Ch. 12 - Proofs in Calculus

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12.1 LIMITS OF SEQUENCES

Sequence: A real valued function defined on the set of natural numbers $f: \mathbb{N} \to \mathbb{R}$

$\left\{ \frac{n}{2n+1} \right\}$ is the sequence $\frac{1}{3}, \frac{2}{5}, \frac{3}{7} \ldots$

$n=1$ $n=2$ $n=3$ ...

↳ the larger the value of n, the closer the term is to $\frac{1}{2}$

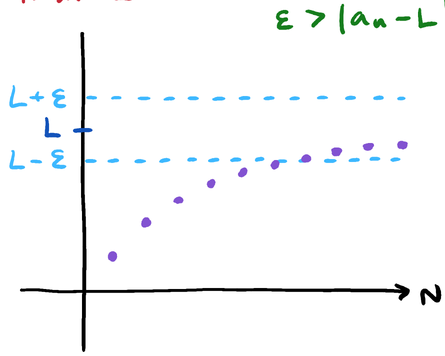
→ the sequence converges to the limit $L = \frac{1}{2}$

↳ if the sequence doesn't converge it diverges, it must be one or the other

The DISTANCE between a and b is $|a-b|$

→ ε is an arbitrarily small distance we choose from L

$\varepsilon > |a_n - L|$



Proof: a sequence $\{a_n\}$ converges to L if for every $\varepsilon > 0$, there exists a positive integer N such that if $n > N$, then $|a_n - L| < \varepsilon$.

Symbolically: $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t $(n > N) \Rightarrow (|a_n - L| < \varepsilon)$

Intuitively: for any distance from the limit, we can find an n^{th} term within that distance

Ex) Prove $\{\frac{1}{n}\}$ converges to 0

Let $\varepsilon > 0$. Choose $N = \lceil \frac{1}{\varepsilon} \rceil$ and take $n > N$.
Thus $n > \frac{1}{\varepsilon}$, so $\left|\frac{1}{n} - L\right| = \left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| = \frac{1}{n} < \varepsilon$

↳ divergence is the negation of convergence

Symbolically: $\forall L \in \mathbb{R},\ \exists \varepsilon > 0$ s.t. $\forall N \in \mathbb{N},\ \exists n > N,\ |a_n - L| \geq \varepsilon$

Ex) Prove $\left\{3 + \frac{2}{n^2}\right\}$ converges to 3

↳ Divergence to infinity

Symbolically: $\forall M \in \mathbb{R},\ \exists N \in \mathbb{N}$ s.t. $\forall n > N,\ a_n > M$

Let $\varepsilon > 0$, $\left\lceil \sqrt{\frac{2}{\varepsilon+3}} \right\rceil$ $\quad n > \sqrt{\frac{2}{\varepsilon+3}}$

↳ for every number, we can find a bigger one

$$|a_n - L| = \left|\left(3 + \frac{2}{n^2}\right) - 3\right| = \left|\frac{2}{n^2}\right| = \frac{2}{n^2} < \varepsilon$$

Ex) Prove $\left\{\frac{n}{2n+1}\right\}$ converges to $\frac{1}{2}$

Let $\varepsilon > 0$. Choose $N = \left\lceil \frac{\frac{1}{\varepsilon} - 2}{4} \right\rceil$

If $n > N$, then $n > \frac{\frac{1}{\varepsilon} - 2}{4}$

$$\left|\frac{n}{2n+1} - \frac{1}{2}\right| = \left|\frac{-1}{4n+2}\right| = \frac{1}{4n+2} < \varepsilon$$

SIDE WORK

$$\frac{n}{2n+1} - \frac{1}{2} < \varepsilon$$

$$\frac{2n}{2(2n+1)} - \frac{2n+1}{2(2n+1)} < \varepsilon$$

$$\frac{2n - 2n - 1}{2(2n+1)} = \frac{1}{4n+2} < \varepsilon$$

$$\frac{1}{\varepsilon} < 4n + 2$$

$$\frac{\frac{1}{\varepsilon} - 2}{4} < n$$

12.2 INFINITE SERIES

For real numbers, we write $\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \ldots$ to denote infinite series

We denote $\{S_n\}$ as the sequence of partial sums

$S_1 = a_1 \qquad S_2 = a_1 + a_2 \qquad S_3 = a_1 + a_2 + a_3 \qquad S_n = \sum_{k=1}^{n} a_k$

The series $\sum_{k=1}^{\infty} a_k$ converges to L if and only if the sequence of partial sums $\{S_n\}$ converges to L.

↳ we write $\sum_{k=1}^{\infty} a_k = L$

Ex) Prove the series $\sum_{k=1}^{\infty}\left(\frac{1}{k+1} - \frac{1}{k+2}\right)$ converges to $\frac{1}{2}$

$$S_1 = \frac{1}{2} - \frac{1}{3}$$

$$S_2 = \left(\frac{1}{2} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \frac{1}{4}\right)$$

"telescoping sums" cancel on themselves like this

$$S_3 = \left(\frac{1}{2} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \frac{1}{5}\right)$$

$$\cdots$$

$$S_n = \left(\frac{1}{2} - \frac{1}{n+2}\right)$$

$$\lim_{n\to\infty} S_n = \frac{1}{2}$$

PROOF

Let $\varepsilon > 0$. Consider $N = \left\lceil \frac{1}{\varepsilon} \right\rceil$. For any $n > N$, $n > \frac{1}{\varepsilon}$

$$\left|S_n - \frac{1}{2}\right| = \left|\frac{1}{2} - \frac{1}{n+2} - \frac{1}{2}\right| = \left|\frac{-1}{n+2}\right| = \frac{1}{n+2} < \frac{1}{n} < \varepsilon$$

Hence $\lim_{n\to\infty} S_n = \frac{1}{2}$

NOTE: HARMONIC SERIES $\sum_{k=1}^{\infty} \frac{1}{k}$ DIVERGES