Assignment 10 (Ch. 10, 6, 12)

Tuesday, August 9, 2016 12:12 PM

Homework 10 (due Aug 11)

- 10.34, 10.42
- 6.8, 6.10, 6.12, 6.20, 6.22, 6.24, 6.26, 6.30(part (a) only)
- 12.4, 12.8

10.34. Prove that $|\mathbf{Q} - \{q\}| = \aleph_0$ for every rational number q and $|\mathbf{R} - \{r\}| = c$ for every real number r.

$\aleph_0 = |\mathbb{N}| = |A|$ for any denumerable set A. So we must prove $\mathbb{Q} - \{q\}$ is denumerable.

$\mathbb{Q} - \{q\}$ is a subset of the denumerable set $\mathbb{Q}$ and $\mathbb{Q} - \{q\}$ is infinite, so by theorem 10.3 $\mathbb{Q} - \{q\}$ is denumerable.

Thus $\mathbb{Q} - \{q\} = |\mathbb{N}| = \aleph_0$ as required. ■

$c = |\mathbb{R}|$, so $c = |\mathbb{R} - \{r\}|$ if there exists a bijection $f: \mathbb{R} - \{q\} \to \mathbb{R}$

By theorem 10.19, since $\mathbb{R} - \{r\} \in \mathbb{R}$ if there exists an injection $g: \mathbb{R} - \{q\} \to \mathbb{R}$, then there exists bijection f.

Consider $g(x) = x + r$, then every $a \in \mathbb{R} - \{q\}$ will be mapped to an element $b \in \mathbb{R}$.

To prove g is injective, consider $g(a) = g(b)$. Then $a + r = b + r$. subtracting r from both sides yields $a = b$, so g is injective.

Thus there exists a bijection f and so $|\mathbb{R} - \{r\}| = |\mathbb{R}| = c$, as required. ■

10.42. Let S and T be two sets. Prove that if $|S - T| = |T - S|$, then $|S| = |T|$.

Assume $|S - T| = |T - S|$, then there exists a bijection $f: S - T \to T - S$.
So equivalently there exists a bijection $f: S \cap \overline{T} \to T \cap \overline{S}$

6.8. Find a formula for $1 + 4 + 7 + \cdots + (3n - 2)$ for positive integers n, and then verify your formula by mathematical induction.

FINDING FORMULA:

Let
$$\begin{aligned} S &= 1 + 4 + 7 + \ldots + (3n-2) \\ &= 1 + 4 + 7 + \cdots + (3n-11) + (3n-8) + (3n-5) + (3n-2) \\ &= [1 + (3n-2)] + [4 + (3n-5)] + [7 + (3n-8)] + \ldots \\ &= (3n-1) + (3n-1) + (3n-1) + \ldots \end{aligned}$$

this is for only $\frac{1}{2}n$ elements, because we have merged every other element, so

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$$= \frac{1}{2}n(3n-1)$$

PROVING BY INDUCTION

Take base case $n=1$ so $S_1 = 1 = \frac{1}{2}(1)(3-1) = 1$, which is true.

Assume $1+4+7+\dots+(3k-2) = \frac{1}{2}k(3k-1)$ for some $k \in \mathbb{Z}^+$

Then for by the induction hypothesis

$$1+4+7+\dots+(3(k+1)-2) = \underbrace{1+4+7+\dots+(3k-2)} + \underbrace{(3(k+1)-2)}$$

$$= \frac{1}{2}k(3k-1) + 3k+1$$

$$= \frac{1}{2}(3k^2 - k + 6k + 2)$$

$$= \frac{1}{2}(3k^2 + 5k + 2)$$

$$= \frac{1}{2}(3k+2)(k+1)$$

$$= \frac{1}{2}(k+1)(3(k+1)-1)$$

which is our original equation with $k+1$ instead of k, so by mathematical induction, the formula is true.

6.10. Let $r \neq 1$ be a real number. Use induction to prove that $a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$ for every positive integer n.

Base case: $n=1$, then $a = a\left(\frac{1-r^1}{1-r}\right) = a$, which is true

Assume $a + ar + ar^2 + \dots + ar^{k-1} = a\left(\frac{1-r^k}{1-r}\right)$ for some $k \in \mathbb{Z}^+$

Then for $k+1$ by the induction hypothesis

$$a + ar + ar^2 + \dots + ar^{(k+1-1)} = \underbrace{a + ar + ar^2 + \dots + ar^{k-1}} + ar^{k+1-1}$$

$$= a\frac{1-r^k}{1-r} + ar^k$$

$$= \frac{1}{1-r}\left[a(1-r^k) + ar^k(1-r)\right]$$

$$= \frac{1}{1-r}\left[a - \cancel{ar^k} + \cancel{ar^k} - ar^{k+1}\right]$$

$$= \frac{a}{1-r}\left[1 - r^{k+1}\right] = \frac{a(1-r^{k+1})}{1-r}$$

which is our original equation with $k+1$ instead of k, so by mathematical induction, the formula is true.

6.12. Consider the open sentence $P(n)$: $9 + 13 + \cdots + (4n + 5) = \frac{4n^2+14n+1}{2}$, where $n \in \mathbf{N}$.

(a) Verify the implication $P(k) \Rightarrow P(k + 1)$ for an arbitrary positive integer k.
(b) Is $\forall n \in \mathbf{N}$, $P(n)$ true?

(a) Assume $9+13+\ldots+(4k+5) = \frac{4k^2+14k+1}{2}$ for some $k \in \mathbb{Z}^+$

Then for $k+1$, by the induction hypothesis

$$9+13+\ldots+(4k+5) = \underbrace{9+13+\ldots+(4k+5)}_{\frac{4k^2+14k+1}{2}} + \underbrace{(4(k+1)+5)}_{4k+9}$$

$$= \frac{1}{2}(4k^2+22k+19)$$

$$= \frac{1}{2}[(4k^2+8k+4)+(14k+14)+1]$$

$$= \frac{1}{2}[4(k+1)^2+14(k+1)+1]$$

$$= \frac{4(k+1)^2+14(k+1)+1}{2}$$

which is our original equation with $k+1$ instead of k, so by mathematical induction, the formula is true.

(b) Take base case $n=1$, $S_1 = 9$

$$\frac{4(1)^2+14(1)+1}{2} = \frac{19}{2} = 9.5$$

But $9 \neq 9.5$, so the base case fails and the statement is <u>false</u>

6.20. (a) Use mathematical induction to prove that every finite nonempty set of real numbers has a largest element.
(b) Use (a) to prove that every finite nonempty set of real numbers has a smallest element.

We will prove this by induction

(a) Base case: For some $S=\{a_1\}$, then the largest element is a_1

Assume $S_1 = \{a_1, a_2, a_3 \ldots a_k\}$ for some set with k elements has some biggest element a

By the induction hypothesis $S_2 = \{a_1, a_2, a_3 \ldots a_k, a_{k+1}\}$ and has $k+1$ elements and a largest element b

Then $S_2 - \{b\}$ has k elements, and by our assumption, then has some largest element b. This can be repeated for any size finite nonempty set of real numbers.

(b) By repeating the above process until the set only has one element, that element will be the smallest element of the set.

6.22. Prove that $3^n > n^2$ for every positive integer n.

We will prove by induction

6.22. Prove that $3^n > n^2$ for every positive integer n.

We will prove by induction

Base case: $n=1$, $3^1 > 1^2$, $3 > 1$ which is true

Also consider $n=2$, $3^2 > 2^2$, $9 > 4$ which is also true

Since we have shown $n=1$ and $n=2$ is true, so we will only now consider cases where $n > 2$

Assume $3^k > k^2$ for some positive integer k

Then for $k+1$ by the induction hypothesis

$$3^{k+1} = 3(3^k) > 3k^2 = k^2 + 2k^2$$
$$= \underbrace{k^2 + 2k \cdot k > k^2 + 2k}_{\text{since we know } k>2} + 1 = (k+1)^2$$

Thus $3^{k+1} > (k+1)^2$ as required.

6.24. Prove Bernoulli's Identity: For every real number $x > -1$ and every positive integer n,

$$(1+x)^n \geq 1 + nx.$$

We will prove this by induction

Base case: $n=1$, $(1+x)^1 \geq 1 + (1)x$
$1 + x \geq 1 + x$ which is true

Assume $(1+x)^k \geq 1 + kx$ for some $k \in \mathbb{Z}^+$

By the induction hypothesis

$$(1+x)^{k+1} = (1+x)(1+x)^k \geq (1+x)(1+kx) \quad \text{By our original assumption}$$
$$= 1 + kx + x + kx^2$$
$$= \underbrace{kx^2 + 1 + (k+1)x \geq 1 + (k+1)x}_{\text{because } k>0 \text{ and } x^2>0}$$

Thus $(1+x)^{k+1} \geq 1 + (k+1)x$

which is our original equation with $k+1$ instead of k, so by mathematical induction, the formula is true.

6.26. Prove that $81 \mid (10^{n+1} - 9n - 10)$ for every nonnegative integer n.

We will prove this by induction.

Base case: $n=1$, $81 \mid (10^{1+1} - 9(1) - 10)$, $81 \mid 81$ so this is true

Assume $81 \mid (10^{k+1} - 9k - 10)$ for some $k \in \mathbb{Z}^+$

By our induction hypothesis

$$81 \mid (10^{k+1+1} - 9(k+1) - 10)$$

$$81 \mid (10 \cdot 10 \cdot 10^k - 9k - 9 - 10)$$

$$81 \mid (10 \cdot 10^{k+1} - 9k - 19)$$

$$81 \mid (10 \cdot 10 \cdot 10^k - 9k - 9 - 10)$$

$$81 \mid (10 \cdot 10^{k+1} - 9k - 19)$$

By our original assumption, we know $81 \mid (10^{k+1} - 9k - 10)$
so $81m = 10^{k+1} - 9k - 10$ for some $m \in \mathbb{Z}$

So $10^{k+1} = 81m + 9k + 10$, and substituting this we get

$$81 \mid (10(81m + 9k + 10) - 9k - 19)$$

$$81 \mid (810m + 90k + 100 - 9k - 19)$$

$$81 \mid (810m + 81k + 81)$$

$$81 \mid 81(10m + k + 1)$$

Since $10m + k + 1$ is an integer, by mathematical induction the original statement is true.

6.30. Recall for integers $n \geq 2, a, b, c, d$, that if $a \equiv b \ (mod\ n)$ and $c \equiv d \ (mod\ n)$, then both $a + c \equiv b + d \ (mod\ n)$ and $ac \equiv bd \ (mod\ n)$. Use these results and mathematical induction to prove the following: For any $2m$ integers $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_m$ for which $a_i \equiv b_i \ (mod\ n)$ for $1 \leq i \leq m$,

(a) $a_1 + a_2 + \cdots + a_m \equiv b_1 + b_2 + \cdots + b_m \ (mod\ n)$ and
(b) $a_1 a_2 \cdots a_m \equiv b_1 b_2 \cdots b_m \ (mod\ n)$.

We will prove this by induction

Base case: $m = 1$, $a_1 \equiv b_1 \ (\text{mod } n)$. Since this is a given in the problem, it is true.

Assume $a_1 + a_2 + \ldots + a_k \equiv b_1 + b_2 + \ldots + b_k \ (\text{mod } n)$ for some $k \in \mathbb{Z}^+$

By the induction hypothesis consider

$$\begin{aligned} a_1 + a_2 + \cdots + a_k \equiv \ & \underbrace{a_1 + a_2 + \cdots + a_k}_{} + a_{k+1} \\ \equiv \ & b_1 + b_2 + \cdots + b_k \ (\text{mod } n) \ + \ a_{k+1} \\ \equiv \ & b_1 + b_2 + \cdots + b_k + b_{k+1} \ (\text{mod } n) \end{aligned}$$

And so $a_1 + a_2 + \cdots + a_k + a_{k+1} \equiv b_1 + b_2 + \cdots + b_k + b_{k+1} \ (\text{mod } n)$

which is our original equation with $k+1$ instead of k, so by mathematical induction, the formula is true.

12.4. Prove that the sequence $\left\{\frac{1}{n^2+1}\right\}$ converges to 0.

For some $\varepsilon > 0$, we must show there is an $N \in \mathbb{N}$ such that if $n > N$, then

$$\left| \frac{1}{n^2+1} - L \right| = \left| \frac{1}{n^2+1} - 0 \right| = \frac{1}{n^2+1} < \varepsilon$$

Let $\varepsilon > 0$. Consider $N = \left\lceil \sqrt{\frac{1}{\varepsilon} - 1} \right\rceil$ and let $n \in \mathbb{Z}$, $n > N$

Thus $n > \left\lceil \sqrt{\frac{1}{\varepsilon} - 1} \right\rceil$ and so

$$\frac{1}{n^2+1} < \frac{1}{\left\lceil \sqrt{\frac{1}{\varepsilon} - 1} \right\rceil^2 + 1} = \frac{1}{\frac{1}{\varepsilon} - 1 + 1} = \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$$

Thus $n > \left\lceil \sqrt{\frac{1}{\varepsilon} - 1} \right\rceil$ and so

$$\frac{1}{n^2+1} < \frac{1}{\left\lceil \sqrt{\frac{1}{\varepsilon}-1} \right\rceil^2 + 1} = \frac{1}{\frac{1}{\varepsilon} - 1 + 1} = \frac{1}{1/\varepsilon} = \varepsilon$$

Hence $\frac{1}{n^2+1} < \varepsilon$ as required.

12.8. Show that the sequence $\{n^4\}$ diverges to infinity.

$\{n^4\}$ diverges to infinity if and only if for every real number $M > 0$, there is some positive integer N such that if $n > N$, then $a_n > M$

Consider $N = n - 1$. Then $n > N$, as required. And so $n = N + 1$