Assignment 9 (Ch. 10)

Saturday, July 30, 2016 9:33 PM

- 10.4, 10.6, 10.12, 10.16, 10.20, 10.24, 10.26 (just state whether the given statements are true or false, no need to prove/disprove), 10.28, 10.32

First we will prove the lemma L1 that states:
"The union of two disjoint denumerable sets is denumerable."

Let A and B be denumerable disjoint sets.
So there exist functions $f: A \to N$ and $g: B \to N$.

So let $A = \{a_1, a_2, a_3, a_4 \dots\}$ and $B = \{b_1, b_2, b_3, b_4 \dots\}$

We can define $h: A \cup B \to N$ such that

$$h = \{(a_1, 1), (b_1, 2), (a_2, 3), (b_2, 4), (a_3, 5), (b_3, 6) \dots\}$$

So $A \cup B$ is also denumerable.

10.4. Let $\mathbf{R}^+$ denote the set of positive real numbers and let A and B be denumerable subsets of $\mathbf{R}^+$. Define $C = \{x \in \mathbf{R} : -x \in B\}$. Show that $A \cup C$ is denumerable.

Since B is denumerable and $C = \{x \in R, -x \in B\}$, then $|C| = |B|$, so C is also denumerable.

Additionally since $A \subseteq R^+$ and $B \subseteq R^+$, then every element of A and every element of B is positive.

Since every element of B is positive, then every element of C is negative. Thus A and C are disjoint.

By lemma L1, the union of two denumerable disjoint elements is denumerable, so $A \cup C$ is denemerable as required. ■

10.6. (a) Prove that the function $f : \mathbf{R} - \{1\} \to \mathbf{R} - \{2\}$ defined by $f(x) = \frac{2x}{x-1}$ is bijective.
(b) Explain why $|\mathbf{R} - \{1\}| = |\mathbf{R} - \{2\}|$.

a) will will prove that

i) $f(x)$ is injective.

Assume $f(a) = f(b)$. Then $\frac{2a}{a-1} = \frac{2b}{b-1}$, so

$$2a(b-1) = 2b(a-1)$$
$$2ab - 2a = 2ab - 2b$$
$$2b = 2ab - 2ab + 2a$$
$$2b = 2a$$
$$a = b$$

Hence $f(x)$ is injective.

ii) $f(x)$ is surjective

For every b in $\{R\} - 2$, there exists some a in $R - \{1\}$ such that $f(a) = b$

Consider $a = \frac{b}{b-2}$, then

$2(\frac{b}{b-2})$ $\frac{2b}{}$ $2b$

Consider $a = \frac{b}{b-2}$, then

$$f(a) = \frac{2\left(\frac{b}{b-2}\right)}{\frac{b}{b-2} - 1} = \frac{\frac{2b}{b-2}}{\frac{2}{b-2}} = \frac{2b}{2} = b$$

Hence $f(x)$ is surjective

By proving (i) and (ii) we have proved $f(x)$ is bijective. ■

b) We know $\mathbb{R}$ is uncoutable, and $\mathbb{R}$ minus a finite number of elements is still uncountable.

So $|\mathbb{R} - \{1\}| = |\mathbb{R} - \{2\}|$

10.12. Prove that the set of all 2-element subsets of **N** is denumerable.

Let A be the set of all 2-element subsets of $\mathbb{N}$.

To prove A is denumerable, we will show there exists a bijection $f: \mathbb{N} \to A$

We will do this similar to the proof used in Result 10.6, by constructing the following infinite table.

	1	2	3	4	
1	$\{1,1\}$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\cdots$
2	$\{2,1\}$	$\{2,2\}$	$\{2,3\}$	$\{2,4\}$	$\cdots$
3	$\{3,1\}$	$\{3,2\}$	$\{3,3\}$	$\{3,4\}$	$\cdots$
4	$\{4,1\}$	$\{4,2\}$	$\{4,3\}$	$\{4,4\}$	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

Define the function $f: \mathbb{N} \to A$ as continuous sweeping successive diagonals that <u>skip repeats</u>

By this definition, f will reach every possible 2 element subset and there will be no repeats so f is bijective.

Hence, the set A is denumerable. ■

10.16. Let $A_1, A_2, A_3, \ldots$ be pairwise disjoint denumerable sets. Prove that $\cup_{i=1}^{\infty} A_i$ is denumerable.

By lemma L1, we know the union of two disjoint denumerable sets is denumerable. Since A_1 and A_2 are disjoint and denumerable, then

$A_1 \cup A_2$ is denumerable

Since A_1, A_2, and A_3 are disjoint, then $A_1 \cup A_2$ and A_3 also disjoint. So

$(A_1 \cup A_2) \cup A_3$

Since A_1, A_2, and A_3 are disjoint, then $A_1 \cup A_2$ and A_3 also disjoint. So

$(A_1 \cup A_2) \cup A_3$ is denumerable.

WLOG and repeating this process, then

$A_1 \cup A_2 \cup A_3 \cup A_4 \ldots$ is denumerable

So $\bigcup_{i=1}^{\infty} A_i$ is denumerable

10.20. Prove that the set of irrational numbers is uncountable.

Assume to the contrary that the set of irrational numbers, $\mathbb{I}$, is countable.

Case 1) Assume $\mathbb{I}$ is denumerable

We also know $\mathbb{Q}$ is denumerable from result 10.8
Additionally we know the sets $\mathbb{I}$ and $\mathbb{Q}$ are disjoint.

By lemma L1, the union of two denumerable disjoint sets is denumerable. So $\mathbb{I} \cup \mathbb{Q}$ is denumerable.

And since $\mathbb{I} \cup \mathbb{Q} = \mathbb{R}$, the set of reals is denumerable.
However, this is a contradiction to Corollary 10.11

So $\mathbb{I}$ must be uncountable.

Case 2) Assume $\mathbb{I}$ is countable and finite

Then since $\mathbb{Q}$ is denumerable, $\mathbb{I} \cup \mathbb{Q}$ is denumerable.
So $\mathbb{R}$ is denumerable, which is again a contradiction.

So in all cases, we arrive at a contradiction. Thus the set of irrationals must be uncountable. ■

10.24. Prove that **R** and **R**$^+$ are numerically equivalent.

$\mathbb{R}$ and $\mathbb{R}^+$ are numerically equivalent if there exists a bijection between them.

By theorem 10.19, if A and B are non-empty sets such that B is a subset of A and there exists an injective function from A to B, then there exists a bijective function from A to B.

Both $\mathbb{R}$ and $\mathbb{R}^+$ are non-empty and $\mathbb{R}^+$ is a subset of $\mathbb{R}$. So we will prove there exists an injective function $f: \mathbb{R} \to \mathbb{R}^+$.

Consider $f(x) = e^x$ and assume $f(a) = f(b)$. Then

$$
\begin{aligned}
e^a &= e^b \\
\ln(e^a) &= \ln(e^b) \\
\ln(e)a &= \ln(e)b \\
a &= b
\end{aligned}
$$

So f is injective and maps from $\mathbb{R}$ to $\mathbb{R}^+$

Thus by Theorem 10.19, there exists a bijection from $\mathbb{R}$ to $\mathbb{R}^+$, so $|\mathbb{R}| = |\mathbb{R}^+|$, as required.

10.26. Prove or disprove the following:

(a) If A is an uncountable set, then $|A| = |\mathbf{R}|$.
(b) There exists a bijective function $f : \mathbf{Q} \to \mathbf{R}$.
(c) If A, B and C are sets such that $A \subseteq B \subseteq C$ and A and C are denumerable, then B is denumerable.
(d) The set $S = \left\{ \frac{\sqrt{2}}{n} : n \in \mathbf{N} \right\}$ is denumerable.
(e) There exists a denumerable subset of the set of irrational numbers.
(f) Every infinite set is a subset of some denumerable set.
(g) If A and B are sets with the property that there exists an injective function $f : A \to B$, then $|A| = |B|$.

a) conside $A = \mathcal{P}(\mathbb{R})$, then A is uncountable but $|A| > |\mathbb{R}|$

b) $\mathbb{Q}$ is countable and $\mathbb{R}$ is uncountable, so $|\mathbb{Q}| \neq |\mathbb{R}|$, so FALSE

c) If $A \subseteq B$ and A is infinite, then B is infinite. If $B \subseteq C$ and C is countable, then B is countable. so TRUE

d) There exists a bijection $f(n) = \frac{\sqrt{2}}{n}$ from $\mathbb{N}$ to S, so TRUE

e) consider $S = \{\pi + n : n \in \mathbb{N}\}$, then S is denumerable and irrational TRUE

f) Uncountable sets are infinite sets and are not denumerable so FALSE

g) f has to be surjective or $B \subseteq A$, so FALSE

10.28. Prove or disprove: If A and B are two sets such that A is countable and $|A| < |B|$, then B is uncountable.

Disprove:

Conside $A = \{0, 1, 2\}$ and $B = \{0, 1, 2, 3\}$.

A is countable, and $|A| = 3$ and $|B| = 4$, so $|A| < |B|$.

However, B is still countable. So the statement is disproved.

10.32. Prove that if A, B and C are nonempty sets such that $A \subseteq B \subseteq C$ and $|A| = |C|$, then $|A| = |B|$.

Since $A \subseteq B$, $|A| \leq |B|$.

And since $B \subseteq C$, $|B| \leq |C|$. And $|A| = |C|$, so $|B| \leq |A|$.

Thus $|A| \leq |B|$ and $|B| \leq |A|$, so by the Shroder-Bernstein theorem, $|A| = |B|$, as required.