# Assignment 5 (Ch. 4-5)

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Homework 5 (due July 7)

- 4.28, 4.34, 4.36, 5.2, 5.4, 5.6, 5.12, 5.18, 5.28, 5.32, 5.36, 5.38

4.28. Prove that if $r$ is a real number such that $0 < r < 1$, then $\frac{1}{r(1-r)} \geq 4$.

Note that $(2r-1)^2 \geq 0$ for all $r \in \mathbb{R}$. Then by expanding we get

$$4r^2 - 4r + 1 \geq 0$$
$$4(r^2 - r) \geq -1$$
$$4 \geq \frac{-1}{r^2 - r}$$
$$4 \geq \frac{1}{r(1-r)} \quad \text{where } r \neq 0 \text{ and } r \neq 1$$

So the inequality is true for all $r \in \mathbb{R}$ <u>except</u> where $r=0$ and $r=1$. Thus the equality is also true for the interval $0 < r < 1$. $\blacksquare$

4.34. Prove for every three real numbers $x$, $y$ and $z$ that $|x - z| \leq |x - y| + |y - z|$.

There exist real numbers $a$ and $b$ such that

$a = x - y$ and $b = y - z$

And by Theorem 4.17, we know $|a+b| \leq |a| + |b|$

Thus by substituting $a$ and $b$, $|x-z| \leq |x-y| + |y-z|$, as required. $\blacksquare$

4.36. Prove for every positive real number $x$ that $1 + \frac{1}{x^4} \geq \frac{1}{x} + \frac{1}{x^3}$.

We will start by rearranging the equation. Note $x \neq 0$.

$$1 + \frac{1}{x^4} \geq \frac{1}{x} + \frac{1}{x^3} = \frac{x^2}{x^3} + \frac{1}{x^3} = \frac{x^2+1}{x^3} = \frac{x^3+x}{x^4}$$
$$1 + \frac{1}{x^4} = \frac{x^4}{x^4} + \frac{1}{x^4} = \frac{x^4+1}{x^4} \geq \frac{x^3+x}{x^4}$$

And so $x^4 + 1 \geq x^3 + x$, thus

$$x^4 - x^3 - x + 1 \geq 0$$
$$(x^2 + x + 1)(x-1)^2 \geq 0$$

Since $x > 0$, we know $x^2 + x + 1 > 0$ and that $(x-1)^2 \geq 0$.

Thus for all $x > 0$, $(x^2+x+1)(x-1)^2 \geq 0$.

and that $(x-1)^2 \geq 0$.

Thus for all $x>0$, $(x^2+x+1)(x-1)^2 \geq 0$.
So the inequality is true, as required. ∎

5.2. Disprove the statement: If $n \in \{0, 1, 2, 3, 4\}$, then $2^n + 3^n + n(n-1)(n-2)$ is prime.

When $n=4$, $2^n + 3^n + n(n-1)(n-2) = 16 + 81 + 4(3)(2) = 121$

$121 = 11 \cdot 11$, so $121$ is not prime, thus the statement is disproved. ∎

5.4. Disprove the statement: Let $n \in \mathbf{N}$. If $\frac{n(n+1)}{2}$ is odd, then $\frac{(n+1)(n+2)}{2}$ is odd.

When $n=2$, $\frac{n(n+1)}{2} = \frac{2(3)}{2} = 3$, which is odd

$\frac{(n+1)(n+2)}{2} = \frac{3 \cdot 4}{2} = 6$, which is even

$n \in \mathbb{N}$, thus the statement is disproved. ∎

5.6. Let $a, b \in \mathbf{Z}$. Disprove the statement: If $ab$ and $(a+b)^2$ are of opposite parity, then $a^2b^2$ and $a+ab+b$ are of opposite parity.

if $a=1$ and $b=1$, $ab=1$ and $(a+b)^2=4$, which are of opposite parity
however, $a^2b^2=1$ and $a+ab+b=3$, which are of the same parity
Thus the statement is disproved ∎

5.12. Prove that 200 cannot be written as the sum of an odd integer and two even integers.

Proof by contradiction: "200 can be written as the sum of and odd integer $a$, and two even integers $b$ and $c$."

Since $a$ is odd, $a = 2k+1$ where $k \in \mathbb{Z}$
Since $b, c$ are even, $b = 2n$ and $c = 2m$ where $n, m \in \mathbb{Z}$

$200 = a+b+c = 2k+1 + 2n + 2m = 2(k+n+m)+1$

Since $k+n+m$ is an integer, then $200$ is odd. However, this is a contradiction, thus the original statement is proved. ∎

5.18. Let $a$ be an irrational number and $r$ a nonzero rational number. Prove that if $s$ is a real number, then either $ar+s$ or $ar-s$ is irrational.

We will first prove the lemma L1: The product of a non-zero rational number and an irrational number is an irrational number.

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We will prove this lemma by contradiction. Let $a$ be an irrational number and $r$ be a nonzero rational number. Assume their product $ar$ is rational number $q$, $ar = q$, so

$a = r/q$. Since $r$ and $q$ are rational, then

$r = n/m$, $q = j/k$ for $n, m, j, k \in \mathbb{Z}$

so $a = \frac{nk}{mj}$

Since $nk$ and $mj$ are integers, $a$ is rational. However, this is a contradiction and so the lemma L1 is proved. ∎

Proof of 5.18

Assume $a$ is irrational and $r$ is a nonzero rational number. Then by lemma L1, the product $ar$ is irrational.

By result 5.15 in the textbook, the sum of a rational number and an irrational number is irrational. So $ar - s$ and $ar + s$ are irrational, as required. ∎

5.28. Prove that there do not exist positive integers $m$ and $n$ such that $m^2 - n^2 = 1$.

Proof by contradiction: there exist positive integers $m$ and $n$ such that $m^2 - n^2 = 1$.

Then $m^2 - n^2 = (m-n)(m+n) = 1$

Since $m-n$ and $m+n$ are integers, for $(m-n)(m+n) = 1$
$(m-n) = 1$ and $(m+n) = 1$

However, since $m$ an $n$ are positive integers, $m \geq 1$ and $n \geq 1$ so $m+n \geq 2$, this contradicts our statement that $m+n = 1$

So the original statement is proved. ∎

5.32. Prove that there exist no positive integers $m$ and $n$ for which $m^2 + m + 1 = n^2$.

Proof by contradiction: there exist positive integers $m$ and $n$ such that $m^2 + m + 1 = n^2$

Because $m \geq 1$, $m^2 < m^2 + m + 1$
so $m^2 < n^2$

Also $n^2 = m^2 + m + 1 < m^2 + 2m + 1$

so $m^2 < n^2$

Also $n^2 = m^2 + m + 1 < m^2 + 2m + 1$

so $n^2 < (m+1)^2$

Thus $m^2 < n < (m+1)^2$, so $m < n < m+1$. However, since $n$ is an integer, this cannot be so and is a contradiction. So the original statement is proved. $\blacksquare$

5.36. Let $a, b \in \mathbf{R}$. Prove that if $ab \neq 0$, then $a \neq 0$ by using as many of the three proof techniques as possible.

Direct proof

Assume $ab \neq 0$, thus $a \neq 0$ and $b \neq 0$.
So $a \neq 0$, as required.

Proof by contrapositive "if $a = 0$, then $ab = 0$"

Assume $a = 0$, then for any $b \in \mathbb{R}$ $ab = 0$.

Thus by proving the contrapositive we have proved the original statement.

Proof by contradiction "if $ab \neq 0$, then $a = 0$"

Assume $ab \neq 0$, thus $a \neq 0$ and $b \neq 0$. However, this contradicts our conclusion that $a = 0$, so the original statement is proved.

5.38. Prove the following statement using more than one method of proof.
Let $a, b \in \mathbf{Z}$. If $a$ is odd and $a + b$ is even, then $b$ is odd and $ab$ is odd.

Direct proof

Assume $a$ is odd and $a+b$ is even, then $a = 2k+1$ and $a+b = 2n$, for integers $k$ and $n$. Then

$b = 2n - a = 2n - 2k - 1 = 2(n-k-1)+1$. Since $n-k-1$ is an integer, $b$ is odd, as required

Since $b$ is odd, $b = 2m+1$ for some integer $m$.

Thus $ab = (2k+1)(2m+1) = 4km + 2k + 2m + 1 = 2(2km + k + m) + 1$

Since $2km + k + m$ is an integer, $ab$ is odd, as required.

Proof by contradiction "if $a$ is odd and $a+b$ is even, then $b$ is even or $ab$ is even."

Assume $a$ is odd and $a+b$ is even. Then $a = 2k+1$ and $a+b = 2n$ for integers $k$ and $n$.

$b = 2n - a = 2n - 2k - 1 = 2(n-k-1)+1$. Since $n-k-1$ is an integer, then $b$ is odd.

However, this contradicts our original claim that $b$ is even. Thus the original statement is proved. $\blacksquare$