Regla de L'Hôpital

52.- Calcula los siguientes límites, aplicando la regla de L'Hôpital sólo cuando sea conveniente.

i) $\lim_{x\to 3} \dfrac{x^4-20x-21}{x^2+11x-42} = \dfrac{81-60-21}{9+33-42} = \dfrac{0}{0}$ IND.

Factorizando

$= \lim_{x\to 3} \dfrac{\cancel{(x-3)}(x+1)(x^2+3x+7)}{\cancel{(x-3)}(x+14)} = \dfrac{4(9+9+7)}{17} = \dfrac{100}{17}$

L.H.

$\lim_{x\to 3} \dfrac{4x^3-20}{2x+11} = \dfrac{108-20}{17} = \dfrac{88}{17}$

$\frac{\infty}{\infty}$ IND

ii) $$\lim_{x\to\infty}\frac{x^3-7x+3}{3-x+5x^3} = \lim_{x\to\infty}\frac{1-\frac{7}{x^2}+\frac{3}{x^3}}{\frac{3}{x^3}-\frac{1}{x^2}+5} = \frac{1}{5}$$

L.H.

$$\lim_{x\to\infty}\frac{3x^2-7}{15x^2-1} = \lim_{x\to\infty}\frac{3-\frac{7}{x^2}}{15-\frac{1}{x^2}} = \frac{3}{15} = \frac{1}{5}$$

$\frac{\infty}{\infty}$

L'H

$$\lim_{x\to\infty}\frac{6x}{30x} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x\to\infty}\frac{6}{30} = \frac{1}{5}$$

iii) $\lim\limits_{x\to 0}\dfrac{\operatorname{sen} x}{x} = \dfrac{0}{0}$ IND.

$$\lim_{x\to 0}\frac{\operatorname{sen} x}{x} \underset{L.H}{=} \lim_{x\to 0}\frac{\cos x}{1} = \cos 0 = 1$$

$\frac{0}{0}$ IND

iv) $\lim\limits_{x\to 0}\dfrac{e^x - 1}{x} \underset{L.H}{=} \lim\limits_{x\to 0}\dfrac{e^x}{1} = 1$

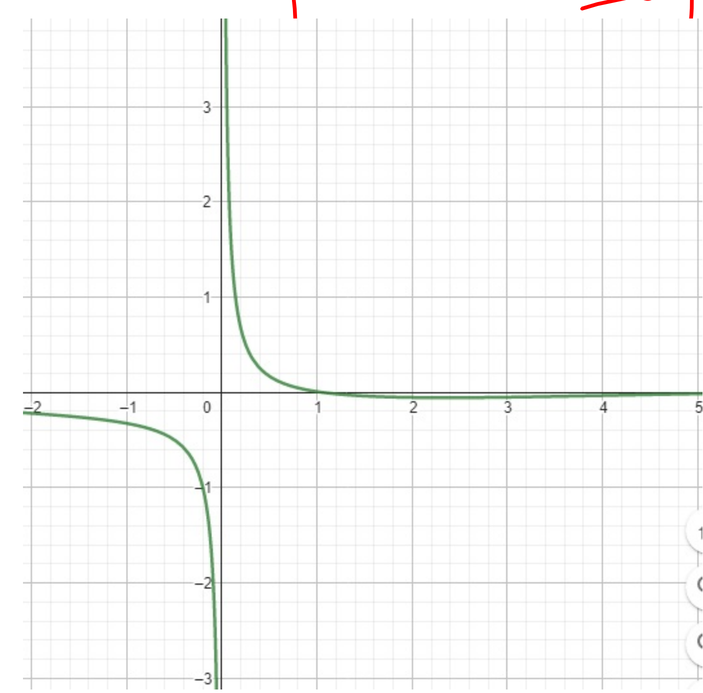
v) $\lim\limits_{x\to 0}\frac{1-\cos x}{x^2} \underset{L'H}{\overset{\frac{0}{0}}{=}} \lim\limits_{x\to 0}\frac{\operatorname{sen} x}{2x} \underset{L'H}{\overset{\frac{0}{0}}{=}} \lim\limits_{x\to 0}\frac{\cos x}{2} = \frac{1}{2}$

vi) $\lim\limits_{x\to 0}\frac{\operatorname{sen} x - x + \frac{x^2}{6}}{x^3} \underset{L'H}{\overset{\frac{0}{0}}{=}} \lim\limits_{x\to 0}\frac{\cos x - 1 + \frac{x}{3}}{3x^2} \underset{L'H}{\overset{\frac{0}{0}}{=}} \lim\limits_{x\to 0}\frac{-\operatorname{sen} x + 1/3}{6x} = \frac{1/3}{0}$

$\frac{K}{0}$

(Límites laterales)

$$\left.\begin{array}{l} \lim\limits_{x\to 0^+}\frac{-\operatorname{sen} x + 1/3}{6x} = \frac{1/3}{0^+} = +\infty \\ \lim\limits_{x\to 0^-}\frac{-\operatorname{sen} x + 1/3}{6x} = \frac{1/3}{0^-} = -\infty \end{array}\right\} \Rightarrow \nexists \lim$$



vii) $\displaystyle \lim_{x\to 0} \frac{\operatorname{arc\,tg} x - x + \frac{x^3}{3}}{x^3} \overset{\frac{0}{0}}{\underset{L'H}{=}} \lim_{x\to 0} \frac{\frac{1}{1+x^2} - 1 + \frac{\cancel{3}x^2}{\cancel{3}}}{3x^2} \overset{\frac{0}{0}}{\underset{L'H}{=}}$

$\displaystyle \underset{L'H}{=} \lim_{x\to 0} \frac{\frac{-2x}{(1+x^2)^2} + 2x}{6x} \overset{\frac{0}{0}}{\underset{L'H}{=}} \lim_{x\to 0} \frac{\frac{-2(1+x^2)^2 + 2x\, 2(1+x^2)\cdot 2x}{(1+x^2)^4} + 2}{6} =$

$\displaystyle = \frac{-2+2}{6} = \frac{0}{6} = 0$

viii) $\lim_{x \to 0} \frac{x - \text{sen}\, x}{x.\text{sen}\, x} \underset{L'H}{\overset{\frac{0}{0}}{=}} \lim_{x \to 0} \frac{1 - \cos x}{\text{sen}\, x + x \cdot \cos x} \underset{L'H}{\overset{\frac{0}{0}}{=}} \lim_{x \to 0} \frac{\text{sen}\, x}{\cos x + \cos x - x\, \text{sen}\, x} =$

$$= \frac{0}{1 + 1 - 0} = \frac{0}{2} = 0$$

ix) $\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{\text{sen}\, x} \right) \underset{\infty - \infty}{=} \lim_{x \to 0} \frac{\text{sen}\, x - x}{x \cdot \text{sen}\, x} \underset{L'H}{\overset{\frac{0}{0}}{=}} \lim_{x \to 0} \frac{\cos x - 1}{\text{sen}\, x + x \cos x} \overset{\frac{0}{0}}{=}$

$$\underset{L'H}{=} \lim_{x \to 0} \frac{-\text{sen}\, x}{\cos x + \cos x - x\, \text{sen}\, x} = \frac{0}{1 + 1 - 0} = \frac{0}{2} = 0$$

x) $\lim_{x\to 0} \frac{1}{x} \cdot \log\left(\frac{\operatorname{sen} x}{x}\right)^{\infty \cdot 0} = \lim_{x\to 0} \frac{\log\left(\frac{\operatorname{sen} x}{x}\right)}{x} \overset{\frac{0}{0}}{\underset{L'H}{=}} \lim_{x\to 0} \frac{\frac{x\cos x - \operatorname{sen} x}{\cancel{x^2}}}{\frac{\operatorname{sen} x}{\cancel{x}}} =$

$= \lim_{x\to 0} \frac{x\cos x - \operatorname{sen} x}{x \operatorname{sen} x} \overset{\frac{0}{0}}{\underset{L'H}{=}} \lim_{x\to 0} \frac{\cancel{\cos x} - x\operatorname{sen} x - \cancel{\cos x}}{\operatorname{sen} x + x\cos x} \overset{\frac{0}{0}}{=}$

$\underset{L'H}{=} \lim_{x\to 0} \frac{-\operatorname{sen} x - x\cos x}{\cos x + \cos x - x\operatorname{sen} x} = \frac{0}{1+1-0} = \frac{0}{2} = 0$

xi) $\lim\limits_{x\to\infty}\left(1-e^{-x}\right)^{x^2} \overset{1^\infty}{=} e^{\lim\limits_{x\to\infty} x^2\left(\cancel{1}-e^{-x}-\cancel{1}\right)} = e^{0} = 1$

$$\lim_{x\to\infty} x^2\left(-e^{-x}\right) = \lim_{x\to\infty} \frac{-x^2}{e^x} \underset{L'H}{\overset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{-2x}{e^x} \overset{\frac{\infty}{\infty}}{=}$$

$$\underset{L'H}{=} \lim_{x\to\infty} \frac{-2}{e^x} = \frac{-2}{\infty} = 0$$

xii) $\lim_{x\to 0}(\cos 2x)^{\frac{3}{x^2}} \overset{1^\infty}{=} e^{\lim_{x\to 0}\frac{3}{x^2}(\cos 2x - 1)} = e^{-6}$

$$\lim_{x\to 0}\frac{3}{x^2}(\cos 2x - 1) = \lim_{x\to 0}\frac{3(\cos 2x - 1)}{x^2} \underset{L'H}{\overset{\frac{0}{0}}{=}}$$

$$= \lim_{x\to 0}\frac{3(-2\cdot \operatorname{sen} 2x)}{2x} \underset{L'H}{\overset{\frac{0}{0}}{=}} \lim_{x\to 0}\frac{-6\cdot\cancel{2}\cdot\cos 2x}{\cancel{2}} = -6$$

xiii) $\lim_{x\to 1}\left(\operatorname{tg}\frac{\pi x}{4}\right)^{\operatorname{tg}\frac{\pi x}{2}} \overset{1^\infty}{=} e^{\lim_{x\to 1}\operatorname{tg}\frac{\pi x}{2}\left(\operatorname{tg}\frac{\pi x}{4}-1\right)} \overset{(*)}{=} e^{-1}$

$(*)\ \lim_{x\to 1}\operatorname{tg}\frac{\pi x}{2}\left(\operatorname{tg}\frac{\pi x}{4}-1\right)^{\infty\cdot 0} = \lim_{x\to 1}\dfrac{\operatorname{tg}\frac{\pi x}{4}-1}{\dfrac{1}{\operatorname{tg}\frac{\pi x}{2}}} \underset{L'H}{\overset{\frac{0}{0}}{=}} \lim_{x\to 1}\dfrac{\left(1+\operatorname{tg}^2\frac{\pi x}{4}\right)\cdot\frac{\cancel{\pi}}{\cancel{4}\,2}}{-\dfrac{1}{\operatorname{tg}^2\frac{\pi x}{2}}\left(1+\operatorname{tg}^2\frac{\pi x}{2}\right)\cdot\frac{\cancel{\pi}}{\cancel{2}}} =$

$= \lim_{x\to 1}\dfrac{1+\operatorname{tg}^2\frac{\pi x}{4}}{-2\left(\dfrac{1}{\operatorname{tg}^2\frac{\pi x}{2}}+1\right)} = \dfrac{2}{-2\cdot 1} = -1$

xiv) $\lim\limits_{x\to 1} x^{\frac{1}{1-x}} \overset{1^{\infty}}{=} e^{\lim\limits_{x\to 1} \frac{1}{1-x}(x-1)} = e^{-1}$

$$\lim_{x\to 1} \frac{1}{1-x}(x-1) = \lim_{x\to 1} -1 = -1$$

xv) $\lim\limits_{x\to 0} \frac{1}{x}\log\left(\frac{\text{tg } x}{x}\right) \overset{\infty\cdot 0}{=} \lim\limits_{x\to 0} \frac{\ln\left(\frac{\text{tg}x}{x}\right)}{x} \underset{L'H}{\overset{\frac{0}{0}}{=}} \lim\limits_{x\to 0} \frac{\frac{x(1+\text{tg}^2x)-\text{tg}x}{x^2}}{\frac{\text{tg}x}{x}} =$

$$= \lim_{x\to 0} \frac{x(1+\text{tg}^2x)-\text{tg}x}{x\,\text{tg}x} \underset{L'H}{\overset{\frac{0}{0}}{=}} \lim_{x\to 0} \frac{\cancel{(1+\text{tg}^2x)} + x(2\text{tg}x)(1+\text{tg}^2x) - \cancel{(1+\text{tg}^2x)}}{x(1+\text{tg}^2x)+\text{tg}x} =$$

$$= \lim_{x\to 0} \frac{x\cdot 2\text{tg}x\,(1+\text{tg}^2x)}{x(1+\text{tg}^2x)+\text{tg}x} \underset{L'H}{\overset{\frac{0}{0}}{=}} \lim_{x\to 0} \frac{2\text{tg}x\,(1+\text{tg}^2x) + x\left(2(1+\text{tg}^2x)^2 + 4\text{tg}^2x\,(1+\text{tg}^2x)\right)}{2\cdot(1+\text{tg}^2x) + x\,(2\text{tg}x)(1+\text{tg}^2x)}$$

$$= \lim_{x\to 0} \frac{\cancel{(1+\text{tg}^2x)}(2\text{tg}x + x(2(1+\text{tg}^2x)+4\text{tg}^2x))}{\cancel{(1+\text{tg}^2x)}\,2 + x(2\text{tg}x)} = \frac{0}{2} = 0$$

53.- Calcula los siguientes limites aplicando, cuando sea oportuno, la regla de L'Hôpital:

i) $$\lim_{x\to 0} \frac{\ln\left(e^x + x^3\right)}{x} \underset{L'H}{\overset{\frac{0}{0}}{=}} \lim_{x\to 0} \frac{\frac{e^x+3x^2}{e^x+x^3}}{1} = 1$$

ii) $\lim_{x\to 0}\left(1+\frac{1}{x}\right)^{\frac{1}{2}}-\left(1+\frac{4}{x}\right)^{\frac{1}{2}} = \lim_{x\to 0} \sqrt{1+\frac{1}{x}}-\sqrt{1+\frac{4}{x}} =$

$$= \lim_{x\to 0} \frac{1+\frac{1}{x}-\left(1+\frac{4}{x}\right)}{\sqrt{1+\frac{1}{x}}+\sqrt{1+\frac{4}{x}}} = \lim_{x\to 0} \frac{-\frac{3}{x}}{\sqrt{1+\frac{1}{x}}+\sqrt{1+\frac{4}{x}}} = \lim_{x\to 0} \frac{-3}{x\left(\sqrt{1+\frac{1}{x}}+\sqrt{1+\frac{4}{x}}\right)} =$$

$$= \lim_{x\to 0} \frac{-3}{\sqrt{\underbrace{x^2+x}_{x(x+1)}}+\sqrt{x^2+4x}} =$$

$x(x+1)$ $\qquad$ $x(x+4)$

$x<-1$ $\qquad$ $x<-4$

$\underbrace{\qquad\qquad}$

$x<-4$

$\lim_{x\to 0^+} \frac{-3}{0^+} = -\infty$

$\nexists \lim_{x\to 0^-}$

iii) $\lim\limits_{x\to 0}\frac{a^x - b^x}{x} \underset{L'H}{\overset{0/0}{=}} \lim\limits_{x\to 0}\frac{a^x \ln a - b^x \ln b}{1} = \ln a - \ln b = \ln\frac{a}{b}$

iv) $\lim\limits_{x\to 0}\frac{\ln\cos 3x}{x^2} \underset{L'H}{\overset{0/0}{=}} \lim\limits_{x\to 0}\frac{-3\,\mathrm{sen}\,3x}{2x\cos 3x} \underset{L'H}{\overset{0/0}{=}} \lim\limits_{x\to 0}\frac{-9\cos 3x}{2\cos 3x - 6x\cdot\mathrm{sen}\,3x} = -\frac{9}{2}$

v) $\lim\limits_{x\to 0}\frac{\mathrm{sen}\,x - \cos x}{1-\cos x} = \frac{0-1}{1-1} = \frac{-1}{0}$

$\lim\limits_{x\to 0^+}\frac{0^+ - 1}{1-1} = \frac{-1}{0^+} = -\infty$

$\lim\limits_{x\to 0^-}\frac{0^- - 1}{1-1} = \frac{-1}{0^+} = -\infty$

$\Rightarrow \lim\limits_{x\to 0}\frac{\mathrm{sen}\,x - \cos x}{1-\cos x} = -\infty$

vi) $\lim\limits_{x\to 0}\frac{\ln(1+x)}{\sqrt[4]{x^3}} \underset{L'H}{\overset{0/0}{=}} \lim\limits_{x\to 0}\frac{1}{(1+x)\,\frac{3}{4}\,x^{-1/4}} = \lim\limits_{x\to 0}\frac{3\sqrt[4]{x}}{4(1+x)} \Rightarrow$

$\lim\limits_{x\to 0^+}\frac{3\sqrt[4]{x}}{4(1+x)} = \frac{0^+}{4} = +\infty$

$\lim\limits_{x\to 0^-}\frac{3\sqrt[4]{x}}{4(1+x)} \quad \nexists$

ix) $\lim\limits_{x\to-1} \dfrac{x^3+1}{x^2-3x+4} = \dfrac{0}{1+3+4} = \dfrac{0}{8} = 0$

x) $\lim\limits_{x\to 0}\left(\dfrac{x - \operatorname{sen} x}{\operatorname{tg} x - x}\right) \overset{0/0}{\underset{L'H}{=}} \lim\limits_{x\to 0} \dfrac{1-\cos x}{(1+\operatorname{tg}^2 x)-1} \overset{0/0}{\underset{L'H}{=}} \lim\limits_{x\to 0} \dfrac{\operatorname{sen} x}{2\operatorname{tg} x \cdot (1+\operatorname{tg}^2 x)} \overset{0/0}{\underset{L'H}{=}}$

$= \lim\limits_{x\to 0} \dfrac{\cos x}{2(1+\operatorname{tg}^2 x)^2 + 4\operatorname{tg}^2 x\,(1+\operatorname{tg}^2 x)} = \dfrac{1}{2}$

xi) $\lim\limits_{x\to 0} \dfrac{1-\cos x}{\operatorname{sen} x + \cos x} = \dfrac{1-1}{0+1} = \dfrac{0}{1} = 0$

54.- Calcula los siguientes limites aplicando, cuando sea oportuno, la regla de L'Hôpital:

i) $\lim\limits_{x \to \frac{\pi}{2}} \cos x . \ln(\text{tg } x) \overset{0 \cdot \infty}{=} \lim\limits_{x \to \frac{\pi}{2}} \dfrac{\ln(\text{tg } x)}{\dfrac{1}{\cos x}} \underset{L'H}{\overset{\frac{\infty}{\infty}}{=}} \lim\limits_{x \to \frac{\pi}{2}} \dfrac{\dfrac{1}{\text{tg } x \cdot \cancel{\cos^2 x}}}{\dfrac{\text{sen } x}{\cancel{\cos^2 x}}} = \lim\limits_{x \to \frac{\pi}{2}} \dfrac{1}{\text{tg } x \cdot \text{sen } x} = \dfrac{1}{\infty \cdot 1} = 0$

ii) $\lim\limits_{x \to 0} (\cos x + \text{sen } x)^{\frac{1}{x}} \overset{1^\infty}{=} e^{\lim\limits_{x \to 0} \frac{1}{x}(\cos x + \text{sen } x - 1)} = e^1 = e$

$$\lim_{x \to 0} \frac{1}{x}(\cos x + \text{sen } x - 1) \overset{\infty \cdot 0}{=} \lim_{x \to 0} \frac{\cos x + \text{sen } x - 1}{x} \underset{L'H}{\overset{\frac{0}{0}}{=}} \lim_{x \to 0} \frac{\cos x - \text{sen } x}{1} = 1$$

iii) $\lim\limits_{x \to \frac{\pi}{2}} (\text{tg } x)^{\cos x} \overset{\infty^0}{=} \underbrace{K} \Rightarrow \ln\left(\lim\limits_{x \to \frac{\pi}{2}} (\text{tg } x)^{\cos x}\right) = \ln K$

$$\lim_{x \to \frac{\pi}{2}} \ln(\text{tg } x^{\cos x}) = \lim_{x \to \frac{\pi}{2}} \cos x \ln \text{tg } x \overset{0 \cdot \infty}{=} \lim_{x \to \frac{\pi}{2}} \frac{\ln \text{tg } x}{\frac{1}{\cos x}} \underset{L'H}{\overset{\frac{\infty}{\infty}}{=}} \lim_{x \to \frac{\pi}{2}} \frac{1}{\text{tg } x \cdot \cos^2 x \cdot \left(-\frac{1}{\cos^2 x} \cdot (-\text{sen } x)\right)} =$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{1}{\text{tg } x \cdot \text{sen } x} = \lim_{x \to \frac{\pi}{2}} \frac{\cos x}{\text{sen}^2 x} = \frac{0}{1} = 0 \Rightarrow$$

Si $\ln K = 0$

$K = e^0$

$\boxed{K = 1}$

iv) $\lim_{x\to 0}\left(e^x + x^3\right)^{\frac{1}{x}} \overset{1^\infty}{=} e^{\lim_{x\to 0}\frac{1}{x}\left(e^x+x^3-1\right)} = e^1 = e$

$\lim_{x\to 0}\frac{e^x+x^3-1}{x} \underset{L'H}{\overset{\frac{0}{0}}{=}} \lim_{x\to 0}\frac{e^x+3x^2}{1} = \frac{1+0}{1} = 1$

v) $\lim_{x\to 0}\left[\left(e^{\operatorname{sen} x}-1\right)\operatorname{tg}\left(x+\frac{\pi}{2}\right)\right] \overset{0\cdot\infty}{=} \lim_{x\to 0}\frac{e^{\operatorname{sen} x}-1}{\frac{1}{\operatorname{tg}\left(x+\frac{\pi}{2}\right)}} \underset{L'H}{\overset{\frac{0}{0}}{=}} \lim_{x\to 0}\frac{e^{\operatorname{sen} x}\cdot\cos x}{-\frac{1}{\operatorname{tg}^2\left(x+\frac{\pi}{2}\right)}\cdot\left(1+\operatorname{tg}^2\left(x+\frac{\pi}{2}\right)\right)} =$

$= \lim_{x\to 0}\frac{e^{\operatorname{sen} x}\cdot\cos x}{-\frac{1}{\operatorname{tg}^2\left(x+\frac{\pi}{2}\right)}-1} = \frac{1\cdot 1}{0-1} = \frac{1}{-1} = -1$

vi) $\lim\limits_{x\to\infty}(1+x)^{\frac{1}{x}} \overset{\infty^0}{=} \underbrace{K} \Rightarrow \lim\limits_{x\to\infty} \frac{1}{x}\ln(1+x) = \ln K$

$$\lim_{x\to\infty} \frac{\ln(1+x)}{x} \underset{L'H}{\overset{\frac{\infty}{\infty}}{=}} \lim_{x\to\infty} \frac{1}{(1+x)\cdot 1} = \frac{1}{\infty} = 0 \Rightarrow \ln K = 0 \Rightarrow K = e^0$$

$$\boxed{K = 1}$$

vii) $\lim\limits_{x\to\infty} x.\ln\left(\frac{1+x}{x}\right) \overset{\infty\cdot 0}{=} \lim\limits_{x\to\infty} \frac{\ln\left(\frac{1+x}{x}\right)}{\frac{1}{x}} \underset{L'H}{\overset{\frac{0}{0}}{=}} \lim\limits_{x\to\infty} \frac{\frac{x-(1+x)}{x^2}}{\left(\frac{1+x}{x}\right)\cdot\left(-\frac{1}{x^2}\right)} =$

$$= \lim_{x\to\infty} \frac{1}{\frac{1+x}{x}} = \lim_{x\to\infty} \frac{x}{1+x} = 1$$