## I. CAPACITANCE MATRIX

Here we derive the capacitance matrix using the main part of the paper:

$$\vec{C} = |C| \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2+\alpha & -1 \\ 0 & -1 & 2 \end{pmatrix} \tag{1}$$

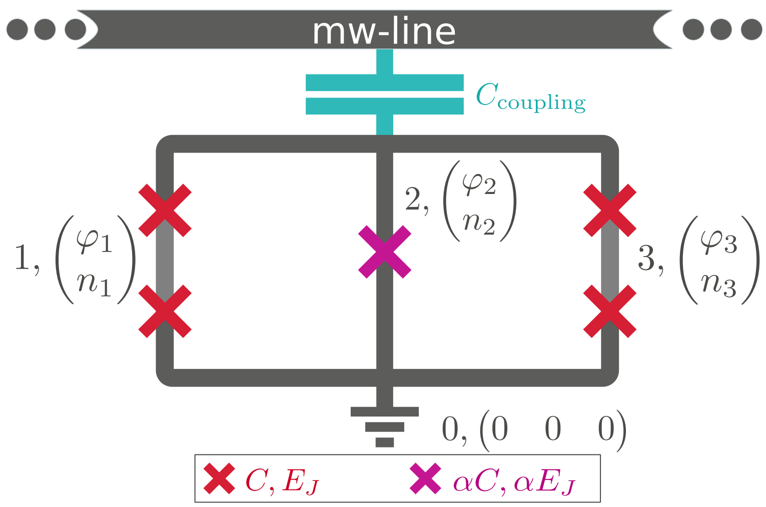


FIG. 1. **Topology of the twin flux qubit.**

The capacitance matrix is found from expressing the charges on the 3 different islands on the system, Fig 1. The voltage on the ground island is assumed to be $V_0 = 0$.

$$\vec{Q} = C\vec{V} \equiv \begin{bmatrix} |C|(V_1 - 0) + |C|(V_1 - V_2) \\ |C|(V_2 - V_1) + |C|(V_2 - V_3) + |C|(V_2 - 0) \\ |C|(V_3 - V_2) + |C|(V_3 - 0) \end{bmatrix} = |C| \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2+\alpha & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3. \end{pmatrix} \tag{2}$$

## II. REPRESENTATION OF HAMILTONIAN IN THE CHARGE BASIS

Here, we show how to represent the Hamiltonian of the system in matrix form for the charge basis, Fig 2.

The Hamiltonian

$$\begin{aligned}\mathcal{H} &= T + U \\ &= E_C\,|C|\left\langle \hat{C}^{-1}\right\rangle_{|n_1,n_2,n_3\rangle} \\ &\quad + E_J\big[4 + \alpha - \alpha\cos(\varphi_2) - \cos(\varphi_1) - \cos(\varphi_3) - \\ &\qquad \cos(\varphi_2 - \varphi_1 - \varphi_{\text{ext}}) - \cos(\varphi_2 - \varphi_3 + \eta\varphi_{\text{ext}})\big]\end{aligned} \tag{3}$$

in the charge basis takes the form shown in Fig. 2, where a state $|-1,0,1\rangle$ would correspond to a CP-occupation of -1, 0 and 1 on islands 1, 2 and 3. Kinetic terms ($T(n_1,n_2,n_3)$ naturally fall on the diagonal axis of the matrix. The phase-dependent term ($U(\varphi_1,\varphi_2,\varphi_3,\varphi_{\text{ext}})$) are represented using the following procedure:

1. Derive the commutation relation between the number, $\hat{n}$, and the exponential phase, $e^{\pm i\hat{\varphi}}$, by using the standard relation $[\hat{n},\hat{\varphi}] = 1$:

$$\begin{aligned}\left[\hat{n}, e^{\pm i\hat{\varphi}}\right] &= \left[\hat{n}, \sum_{\alpha=0}^{\infty}\frac{(\pm i\hat{\varphi})^{\alpha}}{\alpha!}\right] = \sum_{\alpha=0}^{\infty}(\pm i)^{\alpha}\frac{[\hat{n},\hat{\varphi}^{\alpha}]}{\alpha!} \\ &= \sum_{\alpha=0}^{\infty}(\pm i)^{\alpha}\frac{-\alpha i\hat{\varphi}^{\alpha-1}}{\alpha!} = \pm\sum_{\alpha=1}^{\infty} i^{\alpha-1}\frac{(\pm\hat{\varphi})^{\alpha-1}}{(\alpha-1)!} = \pm e^{\pm i\hat{\varphi}}.\end{aligned} \tag{4}$$

2. Operating with the number operator on state $e^{\pm i\hat{\varphi}}|n\rangle$ and using the commutation result

$$\begin{aligned}\hat{n}\left[e^{\pm i\hat{\varphi}}|n\rangle\right] &= \left[\pm e^{\pm i\hat{\varphi}} + e^{\pm i\hat{\varphi}}\hat{n}\right]|n\rangle \\ &= (n\pm 1)\left[e^{\pm i\hat{\varphi}}|n\rangle\right].\end{aligned} \tag{5}$$

3. Evidently, the exponential phase operator is a ladder operator for the $|n\rangle$ state:

$$e^{\pm i\hat{\varphi}}|n\rangle = |n\pm 1\rangle \Rightarrow e^{\pm i\varphi} = \sum_n |n\pm 1\rangle\langle n|. \tag{6}$$

4. Thus operator $\cos(\hat{\varphi}_2 - \hat{\varphi}_1 - \hat{\varphi}_{\text{ext}})$ can be expressed in the number basis $\{n_1, n_2, n_3\}$ as:

$$\begin{aligned}
\cos(\hat{\varphi}_2 - &\hat{\varphi}_1 - \hat{\varphi}_{\text{ext}}) = \\
&= \frac{1}{2}\left(e^{i\hat{\varphi}_2} e^{-i\hat{\varphi}_1} e^{-i\hat{\varphi}_{\text{ext}}} + \text{c.c.}\right) \\
&= \frac{1}{2}\left(\left[\sum_{n_2} |n_2+1\rangle\langle n_2|\right]\right. \\
&\qquad \left.\otimes \left[\sum_{n_1} |n_1-1\rangle\langle n_1|\right] \otimes \mathbb{I}^{(3)}\right) e^{-i\varphi_{\text{ext}}} + \text{c.c.} \\
&= \frac{1}{2} e^{-i\varphi_{\text{ext}}} \sum_{n_{1,2,3}} |n_1-1, n_2+1, n_3\rangle \langle n_1, n_2, n_3| + \text{c.c.}
\end{aligned} \tag{7}$$

5. Physically this corresponds to a CP exchange between island 1 and island 2. An example of a term could be

$$\frac{1}{2} e^{-i\varphi_{\text{ext}}} |-1, 1, 0\rangle \langle 0, 0, 0| + \frac{1}{2} e^{+i\varphi_{\text{ext}}} |0, 0, 0\rangle \langle -1, 1, 0|, \tag{8}$$

which would be a pair of symmetrical off-diagonal elements in the matrix of Fig. 2.

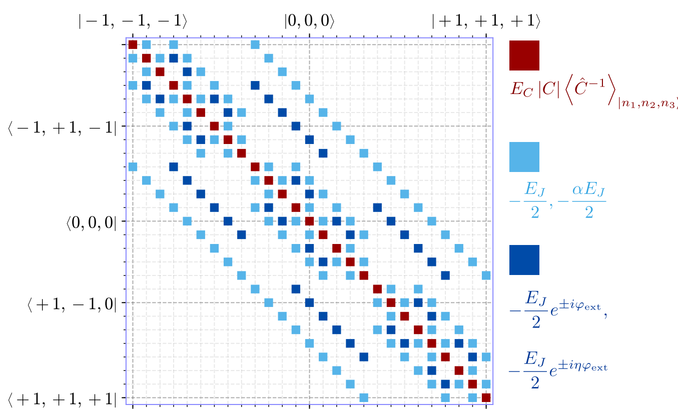


FIG. 2. **Hamiltonian in the CP-basis representation for 3-CP-states-per-island (27 system states):** Purple square denote the kinetic terms that all fall on the main diagonal. Light blue squares denote simple off-diagonal terms distributed symmetrically about the main diagonal, arising from e.g. $\cos(\varphi_2)$. Dark blue squares are have an additional flux dependence $e^{i\varphi_{\text{ext}}}, e^{i\eta\varphi_{\text{ext}}}$, arising from e.g. $\cos(\varphi_2 - \varphi_1 - \varphi_{\text{ext}})$.

To decide on the number of states for the simulation, we took a sufficiently complete system state of 19 interacting CPs, and methodically "switched off" interactions between the high-CP-number states. I logged the deviation of the energy spectra of Fig. 3, where it shows that 9-CP-states describes the system almost as well as with the complete system state.

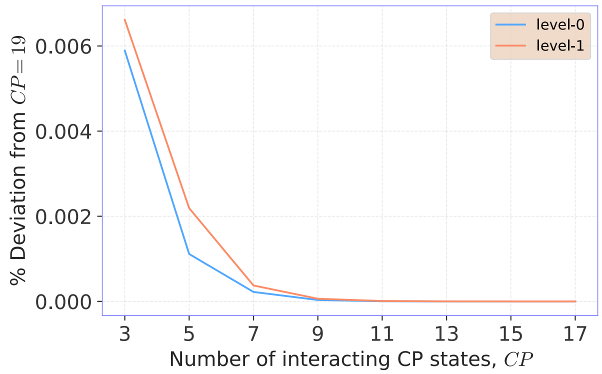


FIG. 3. **Choosing the lowest number of interacting CP, that would capture the nature of perturbation effects:** The energy levels for state $|0\rangle$ (blue) and $|1\rangle$ (red) were simulated for different numbers of interacting CP and compared to the simulation with 19CP (which we assume to be the most accurate and most intense simulation for the system). Consistency was reached for simulations with 9CP.

## III. TRANSITION MATRIX ELEMENTS

Here we derive the transition matrix element used in the main paper

$$\langle g| \hat{V}_2 |e\rangle \equiv \frac{E_C}{2\,|e|\,(1+\alpha)} \langle g| \left[\hat{n}_1 + 2\hat{n}_2 + \hat{n}_3\right] |e\rangle \,. \tag{9}$$

Microwaves in the transmission line, with voltage $V_{\mathrm{mw}} = |V_{\mathrm{mw}}| \cos(\omega_{21} t)$ are coupled via capacitor $\mathrm{C}_{\mathrm{coupling}}$ to the qubit. Transitions $|1\rangle \leftrightarrow |2\rangle$ stimulated by this driving generate a qubit voltage of $V_2 = \langle 1| \hat{V}_2 |2\rangle$

Expressing the voltage on the different islands:

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \vec{V} = \vec{Q}C^{-1} = 2eC^{-1}\vec{n} = \frac{2e}{|C|}\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2+\alpha & -1 \\ 0 & -1 & 2 \end{pmatrix}^{-1}\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$$
$$= \frac{2e}{|C|}\frac{1}{4+4\alpha}\begin{pmatrix} 3+2\alpha & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3+2\alpha \end{pmatrix}\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \tag{10}$$

one can read off

$$\hat{V}_2 = \frac{e}{|C|\,(1+\alpha)}\left[\hat{n}_1 + 2\hat{n}_2 + \hat{n}_3\right]. \tag{11}$$

Thus, the transition matrix element

$$\langle g|\,\hat{V}_2\,|e\rangle \equiv \frac{E_C}{2\,|e|\,(1+\alpha)}\,\langle g|\left[\hat{n}_1 + 2\hat{n}_2 + \hat{n}_3\right]|e\rangle\,. \tag{12}$$