $A_4 = l_2 \cap l_4 \cap l_5$.

So for the path from node 1 to 4:

$$P(R_{14}) = \sum_{J\subseteq\{1,2,3,4\},J\neq\emptyset} (-1)^{|J|-1} P\left(\bigcap_{j\in J} A_j\right)$$

$$\begin{aligned} = \quad & \big[(-1)^{4-1}P(\cap_{j\in J}A_j)\big] && \text{One subset with 4 elements} \\ & + 4\big[(-1)^{3-1}P(\cap_{j\in J}A_j)\big] && \text{Four subsets with 3 elements} \\ & + 6\big[(-1)^{2-1}P(\cap_{j\in J}A_j)\big] && \text{Six subsets with 2 elements} \\ & + 4\big[(-1)^{1-1}P(\cap_{j\in J}A_j)\big] && \text{Four subsets with 1 element} \end{aligned}$$

Because the links are independent, $P(l_1 \cap l_5) = P(l_1)P(l_5) = p_1p_5$, so $P(\cap_{j\in J}A_j) = \Pi_{j\in J}(p_j)$. Note still that $J \subseteq \{1, 2, 3, 4\}$.

$$= -P(\cap_{j\in J}A_j) + 4[P(\cap_{j\in J}A_j)] - 6[P(\cap_{j\in J}A_j)] + 4[P(\cap_{j\in J}A_j)]$$

$$= \boxed{-\Pi_{j\in J}(p_j) + 4[\Pi_{j\in J}(p_j)] - 6[\Pi_{j\in J}(p_j)] + 4[\Pi_{j\in J}(p_j)]}$$

[1] I am assuming that there will be no situations in which the message will retravel the path between n_2 and n_3

(b) Here we can use, for instance, $p_1 p_5 = p^2$, so

$$\begin{aligned} P(R_{14}) &= -P(\cap_{j\in J} A_j) + 4[P(\cap_{j\in J} A_j)] - 6[P(\cap_{j\in J} A_j)] + 4[P(\cap_{j\in J} A_j)] \\ &= -\big(p^4\big) + 4\big(p^3\big) - 6\big(p^2\big) + 4\big(p^1\big) \\ &= \boxed{-p^4 + 4p^3 - 6p^2 + 4p} \end{aligned}$$

Graphing this function, it appears to be correct because as the probability of success increases for a single link, the reliability of the entire network increases.

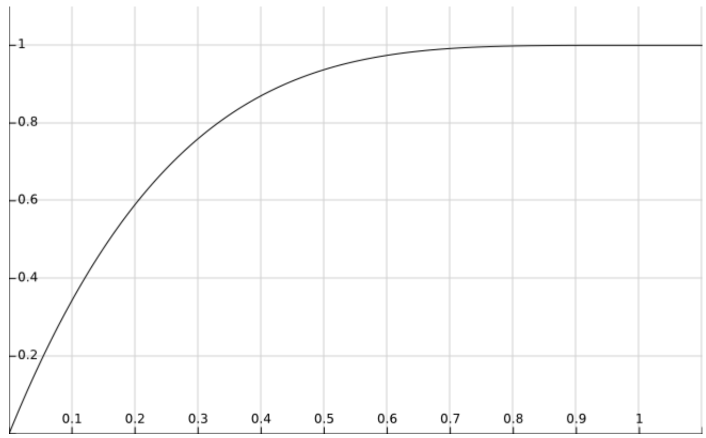

Figure 1: p on X-axis, $P(R_{14})$ on Y-axis

Question 4

(a) For a uniform distribution, pdf is $f_X(x) = \frac{1}{b-a}$

$$E(X) = \frac{1}{b-a}\int_a^b x dx = \boxed{\frac{a+b}{2}} \qquad E(X^2) = \frac{1}{b-a}\int_a^b x^2 dx = \frac{a^2+ab+b^2}{3}$$

$$Var(X) = E(X^2) - E(X)^2 = \left(\frac{a^2+ab+b^2}{3}\right) - \left(\frac{a+b}{2}\right)^2 = \boxed{\frac{(b-a)^2}{12}}$$

(b) For an exponential distribution, pdf is $f_X(x) = \lambda e^{-\lambda x}$

$$\begin{aligned}
E(X) &= \int_0^\infty x\lambda e^{-\lambda x} dx = \lambda\left(\frac{-xe^{-\lambda x}}{\lambda}\bigg|_0^\infty + \frac{1}{\lambda}\int_0^\infty e^{-\lambda x} dx \right) \\
&= \lambda\left(0 + \frac{-e^{-\lambda x}}{\lambda^2}\bigg|_0^\infty\right) = \lambda\left(\frac{1}{\lambda^2}\right) = \boxed{\frac{1}{\lambda}}
\end{aligned}$$

$$E(X^2) = \int_0^\infty x^2\lambda e^{-\lambda x} dx = \frac{2}{\lambda^2}$$

$$Var(X) = E(X^2) - E(X)^2 = \left(\frac{2}{\lambda^2}\right) - \left(\frac{1}{\lambda}\right)^2 = \boxed{\frac{1}{\lambda^2}}$$

(c) For a Gaussian distribution[2], pdf is $f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$.

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} xf(x)dx \\
&= \int_{-\infty}^{\infty} x\frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx \\
&= \int_{-\infty}^{\infty} [(x-\mu)+\mu]\frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx && \text{Setting } x = x - \mu + \mu \\
&= \int_{-\infty}^{\infty} (x-\mu)\frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx + \int_{-\infty}^{\infty} \mu\frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx && \text{Distributing} \\
&= \int_{-\infty}^{\infty} \frac{-(x-\mu)}{\sigma^2}\frac{-\sigma^2}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx + \int_{-\infty}^{\infty} \mu\frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx && \text{Rearranging so left integral cancels} \\
&= \frac{-\sigma^2}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \frac{-(x-\mu)}{\sigma^2} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx + \mu\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx && \text{Factoring constants} \\
&= \frac{-\sigma^2}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \frac{-(x-\mu)}{\sigma^2} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx + \mu\cancelto{1}{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx} && \text{From } \int_{-\infty}^{\infty} f(x)dx = 1 \\
&= \frac{-\sigma^2}{\sqrt{2\pi}\sigma}\left[e^{\frac{-(x-\mu)^2}{2\sigma^2}}\right]_{-\infty}^{\infty} + \mu && \text{Integrating and canceling} \\
&= \frac{-\sigma^2}{\sqrt{2\pi}\sigma}[0-0] + \mu \\
&= \cancelto{0}{\frac{-\sigma^2}{\sqrt{2\pi}\sigma}[0-0]} + \mu \\
&= \boxed{\mu}
\end{aligned}$$

[2] *Disclosure:* To derive the mean and variance of the Gaussian distribution, I followed the methods found by (Wikibooks, 2016) at `https://en.wikibooks.org/wiki/Statistics/Distributions/Normal_(Gaussian)`

$$
\begin{aligned}
Var(X) &= \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx \\
&= \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\cdot\left(\frac{x-\mu}{\sigma}\right)^2} dx \\
&= \int_{-\infty}^{\infty} \sigma^2 2w^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-w^2} \sigma\sqrt{2} dw && \text{Substituting } w = \frac{x-\mu}{\sigma\sqrt{2}} \\
&= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} w \cdot w e^{-w^2} dw && \text{Factoring out constants} \\
&= \frac{2\sigma^2}{\sqrt{\pi}} \left(\left[w \frac{-1}{2} e^{-w^2} \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{-1}{2} e^{-w^2} dw \right) && \text{Integrate by parts } u = w,\ v = -\frac{1}{2} e^{-w^2} \\
&= \frac{2\sigma^2}{\sqrt{\pi}} \left(\cancelto{0}{\left[w \frac{-1}{2} e^{-w^2} \right]_{-\infty}^{\infty}} - \int_{-\infty}^{\infty} \frac{-1}{2} e^{-w^2} dw \right) && \text{Appling L'Hopital to } \left[w \frac{-1}{2} e^{-w^2} \right]_{-\infty}^{\infty} \\
&= \frac{2\sigma^2}{\sqrt{\pi}} \left(\frac{1}{2} \int_{-\infty}^{\infty} e^{-w^2} dw \right) && \text{Factoring out} \frac{-1}{2} \\
&= \frac{2\sigma^2}{\sqrt{\pi}} \left(\frac{1}{2} \sqrt{\pi} \right) \\
&= \boxed{\sigma^2}
\end{aligned}
$$

Question 5

X_1 and X_2 have a *bivariate normal distribution* if their joint probability density function, $f(x, y) = P(X \leq x, Y \leq y)$ is:

$$
f_{X_1,X_2}(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} e^{\frac{1}{2(1-r^2)}\left(\frac{(x_1-a)^2}{\sigma_1^2} - \frac{2r(x_1-a)(x_2-b)}{\sigma_1\sigma_2} + \frac{(x_2-b)^2}{\sigma_2^2}\right)} \qquad \sigma_{1,2} > 0, \quad r \in (-1, 1)
$$

(a) From (Cox, 1997), $\text{Cov}(X, X) = \text{Var}(X)$ and:

$$
\text{Correlation coefficient}, r = \frac{\text{Cov}(X_1, X_2)}{\sigma_1\sigma_2}
$$

So where $X_1 = X_2 = X$:

$$
r = \frac{\text{Cov}(X_1, X_2)}{\sigma_1\sigma_2} = \frac{\text{Var}(X)}{\sigma^2} = \frac{\sigma^2}{\sigma^2} = 1
$$

Then the covariance matrix Σ of the bivariate normal distribution is:

$$
\Sigma = \begin{bmatrix} \text{Cov}(X_1, X_1) & \text{Cov}(X_1, X_2) \\ \text{Cov}(X_2, X_1) & \text{Cov}(X_2, X_2) \end{bmatrix} = \begin{bmatrix} r\sigma_1\sigma_1 & r\sigma_1\sigma_2 \\ r\sigma_2\sigma_1 & r\sigma_2\sigma_2 \end{bmatrix} = \boxed{\begin{bmatrix} \sigma_1^2 & r\sigma_1\sigma_2 \\ r\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}}
$$

Note that this matrix is symmetric.

(b) Because $X_1 \sim N(a, \sigma_1^2)$ and $X_2 \sim N(b, \sigma_2^2)$:

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1, x_2) dx_2 = \boxed{\frac{1}{\sqrt{2\pi}\sigma_1} e^{\frac{-(x-a)^2}{2\sigma_1^2}}}$$

$$f_{X_2}(x_2) = \int_{-\infty}^{\infty} f_{X_1,X_2}(x_1, x_2) dx_1 = \boxed{\frac{1}{\sqrt{2\pi}\sigma_2} e^{\frac{-(x-b)^2}{2\sigma_2^2}}}$$

(c) X_1 and X_2 are independent if $P(X_1 \cap X_2) = P(X_1)P(X_2)$, so they are independent if:

$$f_{X_1}(x_1) f_{X_2}(x_2) = f_{X_1,X_2}(x_1, x_2)$$

$$\left(\frac{1}{\sqrt{2\pi}\sigma_1} e^{\frac{-(x-a)^2}{2\sigma_1^2}}\right)\left(\frac{1}{\sqrt{2\pi}\sigma_2} e^{\frac{-(x-b)^2}{2\sigma_2^2}}\right) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} e^{\frac{1}{2(1-r^2)}\left(\frac{(x_1-a)^2}{\sigma_1^2} - \frac{2r(x_1-a)(x_2-b)}{\sigma_1\sigma_2} + \frac{(x_2-b)^2}{\sigma_2^2}\right)}$$

And because I'm lazy, I'm just going to show that this is false with a simple counterexample. Consider $x_1 = x_2 = r = \sigma_1 = \sigma_2 = 1/2$, and $a = 0$, $b = 1$.

$$\frac{1}{\sqrt{2\pi}(0.5)} e^{\frac{-((0.5)-0)^2}{2(0.5)^2}} \frac{1}{\sqrt{2\pi}(0.5)} e^{\frac{-((0.5)-1)^2}{2(0.5)^2}} = \frac{1}{2\pi(0.5)(0.5)\sqrt{1-(0.5)^2}} e^{\frac{1}{2(1-(0.5)^2)}\left(\frac{(0.5)^2}{(0.5)^2} - \frac{2(0.5)(0.5)(0.5-1)}{(0.5)(0.5)} + \frac{(0.5-1)^2}{(0.5)^2}\right)}$$

$$\frac{1}{\sqrt{2\pi}(0.5)} e^{-0.5} \frac{1}{\sqrt{2\pi}(0.5)} e^{-0.5} = \frac{1}{(0.5)\pi\sqrt{1-(0.5)^2}} e^{2}$$

$$0.234199... = 5.43173...$$

This is obviously false, so they are $\boxed{\textbf{not independent}}$.

This depends on r because if I set $r = 0$ then $\text{Cov}(X_1, X_1) = 0$ because:

$$r = \frac{\text{Cov}(X_1, X_2)}{\sigma_1\sigma_2}$$

And since X_1 and X_2 are *jointly normal*, if $\text{Cov}(X_1, X_2) = 0$ then they would be independent.

(d) For the reasons discussed above, X_1 and X_2 would be uncorrelated if $r = 0$. But since this is not a necessary condition, they are $\boxed{\textbf{not uncorrected}}$.

References

Cox, D. (1997). *Covariance and correlation.* Retrieved 2017-03-05, from http://www.stat.rice.edu/~dcox/Stat421/Supp2/node3.html

Wikibooks. (2016). *Statistics/distributions/normal (gaussian) — wikibooks, the free textbook project.* Retrieved 2017-03-05, from https://en.wikibooks.org/w/index.php?title=Statistics/Distributions/Normal_(Gaussian)&oldid=3147529