Assignment 3 (Ch. 4)

Sunday, June 12, 2016 12:14 PM

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MATH 220

4.46. Let A and B be sets. Prove that $A \cup B = A \cap B$ if and only if $A = B$.

IT IS A BICONDITIONAL, SO WE MUST PROVE BOTH 1) IF $A \cap B = A \cup B$ THEN, $A = B$ and 2) IF $A = B$ THEN, $A \cup B = A \cap B$.

PROOF OF 1) BY CONTRAPOSITIVE "IF $A \neq B$, then $A \cap B \neq A \cup B$"

Assume $A \neq B$. So then either i) $x \in A$, $x \notin B$ or ii) $x \in B$, $x \notin A$ for some x.

case i) If $x \in A$, then $x \in A \cup B$. But if $x \notin B$, then $x \notin A \cap B$. Therefore $A \cup B \neq A \cap B$.

case ii) WLOG, SAME AS PROOF OF CASE i)

PROOF OF 2)

Assume $A = B$. So then $A \cup B = A = B$ and $A \cap B = A = B$. Thus $A \cup B = A \cup B$.

$\blacksquare$

4.54. Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$ for every two sets A and B (Theorem 4.22(4b)).

We must show 1) $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and 2) $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$.

1) Let $x \in \overline{A \cap B}$, so $x \notin A \cap B$. Therefore $x \notin A$ or $x \notin B$. So $x \in \overline{A}$ or $x \in \overline{B}$. Hence $x \in \overline{A} \cup \overline{B}$.

2) Let $x \in \overline{A} \cup \overline{B}$, so $x \in \overline{A}$ or $x \in \overline{B}$. Therefore $x \notin A$ or $x \notin B$. So $x \notin A \cap B$. Hence $x \in \overline{A \cap B}$. $\blacksquare$

4.56. Let A, B and C be sets. Prove that $(A - B) \cup (A - C) = A - (B \cap C)$.

$$
\begin{aligned}
(A-B) \cup (A-C) &= (A \cap \overline{B}) \cup (A \cap \overline{C}) && \text{By result 4.19} \\
&= A \cap (\overline{B} \cup \overline{C}) && \text{By distributive law} \\
&= A - (\overline{\overline{B} \cup \overline{C}}) && \text{By result 4.19} \\
&= A - (\overline{\overline{B}} \cap \overline{\overline{C}}) && \text{By DeMorgan's} \\
&= A - (B \cap C) && \text{By double negation}
\end{aligned}
$$

4.60. For $A = \{x, y\}$, determine $A \times \mathcal{P}(A)$.

$$
\begin{aligned}
&\{x, y\} \times \mathcal{P}(A) \\
&= \{x, y\} \times \{\{x, y\}, \{x\}, \{y\}, \emptyset\} \\
&= \{(x, \{x, y\}), (x, \{x\}), (x, \{y\}), (x, \emptyset), \\
&\quad (y, \{x, y\}), (y, \{x\}), (y, \{y\}), (y, \emptyset)\}
\end{aligned}
$$

4.62. Let A and B be sets. Prove that $A \times B = \emptyset$ if and only if $A = \emptyset$ or $B = \emptyset$.

1) If $A = \emptyset$ or $B = \emptyset$ then $A \times B = \emptyset$

$A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$

Since A or B has no elements, $A \times B = \phi$

2) if $A \times B = \phi$ then $A = \phi$ or $B = \phi$

Proof by contrapositive: "if $A \neq \phi$ and $B \neq \phi$ then $A \times B \neq \phi$"

Assume $A \neq \phi$ and $B \neq \phi$ then there exists $a \in A$ and $b \in B$.
So $A \times B = \{(a,b)\} \neq \phi$.

Thus by proving the contrapositive, we have proved 2). $\blacksquare$

4.66. Result 4.23 states that if A, B, C and D are sets such that $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$.

(a) Show that the converse of Result 4.23 is false.

(b) Under what added hypothesis is the converse true? Prove your assertion.

a) converse: "if $A \times B \subseteq C \times D$, then $A \subseteq C$ and $B \subseteq D$."

If $A = \phi$, then $A \times B = \phi$ so $A \times B \subseteq C \times D$.
However, its not necessarily true that $B \subseteq D$.

b) "if $A \times B \subseteq C \times D$ and A and B are not empty sets, then $A \subseteq C$ and $B \subseteq D$"

4.70. Let A and B be sets. Show, in general, that $\overline{A \times B} \neq \overline{A} \times \overline{B}$.

Consider some universal set, U. Then

$\overline{A \times B} = U - (A \times B)$, $\overline{A} = U - A$, $\overline{B} = U - B$

Let $U = \{a,b\}$ $A = \{a\}$ $B = \{b\}$ $\overline{A} = \{b\}$ $\overline{B} = \{a\}$

$U - (A \times B) = \{(a,a),(a,b),(b,a),(b,b)\} - \{(a,b)\} = \{(a,a),(b,a),(b,b)\}$

$\overline{A} \times \overline{B} = \{(b,a)\}$ So, in general $\overline{A \times B} \neq \overline{A} \times \overline{B}$ $\blacksquare$

4.4. Let $x, y \in \mathbf{Z}$. Prove that if $3 \nmid x$ and $3 \nmid y$, then $3 \mid (x^2 - y^2)$.

Assume $3 \nmid x$ and $3 \nmid y$, then ~~$x=3n$~~, $x = 3n+1$ or $x = 3n+2$ and ~~$y=3n$~~ $y = 3m+1$ or $y = 3m+2$ for some integer n and m.

case 1: $x = 3n+1$ and $y = 3m+1$. So

$$\begin{aligned} x^2 - y^2 &= (3n+1)^2 - (3m+1)^2 \\ &= 9n^2 + 6n + 1 - 9m^2 - 6m - 1 \\ &= 3(3n^2 + 2n - 3m^2 - 2m) \end{aligned}$$

$3n^2 + 2n - 3m^2 - 2m$ is an integer, so $3 \mid (x^2 - y^2)$

case 2: $x = 3n+2$ and $y = 3m+2$. So

$$\begin{aligned} x^2 - y^2 &= (3n+2)^2 - (3m+2)^2, \text{ so} \\ &= 9n^2 + 12n + 4 - 9m^2 - 12m - 4 \\ &= 3(3n^2 + 6n - 3m^2 - 6m) \end{aligned}$$

$3n^2 + 6n - 3m^2 - 6m$ is an integer, so $3 \mid (x^2 - y^2)$

case 3: $x = 3n+2$ and $y = 3m+1$. So

$$\begin{aligned} x^2 - y^2 &= (3n+2)^2 - (3m+1)^2 \\ &= 9n^2 + 12n + 4 - 9m^2 - 6m - 1 \\ &= 3(3n^2 + 4n - 3m^2 - 2m + 1) \end{aligned}$$

$3n^2 + 6n - 3m^2 - 2m + 1$ is an integer, so $3 \mid (x^2 - y^2)$.

case 4: $x = 3n+1$ and $y = 3m+2$. So

$$\begin{aligned} x^2 - y^2 &= (3n+1)^2 - (3m+2)^2 \\ &= 9n^2 + 6n + 1 - 9m^2 - 12m - 4 \\ &= 3(3n^2 + 2n - 3m^2 - 4m - 1) \end{aligned}$$

$3n^2 + 2n - 3m^2 - 4m - 1$ is an integer, so $3 \mid (x^2 - y^2)$ $\blacksquare$

4.10. Let $n \in \mathbf{Z}$. Prove that $2 \mid (n^4 - 3)$ if and only if $4 \mid (n^2 + 3)$.

To prove the biconditional, we must prove 1) if $2 \mid (n^4 - 3)$ then $4 \mid (n^2 + 3)$ and 2) if $4 \mid (n^2 + 3)$ then $2 \mid (n^4 - 3)$.

Proof of 1)

Assume $2 \mid (n^4 - 3)$. Since $n^4 - 3$ is divisible by 2, $n^4 - 3$ is even. So $n^4 - 3 = 2k$ for some integer k.

$n^4 - 3 = 2k$, $n^4 = 2k + 3 = 2(k+1) + 1$

since $k+1$ is an integer, then n^4 is odd. By theorem 3.12 n^2 is odd, and applying the same logic again show n is odd.

So $n = 2m+1$ for some integer m.

Thus
$$\begin{aligned} n^4 + 3 &= (2m+1)^4 + 3 \\ &= 16m^4 + 32m^3 + 24m^2 + 8m + 4 \\ &= 4(4m^4 + 8m^3 + 6m^2 + 2m + 1) \end{aligned}$$

Since $4m^4 + 8m^3 + 6m^2 + 2m + 1$ is an integer, 4 divides $n^4 + 3$. So $4 \mid (n^4 + 3)$

Proof of 2)

Assume $4 \mid (n^2 + 3)$, then $n^2 + 3 = 4k$ for some integer k.

Then $n^2 = 4k - 3$. $n^4 - 3 = (4k-3)^2 - 3 = 16k^2 - 24k + 9 - 3$
$= 2(8k^2 - 12k + 3)$

Since $8k^2 - 12k + 3$ is an integer, so $2 \mid (n^4 - 3)$

$\blacksquare$

4.16. Let $a, b \in \mathbf{Z}$. Prove that if $a^2 + 2b^2 \equiv 0 \pmod 3$, then either a and b are both congruent to 0 modulo 3 or neither is congruent to 0 modulo 3.

We will prove this using the contrapositive "if only one of a or b is congruent to 0 modulo 3 then $a^2 + 2b^2 \not\equiv 0 \bmod 3$"

case 1) $a \equiv 0 \bmod 3$, $b \not\equiv 0 \bmod 3$

So $a=3k$ for some $k\in\mathbb{Z}$ and there are three possibilities for b where $n\in\mathbb{Z}$

i) $b=3n+1$

Then $a^2+2b^2 = 9k^2 + 18n^2+12n+2$
$= 3(3k^2+6n^2+4n)+2$

Since $3k^2+6n^2+4n \in\mathbb{Z}$, $3\nmid(a^2+2b^2)$

ii) $b=3n+2$

Then $a^2+2b^2=9k^2+18n^2+24n+8$
$=3(3k^2+6n^2+8n+2)+2$

Since $3k^2+6n^2+8n+2 \in\mathbb{Z}$, $3\nmid(a^2+2b^2)$

case 2) $b\equiv 0 \bmod 3$, $a\not\equiv 0 \bmod 3$

So $b=3k$ for some $k\in\mathbb{Z}$ and there are two possibilities for a where $n\in\mathbb{Z}$

i) $a=3n+1$

Then $a^2+2b^2= 18k^2+9n^2+6n+1$
$= 3(6k^2+3n^2+2n)+1$

Since $6k^2+3n^2+2n \in\mathbb{Z}$, $3\nmid(a^2+2b^2)$

ii) $a=3n+2$

Then $a^2+2b^2= 18k^2+9n^2+12n+4$
$=3(6k^2+3n^2+4n+1)+1$

Since $6k^2+3n^2+4n+1 \in\mathbb{Z}$, $3\nmid(a^2+2b^2)$

Thus by proving the contrapositive, we have proved the original case. ■

4.24. Let x and y be even integers. Prove that $x^2 \equiv y^2 \pmod{16}$ if and only if either (1) $x \equiv 0 \pmod 4$ and $y \equiv 0 \pmod 4$ or (2) $x \equiv 2 \pmod 4$ and $y \equiv 2 \pmod 4$.

1) if ($x\equiv 0 \bmod 4$ and $y\equiv 0 \bmod 4$) or ($x\equiv 2 \bmod 4$ and $y\equiv 2 \bmod 4$) then, $x^2\equiv y^2 \bmod 16$

case 1) $x\equiv 0 \bmod 4$ and $y\equiv 0 \bmod 4$

$x=4n$, $y=4m$ for some $n,m\in\mathbb{Z}$

$x^2-y^2=(4n)^2-(4m)^2 = 16n^2-16m^2 = 16(n^2-m^2)$

Since n^2-m^2 is an integer $16\mid x^2-y^2$ so $x^2\equiv y^2 \bmod 16$

case 2) $x\equiv 2 \bmod 4$ and $y\equiv 2 \bmod 4$

$x = 4n+2$, $y = 4m+2$ for some $n, m \in \mathbb{Z}$

$x^2 - y^2 = 16n^2 + 16n \cancel{+4} - 16m^2 - 16m \cancel{-4}$
$= 16(n^2 + n - m^2 - m)$

$n^2 + n - m^2 - m$ is an integer so $x^2 \equiv y^2 \bmod 16$

2) if $x^2 \equiv y^2 \bmod 16$ then $(x \equiv 0 \bmod 4$ and $y \equiv 0 \bmod 4)$
or $(x \equiv 2 \bmod 4$ and $y \equiv 2 \bmod 4)$

Assume $x^2 \equiv y^2 \bmod 16$, then $x^2 - y^2 = 16k$ for some $k \in \mathbb{Z}$

so $x^2 = 16k + y^2$

Since y is even, $y = 2m$ for some $m \in \mathbb{Z}$, so $y^2 = 4m^2$

So $x^2 = 16k + 4m^2 = 4(4k + m^2)$

Since $4k + m^2$ and x^2 are integers, $4 \mid (4x + m^2)$

Also, $y^2 = x^2 - 16k$ and x is even, so $x = 2m$ for $m \in \mathbb{Z}$

So $y^2 = x^2 - 16k = 4m^2 - 16k = 4(m^2 - 4k)$

Since $m^2 - 4k$ and y^2 are integers, $4 \mid (m^2 - 4k)$