The finite volume method for conservative nonlinear hyperbolic systems: the Saint-Venant equations

mad max

July 28, 2022

Abstract

1 Generalities

This document is heavily inspired by the lecture notes of Lagrée 2022

1.1 Saint-Venant (SV) or the shallow water (SW) equations

Adhémar Jean-Claude Barré de Saint-Venant proposed 150 years ago the following system of nonlinear hyberbolic partial differential equations (PDE's) for a one-dimensional steady fluid motion. They are derived from depth-integrating the Navier-Stokes equations under simplifying assumptions[1] (see Toro 2013, pp. 35). Such system reads as

$$\begin{cases} \partial_t h + \partial_x Q = 0 \\ \partial_t Q + \partial_x \left(\Gamma \frac{Q^2}{h} + g \frac{h^2}{2} \right) = -gh\partial_x z - \frac{\tau}{\rho}, \end{cases} \tag{1.1}$$

with $h(x,t)$ the water height, $Q(x,t) = h(x,t)u(x,t)$ the flow discharge, $u(x,t)$ the depth-averageg velocity, Γ a shape factor (Boussinesq coefficient), τ the basal shear resistance law (given by Q and oriented as $-Q/|Q|$), $z(x,t)$ the (bed) topography and g the gravity. The reader should keep in mind that other formulations exist in the litterature for more specific applications[2], *e.g.,*

[1] One of these is the hydrostatic pressure distribution, *i.e.,* $\partial_z p + \rho g = 0$ (see Castro-Orgaz and Hager 2019, pp. 22). This yields a bottom pressure $p|_z = \rho g h$, *i.e.,* the weight of the water column above, and a linear pressure distribution.

[2] As an example, for unsteady open channel flow (Kerger et al. 2011; Hodges 2019), the system 1.1 is often expressed (after area-integrating of the Navier-Stokes equations) as

$$\begin{cases} \partial_t A + \partial_x Q = 0 \\ \partial_t Q + \partial_x \left(\frac{Q^2}{A} + g I_1 \right) = g A (S_0 - S_f) + g I_2, \end{cases} \tag{1.2}$$

completed by the following

$$I_1(h) = \int_{-h_b}^{h_{fs}} (h-\xi) l(x,\xi) \mathrm{d}\xi \ , \ I_2(h) = \int_{-h_b}^{h_{fs}} (h-\xi) \partial_x l(x,\xi) \mathrm{d}\xi \tag{1.3}$$

numerical solutions to channel routing for hydraulic application, see `RS MINERVE` (https://crealp.ch/rs-minerve/).

1.1.1 Closure of the system

For rivers and streams, one can choose the following expressions for Γ and τ

$$\Gamma = 1 \quad \text{and} \quad \tau = \rho c |Q| \frac{Q}{h^\beta}, \tag{1.5}$$

where $\Gamma = 1$ because a sufficiently smooth bottom is assumed and $\beta = 2$ (the Chezy or Darcy Weissbach law) or $\beta = 7/3$ (the Manning-Strickler law).

However, other expressions can be defined. As an example, for complex flows such as glacier flows (the Glen law gives $n = 1/3$) or mud flows ($0.1 < n < 0.4$), we have

$$\Gamma = \frac{2(1+2n)}{2+3n} \quad \text{and} \quad \tau = c_n \mu_n \left(\frac{Q}{h^2}\right)^n \frac{|Q|}{Q}, \tag{1.6}$$

where μ_n is the friction coefficient. For the snow, the Voellmy law can be adopted and reads

$$\Gamma = 1 \quad \text{and} \quad \tau = \rho g h \left(\mu_0 + \frac{1}{\xi}\frac{Q^2}{h^3}\right) \frac{|Q|}{Q}, \tag{1.7}$$

where, typically, $\mu_0 = 0.2$ and $\xi = 500$ m·s^{-1}.

1.2 Compact conservative form of the system

Let us rewrite the system of hyperbolic PDE's 1.1 as a compact and conservative system of equations[3] in differential form, such as

$$\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = \mathbf{S}(\mathbf{U}), \tag{1.8}$$

where $\mathbf{U}$ is the vector of conservative variables, $\mathbf{F}(\mathbf{U})$ is the flux function vector and $\mathbf{S}(\mathbf{U})$ the source term vector

$$\mathbf{U} = \begin{bmatrix} h \\ Q \end{bmatrix}, \quad \mathbf{F}(\mathbf{U}) = \begin{bmatrix} Q \\ \frac{Q^2}{h} + g\frac{h^2}{2} \end{bmatrix}, \quad \mathbf{S}(\mathbf{U}) = \begin{bmatrix} 0 \\ -gh\partial_x z - \frac{\tau}{\rho} \end{bmatrix}. \tag{1.9}$$

1.2.1 The Jacobian of the flux function

Let us consider $\mathbf{S}(\mathbf{U}) := \mathbf{0}$ in 1.8 and then derive the flux function vector $\mathbf{F}(\mathbf{U})$ using the chain rule. Then, one obtains that $\partial_x \mathbf{F}(\mathbf{U}) = \partial_{\mathbf{U}} \mathbf{F}(\mathbf{U}) \cdot \partial_x \mathbf{U}$, which

where A is the flow area, $Q = Au$ is the flow discharge, S_0 is the river bed slope, S_f is the friction term resulting from the resistance law, h is the water height, l is the free-surface width, h_{fs} is the free surface elevation and h_b is the bottom elevation.

The friction term S_f is assumed to be given by the Manning-Strickler relation, which reads

$$S_f = \frac{Q|Q|}{A^2 K^2 R_h^{4/3}}, \tag{1.4}$$

where K is the Strickler coefficient and R_h is the hydraulic radius.

[3] Assuming $\Gamma = 1$, which is widely accepted in the litterature.

yields to

$$\partial_x \mathbf{F}(\mathbf{U}) = \begin{bmatrix} \partial_h Q & \partial_Q Q \\ \partial_h(\frac{Q^2}{h} + g\frac{h^2}{2}) & \partial_Q(\frac{Q^2}{h} + g\frac{h^2}{2}) \end{bmatrix} \cdot \partial_x \begin{bmatrix} h \\ Q \end{bmatrix}, \tag{1.10}$$

$$\partial_x \mathbf{F}(\mathbf{U}) = \begin{bmatrix} 0 & 1 \\ gh - \frac{Q^2}{h^2} & \frac{2Q}{h} \end{bmatrix} \cdot \partial_x \begin{bmatrix} h \\ Q \end{bmatrix}, \tag{1.11}$$

$$\partial_x \mathbf{F}(\mathbf{U}) = \begin{bmatrix} 0 & 1 \\ gh - u^2 & 2u \end{bmatrix} \cdot \partial_x \begin{bmatrix} h \\ Q \end{bmatrix}, \tag{1.12}$$

$$\partial_x \mathbf{F}(\mathbf{U}) = \mathbf{J}(\mathbf{U}) \partial_x \mathbf{U}. \tag{1.13}$$

Now, the system 1.8 can be rewritten with such alternative definition of the flux vector function, such as

$$\partial_t \mathbf{U} + \mathbf{J}(\mathbf{U}) \partial_x \mathbf{U} = \mathbf{0}, \tag{1.14}$$

The Jacobian[4] of the flux function $\mathbf{J}(\mathbf{U}) \equiv \partial_{\mathbf{U}} \mathbf{F}(\mathbf{U})$ has two eigenvalues, *i.e.,* $\lambda_{\mp}^{(1,2)}$, and two eigenvectors, *i.e.,* $\boldsymbol{\lambda}^{(1,2)}$. They can be found by solving

$$|\mathbf{J}(\mathbf{U}) - \lambda \mathbf{I}| = \det(\mathbf{J}(\mathbf{U}) - \lambda \mathbf{I}) = 0, \tag{1.15}$$

with the solutions $\lambda_{(1,2)}^{\mp} = u \mp c$. These eigenvalues will be further used to reconstruct or estimate numerical fluxes[5] at the interface between two control volumes[6], *i.e.,* where a local Riemann problem exists (we will come back later on this concern in 2.4). The eigenvalues $\lambda_{(1,2)}^{\mp}$ are given by

$$\lambda_{(1)}^{-} = \frac{Q}{h} - \sqrt{gh} \quad \text{and} \quad \lambda_{(2)}^{+} = \frac{Q}{h} + \sqrt{gh}, \tag{1.16}$$

whereas the eigenvector are expressed as

$$\boldsymbol{\lambda}^{(1)} = \begin{bmatrix} -c/h \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\lambda}^{(2)} = \begin{bmatrix} c/h \\ 1 \end{bmatrix}, \tag{1.17}$$

where $c = \sqrt{gh}$ is the wave propagation speed or *celerity*, hence denoted by c. It could be think of as a *gravity* wavespeed (Kerger et al. 2011).

Such system is called *hyperbolic*[7] since the Jacobian $\mathbf{J}(\mathbf{U})$ is i) diagonalizable and, ii) has real eigenvalues (LeVeque 2002, see pp. 31, chap. 2.9 Hyperbolicity of Linear Systems).

[4] In the classical textbooks LeVeque 2002; Toro 2013 about hyperbolic systems, the Jacobian of the flux function vector is often referred as $\mathbf{A} \equiv \mathbf{A}(\mathbf{U})$.

[5] It should not be understood as physical fluxes.

[6] Actually, this is the key point of any finite volume approximation. Estimating or approximating these inter-cell numerical fluxes at the interface is of utmost importance and is a real crux of the method. Particularly, finding an accurate solution to the local Riemann problem is central in every finite volume approximations.

[7] In general, a system "is said to be hyperbolic at a point (x, t) if $\mathbf{A}$ has m real eigenvalues $\lambda_1, ..., \lambda_m$ and a corresponding set of m linearly independent right eigenvectors $\mathbf{K}^{(1)}, ..., \mathbf{K}^{(m)}$. The system is said to be strictly hyperbolic if the eigenvalues λ_i are all distinct" (Toro 2013, see pp. 45, chap. 2.1 Quasi-Linear Equations: Basic Concepts).

2 Numerical resolution: the finite volume approximation

As stated in LeVeque 2002, pp. 64, finite volume methods "are closely related to finite difference methods, and a finite volume method can often be interpreted directly as a finite difference approximation to the differential equation. However, finite volume methods are derived on the basis of the integral form of the conservation law, a starting point that turns out to have many advantages".

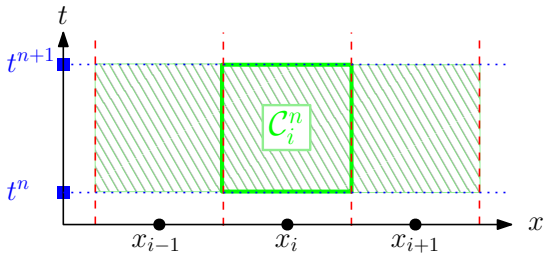

Figure 1 — Typical finite volume discretization in the $x-t$ plane, with a volume control $\mathcal{C}_i$.

In the following, we will see how to express the differential form of the hyperbolic system 1.8 into an integral form of such conservative system.

2.1 Finite volume discretization (space and time)

The first step for any numerical approximation is to discretize the continuous domain (in space and time).

Let us consider a *uniform* discretization of the continuous domain $[x_L, x_R]$ into a regular grid composed of discrete points. These are denoted as

$$x_i = x_L + (i + 1/2)\Delta x \quad \text{for} \quad i = 0, ..., n, \tag{2.1}$$

with $\Delta x = \frac{x_R - x_L}{n+1}$. Let us also define midpoint or edge values

$$x_{i-\frac{1}{2}} = x_i - \Delta x/2 \quad \text{for} \quad i = 0, ..., n+1. \tag{2.2}$$

These values define the *control volumes* or cells or elements such that we have the following definition

$$\mathcal{C}_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}], \tag{2.3}$$

and they are naturally centered at the x_i location. Adding a temporal discretization for a time interval $t \in [t_0, t]$, and, one obtain the following discretization for a given control volume $\mathcal{C}_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [t_n, t_{n+1}]$ (see Figs. 1 & 2).

2.2 Integral form of conservative hyperbolic systems

Starting with the finite volume discretization, *i.e.*, $\mathcal{C}_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, the value $\mathbf{U}_i^n$ will approximate the average value (or the cell average[8]) over the ith interval

[8] The reader should observe that one can choose a piece-wise constant approximation to $\mathbf{U}_i^n$, *i.e.*, $\mathbf{U}_i^n \approx$ cst in the control volume $\mathcal{C}_i$.

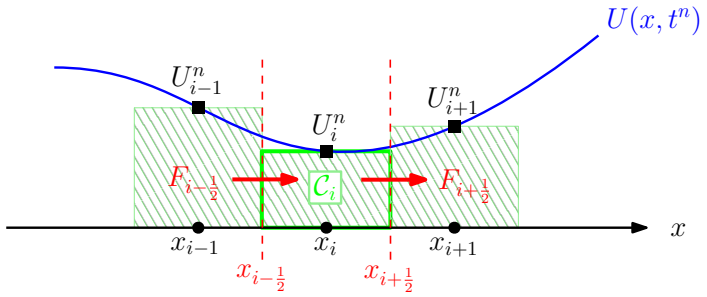

Figure 2 — Finite volume discretization (only spatial discretizazion is shown here), inspired from LAGRÉE 2022. The continuous solution U^n is discretized into a piecewise constant distribution (GODUNOV and BOHACHEVSKY 1959) of data U^n_i at the center of the control volume $\mathcal{C}_i$, where n indicates the temporal discretization and i the spatial discretization.

at a time t_n, such as

$$\mathbf{U}_i \approx \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathbf{U}(x,t)\mathrm{d}x \equiv \frac{1}{\Delta x}\int_{\mathcal{C}_i} \mathbf{U}(x,t)\mathrm{d}x. \tag{2.4}$$

After defining a cell-average description of $\mathbf{U}(x,t)$ over a control volume $\mathcal{C}_i$, let us have a look to the integral form of the conservative system. The integral form of the conservative system 1.8, now reads as

$$\int_{\mathcal{T}}\int_{\mathcal{C}_i} [\partial_t \mathbf{U}(x,t) + \partial_x \mathbf{F}(\mathbf{U}(x,t))]\mathrm{d}x\mathrm{d}t = \int_{\mathcal{T}}\int_{\mathcal{C}_i} \mathbf{S}(\mathbf{U})\mathrm{d}x\mathrm{d}t, \tag{2.5}$$

where $\mathcal{T} = [t^n, t^{n+1}]$ and $\mathcal{C} = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$, with $t^{n+1} = t^n + \Delta t$, which generates, after integration and when considering $\mathbf{S}(\mathbf{U}) := \mathbf{0}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{C}_i} \mathbf{U}(x,t)\mathrm{d}x = \mathbf{F}(\mathbf{U}(x_{i-\frac{1}{2}},t)) - \mathbf{F}(\mathbf{U}(x_{i+\frac{1}{2}},t)). \tag{2.6}$$

Then, integrating in time from t_n to t_{n+1}, rearranging and dividing by Δx yields to a general numerical scheme of the form

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{\Delta x}(\mathbf{F}^n_{i+\frac{1}{2}} - \mathbf{F}^n_{i-\frac{1}{2}}), \tag{2.7}$$

where $\mathbf{F}^n_{i\pm\frac{1}{2}}$ is some approximation of the average numerical fluxes at the inter-cells $x = x_{i\pm\frac{1}{2}}$, as

$$\mathbf{F}^n_{i\pm\frac{1}{2}} \approx \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \mathbf{F}(\mathbf{U}(x_{i\pm\frac{1}{2}},t))\mathrm{d}t. \tag{2.8}$$

The general concern of any finite volume method is to find a good enough approximation of 2.8 and, we will see in the following different approximations of these inter-cell numerical fluxes to solve for the discrete conservative hyperbolic system in integral form.

2.3 Solution strategy (with or without a source term)

To solve numerically for the system of hyperbolic partial differential equations (PDE's, see 1.8) with initial conditions (IC's) of considering a source term, a splitting scheme (or approach) (Toro 2001; Castro-Orgaz and Hager 2019) can be used. The general problem is to determine the conservative variables vector $\mathbf{U}$ at time t^{n+1}, considering the effect of the source term(s), and this reads as

$$\left.\begin{array}{ll} \text{PDE's} & \partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = \mathbf{S}(\mathbf{U}) \\ \text{IC's} & \mathbf{U}(x,t) = \mathbf{U}_i^n \end{array}\right\} \Rightarrow \mathbf{U}_i^{n+1}(x_i, t^{n+1}). \tag{2.9}$$

The splitting approach solves the general problem 2.9 in two consecutive steps, which are given in the following.

(*Step 1*) Let us solve the homogeneous part of 2.9 using an appropriate numerical scheme, *i.e.,* a Godunov-type scheme, such as

$$\left.\begin{array}{ll} \text{PDE's} & \partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = \mathbf{0} \\ \text{IC's} & \mathbf{U}(x,t) = \mathbf{U}_i^n \end{array}\right\} \Rightarrow \mathbf{U}_i^{\text{adv}}(x_i, t^{n+1}), \tag{2.10}$$

where the temporary solution $\mathbf{U}_i^{\text{adv}}(x_i, t^{n+1})$ accounts for the effects of advection. This step can be regarded as a predictor step (or transient) to approximate a solution due to advective transport, outlooking the effects of any source terms. The transient solution due to advection (within a proper conservative numerical scheme) is given by

$$\mathbf{U}_i^{\text{adv}} = \mathbf{U}_i^n - \frac{\Delta t}{\Delta x}(\mathbf{F}_{i+\frac{1}{2}}^n - \mathbf{F}_{i-\frac{1}{2}}^n), \tag{2.11}$$

and, if initially considering no source terms, *i.e.,* $\mathbf{S}(\mathbf{U}) := \mathbf{0}$, then the solution procedure is over and one has $\mathbf{U}_i^{n+1}(x_i, t^{n+1}) := \mathbf{U}_i^{\text{adv}}$.

(*Step 2*) Update (or correct) the transient solution including effects of the source terms, such as

$$\left.\begin{array}{ll} \text{ODE's} & \partial_t \mathbf{U} = \mathbf{S}(\mathbf{U}) \\ \text{IC's} & \mathbf{U}(x,t) = \mathbf{U}_i^{\text{adv}} \end{array}\right\} \Rightarrow \mathbf{U}_i^{n+1}(x_i, t^{n+1}), \tag{2.12}$$

which can be solved with a variety of ODE solvers. This step can be regarded as a correction step, as opposed to the predictor step in the first stage of the solution strategy. A first-order forward Euler scheme is a fair choice, and it reads

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^{\text{adv}} + \Delta t \mathbf{S}(\mathbf{U}), \tag{2.13}$$

but a better approximation is given by

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^{\text{adv}} + \Delta t \mathbf{S}(\mathbf{U}_\mathbf{i}^{\text{adv}}), \tag{2.14}$$

where the source terms take into account the effects of advection over the conservative variable vector.

Other schemes exist, such as first-order backward Euler scheme and the second-order or third-order TVD Runge-Kutta scheme to name a few.

2.4 The Riemann problem

A Riemann problem[9] is a specific initial value problem (IVP) composed of a conservation equation together with piece-wise constant data reconstruction (Godunov and Bohachevsky 1959), which has a single discontinuity in the domain being considered. It is a widely encountered problem in computational fluid dynamics, especially when a piece-wise constant distribution is assumed (*e.g.,* the Godunov-type methods, see Guinot 2012; Toro 2013).

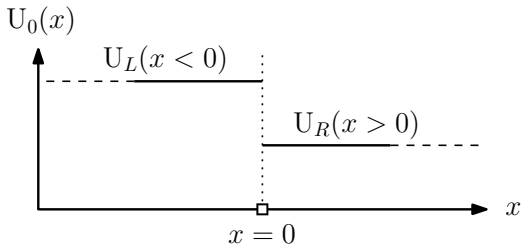

Figure 3 — Sketch of the initial data (constant distribution with a single discontinuity) for the Riemann problem. The initial data consist of two constant states $\mathrm{U}_{L,R}$ separated by a discontinuity located at $x=0$.

The local Riemann problem for the Saint-Venant equations under a piece-wise constant distribution (see 2.5.1 and Fig. 2) is formally defined as an initial-value problem and reads

$$\left.\begin{aligned} &\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = \mathbf{0}, \\ &\mathbf{U}(x,0) = \mathbf{U}_0(x) = \begin{cases} \mathbf{U}_L & \text{for } x<0, \\ \mathbf{U}_R & \text{for } x>0, \end{cases} \end{aligned}\right\} \tag{2.15}$$

where the discontinuity initially located at $x=0$ at time $t=0$ is moving throughout time.

A Riemann solver aims at finding the solution (approximate or exact) to the local Riemann problem, which can be further used to determine numerical fluxes. The solution to the local Riemann problem $\mathbf{U}(x,t)$ subject to 2.15 is to be determined from

$$\partial_t \mathbf{U} + \partial_x \mathbf{F} = \mathbf{0}. \tag{2.16}$$

We will now refer to the Riemann problem as $\mathcal{R}(\mathbf{U}_L, \mathbf{U}_R)$. The solution can be exactly calculated (see Castro-Orgaz and Hager 2019; Toro 2013; LeVeque 2002), but we will see, in the following, numerical techniques to approximate the solution to the Riemann problem $\mathcal{R}(\mathbf{U}_L, \mathbf{U}_R)$.

[9] For a clear discussion of a Riemann problem, one can refer to the classical textbook of Toro 2013. In particular, a complete description is given in Toro 2013, see pp. 49, chap. 2.2 The Linear Advection Equation

2.5 Godunov-type methods for the local Riemann problems

2.5.1 The Godunov method

The method was first proposed by Godunov and Bohachevsky 1959 and it is a first-order scheme (Toro 2013). The numerical fluxes need to be calculated at the interface between two control volumes $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ to satisfy a conservative numerical scheme. Exploiting the definition of cell average (see 2.4) and constucting a piece-wise constant distribution, a local Riemann problem arises at each interface between adjacent cells, resulting in a single discontinuity. Each local Riemann problem $\mathcal{R}(\mathbf{U}_i, \mathbf{U}_{i+1})$ is defined as

$$\left.\begin{aligned} &\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = \mathbf{0}, \\ &\mathbf{U}_0(x) = \begin{cases} \mathbf{U}_L & \text{for } \; x_{i-0} < x < x_{i+\frac{1}{2}}, \\ \mathbf{U}_R & \text{for } \; x_{i+\frac{1}{2}} < x < x_{i+1}, \end{cases} \end{aligned}\right\} \tag{2.17}$$

and, the inter-cell numerical fluxes $\mathbf{F}_{i+\frac{1}{2}}$ are computed using solutions of local Riemann problems. One generally expresses the *Godunov* inter-cell numerical fluxes as

$$\mathbf{F}^{\text{god}}_{i\pm\frac{1}{2}} = \mathbf{F}(\mathbf{U}_{i\pm\frac{1}{2}}(0)), \tag{2.18}$$

where $\mathbf{U}_{i\pm\frac{1}{2}}(0)$ denotes the solution of the Riemann problem $\mathcal{R}(\mathbf{U}_i, \mathbf{U}_{i+1})$.

The implementation of the Godunov's scheme is given in the following procedure:

- start at t^n with known cell-averaged values $\mathbf{U}_i$ over the computational domain,
- set the boundary conditions,
- solve (exact or approximate) $\mathcal{R}(\mathbf{U}_i, \mathbf{U}_{i+1})$ at inter-cell location $i + \frac{1}{2}$, and for any control volume $\mathcal{C}_i$,
- update $\mathbf{U}_i^n$ to $\mathbf{U}_i^{n+1}$ using 2.7,
- repeat.

2.5.2 The HLL (Harten, Lax, van Leer) approximate Riemann solver

An approximate Riemann solver was proposed by Harten, Lax and van Leer in 1983 and was coined HLL[10] solver (Harten et al. 1983) and it requires estimates for the fastest signal velocities emerging at the local Riemann problem, *i.e.,* the discontinuity. This resulted in a *two-wave model*.

Formally, the approximate numerical flux is given by the following (Toro 2001; Huang et al. 2013; Toro 2013; Toro 2019):

$$\mathbf{F}_{i+\frac{1}{2}} = \begin{cases} \mathbf{F}_L & \text{if} \quad s_L \geq 0, \\ \mathbf{F}^{\text{hll}} = \frac{s_R \mathbf{F}_L - s_L \mathbf{F}_R + s_R s_L (\mathbf{U}_R - \mathbf{U}_L)}{s_R - s_L} & \text{if} \quad s_L \leq 0 \leq s_R, \\ \mathbf{F}_R & \text{if} \quad s_R \leq 0, \end{cases} \tag{2.19}$$

[10] A more accurate method is the HLLC solver, proposed by Toro and co-workers in 1992, see Toro 2001; Toro 2019. It considers a *three-wave model* and better resolve intermediate waves at the discontinuity.

where a reliable estimate for the wave speeds $s_{L,R}$ at the discontinuity (considering wet-dry transition) is needed, *i.e.,*

$$s_L = \begin{cases} u_R - 2a_R & \text{if} \quad h_L = 0, \\ \min(u_L - a_L, u_\star - a_\star) & \text{if} \quad h_L > 0, \end{cases} \tag{2.20}$$

and,

$$s_R = \begin{cases} u_L + 2a_L & \text{if} \quad h_R = 0, \\ \max(u_R + a_R, u_\star - a_\star) & \text{if} \quad h_R > 0, \end{cases} \tag{2.21}$$

where

$$q_{L,R} = \begin{cases} \sqrt{\frac{1}{2}\left(\frac{(h_\star + h_{L,R})h_\star}{h_{L,R}^2}\right)} = 0 & \text{if} \quad h_\star > h_{L,R}, \\ 1 & \text{if} \quad h_\star \leq h_{L,R}, \end{cases} \tag{2.22}$$

with

$$h_\star = \frac{1}{g}\left(\frac{1}{2}(a_L + a_R) + \frac{1}{4}(u_L - u_R)\right)^2, \tag{2.23}$$

$$u_\star = \frac{1}{2}(u_L + u_R) + (a_L - a_R), \tag{2.24}$$

$$a_{L,R,\star} = \sqrt{g h_{L,R,\star}}. \tag{2.25}$$

2.5.3 The Rusanov flux

Another estimate for wave speed velocities is the following:

$$s_L = -s^+ \quad , \quad s_R = s^+, \tag{2.26}$$

where the estimate $s^+ = \max(|u_L| + a_L, |u_R| + a_R)$. If s_L, R are substituted in 2.19 following this maximum wave speed estimate, one obtain the Rusanov flux, *i.e.,*

$$\mathbf{F}^{\text{Rus}}_{i\pm\frac{1}{2}} = \frac{1}{2}(\mathbf{F}_L + \mathbf{F}_R) - \frac{1}{2}s^+(\mathbf{U}_R - \mathbf{U}_L). \tag{2.27}$$

If one selects $s^+ = \frac{\Delta x}{\Delta t}$, this results in the Lax-Friedrich flux

$$\mathbf{F}^{\text{LF}}_{i\pm\frac{1}{2}} = \frac{1}{2}(\mathbf{F}_L + \mathbf{F}_R) - \frac{\Delta x}{2\Delta t}(\mathbf{U}_R - \mathbf{U}_L). \tag{2.28}$$

2.5.4 The Courant Friedrich Levy condition (CFL)

To ensure numerical stability of the solution, the CFL condition requires that

$$\max(|\lambda_i|)\frac{\Delta t}{\Delta x} \leq 1, \tag{2.29}$$

and if this condition is satisfied, no characteristics waves will travel across discontinuities of neighbouring local Riemann problems. As a result, local solutions will not be influenced by waves travelling across the domain.

3 Channel routing in RS MINERVE

3.1 Saint Venant type

3.2 Muskingum-Cunge type

3.3 Kinematic wave type

References

Lagrée, P.-Y. (Jan. 24, 2022). *Résolution numérique des équations de Saint-Venant,mise en oeuvre en volumes finis par un solveur de Riemann bien balancé*. URL: http://www.lmm.jussieu.fr/~lagree/COURS/MFEnv/code_C_saintvenant.pdf.

Castro-Orgaz, O. and W.H. Hager (2019). *Shallow Water Hydraulics*. Springer International Publishing. ISBN: 9783030130732. DOI: 10.1007/978-3-030-13073-2. URL: https://link.springer.com/book/10.1007/978-3-030-13073-2.

Hodges, B. R. (2019). "Conservative finite-volume forms of the Saint-Venant equations for hydrology and urban drainage". In: *Hydrology and Earth System Sciences* 23.3, pp. 1281–1304. DOI: 10.5194/hess-23-1281-2019. URL: https://hess.copernicus.org/articles/23/1281/2019/.

Toro, E. F. (2019). "The HLLC Riemann solver". In: *Shock Waves* 29.8, pp. 1065–1082. ISSN: 1432-2153. DOI: 10.1007/s00193-019-00912-4. URL: https://doi.org/10.1007/s00193-019-00912-4.

Huang, Yuxin et al. (2013). "Well-Balanced Finite Volume Scheme for Shallow Water Flooding and Drying Over Arbitrary Topography". In: *Engineering Applications of Computational Fluid Mechanics* 7.1, pp. 40–54. DOI: 10.1080/19942060.2013.11015452. eprint: https://doi.org/10.1080/19942060.2013.11015452. URL: https://doi.org/10.1080/19942060.2013.11015452.

Toro, Eleuterio F (2013). *Riemann solvers and numerical methods for fluid dynamics: a practical introduction*. Springer Science & Business Media. DOI: 10.1007/b79761.

Guinot, Vincent (2012). *Wave propagation in fluids: models and numerical techniques*. John Wiley & Sons.

Kerger, F. et al. (2011). "An exact Riemann solver and a Godunov scheme for simulating highly transient mixed flows". In: *Journal of Computational and Applied Mathematics* 235.8, pp. 2030–2040. ISSN: 0377-0427. DOI: https://doi.org/10.1016/j.cam.2010.09.026. URL: https://www.sciencedirect.com/science/article/pii/S0377042710005558.

LeVeque, Randall J. (2002). *Finite Volume Methods for Hyperbolic Problems*. Cambridge Texts in Applied Mathematics. Cambridge University Press. DOI: 10.1017/CBO9780511791253.

Toro, Eleuterio (2001). *Shock Capturing Methods for Free Surface Shallow Flows*. Wiley and Sons. ISBN: 978-0-471-98766-6.

Harten, Amiram et al. (1983). "On Upstream Differencing and Godunov-Type Schemes for Hyperbolic Conservation Laws". In: *SIAM Review* 25.1, pp. 35–61. DOI: 10.1137/1025002. eprint: https://doi.org/10.1137/1025002. URL: https://doi.org/10.1137/1025002.

Godunov, Sergei K. and I. Bohachevsky (1959). "Finite difference method for numerical computation of discontinuous solutions of the equations of fluid dynamics". In: *Matematičeskij sbornik* 47(89).3, pp. 271–306. URL: https://hal.archives-ouvertes.fr/hal-01620642.