Ch. 5 - Existence and Contradiction

Saturday, July 16, 2016 10:43 AM

5.1 COUNTEREXAMPLES

For all statements ($\forall x \in S, P(x)$) can be disproved using a counterexample.

↳ recall the negation is where $\exists x \in S, \sim P(x)$

Ex) Disprove the statement: if $x \in \mathbb{R}$, then $(x^2-1)^2 > 0$

Consider $x=1$, $(x^2-1)^2 = (1-1)^2 = 0$. Thus $(x^2-1)^2 > 0$ is false.

Ex) Disprove the statement: for every positive integer n, $3 \mid (n^2-1)$

$$k = \frac{n^2-1}{3} \quad k,n \in \mathbb{Z} \quad n > 0$$

consider $n=3$: $k = \frac{3^2-1}{3} = \frac{8}{3}$

$\frac{8}{3} \notin \mathbb{Z}$, so the statement is false.

5.2 PROOF BY CONTRADICTION

Given statement to prove such as $\forall x \in S, P(x) \rightarrow Q(x)$

Proceed like so:

Assume to the contrary that there exists some $x \in S$ such that $P(x)$ and $\sim Q(x)$.

Then arrive at a contradiction to a known fact or an assumption made along the way.

Ex) Prove no odd integer can be expressed as the sum of three even integers.

Assume to the contrary that an odd integer, a, can be expressed as the sum of three even integers, b, c, d.

$$a = b+c+d = 2n+2m+2k \quad n,m,k \in \mathbb{Z}$$
$$= 2(n+m+k)$$

Since $n+m+k \in \mathbb{Z}$, then a is even. However, this is a contradiction to our assumption.

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Ex) Prove $\boxed{\text{the sum of a rational number and an irrational number is irrational.}}$

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Assume to the contrary that the sum of a rational number, a, and an irrational number, b is a rational number, c

$a+b=c$ where $a=\frac{n}{m}$, $c=\frac{p}{q}$ $n,m,p,q \in \mathbb{Z}$

$$b=c-a=\frac{p}{q}-\frac{n}{m}=\frac{pm-qn}{qm}$$

However, since $pm-qn$ and $qm \in \mathbb{Z}$, then b is rational. But this is a contradiction.

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5.3 REVIEW OF PROOF TECHNIQUES

$\forall x \in S,\ P(x) \rightarrow Q(x)$

DIRECT	Assume $P(x)$ for some $x \in S$, show $Q(x)$
CONTRAPOSITIVE	Assume $\sim Q(x)$ for some $x \in S$, show $\sim P(x)$
CONTRADICTION	Assume $P(x)$ and $\sim Q(x)$ for some $x \in S$, show contradiction

5.4 EXISTENCE PROOFS

$\exists x \in S$ s.t. $R(x)$

↳ prove a statement by proving existence

Ex) Prove there exist irrational numbers a and b such that a^b is rational

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Consider $a=\sqrt{2}$ $b=\sqrt{2}$, $a^b=\sqrt{2}^{\sqrt{2}}$

Case i) $\sqrt{2}^{\sqrt{2}}$ is rational

Then we have an a and b as required.

case ii) $\sqrt{2}^{\sqrt{2}}$ is irrational

Then $\sqrt{2}^{\sqrt{2}^{\sqrt{2}}} = \sqrt{2}^{\sqrt{2}\sqrt{2}} = \sqrt{2}^2 = 2$

So we have an irrational number to an irrational power such that the result is rational, as required.

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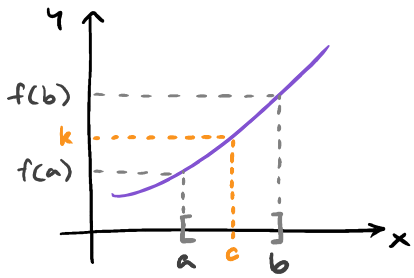
Intermediate value theorem (IVT)

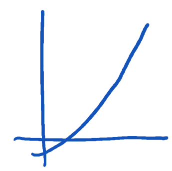
↳ $\boxed{\text{on a function } f \text{ that is continuous from } [a,b] \text{ and } f(a)<k<f(b)\text{, then there exists } c\in(a,b) \text{ such that } f(c)=k.}$



Ex) Prove $x^5+2x-5=0$ has a real solution $x=a$ between $x=1$ and $x=2$

Let $f(x)=x^5+2x-5$

$f(1)=1+2-5=-2$
$f(2)=2^5+4-5=32-1=31$



Since 0 is between $f(1)$ and $f(2)$ and $f(x)$ is continuous, by the IVT we know there is a solution between 1 and 2.

Unique → assume a & b are solutions, prove $a=b$

5.5 DISPROVING EXISTENCE STATEMENTS

→ cannot disprove existence statements with counterexample

$\sim(\exists x\in S \text{ s.t } P(x)) \equiv \forall x\in S, \sim P(x)$

↑ existence statement is disproved if $P(x)$ is false for all $x\in S$

Ex) Disprove: there exists an odd integer n s.t. n^2+2n+3 is odd

Assume n is odd, then $n=2k+1$ $k\in\mathbb{Z}$

$$n^2+2n+3 = 4k^2+4k+1+4k+2+3 = 4k^2+8k+6$$
$$= 2(2k^2+4k+3)$$

Since $2k^2+4k+3\in\mathbb{Z}$, n^2+2n+3 is even for all odd n.