# Midterm 1 Review Worksheets

Saturday, June 4, 2016 2:37 PM

1. (a) Find $\mathcal{P}(A)$ and $|\mathcal{P}(\mathcal{P}(A))|$ for $A = \{a, b\}$

$$\boxed{\mathcal{P}(A) = \{ \{a\}, \{b\}, \{a,b\}, \phi \}}$$

$$|\mathcal{P}(A)| = 2^{|A|} = 2^2 = 4$$

$$|\mathcal{P}(\mathcal{P}(A))| = 2^{|\mathcal{P}(A)|} = 2^4 = \boxed{16}$$

(b) Find $\mathcal{P}(A)$ and $|\mathcal{P}(A)|$ for $A = \{1, \{2\}, \emptyset\}$

$$\mathcal{P}(A) = \{ \{1\}, \{\{2\}\}, \{\phi\}, \{1, \phi\}, \{1, \{2\}\}, \{\{2\}, \phi\}, \{1, \{2\}, \phi\}, \phi \}$$

$$|\mathcal{P}(A)| = 2^{|A|} = 2^3 = \boxed{8}$$

2. Give an example of a set $S$ such that

(a) $S \in \mathcal{P}(\mathbb{Z})$ and $|S| = 3$.

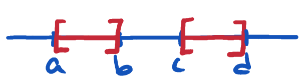
$$\mathcal{P}(\mathbb{Z}) = \{ \{1\}, \{1,2\}, \{1,2,3\} \ldots \}$$

$$\boxed{S = \{1, 2, 3\}} \quad |S| = 3$$

(b) $S \subseteq \mathcal{P}(\mathbb{Z})$ and $|S| = 3$.

$$\boxed{S = \{ \{1\}, \{2\}, \{1,2\} \}} \quad |S| = 3$$

3. Let $a, b, c, d$ be real numbers with $a < b < c < d$. Express the set $[a, b] \cup [c, d]$ as the difference of two sets.




$$\boxed{[a, d] - (b, c)}$$

4. Let

$$A = \{n \in \mathbb{N} : n = 4k \text{ for some } k \in \mathbb{N}, 1 \le k \le 10\}$$

$$B = \{n \in \mathbb{N} : n = 3k \text{ for some } k \in \mathbb{N}, 1 \le k \le 30\}$$

Compute the following:

(a) $|A|$

(b) $|B|$

(c) $|A \cup B|$

(d) $|A \cap B|$

(e) Find a formula for $|A \cup B|$ in terms of $|A|$, $|B|$, and $|A \cap B|$. Use a Venn diagram to guide you.

$$A = \{ 4000, 8000, 12000, \ldots 40000 \}$$

$$|A| = 10$$

$B = \{3000, 6000, 9000, \dots 60000\}$

$|B| = 30$ $\qquad$ $\underbrace{4, 8, \cancel{12}, 16, 20, \cancel{24}, 28, 32, \cancel{36}, 40}_{\text{3 common multiples}}$

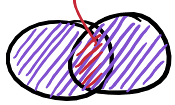
$|A \cup B| = 10 + 30 - 3 = 37$

$|A \cap B| = 3$

$|A \cup B| = |A| + |B| - |A \cap B|$

subtract b/c double counted $A \cap B$

$A \cap B$



2. For each $n \in \mathbb{N}$, define the half-open interval $S_n$ by

$$S_n = (n, n+1]$$

(a) Find $\bigcup_{n=1}^{\infty} S_n$

(b) Find $\bigcap_{n=1}^{\infty} S_n$

$(1,2] \cup (2,3] \cup (3,4] \cdots \cup (\infty, \infty] = \boxed{(1, \infty)}$

$(1,2] \cup (2,3] \cup (3,4] \cdots \cup (\infty, \infty] = \boxed{\emptyset}$

3. Let $C$ be the circle of radius 1 centered at the origin in the $xy$-plane. Express this set in terms of its points $(x, y)$ and some property $p(x, y)$.

$S = \{(x,y) : \cancel{x^2 + y^2} = 1, \ x, y \in \mathbb{R}\}$

$\boxed{S = \{(x,y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 1\}}$

4. Negate the following statements:

$P : \pi < 4, \qquad Q : \ln(e^2x^2) = 2(1 + \ln(x)), \qquad R : 9$ is an even integer

$\pi \geq 4$ $\qquad$ $\ln(e^2x^2) \neq 2(1 + \ln x)$ $\qquad$ 9 is not an even integer

if we new domain was integers or naturals, we could say odd

5. Construct the truth tables for $P \vee Q$ and $P \wedge Q$.

| P | Q | $P \vee Q$ | $P \wedge Q$ |
|---|---|---|---|
| T | T | T | T |
| T | F | T | F |

F
F

2. Let the statements $P$ and $Q$ be defined by

$$P: 4 \text{ is an even integer}, \qquad Q: 3 < \sqrt{17}$$

T

Write each of the statements below symbolically and determine its truth value

(a) 4 is an even integer and $3 < \sqrt{17}$.

(b) 4 is an even integer and $3 \geq \sqrt{17}$.

(c) 4 is an odd integer and $3 \geq \sqrt{17}$.

(d) 4 is an even integer or $3 < \sqrt{17}$.

(e) 4 is an odd integer or $3 < \sqrt{17}$.

(f) 4 is an odd integer or $3 \geq \sqrt{17}$.

a) $P \wedge Q \equiv T$

b) $P \wedge \sim Q \equiv F$

c) $\sim P \wedge \sim Q \equiv F$

d) $P \vee Q$ T

e) $\sim P \vee Q \equiv T$

f) $\sim P \vee \sim Q \equiv F$

3. Which of the following statements mean the same thing as "if 3 is prime, then 4 is even"?

if P then Q

✗ 4 is even only if 3 is prime    Q only if P

✓ For 4 to be even it is sufficient that 3 be prime    for Q, suff. $P \equiv P \to Q$

✓ For 3 to be prime it is necessary that 4 be even    for P, nec. $Q \equiv P \to Q$

✗ For 4 to be even, 3 must be prime    for Q, $P \equiv Q \to P$

✓ 4 is even when 3 is prime    Q when $P \equiv P \to Q$

✗ 3 is prime if 4 is even    P if $Q \equiv Q \to P$

4. Determine the values of $a$ and/or $b$ for which the statement is true:

(a) $3 < 2$ and $b = 6$    any a

(b) $a = 4$ or $2 < 3$    any a, any b

(c) $a = 4$ or $3 < 2$    , any b

(d) If $a = 4$, then $2 < 3$    any a, any b

(e) If $a = 4$, then $3 < 2$    $a \neq 4$, any b

(f) If $2 < 3$, then $b = 6$    $=6$, any a

(g) If $3 < 2$, then $b = 6$    any a, any b

(g) If $3 < 2$, then $b = 6$ any a, any b

5. Express each of the following statements as a conditional statement in "if-then" form. Also write the negation, converse and contrapositive for each.

$P \to Q$

(a) Every odd number is prime

$P \to \sim Q$ $Q \to P$ $\sim Q \to \sim P$

odd is suff. for prime

if odd, then prime

if odd, then not prime

if P, then Q

P is sufficient for Q

Q is necessary for P

(b) Passing the test requires solving all the problems

Q is necess. for P

if solve all problems, pass test

if solve all problems, fail test

(c) Being first in line guarantees a good seat

P suff. for Q

if first in line, then good seat

if first in line, then bad seat

(d) I get mad when you whistle

whistling sufficient for mad

P Q

if whistle, then I get mad

if whistle, then

Consider the following open sentence: "$x$ is even if and only if $2x$ is even" over the domain of natural numbers. What is the largest subset $S$ of the domain that makes this open sentence always true? Justify your answer.

if and only if $\therefore$ $P \to Q \wedge Q \to P$ $x \in \mathbb{N}$

Where $a, b \in \mathbb{Z}$

if $x = 2a$, then $2x = 2b$, $x = b$

if $2x = 2b$, then $x = 2a$

$x = b$

$S = \{2n : n \in \mathbb{N}\} = \{2, 4, 6, 8, 10 \ldots\}$

3. For statements $P$, $Q$ and $R$ prove that the following statement is a tautology:

$$[(P \Rightarrow Q) \Rightarrow R] \vee [(\sim P) \vee Q].$$

$[(P \to Q) \to R] \vee (\sim P \vee Q)$

$= [(\sim P \vee Q) \to R] \vee (\sim P \vee Q)$ by theorem 2.17, $P \to Q \equiv \sim P \vee Q$

$= [\sim(\sim P \vee Q) \vee R] \vee (\sim P \vee Q)$ by theorem 2.17

$= [(P \wedge \sim Q) \vee R] \vee (\sim P \vee Q)$ by DeMorgan's, $\sim(P \vee Q) \equiv (\sim P \wedge \sim Q)$

$= [(P \vee R) \wedge (\sim Q \vee R)] \vee (\sim P \vee Q)$ by distributive law

$= ([(P \vee R) \wedge (\sim Q \vee R)] \vee \sim P) \vee Q$ by associative law

$= [(\sim P \vee (P \vee R)) \wedge (\sim P \vee (\sim Q \vee R))] \vee Q$ by distributive law

$= [(\sim P \vee P \vee R) \wedge (\sim P \vee \sim Q \vee R)] \vee Q$ by associative law

$= (\sim P \vee P \vee R \vee Q) \wedge (\sim P \vee \sim Q \vee R \vee Q)$ by distributive law

$= (T \vee R \vee Q) \wedge (T \vee R \vee \sim P)$ by inverse law

$= T \wedge T$ by domination law

$= T$ by tautology

Always true, thus a tautology.

1. Prove the logical equivalence $(P \wedge (\sim Q)) \vee Q \equiv P \vee Q$

$(P \wedge \sim Q) \vee Q \equiv P \vee Q$

$Q \vee (P \wedge \sim Q) \equiv P \vee Q$ by commutative law

$(Q \vee P) \wedge (Q \vee \sim Q) \equiv P \vee Q$ by distributive law

$(Q \vee P) \wedge T \equiv P \vee Q$ by inverse law

$(P \vee Q) \wedge T \equiv P \vee Q$ by commutative law

$P \vee Q \equiv P \vee Q$ by identity law

2. The statement "For every integer $m$, either $m \leq 1$ or $m^2 \geq 4$" can be expressed using symbols as

$$\forall m \in \mathbb{Z}, m \leq 1 \text{ or } m^2 \geq 4$$

For each of the statements below:

(a) Express the negation of the statement in symbols

(b) Express the negation of the statement in words

1. There exists integers $a$ and $b$ such that both $ab < 0$ and $a + b > 0$.

$\sim (\exists\, a, b \in \mathbb{Z} \text{ s.t. } ab < 0 \wedge a + b > 0)$

$\forall\, a, b \in \mathbb{Z},\ ab \geq 0 \vee a + b \leq 0$

for all integers a and b, $ab \geq 0$ or $a + b \leq 0$

2. For all real numbers $x$ and $y$, $x \neq y$ implies that $x^2 + y^2 > 0$.

$\sim (\forall\, x, y \in \mathbb{R},\ x \neq y \rightarrow x^2 + y^2 > 0)$

$\exists\, x, y \in \mathbb{R} \text{ s.t. } \sim (x \neq y \rightarrow x^2 + y^2 > 0)$ $\sim (P \rightarrow Q) \equiv P \wedge \sim Q$

$\exists\, x, y \in \mathbb{R} \text{ s.t. } x \neq y \wedge x^2 + y^2 \leq 0$

there exists real numbers $x, y$ such that $x \neq y$ and $x^2 + y^2 \leq 0$

3. State and prove the negation of the following statement:

$$\exists k \in \mathbb{Z} \text{ s.t. } k \text{ is odd and } k^2 \text{ is even.}$$

$\sim (\exists k \in \mathbb{Z} \text{ s.t. } k \text{ is odd and } k^2 \text{ is even})$

$\forall k \in \mathbb{Z}$, $k$ is even or $k^2$ is odd

Proof

Assume $k$ is odd, thus $k = 2a + 1$ for some integer $a$,

then $k^2 = (2a+1)^2 = 4a^2 + 4a + 1 = 2(2a^2 + 2a) + 1$

$2a^2 + 2a$ is an integer, thus $k^2$ is odd.

Hence when $k$ is odd, $k^2$ is odd

4. Let $p(x)$ be a polynomial of degree $d$ with integer coefficients, and constant term equal to zero (i.e. $p(0) = 0$). Let $n \in \mathbb{N}$. Prove that if $n$ is even, then $p(n)$ is even.

$p(x) = 0x^0 + a_1x^1 + a_2x^2 + \ldots a_d x^d$

Assume $n$ is even, $n = 2k$ for some integer $k$

$$p(2k) = 0 + a_1(2k)^1 + a_2(2k)^2 + a_3(2k)^3 + \cdots + a_i(2k)^i$$
$$= 2a_1k + 4a_2k^2 + 8a_3k^3 + \cdots + 2^i a_i k^i$$
$$= 2(a_1k + 2a_2k^2 + 4a_3k^3 + \ldots 2^{i-1} a_i k^i)$$

Since all coefficients $a$ and $k$s are integers, $p(2k) = p(n)$ is an even number.

$\blacksquare$

1. Let $x$ and $y$ be integers. Prove that if $x$ and $y$ are not of the same parity, then $(x+y)^2$ is odd.

Proof by cases where $x$ and $y$ have opposite parity.

Case 1: Assume $x$ is odd and $y$ is even.

So $x = 2a + 1$ and $y = 2b$ for some integers $a$ and $b$. Then

$$(x+y)^2 = (2a+2b+1)^2 = (2(a+b)+1)(2(a+b)+1)$$
$$= 4(a+b)^2 + 4(a+b) + 1$$
$$= 2[2(a+b)^2 + 2a + 2b] + 1$$

Since $2(a+b)^2 + 2a + 2b$ is an integer, $(x+y)^2$ is odd.

Case 2: WLOG the same proof can be applied when $x$ is even and $y$ is odd.

$\blacksquare$

2. Prove that $3n+2$ is odd implies $n$ is odd for all $n \in \mathbb{Z}$, first directly and then by contrapositive.

If $3n+2$ is odd, then $n$ is odd for all $n \in \mathbb{Z}$

We will first prove the contrapositive "if $n$ is even, then $3n+2$ is even for all $n \in \mathbb{Z}$".

If $n$ is even, $n = 2k$ for some integer $k$. Then $3n+2 = 6k+2 = 2(3k+1)$. Since $3k+1$ is an integer, $3n+2$ is even.

Thus by proving the contrapositive, we have proved the original statement.

$\blacksquare$

Not going to bother with direct proof b/c it will be a pain.

3. Let $a \in \mathbb{Z}$. Prove that $a$ is even if and only if $a^3$ is even.

"if and only if" indicates the biconditional. Thus we must prove both 1) if $a$ is even, then $a^3$ is even and 2) if $a^3$ is even, then $a$ is even.

1) assume $a$ is even, then $a = 2k$ for some integer $k$.

Then $a^3 = (2k)^3 = 8k^3 = 2(4k^3)$. Since $4k^3$ is an integer, $a^3$ is even.

2) we will prove this by the contrapositive "if $a$ is odd, then $a^3$ is odd."

assume $a$ is odd, then $a = 2k+1$ for some integer $k$. Then

$$\begin{aligned} a^3 &= (2k+1)^3 = (2k+1)(2k+1)(2k+1) \\ &= (4k^2+4k+1)(2k+1) \\ &= 8k^3 + 8k^2 + 2k + 4k^2 + 4k + 1 \\ &= 8k^3 + 12k^2 + 6k + 1 \\ &= 2(4k^3 + 6k^2 + 3k) + 1 \end{aligned}$$

Since $4k^3 + 6k^2 + 3k$ is an integer, $a^3$ is odd.

Thus we have proved the contrapositive and also the original statement.

Having proved both implications, we have proved the biconditional. ∎

1. Prove part (4) of the "sets lemma." Let $A$ and $B$ be sets.

"If $x \notin A \cap B$, then $x \notin A$ or $x \notin B$."

~~we will prove this by the contrapositive "if~~ $\cancel{x \in A}$ ~~and~~ $\cancel{x \in B}$, ~~then~~ $\cancel{A \in A \cap B}$~~"~~

~~Assume~~ $\cancel{x \in A \cap x \in B}$

$\sim(x \in S) \equiv x \notin S$

Assume $x \notin A \cap B$. Then $\sim(x \in A \cap B)$, and so $\sim(x \in A \wedge x \in B)$. By DeMorgan's law then $\sim(x \in A) \vee \sim(x \in B)$, and so $x \notin A \vee x \notin B$ ∎

2. Let $A$ and $B$ be sets. Prove that

$$(A \cap B = A) \Leftrightarrow (A \subseteq B)$$

we must prove two implications to prove the biconditional.

case 1: $A \cap B = A \rightarrow A \subseteq B$

Assume $A \cap B = A$. Let $x \in A$, since $A = A \cap B$, $x \in A \cap B$. Therefore $x \in A \wedge x \in B$

Since $x \in A$ and $x \in B$, therefore $A \subseteq B$. Q

case 2: $A \subseteq B \rightarrow A \cap B = A$ $A \cap B = A$, so $A \cap B \subseteq A$ and $A \subseteq A \cap B$

If $A \cap B = A$, then 1) $A \cap B \subseteq A$ and 2) $A \subseteq A \cap B$.

1) Assume $A \subseteq B$. Take $x \in A$. Then also $x \in B$. so $x \in A$ and $x \in B$, thus $x \in A \cap B$. So $A \subseteq A \cap B$. Q

2) Assume $A \subseteq B$. Take $x \in A$, Then also $x \in B$. So $x \in A$

3. Let $A$ and $B$ be sets. Prove that

$$\underbrace{(A \times B) \cap (B \times A)}_{LHS} = \underbrace{(A \cap B) \times (B \cap A)}_{RHS}$$

If $RHS = LHS$, then $RHS \subseteq LHS \wedge LHS \subseteq RHS$

Let $a \in A$, $b \in B$.

Then $A \times B = (a,b)$ and $B \times A = (b,a)$.
So $(A \times B) \cap (B \times A) = \emptyset$.

Then $A \cap B = \emptyset$ and $B \cap A = \emptyset$. So $A \times B = \emptyset$.

Let $x \in LHS$ and $y \in RHS$.

if $x \in (A \times B) \cap (B \times A)$, then $x \in (A \times B) \wedge x \in (B \times A)$