Ch. 3 Direct Proof and Proof by Contrapositive

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Things you don't have to prove

AXIOMS → something we accept as truth without proof
↳ "the sum of two integers is an integer"

THEOREMS → true statements that can be verified
→ they're also usually "significant or interesting"
↳ Pythagorem theorem ↳ usually stated as implications

COROLLARY → a result deduced from a theorem
↳ "the hypotenus of a right triangle is longer than its side"

LEMMA → a true statement useful for proving something else

3.1 TRIVIAL/VACUOUS PROOFS

↳ most implications expressed as a theorem are of the form $\forall x \in S,\ P(x) \to Q(x)$

TRIVIAL PROOF → can prove above implication by showing $Q(x)$ is true for all $x \in S$ because by definition, if $Q(x)$ is true, the implication is true

P	Q	P→Q
T	T	T
T	F	F
F	T	T
F	F	T

Ex) Prove the following: Let $x \in \mathbb{R}$. If $x<0$, then $x^2+1>0$.

By a trival proof, we only must show $x^2+1>0$ for all $x \in \mathbb{R}$.

Since $x^2 \geq 0$ for all real numbers, then $x^2+1 > x^2 \geq 0$. Thus, $x^2+1>0$. ■

VACUOUS PROOF → where $\forall x \in S,\ P(x) \to Q(x)$ the implication can be proved if it can be shown $P(x)$ is F for all $x \in S$

Ex) Prove the following: Let $x \in \mathbb{R}$. If $x^2-2x+2 \leq 0$, then $x^3 \geq 8$

By a vacuous proof, it must be shown $x^2-2x+2 \leq 0$ is F for all $\mathbb{R}$.

$x^2-2x+2 \geq 0$

$(x-1)(x-1)+1 \geq 0$

since $(x-1)^2 \geq 0$, then $(x-1)^2+1 \geq 0$ thus $x^2-2x+2 \leq 0$ is fal for all $x \in \mathbb{R}$, thus the implication is true. ■

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Ex) Let $S = \{n \in \mathbb{Z} : n \geq 2\}$ and let $n \in S$.
If $2n + \frac{2}{n} < 5$, then $4n^2 + \frac{4}{n^2} < 25$

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By vacuous proof, if we show $2n + \frac{2}{n} \geq 5$ for all $n \in \mathbb{Z} : n \geq 2$, we show implication is true for all $n \in \mathbb{Z} : n \geq 2$

case $n = 2$: $2n + \frac{2}{n} = 5 \geq 5$

case $n \geq 3$: $2n \geq 6$ and $0 < \frac{2}{n} \leq \frac{2}{3}$

thus, $2n + \frac{2}{n} \geq 5$

As demonstrated, $2n + \frac{2}{n} \geq 5$ for all $n \in \mathbb{Z} : n \geq 2$ thus the hypothesis of the implication is always false, thus the implication is true. ■

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3.2 DIRECT PROOF

↳ unlike trivial proofs, most implications $P(x) \to Q(x)$ have a $Q(x)$ that depends on the value of $P(x)$ or the $P(x)$ depends on $Q(x)$

DIRECT PROOF → assume $P(x)$ is true for arbitrary $x \in S$ and show $Q(x)$ is true for this x

HELPFUL AXIOMS OF INTEGERS

- the negative of every integer is an integer
- the sum/difference of every two integers is an integers
- the product of every two integers is an integer

EVEN/ODD

- n is even if $n = 2k$ for some integer k
- n is odd if $n = 2k+1$ for some integer k

an integer is even if and only if its square is even

$\{2k : k \in \mathbb{Z}\} \cup \{2k+1 : k \in \mathbb{Z}\} = \mathbb{Z}$

$\{2k : k \in \mathbb{Z}\} \cap \{2k+1 : k \in \mathbb{Z}\} = \phi$

Result 3.4 *If n is an odd integer, then $3n + 7$ is an even integer.*

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By direct proof assume n is odd, thus $n = 2k+1$ for some integer k.

$$3n+7 = 3(2k+1)+7 = 6k+10 = 2(3k+10)$$

Since $3k+10$ is an integer, $2(3k+10)$ is of the form $2m$ where m is an integer, therefore is even.

$\blacksquare$

Result 3.6 *If n is an odd integer, then $4n^3 + 2n - 1$ is odd.*

Since n is an odd integer, it is of the form $n = 2k+1$ for some integer k.

Thus
$$\begin{aligned} 4n^3+2n-1 &= 4(2k+1)^3 + 2(2k+1) - 1 \\ &= 4(2k+1)^3 + 4k + 2 - 1 \\ &= 2\left[\,2(2k+1)^3 + 2k\,\right] + 1 \end{aligned}$$

$2(2k+1)^3 + 2k$ is an integer, thus $4n^3+2n-1$ is an odd number.

$\blacksquare$

Result 3.7 *Let $S = \{1, 2, 3\}$ and let $n \in S$. If $\frac{n(n+3)}{2}$ is even, then $\frac{(n+2)(n-5)}{2}$ is even.*

By proof by cases, we will address each possibility of n.

case $n=1$: $\quad \frac{n(n+3)}{2} = 2$, $\quad \frac{(n+2)(n-5)}{2} = -6$

$T \to T$ therefore implication is T.

case $n=2$ $\quad \frac{n(n+3)}{2} = 5$, hypothesis is false, therefore implication is T.

case $n=3$ $\quad \frac{n(n+3)}{2} = 9$, hypothesis is false, therefore implication is T.

The implication is true for the entire domain thus the statement is true.

$\blacksquare$

Result 3.8 *If n is an even integer, then $3n^5$ is an even integer.*

Assume n is an even integer, thus $n = 2k$ for some integer k. Then

$$3n^5 = 3(2k)^5 = 2^5 = 96k^5 = 2(48k^5)$$

Since $48k^5$ is an integer, $2(48k^5)$ is even.

$\blacksquare$

3.3 PROOF BY CONTRAPOSITIVE

theorem 3.9

↳ CONTRAPOSITIVE $\quad \boxed{P \to Q \equiv {\sim}Q \to {\sim}P}$

↳ "if 3 is odd, then 57 is prime" contrapositive is "if 57 is not prime, then 3 is even."

Sometimes it is easier to prove the contrapositive than it is to prove the original implication.

Result 3.10 *Let $x \in \mathbf{Z}$. If $5x - 7$ is even, then x is odd.*

Using proof by contrapositive, we will prove the contrapositive "If x is even, $5x-7$ is odd."

Assuming x is even, $x = 2k$ for some integer k.

Thus, $5x-7 = 10k-7 = 10k-8+1 = 2(5k-4)+1$

Since $5k-4$ is an integer, $5x-7$ is odd.

Thus, by proving the contrapositive we have proved the original statement. ■

Result 3.11 *Let $x \in \mathbf{Z}$. Then $11x - 7$ is even if and only if x is odd.*

"If and only if" represents the biconditional. We must prove $P \to Q$ and $Q \to P$

First we will do a direct proof of

if x is odd then $11x-7$ is even

Assume $x = 2k+1$ for some integer k, then $11x-7 = 22k+4 = 2(11k+2)$.

Since $11k+2$ is an integer, $11x-7$ is even

Second we will prove if $11x-7$ is even then x is odd by contrapositive "if x is even then $11x-7$ is odd"

Assume $x=2k$ for some integer k, then $11x-7 = 22k-7 = 2(11k-4)+1$

Since $11k-4$ is an integer, $11x-7$ is odd

Thus, by proving the contrapositive, we have proved the original.

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Result to Prove Let $x \in \mathbf{Z}$. If $5x - 7$ is odd, then $9x + 2$ is even.

To prove this we must figure out what x is, we can tell that if $5x-7$ is odd then x is even.

↳ by contrapositive, we will prove if x is odd then $5x-7$ is even.

$x = 2k+1$ for some integer k.

$5x-7 = 10k-2 = 2(5k-1)$

$5k-1$ is an integer, therefore $5x-7$ is even.

Thus by proving the contrapositive, we have proved if $5x-7$ is odd, then x is even.

Continuing upon that, we must prove if x is even then $9x+2$ is even.

$x=2k$, then $9x+2 = 18k+2 = 2(9k+1)$

since $9k+1$ is an integer, $9x+2$ is even.

Thus we have proved if $5x-7$ is odd, then $9x+2$ is even. ■

3.4 PROOF BY CASES

PROOF BY CASES

↳ sometimes its convienient to divide a proof into different cases and prove them individually.

↳ THERE MUST BE A CASE FOR EVERY POSSIBLE SITUATION

PARITY → integers are of same parity when both even or both odd, opposite parity when one is even & one is odd

theorem 3.16 → $\boxed{\text{x and y have same parity if and only if } x+y \text{ is even}}$

Result 3.15 *If $n \in \mathbf{Z}$, then $n^2 + 3n + 5$ is an odd integer.*

Proof by cases where n is even/odd.

case 1: n is even, $n=2k$ for some integer k

$$n^2+3n+5 = 4k^2+6k+4+1 = 2(2k^2+3k+2)+1$$

$2k^2+3k+2$ is an integer, so n^2+3n+5 is odd

case 2: n is odd, $n=2k+1$ for some integer k

$$\begin{aligned} n^2+3n+5 &= (2k+1)^2 + 3(2k+1) + 5 \\ &= 4k^2+4k+1+6k+3+5 \\ &= 4k^2+10k+8+1 \\ &= 2(2k^2+5k+4)+1 \end{aligned}$$

$2k^2+5k+4$ is an integer, so n^2+3n+5 is odd

As shown, n^2+3n+5 is odd for all integers n. ■

As shown, n^2+3n+5 is odd for all integers n. $\blacksquare$

WITHOUT LOSS OF GENERALITY (WLOG)

- → used when two cases have the same proof, only difference is swapping variable names
- → best to work out details of one case first to be sure

NOTE: SOURCE OF MANY ERRORS

Theorem 3.16 *Let $x, y \in \mathbf{Z}$. Then x and y are of the same parity if and only if $x + y$ is even.*

We will solve by addressing cases for x and y.

case 1: x and y are even, $x=2a$, $y=2b$ where $a,b \in \mathbb{Z}$

$x+y = 2a+2b = 2(a+b)$, $a+b$ is an integer
thus, $x+y$ is even

case 2: x and y are odd, $x=2a+1$, $y=2b+1$ where $a,b \in \mathbb{Z}$

$x+y = 2a+1+2b+1 = 2(a+b+1)$, $a+b+1 \in \mathbb{Z}$
thus, $x+y$ is even

case 3: x is even and y is odd, $x=2a$, $y=2b+1$ where $a,b \in \mathbb{Z}$

$x+y = 2(a+b)+1$, $a+b$ is an integer, thus $x+y$ is odd.

case 4: x is odd and y is even, WLOG, $x+y$ is odd. $\blacksquare$

$\boxed{\text{Let } a \text{ and } b \text{ be integers. Then } ab \text{ is even if and only if } a \text{ is even or } b \text{ is even.}}$ ↰ theorem 3.17