Assignment 8 (Ch. 9)

CHARLES CLAYTON
#21518139

Sunday, July 24, 2016 6:40 PM

Homework 8 (due July 28)

- 9.18, 9.26, 9.28, 9.30, 9.32, 9.36, 9.40, 9.42, 9.46, 9.54, 9.58

9.18. Let $A = \{w, x, y, z\}$ and $B = \{r, s, t\}$. Give an example of a function $f : A \to B$ that is neither one-to-one nor onto. Explain why f fails to have these properties.

Consider $f = \{(w,r),(x,r),(y,r),(z,r)\}$

This is not one-to-one because $w, x, y,$ and z map all map to r.
This is not onto because s and t are not mapped to.

9.26. Give an example of a function $f : \mathbf{N} \to \mathbf{N}$ that is
(a) one-to-one and onto (b) one-to-one but not onto
(c) onto but not one-to-one (d) neither one-to-one nor onto.

a) consider $f(x) = x$ where $x \in \mathbb{N}$

b) consider $f(x) = 2x$

c) consider $f(x) = x^3 - x$

d) consider $f(x) = x^2$

9.28. Let $A = \{2, 4, 6\}$ and $B = \{1, 3, 4, 7, 9\}$. A relation f is defined from A to B by $a\ f\ b$ if 5 divides $ab + 1$. Is f a one-to-one function?

$f = \{(a,b) : 5 \mid (ab+1)\}$ $a \in A, b \in B$

$f = \{(2,7),(4,1),(6,4),(6,9)\}$

This function is not one-to-one because for $a=6, b=4$ $5 \mid (6 \cdot 4 + 1)$
and for $a=6, b=9$ $5 \mid (6 \cdot 9 + 1)$

So $a=6$ maps to two elements in B.

9.30. Prove that the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 7x - 2$ is bijective.

We must prove

1) f is injective

Assume $f(a) = f(b)$ for some $a, b \in \mathbb{R}$. So $7a - 2 = 7b - 2$.
Thus $7a = 7b$. So finally $a = b$. Thus f is injective.

2) f is surjective

Let $a \in \mathbb{R}$. We must show there exists $x \in \mathbb{R}$ such that $f(x) = a$. Consider $x = \frac{a+2}{7}$, then for every $x \in \mathbb{R}$

$$f(x) = 7x - 2 = 7\frac{a+2}{7} - 2 = a + 2 - 2 = a$$

Thus f is surjective.

Since 1) and 2) are true, f is bijective.

Thus f is surjective.

Since 1) and 2) are true, f is bijective.

9.32. Prove that the function $f : \mathbf{R} - \{2\} \to \mathbf{R} - \{5\}$ defined by $f(x) = \frac{5x+1}{x-2}$ is bijective.

We must prove

1) f is injective

Assume $f(a) = f(b)$ for some $a, b \in \mathbb{R} - \{2\}$. Then $\frac{5a+1}{a-2} = \frac{5b+1}{b-2}$
So $(5a+1)(b-2) = (5b+1)(a-2)$. So $a = b$, thus f is injective.

2) f is surjective

Let $a \in \mathbb{R} - \{5\}$. We must show there exists $x \in \mathbb{R} - \{2\}$ such that $f(x) = a$. Consider $x = \frac{2a+1}{a-5}$, then for every $x \in \mathbb{R} - \{2\}$

$$f(x) = \frac{5x+1}{x-2} = \frac{5 \cdot \frac{2a+1}{a-5}}{\frac{2a+1}{a-5} - 5} = \frac{\frac{11a}{a-5}}{\frac{11}{a-5}} = \frac{11a}{11} = a$$

Thus f is surjective.

Since 1) and 2) are true, f is bijective.

9.36. Let $A = \{a, b, c, d, e, f\}$ and $B = \{u, v, w, x, y, z\}$. With each element $r \in A$, there is associated a list or subset $L(r) \subseteq B$. The goal is to define a "list function" $\phi : A \to B$ with the property that $\phi(r) \in L(r)$ for each $r \in A$.

(a) For $L(a) = \{w, x, y\}$, $L(b) = \{u, z\}$, $L(c) = \{u, v\}$, $L(d) = \{u, w\}$, $L(e) = \{u, x, y\}$, $L(f) = \{v, y\}$, does there exist a bijective list function $\phi : A \to B$ for these lists?

(b) For $L(a) = \{u, v, x, y\}$, $L(b) = \{v, w, y\}$, $L(c) = \{v, y\}$, $L(d) = \{u, w, x, z\}$, $L(e) = \{v, w\}$, $L(f) = \{w, y\}$, does there exist a bijective list function $\phi : A \to B$ for these lists?

****

a) $\phi(a) \in \{w, x, y\}$ $\phi(b) \in \{u, z\}$ $\phi(c) \in \{u, v\}$ $\phi(d) \in \{u, w\}$ $\phi(e) = \{u, x, y\}$ $\phi(f) = \{v, y\}$

Yes, there is a bijective function $\phi : A \to B$. Consider for example

$\phi = \{(a, x), (b, z), (c, v), (d, w), (e, u), (f, y)\}$

b) $\phi(a) \in \{u, v, x, y\}$ $\phi(b) \in \{v, w, y\}$ $\phi(c) \in \{v, y\}$ $\phi(d) \in \{u, w, x, z\}$ $\phi(e) \in \{v, w\}$ $\phi(f) \in \{w, y\}$

Yes, there is a bijective function $\phi : A \to B$. Consider for example

$\phi = \{(a, x), (b, v), (c, y), (d, z), (e, v), (f, w)\}$

9.40. Let A and B be nonempty sets. Prove that if $f : A \to B$, then $f \circ i_A = f$ and $i_B \circ f = f$.

$i_A = \{(a, a) : a \in A\}$ so $i_A(a) = a$, $a \in A$

$i_A = \{(a,a) : a \in A\}$ so $i_A(a) = a$, $a \in A$

Thus $f \circ i_A = f(i_A(a)) = f(a)$

$i_B = \{(b,b) : b \in A\}$ so $i_B(b) = b$, $b \in B$

Thus $i_B \circ f = i_B(f(a)) = f(a)$

So $f \circ i_A = f$ and $i_B \circ f = f$

9.42. Prove or disprove the following:

(a) If two functions $f : A \to B$ and $g : B \to C$ are both bijective, then $g \circ f : A \to C$ is bijective.
(b) Let $f : A \to B$ and $g : B \to C$ be two functions. If g is onto, then $g \circ f : A \to C$ is onto.
(c) Let $f : A \to B$ and $g : B \to C$ be two functions. If g is one-to-one, then $g \circ f : A \to C$ is one-to-one.
(d) There exist functions $f : A \to B$ and $g : B \to C$ such that f is not onto and $g \circ f : A \to C$ is onto.
(e) There exist functions $f : A \to B$ and $g : B \to C$ such that f is not one-to-one and $g \circ f : A \to C$ is one-to-one.

a) Assume $f : A \to B$ and $g : B \to C$ are bijective. We must prove

1) $g \circ f$ is injective

Since f and g are bijective, we know f and g are injective.
Let $(g \circ f)(a_1) = (g \circ f)(a_2)$ for $a_1, a_2 \in A$. So $g(f(a_1)) = g(f(a_2))$.
Since g is injective, then $f(a_1) = f(a_2)$. Since f is injective, then $a_1 = a_2$.

Thus $g \circ f$ is injective.

2) $g \circ f$ is surjective

Since f and g are bijective, we know f and g are surjective.
Let $c \in C$. Since g is surjective, for any c there exists $b \in B$
such that $g(b) = c$.

Additionally since f is surjective, there exists $a \in A$ such
that $f(a) = b$. So $(g \circ f)(a) = g(f(a)) = g(b) = c$.

Thus $g \circ f$ is surjective.

Since 1) and 2) are true $g \circ f$ is bijective.

b) Consider the function $g(x) = x^3 - x^2$ which is surjective from $g : \mathbb{R} \to \mathbb{R}$
and the function $f(x) = x^2$.

Then $(g \circ f)(x) = g(f(x)) = (x^2)^3 - (x^2)^2 = x^6 - x^4$

However $(g \circ f)(x)$ is not surjective. So the statement is disproved.

c) Consider the function $g(x) = x$ which is injective, and the function $f(x) = x^2$

Then $(g \circ f)(x) = g(f(x)) = x^2$, which is not injective

So $(g \circ f)(x)$ is not injective. So the statement is disproved.

d) Consider the non-surjective function $f(x) = |x|$ and the function $g(x) = \log(x)$
Then $(g \circ f)(x) = g(f(x)) = \log(|x|)$

d) Consider the non-surjective function $f(x)=|x|$ and the function $g(x)=\log(x)$
Then $(g\circ f)(x)=g(f(x))=\log(|x|)$
which is surjective. So the statement is true.

e) Assume f is not injective. Then there exist $a_1, a_2 \in A$ s.t $f(a_1)=f(a_2)$
but $a_1 \neq a_2$. Because $f(a_1)=f(a_2)$, then $g(f(a_1))=g(f(a_2))$.
So $(g\circ f)(a_1)=(g\circ f)(a_2)$, but since $a_1 \neq a_2$, $g\circ f$ is not injective

Thus the statement is disproved

9.46. Let A be the set of odd integers and B the set of even integers. A function $f : A \times B \to A \times A$ is defined by $f(a, b) = (3a - b, a + b)$ and a function $g : A \times A \to B \times A$ is defined by $g(c, d) = (c - d, 2c + d)$.

(a) Determine $(g \circ f)(3, 8)$.
(b) Determine whether the function $g \circ f : A \times B \to B \times A$ is one-to-one.
(c) Determine whether $g \circ f$ is onto.

a) $f(3,8)=(3\cdot 3-8,\ 3+8)=(1,11)$
$g(1,11)=(1-11,\ 2\cdot 1+11)=(-10,13)$
So $(g\circ f)(3,8)=(-10,13)$

b) Consider $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Let $(g\circ f)(a_1,b_1)=(g\circ f)(a_2,b_2)$
$g(3a_1-b_1,\ a_1+b_1)=(3a_1-b_1-a_1-b_1,\ 2(3a_1-b_1)+a_1+b_1)=(2a_1-2b_1,\ 7a_1-b_1)$

So $(2a_1-2b_1, 7a_1-b_1)=(2a_2-2b_2,\ 7a_2-b_2)$
Thus $2a_1-2b_1=2a_2-2b_2$ and $7a_1-b_1=7a_2-b_2$
So $a_1=a_2$ and $b_1=b_2$. So $g\circ f$ is injective.

c) It is not onto because if $a_2, a_3 \in A$.
There do not exist $a_1 \in A$ $b_1 \in B$ such that $(g\circ f)(a_1,b_1)=(a_2,a_3)$
for all a_1 and b_1

9.54. Let the functions $f : \mathbf{R} \to \mathbf{R}$ and $g : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 2x + 3$ and $g(x) = -3x + 5$.

(a) Show that f is one-to-one and onto.
(b) Show that g is one-to-one and onto.
(c) Determine the composition function $g \circ f$.
(d) Determine the inverse functions f^{-1} and g^{-1}.
(e) Determine the inverse function $(g \circ f)^{-1}$ of $g \circ f$ and the composition $f^{-1} \circ g^{-1}$.

a)

1) f is injective

Let $f(a)=f(b)$ $a,b\in\mathbb{R}$ Then $2a+3=2b+3$
$2a=2b$
$a=b$

So f is injective

2) f is surjective

Let $x, a \in \mathbb{R}$. Consider $x=\frac{a-3}{2}$. Then for every $x\in\mathbb{R}$

$f(x)=2x+3=2\,\frac{a-3}{2}+3=a-3+3=a$

$f(x) = 2x + 3 = 2\,\frac{a-3}{2} + 3 = a - 3 + 3 = a$

Thus f is surjective.

b)

1) g is injective

Let $g(a) = g(b)$ $a, b \in \mathbb{R}$ Then $-3a + 5 = -3b + 5$
$-3a = -3b$
$a = b$

So g is injective.

2) g is surjective

Let $x, a \in \mathbb{R}$. Consider $x = \frac{a-5}{-3}$. Then for every $x \in \mathbb{R}$

$$g(x) = -3x + 5 = -3\,\frac{a-5}{-3} + 5 = a - 5 + 5 = a$$

So g is surjective.

c) $(g \circ f)(x) = g(f(x)) = -3(2x+3) + 5 = -6x - 9 + 5 = -6x - 4$

d) $f^{-1} = \frac{x-3}{2}$ $\qquad g^{-1} = \frac{-x+5}{3}$

e) $(g \circ f)^{-1} = \frac{x+4}{-6} = \frac{-x-4}{6}$

$$f^{-1} \circ g^{-1} = f^{-1}(g^{-1}(x)) = \frac{\left(\frac{-x+5}{3}\right) - 3}{2} = \frac{\frac{-x+5}{3} - \frac{9}{3}}{2} = \frac{-x-4}{6}$$

9.58. Suppose, for a function $f : A \to B$, that there is a function $g : B \to A$ such that $f \circ g = i_B$. Prove that if g is surjective, then $g \circ f = i_A$.

Assume g is surjective and $f \circ g = i_B$.

Then $f(g(x)) = i_B(x)$ so $g(x) = i_B(x)$