INTEGRAL DEFINIDA. APLICACIONES

1.- Halla las siguientes integrales definidas:

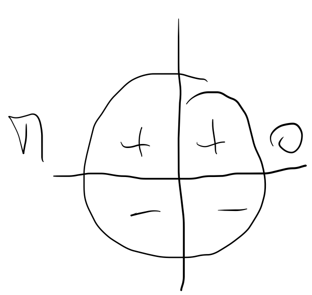
i) $\int_{-2}^{2}\left(x^3+1\right)dx = \left[\frac{x^4}{4}+x\right]_{-2}^{2} = \left(\frac{2^4}{4}+2\right)-\left(\frac{(-2)^4}{4}-2\right) = 4$

ii) $\int_{-\pi}^{2\pi}\left|\operatorname{sen} x\right|dx = \int_{-\pi}^{0} -\operatorname{sen} x\,dx + \int_{0}^{\pi} \operatorname{sen} x\,dx + \int_{\pi}^{2\pi} -\operatorname{sen} x\,dx = \left[\cos x\right]_{-\pi}^{0} + \left[-\cos x\right]_{0}^{\pi} +$

$+\left[\cos x\right]_{\pi}^{2\pi} = 1-(-1)+1-(-1)$

$+1-(-1) = 6$

$$\left|\operatorname{sen} x\right| = \begin{cases} \operatorname{sen} x, & x \in [0,\pi] \\ -\operatorname{sen} x, & x \in (-\pi,0)\cup(\pi,2\pi) \end{cases}$$



senx

iii) $\int_{-1}^{1} f(x)\,dx$, siendo $f(x) = \begin{cases} -x-1 & \text{si } x \le 0 \\ x^2+1 & \text{si } x > 0 \end{cases}$

$$\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx = \int_{-1}^{0} (-x-1)\,dx + \int_{0}^{1} (x^2+1)\,dx =$$

$$= \left[-\frac{x^2}{2} - x\right]_{-1}^{0} + \left[\frac{x^3}{3} + x\right]_{0}^{1} = 0 - \left(-\frac{1}{2}+1\right) + \left(\frac{1}{3}+1\right) - 0 =$$

$$= -\frac{1}{2} + \frac{4}{3} = \frac{5}{6}$$

iv) $\int_{4}^{12} \frac{x}{\sqrt{x-3}}\,dx$

$$= \int_{1}^{3} \frac{(t^2+3)\,2\cancel{t}}{\cancel{t}}\,dt = 2\int_{1}^{3} (t^2+3)\,dt = \left[\frac{2t^3}{3} + 6t\right]_{1}^{3} =$$

$x-3 = t^2 \Rightarrow dx = 2t\,dt$

$x = t^2+3$

$x = 12 \Rightarrow t = 3$

$x = 4 \Rightarrow t = 1$

$$= 18 + 18 - \frac{2}{3} - 6 = 30 - \frac{2}{3} = \frac{88}{3}$$

3.- Utiliza la regla de Barrow para calcular :

(a) $\int\limits_0^3 \left(3x^2 - 6\right)\, dx$

Sol: $9 \ (F(x) = x^3 - 6x)$

$$\int_0^3 (3x^2-6)\,dx = \left[\frac{\cancel{3}x^3}{\cancel{3}} - 6x\right]_0^3 = 27 - 18 - 0 + 0 = 9$$

(b) $\int\limits_1^2 \frac{1}{x}\, dx = \left[\ln|x|\right]_1^2 = \ln 2 - \ln 1 = \ln\frac{2}{1} = \ln 2$

Sol: $\log(2) \ (F(x) = \log(x))$

$$\int_0^1 \frac{5}{7x^2+7}\,dx$$

Sol: $\frac{5\pi}{28}$ $\left(F(x) = \frac{5\operatorname{atan}(x)}{7}\right)$

$$\int_2^3 \frac{1}{x\log(x)}\,dx$$

Sol: $\log\left(\frac{\log(3)}{\log(2)}\right)$ $\left(F(x) = \log(\log(x))\right)$

$$\int_{\frac{\pi}{2}}^{2\pi} \sin^5(x)\cos(x)\,dx$$

Sol: $-\frac{1}{6}$ $\left(F(x) = \frac{\sin^6(x)}{6}\right)$

$$\int_2^5 e^x x \, dx$$

$$\boxed{\textbf{Sol:}\ -e^2 + 4e^5 \ \ (F(x) = (x-1)\,e^x)}$$

4.- Calcula $\int_0^5 f(x)\,dx$ para $f(x) = \begin{cases} x+1 & \text{si } x \le 1 \\ 3-x & \text{si } 1 < x \le 3 \\ x-3 & \text{si } x > 3 \end{cases}$. Representa gráficamente la función, y explica el significado geométrico de la integral que has calculado.

$$\boxed{\textbf{Sol: } \frac{11}{2}\ \left(F(x) = \begin{cases} \frac{x^2}{2} + x & \text{for } x < 1 \\ -\frac{x^2}{2} + 3x - 1 & \text{for } x \le 3 \\ \frac{x^2}{2} - 3x + 8 & \text{otherwise} \end{cases}\right)}$$

$$\int_0^5 f(x)\,dx = \left[\frac{x^2}{2} + x\right]_0^1 + \left[-\frac{x^2}{2} + 3x - 1\right]_1^3 + \left[\frac{x^2}{2} - 3x + 8\right]_3^5 =$$

$$= \left(\frac{1}{2} + 1\right) - 0 + \left(-\frac{9}{2} + 9 - 1\right) - \left(-\frac{1}{2} + 3 - 1\right) + \left(\frac{25}{2} - 15 + 8\right) - \left(\frac{9}{2} - 9 + 8\right) =$$

$$= \frac{3}{2} - \frac{9}{2} + \frac{16}{2} - \frac{3}{2} + \frac{25}{2} - \frac{14}{2} - \frac{9}{2} + \frac{2}{2} = \frac{11}{2}$$

$$\int_{-5}^{5} |x| \, dx = \int_{-5}^{0} -x \, dx + \int_{0}^{5} x \, dx = \left[-\frac{x^2}{2}\right]_{-5}^{0} + \left[\frac{x^2}{2}\right]_{0}^{5} =$$

Sol: $25 \; \left(F(x) = \int |x| \, dx\right)$

$$= 0 + \frac{25}{2} + \frac{25}{2} - 0 = 25$$

$$|x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$$

$$\int_{\frac{\pi}{2}}^{2\pi} \sin^5(x)\cos(x) \, dx = \int_{1}^{0} t^5 \, dt = \left[\frac{t^6}{6}\right]_{1}^{0} = 0 - \frac{1}{6} = -\frac{1}{6}$$

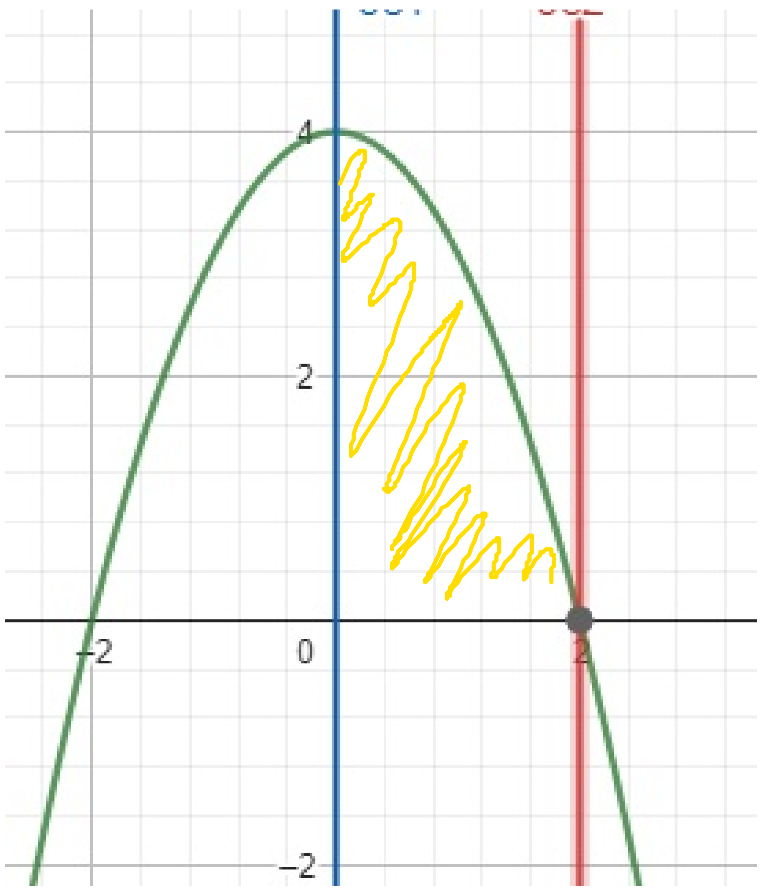
Sol: $-\frac{1}{6} \; \left(F(x) = \frac{\sin^6(x)}{6}\right)$

$\Rightarrow t = \sin(x) \quad dt = \cos x \, dx$

$x = 90 \to t = 1$

$x = 2\pi \Rightarrow t = 0$

6.- Halla el área del recinto limitado por la gráfica de la función $f(x) = -x^2 + 4$, el eje de abscisas y las rectas x=0 y x=2.



$$\int_0^2 (-x^2+4)\,dx = \left[-\frac{x^3}{3}+4x\right]_0^2 =$$

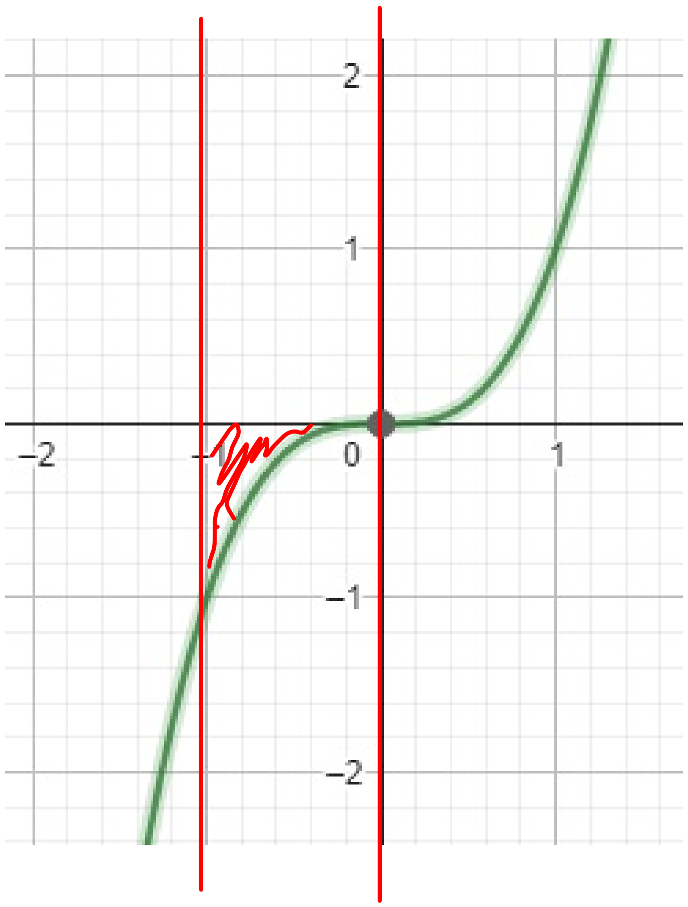
$$= \left(-\frac{8}{3}+8\right) - \left(0+0\right) = \frac{16}{3}$$

7.- Halla el área del recinto limitado por la gráfica de la función $f(x) = x^3$, el eje de abscisas y las rectas $x = -1$ y $x = 0$.

$Dom(f) = \mathbb{R}$

$x^3 = 0 \Rightarrow x = 0$

	$(-\infty, 0)$	$(0, +\infty)$
$f(x)$	$-$	$+$



$$A = \left| \int_{-1}^{0} x^3\,dx \right| = \left| \left[\frac{x^4}{4} \right]_{-1}^{0} \right| =$$

$$= \left| -\frac{1}{4} \right| = \frac{1}{4}$$

8.- Halla el área del recinto limitado por la gráfica de $f(x)=x^4-3x^3-4x^2+12x$ y el eje OX.

$$0=x^4-3x^3-4x^2+12x$$

$$0=x\left(x^3-3x^2-4x+12\right)$$

$$0=x(x-2)(x+2)(x-3)$$

$$\begin{array}{r|rrrr} & 1 & -3 & -4 & 12 \\ 2 & & 2 & -2 & -12 \\ \hline & 1 & -1 & -6 & \boxed{0} \end{array}$$

$$x=\frac{1\pm\sqrt{1+24}}{2}=\begin{cases}3\\-2\end{cases}$$

Raíces $-2, 0, 2, 3$

$$A=\left|\int_{-2}^{0} f(x)\,dx\right|+\left|\int_{0}^{2} f(x)\,dx\right|+\left|\int_{2}^{3} f(x)\,dx\right|=$$

$$=\left|\underbrace{\left[\frac{x^5}{5}-\frac{3x^4}{4}-\frac{4x^3}{3}+6x^2\right]}_{G(x)}{}_{-2}^{0}\right|+$$

$$+\left|\left[G(x)\right]_0^2\right|+\left|\left[G(x)\right]_2^3\right|=$$

$$=\left|-\frac{244}{15}\right|+\left|\frac{116}{15}\right|+\left|\frac{-113}{60}\right|=\frac{1553}{60}$$

9.- Halla el área limitada por las gráficas de $f(x) = -x^2-4x+3$ y $g(x) = x^2-x-6$.

$f(x) = g(x)$ $\qquad -x^2-4x+3 = x^2-x-6$

$$0 = 2x^2+3x-9$$

$$x = \frac{-3 \pm \sqrt{9+72}}{4} = \begin{cases} \frac{3}{2} \\ -3 \end{cases}$$

$$\text{Área} = \left| \int_{-3}^{3/2} \left(f(x) - g(x)\right) dx \right| = \left| \int_{-3}^{3/2} \left[(-x^2-4x+3) - (x^2-x-6)\right] dx = \right.$$

$$= \left| \int_{-3}^{3/2} (-2x^2-3x+9)\, dx \right| = \left| \left[-\frac{2x^3}{3} - \frac{3x^2}{2} + 9x \right]_{-3}^{3/2} \right| = \frac{243}{8}$$

11.- Halla el área limitada por las gráficas de las funciones $f(x) = 5x-9$ y $g(x) = 3x^3-21x^2+47x-33$

$$f(x)=g(x) \Rightarrow 5x-9=3x^3-21x^2+47x-33 \Rightarrow 0=3x^3-21x^2+42x-24$$

$$0=x^3-7x^2+14x-8$$

$$0=(x-1)(x-2)(x-4)$$

$$x=1,\ x=2,\ x=4$$

$$A=\left|\int_1^2 (f(x)-g(x))\,dx\right|+\left|\int_2^4 (f(x)-g(x))\,dx\right|=\left|\int_1^2 (-3x^3+21x^2-42x+24)\,dx\right|+$$

$$+\left|\int_2^4 f-g\right|=\left|\left[\underbrace{-\frac{3x^4}{4}+7x^3-21x^2+24x}_{G(x)}\right]_1^2\right|+\left|\left[G(x)\right]_2^4\right|=\left|-\frac{5}{4}\right|+|8|=\frac{37}{4}$$

u^2