Assignment 1: Due Monday Mar 06 at 1:00 pm in class

Please include your name and student number when submitting the assignments for grading. Additionally, please note that you are to solve the problems below on your own. Collaborative work will be considered as cheating and UBC plagiarism penalties will apply. This assignment is worth 2.5% of your final grade. If you hand in the assignment X days late, then you will be penalized as follows: the max mark you can get is $\max\{5 - X, 0\}$.

The core concepts this assignment is designed to asses are:

- **Axioms of Probability and Event Operations**: Defining events ω, sample space Ω, probability measures $P(\omega)$, and algebra of sets.
- **Event Properties:** Independence and Mutually Exclusive
- **Total Probability Rule**
- **Random variables:** Statistics of random variables, and the three important distributions (uniform, exponential, Gaussian)

Questions

1. In throwing a pair of dice, let E_1 be the event that "the first die turns up odd", E_2 the event that "the second die turns up odd," and E_3 the event that "the total number of spots is odd." Given this information, answer the following:

   (a) Are the events E_1, E_2, and E_3 independent?

   (b) Are the events E_1, E_2, and E_3 mutually exclusive?

2. A robot leaves the point O as illustrated in Fig.1 with the goal of reaching the final point A by traversing through a set of paths. Unfortunately, the robot does not have any *a priori* information about the paths between the points. Therefore, at each point, the robot is equiprobable to take any path. What is the probability of the robot reaching the point A when starting from the point O?

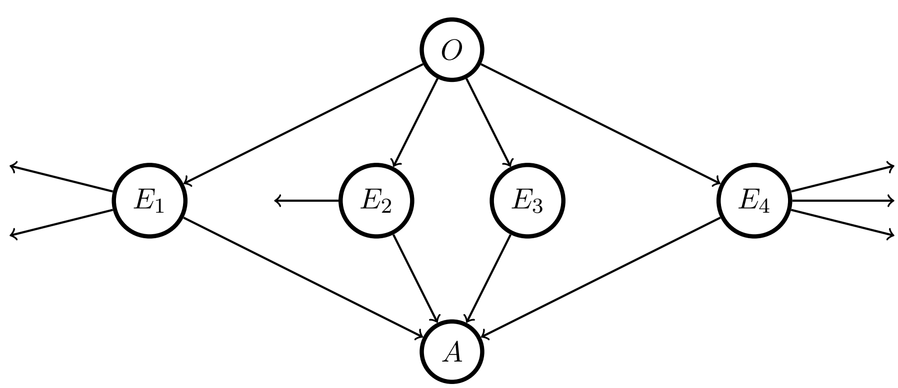

Figure 1: Schematic of the possible paths for a robot to take. Note that the robot can only move forward as indicated by the arrows.

3. Consider the communication network illustrated in Fig.2 which consists of four nodes n_1, n_2, n_3 and n_4, and five directed links $l_1 = (n_1, n_2), l_2 = (n_1, n_3), l_3 = (n_2, n_3), l_4 = (n_3, n_2), l_5 = (n_2, n_4)$, and $l_6 = (n_3, n_4)$. A message is sent from source node n_1 to the destination node n_4. The probability that the link l_i is functioning is given by p_i for $i \in \{1, \ldots, 6\}$. The links

behave physically independent from each other. A path from node n_1 to node n_4 is only functioning if each of its links is functioning. The goal is to construct the probability of their being a functioning path from node n_1 to node n_4.

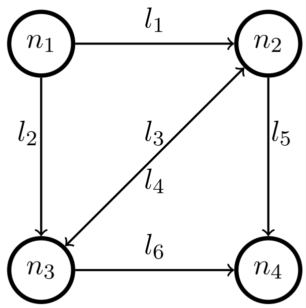

Figure 2: Schematic of the four node Communication Network.

(a) A useful tool for constructing the probability of a functioning path between n_1 to n_4, denoted by $P(R_{14})$, is the *inclusion-exclusion* principle. For a finite set of events $A_1, \ldots, A_n$ the inclusion-exclusion principle is given by:

$$\Big|\bigcup_{i=1}^{n} A_i\Big| = \sum_{\substack{J\subseteq\{1,2,\ldots,n\} \\ J\neq\emptyset}} (-1)^{|J|-1}\Big|\bigcap_{j\in J} A_j\Big|. \tag{1}$$

For $n = 2$, Eq.(1) gives the familiar relation that $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$. Use the inclusion-exclusion formula (1) to construct the probability that there is a functioning path from node n_1 to n_4 (i.e. $P(R_{14})$).

(b) How does the expression for the probability simplify when $p_i = p$ for all $i \in \{1, \ldots, 6\}$?

4. Compute the expected value $E\{X\}$ and variance $\mathrm{Var}(X)$ of the following distributions:

(a) Uniform distribution

$$f_X(x) = \frac{1}{b-a}, \quad a \leq x \leq b.$$

(b) Exponential distribution

$$f_X(x) = \lambda e^{-\lambda x}, \quad 0 \leq x.$$

(c) Gaussian distribution

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty \leq x \leq \infty.$$

5. Two continuous random variables X_1 and X_2 are said to have a *bivariate normal distribution* if their joint probability density function is:

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} \times \exp\Big\{ -\frac{1}{2(1-r^2)}\Big[\frac{(x_1-a)^2}{\sigma_1^2} - 2r\frac{(x_1-a)(x_2-b)}{\sigma_1\sigma_2} + \frac{(x_2-b)^2}{\sigma_2^2}\Big]\Big\}$$

where $\sigma_1 > 0$, $\sigma_2 > 0$, and $r \in (-1, 1)$. Given the bivariate normal distribution, solve the following.

(a) Construct the covariance matrix Σ of the bivariate normal distribution. The covariance matrix is defined by:

$$\Sigma = \begin{bmatrix} \mathrm{Cov}(X_1, X_1) & \mathrm{Cov}(X_1, X_2) \\ \mathrm{Cov}(X_2, X_1) & \mathrm{Cov}(X_2, X_2) \end{bmatrix}.$$

(b) Construct the marginal distributions $f_{X_1}(x_1)$ and $f_{X_2}(x_2)$ from the bivariate normal distribution.

(c) Are the random variables X_1 and X_2 independent? How does this result depend on r?

(d) Are the random variables X_1 and X_2 uncorrelated?