Appendix A

Derivation of Classical Hamiltonians for Qubit Systems

These derivations follow a standard procedure [24] for the writing the Hamiltonians of classical circuits.

A.1 Cooper Pair Box

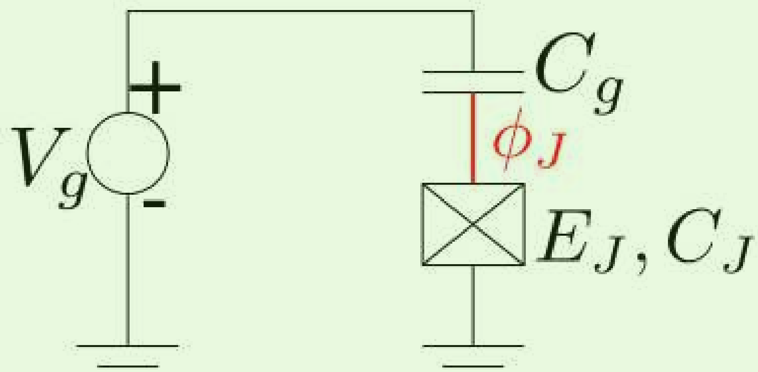

Figure A.1: Circuit for the Cooper Pair Box, with the node flux variable marked.

The Cooper Pair Box circuit is straightforward to model. Once we remove the voltage source, there is only one node other than ground, as shown in Figure A.1. Our kinetic part will include a charging term for both capacitances:

$$T = \frac{C_g}{2}\dot{\phi}_J^2 + \frac{C_j}{2}\dot{\phi}_J^2 = \frac{C_\Sigma}{2}\dot{\phi}_J^2$$

where $C_\Sigma = C_g + C_j$ is the total island capacitance. Our potential terms will include the Josephson term and the external source energy. The energy which the source supplies is V_g times charge on the supply-side of the gate capacitor. This charge can be written as the voltage across the gate capacitor (-$\dot{\phi}_J$) times the gate capacitance C_g. Putting that together,

$$U = -E_j \cos\left(\frac{2\pi}{\Phi_0}\phi_J\right) - V_g C_g \dot{\phi}_J$$

The Lagrangian is then

$$\mathcal{L} = T - U = \frac{C_\Sigma}{2}\dot{\phi}_J^2 + E_j \cos\left(\frac{2\pi}{\Phi_0}\phi_J\right) + V_g C_g \dot{\phi}_J$$

The conjugate momentum is the charge in the island plus an effective offset charge gated by the source.

$$Q_J = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_J} = C_\Sigma \dot{\phi}_J + V_g C_g$$

where

$$C_*^2 = C_B C_g + C_B C_{\text{in}} + C_g C_{\text{in}} + C_B C_r + C_g C_r$$

in agreement with [4]. The first line is the resonator term, the second is the qubit, and the third is all of the intercouplings. In the reasonable limit that C_r is much greater than all other capacitances, this reduces to

$$\begin{aligned}\mathcal{H} = \; & \frac{\phi_r^2}{2L_r} + \frac{Q_r^2}{2C_r} \\ & + \frac{Q_J^2}{2C_\Sigma} - E_J \cos\left(\frac{2\pi}{\hbar}\phi_J\right) \\ & + \beta\frac{Q_r Q_j}{C_r} + \frac{C_{\text{in}} Q_r V_g}{C_r}\end{aligned}$$

where $C_\Sigma = C_g + C_B$ and $\beta = C_g/C_\Sigma$ is an impedance divider ratio which determines how much of the transmission line voltage is seen by the qubit.

The last term of the above expression does not disappear in the limit of large C_r, because Q_r is also large (such that their ratio is the voltage on the resonator). However, the term coupling Q_J and V_g *did* vanish. Naively, this is worrisome because that was the term which we would have expected to provide an effective offset charge (as in the CPB case) which allows use to tune the qubit energy levels.

This trouble appears because our lumped LC model of the resonator is only valid for wavelengths on the scale of the resonator [14]; it does not work at DC.[1] In actuality, the resonator is just a capacitor at DC, and (again assuming C_r to be the largest capacitance in the system), any DC gate voltage will show up at the gate capacitor, and have the same effect it would have in the CPB system. We can add this DC term in to produce the final form of the classical Hamiltonian.

$$\begin{aligned}\mathcal{H} = \; & \frac{\phi_r^2}{2L_r} + \frac{Q_r^2}{2C_r} \\ & + \frac{\left(Q_J - C_g V_g^{\text{DC}}\right)^2}{2C_\Sigma} - E_J \cos\left(\frac{2\pi}{\hbar}\phi_J\right) \\ & + \beta\frac{Q_r Q_j}{C_r} + \frac{C_{\text{in}} Q_r V_g}{C_r}\end{aligned}$$

[1] Applying the LC model at DC would for would force the centerline of the resonator to always have zero DC voltage, because otherwise the current through the "effective inductor" increases to infinity.