## Problem 1 Todo: Standard Deviation

(a) The probability that $m$ additional tests will be taken is the probability that there is a faulty item in the pool. The probability that no items in $m$ are faulty is $(1-p)^m$, so the probability that at least one item in the pool is faulty is $1-(1-p)^m$. Multiply this by $m+1$ to indicate the initial pool test and the subsequent $m$ item tests.

If no items in the pool are faulty, $(1-p)^m$, then only one test will be taken so the average number of tests per pool is:

$$\#\text{ tests per pool} = (m+1)(1-(1-p)^m) + 1(1-p)^m$$

And so the cost per pool is:

$$X = 5\left[(m+1)(1-(1-p)^m) + 1(1-p)^m\right]$$

| **i** | $\mathbf{m_i}$ | $\mathbf{X_i}$ | $P$ |
|---|---|---|---|
| 1 | 1000 | 5004.8 | |
| 2 | 500 | 2488.6 | |
| 3 | 200 | 871.02 | |
| 4 | 100 | 321.98 | |
| 5 | 50 | 103.75 | |
| 6 | 25 | 32.772 | |
| 7 | 20 | 23.209 | |
| 8 | 10 | 9.7809 | |
| 9 | 8 | 8.0902 | |
| 10 | 5 | 6.2252 | |

The mean is the weighted average of all possible values of $X_i$ and their probabilities:

$$\mu =$$

$$\sigma =$$

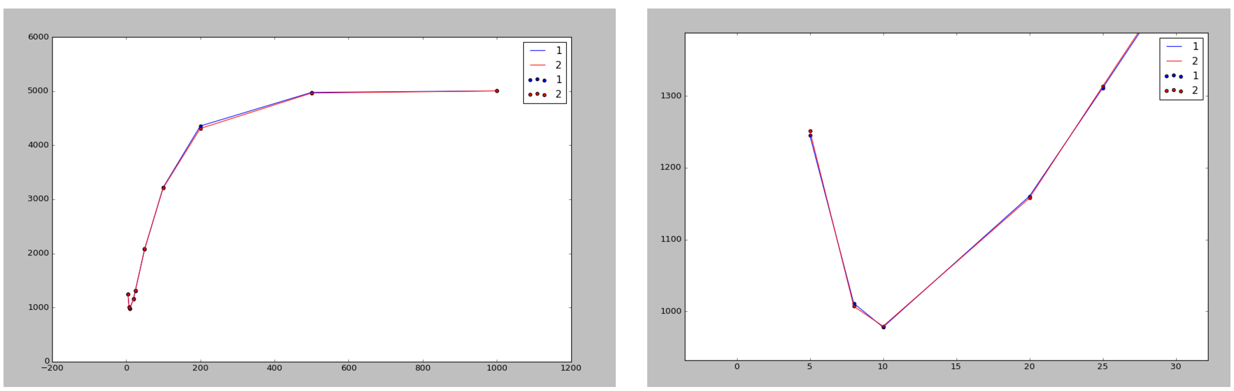
(b) So the average number of tests is the average number of tests per pool times the number of pools, $k$:

$$T_j\left[(m+1)(1-(1-p)^m) + 1(1-p)^m\right]$$

| $\mathbf{j}$ | $\mathbf{k_j}$ | $\mathbf{T_j} = \mu$ | $\sigma$ |
|---|---|---|---|
| 1 | 1 | 5004.78 | |
| 2 | 2 | 4977.15 | |
| 3 | 5 | 4355.10 | |
| 4 | 10 | 3219.84 | |
| 5 | 20 | 2074.97 | |
| 6 | 40 | 1310.89 | |
| 7 | 50 | 1160.47 | |
| **8** | **100** | **978.09** | |
| 9 | 125 | 1011.28 | |
| 10 | 200 | 1245.05 | |

(c) The best strategy is where $m = 10$ and $k = 100$.

Using the Python (code in Appendix A), I ran 1000 simulations for each pool size to confirm these results, comparing both the simulated and the calculated costs and graphing the two curves. In Figure 1, the *blue* curve[1] represents the costs as calculated using the above formula, and the *red* curve represents the cost as determined by averaging the results the simulations.




(a) range $m = [0, 1000]$ (b) range $m = [5, 30]$

Figure 1: Pooling size, $m$ (x-axis) vs. Total cost, $T$ (y-axis)

## Problem 2

Let $Y$ represent the number of traffic accidents in a given time $[Y \approx P(\lambda)]$ where $\lambda = 5/\text{day} = 35/week = \frac{5}{24}/hour$

(a) For a Poisson distribution:

$$\mu = \sigma^2 = \lambda = 35/week$$

[1] If this has been printed in black and white, the observation is that the two curves are almost identical.

(b)

$$P(X > 40) = 1 - \sum_{i=0}^{40} P(X = i) = 1 - \sum_{i=0}^{40} \frac{e^{-35}(35^i)}{i!} = \boxed{0.17506}$$

(c) The probability of waiting less than four hours is the same as the probability that an accident happens in the $0th$, $1st$, $2nd$, or $3rd$ hour.

$$P(X < 4) = \int_0^4 \frac{5}{24} e^{-\frac{5}{24}t} dt = 1 - e^{-\frac{5}{6}} = \boxed{0.5654}$$

(d)

$$\frac{n}{\lambda} = \frac{4}{5/24} = \boxed{19.2 \text{ hours}}$$

**Problem 3**

Where $U$ is a uniform random variable on the interval (0, 1) and $X = -\ln(1 - U)/\lambda$

(a) *Proof.* Notice that the range of $X$ is $(-\infty, 1)$, so for $x > 1$:

$$F_X(x) = 0$$

For $x \leq 1$:

$$\begin{aligned} F_X(x) &= P(X \leq x) \\ &= P(-\ln(1 - U)/\lambda \leq x) \\ &= P(U \leq 1 - e^{-\lambda x}) \\ &= F_U(1 - e^{-\lambda x}) \end{aligned} \tag{1}$$

Since $U$ is Unif(0, 1), then $F_U(u) = u,\ 0 \leq u \leq 1$, so:

$$\boxed{F_X(x) = 1 - e^{-\lambda x}}$$

$$f_X(x) = \frac{\partial}{\partial x} F_X(x) = \lambda e^{-\lambda x}$$

Mean:

$$\mu = E(X) = \int_0^\infty x f_X(x) dx = \int_0^\infty x\lambda e^{-\lambda x} dx$$

$$= -xe^{-\lambda x}\Big|_0^\infty + \frac{-1}{\lambda} e^{-\lambda x}\Big|_0^\infty = \boxed{\frac{1}{\lambda}}$$

Variance:

$$\sigma^2 = Var(X) = E(X^2) - E(X)^2$$

$$= \left( \int_0^\infty x^2 \lambda e^{-\lambda x} dx \right) - \left( \frac{1}{\lambda} \right)^2$$

$$= \left( \frac{2}{\lambda^2} \right) - \left( \frac{1}{\lambda} \right)^2 = \boxed{\frac{1}{\lambda^2}}$$

□

(b)

$$Y = e^{\lambda X} = e^{-\lambda \ln(1-U)/\lambda} = e^{-\ln(1-U)} = \frac{1}{1-U}$$

Notice that the range of $Y$ is $(-\infty, 1)$, so for $y > 1$:

$$F_Y(y) = 0$$

For $x \leq 1$:

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P\left( \frac{1}{1-U} \leq y \right) \\ &= P\left( U \leq \frac{y-1}{y} \right) \\ &= F_U\left( \frac{y-1}{y} \right) \end{aligned} \tag{2}$$

Since $U$ is Unif$(0, 1)$, then $F_U(u) = u, \; 0 \leq u \leq 1$, so:

$$\boxed{F_Y(y) = \frac{y-1}{y}}$$

$$\boxed{f_Y(y) = \frac{\partial}{\partial x} F_Y(y) = \frac{1}{y^2}}$$

**Problem 4 (Incomplete)**

Let $Z$ be the standard normal variable with density:

$$\varphi(z) = f(z) = \frac{1}{\sqrt{2\pi}} e^{-(z^2/2)}$$

(a) *Proof.*

$$
\begin{aligned}
I^2 =& \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} \; dxdy \\
=& \frac{1}{2\pi} \int_{-\pi}^{\pi} \int_{0}^{\infty} e^{-r^2/2} \; rdrd\theta \\
=& \frac{1}{2\pi} \int_{0}^{2\pi} \int_{0}^{\infty} e^{-r^2/2} \; rdrd\theta \\
=& \frac{1}{2\pi} 2\pi \Big[ - e^{-r^2/2} \; \Big|_0^{\infty} \Big] \\
=& \frac{1}{2\pi} 2\pi [0 - (-1)] \\
=& 1
\end{aligned}
$$

Since $I^2 = 1$, then $I = \pm 1$. Since $e^{-(z^2/2)} > 0$ for all $z$, then $I = 1$, as required. $\square$

(b)

$$
\begin{aligned}
\varphi'(z) &= -z\varphi(z) \\
z\varphi'(z) &= -z\varphi(z)
\end{aligned}
$$

*Proof.* Mean:

$$
\begin{aligned}
\mu = E(Z) &= \int_{-\infty}^{\infty} zf(z)dz \\
&= \int_{-\infty}^{\infty} z \frac{1}{\sqrt{2\pi}} e^{-(z^2/2)} dz \\
&= \int_{-\infty}^{0} z \frac{1}{\sqrt{2\pi}} e^{-(z^2/2)} dz + \int_{0}^{\infty} z \frac{1}{\sqrt{2\pi}} e^{-(z^2/2)} dz \qquad addmoredetail \\
&= \frac{-1}{2\pi} + \frac{1}{2\pi} \\
&= 0
\end{aligned}
$$

$\square$

*Proof.* Variance:

$$
\begin{aligned}
\sigma^2 = Var(Z) = E(Z^2) - \cancelto{0}{E(Z)^2} &= E(Z^2) \\
&= \int_{-\infty}^{\infty} z^2 f(z) dz \\
&= \int_{-\infty}^{\infty} z^2 \frac{1}{\sqrt{2\pi}} e^{-(z^2/2)} dz \qquad Addmoredetail \\
&= 1
\end{aligned}
$$

$\square$

### Problem 5

Suppose the lifetime $Y$ of a system has failure rate $h(y) = 2y$, $y > 0$.

(a) Assuming $y$ represents time, then the failure rate increases with time so the system gets weaker as it ages.

(b)

$$F_Y(x) = 1 - \exp\left(-\int_0^x h(y)\ dy\right) = 1 - \exp\left(-\int_0^x 2y\ dy\right) = \boxed{1 - e^{-x^2}}$$

$$f_Y(x) = \frac{\partial}{\partial x}\left(1 - e^{-x^2}\right) = \boxed{2xe^{-x^2}}$$

(c)

$$F_Y(m) = \frac{1}{2} = 1 - e^{-m^2} \to m = \pm \ln 2$$

$$\boxed{m = 0.83255}$$

### Problem 6 (Am I allowed to use code here?)

$$X \sim N(5, \sigma^2)$$

$$F(x) = P(X \leq x) = \Phi\left(\frac{x-5}{\sigma}\right)$$

Determine $\sigma$ where:

$$\frac{1}{100} = 1 - P(-0.02 \leq X \leq 0.02) = 1 - \left[\Phi\left(\frac{0.02}{\sigma}\right) - \Phi\left(\frac{-0.02}{\sigma}\right)\right] = 1 - [2\Phi(0.02) - 1]$$

$$\frac{99}{100} = P(-0.02 \leq X \leq 0.02) = \Phi\left(\frac{0.02}{\sigma}\right) - \Phi\left(\frac{-0.02}{\sigma}\right) = 2\Phi\left(\frac{0.02}{\sigma}\right) - 1$$

Doing a direct search with the code in Appendix C to solve for $\sigma$, when $\boxed{\sigma < 0.007764489}$ at most 1% of the ball bearings are out-of-spec.

### Problem 7 (What does actual temperature of the medium mean?)

$\sigma = 0.1\mu$, $\sigma^2 = 0.01\mu^2$.

$$X \sim N(\mu, 0.01\mu^2)$$

$$0.95 = P(-0.1 \leq X \leq 0.1) = 2\Phi\left(\frac{0.1}{0.1\mu}\right) - 1$$

Doing a direct search with the code in Appendix D to solve for $\mu$, when $\boxed{\mu < 0.51021345}$ the probability is larger than or equal to 0.95 that the temperature reading is within 0.1.

**Problem 8 (Not Started)**

Show $U = F_X(X)$ is uniform on interval (0, 1) where $X$ is continuous and has invertible cdf $F_X(x)$.

Call $Q_X(x)$ a quantile of $X$. If $F_X$ is invertible, then $F_X^{-1}(x) = Q_X(x)$.

Given a uniform variable $U$ on the interval [0, 1] and an invertible cdf $F_X$, then the random variable $X = F_X^{-1}(U)$ has distribution $F$.

Since $F$ is continuous, then $F$ is invertible since it is continous and strictly increasing. .

**Problem 9 (Not Started)**

**Problem 10 (Unsure)**

**Using Python:**

```
n = 10000000
x1 = numpy.random.normal(loc=90,  scale=10, size=n)
x2 = numpy.random.normal(loc=100, scale=12, size=n)
x3 = numpy.random.normal(loc=110, scale=14, size=n)
v  = x2 - 1/2*(x1+x3)
print(len([i for i in v if -9 <= i <= 9])/n)

>>> 0.4578041
```

**Using R:**

```
n <- 10000000
x1 <- rnorm(n, mean=90,  sd=10)
x2 <- rnorm(n, mean=100, sd=12)
x3 <- rnorm(n, mean=110, sd=14)
v  <- x2 - 0.5*(x1+x3)
sum(-9 <= v & v <= 9)/n

> [1] 0.457829
```

$$P\left(-9 \leq X_2 - \frac{1}{2}(X_1 + X_3) \leq 9\right) \approx \boxed{0.4578}$$

**Problem 11 (How do we get $x - \mu = \epsilon$???**

(a) *Proof.*

$$P(|X - \mu| \leq \epsilon) \geq 1 - \frac{\sigma^2}{\epsilon^2}, \text{ for all } \epsilon > 0$$

Where $X$ is a random variable with mean $\mu$ and variance $\sigma^2$, and $\epsilon > 0$:

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx$$

As given:

$$\sigma^2 \geq \int_{-\infty}^{\mu-\epsilon} (x - \mu)^2 f(x) dx + \int_{\mu+\epsilon}^{\infty} (x - \mu)^2 f(x) dx$$

Factoring:

$$\sigma^2 \geq (x - \mu)^2 \Big( \int_{-\infty}^{\mu-\epsilon} f(x) dx + \int_{\mu+\epsilon}^{\infty} f(x) dx \Big)$$

By CDF definition:

$$\begin{aligned}
\sigma^2 \geq& (x - \mu)^2 \Big( P(X \leq \mu - \epsilon) + P(X \geq \mu + \epsilon) \Big) \\
\sigma^2 \geq& (x - \mu)^2 \Big( P(X - \mu \leq -\epsilon) + P(X - \mu \geq \epsilon) \Big) \\
\sigma^2 \geq& (x - \mu)^2 \Big( P(|X - \mu| \geq \epsilon) \Big)
\end{aligned}$$

Dividing by $(x - \mu)^2$:

$$\begin{aligned}
P(|X - \mu| \geq \epsilon) &\leq \frac{\sigma^2}{(x - \mu)^2} \\
P(|X - \mu| \leq \epsilon) &\geq 1 - \frac{\sigma^2}{(x - \mu)^2}
\end{aligned}$$

Since $x - \mu = \epsilon$ (???):

$$\boxed{P(|X - \mu| \leq \epsilon) \leq 1 - \frac{\sigma^2}{\epsilon^2}}$$

$\square$

(b) *Proof.*

$$\overline{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

So:

$$\begin{aligned}
E(\overline{X}) &= E\left(\frac{X_1 + X_2 + \cdots + X_n}{n}\right) \\
&= \frac{1}{n}\left(E(X_1) + E(X_2) + \cdots + E(X_n)\right) \\
&= \frac{1}{n}\left(\mu_1 + \mu_2 + \cdots + \mu_n\right) \\
&= \frac{1}{n}\left(\mu(n)\right) = \boxed{\mu}
\end{aligned}$$

□

*Proof.*

$$\overline{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

So:

$$\begin{aligned}
Var(\overline{X}) &= Var\left(\frac{X_1 + X_2 + \cdots + X_n}{n}\right) \\
&= Var\left(\frac{X_1}{n} + \frac{X_2}{n} + \cdots + \frac{X_n}{n}\right)
\end{aligned}$$

Since $X_1$, $X_2$, ... , $X_n$ are independent[2], $Var\left(\sum_{i=1}^{n}(X_i)\right) = \sum_{i=1}^{N} Var(X_i)$, so:

$$Var(\overline{X}) = Var\left(\frac{X_1}{n}\right) + Var\left(\frac{X_2}{n}\right) + \cdots + Var\left(\frac{X_n}{n}\right)$$

From $Var(aX) = a^2 Var(X)$:

$$\begin{aligned}
Var(\overline{X}) &= \frac{1}{n^2}Var(X_1) + \frac{1}{n^2}Var(X_2) + \cdots + \frac{1}{n^2}Var(X_n) \\
&= \frac{1}{n^2}\Big(Var(X_1) + Var(X_2) + \cdots + Var(X_n)\Big) \\
&= \frac{1}{n^2}\Big(\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2\Big) \\
&= \frac{1}{n^2}\sigma^2(n) = \boxed{\frac{\sigma^2}{n}}
\end{aligned}$$

□

(c) *Proof.* From (a), for all $\epsilon > 0$:

$$P(|X - \mu| \leq \epsilon) \leq 1 - \frac{\sigma^2}{\epsilon^2}$$

[2] Assuming "independently measured" mean independent

Substituting $\overline{X}$ for $X$:

$$P(|\overline{X} - E(\overline{X})| \leq \epsilon) \leq 1 - \frac{Var(\overline{X})}{\epsilon^2}$$
$$P(|\overline{X} - E(\overline{X})| \geq \epsilon) \leq \frac{Var(\overline{X})}{\epsilon^2}$$

From (b), substituting $E(\overline{X}) = \mu$ and $Var(\overline{X}) = \frac{\sigma^2}{n}$:

$$P(|\overline{X} - \mu| \geq \epsilon) \leq \frac{\frac{\sigma^2}{n}}{\epsilon^2}$$
$$P(|\overline{X} - \mu| \geq \epsilon) \leq \frac{\sigma^2}{n\epsilon^2}$$

And so as $n \to \infty$:

$$P(|\overline{X} - \mu| \geq \epsilon) \leq \cancelto{0}{\frac{\sigma^2}{n\epsilon^2}}$$

And because the probability cannot be negative:

$$\lim_{n\to\infty} P(|\overline{X} - \mu| \geq \epsilon) = 0$$

Therefore, as required:

$$\boxed{\lim_{n\to\infty} P(|\overline{X} - \mu| \leq \epsilon) = 1} \quad \text{for all } \epsilon > 0$$

□

# Appendix

## A: Problem 1 Simulation Code

```
import matplotlib.pyplot as plt
import numpy, random

n = 1000
T = 5
p = 0.01
pool_sizes = [1000, 500, 200, 100, 50, 25, 20, 10, 8, 5]

def main():
    tests = []

    for m in pool_sizes:
        # calculate the total cost using probability
        k = n/m
        number_of_tests_per_pool = (m+1)*(1-(1-p)**m)+1*(1-p)**m
        cost_per_pool = number_of_tests_per_pool*T
        number_of_tests_total = number_of_tests_per_pool*k
        cost_total = number_of_tests_total*T

        # simulate 1000 tests with random sequences of approximately p
        simulation_costs = []
        for i in range(1000):
            simulated_items = create_random_items(n, p)
            simulated_pools = split_into_sublists(simulated_items, m)
            simulation_costs.append(calculate_cost(simulated_pools))

        # save the results
        test_result = {
            "m": m,
            "calculated" : {
                "tests_per_pool":   number_of_tests_per_pool,
                "tests_total":      number_of_tests_total,
                "cost_per_pool":    cost_per_pool,
                "cost_total":       cost_total,
            },
            "simulated" : {
                "cost_total" : numpy.mean(simulation_costs),
                "cost_variance" : numpy.var(simulation_costs),
            }
        }
        tests.append(test_result)

    # print the results
    for test_result in tests:
        print(test_result["m"], "\t", test_result["calculated"]["cost_total"], "\t", test_result["simulated"]["cost_total"])

    # plot the results
    scatter([[t["m"], t["calculated"]["cost_total"], t["simulated"]["cost_total"]] for t in tests], connect_dots=True)

# create an array of booleans, approximately p of which are False (defective)
def create_random_items(n, p):
    arr = []
    for i in range(n):
        num = random.randint(1,int(1/p))
        arr.append( False if num == 1 else True)
    return arr

# convert a list into a list of lists segmented into chunks
def split_into_sublists(arr, size):
    a = [arr[x:x+size] for x in range(0, len(arr), size)]
    if not all(len(i) == len(a[0]) for i in a):
        return None
    return a

# check if all items in an list are True
def all_true(arr):
    return len(arr) == arr.count(True)

def calculate_cost(pools):
    cost = 0
    for pool in pools:
        # cost for initial test of pool
        cost += T
        # if all in pool are true, then only this one test needed to be conducted
        # otherwise, conduct tests again for all members of the pool
        if not all_true(pool): cost += len(pool)*T
    return cost

# scatter plot a 2d array
def scatter(arr, connect_dots=False):
    colors = ["b", "r", "g", "y"]
    x = [i[0] for i in arr]
    for series in range(1, len(arr[0])):
```

```
        y = [i[series] for i in arr]
        plt.scatter(x, y, label=str(series), c=colors[series-1])
        if connect_dots:
            plt.plot(x, y, label=str(series), c=colors[series-1])
    plt.legend()
    plt.show()

if __name__ == "__main__":
    print("Starting...")
    main()
    print("Done.")
```

## B: Problem 2(c) Simulation

```
import numpy, random

def poissonDelay():
    # where log is the natural logarithm
    return -numpy.log(1-random.random())/

def poissonDist(n, ):
    return [poissonDelay() for i in range(n)]

def poissonTimes(n, ):
    dist = poissonDist(n, )
    intervals = []
    for i in range(n):
        intervals.append(sum(dist[:i]))
    return intervals

lam = 5/24.0  # per hour
less_than = 4 # hours

n = 100000
all = []
for i in range(n):
    accident_times = numpy.array(poissonTimes(10, lam))
    accidents_within_time = numpy.array(numpy.where(accident_times < less_than))
    all.append(accidents_within_time.size)

print(1 - all.count(1)/n )

>>>0.56568
```

## C: Problem 6

```
import numpy, scipy.stats

def pnorm(a):
    return scipy.stats.norm.cdf(a)

mean = 5
tolerance = 0.02
n = 1000000

for std_dev in numpy.arange(0.007,0.008,0.000001):
    dist = numpy.random.normal(loc=mean, scale=std_dev, size=n)

    too_sml = numpy.array(numpy.where(dist < mean-tolerance)).size
    too_big = numpy.array(numpy.where(dist > mean+tolerance)).size
    num_in_spec = (n - too_sml) - too_big
```

```
    simulated_percent = num_in_spec/n
    calculated_percent = 2*pnorm(0.02/std_dev)-1

    print(std_dev, simulated_percent, calculated_percent)

    if simulated_percent <= 0.99 and calculated_percent <= 0.99:
        break
```

## D: Problem 7

```
import numpy, scipy.stats, math

def pnorm(a):
    return scipy.stats.norm.cdf(a)

### determine the mean ###

for mean in numpy.arange(0.5102,0.512,0.00000001):
     p = 2*pnorm(0.1/(0.1*mean))-1
     print(mean, p)
     if p < 0.95: break


### confirm the mean ###

mean = 0.51021345
std_dev = 0.1*mean

domain = numpy.arange(0.3, 0.7, 0.00001)
norm   = scipy.stats.norm.pdf(domain, mean, std_dev)

total   = sum(norm)
in_spec = sum([norm[i] for i in range(len(domain)) if (mean-0.1) < domain[i] < (mean+0.1)])
print(in_spec/total)
```