A Survey of Minimal Surfaces and Soap

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Abstract

The primary goal of this paper is to give a broad introduction to the study of minimal surfaces, beginning with a general discussion of the origins of the concept as it first began within the calculus of variations. From there, we will provide a detailed description of what defines a minimal surface from a mathematical perspective before viewing in detail the properties of three particular minimal surfaces: the catenoid, helicoid, and Scherk's First Surface. These concepts will subsequently be connected to the topic of soap films and bubbles.

1 Historical Background

The study of minimal surfaces first began to emerge in the calculus of variations, a subject area in which Joseph-Louis Lagrange and Leonhard Euler were the two most prominent pioneers. While they certainly studied geodesic curves and functionals such as

$$J = \int_a^b f\left(y(\alpha, x), y'(\alpha, x)\right) dx$$

which allow for the minimization of arc length on an arbitrary surface, the basic question regarding the analogous structure for two dimensional surfaces rather than one dimensional lines was the following: Given a closed boundary, what is the surface which has minimal surface area when draped across it? Although generally associated with Euler and Lagrange, this is often dubbed "Plateau's Problem" after Joseph Plateau, a Belgian physicist who did experimental work invoving soap films. The equation that was eventually developed by Lagrange in answer to this quandary, if $z = z(x, y)$, then the area functional

$$A = \int_{\mathbf{S}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dxdy$$

is what needed to be minimized. Here, $\mathcal{L} = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$, so from the two dimensional Euler-Lagrange equation for independent variables,

$$\frac{\partial}{\partial x}\left(\frac{\frac{\partial z}{\partial x}}{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}}\right) + \frac{\partial}{\partial y}\left(\frac{\frac{\partial z}{\partial y}}{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}}\right) = 0.$$

Unfortunately, any information which this equation may contain is, shall we say, veiled. A leap towards a much greater level of insight was made by Jean Baptiste Meusnier in 1776, who made a connection between the above equation and the geometric properties it entails. This made apparent the value of differential geometry, a field to which Euler and Gauss were prominent early contributors, and whose ideas were later furthered by Riemann, Ricci-Curbastro, and Levi-Cevita (Meeks and Perez, 2012).

2 Mathematical Development

2.1 Principal Curvatures

In order to determine the most general requirements for a surface to be minimal, we must approach the topic from a mathematical point of view, beginning with an introduction to the concept of principal curvatures. At any point p on an arbitrary surface $\mathbf{S} \subset \mathbb{R}^3$, there is a space of tangent vectors, say $T_{\mathbf{S}}(p)$; from within this space, choose a tangent vector $\hat{T}$ that is of unit length. Additionally, let $\hat{n}(p)$ be the unit normal at p. The set $\{\hat{T}, \hat{n}(p)\}$ forms an orthonormal basis for what is known as osculating plane. Within this plane there is a circle, known as the osculating circle, which (1) is tangent to $\mathbf{S}$ at p, and (2) has the same Taylor series approximation up to second order of the plane curve formed by the intersection of $\mathbf{S}$ and the osculating plane. This provides us with enough information to formally define the curvature:

Definition 1 Let R be the radius of the osculating circle given. The *curvature* of $\mathbf{S}$ in the plane defined by $\hat{T}$ and $\hat{n}$ is given as $\kappa = \frac{1}{R}$.

An immediate extension of this which can be made without any intermediate steps is the concept of principal curvatures:

Definition 2 Let $\vec{w}_1, \vec{w}_2 \in T_{\mathbf{S}}(p)$ have the property $\vec{w}_1 \cdot \vec{w}_2 = 0$, and with corresponding curvatures k and k'. The *principal normal curvatures* k_1 and k_2 represent the maximum and minimum curvatures at p, where the maximum and minimum are taken over all such orthogonal vectors $\vec{w}_1$ and $\vec{w}_2$.

2.2 Fundamental Forms

There are two quadratic forms which provide crucial information about the characteristics of any surface in $\mathbb{R}^3$ (note the concepts discussed are generalizable to n-dimensional manifolds, but we restrict ourselves to this simpler case), the first of which is the First Fundamental Form, often denoted $\mathbf{I}$. Let $\mathbf{S}(u, v) \subset \mathbb{R}^3$ be a two dimensional surface embedded in three dimensional space. At any point on this surface, we can consider the differential element of the radial vector

$$d\vec{r} = \frac{\partial \vec{r}}{\partial u} du + \frac{\partial \vec{r}}{\partial v} dv.$$

The most intuitive way to introduce $\mathbf{I}$ is to consider the differential area element

$$da = d\vec{r}^2 = \left\| \frac{\partial \vec{r}}{\partial u} \right\|^2 du^2 + 2 \left(\frac{\partial \vec{r}}{\partial u} \cdot \frac{\partial \vec{r}}{\partial v} \right) du dv + \left\| \frac{\partial \vec{r}}{\partial v} \right\|^2 dv^2$$

which is a quadratic form. The coefficients of $\mathbf{I}$ which will be especially useful in our discussion of minimal surfaces are exactly the coefficients of the above quadratic, which we denote

$$\boxed{E = \left\| \frac{\partial \vec{r}}{\partial u} \right\| = \frac{\partial \vec{r}}{\partial u} \cdot \frac{\partial \vec{r}}{\partial u}; \quad F = \frac{\partial \vec{r}}{\partial u} \cdot \frac{\partial \vec{r}}{\partial v}; \quad G = \left\| \frac{\partial \vec{r}}{\partial v} \right\| = \frac{\partial \vec{r}}{\partial v} \cdot \frac{\partial \vec{r}}{\partial v}}.$$

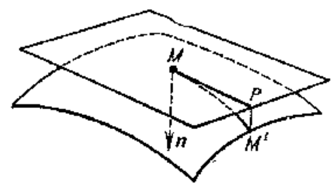
Another way of developing the notion of $\mathbf{I}$ which, although less intuitive, is distinctly valuable in its generality. At every point p on the surface $\mathbf{S}(u, v)$, we may consider the tangent space $T_{\mathbf{S}}(p)$; if $\langle \;, \; \rangle$ represents the Euclidean inner product (the dot product), then we may restrict the inner product to the collection of tangent spaces $T_{\mathbf{S}}$ and call it $P(\;, \;)$. Then for any vectors $w, x \in T_{\mathbf{S}}$, $P(w, x) \triangleq \langle w, x \rangle$. This is what defines $\mathbf{I}$, the inner product restricted to tangent spaces; it may not be immediately obvious, so we add the following comment for clarity. If

$\mathbf{S}(u,v) = (x(u,v), y(u,v), z(u,v))$, then we obtain the all-important coefficients E, F, and G via the symmetric matrix

$$(g_{ij}) = \begin{bmatrix} \left\langle \frac{\partial \mathbf{S}}{\partial u}, \frac{\partial \mathbf{S}}{\partial u} \right\rangle & \left\langle \frac{\partial \mathbf{S}}{\partial u}, \frac{\partial \mathbf{S}}{\partial v} \right\rangle \\ \left\langle \frac{\partial \mathbf{S}}{\partial v}, \frac{\partial \mathbf{S}}{\partial u} \right\rangle & \left\langle \frac{\partial \mathbf{S}}{\partial v}, \frac{\partial \mathbf{S}}{\partial v} \right\rangle \end{bmatrix} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$$

We say that (g_{ij}) is a metric tensor, a mathematical object which induces a Riemannian metric at each point of the surface in question (Ibanov, First Fundamental Form, 2011).

The Second Fundamental Form, frequently referred to simply by **II**, provides a measure of the deviation of a surface from the tangent plane at a given point, as visually depicted by the figure.



An illustration of the deviation of a surface from the tangent plane in a neighborhood around the point of contact (Ibanov, 2011).

Again for any point on the surface we can consider a differential element of the radial vector $d\vec{r} = \frac{\partial \vec{r}}{\partial u} du + \frac{\partial \vec{r}}{\partial v} dv$. At the point at the tail of $d\vec{r}$, we denote the unit normal vector by

$$\hat{n} = \frac{\vec{n}}{\|\vec{n}\|} = \frac{\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}}{\left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\|}.$$

We then define the deviation of the surface from the tangent plane by

$$\mathbf{II} = \langle d\vec{r}, d\hat{n} \rangle = \left\langle \frac{\partial^2 \vec{r}}{\partial u^2}, \hat{n} \right\rangle du^2 + 2 \left\langle \frac{\partial^2 \vec{r}}{\partial u \partial v}, \hat{n} \right\rangle dudv + \left\langle \frac{\partial^2 \vec{r}}{\partial v^2}, \hat{n} \right\rangle dv^2.$$

Generally speaking, the coefficients of this qudratic form are labeled in the following way

$$\boxed{L = \left\langle \frac{\partial^2 \vec{r}}{\partial u^2}, \hat{n} \right\rangle = \frac{\partial^2 \vec{r}}{\partial u^2} \cdot \hat{n}; \quad M = \left\langle \frac{\partial^2 \vec{r}}{\partial u \partial v}, \hat{n} \right\rangle = \frac{\partial^2 \vec{r}}{\partial u \partial v} \cdot \hat{n}; \quad N = \left\langle \frac{\partial^2 \vec{r}}{\partial v^2}, \hat{n} \right\rangle = \frac{\partial^2 \vec{r}}{\partial v^2} \cdot \hat{n}}.$$

Since this form can be applied at any arbitrary point on the surface $\mathbf{S}(u,v)$, it is common to replace $\vec{r}$ with simply $\mathbf{S}$ in each of the above definitions. From here, we are one step away from understanding what a minimal surface truly is (Ibanov, 2011).

2.3 Mean Curvature

The mean curvature at a point $p \in \mathbf{S}(u,v)$, denoted $H(p)$, can be viewed from two different perspectives. It may be written both in terms of the principal curvatures of the surface at p, or in terms of the First and Second Fundamental Forms. If the principal curvatures and principal radii at p are k_1, k_2, R_1, and R_2 respectively then we have (as one might expect)

$$H(p) = \frac{1}{2}(k_1 + k_2) = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

In general, since R_1 and R_2 are difficult to determine, there is a crucial fact which can be proven with a little bit of complex analysis (a proof which is omitted here): the mean curvature at the point p may be written in terms of any two normal curvatures, as it is invariant under the choice of orthogonal vectors in the tangent space. More conveniently, it is possible to formulate the mean curvature in terms of the coefficients of **I** and **II** for ease of computation:

$$H(p) = \frac{GL + EN - 2FM}{2(EG - F^2)}.$$

Note that since p was chosen arbitrarily, provided we not evaluate the coefficients of **I** and **II** at a particular point, we may simply denote the mean curvature by H.

2.4 Minimality

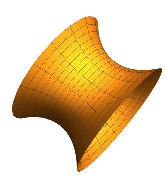
Presented here as concretely as possible are the two definitions of a minimal surface which will be most relevant to our discussion:

Definition 3 A surface $\mathbf{S} \subset \mathbb{R}^3$ is a *minimal surface* if it has mean curvature $H = 0$.

Definition 4 A surface $\mathbf{S} \subset \mathbb{R}^3$ is a *minimal surface* if for every point $p \in \mathbf{S}$, $\exists B(r,p) \subseteq \mathbf{S}$, a neighborhood of p, such that $B(r,p)$ is equal to the idealized soap film with the same boundary. By idealized we mean to say a bubble which has a net pressure difference of zero across its boundary.

The former definition is most useful computationally, and can still be used in settings which may not be physically meaningful; the latter, as we will see more in sections 3 and 3.1, directly links the mathematical definition of minimality to the geometry of soap films. The importance of the word "idealized" will become clear through section 3.1 (Meeks and Perez, 2012).

2.5 The Catenoid



A catenoid with $a = 1$.

One of the first curves one sees when first introduced to the calculus of variations is the catenary curve. Such a curve represents the path taken by a string or similarly flexible object which, when hung between two points A and B while subject to the force of gravity, sags below the straight line connecting the two points. The equation for the catenary curve being $y = a\cosh\left(\frac{x}{a}\right)$, we can obtain the equation for a surface called a catenoid in the parameterized form

$$\mathbf{S}(u,v) = (a\cosh\left(\frac{u}{a}\right)\cos(v), a\cosh\left(\frac{u}{a}\right)\sin(v), u).$$

Now that we have this parameterization, we wish to verify that this is a surface which does in fact have zero mean curvature, i.e. that it satifies the definition of a minimal surface. To do so, as outlined in the previous section we must first compute the tangent vectors. These are given by

$$\frac{\partial \mathbf{S}}{\partial u} = (\sinh\left(\frac{u}{a}\right)\cos(v), \sinh\left(\frac{u}{a}\right)\sin(v), 1); \quad \frac{\partial \mathbf{S}}{\partial v} = (-a\cosh\left(\frac{u}{a}\right)\sin(v), a\cosh\left(\frac{u}{a}\right)\cos(v), 0).$$

To determine the First Fundamental Form, we need the coefficients E, F, and G. These are

$$E = \frac{\partial \mathbf{S}}{\partial u}\cdot\frac{\partial \mathbf{S}}{\partial u} = \sinh^2\left(\frac{u}{a}\right)\cos^2(v) + \sinh\left(\frac{u}{a}\right)\sin^2(v) + 1 = \cosh^2\left(\frac{u}{a}\right);$$

$$F = \frac{\partial \mathbf{S}}{\partial v}\cdot\frac{\partial \mathbf{S}}{\partial u} = -a\cosh\left(\frac{u}{a}\right)\sin(v)\sinh\left(\frac{u}{a}\right)\cos(v) + a\cosh\left(\frac{u}{a}\right)\cos(v)\sinh\left(\frac{u}{a}\right)\sin(v) + 0 = 0;$$

$$G = \frac{\partial \mathbf{S}}{\partial v}\cdot\frac{\partial \mathbf{S}}{\partial v} = a^2\cosh^2\left(\frac{u}{a}\right)\sin^2(v) + a^2\cosh^2\left(\frac{u}{a}\right)\cos^2(v) = a^2\cosh^2\left(\frac{u}{a}\right).$$

We know that the cross product of two tangent vectors must be perpendicular to the surface at that point, so to find the unit normal vector we consider

$$\vec{n} = \frac{\partial \mathbf{S}}{\partial u}\times\frac{\partial \mathbf{S}}{\partial v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \sinh\left(\frac{u}{a}\right)\cos(v) & \sinh\left(\frac{u}{a}\right)\sin(v) & 1 \\ -a\cosh\left(\frac{u}{a}\right)\sin(v) & a\cosh\left(\frac{u}{a}\right)\cos(v) & 0 \end{vmatrix}$$

$$= a\left(-\cosh\left(\frac{u}{a}\right)\cos(v), -\cosh\left(\frac{u}{a}\right)\sin(v), \cos\left(\frac{u}{a}\right)\sinh\left(\frac{u}{a}\right)\right).$$

We want the unit normal, so we divide by the magnitude to find

$$\hat{n} = \frac{a}{\cosh^2\left(\frac{u}{a}\right)}\left(-\cosh\left(\frac{u}{a}\right)\cos(v), -\cosh\left(\frac{u}{a}\right)\sin(v), \cos\left(\frac{u}{a}\right)\sinh\left(\frac{u}{a}\right)\right)$$

We also need the coefficients of the Second Fundamental Form, which are given as previously stated:

$$L = \frac{\partial^2 \mathbf{S}}{\partial u^2}\cdot\hat{n} = -1$$

$$M = \frac{\partial^2 \mathbf{S}}{\partial u \partial v}\cdot\hat{n} = 0$$

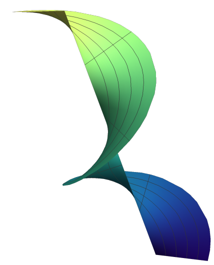
$$N = \frac{\partial^2 \mathbf{S}}{\partial v^2}\cdot\hat{n} = -a^2\frac{\partial^2 \mathbf{S}}{\partial u^2} = a^2.$$

Having in hand all of the information provided by the fundamental forms, we see

$$H = \frac{GL + EN - 2FM}{2(EG - F^2)} = \frac{-a^2\cosh^2\left(\frac{u}{a}\right) + a^2\cosh^2\left(\frac{u}{a}\right)}{2\left(a^2\cosh^4\left(\frac{u}{a}\right)\right)} = 0.$$

From the definition of a minimal surface given in the Mathematical Development section, we see that the catenoid is indeed a minimal surface (Weisstein, n.d.).

2.6 The Helicoid



A helicoid defined by the parameterization $(u\cos(v), u\sin(v), v)$ for $0 \le u \le 2$ and $0 \le v \le 3\pi$.

Another basic example of a two dimensional surface embedded in $\mathbb{R}^3$ which can be shown to be minimal is the helicoid. This surface arises in a natural way; similar to how a catenoid is obtained from revolving a catenary curve about an axis, the helicoid is connected directly to a geodesic curve as well. One may consider the shortest path between any two points on the surface of a cylinder, which is found to be a helix quite easily from the Euler-Lagrange equations with constraints. If we take this curve and connect it to the axis of the cylinder at every point, we obtain the helicoid, a surface with the parameterized form $\mathbf{X}(u, v) = (u\cos(v), u\sin(v), v)$. To show that this is in fact a minimal surface, we follow the same process as in the preceding section:

$$\frac{\partial \mathbf{X}}{\partial u} = (\cos(v), \sin(v), 0), \quad \frac{\partial^2 \mathbf{X}}{\partial u^2} = \vec{0};$$

$$\frac{\partial \mathbf{X}}{\partial v} = (-u\sin(v), u\cos(v), 1), \quad \frac{\partial^2 \mathbf{X}}{\partial v^2} = (-u\cos(v), -u\sin(v), 0); \quad \frac{\partial^2 \mathbf{X}}{\partial u \partial v} = \frac{\partial^2 \mathbf{X}}{\partial v \partial u} = (-\sin(v), \cos(v), 0).$$

As before, we then know

$$E = \frac{\partial \mathbf{X}}{\partial u} \cdot \frac{\partial \mathbf{X}}{\partial u} = \cos^2(v) + \sin^2(v) = 1;$$

$$F = \frac{\partial \mathbf{X}}{\partial v} \cdot \frac{\partial \mathbf{X}}{\partial u} = -u\sin(v)\cos(v) + u\sin(v)\cos(v) = 0;$$

$$G = \frac{\partial \mathbf{X}}{\partial v} \cdot \frac{\partial \mathbf{X}}{\partial v} = u^2\sin^2(v) + u^2\cos^2(v) + 1 = u^2 + 1.$$

We then proceed in obtaining the unit normal and the coefficients of the Second Fundamental Form, finding

$$\vec{n} = \frac{\partial \mathbf{X}}{\partial u} \times \frac{\partial \mathbf{X}}{\partial v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \cos(v) & \sin(v) & 0 \\ -u\sin(v) & u\cos(v) & 1 \end{vmatrix} = (\sin(v), -\cos(v), u);$$

$$\|\vec{n}\|^2 = \sin^2(v) + \cos^2(v) + u^2 = 1 + u^2 \implies \hat{n} = \frac{1}{\sqrt{1+u^2}}(\sin(v), -\cos(v), u).$$

Consequently, the coefficients become

$$L = 0; \quad M = -\frac{1}{\sqrt{1+u^2}}; \quad N = 0.$$

Thus, we arrive at the conclusion

$$H = \frac{GL + EN - 2FM}{2(EG - F^2)} = \frac{0}{2(1+u^2)} = 0$$

and we then know by definition that the helicoid is a minimal surface, as we wished to verify.

2.7 The Minimal Surface Equation

Here, we will introduce a more general condition which a given surface must satisfy in order to be minimal, which is the minimal surface equation. There are two methods of deriving this, each of which is important in its own right, so we will cover both here.

Method I

It is frequently the case that a surface can be described as a function of only two variables within $\mathbb{R}^3$. Without loss of generality, we can assume that of the three coordinates x, y, and z, $z = s(x, y)$. Then, we can write a parameterization (known as a Monge type) of the surface $\mathbf{S}(u, v) = (u, v, s(u, v))$. We want to know when such a surface is minimal; the only approach at our disposal is to find the mean curvature and to impose the restriction that it equal zero. Here, we will introduce the notation $\frac{\partial s}{\partial v} = s_v$, $\frac{\partial s}{\partial u} = s_u$, etc. We then have

$$\frac{\partial \mathbf{S}}{\partial u} = (1, 0, s_u), \quad \frac{\partial^2 \mathbf{S}}{\partial u^2} = (0, 0, s_{uu});$$

$$\frac{\partial \mathbf{S}}{\partial v} = (0, 1, s_v), \quad \frac{\partial^2 \mathbf{S}}{\partial v^2} = (0, 0, s_{vv}), \quad \frac{\partial^2 \mathbf{S}}{\partial v \partial u} = \frac{\partial^2 \mathbf{S}}{\partial u \partial v} = (0, 0, s_{uv}).$$

This leads us to

$$E = 1 + s_u^2; \;\; F = s_u s_v; \;\; G = 1 + s_v^2.$$

It is also quickly determined that

$$\vec{n} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 0 & s_u \\ 0 & 1 & s_v \end{vmatrix} = (-s_u, -s_v, 1) \implies \hat{n} = \frac{1}{\sqrt{1 + s_u^2 + s_v^2}}(-s_u, -s_v, 1).$$

As is now customary, we determine the coefficients of the Second Fundamental Form, namely

$$L = \frac{s_{vv}}{\sqrt{1 + s_u^2 + s_v^2}}; \;\; M = \frac{s_{uv}}{\sqrt{1 + s_u^2 + s_v^2}}; \;\; N = \frac{s_{uu}}{\sqrt{1 + s_u^2 + s_v^2}}.$$

We then see if we want our surface to have the property of zero mean curvature that

$$H = \frac{GL + EN - 2FM}{2(EG - F^2)} = 0 \iff GL + EN - 2FM = 0 \iff s_{vv}(1+s_u^2) + s_{uu}(1+s_v^2) - 2 s_u s_v s_{uv} = 0$$

$$\therefore \boxed{s_{vv}(1 + s_u^2) - 2 s_u s_v s_{uv} + s_{uu}(1 + s_v^2) = 0}.$$

The boxed equation is what we refer to as the minimal surface equation, a quasi-linear elliptic partial differential equation. Unless further restrictions are imposed on $s(u, v)$, this equation is not one that can be easily solved by hand.

2.7.1 Method II

The original equation formulated by Lagrange in answer to the question stated in section 1,

$$\frac{\partial}{\partial x}\left(\frac{\frac{\partial z}{\partial x}}{\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}}\right)+\frac{\partial}{\partial y}\left(\frac{\frac{\partial z}{\partial y}}{\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}}\right)=0$$

can be expanded in an obvious way, taking the derivatives via the quotient rule. In keeping with the notation $\frac{\partial z}{\partial x}=z_x$ and the like, we obtain

$$\frac{z_{xx}\sqrt{1+z_x^2+z_y^2}-\frac{(2z_xz_{xx}+2z_yz_{xy})z_x}{2\sqrt{1+z_x^2+z_y^2}}}{1+z_x+z_y}+\frac{z_{yy}\sqrt{1+z_x^2+z_y^2}-\frac{(2z_xz_{yx}+2z_yz_{yy})z_x}{2\sqrt{1+z_x^2+z_y^2}}}{1+z_x+z_y}=0.$$

At first glance, this seems like a foolish endeavor; thankfully, the simplifications arrive fast and furious. Importantly, we are restricting ourselves to the natural setting of differentiability where $z_{xy}=z_{yx}$. The expression reduces by first multiplying on both sides by $1+z_x^2+z_y^2$, and subsequently by $\sqrt{1+z_x^2+z_y^2}$. We then may write

$$z_{xx}(1+z_x^2+z_y^2)-z_x^2z_{xx}-z_xz_yz_{xy}+z_{yy}(1+z_x^2+z_y^2)-z_xz_yz_{xy}-z_y^2z_{yy}=0$$

$$\therefore \boxed{(1+z_y^2)z_{xx}-2z_xz_yz_{xy}+(1+z_x^2)z_{yy}=0}.$$

Simple inspection shows that the boxed equation found via Method II is equivalent, if not identical, to that found via Method I (Oprea, The Mathematics of Soap Films, 2000).

2.8 Scherk's First Surface

In order to further illuminate the potential power of the minimal surface equation, we will explore here one minimal surface which results from a very simple restriction on the function $s(u,v)$, a surface originally discovered by Heinrich Ferdinand Scherk in 1834. The condition we would like to impose is that of additive separability, meaning $s(u,v)=q(u)+r(v)$. If this is the case, then we can replace $s(u,v)$ in the minimal surface equation; because $q_v=0$ and $r_u=0$, the equation reduces to

$$r_{vv}(1+q_u^2)+q_{uu}(1+r_v^2)=0.$$

For a very brief moment we venture into the realm of solving differential equations with little regard for differential geometry. Note that since q is only dependent on u, $q_u=\frac{\partial q}{\partial u}=\frac{dq}{du}$, and similarly for r_v. It is clear that we can separate the given partial differential equation into two ordinary differential equations in the following, standard way. Letting $\lambda\in\mathbb{R}$ be the constant of separation:

$$r_{vv}(1+q_u^2)=-q_{uu}(1+r_v^2)\implies\frac{1+r_v^2}{r_{vv}}=-\frac{1+q_u^2}{q_{uu}}=\lambda.$$

From here, we first solve $1+r_v^2=\lambda r_{vv}$. To simplify this relation and work only with first order derivatives and below, let $r_v\to\rho$. Then we find that we can solve by direct integration,

$$\int\frac{\lambda d\rho}{1+\rho^2}=\int dv\implies v=\lambda\arctan(\rho)\implies\rho=\tan\left(\frac{v}{\lambda}\right)\implies r(v)=\int\tan\left(\frac{v}{\lambda}\right)dv=-\lambda\log\left[\cos\left(\frac{v}{\lambda}\right)\right].$$

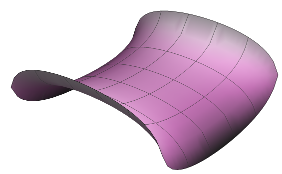
Solving for $q(u)$ is a symmetric calculation leading to

$$q(u)=\lambda\log\left[\cos\left(\frac{u}{\lambda}\right)\right].$$

Thus, returning to our assumed form and using the rules of logarithms, we have

$$s(u,v) = q(u) + r(v) = \lambda \log \left[\frac{\cos(u/\lambda)}{\cos(v/\lambda)} \right].$$

This is the equation for Scherk's first surface in Cartesian coordinates ($u \longleftrightarrow x$, $v \longleftrightarrow y$), one image of which is supplied below, while another can be found in the Appendix showing a greater range of u and v values (Oprea, The Mathematics of Soap Films, 2000).



Scherk's First Surface plotted for the range of values $-\frac{3}{2} \leq u, v \leq \frac{3}{2}$ and $\lambda = 1$.

3 The Geometry of Soap: Films versus Bubbles

Note that the title of this section clearly implies there is a difference between soap *films* and soap *bubbles*, something which becomes apparent when one considers the role of air pressure. In the following section, we will derive the Young-Laplace equation and then use it to restrict our classification of minimal surfaces.

3.1 Young-Laplace Equation

The Young-Laplace equation, developed separately by Thomas Young and Pierre-Simon Laplace between 1805 and 1806, is a relation between the pressure difference across a boundary (e.g. a soap film or bubble separating air into pockets), surface tension, and the shape of the boundary. From this conceptual introduction alone, the relevance of this equation with regards to the study of minimal surfaces is apparent. To derive this equation, we begin with a small, roughly rectangular piece of a bubble with area $A = xy$ (x and y being the side lengths). Adding a uniform pressure p to one side of the surface pushes the membrane outward by a small radial distance δr, doing work on the surface in the process. The work done is

$$W = \sigma \cdot \Delta A$$

where σ represents the surface tension and ΔA is the change in surface area. At the same time, we know that if we let δr be the additional radial distance, then

$$W = F \cdot D = p \cdot A \cdot \delta r = p \cdot xy \cdot \delta r.$$

What we want to do now is to obtain a good approximation for ΔA; suppose that the original surface when viewed along the y-axis is part of a circle of radius R_1, and similarly when viewed along the x-axis is part of a circle of radius R_2. Then the side lengths of the expanded surface can be written as $x + \delta_1 x$ and $y + \delta_2 y$. Consequently, the change in area must be approximately

$$\Delta A = (x + \delta_1 x)(y + \delta_2 y) - xy.$$

But, because x and $x + \delta_1 x$ are parts of circles with the same included angle θ, from the arc length formula $s = R\theta$ we must have

$$\frac{x}{R_1} = \frac{x + \delta_1 x}{R_1 + \delta r} \implies x + \delta_1 x = x \left(\frac{R_1 + \delta r}{R_1} \right) = x \left(1 + \frac{\delta r}{R_1} \right).$$

Effectively the same argument shows that

$$y + \delta_2 y = y\left(1 + \frac{\delta r}{R_2}\right).$$

Now, plugging these into the formula for ΔA we find

$$W = \sigma \cdot xy\left[\frac{\delta r}{R_1} + \frac{\delta r}{R_2} + \frac{(\delta r)^2}{R_1 R_2} + 1\right] - xy$$

Assuming $\delta r << 1$, the $(\delta r)^2$ term is negligible. So, what remains is

$$W = \sigma \cdot xy \cdot \delta r\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

Knowing also that $W = p \cdot xy \cdot \delta r$, we now have

$$\boxed{p = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}.$$

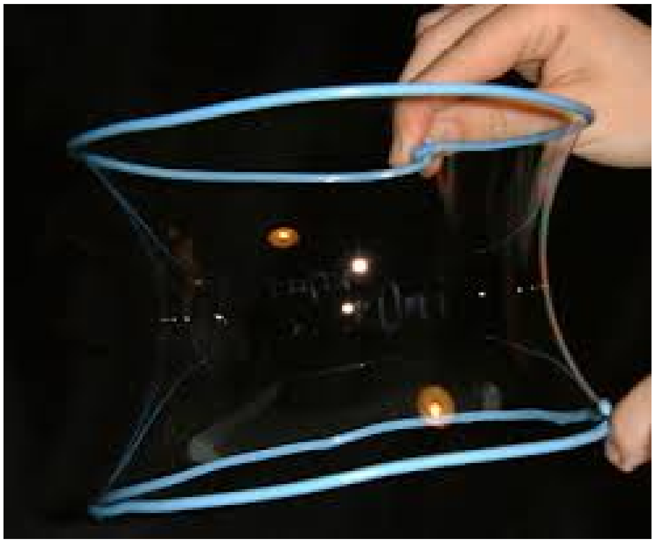
The boxed equation is the Young-Laplace equation. Since by definition x and y represent orthogonal directions, we know that $\frac{1}{R_1}$ and $\frac{1}{R_2}$ satisfy the definition of normal curvatures. Referring back to the first definition of mean curvature we know

$$2H = \frac{1}{R_1} + \frac{1}{R_2}, \quad \therefore p = 2\sigma H.$$

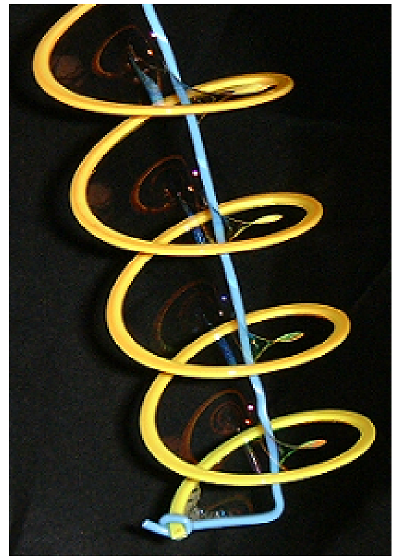
As a result, assuming nonzero surface tension, the surface **S** is minimal for

$$H = 0 \iff p = 0.$$

This provides direct verification of the fact that a soap bubble in an atmosphere such as Earth's is *not* a minimal surface, because the pressure on the inside of the bubble is necessarily greater than on the outside. This is due to the fact that there are two sources of pressure which decrease the size of the bubble-atmospheric pressure and surface tension-while there is only one source of pressure preventing the bubble from shrinking and/or popping, namely the internal air pressure. On the other hand, a soap film necessarily has zero pressure difference across it, provided the same medium is occupying the space on either side, and hence a soap film will take the shape of a minimal surface. The following figures help to illustrate this fact (Oprea, A Quick Trip through Differential Geometry and Complex Variables, 2000).



(a) A soap film in the shape of a catenoid. (Spirit, 2006)

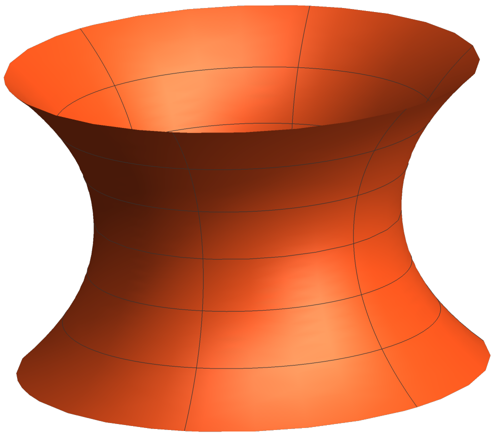


(b) A soap film in the shape of a helicoid (Brasz, 2010).

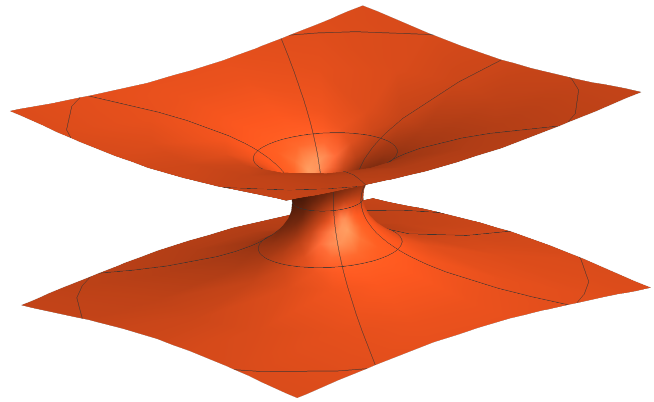
4 The Value of Minimal Surface Theory

Since its beginning roughly 250 years ago with Euler and Lagrange, the study of minimal surfaces has proved to be a fruitful venture. The basic mathematical principles and rudimentary examples outlined throughout this paper, as well as the application to soap films and bubbles, are no more than the tip of a truly immense iceberg. Extrapolating and further developing the same concepts from differential geometry used here will at some point in the (hopefully) near future lead to breakthroughs in general relativistic theory (the two dimensional surface of a so-called wormhole has been related to the catenoid (Rossen Dandolof, 2009)), or topics related to the Positive Mass Theorem. But, the topic that may be of the most intrigue is differential geometry as it relates to high dimensional manifolds, like the Calabi-Yau manifold, in our quest for a way to reconcile the differences between our theories of gravity and quantum mechanics. The study of minimal surfaces is key to the future of physics, and the future is here.

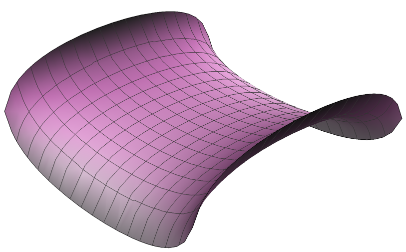
Appendix



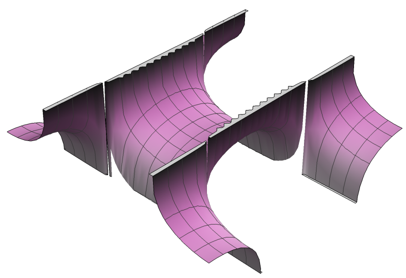
(a) A catenoid plotted on the range $0 \leq v \leq 2\pi$, $-1 \leq u \leq 1$.



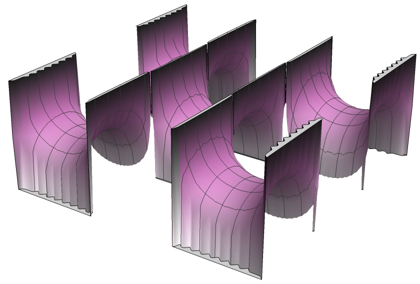
(b) Simply extending the u range of the catenoid plot to $[-4, 4]$ reveals how a wormhole is a minimal surface.



(a) The same image of Scherk's surface as above for reference.



(b) Scherk's surface plotted over the domain $[-\pi, \pi] \times [-\pi, \pi]$. The periodic nature of the surface begins to emerge.



(c) Scherk's surface plotted over the domain $[-\frac{3\pi}{2}, \frac{3\pi}{2}] \times [-\frac{3\pi}{2}, \frac{3\pi}{2}]$; the periodicity is fully realized.