Ch. 10 - Cardinalities of Sets

Sunday, August 14, 2016 11:37 AM

A set S is FINITE if $S = \emptyset$ or if $|S| = n$ for some $n \in \mathbb{N}$

↳ INFINITE if not FINITE

10.1 NUMERICALLY EQUIVALENT SETS

Sets A and B are numerically equivalent, $|A| = |B|$, if either $A = B = \emptyset$ or there exists a bijection b/w A and B

↳ $|A| = |B| \longleftrightarrow$ bijection $f: A \to B \longleftrightarrow f^{-1}$ exists

10.2 DENUMERABLE SETS

Set A is denumerable (countably infinite) if $|A| = |\mathbb{N}|$

→ note $|\mathbb{Z}| = |\mathbb{N}|$ so $\mathbb{Z}$ is denumerable
→ $\mathbb{Q}$ is denumerable
→ if A and B are denumerable, $A \times B$ is denumerable

THEOREM 10.4 Every infinite subset of a denumerable set is denumerable

10.3 UNCOUNTABLE SETS

Set A is uncountable if there does not exist any bijection $f: \mathbb{N} \to A$, $|A| = |\mathbb{R}|$

Common uncountable sets: $\mathbb{R}$, interval (a,b) for some $a, b \in \mathbb{R}$

Axioms about real numbers

↳ every $q \in \mathbb{Q}$ has repeating decimal expansion

$\frac{1}{3} = 0.333\ldots$ $\quad \frac{2}{11} = 0.1818\ldots$ $\quad \frac{1}{2} = 0.5000\ldots$

↳ some rational numbers have two repeating decimal expansions

$\frac{1}{5} = 0.2000\ldots = 1.999\ldots$ ↳ only if denominator has 2s or 5s

↳ every $i \in \mathbb{I}$ has unique non-repeating decimal expansion

THEOREM 10.10 Let A and B be sets, $A \subseteq B$. If A is uncountable, then B is uncountable.

10.4 COMPARING CARDINALITIES OF SETS

If $|A| < |B|$, then an injection $f: A \to B$ exists but not a surjection

↳ there are more elements in A so not all elements can be mapped to B by a well-defined function

THEOREM 10.16 For nonempty set A, $|P(A)| = |2^A|$

$|\mathbb{N}| = \aleph_0$ "aleph null"

$\aleph_0 < c$

$|\mathbb{R}| = c$ "the continuum"

CONTINUUM HYPOTHESIS There exists no set A such that $\aleph_0 < |A| < c$

↳ actually impossible to both prove and impossible to disprove

THEOREM 10.17 For some set A, $|A| < |P(A)|$

↳ so $|A| < |P(A)| < |P(P(A))| \ldots$

→ there are a denumerable number of different sized infinite sets

Note, there exists set $A \subset \mathbb{R}$ such that $|A| = |\mathbb{R}|$

↳ consider $|A| = |(0,1)| = |\mathbb{R}|$

10.5 SCHRÖDER-BERNSTEIN THEOREM

THEOREM 10.19 If $B \subseteq A$ and there exists an injection $f: A \to B$, then there exists a bijection $g: A \to B$

↳ to prove $|A| = |B|$ where $B \subseteq A$, only must show there's an injection from $A \to B$

SCHRÖDER-BERNSTEIN THEOREM

If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$

↳ i.e. If there are injections $f: A \to B$ and $g: B \to A$, then $|A| = |B|$

↑ bijection between A and B

THEOREM 10.21 $P(\mathbb{N})$ and $\mathbb{R}$ are numerically equivalent

↳ i.e. $2^{\mathbb{N}}$ and $\mathbb{R}$ are numerically equivalent