Journal of Hydraulic Research
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tjhr20

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Eleuterio F. Toro [a] & Pilar Garcia-Navarro [b]

[a] Laboratory of Applied Mathematics, Department of Civil and Environmental Engineering, University of Trento, Trento, Italy E-mail:

[b] Fluid Mechanics, CPS, University of Zaragoza, Spain E-mail:

Published online: 26 Apr 2010.

Journal of Hydraulic Research Vol. 45, No. 6 (2007), pp. 736–751

Godunov-type methods for free-surface shallow flows: A review

Méthodes de type Godunov pour les écoulements peu profonds à surface libre: Une revue

ELEUTERIO F. TORO, *Laboratory of Applied Mathematics, Department of Civil and Environmental Engineering, University of Trento, Trento, Italy. E-mail: toro@ing.unitn.it; http://www.ing.unitn.it/toro*

PILAR GARCIA-NAVARRO, *Fluid Mechanics, CPS, University of Zaragoza, Spain. E-mail: pigar@unizar.es*

ABSTRACT
This review paper concerns the application of numerical methods of the Godunov type to the computation of approximate solutions to free-surface gravity flows modelled under a shallow-water type assumption. In the absence of dissipative processes the resulting governing equations are, with rare exceptions, of hyperbolic type. This mathematical property has, in the main, been responsible for the transfer of the Godunov-type numerical methodology, initially developed for the compressible Euler equations of gas dynamics in the aerospace community, to hydraulics and related areas of application. Godunov methods offer distinctive advantages over other methods. For example, they give correct representation of discontinuous waves (bores); this means the correct propagation speed (the methods are conservative), sharp definition of transitions and absence of unphysical oscillations in the vicinity of the wave. Future trends include (i) the use of these methods to deal with physically more complete models without the shallow water assumption and (ii) implementation of very-high order versions of these methods.

RÉSUMÉ
Cet exposé synoptique concerne l'application des méthodes numériques de type Godunov au calcul de solutions approchées des écoulements gravitaires à surface libre modélisés avec l'hypothèse de faible profondeur. En l'absence de processus dissipatifs, les équations qui régissent ces écoulements sont, à de rares exceptions près, de type hyperbolique. Cette propriété mathématique, a été principalement responsable de la transposition à l'hydraulique et aux domaines annexes, de la méthodologie numérique de type Godunov, initialement développée dans la communauté aérospatiale pour les équations compressibles d'Euler de la dynamique des gaz. Les méthodes de Godunov offrent des avantages distinctifs par rapport à d'autres méthodes. Par exemple, ils donnent une représentation correcte des ondes de discontinuité (ressauts); ceci signifie une vitesse correcte de propagation (les méthodes sont conservatives), une définition précise des transitions et l'absence d'oscillations parasites à proximité de l'onde. Les futures tendances incluent (i) l'utilisation de ces méthodes pour traiter des modèles physiquement plus complets sans l'hypothèse de faible profondeur et (ii) leur développement en versions d'ordres très élevés.

Keywords: Shallow water flows, hyperbolic systems, shock waves or bores, Godunov's method, Riemann solvers.

1 Introduction

Computational hydraulics is a broad branch of Computational Fluid Dynamics (CFD) whose recent advances are becoming difficult to follow and virtually impossible to review comprehensibly. Within this broad subject let us move into the area of free surface flows of interest for instance in channels, rivers, estuaries and coastal areas. Much effort has been recently devoted to the development of numerical techniques for free surface flows, of which, those concerned with the resolution of unsteady shallow water flow, have been strongly influenced by the upwind philosophy, initially introduced in gas dynamics. There has been much research into the efficient solution of homogeneous systems of conservation laws. The main focus has been put on the accurate representation of large spatial gradients and of discontinuities, such as shock waves in gases or hydraulic jumps in shallow flows. These methods are specially suited to advection dominated problems and their implementation is not straightforward when source terms are relevant, for instance, source terms to account for bed variations. It has become accepted that the discretization of source terms is as important as that of advection terms.

As numerical models become more complex and the areas of application of these methods grow, it has become important that other aspects of the discretization be given due attention.

This is certainly true in the field of computational hydraulics where the modelling can be dominated by the effects not only of bed slope and roughness source terms. The shallow water, or St. Venant equations, are accepted for many practical applications as proper models for unsteady flow of water in one and two space

Revision received September 8, 2006/Open for discussion until June 30, 2008.

dimensions. The equations express the physical conservation principles of mass and momentum (Abbott, 1992; Holly *et al.*, 1980).

Many numerical techniques have been applied to the simulation of such kind of flows. Amongst them, there exist methods developed to deal with the Euler equations that are able to cope with complex systems of discontinuities and shock waves (Hirsch, 1990; Roe, 1989; Toro, 1997). Because of the non-linearity of the hyperbolic shallow water equations, their solution may become discontinuous, even if the initial conditions are smooth. These phenomena reflect physical processes such as hydraulic jumps and surges. The numerical technique adopted determines the quality of the results. For instance, it is well known that linear second order schemes show oscillatory behaviour near discontinuities, or simply large spatial gradients; this is recognized theoretically by Godunov's theorem (Godunov, 1959). Early applications of Godunov-type schemes to channel flows have been reported to be successful (Alcrudo and García-Navarro, 1992; García-Navarro and Alcrudo, 1992; Glaister, 1992; Toro, 1992). However, the application of these schemes to river flow and complex geometries is not so common in the literature. The presence of extreme slopes, high roughness and strong changes in the irregular geometry represent a great difficulty that can lead to appreciable numerical errors, presumably arising from the source terms of the equations.

The rest of this paper is organized as follows: In section 2 we give a broad overview of Godunov methods, with a discussion on all the main aspects of the methods. Then, in the remaining part of the paper we discuss in more detail some of these aspects. The paper concludes with a short summary and a brief discussion on future trends.

2 Overview of Godunov methods

In this section, we present an informal review of Godunov methods for the computation of approximate solutions to a class of physical problems characterized by a free-surface under gravity and assumed to be of the shallow-water type. Under the further assumption that dissipative processes are negligible the resulting non-linear first order partial differential equations are of hyperbolic type; a possible exception arises in the so-called multi-layer models, in which the equations can be of mixed elliptic-hyperbolic type. Within the class of problems of interest here the physical situation of water flows, on fixed or movable beds, is probably the one that attracts the widest attention amongst engineers and environmental scientists. However it is worth mentioning that shallow-water type models are also used as simple models for atmospheric flows, for the study of heavy gas dispersion, snow avalanches, debris flows, and others. At this point we refer the reader to (Toro, 2001), which contains a wealth of information about Godunov methods in general and also a number of specific applications to shallow water type flows.

The approach taken in this section is informal and aims at identifying a number of characterizing features of Godunov-type methods. The subsequent sections will discuss in more detail some of these features.

2.1 *The finite volume framework*

For the purpose of this section we consider a generic $m \times m$ system of hyperbolic balance laws

$$\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = \mathbf{S}(\mathbf{U}) \tag{1}$$

with $\mathbf{U}(x, t)$ the vector of conserved variable, $\mathbf{F}(\mathbf{U})$ the vector of fluxes and $\mathbf{S}(\mathbf{U})$ the vector of source terms. It is understood that the vector of source terms does not contain derivatives of the unknown vector $\mathbf{U}$. The m components of $\mathbf{U}$, $\mathbf{F}$ and $\mathbf{S}$ are respectively denoted by u_i, f_i, s_i, with $i = 1, \ldots, m$. The system is said to be hyperbolic if the Jacobian matrix of the system

$$\mathbf{A}(\mathbf{U}) = \frac{\partial \mathbf{F}}{\partial \mathbf{U}} \tag{2}$$

has m real eigenvalues $\lambda_i(\mathbf{U})$ and a corresponding set of m linearly independent right eigenvectors $\mathbf{R}_i(\mathbf{U})$. Recall that solutions of non-linear hyperbolic systems, such as shallow water type equations, admit discontinuous solutions. These can be developed by the non-linearity of the equations, even when the initial conditions are smooth. Discontinuities are generally called *shocks*, independently of the particular field of application. In shallow water problems one also uses the terminology *bore*, for a shock. One is then faced with the problem of solving numerically equations that will admit shocks.

The mathematical property of hyperbolicity of the shallow water type equations has, in the main, been responsible for the transfer of numerical methodology, initially developed in the aerospace community, to hydraulics and related areas in geophysics. Godunov's method (Godunov, 1959) was first formulated for the solution of the compressible Euler equations of gas dynamics. Two distinctive features of the Godunov method are (i) the use of the conservative form of the equations to produce a relation between integral averages of conserved variables and intercell fluxes and (ii) the use of wave propagation information, *upwinding*, into the discretization scheme to compute intercell fluxes and thus produce a numerical scheme.

Let us consider a control volume

$$D = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [t^n, t^{n+1}]$$

of dimensions

$$\Delta x = x_{i-\frac{1}{2}} - x_{i+\frac{1}{2}}, \quad \Delta t = t^{n+1} - t^n.$$

Integration of the governing equations (1) in space and time on this control volume yields the exact formula

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{\Delta x}[\mathbf{F}_{i+\frac{1}{2}} - \mathbf{F}_{i-\frac{1}{2}}] + \Delta t \mathbf{S}_i, \tag{3}$$

where

$$\left.\begin{aligned} \mathbf{U}_i^n &= \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathbf{U}(x, t^n)\mathrm{d}x \\ \mathbf{F}_{i+\frac{1}{2}} &= \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} \mathbf{F}(\mathbf{U}(x_{i+\frac{1}{2}}, t))\mathrm{d}t \\ \mathbf{S}_i &= \frac{1}{\Delta x \Delta t}\int_{t^n}^{t^{n+1}}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathbf{S}(\mathbf{U}(x, t))\mathrm{d}x\,\mathrm{d}t \end{aligned}\right\} \tag{4}$$

A numerical scheme is determined from formula (3) by defining approximations to the flux and source integrals in Eq. (4), leading to a *numerical flux* and a *numerical source*, which we still denote by $\mathbf{F}_{i+\frac{1}{2}}$ and $\mathbf{S}_i$ respectively.

In the original formulation of the Godunov method the wave propagation information was furnished via local exact solutions of the governing equations subject to special initial conditions consisting of two constant states separated by a discontinuity. This particular initial-value problem is called the Riemann problem, which is a generalization of the shock tube problem in gas dynamics and of the dam-break problem in shallow water flows. Formally, the relevant Riemann problem is defined as

$$\left.\begin{aligned} \partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = 0 \\ \mathbf{U}(x, 0) = \begin{cases} \mathbf{U}_i^n & \text{if } x < 0 \\ \mathbf{U}_{i+1}^n & \text{if } x > 0 \end{cases} \end{aligned}\right\} \tag{5}$$

where $\mathbf{U}_i^n$ and $\mathbf{U}_{i+1}^n$ are two constant vectors. The exact, similarity solution of this problem is denoted by $\mathbf{U}_{i+\frac{1}{2}}(x/t)$. The sought Godunov numerical flux $\mathbf{F}_{i+\frac{1}{2}}$ may be determined by direct computation of the time integral average of the flux function in Eq. (4) evaluated at the solution of the Riemann problem (5) at the interface position $x = 0$. Given that $\mathbf{U}_{i+\frac{1}{2}}(x/t)$ is constant along $x/t = 0$, exact evaluation of the integral gives the Godunov flux as

$$\mathbf{F}_{i+\frac{1}{2}}^{\text{Godunov}} = \mathbf{F}(\mathbf{U}_{i+\frac{1}{2}}(0)). \tag{6}$$

At this point we emphasize that the Godunov discretization approach uses, locally, solutions of the conservation laws. The wave propagation information comes through the local Riemann problem solutions. Up to this point nothing has been said about the source term. The relevant Riemann problem is stated and solved for the homogeneous equations, no source terms being considered. Over the years, special approaches have been developed to account for the presence of source terms.

2.2 *The Riemann problem and Riemann solvers*

For a number of hyperbolic systems, including the shallow water equations, it is possible to find the exact solution of the Riemann problem. However in most practical situations nowadays one resorts to approximate solutions, either for the state $\mathbf{U}_{i+\frac{1}{2}}$ or directly for the flux $\mathbf{F}_{i+\frac{1}{2}}^{\text{Godunov}}$. Exact and approximate Riemann solvers for shallow water flows are given in Toro (2001); see also Toro (1999).

We warn the reader that great care must be exercised when selecting a so-called Riemann solver. First, we emphasize that for some systems of the shallow water type the available exact Riemann solvers are only about 25% more expensive than a typical approximate solver, and therefore the search for cheaper alternatives is not always justified. Small savings in the Riemann solver become insignificant when dealing with realistic problems, for which the solution methods include, amongst other costly tasks, grid generation and reconstruction procedures to obtain higher accuracy.

In search of simplicity and efficiency the user is often driven to the use of linearized Riemann solvers. However, these are prone to a number of shortcomings. First, in the presence of transcritical flow, or sonic flow, one must incorporate a so-called *entropy fix* in order to avoid the computation of unphysical solutions. Then, in the vicinity of very shallow flows, that may come about for example from the action of strong rarefaction waves, linearized Riemann solvers will compute negative water depths, leading to a failure of the methods. Moreover, in the presence of strong wave interaction, linearized Riemann solvers generally lack robustness.

Therefore, the use of non-linear Riemann solvers is strongly recommended, noting nonetheless that for many practical applications the use of sophisticated linearized Riemann solvers, such as Roe's Riemann solver (Roe, 1981), leads to satisfactory schemes.

In searching for non-linear solvers that avoid the listed shortcomings of linear solvers there are other potential difficulties. Sought simplifications can be obtained by reducing the number of wave families present in the structure of the exact solution of the Riemann problem. For example, a popular approach is that proposed by Harten, *et al.* (1983), today known as the Harten, Lax and van Leer (HLL) approach. An HLL Riemann solver assumes a simplified wave structure of the solution, admitting only two wave families. This structure is only correct for a 1D system with two equations. For a larger system, as for example, the conventional 1D shallow water equations along with a third equation to model the concentration of some species, the HLL approach results in a local Riemann solution that ignores the intermediate wave associated with the concentration variable. Numerically, the consequence of the HLL simplification is excessive numerical dissipation for the concentration wave.

There is a known way of correcting this shortcoming of the HLL approach. It is possible to restore the missing wave in the structure of the solution of the Riemann problem, leading to the HLLC Riemann solver (Toro and Chakraborty, 1994; Toro *et al.*, 1992, 1994). See Fraccarollo and Toro (1993, 1995) for applications of HLLC to the shallow water equations in two space dimensions. We use the terminology *complete Riemann solvers* for those solvers (linear or non-linear) that retain the correct number of wave families in their solution structure. For practical calculations we recommend the use of non-linear and complete Riemann solvers. For general background on Riemann solvers see for instance the textbooks (Godlewski and Raviart, 1996; Toro, 1997, 1999), and specifically for shallow water flows see Toro (2001).

2.3 *Conservative and non-conservative schemes*

An important practical, and theoretical, issue is conservation. First, at the level of the governing equations, these can usually be expressed in *conservation or divergence form* or in *non-conservative form*. Note that even when the vector of unknowns is the vector of physically conserved variables the equations are not necessarily written in conservation form. On the other hand, for the shallow water equations, it is possible to write a system in conservation or divergence form using the depth and depth averaged flow velocity as the unknowns. This *conservative* form of the equations is physically incorrect, as they state conservation of mass (correct) and velocity (incorrect). These equations are only valid for smooth flows; in the presence of a shock the corresponding jump conditions give a shock with the wrong speed. See section 3.9, chapter 9 of Toro (2001) for details.

On the the other hand, there are practical physical situations for which the governing equations do not possess a conservative form, or divergence form. For such systems discontinuous solutions determined through Rankine-Hugoniot jump conditions remain undefined. Then, at the numerical level, one cannot construct a conservative numerical method if the equations do not have a conservative form. One can still construct upwind schemes for solving the equations in non-conservative form. Schemes of this kind are given in Toro (1998). Non-upwind (or centred) schemes for solving equations in non-conservative form are presented in Toro and Siviglia (2003). Non-conservative methods are known for giving incorrect wave propagation speeds. For shallow water flows this fact is shown empirically in Toro and Siviglia (2003).

At the theoretical level, Lax and Wendroff (1960) proved that conservative methods that are convergent converge to the weak solution of the conservation laws (with correct wave speeds). Moreover, Hou and LeFloch (1994) proved that a convergent non-conservative method, in the presence of a shock wave, converges to the incorrect solution. The practical experience and the theoretical results available indicate the importance of the issue of conservation.

Recent advances regarding non-conservative systems and numerical methods have been reported by Pares and collaborators (Castro *et al.*, 2006a, b; Pares, 2006).

2.4 *Accuracy*

The original Godunov upwind method is first order accurate in space and time. The method is also *monotone*; this is a fundamental property, only applicable to the scalar case, used to measure the quality of a numerical scheme. Within the family of first-order schemes that are also monotone, Godunov's scheme is the best, it has the smallest truncation error. The reader is warned against the belief that if a method is first order accurate then it is monotone.

The accuracy of the Godunov scheme has been increased from first order to second order following the seminal works of Kolgan (1972), van Leer (1973), Roe (1983) and others. See also Colella (1985), Colella and Woodward (1984), Sweby (1984) and Toro (1989). The step from first to second order is non trivial. As is well known in practice, common second-order schemes, such as the Lax–Wendroff scheme (Lax and Wendroff, (1960)), are oscillatory; they produce unphysical oscillations in the vicinity of large spatial gradients, which is sometimes known as the Gibbs phenomenon. Theoretically this phenomenon is embodied in Godunov's theorem (1959) which states that linear schemes of order greater than one are oscillatory. Therefore, a necessary condition that non-oscillatory schemes of second or higher order must satisfy is non-linearity, even when applied to linear problems.

A class of non-linear second-order Godunov type schemes that have been very successful are the so called Total Variation Diminishing (TVD) methods. These methods are able to capture large gradients of the solution, or even discontinuities, without (or much reduced) spurious oscillations. These methods are also called high-resolution methods (Harten, 1983). The application of these modern versions of Godunov method to solve practical problems is gaining increasing popularity in many fields of applications, including shallow water type flows (Toro, 2001). Following these developments it has also been possible to modify conventional (oscillatory) schemes so as to satisfy a TVD criterion (for the scalar case). A prominent example is the so-called TVD Mac Cormack scheme, which was applied to the shallow water equations by Garcia-Navarro and Alcrudo (1992).

A discussion on schemes of accuracy greater than two is given in section 2.8.

2.5 *Source terms*

Practical problems in shallow water flows include source terms, that is, non-differential terms that are functions of the unknowns of the problem. These source terms account for bottom variation, wind forces, Coriolis forces and many other physical effects. For a number of years it has been recognized that the treatment of the non-differential terms in the equations can be as challenging as that of treating differential terms. Roe (1986) advanced the idea that these terms could also be discretized following the Godunov philosophy, that is using wave propagation information, or upwinding. Preliminary numerical results on upwinding source terms following Roe's idea were reported by Toro for the Euler equations (1986) and by Glaister for the shallow water equations (1987).

For shallow water flows the upwinding approach was put on a firm basis by the work of Bermúdez and Vázquez-Cendón (1994). We note however that upwinding by itself is not sufficient to correctly treat source terms. It has been recognized that for nearly steady solutions the correct balance between fluxes and source terms is important. It appears as if this fact was first recognized by Glimm and collaborators in the context of the Random Choice Method (Glimm, Marshall and Plohr, 1984). See also the work of Marshall and Méndez (1981). Schemes that balance the effect of fluxes and sources have been termed *well-balanced schemes*. See also the works LeVeque (1998) and Vázquez-Cendón (1999).

The construction of well-balanced schemes has become a popular research topic (Burguete and Garcia-Navarro, 2001; Garcia-Navarro and Vázquez-Cendón, 2000; Hubbard and Garcia-Navarro, 2001).

2.6 *Dry fronts*

Water flows admit the possibility that in parts of the domain the depth of the water be zero, that is, there can be dry regions, commonly called dry-bed or dry-bottom regions. Naturally in dry-bed regions no flow occurs and the governing equations should behave accordingly. The numerical problem is that of dealing with the interface between wet and dry regions. If one considers the local Riemann problem for the shallow water equations in which one state is wet and the other is dry, the exact solution is direct (no iteration) and consists of a single rarefaction wave in which the dry-front path coalesces with that of the tail of the rarefaction wave and mathematically behaves as a contact wave (Stoker, 1992). Even if one uses of the exact solution of this particular type of Riemann problems in Godunov-type methods, there are still of number of potential numerical difficulties. One such difficulty, for explicit methods, is that of stability. Suppose a test problem of the Riemann type in which u_{L} and a_{L} denote the particle velocity and the celerity on the wet side of the interface. Then, the speed of the dry front is $s = u_{\mathrm{L}} + 2a_{\mathrm{L}}$, which is larger than any of the characteristic speeds of the system. This situation may result in the computation of time steps, based on eigenvalues, that are too large and violate the linear stability condition of the method.

Another difficulty is that of computing the particle velocity from the ratio of updated values for the momentum and the water depth, both being very small quantities near the dry front; this operation results in large errors. The particle velocity is the quantity that more clearly reveals the numerical difficulties associated with dry fronts.

The typical behaviour of the numerically computed dry front is characterised by (i) unphysical oscillations (even for the so-called monotone methods) behind the dry front, more visible in velocity snapshots, (ii) positional error of the front, that grows as a function of time, potentially leading to erroneous predictions for arrival times in dam-break simulations. These difficulties are present even when using the exact Riemann problem solution. The introduction of approximate Riemann solvers will not necessarily make things easier, as these have generally been developed under the assumption of two wet states. A widely spread practice is to *wet the bed* by adding a small amount of water to the dry cells in the vicinity of the detected dry front. The resulting local Riemann problem has a different structure to that of the exact problem and by wetting the bed one therefore changes the wave speeds. If one then adds source terms due to slope of the bed, higher order of accuracy and other effects, the situations becomes even more complicated. The topic of dry fronts is one that, in our opinion, deserves further attention. See discussion in section 10.8, chapter 8 of Toro (2001).

2.7 *Multiple space dimensions*

For multi-dimensional problems there are many ways of interpreting Godunov's method. The simplest extension of the method utilizes dimensional splitting, whereby one solves sequences of augmented 1D problems in each coordinate direction. A complete sequence in the *x* direction for instance is called an *x*-sweep. See Strang (1968) for various ways of applying dimensional splitting, a technique that is not restricted to Godunov-type methods. When using splitting methods, in each coordinate sweep one computes the solution of a 1D Riemann problem in which the equations includes also those for the tangential velocity components. The dimensional splitting approach is very simple to implement and is directly applicable to Cartesian meshes or to non-Cartesian structured meshes. One point to note is that splitting may retain the stability limit of unity for the complete method.

The finite volume approach provides another way of interpreting Godunov's method in multiple space dimensions, on grids of any type. The simplest interpretation starts from a conservative update formula analogous to (3), in which the numerical flux appears as a summation of numerical fluxes on each edge of the control volume. Then the numerical fluxes for each volume edge are found as in one space dimension; one computes a Riemann problem solution in the direction normal to the cell boundary, including the equations for the velocity components in the tangential directions. This approach is very simple but results in a reduced stability limit from 1 in 1D, to 1/2 in two dimensions and to 1/3 in three dimensions. Sometimes the definition of the linear stability range of the scheme causes confusion; on occasions authors redefine the Courant number with a hidden factor of 2 or 3 somewhere, stating erroneously that the stability condition of the scheme is still 1.

A more sophisticated interpretation of Godunov's method for multi-dimensional problems was put forward by Colella (1990); he proposed the so called *Corner Transport Scheme* (the CTU scheme), which recognizes more precisely wave propagation direction in two and three space dimensions. Moreover, the stability limit of 1 is recovered. The CTU scheme is also reproduced by the multi-dimensional version of the WAF approach (Billett and Toro, 1997), which automatically produces a second-order version of the CTU scheme. Even more sophisticated upwind approaches in multiple space dimensions were proposed by Roe (1986), leading to what today is known as *multi-dimensional upwinding*, and has been developed in the setting of triangular unstructured meshes.

Unstructured grids have many advantages for multi-dimensional flow analysis, particularly their flexibility when constructing boundary fitted grids for complex geometries and the general absence of preferential grid directions. Nonetheless, conventional schemes on unstructured meshes show that results can depend substantially on the mesh. Researchers realized that it was necessary to incorporate the so-called *genuinely multi-dimensional physics* into the finite volume algorithms. The first step was taken by Davis (1984), who suggested that the shock capturing capabilities of upwind methods could be improved by rotating the Riemann problem to align it with the direction of physically important flow gradients. Common to these methods is the fact that the multi-dimensional physics is added at the cell interfaces, thus retaining some one-dimensional aspects. There is another family of genuinely multidimensional methods which however does not fit into the standard finite volume approach, where the representation of the unknowns is considered to be only piecewise continuous. In this respect these schemes are much closer to finite element methods based on linear elements, with which they share a continuous piecewise linear representation over the cells. On the other hand, they share with upwind methods the properties of asymmetric upwinded stencils and control of monotonicity across discontinuities, and they can be considered as truly multidimensional generalizations of the 1D TVD upwind methods.

The basis of one of the original groups of these multi-dimensional upwinding techniques is the assumption that any observed gradients in the initial data at the start of a time step are linked to the presence of simple waves in the flow. Since an infinite number of simple wave patterns could be responsible for the same observed gradients, it is necessary to hypothesize the number and nature of the waves present: this is known as a wave model. Deconinck *et al*. (1991, 1993) are amongst the earlier contributors to this approach, which was applied to the Euler equations. Unsteady problems have only really been studied in depth more recently, and then mainly in the scalar case (Baines and Hubbard 1998; Hubbard, 1995). Application of multi-dimensional upwind methods to the shallow water equations can be found in Hubard *et al.* (1995) and Hubbard and Baines (1997). The issue of approximating source terms is also discussed in Brufau and Garcia-Navarro (2003).

2.8 *Higher-order Godunov methods*

So far, schemes of second order of accuracy of the TVD type, as briefly discussed in section 2.3, are routinely applied to shallow water type flows, although great care is required in defining accuracy. It is quite common in the literature to speak of second (or even higher) order of accuracy when the scheme is second-order for the model scalar linear homogeneous equation in one space dimension. This is the easy part. To ensure second-order of accuracy for the full equations, including source terms, non-linearity, multiple space dimensions and irregular geometries is quite another. Schemes of higher order of accuracy are beginning to see their way through to applications to shallow water flows. The Essentially Non-Oscillatory (ENO) approach introduced by Harten *et al*. (1987) and Harten and Osher (1987) allows the construction of numerical schemes for hyperbolic conservation laws of accuracy greater than two. These schemes are applicable to the shallow water type equations, see Crnjaric-Zic *et al.*, (2004), Noelle *et al.* (2006) and Xing and Shu (2005).

The ADER approach (Toro *et al.*, 2001) also allows the construction of Godunov-type schemes of arbitrary order of accuracy in space and time. The ADER approach is based on the solution of a generalized Riemann problem in which the two vectors that define the initial conditions, see Eq. (5), are two vectors the component of which are smooth functions, with discontinuities at the origin. This Riemann problem has been termed the Derivative Riemann Problem, or DRP (Toro and Titarev, 2006), see also Toro and Titarev (2002). The ADER approach is capable of yielding numerical schemes of arbitrary order of accuracy for non-linear hyperbolic systems with source terms, in one, two and three space dimensions on unstructured meshes. See also Dumbser (2005), Dumbser and Munz (2005), Käser (2003, 2004), Käser and Iske (2005), Takakura and Toro (2002), Titarev and Toro (2002, 2005) and Toro and Titarev (2005). Applications of the ADER methodology to aeroacoustics are found in Schwartzkopff *et al.* (2002, 2004). An application of ADER methods to the 2D shallow water equations is reported in Toro and Titarev (2005).

Limited experience with schemes of very high order of accuracy shows that many numerical difficulties are due to low order of accuracy, or said in another way, these numerical difficulties could be eliminated, or greatly reduced, by simply using high order methods. Very-high order methods are the next generation of numerical schemes to be used in shallow water type flows, and indeed other types of problems.

2.9 *Schemes related to Godunov's method*

The building block of Godunovs upwind method is the solution of the Riemann problem. Such solutions can also be applied to the construction of other numerical methods, some of which are briefly reviewed in what follows.

Glimm's method (1965), or Random Choice Method (RCM), is directly related to Godunov's method. RCM has been applied to the shallow water equations and it appears as if the first ones to do so were Marshall and Mendez (1981) and Li and Holt (1981). There are two versions of RCM: the staggered-grid version and the non-staggered grid version. Here we consider the latter version. In Godunov's method, for each cell i one computes two Riemann problems to obtain two fluxes that are then used to update the solution in cell i to the next time level, see Eq. (3). In the non-staggered RCM one uses these two Riemann problem solutions more directly. At the new time level one samples randomly the interval that goes from the left interface to the right interface using a sequence of random numbers. The solution allocated to the cell i for the next time level is that picked up at random in the sampled interval. It is found that the solution depends quite crucially on the particular sequence of random numbers used. It is recommended to use the so called van der Corput sequences Colella (1982). See chapter 9 of Toro (2001) for details and numerical results for the shallow water equations. The RCM is directly applicable to systems in two independent variables. The method has been applied to the time-dependent one-dimensional shallow water equations, in which the independent variables are time and distance. The most attractive feature of RCM is its ability to resolve discontinuities as true discontinuities, although the position of the discontinuity suffers from an error which is random in nature. RCM is not conservative and thus positions of discontinuous waves are in error, although on average, wave positions are quite accurate. In the context of the shallow water equations, RCM can capture dry fronts as no other method can, except for the front tracking method.

In principle RCM should also be applicable to the 2D steady supercritical equations, in which the independent variables are x and y say; we are not aware of any publication on this topic. For time-dependent 2D problems one can retain some of the good features of RCM for linear degenerate fields, e.g. contact waves, shear waves and dry fronts. Ivings *et al.* (2003) have exploited this feature of RCM for the 2D time-dependent shallow water equations, in which a hybrid method is constructed, such that RCM is used only for capturing selected features of the solution.

The discontinuous Galerkin finite element (DGFE) method, see for instance (Cockburn and Shu, 1989), also utilizes solutions of local Riemann problems to compute numerical fluxes (Dumbser, 2005), just as in finite volume Godunov-type methods.

The Smooth Particle Hydrodynamics (SPH), see for example Monagham (1982), has also been applied to shallow water type flows, see for example the work of Zoppou and Roberts (2003) and that of Colagrossi and Landrinni (2003). This gridless method has a number of attractive features, including the ability to treat complicated physical situations involving more than one material or phases. As reported by Ben Moussa (2001) and Ben Moussa and Vila (2000) the Smooth Particle Hydrodynamics method could also use local solutions to Riemann problems, making in this manner a connection to Godunov type methods.

2.10 *Godunov methods for more complete water flow models*

The applicability of Godunov methods is not restricted to hyperbolic problems. Dissipative, and even dispersive terms, can also be included in shallow water type mathematical models. Moreover, its is also possible to use Godunov-type methods to solve more complicated equations that do not contain the shallow water approximation, for example, the incompressible Navier–Stokes equations with a free surface. One possible approach is to use Godunov-type methods for the artificial compressibility formulation of Chorin (1967), along with the volume of fluid (VOF) method to deal with the free surface.

2.11 *Godunov methods without explicit upwinding*

Scheme (3) can be applied with the numerical flux (6) substituted by a so called centred flux. A particular example of a centred flux is the Lax-Friedrichs flux. This gives the simplest possible numerical method, which unfortunately is also the least accurate stable scheme. A more accurate centred flux is the FORCE scheme (Toro and Billett, 1996, 2000). For the shallow water equations with a source term due to bottom variation, Chen and Toro (2004) have recently proved the FORCE scheme to be convergent.

It is possible to add a minimum of wave propagation information to the numerical flux. A classical way of doing this is by admitting a single wave speed in the local solution of the Riemann problem. The resulting method is the Rusanov scheme (Rusanov, 1961), sometimes also known as the generalized Lax–Friedrichs scheme. The Rusanov scheme may also be interpreted as an HLL flux in which the two-wave pattern in the local Riemann problem is symmetric.

Godunov-type schemes can then be seen to admit a full hierarchy of numerical fluxes, with various degrees of explicit wave propagation information, or upwinding. At the top of the hierarchy are exact Riemann solvers and complete non-linear approximate Riemann solvers. Then there are solvers of the HLL type with reduced upwinding, that is with a reduced number of waves being admitted in the Riemann solution structure. The simplest upwind scheme is then the Rusanov scheme, admitting just a single wave. Finally there are schemes with no explicit upwinding, such as the FORCE scheme and the Lax-Friedrichs scheme. The most accurate schemes are of course those based on complete, ideally non-linear, Riemann solvers.

3 Shallow-water type approximations

Many engineering and environmental problems involve the study of unsteady water flows characterized by the presence of a vertical scale much smaller than the horizontal ones;these can be described by a shallow water model (Cunge *et al.*, 1980), which forms a set of non-linear hyperbolic equations. The prediction of unsteady flows in a river is important because of the huge potential impacts on property, human life and environment. Also the distribution of the substances transported by the flow is of great importance.

3.1 *1D mathematical model*

Many hydraulic situations can be described by means of a one-dimensional model, either because a more detailed resolution is unnecessary or because the flow is markedly 1D. The equations governing the 1D model can be derived by simple supression of the y components from the 2D equations (see next subsection). This, however, leads to a formulation containing only information per unit width, thus being valid only for simple cases. If a more general 1D model is sought in which the full geometry is retained, a second average on the width is required. Alternatively, the 1D formulation can be derived from mass and momentum control volume analysis (Cunge *et al.*, 1980). The 1D unsteady flow equations can then be written in the form

$$\partial_t \mathbf{U} + \partial_x \mathbf{F}(x, \mathbf{U}) = \mathbf{H}(\mathbf{U}) \tag{7}$$

where the flux has explicit dependence on the spatial variable x,

$$\mathbf{U} = \begin{bmatrix} A \\ Q \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} Q \\ Q^2/A + gI_1 \end{bmatrix},$$
$$\mathbf{H} = \begin{bmatrix} 0 \\ gI_2 + gA(S_0 - S_f) \end{bmatrix}.$$

A is the wetted cross sectional area, Q is the discharge and I_1 and I_2 represent hydrostatic pressure force integrals. I_2 accounts for the pressure forces in a volume of constant depth h due to longitudinal width variations. The friction term represents the action of the shear between the fluid and the solid walls. S_f stands for the energy grade line and is defined, for example, in terms of the Manning's roughness coefficient n. Other forms can equally well be used. From the equations in conservative form

Eq. (7), it is possible to pass to an associated non-conservative form

$$\partial_t \mathbf{U} + \mathbf{J}\partial_x \mathbf{U} = \mathbf{H}', \quad \mathbf{H}' = \mathbf{H} - \partial_x \mathbf{F} \tag{8}$$

where

$$\mathbf{J}(\mathbf{U}) = \frac{\partial \mathbf{F}}{\partial \mathbf{U}} = \begin{bmatrix} 0 & 1 \\ c^2 - u^2 & 2u \end{bmatrix} \tag{9}$$

is the Jacobian matrix of the system, and note that the source term is redefined. The new source term can be manipulated to

$$\mathbf{H}' = \begin{bmatrix} 0 \\ gA\left(S_0 - S_f - \dfrac{\mathrm{d}h}{\mathrm{d}x} + \dfrac{1}{B}\dfrac{\mathrm{d}A}{\mathrm{d}x}\right) \end{bmatrix} \tag{10}$$

with B denoting the width at the free surface; the important point about this form is the presence of total space derivatives only. This is crucial when moving to the discrete level.

The characteristic form of the equations is important for the correct formulation of upwind schemes and boundary conditions. This form is obtained from a diagonalization of the Jacobian in Eq. (8). Denoting by $\mathbf{P}$ and $\mathbf{P}^{-1}$ the matrices that diagonalize $\mathbf{J}$ one has the relation $\mathbf{\Lambda} = \mathbf{P}^{-1}\mathbf{J}\mathbf{P}$. The diagonal matrix $\mathbf{\Lambda}$ is formed by the eigenvalues of $\mathbf{J}$ and $\mathbf{P}$ if formed by the right eigenvectors. The eigenvalues and right eigenvectors are

$$\begin{aligned} &\lambda^1 = u - c, \quad \lambda^2 = u + c, \\ &\mathbf{e}^1 = \begin{bmatrix} 1 \\ u - c \end{bmatrix}, \quad \mathbf{e}^2 = \begin{bmatrix} 1 \\ u + c \end{bmatrix} \end{aligned} \tag{11}$$

Let $\mathbf{W}$ be the set of variables (characteristic variables) that verify

$$d\mathbf{U} = \mathbf{P}d\mathbf{W}, \quad d\mathbf{W} = \mathbf{P}^{-1}d\mathbf{U} \tag{12}$$

Then, the characteristic formulation gives a set of decoupled equations

$$\partial_t \mathbf{W} + \mathbf{\Lambda}\partial_x \mathbf{W} = \mathbf{P}^{-1}\mathbf{H}' \tag{13}$$

3.2 2D mathematical model

The 2D shallow water equations, which represent mass and momentum conservation in plane, can be obtained by depth averaging the Navier–Stokes equations. Neglecting diffusion of momentum due to viscosity and turbulence, wind effects and the Coriolis term, they form the following system of equations:

$$\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) + \partial_y \mathbf{G}(\mathbf{U}) = \mathbf{H}(\mathbf{U}) \tag{14}$$

in which

$$\begin{aligned} &\mathbf{U} = \begin{bmatrix} h \\ q_x \\ q_y \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} q_x \\ q_x^2/h + gh^2/2 \\ q_x q_y/h \end{bmatrix}, \\ &\mathbf{G} = \begin{bmatrix} q_x \\ q_x q_x/h \\ q_x^2/h + gh^2/2 \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} 0 \\ gh(s_{0x} - s_{fx}) \\ gh(s_{0y} - s_{fy}) \end{bmatrix} \end{aligned} \tag{15}$$

where $q_x = uh$ and $q_y = vh$. The variable h represents the water depth, g is the acceleration of the gravity and u, v are the depth averaged components of the velocity along the x and y coordinates respectively. The source terms in the momentum equation are the bed slopes and the friction losses along the two coordinate directions. It is useful to rewrite (14) as

$$\partial_t \mathbf{U} + \nabla \mathbf{E} = \mathbf{H}, \quad \mathbf{E} = \begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix} \tag{16}$$

as this displays the conservative character of the system in the absence of source terms, and also in order to introduce the integral form of the equations over a fixed volume Ω

$$\begin{aligned} \frac{\partial}{\partial t}\mathbf{U}\,\mathrm{d}\Omega + \int \nabla \mathbf{E}\,\mathrm{d}\Omega = \int \mathbf{H}\mathrm{d}\Omega \rightarrow \frac{\partial}{\partial t}\int \mathbf{U}\,\mathrm{d}\Omega + \int \mathbf{E}_n\,\mathrm{d}S \\ = \int \mathbf{H}\,\mathrm{d}\Omega \end{aligned} \tag{17}$$

where S denotes the surface surrounding the volume Ω, n is the unit outward normal vector and the Gauss theorem has been used.

In order to formulate cell centred finite volume methods over a control volume where the dependent variables of the system are represented as piecewise constants, the conservation equations can be written for every cell i and Riemann solvers are oriented perpendicularly to the edges of the grid cells, in much the same way as in one space dimension. For that purpose, the normal flux $\mathbf{E}_n$ and its Jacobian are of interest. The Jacobian matrix, of the normal flux is evaluated as

$$\mathbf{J}_n = \frac{\partial \mathbf{E}_n}{\partial \mathbf{U}} = \frac{\partial \mathbf{F} n_x}{\partial \mathbf{U}} + \frac{\partial \mathbf{G} n_y}{\partial \mathbf{U}} \tag{18}$$

and can be diagonalized as $\Lambda_n = \mathbf{P}^{-1}\mathbf{J}_n\mathbf{P}$, making the schemes rely on its eigenvalues and eigenvectors.

3.3 Numerical approaches

A great deal of work has been devoted to develop one and two-dimensional numerical models for unsteady shallow flows in the last decades and various computational techniques using finite difference, finite element and finite volume methods have been reported (Alcrudo and Garcia-Navarro, 1993; Bermúdez *et al.*, 1998; Cunge *et al.*, 1980; Garcia-Navarro and Vazquez-Cendon, 2000; Sleigh *et al.*, 1998; Toro, 1989). Several numerical difficulties must be adequately treated to obtain an accurate solution without numerical errors. Zhao *et al.* (1994) provided a good historical review and the features required for a two-dimensional river flow simulation model: it should be able to handle complex topography, dry bed advancing fronts, wetting-drying moving boundaries, high roughness values, steady or unsteady flow and subcritical or supercritical conditions.

Amongst the numerical techniques reported, those belonging to the category of conservative methods have gained acceptance in recent years for their important property of providing a proper discrete representation of the physical conservation laws. Essentially imported from gas dynamics, they have been extended to shallow water problems trying to overcome the relevant differences existing between these two applications. Natural topographies is the main challenge. Dominant source terms and open boundaries are two important difficulties to face when using a conservative method since they both can damage the conservative character of the solution.

Upwind methods in particular are becoming increasingly popular in the hydraulics literature and have proved a suitable way to discretize the shallow water equations Billett and Toro (1997), Brufau *et al.* (2002), Burguete and Garcia-Navarro (2001), Hubbard and Garcia-Navarro (2000) and Toro (2001). Being a hyperbolic system of conservation laws, they are a good candidate for application of the techniques developed for the Euler equations in gas dynamics.

The discretization of systems of hyperbolic conservation laws that include source terms following steady state equilibrium criteria is a fundamental step in numerical simulation. Bed slope and friction source terms are of special relevance in hydraulic applications based on a shallow flow model. For that reason, a considerable effort has been recently devoted to this topic in a search for the correct source term discretization given a particular numerical scheme with good properties for the homogeneous case. Following this purpose, LeVeque (1998) incorporated the modelling of source terms to his wave propagation algorithm, while Roe's scheme (Roe, 1981) was modified by a number of authors to include source terms, Bermúdez and Vázquez (1994), Glaister (1992) and Vázquez-Cendón (1999). Following Roe's scheme, the extension from first to second order in the case of systems of equations was explored by Hubbard and Garcia-Navarro (2001) noting that and adequate evaluation of the source terms ensured a correct balance at least in situations of still water by means of an upwind technique based on the Jacobian matrix diagonalization.

Another challenge is the extension of the cited methods to higher order resolution schemes, able to capture shocks and discontinuities more accurately. The extension of numerical methods to second order in 2D problems is not straightforward for different reasons. One of them is the strong influence of the grid used on the properties of the scheme chosen to be moved from one dimensional to 2D cases and, therefore, on the quality of the expectable numerical results. Triangular unstructured adaptive meshes offer, not only the freedom to fit the computational domain to the shape of the physical region of interest, but also the possibility to eliminate as much as possible preferential directions in the numerical solution. On the other hand, triangular grids are a challenge for high order numerical methods since, in some cases, their methodology relays on the structured enlargement of the stencil and the step to pass from structured to anisotropic unstructured meshes is restrained by the difficulty in constructing analogous extensions to higher-order accuracy and sometimes is not directly movable to the context of unstructured meshes.

Murillo *et al.* (2006) put the focus on the hyperbolic nature of the system of conservation laws that describes the shallow water and the solute transport equations allows a decomposition in single and independent components. The analysis of these component is the key to provide solutions to the numerical problems that arise when dealing with realistic cases.

Among cell centred upwind finite volume methods (FV), Roe's scheme (Roe, 1986) can be chosen as the basic first order explicit scheme to build improved versions able to be used in 2D problems. The alternative is to move from the first order piecewise constant representation to the reconstruction of the solution by means of slope limiter functions that can be directly formulated on triangular cells (Batten, 1997). Those methods are said to achieve an order greater than one in regions of smooth solution and to be devoid of oscillations at discontinuities. Among them, the Limited Central Difference (LCD) approach (Batten, 1997), the compressive limiter proposed by Durlofsky *et al.* (1992) and the Maximum Limited Gradient (MLG) (Batten, 1997), are well known. A new reconstruction function with excellent properties was proposed by Wierse (1997).

Finite difference schemes for time-dependent advective equations are traditionally divided in two main groups, according to the way of discretizing the time derivative, as explicit and implicit. Implicit schemes offer numerical stability (not always unconditional, however) at the extra cost of having to deal with the resolution of an algebraic, and often non-linear, system with as many unknowns as grid points at every time step. The allowable time step size is nevertheless restricted in the explicit case by stability reasons to fulfill the Courant-Freidrichs-Levy (CFL) condition (Courant *et al.*, 1952). It is possible to relax the condition over the time step size when using explicit schemes. A generalization of the first order explicit upwind and Roe's method, modified to allow large time steps, was explored in Leveque (1982) and (1983) first in the scalar non-linear case and then adapted to systems of equations. See also Casulli and Toro (1990) and Toro and Billett (1997) for large-time step explicit schemes.

Murillo *et al.* (2005) presented an explicit extension of the two-dimensional upwind finite volume scheme to values of CFL greater than one is defined, where conceptual simplicity is the most valuable characteristic as the variables at a future time can be independently evaluated at every single point. A different philosophy is used in the implicit discretization that, basically, allows for almost unconditional stability at the cost of an extra algebraic effort. This option can be very efficient but, at the same time can meet important difficulties in non-linear applications Burguete and Garcia-Navarro (2004).

The basic ideas can be illustrated by examining a scalar conservation equation

$$\frac{\partial u}{\partial t} + \frac{\partial f(u, x)}{\partial x} = s(u, x) \tag{19}$$

modeling the evolution of a single variable u in one space dimension. The conservation law (19) contains an important physical meaning. Spatial integration expresses that the time variation of the conserved variable in a given volume is equal to the difference between the incoming and the outgoing fluxes plus the contribution of the source term. When discretizing a conservation law of this kind, incorrect numerical approximations can lead to bad behaviour in the solution and unacceptable errors, which make them useless techniques. Schemes properly approximating the conservation equation are called conservative schemes (Toro, 2001). Conservative schemes produce a good approximation of Eq. (19) cancelling the contributions of the flux at the grid interfaces, being the global variation of the conserved variable due only to the source terms and to the flux at the boundaries. They will be built looking for (1) a way of incorporating the x-dependence of the flux into the numerical scheme, and (2) a discretization which maintains equilibria at steady state. In essence this requires the flux derivatives and the source terms to be discretized in a similar manner. However, the precise form this implies for the discrete source term is not always obvious from the flux approximation. As an example, consider the first order upwind scheme with forward Euler time-stepping for the homogeneous equation ($s = 0$) which, in fluctuation-signal form can be written as

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x_i}(\Delta f_{i-\frac{1}{2}}^{+} + \Delta f_{i+\frac{1}{2}}^{-}) \tag{20}$$

In the case, where f depends on both u and x

$$\Delta f = \frac{\partial f}{\partial u}\Delta u + \frac{\partial f}{\partial x}\Delta x = \tilde{\lambda}\Delta u + \tilde{V} \tag{21}$$

in which $\tilde{\lambda}$ is the advection velocity and $\tilde{V}$ is a term due to the independent spatial variation, so that the scheme (21) is completed by setting

$$\Delta f^{\pm} = \tilde{\lambda}^{\pm}\Delta u + \frac{1}{2}(1 \pm \mathrm{sgn}(\tilde{\lambda}))\tilde{\mathrm{V}} \tag{22}$$

Straightforward algebraic manipulation converts (20) to an equivalent flux-based finite volume scheme

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x_i}(f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}) \tag{23}$$

in which the numerical flux (denoted by an asterisk) for first order upwinding for instance is given by

$$\begin{aligned} f_{i+\frac{1}{2}} &= \frac{1}{2}(f_i + f_{i+1}) - \frac{1}{2}\mathrm{sgn}(\tilde{\lambda}_{i+\frac{1}{2}})(f_{i+1} - f_i) \\ &= \frac{1}{2}(f_i - f_{i+1}) - \frac{1}{2}(|\tilde{\lambda}|\Delta u + \mathrm{sgn}(\tilde{\lambda})\tilde{V})_{i+\frac{1}{2}} \end{aligned} \tag{24}$$

The dependence of the flux on x is incorporated within the evaluation of the numerical flux. Note that the asterisk indicates an approximation of an integrated quantity, generally the flux across a cell face or the source over a cell volume. In the non-homogeneous case, the numerical scheme becomes

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x_i}(\Delta f_{i-\frac{1}{2}}^{+} + \Delta f_{i+\frac{1}{2}}^{-}) + \frac{\Delta t}{\Delta x_i}s_i \tag{25}$$

It follows that in order to maintain the balance at steady state the numerical source term must take the form of two components to balance the two flux difference contributions

$$s_i = \tilde{s}_{i-\frac{1}{2}}^{+} + \tilde{s}_{i+\frac{1}{2}}^{-} \tag{26}$$

with

$$\tilde{s}_{i\pm\frac{1}{2}} = \tilde{s}_{i\pm\frac{1}{2}}^{+} + \tilde{s}_{i\pm\frac{1}{2}}^{-} \tag{27}$$

and is evaluated at the same interface state as the numerical fluxes. For the first order upwind scheme defined, the precise balance is attained with

$$s_i = \frac{1}{2}([1 + \mathrm{sgn}(\tilde{\lambda})]\tilde{s})_{i-\frac{1}{2}} + \frac{1}{2}([1 + \mathrm{sgn}(\tilde{\lambda})]\tilde{s})_{i+\frac{1}{2}} \tag{28}$$

The above analysis is not restricted to the first order upwind scheme. It can be reproduced in the same detail for second order methods. Approximations also follow automatically for flux limited TVD schemes (Glaister, 1987; Toro, 1989).

3.4 Shallow water equations in one space dimension

A general formulation for the conservative scheme is proposed

$$\frac{\Delta \mathbf{U}_i^n}{\Delta t} = \left(\mathbf{H} - \frac{\Delta \mathbf{F}}{\Delta x}\right)_{i-\frac{1}{2}}^{+} + \left(\mathbf{H} - \frac{\Delta \mathbf{F}}{\Delta x}\right)_{i+\frac{1}{2}}^{-} \tag{29}$$

which can be rewritten

$$\frac{\Delta \mathbf{U}_i^n}{\Delta t} = \mathbf{G}_{i-\frac{1}{2}}^{+} + \mathbf{G}_{i+\frac{1}{2}}^{-} \tag{30}$$

Defining $\mathbf{G}$ as

$$\mathbf{G}_{i+\frac{1}{2}} = \left(\mathbf{H} - \frac{\Delta \mathbf{F}}{\Delta x}\right)_{i+\frac{1}{2}} = \left(\mathbf{H}' - \mathbf{J}\frac{\Delta \mathbf{U}}{\Delta x}\right)_{i+\frac{1}{2}} \tag{31}$$

where the last equality has to be taken as a key requirement for the correct definition of the discrete source term decomposition (Burguete and Garcia-Navarro, 2001) and

$$\mathbf{G}^{\pm} = \mathbf{P}\begin{bmatrix} \frac{1 \pm \mathrm{sign}\lambda_1}{2} & 0 \\ 0 & \frac{1 \pm \mathrm{sign}\lambda_2}{2} \end{bmatrix}\mathbf{P}^{-1}\mathbf{G} \tag{32}$$

Making an implicit or semi-implicit treatment of the friction source term is usually necessary to avoid the numerical instabilities produced by dominant source terms. The procedure is directly extended to higher order schemes involving flux limiters for numerical oscillation control, and these limiting functions have to depend on local gradients of $\mathbf{G}$.

3.5 Shallow water discretization in two dimensions

Using the finite volume formulation, for the updating of a single cell only the in-going contributions are taken into account when evaluating the contour flux integral.

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{\Omega_i}\left(\sum_k (\mathbf{P}\Lambda_n - \mathbf{P}^{-1}\Delta \mathbf{U}_k dl_k - \Omega \mathbf{H}_k^{-})\right)_i \tag{33}$$

where the sum in k is over the cell sides and the quantities with superscript are calculated at every cell edge from

$$\Lambda_n - = \frac{1}{2}(\Lambda_n - |\Lambda_n|), \quad \mathbf{H}^{-} = \mathbf{P}(1 - |\Lambda_n|\Lambda_{n^{-1}})\mathbf{P}^{-1}\mathbf{H} \tag{34}$$

(For more details see Murillo *et al.*, 2005).

4 Numerical issues

The procedure chosen for the basic scheme is applied for the ordinary cells, that is, those representing points at the interior of the wet domain. The boundaries of the wet domain are defined by the cells not completely surrounded by other cells. All these cells actually require the definition of suitable boundary conditions in order to reach the solution of a problem. However, for transient flows a distinction can be made considering either wet domains fixed in extension, that is, limited by vertical walls, or those whose size changes as time progresses, that is, those involving moving boundaries.

In most cases, boundary conditions are, strictly speaking, applied only at fixed boundaries. The moving boundaries are considered as wetting fronts and hence included in the ordinary cell procedure in a through calculation that assumes zero water depth for the dry cells. This is what is called in this work the wetting/drying front.

Unsteady shallow water flow over dry beds is at present one of the topics of research in computational hydraulics. One way to deal with this kind of flow is to use a moving computational mesh so that computation is only performed in the wet cells and the grid moves as the water front does. Suitable boundary conditions must be applied for the correct front tracking. A different approach consists of a through calculation of the front position as it advances over a computational mesh covering all the physical domain and in which there are both wet and dry cells. Among the authors using this second methodology, and for numerical reasons, it is frequent the use of a very small value for water depth in the dry cells (wetting the bed) in order to avoid zero depth values. The amount of this threshold is reported to be something small, not specified or justified in general.

4.1 Boundary conditions

The theory of characteristics provides a rigorous rule for the number of physical or external boundary conditions required at the upstream and downstream ends of a given flow domain. They tell us, depending both on the value of the normal velocity through the boundary $\mathbf{u} \cdot \mathbf{n} = un_x + vn_y$ and thelocal Froude number $Fr = \mathbf{u} \cdot \mathbf{n}/c$, there are different possibilities and this is valid for both 1D and 2D problems. The rest of the required information at the boundaries has to come through the so called numerical boundary conditions. A second question is related to the procedure used to obtain numerical boundary conditions (Hirsch, 1990). When using a Godunov type finite volume technique, the idea of using a Riemann solver to calculate the flux at the face of a cell can be extended to the boundaries. The variables are stored at the centre of each cell and the boundary conditions are also imposed there. The value of the variables not prescribed are calculated from a usual finite volume balance. For this purpose, the fluxes across the edges lying on the boundary are estimated by means of a 'ghost' cell outside. Usually, the ghost cell just duplicates the boundary cell. When the boundary is a solid wall, the ghost cell is a mirror cell in which the depth of water has the same value that the boundary cell and the velocities are the same with opposite sign. The solution of the corresponding Riemann problem gives the sought zero value for the particle speed and also gives the water depth that corresponds to it.

For 1D calculations, a further improvement has been suggested by Burguete *et al.* (2005). The method is based on a very important physical principle: the increment of mass in the whole system in a time interval is the result of the entering mass flow minus the leaving mass flow during that period of time. When using a conservative numerical scheme, this physical principle provides a way to reach two objectives, first to allow for a null mass balance error and, second, to supply numerical boundary conditions. In one time step, the numerical scheme supplies updated values for all the nodal variables.

The associated volume increment is

$$\Delta M = \sum_{i=1}^{N} (A_i^s - A_i^n)\Delta x \tag{35}$$

Using, for instance, the first order upwind scheme to calculate the updated values A_i^s and these are used in the volume increment calculation, the above expression is

$$\Delta M = (Q_1^n - Q_N^n)\Delta t \tag{36}$$

and the difference respect to the net incoming flow rate is the volume error of the numerical scheme. It can be seen as if the scheme was generating a numerical inflow volume $\mathrm{V}_{\mathrm{in}}^{\mathrm{num}} = Q_1^n \Delta t$ and the numerical outflow volume $\mathrm{V}_{\mathrm{out}}^{\mathrm{num}} = Q_N^n \Delta t$. In order to achieve perfect volume conservation, if the upstream physical boundary condition is Q_1^{n+1} it is assumed that the physical volume entering during one time step is $\mathrm{V}_{\mathrm{in}}^{\mathrm{phy}} = \frac{1}{2}(Q_1^n + Q_1^{n+1})$. Then the corrected value for the upstream wetted section is

$$A_1^{n+1} = A_1^s + \frac{1}{\Delta x}(\mathrm{V}_{\mathrm{in}}^{\mathrm{phy}} - \mathrm{V}_{\mathrm{in}}^{\mathrm{num}}) = A_1^s + \frac{\Delta t}{2\Delta x}(Q_1^{n+1} + Q_1^n) \tag{37}$$

and for the downstream wetted section is

$$A_N^{n+1} = A_N^s - \frac{\Delta t}{2\Delta x}(Q_N^{n+1} + Q_N^n) \tag{38}$$

4.2 Wetting/drying fronts

Another numerical problem is the treatment of wet/dry interfaces between interior cells, that have traditionally represented a difficulty for modelers wanting to solve the shallow water equations over a bed of irregular geometry. Flow over dry bed involves a complicated situation that can be analysed as a boundary condition which is dynamically changing in time with the moving front and continuously expanding or reducing the flow domain. Akanbi and Katopodes (1987) gave a brief summary of problems encountered in the numerical simulation of flood waves propagating over dry bed. The alternative is to include the wet/dry interfaces in the full domain of computation in which there may be wet cells and dry cells at the same time. In this case the numerical scheme chosen for the discretization must be able to cope with them. In general, cells being flooded or dried during the computation tend to introduce numerical instabilities in the solution, resulting for example in negative water depths or unphysically high velocities. Different approaches have been proposed to handle them. These techniques include modified equations in very shallow regions, shock-capturing schemes or the assumption that a cell is dry if water depth is below a small critical value. Beffa and Connel (2001) reported numerical oscillations when cells switch from dry to wet or *vice-versa*. George and Stripling (1995) represented the local bathymetry in each cell by a sloping facet rather than by a flat bed to eliminate the spurious shocks in their finite volume model. Some authors working with finite elements solve the problem allowing the controlled use of negative depths (Heniche *et al.*, 2000; Kawahara and Umetsu, 1986; Khan, 2000). Bradford and Sanders (2002) used Neumann extrapolation of the velocity in partially wet cells to bypass the incorrectly estimation of pressure and body forces in such cells. In Brufau *et al.* (2002, 2004), driven by the interest of controlling numerical stability and global mass conservation, a two-dimensional model was presented for unsteady flow simulation where the main strategy was based on a local redefinition of the bed slope at specific locations.

The wetting front advance over a dry bed is a moving boundary problem in the context of a depth averaged 2D model. As such, the optimum way to deal with it is to find the physical law that best defines the dynamics of the advancing front and use it as physical boundary condition to be plugged into the general procedure. The question about that physical law makes us reconsider the three-dimensional basic equations at the wetting front position. In advance over adverse dry bed the water column tends to zero smoothly and, hence, the free surface and bottom level tend to reduce to one point where both the free surface and bottom boundary conditions apply simultaneously. This line of reasoning, though interesting, does not solve the discrete problem in a simple way but, on the contrary, leads to the generation of an alternative technique for a number of cells that increases in time as the wetting progresses.

In a different approach closer to the discrete solution, wetting fronts over dry surfaces can be reduced to Riemann problems in which one of the initial depths is zero. This problem can be analytically studied for simplified conditions and the solution exists both for horizontal bed (Ritter solution) and for sloping bed (Toro, 2001). The solution in the latter case, when dealing with adverse slopes, identifies a subset of conditions incompatible with fluid motion (stopping flow). On the other hand, numerical technique based on approximate Riemann solvers are adapted to cope with zero depth cells and provide a discrete solution to the problem in all these cases but do not identify correctly the stopping flow conditions. Therefore, this techniques are unable to solve correctly situations of still water in a domain of irregular shape, generating spurious velocities in the wet/dry contour and violating mass conservation.

In an attempt to generate a simple an efficient rule for this situation, the following steps are proposed. First of all, the variable water level $d = h + z$ is compared between any two cells (L and R) defining a wetting front over adverse slope and two situations can be found: (1) $d_L \geq d_R$, nothing has to be done.The basic Riemann solver in the numerical scheme provides a satisfactory solution, (2) $d_L < d_R$ this corresponds to the stopping conditions and hence something has to be done to modify the basic procedure.

Previous works on this topic have reached to this point and some authors working with finite elements solve the problem allowing the controlled use of negative depths. Some authors propose a solid wall treatment at that position. An alternative is suggested considering a one-dimensional case of still water ($u = v = 0$). It can be seen that the discretization of the mass equation leads to the equilibrium condition $\Delta d_{LR} = 0 \rightarrow \Delta z_{LR} = \Delta h_{LR}$.

This condition is not fulfilled between any two cells under the stopping condition due to the piecewise constant representation of the variables in each cell, leading to the appearance of numerical velocities without physical meaning. The problem of solving steady flow is converted into an unsteady one producing movement in water that should be always at steady state and mass conservation is thus lost.

The above requirement can also be written $h_R - h_L = z_L - z_R \rightarrow h_R = h_L - (z_r - z_L) < 0$, thus predicting the appearance of negative depths at the outside of the wetted domain. In order to avoid the numerical error, the technique proposed is to enforce the local redefinition of the bottom level difference at the interface to fulfill the equilibrium condition and therefore mass conservation.

In unsteady cases, that is, wetting fronts advancing over an adverse dry slope, the procedure described is the same. However in this case the numerical representation of the slope between the two adjacent cells may produce a too fast propagation of the front. It is necessary to reduce to zero the velocity components at the previously dry cell.

5 Summary, future trends and recommendations

Godunov-type methods are today applied routinely to compute to shallow water type flows. These methods have a number of attractive features that include:

(1) the methods are conservative (when a conservative form of the equations exists) and thus shock waves, or bores, are computed with the correct propagation speed;
(2) the methods are robust in the presence of strong gradients;
(3) rapid variations of the solution, including discontinuous waves, are represented with a minimum of numerical diffusion;
(4) the numerical representation of discontinuous waves excludes, to a large extent, the generation of the Gibbs phenomenon, that is, of unphysical oscillations near large spatial gradients;
(5) processes of strong wave interaction are accurately represented by these methods.

Amongst the shortcomings of the methods are:

(1) complexity, which in the early 80's discouraged potential users;
(2) expense, which to some extent depends on whether an explicit or implicit version of the schemes is used; the choice may depend on the particular application.

Of course there are other shortcoming that are common to all numerical methods. These include:

(1) treatment of source terms and
(2) treatment of dry fronts.

Future trends will include:

(1) increasing popularity of these methods for shallow water type flows;
(2) implementation of higher order versions of these methods, as they are being developed by numerical analysts, such as WENO methods, ADER methods and DG Finite Element methods;

(3) extension of these methods for solving the governing equations of physically more complete models, without the shallow water assumption.

The methods reviewed in this paper can be very useful when it comes to modelling, simulating and understanding physical processes that concern our environment, its preservation and its dangers to human life.

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