Assignment 6 (Ch. 5, 7, 8)

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Homework 6 (due Thursday July 14th)

- 5.42, 5.46, 5.50, 5.52, 7.4, 7.16, 7.22, 7.26, 7.64, 7.70, 7.74, 7.78, 8.4

5.42. Show that there exist two distinct irrational numbers a and b such that a^b is rational.

Consider $a=\sqrt{2}$ $b=2\sqrt{2}$ $\quad a^b = \sqrt{2}^{2\sqrt{2}} = \sqrt{2}^{\sqrt{2}^2} = \sqrt{2}^{\sqrt{2}}\sqrt{2}^{\sqrt{2}}$

Now we will investigate $\sqrt{2}^{\sqrt{2}}$, this numbe may be rational or irrational.

Case 1: $\sqrt{2}^{\sqrt{2}}$ is rational. Then it can be represented as

$\sqrt{2}^{\sqrt{2}} = \frac{n}{m}$ where $n, m \in \mathbb{Z}$, thus $\sqrt{2}^{\sqrt{2}}\sqrt{2}^{\sqrt{2}} = \frac{n^2}{m^2}$.

Since n^2 and m^2 are integers, $\sqrt{2}^{\sqrt{2}}\sqrt{2}^{\sqrt{2}}$ is rational as required.

Case 2: $\sqrt{2}^{\sqrt{2}}$ is irrational. Then raising $\sqrt{2}^{\sqrt{2}}$ to the irrational power $\sqrt{2}$ would result in

$$\sqrt{2}^{\sqrt{2}^{\sqrt{2}}} = \sqrt{2}^{\sqrt{2}\sqrt{2}} = \sqrt{2}^2 = 2 \quad \text{which is rational.}$$

Thus where $a = \sqrt{2}^{\sqrt{2}}$ and $b=\sqrt{2}$ and both a and b are irrational, we have a^b is rational, as required.

So there exist irrational numbers a and b such that a^b is rational.

5.46. (a) Prove that there exist four distinct positive integers such that each integer divides the sum of the remaining integers.

(b) The problem in (a) should suggest another problem to you. State and solve such a problem.

a)

Consider the integers 1, 2, 3, 6 :

$$1 \mid (2+3+6) = 1 \mid 11$$
$$2 \mid (1+3+6) = 2 \mid 10$$
$$3 \mid (1+2+6) = 3 \mid 9$$
$$6 \mid (1+2+3) = 6 \mid 6$$

All of these integers are distinct, positive, and each divides the sum of the remaining three as shown. So by example the original statement is proved.

b)

Prove or disprove there are an infinite number of combinations of four such integers.

Let $a=k$, $b=2k$, $c=3k$, $d=6k$ for some positive integer k.

Then a, b, c, d are distinct and positive, and

$$a \mid (b+c+d) \rightarrow k \mid (2k+3k+6k) = k \mid 11k$$

$$b \mid (a+c+d) \rightarrow 2k \mid (k+3k+6k) = 2k \mid 10k$$

$$c \mid (a+b+d) \rightarrow 3k \mid (k+2k+6k) = 3k \mid 9k$$

$$d \mid (a+b+c) \rightarrow 6k \mid (k+2k+3k) = 6k \mid 6k$$

Thus there are an infinite number of combinations

5.50. Disprove the statement: There is a real number x such that $x^6 + x^4 + 1 = 2x^2$.

To disprove the statement, we will prove that for any real number x:

$$x^6 + x^4 - 2x^2 + 1 \neq 0$$

$$x^6 + (x^2-1)(x^2-1) \neq 0$$

$$x^6 + (x^2-1)^2 \neq 0$$

Case where $x=0$:

$$x^6 + (x^2-1)^2 = 0 + (0-1)^2 = 1 \neq 0,$$ as required

Case where $x \neq 0$:

Then for a <u>real</u> nonzero number, then $(x^2-1)^2 \geq 0$ and so

$x^6 + (x^2-1)^2 > 0$, thus $x^6 + (x^2-1)^2 \neq 0$ as required

5.52. The integers 1, 2, 3 have the property that each divides the sum of the other two. Indeed, for each positive integer a, the integers a, $2a$, $3a$ have the property that each divides the sum of the other two. Show that the following statement is false.
There exists an example of three distinct positive integers different from a, $2a$, $3a$ for some $a \in \mathbf{N}$ having the property that each divides the sum of the other two.

Assume there are distinct integers a, b, c such that $a \mid (b+c)$, $b \mid (a+c)$, and $c \mid (a+b)$. Thus, there exist integers n, m, k such that:

$\hookrightarrow$ $n = \frac{b+c}{a}$ $\quad m = \frac{a+c}{b}$ $\quad k = \frac{a+b}{c}$

$\hookrightarrow$ $an - b - c = 0$
$a - bm + c = 0$
$a + b - ck = 0$

$a + b - ck = 0$

7.4. It has been stated that the German mathematician Christian Goldbach is known for a conjecture he made concerning primes. We refer to this conjecture as Conjecture A.
Conjecture A *Every even integer at least 4 is the sum of two primes.*
Goldbach made two other conjectures concerning primes.
Conjecture B *Every integer at least 6 is the sum of three primes.*
Conjecture C *Every odd integer at least 9 is the sum of three odd primes.*
Prove that the truth of one or more of these three conjectures implies the truth of one or two of the other conjectures.

Conjecture A:

Let a and b be prime numbers. Then $a+b=k$, where k is an even integer greater than or equal to 4. So

$$a+b=k=2n \text{ for some integer } n \geq 2.$$

From Conjecture A we will derive Conjecture B.

Adding 2 to both sides gives us $a+b+2=k+2$, where $k+2$ is an integer greater than or equal to 6 which we will call m. Since 2 is a prime number, then

$$a+b+2=m, \text{ for some integer } m \geq 6$$

Which is Conjecture B.

7.16. (a) Express the following quantified statement in symbols:
For every integer n, there exists an integer m such that $(n-2)(m-2) > 0$.
(b) Express in symbols the negation of the statement in (a).
(c) Show that the statement in (a) is false.

a) $\forall n \in \mathbb{Z}, \exists m \in \mathbb{Z}$ s.t. $(n-2)(m-2) > 0$

b) $\sim(\forall n \in \mathbb{Z}, \exists m \in \mathbb{Z}$ s.t. $(n-2)(m-2) > 0)$

$\exists n \in \mathbb{Z}$ s.t. $\forall m \in \mathbb{Z}$, $(n-2)(m-2) \leq 0$

c) We will show that (a) is false by showing its negation (b) is true.

If $n=2$, then $(n-2)=0$. Which would make

$$(n-2)(m-2) \leq 0$$

true for all values of m, as required.

7.22. (a) Express the following quantified statement in symbols:

There exist two integers a and b such that for every positive integer n, $a < \frac{1}{n} < b$.

(b) Prove that the statement in (a) is true.

a) $\exists a, b \in \mathbb{Z}$ s.t. $\forall n \in \mathbb{N}$, $a < \frac{1}{n} < b$ $\quad$ ($a < \frac{1}{n}$ and $b > \frac{1}{n}$)

b) The maximum value of $\frac{1}{n}$ is where n is at its minimum value, $n=1$. So maximum value of $1/n = 1/1 = 1$

So choose $b=2$ to satisfy $b > \frac{1}{n}$

The minimum value of $\frac{1}{n}$ is a positive number because n must be positive and the quotient of two positive numbers is positive.

So choose $a=-1$ to satisfy $a < \frac{1}{n}$

Thus there exist two vales $a=-1$ and $b=2$ such that for every positive integer n $a < \frac{1}{n} < b$, as required.

7.26. Prove the following statement. For every positive real number a and positive rational number b, there exist a real number c and irrational number d such that $ac + bd = 1$.

~~We will prove by disproving the negation:~~

~~$\exists a \in \mathbb{R}^+, b \in \mathbb{Q}^+$ s.t. $\forall c \in \mathbb{R}, d \in \mathbb{I}$, $ac + bd \neq 1$~~

~~If we choose $a=1$ and $b = \frac{1-c}{d}$ then~~

~~$ac + bd = c + \frac{1-c}{d} d$~~

~~$= c + 1 - c$~~

~~$= 1$~~

$\forall a \in \mathbb{R} : a > 0$, $b \in \mathbb{Q} : b > 0$, $\exists c \in \mathbb{R}, d \in \mathbb{I}$ s.t. $ac + bd = 1$

$c = \frac{1-bd}{a}$

We will prove this by construction. For any $a \in \mathbb{R}^+$ and $b \in \mathbb{Q}^+$, if we take $d = \pi$ and $c = \frac{1-bd}{a}$

Then $d \in \mathbb{I}$ and $c \in \mathbb{R}$ as required.

Then for each case $\quad ac + bd = a\frac{1-bd}{a} + bd$

$= a\frac{1-b\pi}{a} + b\pi$

$= 1 - b\pi + b\pi$

$= 1$

7.64. There exist an irrational number a and a rational number b such that a^b is irrational.

$\exists a \in \mathbb{I}, a \in \mathbb{Q}$ s.t. $a^b \in \mathbb{I}$

Choose $a = \sqrt{2}$ and $b = 1$, then

$a^b = \sqrt{2}^1 = \sqrt{2}$ which is irrational, as required.

Thus the statement is true.

7.70. Let A, B and C be sets. Then $A \cup (B - C) = (A \cup B) - (A \cup C)$.

Choose $A = \{1\}$ $B = \{1\}$ $C = \{1\}$, then

$A \cup (B - C) = \{1\} \cup \emptyset = \{1\}$

$(A \cup B) - (A \cup C) = \{1\} - \{1\} = \emptyset$

$\{1\} \neq \emptyset$, thus the statement is false.

7.74. Let $a, b, c \in \mathbf{Z}$. Then at least one of the numbers $a + b$, $a + c$ and $b + c$ is even.

We will disprove the contrapositive :

"if all $a+b$, $a+c$, $b+c$ are odd, then $a, b, c \in \mathbb{Z}$"

Assume $a+b = 2m+1$, $m \in \mathbb{Z}$
$a+c = 2k+1$, $k \in \mathbb{Z}$
$b+c = 2n+1$, $n \in \mathbb{Z}$

Then $b = 2m+1-a$, $c = 2k+1-a$

So
$$\begin{aligned} b+c &= 2m+1-a + 2k+1-a \\ &= 2m + 2k - 2a + 2 \\ &= 2(m+k-a+1) \end{aligned}$$

However since $m+k-a+1$ is an integer, this contradicts our assumption $b+c = 2n+1$

Thus the statement is false.

By proving the contrapositive is false, so too have we shown the original statement is false.

7.78. For every odd prime p, there exist positive integers a and b such that $a^2 - b^2 = p$.

$\forall$ odd prime p, $\exists a, b \in \mathbb{Z}^+$ s.t. $a^2 - b^2 = p$

For every odd prime, p is odd. So $p = 2k+1$ $k \in \mathbb{Z}$

If we choose $a = n+1$, $b = n$ $n \in \mathbb{Z}$.

Then $a^2 - b^2 = (a-b)(a+b)$

$= (n+1-n)(n+1+n)$

$= (1)(2n+1)$

$= 2n+1$, as required.

Thus the statement is <u>true</u>.

8.4. Let $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$. Then $R_1 = \{(a, 2), (a, 3), (b, 1), (b, 3), (c, 4)\}$ is a relation from A to B, while $R_2 = \{(1, b), (1, c), (2, a), (2, b), (3, c), (4, a), (4, c)\}$ is a relation from B to A. A relation R is defined on A by $x \; R \; y$ if there exists $z \in B$ such that $x \; R_1 \; z$ and $z \; R_2 \; y$. Express R by listing its elements.

$R_1 : A \to B$, $R_2 : B \to A$, $R : A \to A$

$(x,y) \in R$	$(x,z) \in R_1$	$(z,y) \in R_2$	$\in B$
a a	a z	z a	2
a b	a z	z b	2
a c	a z	z c	3
b a	b z	z a	
b b	b z	z b	1
b c	b z	z c	1
c a	c z	z a	4
c b	c z	z b	
c c	c z	z c	4

$A \times A = \{ (a,a)\,(a,b)\,(a,c)\,\cancel{(b,a)}\,(b,b)\,(b,c)\,(c,a)\,\cancel{(c,b)}\,(c,c) \}$

$R = \{ (a,a)\;(a,b)\;(a,c)\;(b,b)\;(b,c)\;(c,a)\;(c,c) \}$