Ricci Flow Notes (based on Topping)

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1. INTRODUCTION

Let M be a smooth, closed manifold (throughout these notes unless otherwise specified) equipped with a family $g = g(t)$ of smooth (in time and on M) Riemannian metrics. We give the following definition.

Definition 1 (**RF**) We say g satisfies the <u>Ricci flow</u> if $\partial_t g_{ij} = -2\mathrm{Ric}_{ij}(g)$ for all $i, j = 1, \ldots n$. Equivalently this may be expressed as $\partial_t g = -2\mathrm{Ric}(g)$.

Having jumped right in and stated the equation governing Ricci flow, it's only natural to ask: why this PDE? We will not be truly convinced from the standpoint of results or proofs for some time, so an intuitive attempt at an explanation goes as follows. For $p \in M$, in normal coordinates centered at p we can write the following expression (*at p only, not in a neighborhood*):

$$-\frac{3}{2}\Delta g_{ij} = \mathrm{Ric}_{ij}.$$

Proof. Fill in gap later. Comes from the fact that in the normal coordinate neighborhood away from p the metric can be written as

$$g_{ij}(x) = -\frac{1}{3}R_{ikjl}(p)x^k x^l + O(|x|^3).$$

□

With this local coordinate expression, we can write that **RF** becomes $\partial_t g = 3\Delta g$. This appears at face value to be a heat equation, and while it is not precisely a heat equation we will see later that it does exhibit properties analogous to those of the ordinary heat equation. Of course this is no accident, Hamilton [Ham82] intentionally chose an evolution equation which would possess some diffusivity properties w.r.t. curvature.

1.1 SPECIAL CASES

1.1.1 EINSTEIN MANIFOLDS

Recall the following definition:

Definition 2 A Riemannian manifold (M, g_0) is <u>Einstein</u> if for some $\lambda \in \mathbb{R}$ we have $\mathrm{Ric}(g_0) = \lambda g_0$.

In this restrictive class of manifolds we find our first simple, yet useful, examples. First though we prove a lemma.

Lemma 1 Ricci curvature is invariant under constant scaling of the metric. That is if (M, g) is a Riemannian manifold and $c \in \mathbb{R}_{>0}$, then

$$\mathrm{Ric}(cg) = \mathrm{Ric}(g).$$

Proof. (Simple method with Christoffel symbols) Let $\widetilde{g} = cg$. In components, $\widetilde{g}_{ij} = cg_{ij}$. Then we can determine the Christoffel symbols $\widetilde{\Gamma}_{ij}^k$ of the Levi-Civita connection associated with $\widetilde{g}$ directly, using the fact that $\widetilde{g}^{ij} = \frac{1}{c}g^{ij}$.

$$\widetilde{\Gamma}_{ij}^k = \frac{1}{2}\widetilde{g}^{kl}(\widetilde{g}_{jl,i} + \widetilde{g}_{il,j} - \widetilde{g}_{ij,l}) = \frac{1}{2c}g^{kl}(cg_{jl,i} + cg_{il,j} - cg_{ij,l}) = \Gamma_{ij}^k.$$

In the above we merely use the fact that the scaling factor can be linearly factored out through derivatives and ultimately cancels with the factor from the inverse. Thus the Christoffel symbols do not change under constant scaling of the metric. Since Ricci curvature may be written solely in terms of Christoffel symbols, the claim follows.

(More general method, [Besse]) From Besse's textbook on Einstein manifolds we cite a theorem which states that if one *conformally* changes the metric g by defining $\widetilde{g} = e^{2f}g$ for some smooth function f on M, then

$$\mathrm{Ric}(\widetilde{g}) = \mathrm{Ric}(g) - (n-2)(\nabla df - (df)^2) + (\Delta f - (n-2)|df|^2)g.$$

In particular if $f = \frac{1}{2}\log c$ is a suitably chosen constant map, all terms involving any kind of derivative(s) of f vanish, leaving us with the desired conclusion. $\square$

Remark: It might seem that from the above proposition it follows that the scalar curvatures $R_g, R_{\widetilde{g}}$ of g and $\widetilde{g}$ respectively are also equal, but be careful with traces/contractions! Observe that

$$R_{\widetilde{g}} = \widetilde{g}^{ij}\mathrm{Ric}_{ij}(\widetilde{g}) = \frac{1}{c}g^{ij}\mathrm{Ric}_{ij}(g) = \frac{1}{c}R_g$$

so the scalar curvature of the new metric differs from the original by $\frac{1}{c}$.

Now we can prove a first little proposition.

Proposition 1 Let (M, g_0) be an Einstein manifold with constant λ. Then the family of metrics

$$g(t) = (1-2\lambda t)g_0$$

defined on the interval $[0,T)$ constitutes a Ricci flow of M satisfying $g(0) = g_0$, where T depends on λ in that we need $1 - 2\lambda t > 0$. If $\lambda > 0$, then $1 - 2\lambda t > 0$ exactly when $t < \frac{1}{2\lambda}$, so the interval of existence is $[0, 1/2\lambda)$; if $\lambda < 0$, then $1 - 2\lambda t > 0$ for all t, so the interval of existence is $[0, \infty)$.

Proof. Just compute here, there's only one subtle point which follows from the lemma (it may not look like it because there's a t there, but t is not a function on M, it's only a scaling factor).

$$\partial_t g(t) = \partial_t((1-2\lambda t)g_0) = -2\lambda g_0 = -2\mathrm{Ric}(g_0) = -2\mathrm{Ric}(g(t)).$$

In order for $g(t)$ to be a Riemannian metric, in particular positive, it is necessary that $1 - 2\lambda t > 0$ so we see $t < \frac{1}{2\lambda} := T$. $\square$

Example: (Round Sphere) A concrete example of the above proposition in action is the sphere (S^n, g_{S^n}). We know that $\mathrm{Ric}(S^n) = (n-1)g_{S^n}$, so the family of metrics

$$g(t) = (1 - 2(n-1)t)g_{S^n}, \qquad t \in [0, 1/2(n-1))$$

is a solution to the initial data problem

$$\begin{cases} \partial_t g(t) = -2\mathrm{Ric}(g(t)), \\ g(0) = g_{S^n}. \end{cases}$$

Note that as $t \to \frac{1}{2(n-1)}$, $Vol_{g(t)}(S^n) \to 0$. This is easily seen by $Vol_{g(t)}(S^n) = \int dvol(t) \leq C|1 - 2(n-1)t|$.

Example (Hyperbolic space) Let $(\mathbb{H}^n, g_0)$ be hyperbolic space with standard metric. Then $\text{Ric}(g_0) = -(n-1)g_0$, so again applying the proposition we see

$$g(t) = (1 + 2(n-1)t)g_0$$

is a solution to the corresponding Ricci flow problem with $g(0) = g_0$. Note here that since the Einstein constant is negative, the solution is defined for all $t \in [0, \infty)$. Moreover, the space expands uniformly as $t \to \infty$ (similar volume calculation to the above).

1.1.2 RICCI SOLITONS

We saw in the case of Einstein manifolds that Ricci flows were self-similar (scaling the metric) and either expanded or contracted via $f(t)g_0$ where $f(t)$ was either strictly increasing or strictly decreasing as a function of t (depending on the sign of the Einstein constant). The next natural thing is to generalize this phenomenon to some extent.

Definition 3 A manifold (M, g_0) is a <u>Ricci soliton</u> if there exists a smooth vector field Y and $\lambda \in \mathbb{R}$ (both independent of t) such that

$$-2\text{Ric}(g_0) = \mathcal{L}_Y g_0 - 2\lambda g_0.$$

One immediate observation and question from this definition:

1) If (M, g_0) were Einstein we would directly have $-2\text{Ric}(g_0) = -2\lambda g_0$, so the above definition is natural at least in the sense that it is a perturbation of the equation defining an Einstein manifold.

2) Why is the term $\mathcal{L}_Y g_0$ the appropriate term to add to the expression, and what does it gain us?

To gain some semblance of understanding, we proceed in the following direction: supposing (M, g_0) is a Ricci soliton, given that it appears similar to an Einstein manifold one would hope there is a solution to the Ricci flow on M. Indeed, given Y and λ as in the definition, define a *positive* real-valued function of time (restricting domain as necessary)

$$\sigma(t) = 1 - 2\lambda t$$

and time dependent vector field

$$X(t) = \frac{1}{\sigma(t)} Y.$$

Since X is a smooth vector field, it generates a family of diffeomorphisms ψ_t (its integral curves). With these quantities defined, we may state the following proposition.

Proposition 2 Let $\widetilde{g}(t)$ be the family of metrics on M defined by

$$\widetilde{g}(t) = \sigma(t)\psi_t^*(g_0).$$

Then $\widetilde{g}$ satisfies the Ricci flow with initial data $\widetilde{g}(0) = g_0$.

The proof will follow from a direct calculation using a brief lemma.

Lemma 2 Let $\sigma(t)$ be a smooth, real-valued function of time, $X(t)$ a time dependent vector field generating family of diffeomorphisms ψ_t. Then $\widetilde{g}(t) = \sigma(t)\psi_t^*(g(t))$, where $g(t)$ is a family of smooth metrics, satisfies

$$\partial_t \widetilde{g} = \sigma'(t)\psi_t^*(g) + \sigma(t)\psi_t^*(\partial_t g) + \sigma(t)\psi_t^*(\mathcal{L}_X g).$$

Proof. From usual product rule, we see right away

$$\partial_t \widetilde{g} = \partial_t \left[\sigma(t)\psi_t^*(g(t))\right] = \sigma'(t)\psi_t^*(g(t)) + \sigma(t)\partial_t \left[\psi_t^*(g(t))\right].$$

It then suffices to show that

$$\partial_t \left[\psi_t^*(g(t))\right] = \psi_t^*(\partial_t g) + \psi_t^*(\mathcal{L}_X g)$$

which follows from the limit definition of the Lie derivative via flows (I'm sweeping this under the rug for now). □

Proof. (Prop 2) In this instance, $g(t) = g_0$ is independent of t, so applying the lemma gives

$$\begin{aligned}\partial_t \widetilde{g} &= \sigma'(t)\psi_t^*(g_0) + \sigma(t)\psi_t^*(\mathcal{L}_X g_0) = -2\lambda\psi_t^*(g_0) + \sigma(t)\psi_t^*(\mathcal{L}_{\frac{1}{\sigma(t)}Y} g_0) \\ &= \psi_t^*(-2\lambda g_0 + \mathcal{L}_Y g_0) = \psi_t^*(-2\mathrm{Ric}(g_0)) = -2\mathrm{Ric}(\psi_t^*(g_0)) \\ &= -2\mathrm{Ric}\left(\frac{1}{\sigma(t)}\widetilde{g}\right) = -2\mathrm{Ric}(\widetilde{g}).\end{aligned}$$

□

Special subclasses of Ricci solitons exist which are more computationally nice to write down.

Definition 4 A Ricci soliton defined by a vector field $Y = \nabla f$ where $f \in C^\infty(M, \mathbb{R})$ is called a gradient Ricci soliton.

Example (Hamilton's Cigar) Define the metric $g = \rho^2(dx^2 + dy^2)$ on $\mathbb{R}^2$. We can compute the Gauss curvature K as

$$K = -\frac{1}{\rho^2}\Delta \ln \rho$$

where $\Delta = \partial_x^2 + \partial_y^2$ is the classical Laplacian. Then if we choose $\rho^2 = \frac{1}{1+x^2+y^2}$,

$$K = \frac{2}{1+x^2+y^2} \implies \mathrm{Ric}(g_0) = \frac{2}{1+x^2+y^2} g_0.$$

Claim: The family of metrics $g(t) = \frac{dx^2+dy^2}{e^{4t}+x^2+y^2}$ satisfies the Ricci flow on (M, g_0).

Proof. (Claim) This is just a matter of taking derivatives, since clearly $g(0) = g_0$:

$$\partial_t g(t) = \frac{-4e^{4t}}{e^{4t}+x^2+y^2}(dx^2+dy^2) = -2\mathrm{Ric}(g(t))$$

since we can compute $K = K(g(t))$ for each t by letting $\rho^2(t) = \frac{1}{e^{4t}+x^2+y^2}$:

$$\begin{aligned}K = -\frac{1}{\rho^2}\Delta \ln \rho &= -(e^{4t}+x^2+y^2)\left[\frac{-e^{4t}-x^2+y^2}{(e^{4t}-x^2+y^2)^2} + \frac{-e^{4t}+x^2-y^2}{(e^{4t}+x^2+y^2)^2}\right] \\ &= \frac{2e^{4t}}{e^{4t}+x^2+y^2}.\end{aligned}$$

□

Remark: In the case of a gradient soliton note that $\mathcal{L}_Y g_0 = \mathcal{L}_{\nabla f} g_0 := 2\text{Hess}_{g_0} f = 2\nabla^2 f$ (definition of Hessian given in Petersen's book), so

$$-\text{Ric}(g_0) = \mathcal{L}_Y g_0 - 2\lambda g_0 \iff \text{Ric}(g_0) + \nabla^2 f = \lambda g_0.$$

2. RIEMANNIAN GEOMETRY: BACKGROUND, NOTATION

Denote by ∇ the Levi-Civita connection on (M, g) unless context indicates it should be the gradient. Recall the notation for vectors X, Y the second covariant derivative

$$\nabla^2_{X,Y} = \nabla_X \nabla_Y - \nabla_Y \nabla_{\nabla_X Y}.$$

We can write for any (p, q) tensor field A that $\nabla^2 A$ is a $(p+2, q)$ tensor defined by the above as

$$\nabla^2_{X,Y} A = \nabla_X \nabla_Y A - \nabla_{\nabla_X Y} A \equiv (\nabla^2 A)(X, Y, \cdots).$$

In particular if $A = f$ is a function, then

$$\nabla^2 f = \nabla df = \text{Hess}(f).$$

From this one can further define the rough or connection Laplacian of a tensor field by tracing over the first two entries of

$$\Delta A = tr_{12} \nabla^2 A.$$

We adopt the notation

$$R(X, Y) = \nabla^2_{Y,X} - \nabla^2_{X,Y},$$
$$Rm(X, Y, W, Z) = g(R(X, Y)W, Z).$$

Then define the curvature operator

$$\mathcal{R} : \wedge^2 TM \to \wedge^2 TM, \qquad g(\mathcal{R}(X \wedge Y), W \wedge Z) = Rm(X, Y, W, Z).$$

Perhaps most confusing, we define $*$-notation as a way of simplifying expressions. Given any two tensor fields A, B, write $A * B$ to mean *any tensor field which is a linear combination of tensor fields, each of which is formed by taking* $A \otimes B$ *and using* g *to raise/lower indices, making any number of contractions, switching any number of components.* This notation is not my favorite since everything from $R^2 = Rm * Rm$ to $R = Rm * 1$ goes through, so I will likely use it sparingly.

Define the divergence operator $\delta : \Gamma(\otimes^k T^*M) \to \Gamma(\otimes^{k-1} T^*M)$ by $\delta(\omega) = -tr_{12} \nabla \omega$. The adjoint of this operator is the covariant derivative ∇, but by restricting δ to $\Gamma(\wedge^k T^*M)$ (sections of space of k forms as opposed to sections of more general $(0, k)$-tensors), the adjoint is the exterior derivative d.

Last but not least, the gravitation tensor $G(T)$ is defined for $T \in \Gamma(\text{Sym}^2 T^*M)$ by

$$G(T) = T - \frac{1}{2}(trT)g$$

which in the familiar case of $T = \text{Ric}$ yields the usual $G = \text{Ric} - \frac{1}{2} Rg$ from Einstein's field equations in general relativity, hence the name.

2.1 EVOLUTION OF CONNECTION, CURVATURE

Now that we have briefly reviewed some Riemannian geometry definitions we can state and prove foundational results concerning the evolution of geometric quantities as $g = g(t)$ varies. We will only assume

$g(t)$ is smooth at first, then add later the assumption that it satisfies the Ricci flow. For notation, let $g(t) \in \Gamma(\mathrm{Sym}^2 T^*M)$ be defined for $t \in (a,b)$ and $\partial_t g = h$.

Proposition 3 (Connection) Let X, Y, Z be time independent vector fields and ∇ the Levi-Civita connection associated with $g(t)$ (rather $\nabla = \nabla(t)$ is a family of connections). Then

$$2g\left(\partial_t \nabla_X Y, Z\right) = \nabla_Y h(X,Z) + \nabla_X h(Y,Z) - \nabla_Z h(X,Y).$$

Proof. There are a couple ways to do this, one being to differentiate the Kozsul formula. Or, letting $\Pi(X,Y) = \partial_t \nabla_X Y$ we use the product rule and the fact that Z is independent of time to see

$$\begin{aligned} \partial_t g(\nabla_X Y, Z) &= h(\nabla_X Y, Z) + g(\Pi(X,Y), Z) \\ \implies g(\Pi(X,Y), Z) = \partial_t[Xg(Y,Z) - g(Y, \nabla_X Z)] - h(\nabla_X Y, Z) &= [Xh(Y,Z) - h(Y, \nabla_X Z) - g(Y, \Pi(X,Z))] - h(\nabla_X Y, Z) \\ &= (\nabla_X h)(Y,Z) - \underbrace{g(\Pi(Z,X), Y)}_{(*)}. \end{aligned}$$

In the second and third equalities we use metric compatibility with the connection. Now repeat this process on $(*)$, then once more on analogous resulting term. Feels very similar to proof of Koszul itself. $\square$