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# CHAPTER 1

# FUNDAMENTAL ALGEBRAIC LAWS AND OPERATIONS

**Basic Attacks and Strategies for Solving Problems in this Chapter. See pages 1 to 8 for step-by-step solutions to problems.**

There are fundamental algebraic properties that describe the way arithmetic and algebraic operations involving addition and multiplication of numbers and variable expressions are handled. These properties for the addition operation include the

commutative property of addition

$[a + b = b + a]$,

associative property of addition

$[a + (b + c) = (a + b) + c]$,

inverse property of addition

$[a + (-a) = 0]$,

and identity property of addition

$[a + 0 = a]$.

For the multiplication operation, the properties are the

commutative property of multiplication

$[ab = ba]$,

associative property of multiplication

$[a(bc) = (ab)c]$,

multiplicative inverse property

$[a(1/a) = (1/a)a = 1, a \neq 0]$,

and the multiplicative identity property

$$[a \cdot 1 = 1 \cdot a = a].$$

One general law that governs the operation of multiplication over addition is the distributive property

$$[a(b + c) = ab + ac].$$

In determining the sum of two real numbers observe whether they have the same or different signs. If the signs of the numbers are the same, add the absolute values of the numbers and attach the sign of the addends. If the signs are unlike, then find the difference between the absolute values of the numbers and attach the sign of the number with the greatest absolute value. If subtraction is involved, the first step is to rewrite the subtraction process by using addition of the opposite, that is, $x - y = x + (-y)$. Then, proceed as indicated with addition.

With respect to multiplication and division of two real numbers, the pattern is simply that the product or quotient is positive if both numbers involved have like signs. On the other hand, the product or quotient is negative if the sign of the numbers involved are unlike.

When evaluating a mathematical expression, it is important to perform the operations in an order that begins with the expression in the innermost parentheses, brackets, or braces first and working outward. In the process of working outward, a hierarchical order must be observed. First, simplify all numbers or expressions with exponents, working from left to right if more than one of these expressions is present. Second, do all multiplications and divisions from left to right; and then perform all additions and subtractions from left to right. For instance, to simplify the expression given by

$$3[-4(2 + 3)^2 + 2(2 - 4)^3] + 10,$$

first perform the exponential operations of the expressions in the innermost symbols of grouping, from left to right. Thus,

$$\begin{aligned} 3[-4(2 + 3)^2 + 2(2 - 4)^3] + 10 &= 3[-4(5)^2 + 2(-2)^3] + 10 \\ &= 3[-4(25) + 2(-8)] + 10. \end{aligned}$$

Next, simplify within the brackets, from left to right, the multiplication operations to obtain

$$= 3[-100 + (-16)] + 10.$$

Next, the distributive property can be used to multiply 3 by the terms in the brackets to obtain

$$= -300 + (-48) + 10.$$

Finally, add the above terms, from left to right, to get the results as follows:

$$= -348 + 10 = -338$$

# Step-by-Step Solutions to Problems in this Chapter, "Fundamental Algebraic Laws and Operations"

• **PROBLEM 1**

Find the sum 8 + (- 3).

Solution: The sum of 8 + (- 3) can be obtained by using facts from arithmetic and the associative law:

$$8 + (-3) = (5 + 3) + (-3)$$

Use the associative law of addition (a + b) + c = a + (b + c):

$$= 5 + [3 + (-3)]$$

Using the additive inverse property, a + (- a) = 0:

$$= 5 + 0$$

Using the additive identity property, a + 0 = a

$$= 5.$$

• **PROBLEM 2**

Show that (- 2) + (- 3) = - 5.

Solution: This small problem illustrates some of the basic ideas involved in mathematical proof. We know that (- 2) + (- 3) is an integer because the integers are closed under addition. To show that this integer is - 5, we ask ourselves what property is characteristic of - 5.

$$5 + (-5) = 0,$$

by the additive inverse property, a + (- a) = 0. Moreover, - 5 is the only number which when added to 5 gives 0; for if (5 + b) = 0, by the additive identity, a = a + 0, - 5 = - 5 + 0 = - 5 + (5 + b), by our hypothesis, 5 + b = 0

| | |
|---|---|
| $= (-5 + 5) + b$ | by associative law of addition $a + (b+c) = (a+b)+c$. |
| $= 0 + b$ | by additive inverse property, $a + (-a) = 0$ |
| $= b + 0$ | by commutative law of addition , $a + b = b + a$ |
| $= b$ | by additive identity , $a + 0 = a$ |

Thus, $-5 = b$, proving that $-5$ is the only number which when added to 5 gives 0.

We therefore see that $(-2) + (-3) = -5$ if and only if $5 + [(-2) + (-3)] = 0$. We show below that this sum is zero.

$$5 + [(-2) + (-3)] = [3 + 2] + [(-2) + (-3)]$$

$$= 3 + \{2 + [(-2) + (-3)]\},$$

by associative law of addition, $(a + b) + c = a + (b + c)$;
$$= 3 + [0 + (-3)],$$

by additive inverse property, $a + (-a) = 0$; $= 3 + [(-3) + 0]$, by commutative law of addition, $a + b = b + a$;

$$= 3 + (-3),$$

by additive identity property, $a + 0 = a$; $= 0$, by additive inverse property, $a + (-a) = 0$.

Thus we have shown (a) $5 + (-5) = 0$

(b) $(-5)$ is the only number which when added to 5 equals 0.

(c) $5 + (-5) = 0 = 5 + [(-2) + (-3)]$ and therefore

$$(-5) = (-2) + (-3),$$

completing our proof.

**• PROBLEM 3**

Find the quotient $q$ and remainder $r$ upon dividing 575 by 21.

Solution: If we divide 575 by 21, we obtain

$$\begin{array}{r} 27 \\ 21\overline{)575} \\ \underline{42} \\ 155 \\ \underline{147} \\ 8 \end{array}$$

The quotient is 27 and the remainder is 8.

Check: To check the quotient and the remainder obtained, multiply the quotient by the divisor and then add the remainder to this product. The sum should be equal to the dividend.

$$(27)(21) + 8 = 567 + 8 = 575.$$

Hence, the sum is equal to the dividend, 575. Therefore, the quotient and the remainder obtained are correct.

**• PROBLEM 4**

Evaluate $2 - \{5 + (2 - 3) + [2 - (3 - 4)]\}$

Solution: When working with a group of nested parentheses, we evaluate the innermost parenthesis first.

Thus, $2 - \{5 + (2 - 3) + [2 - (3 - 4)]\}$

$$= 2 - \{5 + (2 - 3) + [2 - (-1)]\}$$

$$= 2 - \{5 + (-1) + [2 + 1]\}$$

$$= 2 - \{5 + (-1) + 3\}$$

$$= 2 - \{4 + 3\}$$

$$= 2 - 7$$

$$= -5.$$

• **PROBLEM 5**

Simplify $4[-2(3 + 9) \div 3] + 5$.

Solution: To simplify means to find the simplest expression. We perform the operations within the innermost grouping symbols first. That is $3 + 9 = 12$.

Thus, $4[-2(3 + 9) \div 3] + 5 = 4[-2(12) \div 3] + 5$

Next we simplify within the brackets:

$$= 4[-24 \div 3] + 5$$

$$= 4 \cdot (-8) + 5$$

We now perform the multiplication, since multiplication is done before addition:

$$= -32 + 5$$

$$= -27$$

Hence, $4[-2(3 + 9) \div 3] + 5 = -27$.

• **PROBLEM 6**

Is the set of all natural numbers from 1 to 10 a closed system under addition?

Solution: For $\{1,2,3,\ldots 10\}$ to be closed with respect to addition, the sum of any two numbers in this set must also be a member of this set. The set of all natural numbers from 1 to 10, inclusive, is therefore not a closed system under addition for it would not be correct to say that given any two numbers in the set there is a number in the set called their sum. For instance, 4 and 7 are in the set but their sum, 11, is not.

• **PROBLEM 7**

Simplify the following expressions, removing the parentheses.

1) $a + (b - c)$

2) $ax - (by - c)$

3) $2 - (-x + y)$.

Solution: 1) Place a factor of 1 between the + (plus) sign and the term in the parenthesis. This procedure does not change the value of the entire expression. Hence, $a + (b-c) = a + 1(b-c)$

$$= a + 1(b) + 1(-c) \text{ distributing}$$
$$= a + b - c$$

2) Again, place a factor of 1 between the - (minus) sign and the term in the parenthesis. Again, this procedure does not change the value of the entire expression. Hence,

$$\begin{aligned} ax - (by - c) &= ax - 1(by - c) \\ &= ax - 1(by) - 1(-c) \text{ distributing} \\ &= ax - by + c \end{aligned}$$

3) Again, place a factor of 1 between the - (minus) sign and the term in parenthesis. Again, this procedure does not change the value of the entire expression. Hence,

$$\begin{aligned} 2 - (-x + y) &= 2 - 1(-x + y) \\ &= 2 - 1(-x) - 1(y) \text{ distributing} \\ &= 2 + x - y \end{aligned}$$

**• PROBLEM 8**

Evaluate $3s - [5t + (2s - 5)]$

Solution: We always evaluate the expression within the innermost parentheses first, when working with a group of nested parentheses. Thus,

$$\begin{aligned} 3s - [5t + (2s - 5)] &= 3s - [5t + 2s - 5] \\ &= 3s - 5t - 2s + 5 \\ &= 3s - 2s - 5t + 5 \\ &= s - 5t + 5. \end{aligned}$$

**• PROBLEM 9**

Simplify

$$8x^2 - [7x - (x^2 - x + 5y)] + (2x - 3y).$$

Solution: Where a succession of arithmetic operations is involved, appropriate grouping symbols indicate clearly how these algebraic operations are to be performed; that is, we perform them before other operations. In this problem we also have grouping symbols within grouping symbols. Therefore, we perform the operation in the innermost parentheses first. Hence, multiply the terms in the parentheses by minus one in order to remove the parentheses. Furthermore, they can be removed from the last two terms.

$$8x^2 - [7x - (x^2 - x + 5y)] + (2x - 3y) = 8x^2 - [7x - x^2 + x - 5y] + 2x - 3y$$

Remove the brackets by multiplying the terms inside by minus one. Then,

$$8x^2 - \left[7x - (x^2 - x + 5y)\right] + (2x - 3y) = 8x^2 - 7x + x^2 - x + 5y + 2x - 3y$$

Now group like terms. Then perform the indicated operations from left to right. Thus, we obtain:

$$8x^2 - \left[7x - (x^2 - x + 5y)\right] + (2x - 3y) = 8x^2 + x^2 - 7x - x + 2x + 5y - 3y$$

$$= 9x^2 - 6x + 2y.$$

In this example, we have found the algebraic sum of these three quantities: $8x^2$ and $-\left[7x - (x^2 - x + 5y)\right]$ and $(2x - 3y)$.

**• PROBLEM 10**

Simplify: $3a - 2\{3a - 2[1 - 4(a - 1)] + 5\}$.

Solution: When working with several sets of brackets and, or parentheses, we work from the inside out. That is, we use the law of distribution throughout the expression, starting from the innermost parentheses, and working our way out. Hence in this case we have: $2\{3a - 2[1 - 4(a - 1)] + 5\}$ and we note that $(a - 1)$ is the innermost parenthesis, so our first step is to distribute the $(-4)$. Thus, we obtain:

$$3a - 2[3a - 2(1 - 4a + 4) + 5].$$

We now find that $(1 - 4a + 4)$ is in our innermost parentheses. Combining terms we obtain:

$$(1 - 4a + 4) = (5 - 4a)\ ;$$

hence, $\quad 3a - 2[3a - 2(1 - 4a + 4) + 5] = 3a - 2[3a - 2(5 - 4a) + 5]$

and since $(5 - 4a)$ is in the innermost parentheses we distribute the $(-2)$, obtaining:

$$3a - 2(3a - 10 + 8a + 5).$$

We are now left with the terms in our last set of parentheses, $(3a - 10 + 8a + 5)$. Combining like terms we obtain:

$$(3a - 10 + 8a + 5) = (11a - 5)$$

hence, $\quad 3a - 2(3a - 10 + 8a + 5) = 3a - 2(11a - 5)$.

Distributing the $(-2)$, $\quad = 3a - 22a + 10$

combining terms, $\quad = -19a + 10$.

Hence $\quad 3a - 2\{3a - 2[1 - 4(a - 1)] + 5\} = -19a + 10$.

**• PROBLEM 11**

(a) Add, $3a + 5a$

(b) Factor, $5ac + 2bc$.

Solution: (a) To add $3a + 5a$, factor out the common factor $a$. Then,

$$3a + 5a = (3 + 5)a = 8a.$$

(b) To factor $5ac + 2bc$, factor out the common factor $c$. Then,

$$5ac + 2bc = (5a + 2b)c.$$

• **PROBLEM 12**

Express each of the following as a single term.

(a) $3x^2 + 2x^2 - 4x^2$ (b) $5axy^2 - 7axy^2 - 3xy^2$

Solution: (a) Factor $x^2$ in the expression.

$$3x^2 + 2x^2 - 4x^2 = (3 + 2 - 4)x^2 = 1x^2 = x^2.$$

(b) Factor $xy^2$ in the expression and then factor a.

$$\begin{aligned} 5axy^2 - 7axy^2 - 3xy^2 &= (5a - 7a - 3)xy^2 \\ &= [(5-7)a - 3]xy^2 \\ &= (-2a - 3)xy^2. \end{aligned}$$

• **PROBLEM 13**

Simplify $x = a + 2[b - (c - a + 3b)]$.

Solution: When working with several groupings, we perform the operations in the innermost parenthesis first, and work outward. Thus, we first subtract $(c - a + 3b)$ from $b$:

$$x = a + 2[b - (c - a + 3b)] = a + 2(b - c + a - 3b)$$

Combining terms,

$$= a + 2(-c + a - 2b)$$

distributing the 2, $= a - 2c + 2a - 4b$

combining terms, $= 3a - 2c - 4b$

To check that $a + 2[b - (c - a + 3b)]$ is equivalent to $3a - 2c - 4b$, replace a,b, and c by any values. Letting $a = 1$, $b = 2$, $c = 3$, the original form

$$\begin{aligned} a + 2[b - (c - a + 3b)] &= 1 + 2[2 - (3 - 1 + 3\cdot 2)] \\ &= 1 + 2[2 - (3 - 1 + 6)] \\ &= 1 + 2[2 - 8] \\ &= 1 + 2(-6) \\ &= 1 + (-12) \\ &= -11 \end{aligned}$$

The final form, $3a - 2c - 4b = 3(1) - 2(3) - 4(2) = 3 - 6 - 8$

$$= -11$$

Thus, both forms yield the same result.

• **PROBLEM 14**

Use the field properties to derive the equation $x = 5$ from the equation $5x - 3 = 2(x + 6)$.

Solution: $5x - 3 = 2(x + 6)$ Given

| | |
|---|---|
| $5x - 3 = 2x + 12$ | distributive property of multiplication over addition |
| $(5x - 3) + (-2x) = 2x + 12 + (-2x)$ | Additive Property (-2x) |
| $3x - 3 = 12$ | Simplifying |
| $(3x - 3) + 3 = 12 + 3$ | additive property (+3) |
| $3x = 15$ | Simplifying |
| $\frac{1}{3}\cdot(3x) = \frac{1}{3}\cdot 15$ | Multiplicative Property($\frac{1}{3}$) |
| $x = 5$ | Simplifying |

We could also derive $5x - 3 = 2(x + 6)$ from $x = 5$ by reversing the steps in the solution. Let us see if 5 will make the equation $5x - 3 = 2(x + 6)$ true.

$$5(5) - 3 \stackrel{?}{=} 2(5 + 6)$$

$$22 = 22$$

Two equations are equivalent if and only if they have the same solution set. Since $5x - 3 = 2(x + 6)$ and $x = 5$ have the same solution set, $\{5\}$, the two equations are equivalent.

**• PROBLEM 15**

Approximate:

$$A = \frac{\pi \times \sqrt{2} \times 2.17}{(6.83)^2 + (1.07)^2}$$

Solution: We use the following approximate values:

$$\pi = 3.1416 \cong 3$$

$$\sqrt{2} = 1.414 \cong 1.5$$

$$2.17 \cong 2$$

$$(6.83)^2 \cong 7^2 = 49 \cong 50$$

$$(1.07)^2 \cong 1^2 = 1$$

Then,

$$A \cong \frac{3 \times 1.5 \times 2}{50 + 1} = \frac{9}{51} \cong \frac{10}{50} = .2$$

**• PROBLEM 16**

Evaluate $p = \dfrac{(a - b)(ab + c)}{(cb - 2a)}$

when $a = +2$, $b = -\frac{1}{2}$, and $c = -3$.

Solution: Inserting the given values of a, b, and c

$$p = \frac{[+2 - (-\frac{1}{2})][(+2)(-\frac{1}{2}) + (-3)]}{[(-3)(-\frac{1}{2}) - 2(+2)]}$$

$$= \frac{[+2 + \frac{1}{2}][-1 - 3]}{[+1\frac{1}{2} - 4]}$$

$$= \frac{(2\tfrac{1}{2})(-\ 4)}{-\ (2\tfrac{1}{2})}$$

The $2\tfrac{1}{2}$ in the numerator cancels the $2\tfrac{1}{2}$ in the denominator.

$$p = \frac{-\ 4}{-\ 1}$$

Multiplying numerator and denominator by - 1

$$p = \frac{+\ 4}{+\ 1}$$

$$p = +\ 4.$$

# CHAPTER 2

# LEAST COMMON MULTIPLE/ GREATEST COMMON DIVISOR

**Basic Attacks and Strategies for Solving Problems in this Chapter. See pages 9 to 11 for step-by-step solutions to problems.**

The *Least Common Multiple* (LCM) of a set of two or more integers is the smallest integer that can be found such that it is divisible by each integer from the given set. To determine the Least Common Multiple of two or more integers, the first step is to factor each of the numbers into their prime factors. Then, select among all the factors a list of those which are unique prime factors of the numbers. Next, determine the exponent to be used for each selected factor by finding the highest number of times the factor appeared in either of the factorizations. The product of the unique prime factors with the appropriate highest exponents is the LCM. For example, the LCM of 12 and 21 is 84. This is determined as follows: prime factors of 12 are

$$2 \times 2 \times 3$$

and for 21 are

$$3 \times 7.$$

The unique prime factors are 2, 3, and 7, respectively, of which the highest number of times 2 appears is twice, 3 appears once, and 7 appears once. So, the product or LCM is:

$$2^2 \times 3 \times 7 = 4 \times 21 = 84.$$

The LCM can be very useful in determining the least common denominator in the addition and/or subtraction of two or more fractions.

The *Greatest Common Divisor* (GCD) of two or more integers is the greatest integer that will divide into all the integers. When finding the GCD, the first step is to factor each number involved into their prime factors. The second step is to select among each of the factorizations the unique common factors. Now determine the exponent to be used for each selected factor by finding the smallest number of times the factor appears. The final step is to find the product of the selected factors with appropriate exponents which is the GCD. For instance, the GCD of 24 and 60 is 12. This solution is found by first factoring 24 and 60 into prime factors as follows:

$$24 = 2 \times 2 \times 2 \times 3 \quad \text{and} \quad 60 = 2 \times 2 \times 3 \times 5.$$

The unique common factors among the two factorizations are 2 and 3 only. The smallest number of times 2 appears among the factorizations is twice and for 3 is once. Thus, the final product is composed of:

$$2^2 \times 3^1 = 4 \times 3 = 12$$

which is the GCD of 24 and 60.

The GCD can be very useful in reducing a fraction to the lowest terms.

# Step-by-Step Solutions to Problems in this Chapter, "Least Common Multiple/ Greatest Common Divisor"

**• PROBLEM 17**

Find the least common multiple (lcm) of 15 and 18.

Solution: Some of the integers divisible by 15 are

$$15, 30, 45, 60, 75, 90, 105, \ldots$$

Some of the integers divisible by 18 are

$$18, 36, 54, 72, 90, 108, \ldots$$

The smallest positive integer divisible by both 15 and 18 is 90. Thus,

$$\text{lcm}\{15, 18\} = 90$$

Another method for finding lcm{15,18} is the following:
Factor 15 and 18 into their prime factors.

$$15 = 3 \cdot 5$$

$$18 = 2 \cdot 3 \cdot 3$$

Now, take the different factors of the two numbers and multiply them together. The exponent to be used for each factor is the highest number of times that the factor appears in either number (15 or 18). The product obtained will be the lcm{15,18}. Hence:

$$\text{lcm}\{15, 18\} = 2^1 \cdot 3^2 \cdot 5^1 = 2(9)(5) = 90.$$

**• PROBLEM 18**

Find the Least Common Multiple, LCM, of 26, 39, and 66.

Solution: Write each number as the product of primes:

$$26 = 2(13), \quad 39 = 3(13), \quad 66 = 2(3)(11)$$

The LCM is obtained by using the greatest power of each prime, only once, to form a product. Thus,

$$\text{LCM} = 2(3)(11)(13)$$

$$= 858.$$

**• PROBLEM 19**

Find the Least Common Multiple, LCM, of 12, 18, 21, 25 and 35.

Solution: We want to express each number as a product of prime factors:

$12 = 2^2(3)$, $18 = 2(3^2)$, $21 = 3(7)$,

$25 = 5^2$, $35 = 5(7)$

Find the LCM by retaining the highest power of each distinct factor and multiplying them together, making sure to use each factor only once regardless of the number of times it appears. Thus,

$$\text{LCM} = (2^2)(3^2)(5^2)(7) = (4)(9)(25)(7)$$

$$= 6300$$

**• PROBLEM 20**

Find the greatest common divisor $\{15,28\}$.

Solution: If 15 and 28 are factored completely into their respective prime factors, $15 = 3 \cdot 5$ and $28 = 2 \cdot 2 \cdot 7$

Since 1 divides every integer, and since 15 and 28 possess no common prime factors, it follows that

$$\gcd\{15,28\} = 1.$$

If the gcd of two integers is 1, then the two integers are said to be relatively prime. Since $\gcd\{15,28\} = 1$, the integers 15 and 28 are relatively prime.

**• PROBLEM 21**

Find the greatest common divisor and the least common multiple of 16 and 12.

Solution: Factor the two given numbers into their prime factors.

$$12 = 2 \cdot 2 \cdot 3$$

$$16 = 2 \cdot 2 \cdot 2 \cdot 2$$

The greatest common divisor of 16 and 12, or $\gcd\{12,16\}$, is the largest number which divides both 16 and 12, (largest common factor).

$$\frac{12}{16} = \frac{2 \cdot 2 \cdot 3}{2 \cdot 2 \cdot 2 \cdot 2} = \frac{(4)(3)}{(4)(2)(2)}$$

Hence, $\gcd\{12,16\} = 4$ .

The least common multiple of 16 and 12, or $\operatorname{lcm}\{12,16\}$, is obtained in the following way. Take the different factors of the two numbers and multiply them together. The exponent to be used for each factor is the highest number of times that the factor appears in either number (12 or 16). The product obtained will be the $\operatorname{lcm}\{12,16\}$. Hence:

$$\operatorname{lcm}\{16,12\} = 2^4 \cdot 3^1 = (16)(3) = 48 .$$

**• PROBLEM 22**

Find the greatest common divisor of 24 and 40. Also, find the least common multiple of 24 and 40.

Solution: To find the greatest common divisor of 24 and 40, or $\gcd\{24,40\}$, we write down the set of all positive integers which divide both 24 and 40. Thus we obtain the two sets

$$\{1,2,3,4,6,8,12,24\} \text{ for } 24$$
$$\{1,2,4,5,8,10,20,40\} \text{ for } 40$$

Those integers dividing both 24 and 40 are in the intersection of these two sets. Thus,

$$\{1,2,3,4,6,8,12,24\} \cap \{1,2,4,5,8,10,20,40\} = \{1,2,4,8\}$$

The largest element in this last set is 8. Thus,

$$8 = \gcd\{24,40\} .$$

Another method for finding the $\gcd\{24,40\}$ is called the factoring technique. Factor the two given numbers into their prime factors.

$$24 = 2\cdot 2\cdot 2\cdot 3$$
$$40 = 2\cdot 2\cdot 2\cdot 5$$

The greatest common divisor of any two numbers is the largest number which divides both of those numbers. Therefore,

$$\frac{24}{40} = \frac{2\cdot 2\cdot 2\cdot 3}{2\cdot 2\cdot 2\cdot 5} = \frac{(8)(3)}{(8)(5)} .$$ Hence,

$$\gcd\{24,40\} = 8 .$$

The following technique is the definition for finding the least common multiple of 24 and 40, or $\operatorname{lcm}\{24,40\}$. To find the $\operatorname{lcm}\{24,40\}$, we write down the set of all positive integer multiples of both 24 and 40. Then we obtain

$$\{24,48,72,96,120,144,168,192,216,240,264,\ldots\} \text{ for } 24$$

$$\{40,80,120,160,200,240,280,\ldots\} \qquad \text{for } 40$$

The integers which are multiples of both 24 and 40; that is, common multiples of 24 and 40, are in the intersection of these two sets. This is the set $\{120,240,\ldots\}$. The smallest element of this set is 120. Hence, $\operatorname{lcm}\{24,40\} = 120$. Another method for finding the $\operatorname{lcm}\{24,40\}$ is called the factoring technique. Factor the two given numbers into their prime factors.

$$24 = 2\cdot 2\cdot 2\cdot 3$$
$$40 = 2\cdot 2\cdot 2\cdot 5$$

Now, take the different factors of the two numbers and multiply them together. The exponent to be used for each factor is the highest number of times that the factor appears in either number (24 or 40). The produce obtained will be the $\operatorname{lcm}\{24,40\}$. Hence:

$$\operatorname{lcm}\{24,40\} = 2^3\cdot 3^1\cdot 5^1 = (8)(3)(5) = (24)(5) = 120.$$

# CHAPTER 3

# SETS AND SUBSETS

**Basic Attacks and Strategies for Solving Problems in this Chapter. See pages 12 to 16 for step-by-step solutions to problems.**

There are two basic operations used to combine sets — *union* and *intersection*. Other operations involving sets include the *complement* and *cartesian* product.

When finding the union of two or more sets, the first step is to examine the sets and determine all common elements that belong to the individual sets and all the unique elements (not common among the sets). Then, the union of the sets, using the roster or list method, is the set of all the common elements written with no repetitions together with all of the unique elements. For example, the union of set $A = \{3, 4, 5\}$ and set $B = \{4, 5, 6\}$ is given by the following set:

$$A \cup B = \{3, 4, 5, 6\}.$$

To find the intersection of two or more sets, simply identify among the sets all the elements which are common. Then, the intersection, using the roster method, is the set of only the common elements among the sets written with no repetitions. For example, the intersection of sets $A$ and $B$ above is given by:

$$A \cap B = \{4, 5\}.$$

In addition to the roster or list method of representing the union and intersection of sets, set builder notation and Venn diagrams are used.

The procedure for finding the complement of a set is to first determine a universal set $U$ and a set $A$ whose elements are a part of $U$. Then, all of the elements of $U$ which do not belong to $A$ form the complement set of $A$, given by $A'$.

The Cartesian product of two sets, say $A$ and $B$ and denoted by $A \times B$, is a set formed by all possible ordered pairs where the first component comes from set $A$ and the second component from set $B$. For example, if set $A = \{1, 2\}$ and set $B = \{1, 4\}$, then $A \times B$ is given as follows:

$$A \times B = \{(1, 1), (1, 4), (2, 1), (2, 4)\}.$$

A subset of a set $A$ is a set in which each of its members belong to the original set. In addition, the empty set is a subset of any set. For example, the set $\{x\}$ and the empty set are two of the subsets of set $A = \{x, y\}$. To find all the possible subsets $N$ of any set, one can use the formula

$$N = 2^n,$$

where $n$ is the number of elements in the original set. The actual listing of subsets is found by forming a set of each possible pairing of the elements in the original set plus the empty set. For example, the set

$$A = \{x, y\}$$

has 4 subsets since

$$N = 2^2 = 4.$$

The subsets are $\{x\}$, $\{y\}$, $\{x, y\}$, and the empty set.

# Step-by-Step Solutions to Problems in this Chapter, "Sets and Subsets"

• **PROBLEM 23**

If $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 3, 4, 5, 6\}$, find $A \cup B$.

Solution: The symbol $\cup$ is used to denote the union of sets. Thus $A \cup B$ (which is read "the union of A and B") is the set of all elements that are in either A or B or both. In this problem, if,

$$A = \{1, 2, 3, 4, 5\} \text{ and } B = \{2, 3, 4, 5, 6\},$$

then

$$A \cup B = \{1, 2, 3, 4, 5, 6\}.$$

• **PROBLEM 24**

If $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 3, 4, 5, 6\}$, find $A \cap B$.

Solution: The intersection of two sets A and B is the set of all elements that belong to both A and B; that is, all elements common to A and B. In this problem, if

$$A = \{1, 2, 3, 4, 5\} \text{ and } B = \{2, 3, 4, 5, 6\},$$

then

$$A \cap B = \{2, 3, 4, 5\}.$$

• **PROBLEM 25**

If $A = \{2,3,5,7\}$ and $B = \{1,-2,3,4,-5,\sqrt{6}\}$, find
(a) $A \cup B$ and (b) $A \cap B$.

Solution: (a) $A \cup B$ is the set of all elements in A or in B or in both A and B, with no element included twice in the union set.

$$A \cup B = \{1,2,-2,3,4,5,-5,\sqrt{6},7\}$$

(b) $A \cap B$ is the set of all elements in both A and B.

$$A \cap B = \{3\}$$

Sometimes two sets have no elements in common. Let $S = \{3,4,7\}$ and $T = \{2,-4,6\}$. What is the intersection of S and T? In this case $S \cap T$ has no elements. Hence $S \cap T = \emptyset$, the empty set. In that case, the sets are said to be disjoint.

The set of all elements entering a discussion is called the universal set, U, When the universal set is not given, we assume it to be the set of real numbers. The set of all elements in the universal set that are not elements of A is called the complement of A, written $\bar{A}$ .

Show that $(A \cap B) \cap C = A \cap (B \cap C)$.

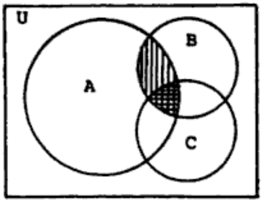



Fig. 1 $(A \cap B) \cap C$

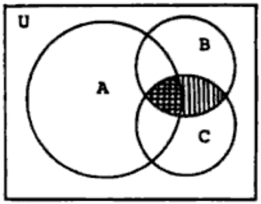



Fig. 2 $A \cap (B \cap C)$

Solution: In Figure 1 the vertically shaded area represents $A \cap B$, and the horizontally shaded area represents the points common to the set $(A \cap B)$ and the set C, that is $(A \cap B) \cap C$. Similarly, in Figure 2, the vertically shaded area represents $B \cap C$, and the horizontally shaded area represents the points common to the set $(B \cap C)$ and the set A, that is $A \cap (B \cap C)$. Since the two horizontally shaded areas in the two figures are the same,

$$(A \cap B) \cap C = A \cap (B \cap C).$$

• PROBLEM 27

If $U = \{1, 2, 3, 4, 5\}$ and $A = \{2, 4\}$, find A'.


Solution: The complement of a set A in U is the set of all elements of U that do not belong to A. The symbol A' (or, sometimes, $\bar{A}$, $\sim A$, or $\tilde{A}$) denotes the complement of A in U. The figure gives a representation of A', the complement of A in U. In this problem, since,

$$U = \{1, 2, 3, 4, 5\}$$

and

$$A = \{2, 4\},$$

$$A' = \{1, 3, 5\}.$$

• PROBLEM 28

$U = \{1,2,3,4,5,6,7,8,9,10\}$, $P = \{2,4,6,8,10\}$, $Q = \{1,2,3,4,5\}$.
Find (a) $\bar{P}$ and (b) $\bar{Q}$.

Solution: $\bar{P}$ and $\bar{Q}$ are the complements of P and Q respectively.

That is, $\bar{P}$ is the set of all elements in the universal set, U, that are not elements of P, and $\bar{Q}$ is the set of elements in U that are not in Q. Therefore,

(a) $\bar{P} = \{1,3,5,7,9\}$; (b) $\bar{Q} = \{6,7,8,9,10\}$

**• PROBLEM 29**

> If U = the set of whole numbers and E = the set of even whole numbers: find $\bar{E}$.

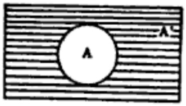
Solution: $\bar{E}$ is called the complement of E. $\bar{E}$ is the set of all elements in the universal set, U, that are not elements of E. Therefore,

$$\bar{E} = \{1,3,5,\ldots\},$$

the set of odd whole numbers.

**• PROBLEM 30**

> Show that the complement of the complement of a set is the set itself.




Solution: The complement of set A is given by A'. Therefore, the complement of the complement of a set is given by (A')'. This set, (A')', must be shown to be the set A; that is, that (A')' = A. In the figure the complement of the set A, or A', is the set of all points not in set A; that is, all points in the rectangle that are not in the circle. This is the shaded area in the figure. Therefore, this shaded area is A'. The complement of this set, or (A')', is the set of all points of the rectangle that are not in the shaded area; that is, all points in the circle, which is the set A. Therefore, the set (A')' is the same as set A; that is,

$$(A')' = A.$$

**• PROBLEM 31**

> If A = {1, 2, 3} and B = {5, 6}, find A x B and B x A.

Solution: The Cartesian product of two sets A and B, denoted by A x B, is the set of all ordered pairs (x, y) such that $x \in A$ and $y \in B$. In this problem, if A = {1, 2, 3} and B = {5, 6}, then the Cartesian product A x B is:

$$A \times B = \{(1, 5), (1, 6), (2,5), (2, 6), (3, 5), (3, 6)\}.$$

The Cartesian product B x A is the set of all ordered pairs (x, y) such that $x \in B$ and $y \in A$. Hence, the Cartesian product B x A is:

$B \times A = \{(5, 1), (5, 2), (5, 3), (6, 1), (6, 2),$
$(6, 3)\}$.

• **PROBLEM 32**

Let $M = \{1, 2\}$ and $N = \{p, q\}$. Find (a) $M \times N$, (b) $N \times M$, and (c) $M \times M$.

Solution: (a) $M \times N$ is the set of all ordered pairs in which the first component is a member of M and the second component is a member of N. Thus,

$$M \times N = \{(1,p), (1, q), (2, p), (2, q)\}.$$

Note that the number of elements in M is 2,

the number of elements in N is 2,

and the number of elements in $M \times N = 2 \times 2 = 4$.

(b) $N \times M$ is the set of all ordered pairs in which the first component is a member of N and the second component is a member of M. Thus,

$$N \times M = \{(p, 1), (q, 1), (p, 2), (q, 2)\}.$$

Once again note that the number of elements in $N \times M$ is $2 \times 2 = 4$.

(c) $M \times M$ is the set of all ordered pairs in which both components are members of M. Thus,

$$M \times M = \{(1, 1), (1, 2), (2, 1), 2, 2)\}.$$

Here too, the number of elements in $M \times M$ is $2 \times 2 = 4$.

• **PROBLEM 33**

List all the subsets of $C = \{1,2\}$.

Solution: $\{1\}$, $\{2\}$, $\{1,2\}$, $\emptyset$, where $\emptyset$ is the empty set. Each set listed in the solution contains at least one element of the set C. The set $\{2,1\}$ is identical to $\{1,2\}$ and therefore is not listed. $\emptyset$ is included in the solution because $\emptyset$ is a subset of every set.

• **PROBLEM 34**

Find four proper subsets of $P = \{n: n \in I, -5 < n \leq 5\}$.

Solution: $P = \{-4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$. All these elements are integers that are either less than or equal to 5 or greater than $-5$. A set A is a proper subset of P if every element of A is an element of B and in addition there is an element of B which is not in A.

(a) $B = \{-4, -2, 0, 2, 4\}$ is a subset because each element of B is an integer greater than $-5$ but less than or equal to 5. B is a proper subset because 3 is an element of P but not an element of B. We can write $3 \in P$ but $3 \notin B$.

(b) $C = \{3\}$ is a subset of P, since $3 \in P$. However, $5 \in P$ but $5 \notin C$. Hence, $C \subset P$.

(c) $D = \{-4, -3, -2, -1, 1, 2, 3, 4, 5\}$ is a proper subset of P, since each element of D is an element of P, but $0 \in P$ and $0 \notin D$.

(d) $\phi \subset P$, since $\phi$ has no elements. Note that $\phi$ is the empty set. $\phi$ is a proper subset of every set except itself.

# CHAPTER 4

# ABSOLUTE VALUES

**Basic Attacks and Strategies for Solving Problems in this Chapter.** **See pages 17 to 18 for step-by-step solutions to problems.**

When solving an equation containing one or more absolute values, the fundamental procedure is the use of the definition of absolute value. The definition states that:

(1) The absolute value of a positive number or positive expression is the positive of the number or expression itself,

(2) The absolute value of a negative number or negative expression is the opposite of the negative number or expression, and

(3) The absolute value of zero is zero.

If only one absolute value is in the given equation, then the application of the definition yields two equations. The next step is to solve each of these equations for the variable and check the results. For instance, to solve the equation

$$|x + a| = c,$$

where $a$ and $c$ are constants, we can write, according to the definition of absolute value, the following two equations:

$$x + a = c \quad \text{and} \quad x + a = -c.$$

Then, to get the solution of the original equation, solve each of the equations and check the results.

Suppose the equation contains two absolute values, such as

$$|ax + b| = |cx + d| + e,$$

where $a$, $b$, $c$, and $d$ are constants and $a$ and $c$ are not zero. Then, there are four possibilities for equations:

$$ax + b = cx + d + e, \tag{1}$$

$$-(ax + b) = cx + d + e, \tag{2}$$

$$ax + b = -(cx + d) + e, \tag{3}$$

$$-(ax + b) = -(cx + d) + e \tag{4}$$

Solve the appropriate equations and check the results in order to get the solution of the original equation.

# Step-by-Step Solutions to Problems in this Chapter, "Absolute Values"

• **PROBLEM 35**

Solve for x when $|x - 7| = 3$.

Solution: This equation, according to the definition of absolute value, expresses the conditions that $x - 7$ must be $3$ or $-3$, since in either case the absolute value is $3$. If $x - 7 = 3$, we have $x = 10$; and if $x - 7 = -3$, we have $x = 4$. We see that there are two values of x which solve the equation.

• **PROBLEM 36**

Solve for x when $|3x + 2| = 5$.

Solution: First we write expressions which replace the absolute symbols in forms of equations that can be manipulated algebraically. Thus this equation will be satisfied if either

$$3x + 2 = 5 \quad \text{or} \quad 3x + 2 = -5.$$

Considering each equation separately, we find

$$x = 1 \quad \text{and} \quad x = -\frac{7}{3}.$$

Accordingly, the given equation has two solutions.

• **PROBLEM 37**

Solve for x when $|5x + 4| = -3$.

Solution: In examining the given equation, it is seen that the absolute value of a number is set equal to a negative value. By definition of an absolute number, however, the number cannot be negative. Therefore the given equation has no solution.

• **PROBLEM 38**

Solve for x when $|5 - 3x| = -2$.

Solution: This problem has no solution, since the absolute value can never be negative and we need not proceed further.

• **PROBLEM 39**

Solve for x in $|2x - 6| = |4 - 5x|$.

Solution: There are four possibilities here. $2x - 6$ and $4 - 5x$

can be either positive or negative. Therefore,

$$2x - 6 = 4 - 5x \qquad (1)$$

$$-(2x - 6) = 4 - 5x \qquad (2)$$

and

$$2x - 6 = -(4 - 5x) \qquad (3)$$

$$-(2x - 6) = -(4 - 5x) \qquad (4)$$

Equations (2) and (3) result in the same solution, as do equations (1) and (4). Therefore it is necessary to solve only for equations (1) and (2). This gives:

$$x = \frac{10}{7}, -\frac{2}{3}.$$

**• PROBLEM 40**

Solve for x when $|2x - 1| = |4x + 3|$.

Solution: Replacing the absolute sysmbols with equations that can be handled algebraically according to the conditions implied by the given equation, we have:

$$2x - 1 = 4x + 3 \quad \text{or} \quad 2x - 1 = -(4x + 3) .$$

Solving the first equation, we have $x = -2$; solving the second, we obtain $x = -\frac{1}{3}$, thus giving us two solutions to the original equation. ( We could also write: $-(2x - 1) = -(4x + 3)$, but this is equivalent to the first of the equations above. )

# CHAPTER 5

# OPERATIONS WITH FRACTIONS

**Basic Attacks and Strategies for Solving Problems in this Chapter. See pages 19 to 34 for step-by-step solutions to problems.**

When simplifying a complex algebraic fraction, the usual procedure is to first combine or eliminate the sum and difference of fractions in the numerator and denominator. This can be achieved by multiplying both numerator and denominator by the least common multiple of the denominators of all fractions involved and then simplify by reducing the results to the lowest terms. This is the premier procedure for simplifying, especially complex algebraic fractions. For example, simplify the following expression by first noting that the LCD is $xy$, which is multiplied by both the numerator and denominator in the rational expression. Thus,

$$\frac{\frac{2}{y}+2}{\frac{1}{x}-3}=\frac{xy\left(\frac{2}{y}\right)+xy(2)}{xy\left(\frac{1}{x}\right)-xy(3)}=\frac{2x+2xy}{y-3xy}.$$

Another procedure for simplifying a complex algebraic fraction is to first find the LCD, then add and/or subtract the fractions in the numerator. Repeat the same procedure for the fractions in the denominator. Then, multiply the resulting fraction in the numerator by the reciprocal of the resulting fraction in the denominator and simplify by reducing the results to the lowest terms. In the example below notice that the expressions in the numerator have an LCD of y and those in the denominator have an LCD of $x$. Thus,

$$\frac{\frac{5}{y}+2}{3-\frac{2}{x}}=\frac{\frac{5}{y}+2\left(\frac{y}{y}\right)}{3\left(\frac{x}{x}\right)-\frac{2}{x}}=\frac{\frac{(5+2y)}{y}}{\frac{(3x-2)}{x}}$$

Now multiply the result in the numerator by the reciprocal of the result in the denominator and reduce to the lowest terms, if necessary, in order to get the final result as follows:

$$= \frac{5+2y}{y} \cdot \frac{x}{3x-2} = \frac{x(5+2y)}{y(3x-2)}.$$

When multiplying two or more algebraic fractions, the procedure is simply to first factor completely the expressions in the numerator and denominator and cancel all possible common factors. The second step is to multiply the numerators together and then the denominators together. Finally, simplify by reducing to the lowest terms. For example, to multiply the following algebraic fractions easily we first factor, cancel, and then reduce to the lowest terms:

$$\frac{3x-6}{5x-20} \cdot \frac{10x-40}{27x-54} = \frac{3(x-2)}{5(x-4)} \cdot \frac{10(x-4)}{27(x-2)} = \frac{3(10)}{5(27)} = \frac{2}{9}$$

Division of algebraic fractions involves multiplying fractions, except take the reciprocal of the fraction designated as the divisor before multiplying. The result after multiplying is simplified by reducing to the lowest terms.

# Step-by-Step Solutions to Problems in this Chapter, "Operations with Fractions"

• **PROBLEM 41**

Simplify $\dfrac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{6}}$ .

Solution: $\dfrac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{6}}$ means $\left(\frac{1}{2}+\frac{1}{3}\right) \div \frac{1}{6}$

Since division by a fraction is equivalent to multiplication by its reciprocal:

$$\left(\frac{1}{2}+\frac{1}{3}\right) \div \frac{1}{6} = \left(\frac{1}{2}+\frac{1}{3}\right) \times 6$$

By the distributive law:

$$\begin{aligned}\left(\frac{1}{2}+\frac{1}{3}\right) \times 6 &= \left(\frac{1}{2}\right)(6) + \left(\frac{1}{3}\right)(6)\\ &= \frac{6}{2}+\frac{6}{3}\\ &= 3 + 2\\ &= 5\end{aligned}$$

• **PROBLEM 42**

Simplify the following expression: $1 - \dfrac{1}{2-\frac{1}{3}}$ .

Solution: In order to combine the denominator, $2 - \frac{1}{3}$, we must convert 2 into thirds. $2 = 2 \cdot 1 = 2 \cdot \frac{3}{3} = \frac{6}{3}$. Thus

$$1 - \frac{1}{2-\frac{1}{3}} = 1 - \frac{1}{\frac{6}{3}-\frac{1}{3}} = 1 - \frac{1}{\frac{5}{3}}$$

Since division by a fraction is equivalent to multiplication by that fraction's reciprocal

$$1 - \frac{1}{\frac{5}{3}} = 1 - (1)\left(\frac{3}{5}\right) = 1 - \frac{3}{5} = \frac{5}{5} - \frac{3}{5} = \frac{2}{5}$$

Therefore, $1 - \dfrac{1}{2-\frac{1}{3}} = \dfrac{2}{5}$.

• **PROBLEM 43**

Simplify the complex fraction $\dfrac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{4}+\frac{1}{5}}$ .

**Solution:** First simplify the expressions in the numerator and denominator by adding the fractions together according to the rule

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

We now obtain $\frac{1}{2} + \frac{1}{3} = \frac{3 + 2}{6} = \frac{5}{6}$

$$\frac{1}{4} + \frac{1}{5} = \frac{4 + 5}{20} = \frac{9}{20}$$

therefore,

$$\frac{\frac{1}{2} + \frac{1}{3}}{\frac{1}{4} + \frac{1}{5}} = \frac{\frac{5}{6}}{\frac{9}{20}}$$

To divide this complex fraction invert the fraction in the denominator and multiply the resulting fraction by the fraction in the numerator:

$$\frac{\frac{1}{2} + \frac{1}{3}}{\frac{1}{4} + \frac{1}{5}} = \frac{5}{6} \times \frac{20}{9} = \frac{100}{54} = \frac{50}{27}$$

• **PROBLEM 44**

Simplify $\dfrac{\frac{2}{3} + \frac{1}{2}}{\frac{3}{4} - \frac{1}{3}}$.

**Solution:** A first method is to just add the terms in the numerator and denominator. Since 6 is the least common denominator of the numerator, $\left(\frac{2}{3} + \frac{1}{2}\right)$, we convert $\frac{2}{3}$ and $\frac{1}{2}$ into sixths:

$$\frac{2}{3} = \frac{2}{3} \cdot 1 = \frac{2}{3} \cdot \frac{2}{2} = \frac{4}{6} \text{ and } \frac{1}{2} = \frac{1}{2} \cdot 1 = \frac{1}{2} \cdot \frac{3}{3} = \frac{3}{6}$$

Therefore $\frac{2}{3} + \frac{1}{2} = \frac{4}{6} + \frac{3}{6} = \frac{7}{6}$

Since 12 is the least common denominator of the denominator, $\left(\frac{3}{4} - \frac{1}{3}\right)$, we convert $\frac{3}{4}$ and $\frac{1}{3}$ into twelfths:

$$\frac{3}{4} = \frac{3}{4} \cdot 1 = \frac{3}{4} \cdot \frac{3}{3} = \frac{9}{12} \text{ and } \frac{1}{3} = \frac{1}{3} \cdot 1 = \frac{1}{3} \cdot \frac{4}{4} = \frac{4}{12}$$

Therefore $\frac{3}{4} - \frac{1}{3} = \frac{9}{12} - \frac{4}{12} = \frac{5}{12}$

Thus, $\dfrac{\frac{2}{3}+\frac{1}{2}}{\frac{3}{4}-\frac{1}{3}} = \dfrac{\frac{7}{6}}{\frac{5}{12}}$

Division by a fraction is equivalent to multiplication by the reciprocal hence $\dfrac{\frac{7}{6}}{\frac{5}{12}} = \frac{7}{6} \cdot \frac{12}{5}$

Cancelling 6 from the numerator and denominator:

$$= \frac{7}{1} \cdot \frac{2}{5} = \frac{14}{5}$$

A second method is to multiply both numerator and denominator by the least common denominator of the entire fraction. Since we have already seen that L.C.D. of the numerator is 6 and the L.C.D. of the denominator is 12, and 12 is divisible by 6, we use 12 as the L.C.D. of the entire fraction. Thus

$$\frac{\frac{2}{3}+\frac{1}{2}}{\frac{3}{4}-\frac{1}{3}} = \frac{12\left(\frac{2}{3}+\frac{1}{2}\right)}{12\left(\frac{3}{4}-\frac{1}{3}\right)}$$

Distribute:

$$= \frac{12\left(\frac{2}{3}\right)+12\left(\frac{1}{2}\right)}{12\left(\frac{3}{4}\right)-12\left(\frac{1}{3}\right)}$$

$$= \frac{4 \cdot 2 + 6}{3 \cdot 3 - 4} = \frac{8+6}{9-4} = \frac{14}{5}$$

**• PROBLEM 45**

If $a = 4$ and $b = 7$ find the value of $\dfrac{a+\frac{a}{b}}{a-\frac{a}{b}}$.

Solution: By substitution, $\dfrac{a+\frac{a}{b}}{a-\frac{a}{b}} = \dfrac{4+\frac{4}{7}}{4-\frac{4}{7}}$.

In order to combine the terms we convert 4 into sevenths:

$$4 = 4 \cdot 1 = 4 \cdot \frac{7}{7} = \frac{28}{7}.$$

Thus, we have:

$$\frac{\frac{28}{7}+\frac{4}{7}}{\frac{28}{7}-\frac{4}{7}} = \frac{\frac{32}{7}}{\frac{24}{7}}.$$

Dividing by $\frac{24}{7}$ is equivalent to multiplying by $\frac{7}{24}$. Therefore,

$$\frac{4 + \frac{4}{7}}{4 - \frac{4}{7}} = \frac{32}{7} \cdot \frac{7}{24}$$

Now, the 7 in the numerator cancels with the 7 in the denominator.

Thus, we obtain: $\frac{32}{24}$, and dividing numerator and denominator by 8,

we obtain: $\frac{4}{3}$.

Therefore $\frac{a + \frac{a}{b}}{a - \frac{a}{b}} = \frac{4}{3}$ when $a = 4$ and $b = 7$.

• **PROBLEM 46**

Perform the following division: $1 \Big/ \frac{x + y}{x^2}$.

Solution: Division by a fraction is equivalent to multiplication by that fraction's reciprocal. Hence,

$$\frac{1}{\frac{x + y}{x^2}} = 1 \cdot \frac{x^2}{x + y} = \frac{x^2}{x + y}.$$

• **PROBLEM 47**

Perform the indicated operation:

$$\frac{3a - 9b}{x - 5} \cdot \frac{xy - 5y}{ax - 3bx}$$

Solution: According to our definition of multiplication, we need only to write the product of the numerators over the product of the denominators. The only remaining step is that of reducing the fraction to lowest terms by factoring the numerator and denominator of the answer and simplifying the result.

$$\frac{3a - 9b}{x - 5} \cdot \frac{xy - 5y}{ax - 3bx} = \frac{(3a - 9b)(xy - 5y)}{(x - 5)(ax - 3bx)}$$

Factor out 3 from $(3a - 9b)$ and $y$ from $(xy - 5y)$. Also, factor out $x$ from $ax - 3bx$.

$$= \frac{3(a - 3b)y(x - 5)}{(x - 5)x(a - 3b)}$$

Group the same terms in numerator and the denominator.

$$= \frac{3y}{x} \cdot \frac{a - 3b}{a - 3b} \cdot \frac{x - 5}{x - 5}$$

Cancel like terms.

$$= \frac{3y}{x} \cdot 1 \cdot 1$$

$$= \frac{3y}{x}$$

This procedure could have been abbreviated in the following manner:

$$\frac{3a-9b}{x-5}\cdot\frac{xy-5y}{ax-3bx}=\frac{3\cancel{(a-3b)}}{\cancel{x-5}}\cdot\frac{y\cancel{(x-5)}}{x\cancel{(a-3b)}}=\frac{3y}{x}$$

**• PROBLEM 48**

Simplify $\dfrac{4x^3+6x^2}{2x}$

Solution: Since the denominator of a fraction cannot equal zero, $2x \neq 0$ or, dividing by 2 we obtain the restriction $x \neq 0$.

Now we proceed to simplify the given expression. First we note that $2x$ may be factored out of the numerator; thus

$$\frac{4x^3+6x^2}{2x}=\frac{2x\left(2x^2+3x\right)}{2x}$$
$$=\frac{2x}{2x}\cdot\left(2x^2+3x\right)$$
$$=2x^2+3x.$$

Thus, $\dfrac{4x^3+6x^2}{2x}=2x^2+3x$, and $x \neq 0$.

**• PROBLEM 49**

Simplify $\dfrac{1+\frac{1}{x}}{1-\frac{1}{x}}$ .

Solution: In order to eliminate the fractions in the numerator and denominator we multiply numerator and denominator by $x$.

$$\frac{1+\frac{1}{x}}{1-\frac{1}{x}}=\frac{x\left(1+\frac{1}{x}\right)}{x\left(1-\frac{1}{x}\right)}=\frac{x+\frac{x}{x}}{x-\frac{x}{x}}=\frac{x+1}{x-1}.$$

**• PROBLEM 50**

Combine into a single fraction in lowest terms.

(a) $\dfrac{6(a+1)}{a+8}-\dfrac{3(a-4)}{a+8}-\dfrac{2(a+5)}{a+8}$

(b) $\dfrac{7x-3y+6}{x+y}-\dfrac{2(x-4y+3)}{x+y}$

(c) $\dfrac{5x+2}{x-6}-\dfrac{3(x+4)}{x-6}-\dfrac{x-7}{x-6}$

Solution: Noting $\dfrac{a}{x}+\dfrac{b}{x}+\dfrac{c}{x}=\dfrac{a+b+c}{x}$ (where a,b,c are any real numbers and x any non-zero real number), we proceed to evaluate these expressions:

(a) $\dfrac{6(a+1)}{a+8}-\dfrac{3(a-4)}{a+8}-\dfrac{2(a+5)}{a+8}=\dfrac{6(a+1)-3(a-4)-2(a+5)}{a+8}$

Distributing, $=\dfrac{6a+6-3a+12-2a-10}{a+8}=\dfrac{6a-3a-2a+6+12-10}{a+8}$

$$= \frac{a + 8}{a + 8} = 1.$$

(b) $$\frac{7x - 3y + 6}{x + y} - \frac{2(x - 4y + 3)}{x + y} = \frac{7x - 3y + 6 - 2(x-4y+3)}{x + y}$$

Distributing, $$= \frac{7x - 3y + 6 - 2x + 8y - 6}{x + y} = \frac{7x-2x-3y+8y+6-6}{x + y}$$

$$= \frac{5x + 5y}{x + y} = \frac{5\cancel{(x + y)}}{\cancel{x + y}} = 5.$$

(c) $$\frac{5x + 2}{x - 6} - \frac{3(x + 4)}{x - 6} - \frac{x - 7}{x - 6} = \frac{5x + 2 - 3(x+4) - (x-7)}{x-6}$$

Distributing, $$= \frac{5x+2-3x-12-x+7}{x-6} = \frac{5x-3x-x+2-12+7}{x-6} = \frac{x - 3}{x - 6}.$$

**• PROBLEM 51**

Combine and simplify $1 + \dfrac{1}{1 + \dfrac{1}{1 - x}}$ .

Solution: First combine the terms in the denominator. Recall $1 = (1-x)/(1-x)$. Thus,

$$1 + \frac{1}{1 + \frac{1}{1 - x}} = 1 + \frac{1}{\frac{1 - x}{1 - x} + \frac{1}{1 - x}}$$

$$= 1 + \frac{1}{\frac{1 - x + 1}{1 - x}}$$

$$= 1 + \frac{1}{\frac{2 - x}{1 - x}}$$

Division by a fraction is equivalent to multiplication by its reciprocal, thus

$$= 1 + 1 \cdot \frac{(1 - x)}{(2 - x)}$$

$$= 1 + \frac{1 - x}{2 - x}$$

Recall $1 = \frac{2 - x}{2 - x}$ , therefore,

$$1 + \frac{1}{1 + \frac{1}{1 - x}} = \frac{2 - x}{2 - x} + \frac{1 - x}{2 - x}$$

$$= \frac{2 - x + 1 - x}{2 - x}$$

$$= \frac{3 - 2x}{2 - x} .$$

Simplify the complex fraction $\dfrac{\dfrac{1}{x} - \dfrac{1}{y}}{\dfrac{1}{x^2} - \dfrac{1}{y^2}}$.

Solution: Add both fractions of the numerator together using the rule: $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$ ; and obtain $\frac{y - x}{xy}$ . Similarly for the denominator, obtain:

$$\frac{y^2 - x^2}{x^2y^2}$$

Now invert the fraction in the denominator and multiply by the numerator:

$$\frac{y - x}{xy} \cdot \frac{x^2y^2}{y^2 - x^2} = \frac{(y - x)}{xy} \cdot \frac{(xy)(xy)}{(y - x)(y + x)}$$

$$= \frac{(y - x)(xy)(xy)}{xy(y - x)(y + x)}$$

$$= \frac{xy}{y + x} \; .$$



Simplify $\dfrac{x + \dfrac{1}{y}}{x - \dfrac{1}{y}}$ .

Solution: Obtain the least common denominator, ℓ.c.d., of the two terms in the numerator and of those that appear in the denominator. This is done by writing down the different factors that appear in the denominators of the terms. The exponent to be used for each factor is the smallest number of times that the factor appears in either of the denominators of the terms. Hence, the ℓ.c.d. of the two terms in the denominator = $(1)^1(y)^1 = 1y = y$. Also, the ℓ.c.d. of the two terms in the numerator is obtained in the same way. Therefore, the ℓ.c.d. of the two terms in the numerator = $(1)^1(y)^1 = 1y = y$. Therefore:

$$\frac{x + \frac{1}{y}}{x - \frac{1}{y}} = \frac{\frac{y(x)}{y} + \frac{1}{y}}{\frac{y(x)}{y} - \frac{1}{y}} = \frac{\frac{yx + 1}{y}}{\frac{yx - 1}{y}}$$

Division is the same as multiplying the numerator by the multiplicative inverse of the denominator. (The multiplicative inverse of a number $a$ is the number $n$, such that $a \cdot n = 1$. This number $n$ is $1/a$ . Hence, $a \cdot 1/a = 1$.) Therefore, the multiplicative inverse of

$$\frac{yx - 1}{y} \text{ is } \frac{y}{yx - 1}.$$

Hence,

$$\frac{x+\frac{1}{y}}{x-\frac{1}{y}} = \frac{\frac{yx+1}{y}}{\frac{yx-1}{y}} = \left(\frac{yx+1}{\cancel{y}}\right)\left(\frac{\cancel{y}}{yx-1}\right) = \frac{yx+1}{yx-1}\ .$$

**• PROBLEM 54**

Simplify the following expressions:

(a) $\dfrac{\frac{a}{b}+\frac{a}{c}}{ab+ac}$ (b) $\dfrac{2-\frac{1}{4}}{\frac{3}{5}+1}$

Solution:

(a) In order to combine the fractions $\frac{a}{b}$ and $\frac{a}{c}$ in the numerator, we convert them into fractions with the same denominator by multiplying a/b by c/c (which is equal to 1) and a/c by b/b (also equal to 1). Multiplication by a fraction equal to 1 does not change the value of the original fractions. Thus,

$$\frac{\left(\frac{a}{b}\right)\left(\frac{c}{c}\right)+\left(\frac{a}{c}\right)\left(\frac{b}{b}\right)}{ab+ac} = \frac{\frac{ac}{bc}+\frac{ab}{bc}}{ab+ac} = \frac{\frac{ac+ab}{bc}}{ab+ac}$$

Multiplying numerator and denominator by bc,

$$= \frac{\frac{ac+ab}{bc}(bc)}{(ab+ac)(bc)} = \frac{ac+ab}{(ac+ab)(bc)} = \frac{1}{bc}\ .$$

(b) $$\frac{2-\frac{1}{4}}{\frac{3}{5}+1} = \frac{(2)\left(\frac{4}{4}\right)-\frac{1}{4}}{\frac{3}{5}+(1)\left(\frac{5}{5}\right)} = \frac{\frac{8}{4}-\frac{1}{4}}{\frac{3}{5}+\frac{5}{5}} = \frac{\frac{7}{4}}{\frac{8}{5}}$$

Since division by a fraction is equivalent to multiplication by that fraction's reciprocal

$$= \frac{7}{4}\cdot\frac{5}{8}$$

$$= \frac{35}{32}\ .$$

**• PROBLEM 55**

Simplify: $\dfrac{x-\frac{2}{y}}{x+\frac{3}{y}}$ .

Solution: The Lowest Common Multiple of the denominators is y .

Since $\frac{y}{y}=1$, multiply numerator and denominator by y . Thus, we obtain:

$$\frac{y\left(x-\frac{2}{y}\right)}{y\left(x+\frac{3}{y}\right)}$$

Distribute:

$$\frac{yx - y\left(\frac{2}{y}\right)}{yx + y\left(\frac{3}{y}\right)}$$

$$= \frac{yx - 2}{yx + 3}$$

therefore,

$$\frac{x - \frac{2}{y}}{x + \frac{3}{y}} = \frac{yx - 2}{yx + 3} \quad .$$

**• PROBLEM 56**

Simplify this expression:

$$\frac{\frac{3}{x} - \frac{2}{y}}{\frac{5}{x} + \frac{6}{y}}$$

Solution: There are two ways to approach this problem. One is to consider it as a division problem:

$$\left(\frac{3}{x} - \frac{2}{y}\right) \div \left(\frac{5}{x} + \frac{6}{y}\right) .$$

Use the least common denominator $xy$ :

$$= \left(\frac{3y}{xy} - \frac{2x}{xy}\right) \div \left(\frac{5y}{xy} + \frac{6x}{xy}\right) .$$

Combine fractions: $$= \frac{3y-2x}{xy} \div \frac{5y+6x}{xy}.$$

Dividing by a fraction is equivalent to multiplying by its reciprocal:

$$= \left(\frac{3y-2x}{xy}\right) \cdot \left(\frac{xy}{5y+6x}\right) .$$

Cancelling out $xy$ : $$= \frac{3y-2x}{5y+6x} .$$

The second approach is to multiply both numerator and denominator by $xy$; this is equivalent to multiplying the fraction by 1:

$$\frac{\frac{3}{x} - \frac{2}{y}}{\frac{5}{x} + \frac{6}{y}} \cdot \frac{xy}{xy} = \frac{3y - 2x}{5y + 6x} \quad .$$

**• PROBLEM 57**

Simplify $$\frac{\frac{2}{x} + \frac{3}{y}}{1 - \frac{1}{x}} .$$

Solution: A first method is to just add the terms in the numerator and denominator, obtaining

$$\frac{\frac{2}{x}+\frac{3}{y}}{1-\frac{1}{x}} = \frac{\frac{2y}{xy}+\frac{3x}{xy}}{\frac{x}{x}-\frac{1}{x}} = \frac{\frac{2y+3x}{xy}}{\frac{x-1}{x}}$$

Since dividing by fraction is equivalent to multiplying by its reciprocal,

$$= \frac{2y+3x}{xy} \cdot \frac{x}{x-1} = \frac{2y+3x}{y(x-1)}$$

A second method is to multiply both numerator and denominator by the least common denominator of the entire fraction, in this case xy:

$$\frac{\frac{2}{x}+\frac{3}{y}}{1-\frac{1}{x}} = \frac{xy\left(\frac{2}{x}+\frac{3}{y}\right)}{xy\left(1-\frac{1}{x}\right)}$$

Distributing,
$$= \frac{xy\left(\frac{2}{x}\right) + xy\left(\frac{3}{y}\right)}{xy(1) - xy\left(\frac{1}{x}\right)}$$

Cancelling like terms,

$$= \frac{2y+3x}{xy-y}$$

Using distributive law,

$$= \frac{2y+3x}{y(x-1)} .$$

• **PROBLEM 58**

Combine $a + b - \frac{2ab}{a+b}$ .

Solution: In order to combine fractions we must transform them into equivalent fractions with a common denominator. In our case we will use $a + b$ as our least common denominator (LCD). Thus

$$\begin{aligned} a+b-\frac{2ab}{a+b} &= \frac{a+b}{a+b}\left(\frac{a+b}{1}\right) - \frac{2ab}{a+b} \\ &= \frac{(a+b)(a+b)}{a+b} - \frac{2ab}{a+b} \\ &= \frac{a^2+2ab+b^2}{a+b} - \frac{2ab}{a+b} \\ &= \frac{a^2+2ab+b^2-2ab}{a+b} \\ &= \frac{a^2+b^2}{a+b} \end{aligned}$$

• **PROBLEM 59**

Add $\frac{2}{x-3}$ and $\frac{5}{x+2}$.

**Solution:** Change these fractions to fractions with a common denominator, then add fractions by adding numerators and placing over the common denominator. Neither of the denominators is factorable; therefore, the LCD is the product of the denominators. LCD = $(x - 3)(x + 2)$. To change $\frac{2}{x - 3}$ to an equivalent fraction with $(x - 3)(x + 2)$ as its denominator, multiply by the unit fraction $\frac{(x + 2)}{(x + 2)}$. To change $\frac{5}{x + 2}$ to an equivalent fraction with the LCD as its denominator, multiply by $\frac{x - 3}{x - 3}$. Then add the resulting fractions as follows:

$$\frac{2}{x - 3} + \frac{5}{x + 2} = \frac{2(x + 2)}{(x - 3)(x + 2)} + \frac{5(x - 3)}{(x - 3)(x + 2)}$$

$$= \frac{2(x + 2) + 5(x - 3)}{(x - 3)(x + 2)}$$

$$= \frac{2x + 4 + 5x - 15}{(x - 3)(x + 2)}$$

$$= \frac{7x - 11}{(x - 3)(x + 2)} \qquad \text{Combining Terms}$$

The numerator is not factorable, so the result can not be reduced.

**• PROBLEM 60**

Combine $\frac{1}{6x} + \frac{1}{3y} - \frac{3x + 2y}{12xy}$ into a single fraction.

**Solution:** Since both 6x and 3y are factors of 12xy, the least common denominator (the LCD) of the given fractions is 12xy. Thus, we wish to convert the given fractions to equal fractions having 12xy as a denominator. We can accomplish this by multiplying each member of the first fraction by 2y and each member of the second by 4x. We thereby obtain

$$\frac{1}{6x} + \frac{1}{3y} - \frac{3x + 2y}{12xy} = \frac{2y \cdot 1}{2y(6x)} + \frac{4x \cdot 1}{4x(3y)} - \frac{3x + 2y}{12xy}$$

$$= \frac{2y}{12xy} + \frac{4x}{12xy} - \frac{3x + 2y}{12xy}$$

$$= \frac{2y + 4x - (3x + 2y)}{12xy}$$

$$= \frac{2y + 4x - 3x - 2y}{12xy}$$

$$= \frac{2y + x - 2y}{12xy}$$

$$= \frac{2y - 2y + x}{12xy}$$

$$= \frac{x}{12xy} .$$

Cancelling x from numerator and denominator,

$$= \frac{1}{12y}$$

Thus, $\frac{1}{6x} + \frac{1}{3y} - \frac{3x + 2y}{12xy} = \frac{1}{12y}$ .

**• PROBLEM 61**

Perform the indicated operations:

$$\frac{3a}{2xy} - \frac{2 - 5x}{y^3} + 6$$

Solution: The first step in adding or subtracting fractions is to convert them into equivalent fractions having like denominators. The simplest method is to find the least common denominator (LCD). The LCD is the product of the unique prime factors of all the original denominators, each factor having as its exponent the positive integer representing the largest number of times the factor appeared in an original denominator. In this example the first denominator is $2 \cdot x \cdot y$, and the second is $y^3$. Since the factor $y$ appears to the third power in the second denominator, the LCD is $2xy^3$. Hence, we multiply the numerator and denominator of each term by the factor necessary to make the denominator equal to $2xy^3$.

$$\frac{3a}{2xy} - \frac{2 - 5x}{y^3} + \frac{6}{1} = \frac{3a(y^2)}{2xy(y^2)} - \frac{(2 - 5x)(2x)}{y^3 \quad (2x)} + \frac{6(2xy^3)}{1(2xy^3)}$$

$$= \frac{3ay^2}{2xy^3} - \frac{(4x - 10x^2)}{2xy^3} + \frac{12xy^3}{2xy^3}$$

$$= \frac{3ay^2 - 4x + 10x^2 + 12xy^3}{2xy^3}$$

**• PROBLEM 62**

Simplify $\dfrac{\dfrac{1}{x - 1} - \dfrac{1}{x - 2}}{\dfrac{1}{x - 2} - \dfrac{1}{x - 3}}$ .

Solution: Simplify the expression in the numerator by using the addition rule:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

Notice bd is the Least Common Denominator, LCD.

We obtain $\frac{x - 2 - (x - 1)}{(x - 1)(x - 2)} = \frac{- 1}{(x - 1)(x - 2)}$ in the numerator.

Repeat this procedure for the expression in the denominator:

$$\frac{x - 3 - (x - 2)}{(x - 2)(x - 3)} = \frac{-1}{(x - 2)(x - 3)}$$

We now have

$$\frac{\dfrac{-1}{(x - 1)(x - 2)}}{\dfrac{-1}{(x - 2)(x - 3)}},$$

which is simplified by inverting the fraction in the denominator and multiplying it by the numerator and cancelling like terms

$$\frac{-1}{(x - 1)(x - 2)} \cdot \frac{(x - 2)(x - 3)}{-1} = \frac{x - 3}{x - 1}.$$

**• PROBLEM 63**

Simplify: $$\frac{\dfrac{1}{a} - \dfrac{1}{a - b}}{\dfrac{1}{a} + \dfrac{1}{a - b}}$$

Solution: The Lowest Common Multiple (L.C.M) of the denominators is $a(a - b)$. Since $\frac{a(a-b)}{a(a-b)} = 1$, multiply numerator and denominator by $a(a - b)$. Thus, we have:

$$\frac{a(a - b)\left(\dfrac{1}{a} - \dfrac{1}{a - b}\right)}{a(a - b)\left(\dfrac{1}{a} + \dfrac{1}{a - b}\right)}.$$

Distribute $a(a - b)$ in numerator and denominator:

$$\frac{a(a - b)\left(\dfrac{1}{a}\right) - a(a - b)\left(\dfrac{1}{a - b}\right)}{a(a - b)\left(\dfrac{1}{a}\right) + a(a - b)\left(\dfrac{1}{a - b}\right)}.$$

Perform the multiplication:

$$\frac{\dfrac{a}{a}(a - b) - a\left(\dfrac{a - b}{a - b}\right)}{\dfrac{a}{a}(a - b) + a\left(\dfrac{a - b}{a - b}\right)}$$

Since $\frac{a}{a} = 1$ and $\frac{a - b}{a - b} = 1$, we have: $\frac{(a - b) - a}{(a - b) + a}$.

Using Associative and Commutative laws of addition, we obtain:

$$\frac{\dfrac{1}{a} - \dfrac{1}{a - b}}{\dfrac{1}{a} + \dfrac{1}{a - b}} = \frac{(a - a) - b}{(a + a) - b} = \frac{-b}{2a - b}.$$

Simplify $\dfrac{\dfrac{x}{x+y} + \dfrac{y}{x-y}}{\dfrac{y}{x+y} - \dfrac{x}{x-y}}$ .

Solution: To eliminate the fractions in this expression, we multiply numerator and denominator by the least common multiple (L.C.M.), the expression of lowest degree into which each of the original expressions can be divided without a remainder. The L.C.M. is the product obtained by taking each factor to the highest degree. In our case the L.C.M. is $(x + y)(x - y)$. Thus, multiplying numerator and denominator by $(x + y)(x - y)$,

$$\frac{\dfrac{x}{x+y} + \dfrac{y}{x-y}}{\dfrac{y}{x+y} - \dfrac{x}{x-y}} = \frac{\dfrac{x}{x+y} + \dfrac{y}{x-y}}{\dfrac{y}{x+y} - \dfrac{x}{x-y}} \cdot \frac{(x+y)(x-y)}{(x+y)(x-y)} .$$

Distributing, $= \dfrac{\left(\dfrac{x}{x+y}\right)(x+y)(x-y) + \left(\dfrac{y}{x-y}\right)(x+y)(x-y)}{\left(\dfrac{y}{x+y}\right)(x+y)(x-y) - \left(\dfrac{x}{x-y}\right)(x+y)(x-y)}$

Cancelling like terms, $= \dfrac{x(x-y) + y(x+y)}{y(x-y) - x(x+y)}$

Distributing, $= \dfrac{x^2 - xy + yx + y^2}{yx - y^2 - x^2 - xy}$

Using the commutative law, $= \dfrac{x^2 - xy + xy + y^2}{xy - y^2 - x^2 - xy}$

Combining terms, $= \dfrac{x^2 + y^2}{-x^2 - y^2}$

Factoring (- 1) from the denominator $= \dfrac{x^2 + y^2}{(-1)(x^2 + y^2)}$

Cancelling $x^2 + y^2$, $= \dfrac{1}{-1}$

$= -1.$

Thus, $\dfrac{\dfrac{x}{x+y} + \dfrac{y}{x-y}}{\dfrac{y}{x+y} - \dfrac{x}{x-y}} = -1.$

• PROBLEM 65

A) If $x = \dfrac{c - ab}{a - b}$, find the value of the expression $a(x + b)$.

B) Also, if $x = \dfrac{c - ab}{a - b}$, find the value of the expression $bx + c$.

Solution: A) Substituting $x = \frac{c - ab}{a - b}$ for $x$ in the expression $a(x + b)$,

$$a(x + b) = a\left(\frac{c - ab}{a - b} + b\right) \tag{1}$$

Obtaining a common denominator of $a - b$ for the two terms in parenthesis; equation (1) becomes:

$$a(x + b) = a\left[\frac{c - ab}{a - b} + \frac{(a-b)b}{a - b}\right]$$

Distributing the numerator of the second term in brackets:

$$a(x + b) = a\left[\frac{c-ab}{a-b} + \frac{ab-b^2}{a-b}\right] = a\left[\frac{c-\cancel{ab}+\cancel{ab}-b^2}{a-b}\right]$$

$$= a\left[\frac{c-b^2}{a-b}\right]$$

$$a(x + b) = \frac{a\left(c-b^2\right)}{a-b}$$

B) Substituting $x = \frac{c-ab}{a-b}$ for $x$ in the expression $bx + c$,

$$bx + c = b\left(\frac{c-ab}{a-b}\right) + c$$

$$= \frac{b(c-ab)}{a-b} + c \tag{2}$$

Obtaining a common denominator of $a - b$ for the two terms on the right side of equation (2):

$$bx + c = \frac{b(c-ab)}{a-b} + \frac{(a-b)c}{a-b}$$

Distributing the numerator of each term on the right side:

$$bx + c = \frac{\cancel{bc}-ab^2}{a-b} + \frac{ac-\cancel{bc}}{a-b}$$

$$= \frac{\cancel{bc}-ab^2+ac-\cancel{bc}}{a - b} = \frac{-ab^2+ac}{a - b}$$

$$= \frac{ac-ab^2}{a-b}$$

Factoring out the common factor of $a$ from the numerator of the right side:

$$bx + c = \frac{a\left(c-b^2\right)}{a-b}$$

**• PROBLEM 66**

When two resistances are installed in an electric circuit in parallel, the reciprocal of the resistance of the system is equal to the sum of the reciprocals of the parallel resistances. If $r_1$ and $r_2$ represent the resistances installed and $R$ the resistance of the system, then

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}$$

What single resistance is the equivalent of resistances of 10 ohms and 25 ohms wired in parallel?

Solution: Let $r_1 = 10$ ohms and $r_2 = 25$ ohms. We are looking for the single resistance R, which is equivalent to $r_1$ and $r_2$.

Here the reciprocal of R $= \frac{1}{R}$

Here the reciprocal of $r_1 = \frac{1}{r_1}$

and the reciprocal of $r_2 = \frac{1}{r_2}$

Now substitute the values for $r_1$ and $r_2$ respectively into the equation. Thus,

$$\frac{1}{R} = \frac{1}{10} + \frac{1}{25}$$

Add the fractions according to the rule

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

$$\frac{1}{R} = \frac{25 + 10}{250} = \frac{35}{250}$$

$$R = \frac{250}{35} = \frac{50}{7} = 7.14 \text{ ohms.}$$

CHAPTER 6

# BASE, EXPONENT, POWER

**Basic Attacks and Strategies for Solving Problems in this Chapter.** **See pages 35 to 52 for step-by-step solutions to problems.**

For any real number $X$ and any positive integer $n$, the $n^{th}$ power of $X$, denoted by $X^n$, is defined to be the product of $n$ factors of $X$. In $X^n$, $X$ is referred to as the base and $n$ is referred to as the exponent. Rules for performing certain operations with powers of any real numbers, say $X$ and $Y$, and any positive integers, $n$ and $m$, are as follows:

(1) To multiply two powers of the same base, simply add their exponents and use this sum as the exponent of the common base

(e.g., $X^n * X^m = X^{n+m}$);

(2) To divide two powers with the same base, simply subtract the exponent of the divisor from the exponent of the dividend and use this difference as the exponent on the common base

(e.g., $X^n / X^m = X^{n-m}$, $n > m > 0$);

(3) To determine the power of a power, simply multiply the exponents and use this product as the exponent of the base

(e.g., $(X^n)^m = X^{nm}$);

(4) The power of a product is the product of the separate powers

(e.g., $(XY)^n = X^n Y^n$); and

(5) The power of a quotient is the quotient of the separate powers

(e.g., $(X/Y)^n = X^n / Y^n$).

Note that powers of sums and differences cannot be taken term by term, rather binomial expansion must be done.

If the power of a nonzero real number $X$ is negative, then the first step before applying the above rules is to change all negative exponents to positive exponents by using the definition

$$X^{-n} = 1 / X^n.$$

For any real number $X > 0$ and any positive rational number, $p/q$ exponent, the aforementioned rules for performing operations with powers apply. On the other hand, if the exponent is a negative rational number, then first use the definition of negative exponents $X^{-p/q}$ and then simplify the resulting fraction.

For any nonzero real number $X$, $X^0 = 1$ but $0^0$ is undefined.

# Step-by-Step Solutions to Problems in this Chapter, "Base, Exponent, Power"

• **PROBLEM 67**

Simplify: (a) $3^{-2}$ (b) $\frac{1}{5^{-2}}$

Solution: (a) Since $x^{-a} = \frac{1}{x^a}$, $3^{-2} = \frac{1}{3^2} = \frac{1}{3 \cdot 3} = \frac{1}{9}$.

(b) Again, recall $\frac{1}{x^a} = x^{-a}$; hence,

$$\frac{1}{5^{-2}} = 5^{-(-2)} = 5^2 = 5 \cdot 5 = 25.$$

• **PROBLEM 68**

Simplify the following expressions:

(a) $-3^{-2}$ (b) $(-3)^{-2}$ (c) $\frac{-3}{4^{-1}}$

Solution:

(a) Here the exponent applies only to 3.
Since $x^{-y} = \frac{1}{x^y}$, $-3^{-2} = -(3^{-2}) = -\frac{1}{3^2} = -\frac{1}{9}$.

(b) In this case the exponent applies to the negative base.
Thus, $(-3)^{-2} = \frac{1}{(-3)^2} = \frac{1}{(-3)(-3)} = \frac{1}{9}$.

(c) $\frac{-3}{4^{-1}} = \frac{-3}{(\frac{1}{4})^1} = \frac{-3}{\frac{1^1}{4^1}} = \frac{-3}{\frac{1}{4}}$.

Division by a fraction is equivalent to multiplication by that fraction's reciprocal, thus

$$\frac{-3}{\frac{1}{4}} = -3 \cdot \frac{4}{1} = -12,$$

and

$$\frac{-3}{4^{-1}} = -12.$$

Evaluate:
(a) $8\left(-\frac{1}{4}\right)^0$ (b) $6^0 + (-6)^0$ (c) $-7(-3)^0$ (d) $9^{-1}$ (e) $7^{-2}$.

Solution: Note $x^0 = 1$ and $x^{-a} = \frac{1}{x^a}$ for all non-zero real numbers x,

(a) $8\left(-\frac{1}{4}\right)^0 = 8(1) = 8$

(b) $6^0 + (-6)^0 = 1 + 1 = 2$

(c) $-7(-3)^0 = -7(1) = -7$

(d) $9^{-1} = \frac{1}{9^1} = \frac{1}{9}$

(e) $7^{-2} = \frac{1}{7^2} = \frac{1}{49}$

• PROBLEM 70

Simplify the expression $\left(3^{-1} + 2^{-1}\right)^{-2}$ .

Solution: Since $x^{-y} = \frac{1}{x^y}$, $3^{-1} = \frac{1}{3^1} = \frac{1}{3}$ and $2^{-1} = \frac{1}{2^1} = \frac{1}{2}$ . Thus,

$$\left(3^{-1} + 2^{-1}\right)^{-2} = \left(\frac{1}{3} + \frac{1}{2}\right)^{-2} .$$

Now, we combine fractions, using 6 as our least common denominator:

$$= \left[\frac{2}{2}\left(\frac{1}{3}\right) + \frac{3}{3}\left(\frac{1}{2}\right)\right]^{-2}$$

$$= \left(\frac{2}{6} + \frac{3}{6}\right)^{-2}$$

$$= \left(\frac{5}{6}\right)^{-2}$$

$$= \frac{1}{\left(\frac{5}{6}\right)^2}$$

$$= \frac{1}{\frac{25}{36}}$$

and since division by a fraction is equivalent to multiplying the numerator by the reciprocal of the denominator, we have:

$$= 1 \times \frac{36}{25}$$

$$= \frac{36}{25} .$$

• PROBLEM 71

Perform the indicated operations:

$(7 \cdot 10^5)^3 \cdot (3 \cdot 10^{-3})^4$.

**Solution:** Since $(ab)^x = a^x b^x$,

$$(7 \cdot 10^5)^3 \cdot (3 \cdot 10^{-3})^4 = (7)^3(10^5)^3 \cdot (3)^4(10^{-3})^4.$$

Recall that $(a^x)^y = a^{xy}$. Thus,

$$= (7^3)(10^{5 \cdot 3}) \cdot (3^4)(10^{-3 \cdot 4})$$

$$= (7^3)(10^{15}) \cdot (3^4)(10^{-12})$$

$$= (7^3)(3^4)(10^{15})(10^{-12}).$$

Since $a^x \cdot a^y = a^{x+y}$, $\quad = (7^3)(3^4)\left[10^{15+(-12)}\right]$

$$= 7^3 3^4 10^3.$$

**• PROBLEM 72**

Simplify:

(a) $2^3 \cdot 2^2$ (b) $a^3 \cdot a^5$ (c) $x^6 \cdot x^4$

**Solution:** If $a$ is any number and $n$ is any positive integer, the product of the $n$ factors $a \cdot a \cdot a \ldots a$ is denoted by $a^n$. $a$ is called the base and $n$ is called the exponent. Also, for base $a$ and exponents $m$ and $n$, $m$ and $n$ being positive integers, we have the law:

$$a^m \cdot a^n = a^{m+n}.$$

Therefore,

(a) $2^3 \cdot 2^2 = (2 \cdot 2 \cdot 2)(2 \cdot 2) = 8 \cdot 4 = 32$

or $2^3 \cdot 2^2 = 2^{3+2} = 2^5 = 32$

(b) $a^3 \cdot a^5 = (a \cdot a \cdot a)(a \cdot a \cdot a \cdot a \cdot a)$

$= (a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a) = a^8$

or $a^3 \cdot a^5 = a^{3+5} = a^8$

(c) $x^6 \cdot x^4 = x^{6+4} = x^{10}$.

**• PROBLEM 73**

Use the laws of exponents to perform the indicated operations:

(a) $5x^5 \cdot 2x^2$ (b) $\left(x^4\right)^6$ (c) $\dfrac{8y^8}{2y^2}$ (d) $\dfrac{x^3}{x^6}\left(\dfrac{7}{x}\right)^2$.

**Solution:** Noting the following properties of exponents:

(1) $a^b \cdot a^c = a^{b+c}$ (2) $\left(a^b\right)^c = a^{b \cdot c}$ (3) $\dfrac{a^b}{a^c} = a^{b-c}$ (4) $\left(\dfrac{a}{b}\right)^c = \dfrac{a^c}{b^c}$

we proceed to evaluate these expressions.

(a) $5x^5 \cdot 2x^2 = 5 \cdot 2 \cdot x^5\, x^2 = 10 \cdot x^5 \cdot x^2 = 10x^{5+2} = 10x^7$

(b) $\left(x^4\right)^6 = x^{4 \cdot 6} = x^{24}$

(c) $\dfrac{8y^8}{2y^2} = \dfrac{8}{2} \cdot \dfrac{y^8}{y^2} = 4 \cdot y^{8-2} = 4y^6$

(d) $\left(\dfrac{x^3}{x^6}\right)\left(\dfrac{7}{x}\right)^2 = \left(\dfrac{x^3}{x^6}\right)\left(\dfrac{7^2}{x^2}\right) = \dfrac{x^3 \cdot 49}{x^6 \cdot x^2} = \dfrac{49x^3}{x^{6+2}} = \dfrac{49x^3}{x^8} = \dfrac{49x^3}{x^{5+3}}$

$$= \frac{49\cancel{x^3}}{x^5 \cdot \cancel{x^3}} = \frac{49}{x^5}$$

**• PROBLEM 74**

Write $5x^{-3}y^0$ without zero or negative exponents.

Solution: Since $a^{-b} = \frac{1}{a^b}$ by definition,

$$x^{-3} = \frac{1}{x^3}$$

and since $x^0 = 1$ by definition (any real non-zero base raised to an exponent of zero equals one),

$$y^0 = 1 .$$

Substituting these values for $x^{-3}$ and $y^0$ we obtain

$$5x^{-3}y^0 = 5 \cdot \frac{1}{x^3} \cdot 1 = \frac{5}{x^3} .$$

**• PROBLEM 75**

Simplify the quotient $\frac{2x^0}{(2x)^0}$ .

Solution: The following two laws of exponents can be used to simplify the given quotient:

1) $a^0 = 1$ where a is any non-zero real number, and

2) $(ab)^n = a^n b^n$ where a and b are any two numbers.

In the given quotient, notice that the exponent in the numerator applies only to the letter x. However, the exponent in the denominator applies to both the number 2 and the letter x; that is, the exponent in the denominator applies to the entire term (2x). Using the first law, the numerator can be rewritten as:

$$2x^0 = 2(1) = 2$$

Using the second law with n = 0, the denominator can be rewritten as:

$$(2x)^0 = 2^0 x^0$$

Using the first law again to further simplify the denominator:

$$\begin{aligned}(2x)^0 &= 2^0 x^0 \\ &= (1)(1) \\ &= 1\end{aligned}$$

Therefore,

$$\frac{2x^0}{(2x)^0} = \frac{2}{1} = 2$$

Write the expression $\left(x + y^{-1}\right)^{-1}$ without using negative exponents.

**Solution:** Since $x^{-a} = \frac{1}{x^a}$, $y^{-1} = \frac{1}{y^1} = \frac{1}{y}$,

$$\left(x + y^{-1}\right)^{-1} = \left(x + \frac{1}{y}\right)^{-1}$$

$$= \frac{1}{x + \frac{1}{y}}$$

Multiply numerator and denominator by $y$ in order to eliminate the fraction in the denominator,

$$\frac{y(1)}{y\left(x + \frac{1}{y}\right)} = \frac{y}{yx + \frac{y}{y}} = \frac{y}{yx + 1}$$

Thus $$\left(x + y^{-1}\right)^{-1} = \frac{y}{yx + 1}$$



Simplify the expression $xy\left(x^{-1} + y^{-1}\right)$.

**Solution:** The following two laws of exponents can be used to simplify the given expression:

1) $a^{-n} = \frac{1}{a^n}$ and

2) $a^m \cdot a^n = a^{m+n}$.

Using the first law,

$$xy\left(x^{-1} + y^{-1}\right) = xy\left(\frac{1}{x^1} + \frac{1}{y^1}\right)$$

$$= xy\left(\frac{1}{x} + \frac{1}{y}\right)$$

Using the distributive property, this last equation becomes:

$$= xy\left(\frac{1}{x}\right) + xy\left(\frac{1}{y}\right)$$

$$= \cancel{x}\left(\frac{1}{\cancel{x}}\right)y + x\cancel{y}\left(\frac{1}{\cancel{y}}\right)$$

$$= y + x$$

Using the second law, we can solve this problem in another way.

$$xy\left(x^{-1} + y^{-1}\right) = xyx^{-1} + xyy^{-1}$$

$$= x^{1+(-1)}y + xy^{1+(-1)}$$

$$= x^0y + xy^0$$
$$= (1 \cdot y) + (x \cdot 1)$$
$$= y + x$$

**• PROBLEM 78**

Express $\left(\frac{a^{-2}}{b^{-3}}\right)^{-2}$ using only positive exponents.

Solution A: By the law of exponents which states that $(x)^{-n} = \frac{1}{x^n}$ where n is a positive integer,

$$\left(\frac{a^{-2}}{b^{-3}}\right)^{-2} = \frac{1}{\left[\frac{a^{-2}}{b^{-3}}\right]^2}.$$

Since $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$, $\left(\frac{a^{-2}}{b^{-3}}\right)^2 = \frac{(a^{-2})^2}{(b^{-3})^2}$. Also, since $(x^m)^n = x^{mn}$, $(a^{-2})^2 = a^{(-2)(2)} = a^{-4}$, $(b^{-3})^2 = b^{(-3)(2)} = b^{-6}$. Hence,

$$\left(\frac{a^{-2}}{b^{-3}}\right)^{-2} = \frac{1}{\left[\frac{a^{-2}}{b^{-3}}\right]^2}$$

$$= \frac{1}{\frac{a^{-4}}{b^{-6}}}$$

$$= \frac{1}{\frac{(a^4)^{-1}}{(b^6)^{-1}}}$$

$$= \frac{1}{\left[\frac{a^4}{b^6}\right]^{-1}}$$

$$\frac{1}{\left[\frac{1}{\frac{a^4}{b^6}}\right]}$$

Note that division is the same as multiplying the numerator by the reciprocal of the denominator. This principle is applied to the term in brackets.

$$\left(\frac{a^{-2}}{b^{-3}}\right)^{-2} = \frac{1}{(1)\left[\frac{b^6}{a^4}\right]} = \left(\frac{1}{\frac{b^6}{a^4}}\right).$$

Applying the same principle to the term in parenthesis on the right side of the equation:

$$\left(\frac{a^{-2}}{b^{-3}}\right)^{-2} = \left(\frac{1}{\frac{b^6}{a^4}}\right) = (1)\left(\frac{a^4}{b^6}\right) = \frac{a^4}{b^6}.$$

Solution B: Since $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$, $\left(\frac{a^{-2}}{b^{-3}}\right)^{-2} = \frac{(a^{-2})^{-2}}{(b^{-3})^{-2}}$. Also, since $(x^m)^n = x^{mn}$, $(a^{-2})^{-2} = a^{(-2)(-2)} = a^4$, and $(b^{-3})^{-2} = b^{(-3)(-2)} = b^6$. Hence,

$$\left(\frac{a^{-2}}{b^{-3}}\right)^{-2} = \frac{a^4}{b^6}.$$

• **PROBLEM 79**

Express $2c^{-2}d^{-1}/3x^{-1}y^3$ as an equal fraction involving only positive exponents.

Solution: Since $a^{-b} = \frac{1}{a^b}$ for all real b,

$$c^{-2} = \frac{1}{c^2},\ d^{-1} = \frac{1}{d},\ x^{-1} = \frac{1}{x}.$$

Hence $\frac{2c^{-2}d^{-1}}{3x^{-1}y^3} = \frac{2\left(\frac{1}{c^2}\right)\left(\frac{1}{d}\right)}{3\left(\frac{1}{x}\right)(y^3)} = \frac{\frac{2}{c^2d}}{\frac{3y^3}{x}}$. Division by a fraction is equivalent to multiplication by its reciprocal, thus

$$\frac{\frac{2}{c^2d}}{\frac{3y^3}{x}} = \left(\frac{2}{c^2d}\right) \times \left(\frac{x}{3y^3}\right) = \frac{2x}{c^2d3y^3} = \frac{2x}{3y^3c^2d}.$$

• **PROBLEM 80**

Convert $a^2b^{-3}c^{-2}/x^{-1}y^3z^{-3}$ into an equal fraction in which all exponents are positive.

Solution: Since $m^{-n} = \frac{1}{m^n}$ for all real n, $b^{-3} = \frac{1}{b^3}$, $c^{-2} = \frac{1}{c^2}$, $x^{-1} = \frac{1}{x}$, and $z^{-3} = \frac{1}{z^3}$. Hence,

$$\frac{a^2b^{-3}c^{-2}}{x^{-1}y^3z^{-3}} = \frac{(a^2)\left(\frac{1}{b^3}\right)\left(\frac{1}{c^2}\right)}{\left(\frac{1}{x}\right)(y^3)\left(\frac{1}{z^3}\right)} = \frac{\frac{a^2}{b^3c^2}}{\frac{y^3}{xz^3}}.$$

Division by a fraction is equivalent to multiplication by its reciprocal, thus

$$\frac{\frac{a^2}{b^3c^2}}{\frac{y^3}{xz^3}} = \left(\frac{a^2}{b^3c^2}\right) \times \left(\frac{xz^3}{y^3}\right) = \frac{a^2xz^3}{b^3c^2y^3}$$

• **PROBLEM 81**

Simplify the quotient $\frac{(x^{-2}y^4)^3}{(xy)^{-3}}$ .

Solution: The following six laws of exponents will be used to simplify the given quotient:

(1) $x^{-n} = \frac{1}{x^n}$ , where n is any positive integer

(2) $\left(\frac{x}{y}\right)^m = \frac{x^m}{y^m}$ ,

(3) $(x^m)^n = x^{m \cdot n}$,

(4) $(xy)^m = x^m y^m$ ,

(5) $x^m \cdot x^n = x^{m+n}$,

(6) $\frac{x^m}{x^n} = x^{m-n}$

Using the first law to simplify the quotient:

$$\frac{(x^{-2}y^4)^3}{(xy)^{-3}} = \frac{\left[\left(\frac{1}{x^2}\right) y^4\right]^3}{\frac{1}{(xy)^3}}$$

$$= \frac{\left(\frac{y^4}{x^2}\right)^3}{\frac{1}{(xy)^3}}$$

Using the second law to simplify the numerator,

$$\frac{(x^{-2}y^4)^3}{(x\text{-}y)^{-3}} = \frac{\left(\frac{y^4}{x^2}\right)^3}{\frac{1}{(xy)^3}}$$

$$= \frac{\frac{(y^4)^3}{(x^2)^3}}{\frac{1}{(xy)^3}}$$

Using the third and fourth laws to simplify both the numerator and the denominator,

$$\frac{(x^{-2}y^4)^3}{(xy)^{-3}} = \frac{\dfrac{(y^4)^3}{(x^2)^3}}{\dfrac{1}{(xy)^3}}$$

$$= \frac{\dfrac{y^{4\cdot 3}}{x^{2\cdot 3}}}{\dfrac{1}{x^3y^3}}$$

$$= \frac{\dfrac{y^{12}}{x^6}}{\dfrac{1}{x^3y^3}}$$

Since multiplying the numerator by the reciprocal of the denominator is equivalent to division, the equation becomes:

$$\frac{(x^{-2}y^4)^3}{(xy)^{-3}} = \frac{y^{12}}{x^6}\ \frac{x^3y^3}{1}$$

$$= \frac{y^{12}x^3y^3}{x^6}$$

Using the fifth law to simplify the numerator:

$$\frac{(x^{-2}y^4)^3}{(xy)^{-3}} = \frac{y^{12+3}x^3}{x^6}$$

$$= \frac{y^{15}x^3}{x^6}$$

Using the sixth law to make the last simplification:

$$\frac{(x^{-2}y^4)^3}{(xy)^{-3}} = \frac{y^{15}x^3}{x^6} = y^{15}\left(\frac{x^3}{x^6}\right) = y^{15}x^{3-6} = y^{15}x^{-3}$$

Hence, $\dfrac{(x^{-2}y^4)^3}{(xy)^{-3}} = y^{15}x^{-3}$ or $\dfrac{y^{15}}{x^3}$.

**• PROBLEM 82**

Evaluate the following expression: $\dfrac{12x^7y}{3x^2y^3}$

Solution: Noting (1) $\dfrac{abc}{def} = \dfrac{a}{d}\cdot\dfrac{b}{e}\cdot\dfrac{c}{f}$, (2) $a^{-b} = \dfrac{1}{a^b}$ and (3) $\dfrac{a^b}{a^c} = a^{b-c}$ for all non-zero real values of a,d,e,f,we proceed to evaluate the expression:

$$\frac{12x^7y}{3x^2y^3} = \frac{12}{3} \cdot \frac{x^7}{x^2} \cdot \frac{y}{y^3} = 4 \cdot x^{7-2} \cdot y^{1-3} = 4x^5y^{-2} = \frac{4x^5}{y^2}.$$

**• PROBLEM 83**

Simplify the quotient $\dfrac{(x^{-2}y^4)^3}{(xy)^{-3}}$ .

**Solution:** Since $(ab)^x = a^xb^x$, and $(a^x)^y = a^{xy}$:

$$\frac{(x^{-2}y^4)^3}{(xy)^{-3}} = \frac{(x^{-2})^3(y^4)^3}{x^{-3}y^{-3}} = \frac{x^{(-2)(3)}y^{4\cdot 3}}{x^{-3}y^{-3}} = \frac{x^{-6}y^{12}}{x^{-3}y^{-3}} .$$

When dividing common bases with different exponents we subtract the exponent of the divisor from the exponent of the dividend

$$\left(\frac{a^x}{a^y} = a^{x-y}\right) ;$$

thus:

$$= x^{-6-(-3)}\ y^{12-(-3)}$$

$$= x^{-6+3}\ y^{12+3}$$

$$= x^{-3}\ y^{15} .$$

**• PROBLEM 84**

Simplify (rewrite without negative exponents, and reduce to a fraction in lowest terms)

$$\frac{7x^{-1}}{x^{-3}+y^{-4}} .$$

**Solution 1:** We see that all negative exponents can be eliminated by multiplying numerator and denominator by $x^3y^4$. Hence,

$$\frac{7x^{-1}}{x^{-3}+y^{-4}} = \frac{7x^{-1}}{x^{-3}+y^{-4}} \cdot \frac{x^3y^4}{x^3y^4} = \frac{7x^{-1}x^3y^4}{x^{-3}x^3y^4+y^{-4}x^3y^4} = \frac{7x^{-1+3}y^4}{x^{-3+3}y^4+x^3y^{-4+4}}$$

$$= \frac{7x^2y^4}{y^4+x^3} .$$

**Solution 2:** Another way to solve this problem is to apply the definition $a^{-n} = 1/a^n$ where $a \neq 0$.

$$\frac{7x^{-1}}{x^{-3}+y^{-4}} = \frac{7\left(\frac{1}{x}\right)}{\frac{1}{x^3} + \frac{1}{y^4}}.$$ Combine $\frac{1}{x^3} + \frac{1}{y^4}$ into one term. The least common denominator is $x^3y^4$.

$$\frac{1}{x^3} + \frac{1}{y^4} = \frac{1}{x^3}\left(\frac{y^4}{y^4}\right) + \frac{1}{y^4}\left(\frac{x^3}{x^3}\right) = \frac{y^4 + x^3}{y^4x^3}$$

Therefore,

$$\frac{7\left(\frac{1}{x}\right)}{\frac{1}{x^3}+\frac{1}{y^4}} = \frac{\frac{7}{x}}{\frac{y^4+x^3}{x^3y^4}}.$$

Division by a fraction is equivalent to multiplication by that fraction's reciprocal. Thus

$$\frac{\frac{7}{x}}{\frac{y^4+x^3}{x^3y^4}} = \frac{7}{x}\cdot\frac{x^3y^4}{y^4+x^3} = \frac{7x^2y^4}{y^4+x^3}$$

• **PROBLEM 85**

Express $\dfrac{3x^{-1} - y^{-2}}{x^{-2} + 2y^{-1}}$ without negative exponents.

<u>Solution:</u> Since $x^{-a} = \frac{1}{x^a}$ for all real $x \neq 0$,

$x^{-1} = \frac{1}{x}$, $\quad y^{-2} = \frac{1}{y^2}$

$x^{-2} = \frac{1}{x^2}$ , and $y^{-1} = \frac{1}{y}$; thus,

$$\frac{3x^{-1} - y^{-2}}{x^{-2} + 2y^{-1}} = \frac{3\left(\frac{1}{x}\right) - \frac{1}{y^2}}{\frac{1}{x^2} + 2\left(\frac{1}{y}\right)}$$

$$= \frac{\frac{3}{x} - \frac{1}{y^2}}{\frac{1}{x^2} + \frac{2}{y}}$$

Multiplying numerator and denominator by the least common multiple, $x^2y^2$,

$$= \frac{x^2y^2\left(\frac{3}{x} - \frac{1}{y^2}\right)}{x^2y^2\left(\frac{1}{x^2} + \frac{2}{y}\right)}$$

Distributing, $= \dfrac{x^2y^2\left(\frac{3}{x}\right) - x^2y^2\left(\frac{1}{y^2}\right)}{x^2y^2\left(\frac{1}{x^2}\right) + x^2y^2\left(\frac{2}{y}\right)}$

Cancelling like terms, $= \dfrac{3xy^2 - x^2}{y^2 + 2x^2y}$

Factoring $x$ from numerator and $y$ from denominator,

$$= \frac{x(3y^2 - x)}{y(y + 2x^2)}$$

Thus, $\dfrac{3x^{-1} - y^{-2}}{x^{-2} + 2y^{-1}} = \dfrac{x(3y^2 - x)}{y(y + 2x^2)}$

• **PROBLEM 86**

Simplify: (a) $\dfrac{a^{-3}b^2}{a^{-2}b^4}$ (b) $\dfrac{3x^{-4}}{y^2} \cdot \dfrac{4x}{9x^2y^{-1}}$

Solution: (a) Since $a^{-n} = \dfrac{1}{a^n}$,

$$\frac{a^{-3}b^2}{a^{-2}b^4} = \frac{\frac{b^2}{a^3}}{\frac{b^4}{a^2}} = \frac{b^2}{a^3} \cdot \frac{a^2}{b^4}$$

Dividing by a fraction is equivalent to multiplying by the reciprocal of the fraction.

$$\frac{b^2}{a^3} \cdot \frac{a^2}{b^4} = \frac{b^2a^2}{a^3b^4} = \frac{a^2}{a^3} \cdot \frac{b^2}{b^4} = \frac{1}{ab^2}$$

$$\text{(b)} \quad \frac{3x^{-4}}{y^2} \cdot \frac{4x}{9x^2y^{-1}} = \frac{3}{y^2x^4} \cdot \frac{4x}{\frac{9x^2}{y}} = \frac{3}{y^2x^4} \cdot \frac{4xy}{9x^2} = \frac{3 \cdot 4}{9yx^5} = \frac{4}{3yx^5}$$

• **PROBLEM 87**

Find the following products.

a. $(3x^2y)(2xy^2)$

b. $(-xy^3)(4xyz)(2yz)$

Solution: Use the following law of exponents to find the indicated products:

$$a^m \cdot a^n \cdot a^x \cdot \ldots = a^{m+n+x+\ldots}$$

a) $(3x^2y)(2xy^2) = 6(x^2 \cdot x)(y \cdot y^2)$

$= 6(x^{2+1})(y^{1+2})$

$= 6x^3y^3$

b) $(-xy^3)(4xyz)(2yz) = -8(x \cdot x)(y^3 \cdot y \cdot y)(z \cdot z)$

$$= -8\left(x^{1+1}\right)\left(y^{3+1+1}\right)\left(z^{1+1}\right)$$

$$= -8(x^2)(y^5)(z^2)$$

$$= -8x^2y^5z^2$$

**• PROBLEM 88**

Use the properties of exponents, to perform the indicated operations in

$$(2^3x^45^2y^7)^5.$$

Solution: Since the product of several numbers raised to the same exponent equals the product of each number raised to that exponent (i.e., $(abcd)^x = a^xb^xc^xd^x$) we obtain,

$$(2^3x^45^2y^7)^5 = (2^3)^5(x^4)^5(5^2)^5(y^7)^5.$$

Recall that $(x^a)^b = x^{a \cdot b}$; thus

$$(2^3x^45^2y^7)^5 = (2^3)^5(x^4)^5(5^2)^5(y^7)^5$$

$$= (2^{3\cdot5})(x^{4\cdot5})(5^{2\cdot5})(y^{7\cdot5})$$

$$= 2^{15}x^{20}5^{10}y^{35}.$$

**• PROBLEM 89**

Perform the indicated operations, and simplify. (Write without negative or zero exponents.) Each letter represents a positive real number.

(a) $\left(7x^{-3}y^5\right)^{-2}$ (b) $\left(5x^7y^{-8}\right)^{-3}$.

Solution: Note that: (1) $(abc)^x = a^xb^xc^x$ (for all real a,b,c), (2) $a^{-x} = \frac{1}{a^x}$ (for all non-zero real a) and (3) $\left(a^b\right)^c = a^{bc}$ (for all real a,b,c). These will be useful in evaluating the given expressions.

(a) $\left(7x^{-3}y^5\right)^{-2} = 7^{-2}\left(x^{-3}\right)^{-2}\left(y^5\right)^{-2} = 7^{-2}(x)^{(-3)(-2)}(y)^{(5)(-2)}$

$= 7^{-2}(x)^6(y)^{-10} = \dfrac{x^6}{(7^2)(y^{10})} = \dfrac{x^6}{49y^{10}}.$

(b) $\left(5x^7y^{-8}\right)^{-3} = 5^{-3}\left(x^7\right)^{-3}\left(y^{-8}\right)^{-3} = 5^{-3}(x)^{(7)(-3)}(y)^{(-8)(-3)}$

$= 5^{-3}(x)^{-21}(y)^{24} = \dfrac{y^{24}}{5^3x^{21}} = \dfrac{y^{24}}{125x^{21}}.$

**• PROBLEM 90**

Evaluate the following expressions:

(a) $\dfrac{-12x^{10}y^9z^5}{3x^2y^3z^6}$  (b) $\dfrac{-16x^{16}y^6z^4}{-4x^4y^2z^7}$.

**Solution:** Noting (a) $\frac{abcd}{efgh} = \frac{a}{e}\cdot\frac{b}{f}\cdot\frac{c}{g}\cdot\frac{d}{h}$ , (2) $a^{-b} = \frac{1}{a^b}$ and (3) $\frac{a^b}{a^c} = a^{b-c}$ for all non-zero real values of a,e,f,g,h, we proceed to evaluate these expressions:

(a) $$\frac{-12x^{10}y^9z^5}{3x^2\,y^3z^6} = \frac{-12}{3}\cdot\frac{x^{10}}{x^2}\cdot\frac{y^9}{y^3}\cdot\frac{z^5}{z^6} = -4\cdot x^{10-2}\cdot y^{9-3}\cdot z^{5-6}$$

$$= -4x^8y^6z^{-1} = \frac{-4x^8y^6}{z^1}.$$

Thus $$\frac{-12x^{10}y^9z^5}{3x^2y^3z^6} = \frac{-4x^8y^6}{z}.$$

(b) $$\frac{-16x^{16}y^6z^4}{-\,4x^4y^2z^7} = \frac{-16}{-4}\cdot\frac{x^{16}}{x^4}\cdot\frac{y^6}{y^2}\cdot\frac{z^4}{z^7}$$

$$= 4x^{16-4}\cdot y^{6-2}\cdot z^{4-7} = 4x^{12}y^4z^{-3} = \frac{4x^{12}y^4}{z^3}.$$

• **PROBLEM 91**

Perform the indicated operations and simplify:

$$\left(\frac{-5b^y}{3^2x^5}\right)^3\left(\frac{3x^7}{5b^y}\right)^2 \quad .$$

**Solution:** $$\left(\frac{-5b^y}{3^2x^5}\right)^3\left(\frac{3x^7}{5b^y}\right) = \frac{(-5b^y)^3}{(3^2x^5)^3}\cdot\frac{(3x^7)^2}{(5b^y)^2} \quad \text{since} \quad \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

$$= \frac{(-5)^3(b^y)^3}{(3^2)^3(x^5)^3}\cdot\frac{3^2(x^7)^2}{5^2(b^y)^2} \quad \text{since } (ab)^x = a^xb^x$$

$$= \frac{-5^3b^{3y}}{3^6x^{15}}\cdot\frac{3^2x^{14}}{5^2b^{2y}} \quad \text{since} \quad (a^x)^y = a^{x\cdot y}$$

$$= \frac{\left(-5^3b^{3y}\right)\left(3^2x^{14}\right)}{\left(3^6x^{15}\right)\left(5^2b^{2y}\right)}$$

$$= \frac{\left(3^2x^{14}\right)\left(-5^3b^{3y}\right)}{\left(3^6x^{15}\right)\left(5^2b^{2y}\right)} \quad \text{using the commutative law of multiplication}$$

$$= \left(3^{2-6}\right)\left(x^{14-15}\right)\left[-\left(5^{3-2}\right)\left(b^{3y-2y}\right)\right] \quad \text{because}$$

$$\frac{x^a}{x^b} = x^{a-b},$$

$$= (3^{-4})(x^{-1})(-5^1)(b^y)$$

$$= \frac{-5b^y}{3^4 x} \quad \text{because} \quad x^{-a} = \frac{1}{x^a}$$

$$= \frac{-5b^y}{3 \cdot 3 \cdot 3 \cdot 3x}$$

$$= \frac{-5b^y}{81x}$$

**• PROBLEM 92**

Determine the value of $(0.0081)^{-3/4}$.

Solution: $(0.0081) = .3 \times .3 \times .3 \times .3 = (.3)^4$,

therefore, $(0.0081)^{-3/4} = (.3^4)^{-3/4}$

Recalling the property of exponents,

$$(a^x)^y = a^{x \cdot y}$$

we have,

$$(.3^4)^{-3/4} = .3^{(4)(-3/4)} = .3^{-3}.$$

Since $\quad a^{-x} = \frac{1}{a^x}$, $.3^{-3} = \frac{1}{.3^3} = \frac{1}{0.027} = \frac{1}{\frac{27}{1000}}$

Division by a fraction is equivalent to multiplication by its reciprocal, thus,

$$\frac{1}{\frac{27}{1000}} = \frac{1000}{27}.$$

Hence,

$$(0.0081)^{-3/4} = \frac{1000}{27}.$$

**• PROBLEM 93**

Simplify $\left[\frac{1600 \times 10,000}{2000}\right]^{1/3}$.

Solution: Observe $1600 = 16 \times 100 = 16 \times 10^2$

$$10,000 = 10^4$$

$$2,000 = 2 \times 10^3 .$$

Thus, $\left[\frac{1600 \times 10,000}{2000}\right]^{1/3} = \left[\frac{(16 \times 10^2)(10^4)}{2 \times 10^3}\right]^{1/3}$ .

Using the associative property,

$$= \left[\frac{16 \times (10^2 \times 10^4)}{2 \times 10^3}\right]^{1/3}$$

Recall: $a^x \cdot a^y = a^{x+y}$, $\quad = \left[\frac{16 \times 10^6}{2 \times 10^3}\right]^{1/3}$

Since $\frac{ab}{cd} = \frac{a}{c} \cdot \frac{b}{d}$,

$$= \left[\frac{16}{2} \times \frac{10^6}{10^3}\right]^{1/3}$$

Recall $\frac{a^x}{a^y} = a^{x-y}$,

$$= \left(8 \times 10^3\right)^{1/3}$$

Since $(ab)^x = a^x b^x$,

$$= 8^{1/3} \times 10^{3(1/3)}$$

$$= 2 \times 10^1$$

$$= 20$$

• **PROBLEM 94**

Determine the value of

$$\frac{5^{3/4}5^{2/3}5^{-5/2}5^{5/3}}{5^{1/3}5^{-5/2}5^{7/4}} .$$

**Solution:** Since $a^x \cdot a^y \cdot a^z = a^{x+y+z}$ for any real base, then,

$$\frac{5^{\frac{3}{4}} \cdot 5^{\frac{2}{3}} \cdot 5^{-\frac{5}{2}} \cdot 5^{\frac{5}{3}}}{5^{\frac{1}{3}} \cdot 5^{-\frac{5}{2}} \cdot 5^{\frac{7}{4}}} = \frac{5^{\frac{3}{4} + \frac{2}{3} - \frac{5}{2} + \frac{5}{3}}}{5^{\frac{1}{3} - \frac{5}{2} + \frac{7}{4}}}$$

The fractional exponents have denominators 4, 3, and 2. Their least common denominator (L.C.D.) is the least common multiple of the denominators, 12. Converting the fractional exponents to twelfths we obtain

$$\frac{5^{\frac{3}{4} + \frac{2}{3} - \frac{5}{2} + \frac{5}{3}}}{5^{\frac{1}{3} - \frac{5}{2} + \frac{7}{4}}} = \frac{5^{\frac{9}{12} + \frac{8}{12} - \frac{30}{12} + \frac{20}{12}}}{5^{\frac{4}{12} - \frac{30}{12} + \frac{21}{12}}}$$

$$= \frac{5^{\frac{7}{12}}}{5^{-\frac{5}{12}}}$$

Since $\frac{a^x}{a^y} = a^{x-y}$ for any real base $(a \neq 0)$

$$\frac{5^{\frac{7}{12}}}{5^{-\frac{5}{12}}} = 5^{\frac{7}{12} - \left(-\frac{5}{12}\right)}$$

$$= 5^{\frac{7}{12} + \frac{5}{12}}$$

$$= 5^{\frac{12}{12}}$$

$$= 5^1$$

$$= 5.$$

Therefore, $\dfrac{5^{\frac{3}{4}} 5^{\frac{2}{3}} 5^{-\frac{5}{2}} 5^{\frac{5}{3}}}{5^{\frac{1}{3}} 5^{-\frac{5}{2}} 5^{\frac{7}{4}}} = 5.$

Express $\left(5^{\frac{1}{2}} + 9^{\frac{1}{8}}\right) \div \left(5^{\frac{1}{2}} - 9^{\frac{1}{8}}\right)$ as an equivalent fraction with a rational denominator.

Solution: The given expression can be rewritten as,

$$\frac{5^{\frac{1}{2}} + 9^{\frac{1}{8}}}{5^{\frac{1}{2}} - 9^{\frac{1}{8}}}\,, \qquad \text{and since}$$

$$9^{\frac{1}{8}} = \left(3^2\right)^{\frac{1}{8}} = 3^{\frac{1}{4}} \qquad \text{we write:}$$

$$\frac{5^{\frac{1}{2}} + 3^{\frac{1}{4}}}{5^{\frac{1}{2}} - 3^{\frac{1}{4}}}\,.$$

To rationalize the denominator, put

$5^{\frac{1}{2}} = x$, $3^{\frac{1}{4}} = y$; then since

$$x^4 - y^4 = \left(5^{\frac{1}{2}}\right)^4 - \left(3^{\frac{1}{4}}\right)^4 = 5^2 - 3 = 25 - 3 = 22,$$

which is rational, we can write

$$x^4 - y^4 = (x - y)\left(x^3 + x^2y + xy^2 + y^3\right),$$

and the factor which rationalizes $x - y$, or $5^{\frac{1}{2}} - 3^{\frac{1}{4}}$ is $x^3 + x^2y + xy^2 + y^3$, and substituting for x and y:

$$\left(5^{\frac{1}{2}}\right)^3 + \left(5^{\frac{1}{2}}\right)^2 \cdot 3^{\frac{1}{4}} + 5^{\frac{1}{2}} \cdot \left(3^{\frac{1}{4}}\right)^2 + \left(3^{\frac{1}{4}}\right)^3$$

$$= \quad 5^{\frac{3}{2}} + 5^{\frac{2}{2}} \cdot 3^{\frac{1}{4}} + 5^{\frac{1}{2}} \cdot 3^{\frac{2}{4}} + 3^{\frac{3}{4}}\,;$$

and the rational denominator is

$$x^4 - y^4 = 5^{\frac{4}{2}} - 3^{\frac{4}{4}} = 5^2 - 3 = 22.$$

Now, since

$$x^4 - y^4 = (x - y)\left(x^3 + x^2y + xy^2 + y^3\right), \qquad \text{then}$$

$$(x - y) = \frac{x^4 - y^4}{x^3 + x^2y + xy^2 + y^3}\,, \qquad \text{and substituting:}$$

$$5^{\frac{1}{2}} - 3^{\frac{1}{4}} = \frac{22}{5^{\frac{3}{2}} + 5^{\frac{2}{2}} \cdot 3^{\frac{1}{4}} + 5^{\frac{1}{2}} \cdot 3^{\frac{2}{4}} + 3^{\frac{3}{4}}}$$

Therefore, the given expression

$$= \frac{5^{\frac{1}{2}} + 3^{\frac{1}{4}}}{\dfrac{22}{5^{\frac{3}{2}} + 5^{\frac{2}{2}} \cdot 3^{\frac{1}{4}} + 5^{\frac{1}{2}} \cdot 3^{\frac{2}{4}} + 3^{\frac{3}{4}}}}$$

$$= \frac{\left(5^{\frac{1}{2}} + 3^{\frac{1}{4}}\right)\left[5^{\frac{3}{2}} + \left(5^{\frac{2}{2}} \cdot 3^{\frac{1}{4}}\right) + \left(5^{\frac{1}{2}} \cdot 3^{\frac{2}{4}}\right) + 3^{\frac{3}{4}}\right]}{22}$$

$$= 5^{\frac{4}{2}} + 5^{\frac{3}{2}}\, 3^{\frac{1}{4}} + 5^{\frac{3}{2}}3^{\frac{1}{4}} + 5^{\frac{2}{2}}3^{\frac{2}{4}} + 5^{\frac{2}{2}}3^{\frac{2}{4}} +$$

$$+ 5^{\frac{1}{2}}3^{\frac{3}{4}} + 5^{\frac{1}{2}}3^{\frac{3}{4}} + 3^{\frac{4}{4}}$$

$$= \frac{5^{\frac{4}{2}} + \left(2 \cdot 5^{\frac{3}{2}} \cdot 3^{\frac{1}{4}}\right) + \left(2 \cdot 5^{\frac{2}{2}} \cdot 3^{\frac{2}{4}}\right) + \left(2 \cdot 5^{\frac{1}{2}} \cdot 3^{\frac{3}{4}}\right) + 3^{\frac{4}{4}}}{22}$$

$$= \frac{5^2 + 2\left(5^{\frac{3}{2}} \cdot 3^{\frac{1}{4}} + 5^{\frac{2}{2}} \cdot 3^{\frac{2}{4}} + 5^{\frac{1}{2}} \cdot 3^{\frac{3}{4}}\right) + 3}{22}$$

$$= \frac{28 + 2\left(5^{\frac{3}{2}} \cdot 3^{\frac{1}{4}} + 5^{\frac{2}{2}} \cdot 3^{\frac{2}{4}} + 5^{\frac{1}{2}} \cdot 3^{\frac{3}{4}}\right)}{22}$$

$$= \frac{14 + 5^{\frac{3}{2}} \cdot 3^{\frac{1}{4}} + 5 \cdot 3^{\frac{1}{2}} + 5^{\frac{1}{2}} \cdot 3^{\frac{3}{4}}}{11}$$

# CHAPTER 7

# ROOTS AND RADICALS

**Basic Attacks and Strategies for Solving Problems in this Chapter. See pages 53 to 78 for step-by-step solutions to problems.**

When simplifying and evaluating roots and radicals, it is essential to first observe the fundamental relation between roots, fractional exponents, and radicals. In particular, for any positive real number $x$ and positive rational number $p/q$ ($q$ is not zero),

$$x^{p/q} = \sqrt[q]{x^p} = \left(\sqrt[q]{x}\right)^p = \left(x^{1/q}\right)^p$$

is the fundamental relation on which all simplifications and evaluations of roots and radicals rest. Also, all the properties involving multiplication and division involving exponents are essential from Chapter 6.

When dealing with negative exponents, it is usually helpful to write the expression as a reciprocal in order to get a positive exponent before further simplification and evaluation of roots. For example, to simplify the expression below we first change the negative exponent to a positive exponent and then apply the aforementioned relationship between the fractional exponents and radicals as follows:

$$8^{-2/3} = 1/8^{2/3} = 1/\left(\sqrt[3]{8}\right)^2 = 1/(2)^2 = 1/4.$$

When dealing with negative values of $x$, it is important to recognize that this leads to complex numbers. In particular, it is important to observe that $\sqrt{-1} = i$, the imaginary part of a complex number. Also, if the index of the radical is an even integer and $x$ is negative, then the results will always include $\sqrt{-1}$. For instance,

$$\sqrt{-4} = \sqrt{4} * \sqrt{-1} = 2i.$$

In the simplification of radicals, it is usually desirable to arrange radicals in a simplest form. A radical with index $n$ is in simplest form if the following conditions hold:

(1) The $n^{th}$ roots of any $n^{th}$ powers in the radicand have been taken. When this is done every exponent in the radicand will be less than $n$.

(2) All denominators have been rationalized; that is, all denominators are free of radicals. (See procedure below.)

(3) The index of the radical is reduced as much as possible.

If a fraction contains one or more radicals in the denominator, then the denominator must be rationalized in order to put it in simplest form. To rationalize a denominator which contains a monomial radical, multiply both the numerator and denominator of the given fraction by an appropriate monomial radical which will enable the results in the denominator to be the $n^{th}$ root of a $n^{th}$ power, which is a result that is free of the radical in the denominator. To rationalize a binominal denominator in which one or both terms is a square root, multiply the numerator and denominator of the fraction by the conjugate of the denominator, i.e., by the same two terms but with the opposite sign between them. For example, to simplify the radical expression below, we need to rationalize the denominator as follows:

$$\frac{\sqrt{2}}{\sqrt{2}-\sqrt{x}}=\frac{\sqrt{2}}{\sqrt{2}-\sqrt{x}}\bullet\frac{\sqrt{2}+\sqrt{x}}{\sqrt{2}+\sqrt{x}}=\frac{2+\sqrt{2x}}{2-x}$$

To simplify a radical in which the index is a multiple of the exponents in the radicand, rewrite the expression by simply dividing the index and all of the exponents under the radical by the GCD. Then, simplify the result.

## Step-by-Step Solutions to Problems in this Chapter, "Roots and Radicals"

### SIMPLIFICATION AND EVALUATION OF ROOTS

**• PROBLEM 96**

Evaluate $\sqrt{400}$.

**Solution:** $400 = 4 \times 100$

Thus, $\sqrt{400} = \sqrt{4 \times 100}$

Since $\sqrt{ab} = \sqrt{a}\ \sqrt{b}$,

$$\begin{aligned} \sqrt{400} &= \sqrt{4}\ \sqrt{100} \\ &= 2 \cdot 10 \\ &= 20 \end{aligned}$$

Check: If $\sqrt{400}$ is 20, then $20^2$ must equal 400, which is true. Hence, 20 is the solution.

**• PROBLEM 97**

Find the value of $\sqrt[4]{-64a^4}$.

**Solution:** We can rewrite $\sqrt[4]{-64^4}$ as,

$$\left[64a^4 \cdot (-1)\right]^{\frac{1}{4}} = \left[(\pm 8a^2)^2 \cdot (-1)\right]^{\frac{1}{4}},$$

by first factoring $-1$ from the expression under the radical, and then using the fact that $64a^4 = \left(\pm 8a^2\right)^2$. Also recall that $\sqrt[4]{x} = x^{\frac{1}{4}}$. Now, since $(ab)^x = a^x \cdot b^x$, and $\left(a^x\right)^y = a^{xy}$, we write:

$$\begin{aligned} \left[(8a^2)^2 \cdot (-1)\right]^{\frac{1}{4}} &= \left[(8a^2)^2\right]^{\frac{1}{4}} \cdot (-1)^{\frac{1}{4}} \\ &= \left[8^2 \cdot (a^2)^2\right]^{\frac{1}{4}} \cdot (-1)^{\frac{1}{4}} \\ &= (8^2)^{\frac{1}{4}} \cdot \left[(a^2)^2\right]^{\frac{1}{4}} \cdot (-1)^{\frac{1}{4}} \\ &= 8^{\frac{1}{2}} \cdot (a^2)^{\frac{1}{2}} \cdot (-1)^{\frac{1}{2} \cdot \frac{1}{2}} \\ &= 8^{\frac{1}{2}} \cdot (a^2)^{\frac{1}{2}} \cdot \left[(-1)^{\frac{1}{2}}\right]^{\frac{1}{2}} \end{aligned}$$

Since $x^{\frac{1}{2}} = \sqrt{x}$, and $(x^{\frac{1}{2}})^{\frac{1}{2}} = \sqrt{\sqrt{x}}$, we write:

$$\sqrt[4]{-64a^4} = \sqrt{\pm 8a^2 \sqrt{-1}} = \sqrt{4a^2 \cdot 2 \cdot \pm \sqrt{-1}},$$

and since

$$\sqrt{a\cdot b\cdot c} = \sqrt{a}\sqrt{b}\sqrt{c}\ ,\quad \sqrt[4]{-64a^4} = \sqrt{4a^2}\cdot\sqrt{2}\cdot\sqrt{\pm\sqrt{-1}}$$

$$= 2a\sqrt{2}\sqrt{\pm\sqrt{-1}}\ .$$

It remains to find the value of $\sqrt{\pm\sqrt{-1}}$. $\sqrt{+\sqrt{-1}}$ is a complex number and can be written as the sum of real and imaginery parts eg. $x+y\sqrt{-1}$ where x is the real part and y is the imaginary part.

Squaring both sides of the equation we obtain:

$$+\sqrt{-1} = (x+y\sqrt{-1})(x+y\sqrt{-1})$$

or, by performing the multiplication,

$$+\sqrt{-1} = x^2 + xy\sqrt{-1} + xy\sqrt{-1} + y\sqrt{-1}\cdot y\sqrt{-1}\ .$$

Combining like terms, and recalling that $\sqrt{-1}\cdot\sqrt{-1} = -1$, we have:

$$+\sqrt{-1} = x^2 - y^2 + 2xy\sqrt{-1}\ .$$

Let us examine the equation,

$$+\sqrt{-1} = x^2 - y^2 + 2xy\sqrt{-1}\ .$$

This equation is only true if $x^2 - y^2 = 0$, and $2xy = 1$, because then the equation becomes:

$$+\sqrt{-1} = 0 + {}^-1\sqrt{-1}$$

$$+\sqrt{-1} = +\sqrt{-1}$$

Therefore, we have the following system of equations:

$$x^2 - y^2 = 0 \quad\text{and}\quad 2xy = 1\ .$$

To solve for x and y we use the method of substitution. Solving for y in the second equation, $2xy = 1$, we have:

$$y = \frac{1}{2x}\ ;$$

and substituting this value into equation one, $x^2 - y^2 = 0$, we obtain:

$$x^2 - \left(\frac{1}{2x}\right)^2 = 0$$

$$x^2 - \frac{1}{4x^2} = 0$$

Multiply both sides by $4x^2$ : $4x^2\left(x^2 - \frac{1}{4x^2}\right) = 4x^2(0)$

Distribute: $4x^4 - 1 = 0$

Add 1 to both sides: $4x^4 = 1$

Divide both sides by 4: $x^4 = \frac{1}{4}$

Now, taking the fourth root of both sides we obtain:

$$x = \sqrt[4]{\tfrac{1}{4}}$$

$$= \left(\tfrac{1}{4}\right)^{\frac{1}{4}}\ ,\ \text{since}\ \sqrt[4]{x} = x^{\frac{1}{4}}$$

$$= \frac{1^{\frac{1}{4}}}{4^{\frac{1}{4}}}\ ,\ \text{since}\ \left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

$$= \frac{1}{(2^2)^{\frac{1}{4}}}$$

$$= \frac{1}{2^{\frac{1}{2}}}\ ,\ \text{since}\ (a^b)^x = a^{bx}$$

$$= \frac{1}{\sqrt{2}}\ ,\ \text{since}\ x^{\frac{1}{2}} = \sqrt{x}$$

Now, since $y = \frac{1}{2x}$, by substitution:

$$y = \frac{1}{2\left(\frac{1}{\sqrt{2}}\right)} = \frac{1}{\frac{2}{\sqrt{2}}} = \frac{\sqrt{2}}{2}$$

Observing the y-value, $\frac{\sqrt{2}}{2}$, closely we see that it is equivalent to the x-value, $\frac{1}{\sqrt{2}}$. This can be seen by multiplying $\frac{\sqrt{2}}{2}$ by $\frac{\sqrt{2}}{\sqrt{2}}$ (which equals 1 and therefore does not alter the value of the fraction).

Thus,

$$\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$$

Therefore,

$$x = \frac{1}{\sqrt{2}},\ y = \frac{1}{\sqrt{2}};\ \text{or}\ \ x = -\frac{1}{\sqrt{2}},\ y = -\frac{1}{\sqrt{2}}\ ;$$

We must include the negative values for $x$ and $y$ as solutions to the equations, because these give us the same results in the equations $x^2 - y^2 = 0$ and $2xy = 1$ as do the positive values for $x$ and $y$, since a negative value squared is positive (equation 1), and a negative multiplied by a negative is positive (equation 2). Therefore, substituting into the equation, $\sqrt{+\sqrt{-1}} = x + y\sqrt{-1}$, we have:

$$\sqrt{+\sqrt{-1}} = \pm\frac{1}{\sqrt{2}} + \left(\pm\frac{1}{\sqrt{2}}\right)\sqrt{-1}\ ,$$

and factoring $\pm\frac{1}{\sqrt{2}}$ from both terms on the right side:

$$\sqrt{+\sqrt{-1}} = \pm\frac{1}{\sqrt{2}}(1 + \sqrt{-1}).$$

Similarly, we assume $\sqrt{-\sqrt{-1}} = x - y\sqrt{-1}$, and proceeding as in the case when $\sqrt{+\sqrt{-1}} = x + y\sqrt{-1}$, we find that the $x$ and $y$ values are again $\pm\frac{1}{\sqrt{2}}$, thus:

$$\sqrt{-\sqrt{-1}} = \pm\frac{1}{\sqrt{2}}(1 - \sqrt{-1})$$

Therefore,

$$\sqrt{\pm\sqrt{-1}} = \pm\frac{1}{\sqrt{2}}(1 \pm \sqrt{-1})\ ;$$

and finally, from the fact that,

$$\sqrt[4]{-64a^4} = 2a\sqrt{2}\sqrt{\pm\sqrt{-1}}\ ,\ \text{and}$$

$$\sqrt{\pm\sqrt{-1}} = \pm\frac{1}{\sqrt{2}}(1 \pm \sqrt{-1}\ )\ ,\ \text{we have:}$$

$$\sqrt[4]{-64a^4} = 2a\sqrt{2} \cdot \left[\pm\frac{1}{\sqrt{2}}\left(1 \pm \sqrt{-1}\right)\right]$$

$$= 2a\left[\pm(1 \pm \sqrt{-1})\right]\ ,\ \text{by cancelling}\ \frac{\sqrt{2}}{\sqrt{2}}\ .$$

Therefore,

$$\sqrt[4]{-64a^4} = \pm 2a(1 \pm \sqrt{-1})\ .$$

**• PROBLEM 98**

Evaluate $16^{-\frac{3}{4}}$.

Solution:
$$16^{-\frac{3}{4}} = \frac{1}{16^{\frac{3}{4}}}$$
$$= \frac{1}{\left(\sqrt[4]{16}\right)^3} \quad .$$

Note that $2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$, hence $\sqrt[4]{16} = 2$. Thus, $16^{-\frac{3}{4}}$
$$= \frac{1}{2^3} = \frac{1}{2 \cdot 2 \cdot 2} = \frac{1}{8} \, .$$

**• PROBLEM 99**

Show that $\sqrt[3]{(-8)^3} = \left(\sqrt[3]{-8}\right)^3$.

Solution: $\sqrt[3]{(-8)^3} = \sqrt[3]{-512} = -8$. Since $(-8)^3 = -512$, $\sqrt[3]{-512} = -8$. $\sqrt[3]{-8} = -2$ since $(-2)^3 = -8$. Therefore $(\sqrt[3]{-8})^3 = (-2)^3 = -8$. $\sqrt[3]{(-8)^3} = -8 = (\sqrt[3]{-8})^3$, hence, $\sqrt[3]{(-8)^3} = (\sqrt[3]{-8})^3$.

**• PROBLEM 100**

Find the indicated roots.

(a) $\sqrt[5]{32}$ (b) $\pm \sqrt[4]{625}$ (c) $\sqrt[3]{-125}$ (d) $\sqrt[4]{-16}$.

Solution: The following two laws of exponents can be used to solve these problems: 1) $\left(\sqrt[n]{a}\right)^n = \left(a^{1/n}\right)^n = a^1 = a$, and 2) $\left(\sqrt[n]{a}\right)^n = \sqrt[n]{a^n}$.

(a) $\sqrt[5]{32} = \sqrt[5]{2^5} = \left(\sqrt[5]{2}\right)^5 = 2$. This result is true because $(2)^5 = 32$, that is, $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 32$.

(b) $\sqrt[4]{625} = \sqrt[4]{5^4} = \left(\sqrt[4]{5}\right)^4 = 5$. This result is true because $\left(5^4\right) = 625$, that is, $5 \cdot 5 \cdot 5 \cdot 5 = 625$.

$-\sqrt[4]{625} = -\left(\sqrt[4]{5^4}\right) = -\left[\left(\sqrt[4]{5}\right)^4\right] = -\left[5\right] = -5$. This result is true because $(-5)^4 = 625$, that is, $(-5) \cdot (-5) \cdot (-5) \cdot (-5) = 625$.

(c) $\sqrt[3]{-125} = \sqrt[3]{(-5)^3} = (\sqrt[3]{-5})^3 = -5$. This result is true because $(-5)^3 = -125$, that is, $(-5) \cdot (-5) \cdot (-5) = -125$.

(d) There is no solution to $\sqrt[4]{-16}$ because any number raised to the fourth power is a positive number, that is, $N^4 = (N) \cdot (N) \cdot (N) \cdot (N) =$ a positive number $\neq$ a negative number, $-16$.

Simplify: (a) $\sqrt[3]{-512}$ (b) $\sqrt[4]{\frac{81}{16}}$ (c) $\sqrt[3]{-16} \div \sqrt[3]{-2}$.

Solution: (a) By the law of radicals which states that $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$ where a and b are any two numbers, $\sqrt[3]{-512} = \sqrt[3]{8(-64)} = \sqrt[3]{8}\sqrt[3]{-64}$. Therefore, $\sqrt[3]{-512} = \sqrt[3]{8}\sqrt[3]{-64} = (2)(-4) = -8$. The last result is true because $(2)^3 = 8$ and $(-4)^3 = -64$.

(b) By another law of radicals which states that $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ where a and b are any two numbers, $\sqrt[4]{\frac{81}{16}} = \frac{\sqrt[4]{81}}{\sqrt[4]{16}}$. Therefore, $\sqrt[4]{\frac{81}{16}} = \frac{\sqrt[4]{81}}{\sqrt[4]{16}} = \frac{3}{2}$. The last result is true because $(3)^4 = 81$ and $(2)^4 = 16$.

(c) By the law of radicals stated in example (b), $\sqrt[3]{-16} \div \sqrt[3]{-2} = \frac{\sqrt[3]{-16}}{\sqrt[3]{-2}} = \sqrt[3]{\frac{-16}{-2}} = \sqrt[3]{8} = 2$. The last result is true because $(2)^3 = 8$.



Show that: (a) $(-8)^{2/3} = (-8^{1/3})^2$

(b) $\left(\frac{1}{64}\right)^{4/3} = \left[\left(\frac{1}{64}\right)^{1/3}\right]^4$

Solution: (a) By the law of exponents which states that $(N)^{a/b} = \sqrt[b]{N^a}$ where N is any number, $(-8)^{2/3} = \sqrt[3]{(-8)^2}$. Therefore, $(-8)^{2/3} = \sqrt[3]{(-8)^2} = \sqrt[3]{64} = 4$. The last result is true because $(4)^3 = 64$. Also, by the law of exponents which states that $\left(N_1^{a_1}N_2^{a_2}\right)^b = N_1^{a_1b}\,N_2^{a_2b}$ where $N_1$ and $N_2$ are any numbers, $\left(-8^{1/3}\right)^2 = \left(-1 \cdot 8^{1/3}\right)^2 = (-1)^2\left(8^{1/3}\right)^2 = 1\left(8^{2/3}\right) = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$. Again, the last result is true because $(4)^3 = 64$. Hence, $(-8)^{2/3} = 4 = \left(-8^{1/3}\right)^2$.

(b) By the first law of exponents stated above, $\left(\frac{1}{64}\right)^{4/3} = \sqrt[3]{\left(\frac{1}{64}\right)^4}$. By another law of exponents which states that $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ where a and b are any two numbers, $\left(\frac{1}{64}\right)^4 = \frac{(1)^4}{(64)^4} = \frac{1}{(4^3)^4}$. (Note the last result is true since

$4^3 = 64$.) Hence, $\left(\frac{1}{64}\right)^4 = \frac{1}{(4^3)^4} = \frac{1}{4^{12}}$. Therefore,

$\left(\frac{1}{64}\right)^{4/3} = \sqrt[3]{\left(\frac{1}{64}\right)^4} = \sqrt[3]{\frac{1}{4^{12}}}$. By a law of radicals which states

that $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$ where a and b are any two numbers, $\sqrt[3]{\frac{1}{4^{12}}} =$

$\frac{\sqrt[3]{1}}{\sqrt[3]{4^{12}}} = \frac{\sqrt[3]{1}}{\sqrt[3]{(4^4)^3}} = \frac{1}{4^4} = \frac{1}{256}$. The expression $\left[\left(\frac{1}{64}\right)^{1/3}\right]^4 =$

$= \left[\sqrt[3]{\frac{1}{64}}\right]^4 = \left[\frac{\sqrt[3]{1}}{\sqrt[3]{64}}\right]^4 = \left[\frac{\sqrt[3]{1}}{\sqrt[3]{(4)^3}}\right]^4 = \left[\frac{1}{4}\right]^4 = \frac{(1)^4}{(4)^4} = \frac{1}{256}$. Hence,

$\left(\frac{1}{64}\right)^{4/3} = \sqrt[3]{\frac{1}{4^{12}}} = \frac{1}{256} = \left[\left(\frac{1}{64}\right)^{1/3}\right]^4$. Therefore, $\left(\frac{1}{64}\right)^{4/3} =$

$\left[\left(\frac{1}{64}\right)^{1/3}\right]^4$.

**• PROBLEM 103**

Find the numerical value of each of the following.

(a) $8^{2/3}$ (b) $25^{3/2}$

Solution:

(a) Since $x^{a/b} = \left(x^{1/b}\right)^a$, $8^{2/3} = \left(8^{1/3}\right)^2 = \left(\sqrt[3]{8}\right)^2 = (2)^2 = 4$

(b) Similarly, $25^{3/2} = \left(25^{1/2}\right)^3 = 5^3 = 125$.

**• PROBLEM 104**

Simplify $\sqrt{12} - \sqrt{27}$.

Solution: Here we have two different radicals, yet when each is simplified, the distributive law gives a simpler form for the expression. Note that 12 and 27 both have a factor 3, hence

$$\sqrt{12} - \sqrt{27} = \sqrt{4 \cdot 3} - \sqrt{9 \cdot 3}$$

Because $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$, $= \sqrt{4} \cdot \sqrt{3} - \sqrt{9} \cdot \sqrt{3}$

$$= 2\sqrt{3} - 3\sqrt{3}$$

Now, we use the distributive law, $= (2 - 3)\sqrt{3}$

$$= (-1)\sqrt{3}$$

$$= -\sqrt{3}$$

**• PROBLEM 105**

Simplify $5\sqrt{12} + 3\sqrt{75}$.

**Solution:** Express 12 and 75 as the product of perfect squares if possible. Thus, $12 = 4 \cdot 3$ and $75 = 25 \cdot 3$; and $5\sqrt{12} + 3\sqrt{75} = 5\sqrt{4 \cdot 3} + 3\sqrt{25 \cdot 3}$.

Since $\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$:
$$= [5 \cdot \sqrt{4} \cdot \sqrt{3}] + [3\sqrt{25} \cdot \sqrt{3}]$$
$$= [(5 \cdot 2)\sqrt{3}] + [(3 \cdot 5)\sqrt{3}]$$
$$= 10\sqrt{3} + 15\sqrt{3}.$$

Using the distributive law:
$$= (10 + 15)\sqrt{3}$$
$$= 25\sqrt{3}.$$

**• PROBLEM 106**

Approximate $\sqrt{23} \times \sqrt{40}$ and $\sqrt{23} \div \sqrt{40}$.

**Solution:** A four-place table of square roots gives

$$\sqrt{23} = 4.796 \qquad \text{and} \qquad \sqrt{40} = 6.325.$$

The product $\sqrt{23} \times \sqrt{40} = (4.796)(6.325)$

$$= 30.334700$$

Thus, rounding off to the nearest one hundredth, we obtain

$$\sqrt{23}\,\sqrt{40} = 30.33, \text{ approximately.}$$

For the division: $\sqrt{23} \div \sqrt{40} = 4.796 \div 6.325$

$$= 6.325\overline{)4.796}$$

Thus,

```
         .7582
6.325)4.7960000
       4 4275
        36850
        31625
        52250
        50600
        16500
        12650
         3850
```

Hence, $\sqrt{23} \div \sqrt{40}$ is approximately 0.7582.

**• PROBLEM 107**

If $a = 3$ and $b = 2$, find $(6a - b)^{-5/4}$.

**Solution:** Substitute $a = 3$ and $b = 2$: $(6 \cdot 3 - 2)^{-5/4}$

Perform the indicated multiplication: $(18-2)^{-5/4}$

$$= 16^{-5/4}$$

$$\text{Since } x^{-y} = \frac{1}{x^y}: \quad = \frac{1}{16^{5/4}}$$

$$= \frac{1}{\left(\sqrt[4]{16}\right)^5}$$

Since $2^4 = 2\cdot2\cdot2\cdot2 = 16$,

$\sqrt[4]{16} = 2$. Hence: $= \frac{1}{2^5}$

$$= \frac{1}{32}$$

• **PROBLEM 108**

Simplify the quotient $\sqrt{x}/\sqrt[4]{x}$. Write the result in exponential notation.

<u>Solution:</u> Since $\sqrt[b]{n^a} = n^{a/b}$, the numerator and the denominator can be rewritten as:

$$\sqrt{x} = x^{1/2} \quad \text{and}$$

$$\sqrt[4]{x} = x^{1/4}$$

Therefore,

$$\frac{\sqrt{x}}{\sqrt[4]{x}} = \frac{x^{1/2}}{x^{1/4}} \qquad (1)$$

According to the law of exponents which states that $\frac{n^a}{n^b} = n^{a-b}$, equation (1) becomes:

$$\frac{\sqrt{x}}{\sqrt[4]{x}} = \frac{x^{1/2}}{x^{1/4}}$$

$$= x^{\frac{1}{2}-\frac{1}{4}}$$

$$= x^{\frac{2}{4}-\frac{1}{4}}$$

$$= x^{\frac{1}{4}}$$

• **PROBLEM 109**

Simplify: (a) $\sqrt{8x^3y}$ (b) $\sqrt{\frac{2a}{4b^2}}$ (c) $\sqrt[4]{25x^2}$ .

<u>Solution:</u>(a) $\sqrt{8x^3y}$ contains the perfect square $4x^2$ . Factoring

out $4x^2$ we obtain,

$$\sqrt{8x^3y} = \sqrt{4x^2 \cdot 2xy} \quad .$$

Recall that $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$ . Thus,

$$= \sqrt{4x^2} \cdot \sqrt{2xy}$$

$$= \sqrt{4}\sqrt{x^2}\ \sqrt{2xy} \quad .$$

Since $\sqrt{x^2} = |x|$,

$$\sqrt{8x^3y} = 2|x|\sqrt{2xy} \quad .$$

(b) $\sqrt{\frac{2a}{4b^2}}$ has a fraction for the radicand, but the denominator is a perfect square.

$$\sqrt{\frac{2a}{4b^2}} = \frac{\sqrt{2a}}{\sqrt{4b^2}} \text{ , since } \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} \text{ ; } \frac{\sqrt{2a}}{\sqrt{4}\sqrt{b^2}} = \frac{\sqrt{2a}}{2|b|} \quad .$$

(c) $\sqrt[4]{25x^2}$ has a perfect square for the radicand.

$$\sqrt[4]{25x^2} = \sqrt[4]{(5x)^2} \quad .$$

Recall that $\sqrt[4]{x} = \sqrt[2]{\sqrt[2]{x}}$ ; hence $\sqrt[4]{(5x)^2} = \sqrt[2]{\sqrt[2]{(5x)^2}}$ . Now, since $\sqrt[2]{(5x)^2} = |5x|$ , $= \sqrt[2]{|5x|}$ . Since $|ab| = |a||b|$ , $= \sqrt[2]{|5||x|} = \sqrt[2]{5|x|} = \sqrt{5|x|}$ .

Radicals with the same index can be multiplied by finding the product of the radicands, the index of the product being the same as the factors.

**• PROBLEM 110**

Find the product $\sqrt[4]{x^3y} \cdot \sqrt[4]{xy^2}$ and simplify.

Solution: Note that $\sqrt[x]{a} \cdot \sqrt[x]{b} = \sqrt[x]{ab}$ ; thus,

$$\sqrt[4]{x^3y} \cdot \sqrt[4]{xy^2} = \sqrt[4]{(x^3y)(xy^2)} \cdot$$

Recall that when multiplying, we add exponents; hence

$$(x^3y^1)(x^1y^2) = (x^{3+1}\ y^{1+2}), \quad \text{and}$$

we obtain,

$$= \sqrt[4]{x^4y^3}$$

$$= \sqrt[4]{x^4}\left(\sqrt[4]{y^3}\right)$$

Now, since $\sqrt[4]{x^4} = (x^{\frac{1}{4}})^4 = x^1 = x$, $\sqrt[4]{x^3y} \cdot \sqrt[4]{xy^2} = x\ \sqrt[4]{y^3}$ .

**• PROBLEM 111**

Perform the indicated operations in the following expression and write the final result without negative or zero exponents:

$$\left(\frac{64a^{-3}\,b^{4/3}}{27a^{-9}b^{-14/3}}\right)^{-2/3} .$$

<u>Solution:</u> Since $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

$$\left(\frac{64a^{-3}b^{4/3}}{27z^{-9}b^{-14/3}}\right)^{-2/3} = \frac{\left(64a^{-3}b^{4/3}\right)^{-2/3}}{\left(27a^{-9}b^{-14/3}\right)^{-2/3}}$$

and $(abc)^n = a^nb^nc^n$ . Thus

$$\frac{(64a^{-3}b^{4/3})^{-2/3}}{\left(27a^{-9}b^{-14/3}\right)^{-2/3}} = \frac{\left(64\right)^{-2/3}\left(a^{-3}\right)^{-2/3}\left(b^{4/3}\right)^{-2/3}}{\left(27\right)^{-2/3}\left(a^{-9}\right)^{-2/3}\left(b^{-14/3}\right)^{-2/3}}$$

Recall $(x^y)^z = x^{yz}$ thus

$$\frac{(64)^{-2/3}(a^{-3})^{-2/3}(b^{4/3})^{-2/3}}{\left(27\right)^{-2/3}\left(a^{-9}\right)^{-2/3}\left(b^{-14/3}\right)^{-2/3}} = \frac{(64)^{-2/3}(a^2)(b^{-8/9})}{\left(27\right)^{-2/3}\left(a^6\right)\left(b^{28/9}\right)}$$

Since $a^6 = a^4 \cdot a^2$, cancel $a^2$ from numerator and denominator,

$$= \frac{(64)^{-2/3}(b^{-8/9})}{\left(27\right)^{-2/3}\left(a^4\right)\left(b^{28/9}\right)}$$

and since $\frac{a^x}{a^y} = a^{x-y}$, $\frac{b^{-8/9}}{b^{28/9}} = b^{-8/9\ -\ 28/9} = b^{-36/9} = \frac{1}{b^{36/9}} = \frac{1}{b^4}$

thus

$$\frac{\left(64\right)^{-2/3}\left(b^{-8/9}\right)}{\left(27\right)^{-2/3}\left(a^4\right)\left(b^{28/9}\right)} = \frac{(64)^{-2/3}}{(27)^{-2/3}\left(a^4\right)\left(b^4\right)}$$

Since

$$x^{a/b} = \left(\sqrt[b]{x}\right)^a$$

$$(64)^{-2/3} = \left(\sqrt[3]{64}\right)^{-2} = (4)^{-2}$$
$$(27)^{-2/3} = \left(\sqrt[3]{27}\right)^{-2} = (3)^{-2}$$

Thus

$$\frac{(64)^{-2/3}}{(27)^{-2/3}a^4b^4} = \frac{(4)^{-2}}{(3)^{-2}a^4b^4}$$

Recall $\qquad x^{-a} = \frac{1}{x^a}$

therefore $\qquad (4)^{-2} = \frac{1}{4^2} = \frac{1}{16}$

and $\qquad (3)^{-2} = \frac{1}{3^2} = \frac{1}{9}$

hence $\qquad \frac{(4)^{-2}}{(3)^{-2}a^4b^4} = \frac{1/16}{\frac{1}{9}\,a^4b^4}$

Multiply numerator and denominator by $16\cdot 9$,

$$= \frac{16\cdot 9(1/16)}{16\cdot 9\left(\frac{1}{9}\right)a^4b^4}$$

$$= \frac{9}{16\, a^4 b^4}$$

thus
$$\left(\frac{64a^{-3}\, b^{4/3}}{27a^{-9}\, b^{-14/3}}\right)^{-2/3} = \frac{9}{16\, a^4 b^4}$$

• PROBLEM 112

Simplify $\sqrt[3]{-81x^3} - 2x\sqrt[3]{3} + 5x\sqrt[3]{24}$.

Solution: Rewrite the expression so it contains similar radicals.

$\sqrt[3]{-81x^3} = \sqrt[3]{(-3)^3x^3 \cdot 3} = \sqrt[3]{(-3x)^3 \cdot 3}$ by the law $(ab)^n = a^n b^n$. Also, since $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$, $\sqrt[3]{(-3x)^3 \cdot 3} = \sqrt[3]{(-3x)^3}\ \sqrt[3]{3}$. Hence, $\sqrt[3]{-81x^3} = \sqrt[3]{(-3x)^3}\sqrt[3]{3}$. Since $(\sqrt[n]{a})^n = (a^{1/n})^n = a^{\frac{1}{n}\cdot n} = a^1 = a$, $\sqrt[3]{(-3x)^3} = -3x$. Therefore, $\sqrt[3]{-81x^3} = -3x\sqrt[3]{3}$. By the same laws, $5x\sqrt[3]{24} = 5x\sqrt[3]{2^3 \cdot 3} = 5x\sqrt[3]{(2)^3}\sqrt[3]{3} = 5x(2)\sqrt[3]{3} = 10x\sqrt[3]{3}$.

Therefore, $\sqrt[3]{-81x^3} - 2x\sqrt[3]{3} + 5x\sqrt[3]{24} = -3x\sqrt[3]{3} - 2x\sqrt[3]{3} + 10x\sqrt[3]{3}$

$$= -5x\sqrt[3]{3} + 10x\sqrt[3]{3} = 5x\sqrt[3]{3}.$$

Hence, $\sqrt[3]{-81x^3} - 2x\sqrt[3]{3} + 5x\sqrt[3]{24} = 5x\sqrt[3]{3}$. Note that the radical used to simplify the given expression was $\sqrt[3]{3}$.

• PROBLEM 113

Find the square root of

$$\frac{3}{2}(x - 1) + \sqrt{2x^2 - 7x - 4}.$$

Solution: The given expression can be rewritten as

$$\frac{3}{2}x - \frac{3}{2} + \sqrt{2x^2 - 7x - 4}.$$

We can eliminate the fractions by factoring ½ from all the terms. To do this we must first multiply the third term by 2 so as not to change the value of this term. Thus, we obtain:

$$\frac{1}{2}\left(3x - 3 + 2\sqrt{2x^2 - 7x - 4}\right)$$

Let us examine the expression under the radical. Notice that this can be rewritten in factored form since

$$2x^2 - 7x - 4 = (2x + 1)(x - 4).$$

Substituting this in the given expression we have:

$$\frac{1}{2}\left(3x - 3 + 2\sqrt{(2x + 1)(x - 4)}\right).$$

Our aim now is to transform the expression into one which is a perfect square. This can be accomplished as follows: Rewrite the first two terms, $3x - 3$ as: $(2x + 1) + (x - 4)$, and substitute this into the expression. Thus, we obtain,

$$\frac{1}{2}\left[(2x + 1) + (x - 4) + 2\sqrt{(2x + 1)(x - 4)}\right],$$

and we are looking for:

$$\sqrt{\frac{1}{2}\left[(2x + 1)+(x - 4) + 2\sqrt{(2x + 1)(x - 4)}\right]}$$

But,

$$(2x + 1) + (x - 4) + 2\sqrt{(2x + 1)(x - 4)}$$

$$= (2x + 1) + (x - 4) + 2\sqrt{2x + 1}\ \sqrt{x - 4}$$

$$= (\sqrt{2x + 1} + \sqrt{x - 4})(\sqrt{2x + 1} + \sqrt{x - 4})$$

$$= (\sqrt{2x + 1} + \sqrt{x - 4})^2;$$

Therefore, our expression becomes:

$$\sqrt{\frac{1}{2}(\sqrt{2x + 1} + \sqrt{x - 4})^2}$$

$$= \sqrt{\frac{1}{2}}(\sqrt{2x + 1} + \sqrt{x - 4})$$

$$= \frac{1}{\sqrt{2}}(\sqrt{2x + 1} + \sqrt{x - 4}).$$

**• PROBLEM 114**

**Find the cube root of $9\sqrt{3} + 11\sqrt{2}$.**

Solution: In this problem we wish to find:

$$\sqrt[3]{9\sqrt{3} + 11\sqrt{2}}$$

Factoring $3\sqrt{3}$ from both terms under the radical, $9\sqrt{3}$ and $11\sqrt{2}$, we obtain:

$$\sqrt[3]{3\sqrt{3}\left(3 + \frac{11}{3}\frac{\sqrt{2}}{\sqrt{3}}\right)}, \qquad \text{or}$$

$$\sqrt[3]{9\sqrt{3} + 11\sqrt{2}}$$

$$= \sqrt[3]{3\sqrt{3}\left(3 + \frac{11}{3}\sqrt{\frac{2}{3}}\right)}.$$

Now, since $\sqrt[3]{3\sqrt{3}} = \sqrt{3}$ (that is, $\sqrt[3]{3\sqrt{3}} = \sqrt[3]{3(3^{\frac{1}{2}})} = \sqrt[3]{3^{(1+\frac{1}{2})}} = \sqrt[3]{3^{\frac{3}{2}}} = (3^{\frac{3}{2}})^{\frac{1}{3}} = 3^{\frac{1}{2}} = \sqrt{3}$) we can write the expression as:

$$= \sqrt{3}\left(\sqrt[3]{3 + \frac{11}{3}\sqrt{\frac{2}{3}}}\right)$$

Observe that $3 + \frac{11}{3}\sqrt{\frac{2}{3}}$ is a perfect cube, since:

$$\left(1 + \sqrt{\frac{2}{3}}\right)\left(1 + \sqrt{\frac{2}{3}}\right)\left(1 + \sqrt{\frac{2}{3}}\right)$$

$$= \left(1 + 2\sqrt{\frac{2}{3}} + \frac{2}{3}\right)\left(1 + \sqrt{\frac{2}{3}}\right)$$

$$= 1 + 2\sqrt{\frac{2}{3}} + \frac{2}{3} + \sqrt{\frac{2}{3}} + 2 \cdot \frac{2}{3} + \frac{2}{3}\sqrt{\frac{2}{3}}$$

$$= \left(1 + \frac{4}{3} + \frac{2}{3}\right) + \left(2\sqrt{\frac{2}{3}} + \sqrt{\frac{2}{3}} + \frac{2}{3}\sqrt{\frac{2}{3}}\right)$$

$$= 3 + \frac{11}{3}\sqrt{\frac{2}{3}}.$$

Thus,

$$\sqrt[3]{3 + \frac{11}{3}\sqrt{\frac{2}{3}}} = 1 + \sqrt{\frac{2}{3}}\text{ , and}$$

the required cube root $= \sqrt{3}\left(1 + \sqrt{\frac{2}{3}}\right)$

$= \sqrt{3} + \sqrt{2}.$

## RATIONALIZING THE DENOMINATOR

**• PROBLEM 115**

Write in fractional exponent form with no denominators.

(a) $\sqrt[3]{\frac{x}{y}}$ (b) $\sqrt[3]{3}$ (c) $\sqrt[4]{x}\ \sqrt[3]{xy^{-1}}$

**Solution:** Noting that $\sqrt[b]{a} = a^{1/b}$ , $\left(\frac{a}{b}\right)^c = \frac{a^c}{b^c}$ , and $a^{-b} = \frac{1}{a^b}$, we proceed to evaluate these expressions.

(a) $\sqrt[3]{\frac{x}{y}} = \left(\frac{x}{y}\right)^{1/3} = \frac{x^{1/3}}{y^{1/3}} = x^{\frac{1}{3}}y^{-\frac{1}{3}}$

(b) $\sqrt[3]{3} \quad = 3^{\frac{1}{3}}$

(c) $\sqrt[4]{x^2}\ \sqrt[3]{xy^{-1}} = (x^2)^{\frac{1}{4}}\left(xy^{-1}\right)^{\frac{1}{3}}$

$= (x^{\frac{2}{4}})(x^{\frac{1}{3}})(y^{-\frac{1}{3}})$, since $(a^b)^c = a^{bc}$ and $(ab^c)^d = a^db^{cd}$

$= (x^{\frac{1}{2}})(x^{\frac{1}{3}})(y^{-\frac{1}{3}})$

$= x^{\frac{1}{2}+\frac{1}{3}}\ y^{-\frac{1}{3}}$, since $(x^a)(x^b) = x^{a+b}$

$= x^{\frac{3}{6}+\frac{2}{6}}\ y^{-\frac{1}{3}}$

$= x^{\frac{5}{6}}\ y^{-\frac{1}{3}}$

**• PROBLEM 116**

Rationalize tne denominator in the quotient $1\Big/\sqrt[5]{x^2}$.

**Solution:** $\sqrt[y]{a^x} = a^{\frac{x}{y}}$, thus $\frac{1}{\sqrt[5]{x^2}} = \frac{1}{x^{\frac{2}{5}}}$. Rationalizing a denominator means eliminating the radical from the denominator, thus we want to eliminate the fractional exponent. When multiplying numbers with the same base, exponents are added, hence multiplying numerator and denominator by $x^{\frac{3}{5}}$ will eliminate the fractional exponent in the denominator:

$$\frac{1}{\sqrt{x^2}} = \frac{1}{x^{\frac{2}{5}}} = \frac{1}{x^{\frac{2}{5}}}\ \frac{x^{\frac{3}{5}}}{x^{\frac{3}{5}}} = \frac{x^{\frac{3}{5}}}{x^{\frac{2}{5}+\frac{3}{5}}} = \frac{x^{\frac{3}{5}}}{x^1} = \frac{\sqrt[5]{x^3}}{x}.$$

**• PROBLEM 117**

Write in radical form without negative exponents, rationalizing denominators. (a) $\left(x^{1/3}\right)^{-3/4}$ (b) $x^{1/6}\big/x^{-2/3}$.

**Solution:** Noting $\left(a^b\right)^c = a^{bc}$, $\frac{a^b}{a^c} = a^{b-c}$, $a^{-b} = \frac{1}{a^b}$, $a^{1/b} = \sqrt[b]{a}$, and $\sqrt[b]{a}\ \ \sqrt[b]{c} = \sqrt[b]{ac}$, we proceed to evaluate these expressions.

(a) $\left(x^{1/3}\right)^{-3/4} = x^{\left(\frac{1}{3}\right)\left(\frac{-3}{4}\right)} = x^{\frac{-3}{12}} = x^{-\frac{1}{4}} = \frac{1}{x^{1/4}} = \frac{1}{\sqrt[4]{x}}$

$$= \frac{1}{\sqrt[4]{x}} \cdot \frac{\sqrt[4]{x^3}}{\sqrt[4]{x^3}} \qquad \left(\text{since } \frac{\sqrt[4]{x^3}}{\sqrt[4]{x^3}} = 1\right)$$

$$= \frac{\sqrt[4]{x^3}}{\sqrt[4]{x}\,\sqrt[4]{x^3}} = \frac{\sqrt[4]{x^3}}{\sqrt[4]{x \cdot x^3}} = \frac{\sqrt[4]{x^3}}{\sqrt[4]{x^4}}.$$

Since $\sqrt[n]{a^n} = (a^n)^{1/n} = a^{n \cdot \frac{1}{n}} = a^{n/n} = a^1 = a$, $\sqrt[4]{x^4} = x$. Thus,

$$\left(x^{\frac{1}{3}}\right)^{-\frac{3}{4}} = \frac{\sqrt[4]{x^3}}{x}$$

(b) $\frac{x^{1/6}}{x^{-2/3}} = x^{\frac{1}{6}-\left(-\frac{2}{3}\right)} = x^{\frac{1}{6}+\frac{2}{3}} = x^{\frac{1}{6}+\frac{4}{6}} = x^{\frac{5}{6}} = \sqrt[6]{x^5}$

**• PROBLEM 118**

Rationalize $\frac{\sqrt[3]{3ax}}{\sqrt[3]{4a^2}}$ .

Solution: Multiply the numerator and the denominator by the radical $\sqrt[3]{(4a^2)^2}$ to eliminate the radical in the denominator:

$$\frac{\sqrt[3]{3ax}}{\sqrt[3]{4a^2}} = \frac{\left[\sqrt[3]{(4a^2)^2}\right]\sqrt[3]{3ax}}{\left[\sqrt[3]{(4a^2)^2}\right]\sqrt[3]{(4a^2)}} = \frac{\sqrt[3]{(4a^2)^2}\ \sqrt[3]{3ax}}{\sqrt[3]{(4a^2)^3}}$$

Note the last result is true because of the law involving radicals which states that $\sqrt[3]{a} \cdot \sqrt[3]{b} = \sqrt[3]{ab}$. Also, since $\sqrt[3]{a^3} = (\sqrt[3]{a})^3 = (a^{1/3})^3 = a^1 = a$, $\sqrt[3]{(4a^2)^3} = (\sqrt[3]{4a^2})^3 = 4a^2$. Hence,

$$\frac{\sqrt[3]{3ax}}{\sqrt[3]{4a^2}} = \frac{\sqrt[3]{(4a^2)^2}\ \sqrt[3]{3ax}}{4a^2} = \frac{\sqrt[3]{16a^4}\ \sqrt[3]{3ax}}{4a^2}.$$

Since $\sqrt[3]{ab} = \sqrt[3]{a}\sqrt[3]{b}$, $\sqrt[3]{16a^4} = \sqrt[3]{(8a^3)(2a)} = \sqrt[3]{8a^3}\sqrt[3]{2a}$

$$= \sqrt[3]{(2a)^3}\sqrt[3]{2a}.$$

Note that the last result is true because $(ab)^x = a^x b^x$; that is, $8a^3 = 2^3a^3 = (2a)^3$. Hence:

$$\frac{\sqrt[3]{3ax}}{\sqrt[3]{4a^2}} = \frac{\sqrt[3]{(2a)^3}\sqrt[3]{2a}\sqrt[3]{3ax}}{4a^2}$$

$$= \frac{2a\sqrt[3]{2a}\sqrt[3]{3ax}}{4a^2}$$

$$= \frac{2a\sqrt{(2a)(3ax)}}{4a^2}$$

$$= \frac{2a\sqrt{6a^2x}}{4a^2}.$$

Therefore, $\dfrac{\sqrt[3]{3ax}}{\sqrt[3]{4a^2}} = \dfrac{\sqrt[3]{6a^2x}}{2a}.$

**• PROBLEM 119**

Express the product $1/\sqrt[3]{x^2} \cdot 1/\sqrt[4]{x}$ in simplest radical form, rationalizing the denominator.

<u>Solution:</u> Since $\dfrac{1}{\sqrt[b]{x^a}} = \dfrac{1}{x^{a/b}}$

$$\frac{1}{\sqrt[3]{x^2}} = \frac{1}{x^{2/3}} \quad \text{and} \quad \frac{1}{\sqrt[4]{x}} = \frac{1}{x^{1/4}}$$

thus,

$$\frac{1}{\sqrt[3]{x^2}} \cdot \frac{1}{\sqrt[4]{x}} = \frac{1}{x^{2/3}} \cdot \frac{1}{x^{1/4}}.$$

Since $a^x \cdot a^y = a^{x+y}$

$$\frac{1}{x^{2/3}} \cdot \frac{1}{x^{1/4}} = \frac{1}{x^{2/3+1/4}} = \frac{1}{x^{8/12+3/12}} = \frac{1}{x^{11/12}}$$

To rationalize the denominator we wish to obtain an integral exponent of $x$ in the denominator, thus we multiply numerator and denominator by $x^{1/12}$,

$$\frac{1}{x^{11/12}} \cdot \frac{x^{1/12}}{x^{1/12}} = \frac{x^{1/12}}{x^{11/12+1/12}} = \frac{x^{1/12}}{x^{12/12}}$$

$$= \frac{\sqrt[12]{x}}{x^1} = \frac{\sqrt[12]{x}}{x}$$

Thus

$$\frac{1}{\sqrt[3]{x^2}} \cdot \frac{1}{\sqrt[4]{x}} = \frac{\sqrt[12]{x}}{x}.$$

**• PROBLEM 120**

Find the factor which will rationalize $\sqrt{3} + \sqrt[3]{5}$.

<u>Solution:</u> We can rewrite $\sqrt{3} + \sqrt[3]{5}$ as,

$$3^{\frac{1}{2}} + 5^{\frac{1}{3}}$$

Observe that both of the above irrational numbers, when raised to the sixth power, become rational $\left[\left(3^{\frac{1}{2}}\right)^6 = 3^3 = 27,\ \left(5^{\frac{1}{3}}\right)^6 = 5^2 = 25\right]$.

Let $x = 3^{\frac{1}{2}}$, $y = 5^{\frac{1}{3}}$; then $x^6$ and $y^6$ are both rational.

Since $x^6$ and $y^6$ are rational, so is $x^6 - y^6$ (and in fact, is equal to $27 - 25 = 2$). To find the factor which rationalizes $x + y = (\sqrt{3} + \sqrt[3]{5})$, we divide $x^6 - y^6$ by $x + y$, and find the quotient to be,

$$x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5.$$

Thus

$$x^6 - y^6 = (x + y)(x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5);$$

and substituting for x and y, the required factor is

$$x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5 =$$

$$\left(3^{\frac{1}{2}}\right)^5 - \left(3^{\frac{1}{2}}\right)^4 5^{\frac{1}{3}} + \left(3^{\frac{1}{2}}\right)^3\left(5^{\frac{1}{3}}\right)^2 - \left(3^{\frac{1}{2}}\right)^2\left(5^{\frac{1}{3}}\right)^3$$

$$+ 3^{\frac{1}{2}}\left(5^{\frac{1}{3}}\right)^4 - \left(5^{\frac{1}{3}}\right)^5 =$$

$$3^{\frac{5}{2}} - 3^{\frac{4}{2}} \cdot 5^{\frac{1}{3}} + 3^{\frac{3}{2}} \cdot 5^{\frac{2}{3}} - 3^{\frac{2}{2}} \cdot 5^{\frac{3}{3}}$$

$$+ 3^{\frac{1}{2}} \cdot 5^{\frac{4}{3}} - 5^{\frac{5}{3}}, \qquad \text{or}$$

$$3^{\frac{5}{2}} - 9 \cdot 5^{\frac{1}{3}} + 3^{\frac{3}{2}} \cdot 5^{\frac{2}{3}} - 15 + 3^{\frac{1}{2}} \cdot 5^{\frac{4}{3}} - 5^{\frac{5}{3}};$$

and the rational product, $\left(3^{\frac{1}{2}}\right)^6 - \left(5^{\frac{1}{3}}\right)^6$, of the above factor and $3^{\frac{1}{2}} + 5^{\frac{1}{3}}$ is

$$3^{\frac{6}{2}} - 5^{\frac{6}{3}} = 3^3 - 5^2 = 2.$$

**• PROBLEM 121**

Rationalize the denominator in the quotient:

$$\frac{1}{\sqrt{x} - \sqrt{y}}.$$

**Solution:** To rationalize a denominator we multiply numerator and denominator by the conjugate of the denominator (recall $a + bi$ and $a - bi$ are complex conjugates). In our example the conjugate of $\sqrt{x} - \sqrt{y}$ is $\sqrt{x} + \sqrt{y}$. Hence, multiplying numerator and denominator by $\sqrt{x} + \sqrt{y}$:

$$\frac{1}{\sqrt{x} - \sqrt{y}} \cdot \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}} = \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x}\sqrt{x} + \sqrt{x}\sqrt{y} - \sqrt{x}\sqrt{y} - \sqrt{y}\sqrt{y}}.$$

Recall that $\sqrt{a}\cdot\sqrt{a} = \sqrt{a^2} = a$, thus $\sqrt{x}\sqrt{x} = x$ and $\sqrt{y}\sqrt{y} = y$; and we obtain:

$$= \frac{\sqrt{x} + \sqrt{y}}{x - y} .$$

**• PROBLEM 122**

Rationalize $\dfrac{\sqrt{3xy}}{\sqrt{2x} - \sqrt{3y}}$ .

**Solution:** To rationalize a fraction, we multiply numerator and denominator by the conjugate of the denominator (where the conjugate of a + b is a - b). In our example, we multiply numerator and denominator by the conjugate of $\sqrt{2x} - \sqrt{3y}$, which is $\sqrt{2x} + \sqrt{3y}$. Thus,

$$\frac{\sqrt{3xy}}{\sqrt{2x} - \sqrt{3y}} = \frac{\sqrt{3xy}}{\sqrt{2x} - \sqrt{3y}} \cdot \frac{\sqrt{2x} + \sqrt{3y}}{\sqrt{2x} + \sqrt{3y}}$$

$$= \frac{\sqrt{3xy}\ (\sqrt{2x} + \sqrt{3y})}{(\sqrt{2x})^2 - (\sqrt{3y})^2} .$$

Since $(\sqrt{a})^2 = \sqrt{a} \cdot \sqrt{a} = \sqrt{a^2} = a$,

$$(\sqrt{2x})^2 = 2x$$

$$(\sqrt{3y})^2 = 3y.$$

Making these substitutions,

$$\frac{\sqrt{3xy}\ (\sqrt{2x} + \sqrt{3y})}{(\sqrt{2x})^2 - (\sqrt{3y})^2} = \frac{\sqrt{3xy}\ (\sqrt{2x} + \sqrt{3y})}{2x - 3y}$$

$$= \frac{\sqrt{3xy} \cdot \sqrt{2x} + \sqrt{3xy} \cdot \sqrt{3y}}{2x - 3y}$$

Since $\sqrt{a}\ \sqrt{b} = \sqrt{ab}$ and $\sqrt{a}\ \sqrt{b}\ \sqrt{c} = \sqrt{abc}$,

$$\frac{\sqrt{3xy} \cdot \sqrt{2x} + \sqrt{3xy} \cdot \sqrt{3y}}{2x - 3y} = \frac{\sqrt{6x^2y} + \sqrt{9xy^2}}{2x - 3y}$$

$$= \frac{\sqrt{x^2}\ \sqrt{6y} + \sqrt{9}\ \sqrt{y^2}\ \sqrt{x}}{2x - 3y}$$

$$= \frac{x\ \sqrt{6y} + 3y\ \sqrt{x}}{2x - 3y} .$$

**• PROBLEM 123**

Reduce $\dfrac{(2 + 3\sqrt{-1})^2}{2 + \sqrt{-1}}$ to the form $A + B\sqrt{-1}$ .

**Solution:** Expanding the numerator, we can rewrite the fraction as,

$$\frac{(2 + 3\sqrt{-1})(2 + 3\sqrt{-1})}{2 + \sqrt{-1}} = \frac{4 + 6\sqrt{-1} + 6\sqrt{-1} + 3\sqrt{-1} \cdot 3\sqrt{-1}}{2 + \sqrt{-1}} =$$

$$\frac{4 + 12\sqrt{-1} + 9(-1)}{2 + \sqrt{-1}} = \frac{-5 + 12\sqrt{-1}}{2 + \sqrt{-1}} .$$

We now rationalize the denominator by multiplying both numerator and denominator by $2 - \sqrt{-1}$. This does not change the value of the fraction, since

$$\frac{2-\sqrt{-1}}{2-\sqrt{-1}} = 1 ,$$

and multiplication by 1 does not change the value of any expression. Thus, we have:

$$\frac{-5+12\sqrt{-1}}{2+\sqrt{-1}}\left(\frac{2-\sqrt{-1}}{2-\sqrt{-1}}\right) = \frac{-10+24\sqrt{-1}+5\sqrt{-1}-12\sqrt{-1}\cdot\sqrt{-1}}{4+2\sqrt{-1}-2\sqrt{-1}-\sqrt{-1}\cdot\sqrt{-1}}$$

$$= \frac{-10+29\sqrt{-1}-12(-1)}{4-(-1)} = \frac{2+29\sqrt{-1}}{5}$$

$$= \frac{2}{5} + \frac{29}{5}\sqrt{-1} ;$$

which is of the required form, $A + B\sqrt{-1}$.

**• PROBLEM 124**

Express with rational denominator $\dfrac{4}{\sqrt[3]{9} - \sqrt[3]{3} + 1}$.

Solution: The given expression can be written as,

$$\frac{4}{9^{\frac{1}{3}} - 3^{\frac{1}{3}} + 1} = \frac{4}{\left(3^2\right)^{\frac{1}{3}} - 3^{\frac{1}{3}} + 1} = \frac{4}{3^{\frac{2}{3}} - 3^{\frac{1}{3}} + 1} .$$

To ratonalize the denominator, multiply both numerator and denominator by $\left(3^{\frac{1}{3}} + 1\right)$. This will not change the value of the fraction because

$$\frac{3^{\frac{1}{3}} + 1}{3^{\frac{1}{3}} + 1} = 1,$$

and multiplication by 1 does not change the value of any given expression.

Thus, multiplying, we obtain:

$$\frac{4\left(3^{\frac{1}{3}} + 1\right)}{\left(3^{\frac{1}{3}} + 1\right)\left(3^{\frac{2}{3}} - 3^{\frac{1}{3}} + 1\right)}$$

$$= \frac{4\left(3^{\frac{1}{3}} + 1\right)}{\left(3^{\frac{1}{3}}\right)\left(3^{\frac{2}{3}}\right) + 3^{\frac{2}{3}} - \left(3^{\frac{1}{3}}\right)\left(3^{\frac{1}{3}}\right) - 3^{\frac{1}{3}} + 3^{\frac{1}{3}} + 1}$$

$$= \frac{4\left(3^{\frac{1}{3}} + 1\right)}{3^{\frac{1}{3}} + 3^{\frac{2}{3}} - 3^{\frac{2}{3}} - 3^{\frac{1}{3}} + 3^{\frac{1}{3}} + 1}$$

$$\frac{4\left(3^{\frac{1}{3}} + 1\right)}{3 + 1} = \frac{4\left(3^{\frac{1}{3}} + 1\right)}{4}$$

$$= 3^{\frac{1}{3}} + 1.$$

## OPERATIONS WITH RADICALS

**• PROBLEM 125**

Find the product $\sqrt[3]{2a^2x} \cdot \sqrt{2x}$

Solution: Since $\sqrt[n]{N} = N^{1/n}$, $\sqrt[3]{2a^2x} = (2a^2x)^{1/3}$ and $\sqrt{2x} = (2x)^{1/2}$. Therefore, $\sqrt[3]{2a^2x} \cdot \sqrt{2x} = (2a^2x)^{1/3}(2x)^{1/2}$. Also, since $(N_1N_2)^{1/n} = (N_1)^{1/n}(N_2)^{1/n}$ where $N_1$ and $N_2$ are any two numbers and n is any positive integer, $(2a^2x)^{1/3} = \left[(2a^2)(x)\right]^{1/3} = (2a^2)^{1/3}(x)^{1/3}$ and $(2x)^{1/2} = 2^{1/2}x^{1/2}$. Therefore:

$$\sqrt[3]{2a^2x} \cdot \sqrt{2x} = (2a^2)^{1/3}(x)^{1/3}(2)^{1/2}(x)^{1/2}$$

$$= (2)^{1/3}(a^2)^{1/3}(x)^{1/3}(2)^{1/2}(x)^{1/2}.$$

By the law of exponents which states that $a^m \cdot a^n = a^{m+n}$, $(2)^{1/3}(2)^{1/2} = 2^{1/3+1/2}$ and $(x)^{1/3}(x)^{1/2} = x^{1/3+1/2}$. Obtaining a least common denominator of 6 for the two fractions in the exponents:

$\frac{1}{3} + \frac{1}{2} = \frac{2(1)}{2(3)} + \frac{3(1)}{3(2)} = \frac{2}{6} + \frac{3}{6} = \frac{5}{6}$. Hence, $(2)^{1/3}(2)^{1/2} = 2^{1/3 + 1/2} = 2^{5/6}$ and $(x)^{1/3}(x)^{1/2} = x^{1/3 + 1/2} = x^{5/6}$.

Therefore, $\sqrt[3]{2a^2x} \cdot \sqrt{2x} = (2)^{1/3}(a^2)^{1/3}(x)^{1/3}(2)^{1/2}(x)^{1/2}$

$$= (2)^{1/3}(2)^{1/2}(x)^{1/3}(x)^{1/2}(a^2)^{1/3}$$

$$= (2^{5/6})(x^{5/6})(a^2)^{1/3}$$

$$= (2x)^{5/6}(a^2)^{1/3}.$$

Since $N^{m/n} = \sqrt[n]{N^m}$, $(2x)^{5/6} = \sqrt[6]{(2x)^5}$. Also, since $(N^m)^n = N^{mn}$, $(a^2)^{1/3} = a^{2/3}$. Therefore,

$$\sqrt[3]{2a^2x}\cdot\sqrt{2x} = \sqrt[6]{(2x)^5}\ a^{2/3} = \sqrt[6]{(2x)^5}\ a^{4/6} = \sqrt[6]{(2x)^5}\ \sqrt[6]{a^4}$$

Since $(xy)^a = x^a y^a$,
$$= \sqrt[6]{(2)^5x^5}\sqrt[6]{a^4}$$
$$= \sqrt[6]{32x^5}\sqrt[6]{a^4}$$
$$= \sqrt[6]{32a^4x^5}\ .$$

• **PROBLEM 126**

Simplify $(\sqrt{3} + \sqrt{2}) \cdot (\sqrt{2} - \sqrt{6})$.

Solution: Using the distributive property the expression on the right can be multiplied by each term in the expression on the left or vice versa. Hence,

$$(\sqrt{3} + \sqrt{2})(\sqrt{2} - \sqrt{6}) = \sqrt{3}(\sqrt{2} - \sqrt{6}) + \sqrt{2}(\sqrt{2} - \sqrt{6})$$

The distributive property again enables us to multiply the terms on the left by their respective right hand members;

$$= (\sqrt{3}\cdot\sqrt{2}) - (\sqrt{3}\cdot\sqrt{6}) + (\sqrt{2}\cdot\sqrt{2})$$
$$- (\sqrt{2}\cdot\sqrt{6})$$

Then since,

$$\sqrt{a}\cdot\sqrt{6} = \sqrt{ab},$$
$$(\sqrt{3} + 2)(\sqrt{2}-\sqrt{6}) = (\sqrt{6}) - (\sqrt{18}) + (2) - (\sqrt{12})$$
$$= \sqrt{6} - 3\sqrt{2} + 2 - 2\sqrt{3}.$$

• **PROBLEM 127**

Multiply $(2\sqrt{3} + 3\sqrt{2})$ by $(3\sqrt{3} - 2\sqrt{2})$.

Solution: Using the following method (foil method) of polynomial multiplication:
$$(x + y)(a + b) = xa + xb + ya + yb,$$
we obtain
$$(2\sqrt{3} + 3\sqrt{2})(3\sqrt{3} - 2\sqrt{2}) = (2\sqrt{3})(3\sqrt{3}) + (2\sqrt{3})(-2\sqrt{2}) + (3\sqrt{2})(3\sqrt{3})$$
$$+ (3\sqrt{2})(-2\sqrt{2})$$
$$= (6)(\sqrt{3})^2 - 4(\sqrt{3}\sqrt{2}) + (9)(\sqrt{3}\sqrt{2}) - 6(\sqrt{2})^2\ .$$

Since $(\sqrt{a})^2 = (a^{\frac{1}{2}})^2 = a^{2/2} = a^1 = a$

$(\sqrt{3})^2 = 3$ and $(\sqrt{2})^2 = 2$

Making these substitutions we obtain,

$$(2\sqrt{3} + 3\sqrt{2})(3\sqrt{3} - 2\sqrt{2}) = (6)(3) + 5(\sqrt{3})(\sqrt{2}) - 6(2)$$
$$= 18 + 5(\sqrt{3})(\sqrt{2}) - 12$$
$$= 6 + 5\sqrt{3}\sqrt{2}$$

Since $\sqrt{a}\cdot\sqrt{b} = \sqrt{ab}$

$$\sqrt{3} \cdot \sqrt{2} = \sqrt{3 \cdot 2} = \sqrt{6}$$

Therefore $(2\sqrt{3} + 3\sqrt{2})(3\sqrt{3} - 2\sqrt{2}) = 6 + 5\sqrt{6}$

**• PROBLEM 128**

Find the product by inspection: $\sqrt{3}(x - \sqrt{5})(x + \sqrt{5})$.

Solution: The formula for the difference of two squares can be used to obtain the product. This formula is: $\left(x^2 - y^2\right) = (x + y)(x - y)$. The product $(x - \sqrt{5})(x + \sqrt{5})$ corresponds to the right side of the formula where x is replaced by x and y is replaced by $\sqrt{5}$. Therefore,

$$3(x - \sqrt{5})(x + \sqrt{5}) = 3\left(x^2 - (\sqrt{5})^2\right)$$

Since $(\sqrt{a})^2 = \sqrt{a}\ \sqrt{a} = \sqrt{a^2} = a$, $(\sqrt{5})^2 = 5$. Thus

$$\sqrt{3}(x - \sqrt{5})(x + \sqrt{5}) = \sqrt{3}\left(x^2 - 5\right).$$

Distributing $\sqrt{3}$,

$$= \sqrt{3}\ x^2 - \sqrt{3}(5)$$

$$= \sqrt{3}\ x^2 - 5\sqrt{3}.$$

Thus $\sqrt{3}(x - \sqrt{5})(x + \sqrt{5}) = \sqrt{3}\ x^2 - 5\sqrt{3}$.

**• PROBLEM 129**

Find the following products:

a. $\sqrt{x}\ (\sqrt{2x} - \sqrt{x})$

b. $(\sqrt{x} - 2\ \sqrt{y})(2\sqrt{x} + \sqrt{y})$

Solution: The following two laws concerning radicals can be used to find the indicated products:

1) $\sqrt{a}\ \sqrt{b} = \sqrt{ab}$

where a and b are any two numbers

2) $\sqrt{a^2} = (\sqrt{a})^2\quad \left(a^{1/2}\right)^2 = a^{(1/2)\,2} = a^1 = a$

or $\sqrt{a^2} = a$

a) Using the distributive property,

$$\sqrt{x}\ (\sqrt{2x} - \sqrt{x}) = \sqrt{x}\ (\sqrt{2x}) - \sqrt{x}\ (\sqrt{x})$$

Using the first law concerning radicals to further simplify this equation,

$$\sqrt{x}\ (\sqrt{2x} - \sqrt{x}) = \sqrt{x \cdot 2x} - \sqrt{x \cdot x}$$

$$= \sqrt{2x^2} - \sqrt{x^2}$$

$$= \sqrt{2}\sqrt{x^2} - \sqrt{x^2}$$

Using, the second law to simplify this equation,

$$\sqrt{x}\ (\sqrt{2x} - \sqrt{x}) = \sqrt{2}\ (x) - x$$

$$= \sqrt{2}x - x$$

b) $(\sqrt{x} - 2\sqrt{y})(2\sqrt{x} + \sqrt{y}) = \sqrt{x}\ (2\sqrt{x}) - 2\sqrt{y}(2\sqrt{x}) + \sqrt{x}(\sqrt{y})$

$- 2\sqrt{y}\ (\sqrt{y})$

Using the first law concerning radicals to simplify this equation,

$$(\sqrt{x} - 2\sqrt{y})(2\sqrt{x} + \sqrt{y}) = 2\sqrt{x \cdot x} - 4\sqrt{x \cdot y} + \sqrt{x \cdot y} - 2\sqrt{y \cdot y}$$

$$= 2\sqrt{x^2} - 4\sqrt{xy} + \sqrt{xy} - 2\sqrt{y^2}$$

$$= 2\sqrt{x^2} - 3\sqrt{xy} - 2\sqrt{y^2}$$

Using the second law to simplify this equation,

$$(\sqrt{x} - 2\sqrt{y})(2\sqrt{x} + \sqrt{y}) = 2(x) - 3\sqrt{xy} - 2(y)$$

$$= 2x - 3\sqrt{xy} - 2y$$

**• PROBLEM 130**

When $x = \dfrac{3 + 5\sqrt{-1}}{2}$, find the value of $2x^3 + 2x^2 - 7x + 72$; and show that it will be unaltered if $\dfrac{3 - 5\sqrt{-1}}{2}$ be substituted for $x$.

Solution: When $x = \dfrac{3 + 5\sqrt{-1}}{2}$,

(eq. 1) $$2x^3 + 2x^2 - 7x + 72 = 2\left(\frac{3+5\sqrt{-1}}{2}\right)^3 + 2\left(\frac{3+5\sqrt{-1}}{2}\right)^2 - 7\left(\frac{3+5\sqrt{-1}}{2}\right) + 72.$$

Simplifying the right side of this equation:

$$\left(\frac{3+5\sqrt{-1}}{2}\right)^2 = \left(\frac{3+5\sqrt{-1}}{2}\right)\left(\frac{3+5\sqrt{-1}}{2}\right) = \frac{9 + 30\sqrt{-1} + 25(-1)}{4}$$

$$= \frac{30\sqrt{-1} - 16}{4}$$

$$\left(\frac{3+5\sqrt{-1}}{2}\right)^3 = \left(\frac{3+5\sqrt{-1}}{2}\right)\left(\frac{3+5\sqrt{-1}}{2}\right)^2 = \left(\frac{3+5\sqrt{-1}}{2}\right)\left(\frac{30\sqrt{-1} - 16}{4}\right)$$

$$= \frac{90\sqrt{-1} + 150(-1) - 48 - 80\sqrt{-1}}{8}$$

$$= \frac{10\sqrt{-1} - 198}{8}$$

Therefore, equation (1) becomes:

$$2x^3 + 2x^2 - 7x + 72 = 2\left(\frac{10\sqrt{-1} - 198}{8}\right) + 2\left(\frac{30\sqrt{-1} - 16}{4}\right) - 7\left(\frac{3+5\sqrt{-1}}{2}\right) + 72$$

$$= \frac{10\sqrt{-1} - 198}{4} + \frac{30\sqrt{-1} - 16}{2} - \frac{21 + 35\sqrt{-1}}{2} + 72$$

$$= \frac{10\sqrt{-1}}{4} - \frac{198}{4} + 15\sqrt{-1} - 8 - \frac{21}{2} - \frac{35}{2}\sqrt{-1} + 72$$

$$= 2\tfrac{1}{2}\sqrt{-1} - 49\tfrac{1}{2} + 15\sqrt{-1} - 8 - 10\tfrac{1}{2} - 17\tfrac{1}{2}\sqrt{-1} + 72$$

Associating,

$$= (2\tfrac{1}{2}\sqrt{-1} + 15\sqrt{-1} - 17\tfrac{1}{2}\sqrt{-1}) - 49\tfrac{1}{2} - 8 - 10\tfrac{1}{2} + 72$$

$$= (17\tfrac{1}{2}\sqrt{-1} - 17\tfrac{1}{2}\sqrt{-1}) - 68 + 72$$

$$= 0 - 68 + 72$$

$$= 4$$

When $x = \frac{3 - 5\sqrt{-1}}{2}$,

(eq. 2) $$2x^3 + 2x^2 - 7x + 72 = 2\left(\frac{3-5\sqrt{-1}}{2}\right)^3 + 2\left(\frac{3-5\sqrt{-1}}{2}\right)^2 - 7\left(\frac{3-5\sqrt{-1}}{2}\right) + 72.$$

Simplifying the right side of this equation:

$$\left(\frac{3-5\sqrt{-1}}{2}\right)^2 = \left(\frac{3-5\sqrt{-1}}{2}\right)\left(\frac{3-5\sqrt{-1}}{2}\right) = \frac{9 - 30\sqrt{-1} + 25(-1)}{4}$$

$$= \frac{-30\sqrt{-1} - 16}{4}$$

$$\left(\frac{3-5\sqrt{-1}}{2}\right)^3 = \left(\frac{3-5\sqrt{-1}}{2}\right)\left(\frac{3-5\sqrt{-1}}{2}\right)^2 = \left(\frac{3-5\sqrt{-1}}{2}\right)\left(\frac{-30\sqrt{-1} - 16}{4}\right)$$

$$= \frac{-90\sqrt{-1} - 48 + 150(-1) + 80\sqrt{-1}}{8}$$

$$= \frac{-10\sqrt{-1} - 198}{8}$$

Therefore, equation (2) becomes:

$$2x^3 + 2x^2 - 7x + 72 = 2\left(\frac{-10\sqrt{-1} - 198}{8}\right) + 2\left(\frac{-30\sqrt{-1} - 16}{4}\right) - 7\left(\frac{3-5\sqrt{-1}}{2}\right) + 72$$

$$= \frac{-10\sqrt{-1}}{4} - \frac{198}{4} - 15\sqrt{-1} - 8 - \frac{21}{2} + \frac{35}{2}\sqrt{-1} + 72$$

$$= -2\tfrac{1}{2}\sqrt{-1} - 49\tfrac{1}{2} - 15\sqrt{-1} - 8 - 10\tfrac{1}{2} + 17\tfrac{1}{2}\sqrt{-1} + 72$$

Associating, $$= \left(-2\tfrac{1}{2}\sqrt{-1} - 15\sqrt{-1} + 17\tfrac{1}{2}\sqrt{-1}\right) - 49\tfrac{1}{2} - 8 - 10\tfrac{1}{2} + 72$$

$$= \left(-17\tfrac{1}{2}\sqrt{-1} + 17\tfrac{1}{2}\sqrt{-1}\right) - 68 + 72$$

$$= 0 - 68 + 72$$

$$= 4$$

Therefore, when $x = \frac{3+5\sqrt{-1}}{2}$, the value of the equation is 4. The equation is unaltered when $3-5\sqrt{-1}/2$ is substituted for $x$, since the value of the equation is also 4 for this substitution.

Find the value of $y^2 - 3y + 1$, for $y = 3 - \sqrt{2}$.

Solution: Substituting $3 - \sqrt{2}$ for y in the expression

$$y^2 - 3y + 1,$$

$$\begin{aligned} y^2 - 3y + 1 &= (3 - \sqrt{2})^2 - 3(3 - \sqrt{2}) + 1 \\ &= (3 - \sqrt{2})(3 - \sqrt{2}) - 3(3 - \sqrt{2}) + 1 \\ &= (9 - 6\sqrt{2} + 2) - 3(3 - \sqrt{2}) + 1 \\ &= (11 - 6\sqrt{2}) - 9 + 3\sqrt{2} + 1 \\ &= 11 - 6\sqrt{2} - 9 + 3\sqrt{2} + 1 \\ &= 11 - 9 + 1 - 6\sqrt{2} + 3\sqrt{2} \\ &= 3 - 3\sqrt{2} \end{aligned}$$



Find the product by inspection:

$$\left(\sqrt[3]{2}a + \sqrt[3]{4}b\right)\left(\sqrt[3]{4}a^2 - 2ab + 2\sqrt[3]{2}b^2\right)$$

Solution: The formula for the sum of two cubes can be used to find the product. This formula is:

$$x^3 + y^3 = (x + y)(x^2 - xy + y^2).$$

The product $\left(\sqrt[3]{2}a + \sqrt[3]{4}b\right)\left(\sqrt[3]{4}a^2 - 2ab + 2\sqrt[3]{2}b^2\right)$ corresponds to the right side of the formula for the sum of two cubes where x is replaced by $\sqrt[3]{2}a$ and y is replaced by $\sqrt[3]{4}b$. Hence,

$$\begin{aligned} \left(\sqrt[3]{2}a + \sqrt[3]{4}b\right)\left(\sqrt[3]{4}a^2 - 2ab + 2\sqrt[3]{2}b^2\right) &= \left(\sqrt[3]{2}a\right)^3 + \left(\sqrt[3]{4}b\right)^3 \\ &= (\sqrt[3]{2})^3a^3 + (\sqrt[3]{4})^3b^3 \end{aligned}$$

since $(ab)^x = a^xb^x$. Also,

$$(\sqrt[n]{x})^n = \left(x^{\frac{1}{n}}\right)^n = x^{\frac{n}{n}} = x^1 = x, \text{ hence}$$

$$(\sqrt[3]{2})^3 = 2 \text{ and } (\sqrt[3]{4})^3 = 4.$$

Therefore $(\sqrt[3]{2}a + \sqrt[3]{4}b)(\sqrt[3]{4}a^2 - 2ab + 2\sqrt[3]{2}b^2) = 2a^3 + 4b^3.$

Find the value of $3x^2 - 4x - 2$, for

$$x = \frac{2 - \sqrt{10}}{3}$$

Solution: Substituting $\frac{2 - \sqrt{10}}{3}$ for x in the expression

$3x^2 - 4x - 2$

$$3x^2 - 4x - 2 = 3\left(\frac{2 - \sqrt{10}}{3}\right)^2 - 4\left(\frac{2 - \sqrt{10}}{3}\right) - 2$$

$$= 3\,\frac{(2 - \sqrt{10})^2}{(3)^2} - 4\left(\frac{2 - \sqrt{10}}{3}\right) - 2$$

$$= \frac{3(2 - \sqrt{10})(2 - \sqrt{10})}{9} - 4\left(\frac{2 - \sqrt{10}}{3}\right) - 2$$

$$= \frac{3(4 - 4\sqrt{10} + 10)}{9} - 4\left(\frac{2 - \sqrt{10}}{3}\right) - 2$$

$$= \overset{1}{\cancel{3}}\,\frac{14 - 4\sqrt{10}}{\underset{3}{\cancel{9}}} - 4\left(\frac{2 - \sqrt{10}}{3}\right) - 2$$

$$= \frac{14 - 4\sqrt{10}}{3} - \frac{(8 - 4\sqrt{10})}{3} - 2$$

$$= \frac{14 - 4\sqrt{10}}{3} - \frac{(8 - 4\sqrt{10})}{3} - \frac{6}{3}$$

$$= \frac{14 - 4\sqrt{10} - 8 + 4\sqrt{10} - 6}{3}$$

$$= \frac{14 - 8 - 6 - \cancel{4\sqrt{10}} + \cancel{4\sqrt{10}}}{3}$$

$$= \frac{6 - 6}{3}$$

$$= \frac{0}{3}$$

$$= 0$$

# CHAPTER 8

# ALGEBRAIC ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION

**Basic Attacks and Strategies for Solving Problems in this Chapter.** **See pages 79 to 89 for step-by-step solutions to problems.**

An algebraic expression is one that contains one or more variables. To evaluate an algebraic expression means to replace the variable or variables in the expression with given numbered values and then simplify the resulting numerical expression.

One type of algebraic expression, in which the terms are all monomials, is called a polynomial. The four fundamental operations can be performed using polynomials. The addition and subtraction of polynomial expressions rely on essentially the same ideas. The commutative, associative, and distributive properties provide the basis for rearranging, regrouping, and combining similar terms in the addition and subtraction of polynomial expressions. To add polynomials, simply combine like terms. To subtract polynomials, simply add the opposite of the second polynomial to the first and combine like terms. The final resulting polynomial in each case should be written in decreasing order according to the power of the variable. Either a vertical or horizontal format can be used for either operation. For example, we can combine each of the following expressions using the indicated operations in vertical or horizontal format.

Simplify

$$(3x^2 - 7x + 1) + (7x^2 - 5) \text{ and} \tag{1}$$

$$(7x^2 - 5) - (3x^2 - 7x + 1). \tag{2}$$

**Vertical Format (Addition)**

$$\begin{array}{r} 3x^2 - 7x + 1 \\ \underline{7x^2 \qquad\; -5} \\ 10x^2 - 7x - 4 \end{array}$$

**Horizontal Format (Subtraction)**

$$
\begin{aligned}
(7x^2 - 5) - (3x^2 - 7x + 1) &= 7x^2 - 5 - 3x^2 + 7x - 1 \\
&= 4x^2 + 7x - 6
\end{aligned}
$$

Multiplying polynomials involves the use of the properties of exponents and the distributive property. In general, to find the product of two polynomials, multiply each term of the first polynomial by each term of the second polynomial, combine similar terms, and write the resulting polynomial in decreasing order. If there are more than two polynomials to be multiplied, multiply the product of the first two by the next polynomial, combine like terms, and write the final product in decreasing order.

To divide a polynomial by a monomial, divide each term in the numerator by the denominator and write the sum of the quotients. To divide a polynomial by a polynomial (degree 2 or more), use a method similar to that used for division of whole numbers. In particular, first arrange the terms in the divisor and dividend in descending powers of the variable, filling in any missing power of the variables in the dividend with +0 times the missing power of the variable. Then, the next step is to divide the first term of the divisor into the first term of the dividend. Multiply the quotient from the division by each term in the divisor and subtract the products of each term from the dividend. This result (difference) is a new dividend. Repeat the last step using the divisor and the new dividend again until we obtain a remainder which is of degree less than that of the divisor or zero. The final quotient is the sum of the quotients obtained from each step plus the remainder expressed as a fraction.

# Step-by-Step Solutions to Problems in this Chapter, "Algebraic Addition, Subtraction Multiplication, and Division"

**• PROBLEM 134**

Evaluate the expression $3x^2y - 2xy^2z + xyz$ when $x = 2$, $y = -1$, and $z = 3$.

Solution: There are two ways to find the value.

(a) We can simplify $3x^2y - 2xy^2z + xyz$ and substitute. Simplifying can be done by applying the distributive property since each term of the expression has a common factor of $xy$. Thus, we have

$$(3x - 2yz + z)xy$$

We now substitute in the expression:

$$[3(2) - 2(-1)(3) + 3](2)(-1) = (6 + 6 + 3)(-2) = -30.$$

(b) We can substitute in the original expression:

$$3(2)^2(-1) - 2(2)(-1)^2(3) + (2)(-1)(3) = -12 - 12 - 6 = -30.$$

In the example above we found that both $(3x - 2yz + z)xy$ and $3x^2y - 2xy^2z + xyz$ have the same value when $x = 2$, $y = -1$, and $z = 3$. It is apparent that both expressions will have equal values for any set of replacements of the variable; they are called equivalent expressions.

**• PROBLEM 135**

Find the value of the polynomial $3x^2y - 2xy^2 + 5xy$ when $x = 1$ and $y = -2$.

Solution: Replace x by 1 and y by $-2$ in the given polynomial to obtain,

$$\begin{aligned} 3x^2y - 2xy^2 + 5xy &= \left[3(1)^2 - (-2)\right] - \left[2(1)(-2)^2\right] \\ &\quad + [5(1)(-2)] \\ &= [(3)(-2)] - [(2)(4)] \\ &\quad + [(5)(-2)] \\ &= -6 - 8 - 10 \\ &= -24. \end{aligned}$$

Thus, when $x = 1$ and $y = -2$, the polynomial

$$3x^2y - 2xy^2 + 5xy = -24.$$

**• PROBLEM 136**

Combine the expressions $2a - 5b - c$ and $8a + 4b - 3c$.

Solution: Whenever two expressions are combined, those two expressions should be added. Therefore, the two expressions $2a-5b-c$ and $8a+4b-3c$ to be combined will be added. $(2a-5b-c) + (8a+4b-3c)$ is the problem. Place all similiar terms together; that is, place the a terms together, the b terms together and the c terms together, Therefore:

$$\begin{aligned}(2a - 5b - c) + (8a + 4b - 3c) &= (2a + 8a) + (-5b + 4b) \\ &\quad + (-c - 3c) \\ &= 10a - b - 4c.\end{aligned}$$

This solution can be written directly, as

$$(2a - 5b - c) + (8a + 4b - 3c) = 10a - b - 4c,$$

or it can be arranged so that like terms appear in columns, as

$$\begin{array}{r} 2a - 5b - \phantom{3}c \\ \underline{8a + 4b - 3c} \\ 10a - \phantom{4}b - 4c \end{array},$$

in which the columns are added.

**• PROBLEM 137**

Add $(3xy^2 + 2xy + 5x^2y) + (2xy^2 - 4xy + 2x^2y)$.

Solution: Use the vertical form, align all like terms, and apply the distributive property.

$$\begin{array}{c} 3xy^2 + 2xy + 5x^2y \\ \underline{2xy^2 - 4xy + 2x^2y} \\ (3 + 2)xy^2 + (2 - 4)xy + (5 + 2)x^2y \end{array}$$

Thus, the sum is $5xy^2 - 2xy + 7x^2y$.

**• PROBLEM 138**

Add the expression $4a^2 - 3 + 5a$, $6a - 2a^2 + 2$, and $2a^2 - 3a + 8$ .

Solution: Arrange each polynomial in decending order of exponents of $a$;

by commutation, $4a^2 - 3 + 5a = 4a^2 + 5a - 3$

similarly, $6a - 2a^2 + 2 = -2a^2 + 6a + 2$

and, $2a^2 - 3a + 8 = 2a^2 - 3a + 8$ .

Since in each column we have the same power of $a$ we may simply add columns to obtain:

$$\begin{array}{r} 4a^2 + 5a - 3 \\ -2a^2 + 6a + 2 \\ \underline{2a^2 - 3a + 8} \\ 4a^2 + 8a + 7 \end{array}$$

Subtract $4y^2 - 5y + 2$ from $7y^2 - 6$.

Solution: The problem is the following:

$$(7y^2 - 6) - (4y^2 - 5y + 2).$$

Whenever a minus sign (-) appears before an expression in parentheses, change the sign of every term in the parentheses.

In this problem, since a minus sign appears before the expression $(4y^2 - 5y + 2)$, the sign of every term in this expression is changed.

$$(4y^2 - 5y + 2) \text{ becomes } -4y^2 + 5y - 2.$$

After the signs have been changed, the new expression $-4y^2 + 5y - 2$ can be added to the expression $7y^2 - 6$ (changing the signs of the expression changes the original problem to an addition problem). Therefore

$$(7y^2 - 6) - (4y^2 - 5y + 2) = 7y^2 - 6 - 4y^2 + 5y - 2.$$

Place the terms with similar powers together. Therefore:

$$(7y^2 - 6) - (4y^2 - 5y + 2) = 7y^2 - 6 - 4y^2 + 5y - 2$$

$$= 7y^2 - 4y^2 + 5y - 6 - 2. \quad (1)$$

Since $7y^2 - 4y^2 = 3y^2$ and $-6- 2 = -8$, equation (1) becomes:

$$(7y^2 - 6) - (4y^2 - 5y + 2) = 3y^2 + 5y - 8,$$

which is the final answer.

• PROBLEM 140

From the sum of $6x^2 + 4xy - 8y^2 - 11$ and $3x^2 - 4y^2 + 8 + 5xy$ subtract $xy - 10 - 5x^2 + 7y^2$.

Solution: First find the sum of $6x^2 + 4xy - 8y^2 - 11$ and $3x^2 - 4y^2 + 8 + 5xy$. Adding these two polynomials together:

$$(6x^2 + 4xy - 8y^2 - 11) + (3x^2 - 4y^2 + 8 + 5xy)$$

$$= 6x^2 + 4xy - 8y^2 - 11 + 3x^2 - 4y^2 + 8 + 5xy$$

Grouping like terms together,

$$(6x^2 + 4xy - 8y^2 - 11) + (3x^2 - 4y^2 + 8 + 5xy)$$

$$= (6x^2 + 3x^2) + (4xy + 5xy) + (-8y^2 - 4y^2) + (-11 + 8)$$

$$= 9x^2 + 9xy + (-12y^2) + (-3) = 9x^2 + 9xy - 12y^2 - 3 \quad (1)$$

Now subtract $xy - 10 - 5x^2 + 7y^2$ from the resultant sum, which is the right side of equation (1), or $9x^2 + 9xy - 12y^2 - 3$. Then,

$$(9x^2 + 9xy - 12y^2 - 3) - (xy - 10 - 5x^2 + 7y^2) =$$

$$= 9x^2 + 9xy - 12y^2 - 3 - xy + 10 + 6x^2 - 7y^2 \quad (2)$$

Grouping like terms together, equation (2) becomes:

$$(9x^2 + 9xy - 12y^2 - 3) - (xy - 10 - 5x^2 + 7y^2)$$

$$= (9x^2 + 5x^2) + (9xy - xy) + (-12y^2 - 7y^2) + (-3 + 10)$$

$$= 14x^2 + 8xy + (-19y^2) + 7$$

$$= 14x^2 + 8xy - 19y^2 + 7,$$

which is the final answer.

**• PROBLEM 141**

Subtract $3x^4y^3 + 5x^2y - 4xy + 5x - 3$ from the polynomial $5x^4y^3 - 3x^2y + 7$.

<u>Solution:</u> $(5x^4y^3 - 3x^2y + 7) - (3x^4y^3 + 5x^2y - 4xy + 5x - 3)$

$$= (5x^4y^3 - 3x^2y + 7) + (-3x^4y^3 - 5x^2y + 4xy - 5x + 3)$$

$$= 5x^4y^3 + (-3x^4y^3) - 3x^2y + (-5x^2y) + 4xy - 5x + 7 + 3$$

$$= 2x^4y^3 - 8x^2y + 4xy - 5x + 10$$

The column form may also be used for subtraction. Here we align the like terms and subtract the coefficients.

$$\begin{array}{rrrrrr} & 5x^4y^3 & - 3x^2y & & & + 7 \\ - & [3x^4y^3 & + 5x^2y & - 4xy & + 5x & - 3] \\ \hline & 2x^4y^3 & - 8x^2y & + 4xy & - 5x & + 10 \end{array}$$

**• PROBLEM 142**

Simplify $3ax(ax^2 - 5bx + c)$.

**Solution:** Using the distributive property,

$$3ax(ax^2 - 5bx + c) = 3ax(ax^2) + 3ax(-5bx) + 3ax(c)$$
$$= 3a^2x^3 - 15abx^2 + 3acx\ .$$

**• PROBLEM 143**

Expand $(x + 5)(x - 4)$.

**Solution:** Distributing the second term:

$$(x + 5)(x - 4) = (x + 5)x + (x + 5)(-4) \qquad (1)$$

Distributing twice on the right side of equation (1):

$$\begin{aligned}(x + 5)(x - 4) &= (x + 5)x + (x + 5)(-4)\\ &= (x^2 + 5x) + (-4x - 20)\\ &= x^2 + 5x - 4x - 20\\ &= x^2 + x - 20.\end{aligned}$$

**• PROBLEM 144**

Find the product $(2x - 5y)(x + 2y)$.

**Solution:** We use the distributive property and simplify.

| | | |
|---|---|---|
| $(2x - 5y)(x + 2y)$ | $= (2x - 5y)x + (2x - 5y)2y$ | Distributive property |
| | $= [x(2x - 5y)]+[2y(2x - 5y)]$ | Commutative property of multiplication |
| | $= [x\cdot 2x + x\cdot(-5y)]+[2y\cdot 2x + 2y\cdot(-5y)]$ | Distributive property |
| | $= 2x^2 - 5xy + 4xy - 10y^2$ | Simplifying |
| | $= 2x^2 - xy - 10y^2$ | Combining like terms |
| $(2x - 5y)(x + 2y)$ | $= 2x^2 - xy - 10y^2$ | |

We can use the properties of a field because the algebraic expressions represent members of the field of real numbers for any replacement of the variables.

**• PROBLEM 145**

Multiply $(4x - 5)(6x - 7)$.

**Solution:** We can apply the FOIL method. The letters indicate the order in which the terms are to be multiplied.

F = first terms
O = outer terms
I = inner terms
L = last terms

Thus,

$$(4x - 5)(6x - 7) = (4x)(6x) + (4x)(-7) + (-5)(6x) + (-5)(-7)$$
$$= 24x^2 - 28x - 30x + 35 = 24x^2 - 58x + 35.$$

Another way to multiply algebraic expressions is to apply the distributive law of multiplication with respect to addition. If a,b, and c are real numbers, then $a(b+c) = ab + ac$. In this case let $a = (4x-5)$ and $b + c = 6x - 7$.

$$(4x - 5)(6x - 7) = (4x - 5)(6x) + (4x - 5)(-7)$$

Then, apply the law again.

$$(4x - 5)(6x - 7) = (4x)(6x) - (5)(6x) + (4x)(-7) + (-5)(-7)$$
$$(4x - 5)(6x - 7) = 24x^2 - 30x - 28x + 35$$

Add like terms.

$$(4x - 5)(6x - 7) = 24x^2 - 58x + 35.$$

**• PROBLEM 146**

Show that $(3x - 2)(x + 5) + 15 = 3x^2 + 13x + 5$ is an identity.

Solution: An equation in x is an identity if it holds for all real values of x. Thus, the given equation is an identity since for each $x \in R$,

$$(3x - 2)(x + 5) + 15 = 3x^2 + 13x - 10 + 15$$

$$= 3x^2 + 13x + 5$$

**• PROBLEM 147**

Multiply $(2x + 3y)(4x - 5y)$.

Solution: Instead of writing one factor beneath the other, we shall use the following process to find the product mentally. Multiply the two first terms, $2x$ and $4x$, to obtain the first term in the product; also, multiply the two last terms, $3y$ and $-5y$, to obtain the last term in the product. Thus, we have $8x^2$ and $-15y^2$ as the first and last terms, respectively. We must now determine the middle term in the desired product. This is done by multiplying the two inner terms, $3y$ and $4x$, and the two outer terms, $2x$ and $-5y$, and adding these. This is shown below, and the middle term is $12xy - 10xy = 2xy$.

$$(2x + 3y)(4x - 5y) = 8x^2 + 2xy - 15y^2$$

12xy

-10xy

It is to be noted that the final result is written immediately with only one intermediate step: the two cross products, $12xy$ and $-10xy$, are kept in mind and added mentally to produce the middle term $2xy$.

**• PROBLEM 148**

Simplify $(5ax + by)(2ax - 3by)$.

Solution: The following formula can be used to simplify the given expression:

$$(N_1 + N_2)N_3 = N_1N_3 + N_2N_3$$

where $N_1, N_2$ and $N_3$ are any three numbers. Note that the distributive property is used in this formula. $N_1$ is replaced by 5ax and by replaces $N_2$. Also, $N_3$ is replaced by (2ax - 3by). Therefore:

$$(5ax + by)(2ax - 3by) = 5ax(2ax - 3by) + by(2ax - 3by)$$

Use the distributive property to simplify the right side of the above equation:

$$\begin{aligned}(5ax + by)(2ax - 3by) &= 5ax(2ax) + 5ax(-3by) + by(2ax) + by(-3by)\\ &= 10a^2x^2 - 15abxy + 2abxy - 3b^2y^2\\ &= 10a^2x^2 - 13abxy - 3b^2y^2\end{aligned}$$

It is often convenient to arrange the two factors vertically as we do in ordinary arithmetic. Hence, the problem is an ordinary multiplication problem.

$$\begin{array}{rr} & 5ax + by \\ \times & 2ax - 3by \\ \hline & -15abxy - 3b^2y^2 \\ + & 10a^2x^2 + 2abxy \qquad\quad \\ \hline & 10a^2x^2 - 13abxy - 3b^2y^2 \end{array}$$

Note that this answer (i.e., product) is the same as the answer obtained above.

**• PROBLEM 149**

Expand $(a + b - 2)^2$ .

Solution: When we enclose a + b in parentheses we may write

$$\begin{aligned}(a + b - 2)^2 &= [(a + b) - 2]^2\\ &= [(a + b) - 2][(a + b) - 2]\end{aligned}$$

Employing our method of polynomial multiplication

$$(x + y)(u + v) = xu + xv + yu + yv$$

Substituting (a + b) for x and u, and (-2) for y and v we obtain

$$[(a + b) - 2]^2 = (a + b)^2 - 4(a + b) + 4$$

Once again employ our method of polynomial multiplication on

$$(a + b)^2 = (a + b)(a + b).$$

Thus $$(a + b - 2)^2 = a^2 + 2ab + b^2 - 4a - 4b + 4$$

**• PROBLEM 150**

Expand $(x + 3y - 5t)^2$.

Solution: It is sometimes convenient to group two or more terms and treat them as a single term. x + 3y will be considered as one term. Hence,

$$\begin{aligned}(x + 3y - 5t)^2 &= [(x + 3y) - 5t]^2\\ &= [(x + 3y) - 5t][(x + 3y) - 5t]\end{aligned}$$

Apply the FOIL method:

$$\begin{aligned}(x + 3y - 5t)^2 &= (x + 3y)^2 - 5t(x + 3y) - 5t(x + 3y) + (-5t)^2\\ &= (x + 3y)^2 + 2(x + 3y)(-5t) + (-5t)^2\\ &= (x + 3y)(x + 3y) + 2(x + 3y)(-5t) + (-5t)^2\end{aligned}$$

$$= x^2 + 3xy + 3xy + 9y^2 + (2x + 6y)(-5t) + 25t^2$$
$$= x^2 + 6xy + 9y^2 - 10tx - 30ty + 25t^2 .$$

**• PROBLEM 151**

Simplify $(x - y)(x^2 + xy + y^2)$.

Solution: By distributing,

$$(x - y)(x^2 + xy + y^2) = x(x^2 + xy + y^2) - y(x^2 + xy + y^2) \qquad (1)$$

Now, distribute the right side of equation (1):

$$(x - y)(x^2 + xy + y^2) = (x^3 + x^2y + xy^2) - (x^2y + xy^2 + y^3).$$

Combining terms:

$$(x - y)(x^2 + xy + y^2) = x^3 + \cancel{x^2y} + \cancel{xy^2} - \cancel{x^2y} - \cancel{xy^2} - y^3$$
$$= x^3 - y^3 .$$

**• PROBLEM 152**

Multiply $3x^2 - 5y^2 - 4xy$ by $2x - 7y$.

Solution: Write one algebraic expression under the other, and multiply the first expression by each term of the second expression, placing similar product terms in the same vertical column.

$$\begin{array}{ll}
3x^2 - 5y^2 - 4xy & \\
2x \;\; - 7y & \\
\hline
6x^3 - 10xy^2 - 8x^2y & \text{multiplying } 3x^2 - 5y^2 - 4xy \text{ by } 2x \\
\qquad 28xy^2 - 21x^2y + 35y^3 & \text{multiplying } 3x^2 - 5y^2 - 4xy \text{ by } -7y \\
\hline
6x^3 + 18xy^2 - 29x^2y + 35y^3 & \text{adding the partial products}
\end{array}$$

**• PROBLEM 153**

Find the product

$$(2x^2 - 3xy + y^2)(2x - y)$$

Solution: Multiplication of polynomials can be carried out very much the same way we multiply numbers. One polynomial is written under the other, and then multiplied term by term. Like terms in the product are arranged in columns and added.

$$\begin{array}{r}
2x^2 - 3xy + y^2 \\
2x - y \\
\hline
4x^3 - 6x^2y + 2xy^2 \\
- 2x^2y + 3xy^2 - y^3 \\
\hline
4x^3 - 8x^2y + 5xy^2 - y^3
\end{array}$$

We are applying the Distributive Law in the following way:

$$\begin{array}{rcl}
(2x^2 - 3xy + y^2)(2x - y) = (2x^2 - 3xy + y^2)(2x) & = & 4x^3 - 6x^2y + 2xy^2 \\
+ (2x^2 - 3xy + y^2)(-y) & = & -2x^2y + 3xy^2 - y^3 \\
\hline
(2x^2 - 3xy + y^2)(2x-y) & = & 4x^3 - 8x^2y + 5xy^2 - y^3
\end{array}$$

Divide $(37 + 8x^3 - 4x)$ by $(2x + 3)$.

Solution: Arrange both polynomials in descending powers of the variable. The first polynomial becomes: $8x^3 - 4x + 37$. The second polynomial stays the same: $2x + 3$. The problem is: $2x + 3\overline{)8x^3 - 4x + 37}$. In the dividend, $8x^3 - 4x + 37$, all powers of x must be included. The only missing power of x is $x^2$. To include this power of x, a coefficient of 0 is used; that is, $0x^2$. This term, $0x^2$, can be added to the dividendwithout changing the dividend because $0x^2 = 0$ (anything multiplied by 0 is 0).

Now to accomplish the division we proceed as follows: divide the first term of the divisor into the first term of the dividend. Multiply the quotient from this division by each term of the divisor and subtract the products of each term from the dividend. We then obtain a new dividend. Use this dividend, and again divide by the first term of the divisor, and repeat all steps again until we obtain a remainder which is of degree lower than that of the divisor or zero. Following this procedure we obtain :

$$
\begin{array}{r l}
 & \phantom{)}4x^2 - 6x + 7 \\
2x + 3 & \overline{)8x^3 + 0x^2 - 4x + 37} \\
 & \phantom{)}\underline{8x^3 + 12x^2} \\
 & \phantom{)8x^3} - 12x^2 - 4x + 37 \\
 & \phantom{)8x^3} - \underline{12x^2 - 18x} \\
 & \phantom{)8x^3 - 12x^2 -} 14x + 37 \\
 & \phantom{)8x^3 - 12x^2 -} \underline{14x + 21} \\
 & \phantom{)8x^3 - 12x^2 - 14x +} 16
\end{array}
$$

The degree of a polynomial is the highest power of the variable in the polynomial.

The degree of the divisor is 1. The number 16 can be written as $16x^0$ where $x^0 = 1$. Therefore, the number 16 has degree 0. When the degree of the divisor is greater than the degree of the dividend, we stop dividing.

Since the degree of the divisor in this problem is 1 and the degree of the dividend (16) is 0, the degree of the divisor is greater than the degree of the dividend. Therefore, dividing is stopped and the remainder is 16. Therefore, the quotient is $4x^2 - 6x + 7$ and the remainder is 16.

In order to verify this,multiply the quotient, $4x^2 - 6x + 7$, by the divisor, $2x + 3$, and then add 16. These two operations should total up to the dividend $8x^3 - 4x + 37$. Thus,

$$(4x^2 - 6x + 7)(2x + 3) + 16 =$$

$$8x^3 - 12x^2 + 14x + 12x^2 - 18x + 21 + 16 =$$

$$8x^3 - 4x + 37,$$

which is the desired result.

**• PROBLEM 155**

> Divide $3x^5 - 8x^4 - 5x^3 + 26x^2 - 33x + 26$ by $x^3 - 2x^2 - 4x + 8$.

Solution: To divide a polynomial by another polynomial we set up the divisor and the dividend as shown below. Then we divide the first term of the divisor into the first term of the dividend. We multiply the quotient from this division by each term of the divisor, and subtract the products of each term from the dividend. We then obtain a new dividend. Use this dividend, and again divide by the first term of the divisor, and repeat all steps again until we obtain a remainder which is of degree lower than that of the divisor or = zero. Following this procedure we obtain:

$$
\begin{array}{r|l}
 & 3x^2 - 2x + 3 \\
\hline
x^3-2x^2-4x+8 & 3x^5 - 8x^4 - 5x^3 + 26x^2 - 33x + 26 \\
 & \underline{3x^5 - 6x^4 - 12x^3 + 24x^2} \\
 & \quad - 2x^4 + 7x^3 + 2x^2 - 33x + 26 \\
 & \quad \underline{- 2x^4 + 4x^3 + 8x^2 - 16x} \\
 & \qquad 3x^3 - 6x^2 - 17x + 26 \\
 & \qquad \underline{3x^3 - 6x^2 - 12x + 24} \\
 & \qquad\qquad - 5x + 2
\end{array}
$$

Thus, the quotient is $3x^2 - 2x + 3$ and the remainder is $- 5x + 2$.

**• PROBLEM 156**

> Find the quotient and remainder when $3x^7 - x^6 + 31x^4 + 21x + 5$ is divided by $x + 2$.

Solution: To divide a polynomial by another polynomial we set up the divisor and the dividend as shown below. Then we divide the first term of the divisor into the first term of the dividend. We multiply the quotient from this division by each term of the divisor, and subtract the products of each term from the dividend. We then obtain a new dividend. Use this dividend, and again divide by the first term of the divisor, and repeat all steps again until we obtain a remainder which is of degree lower than that of the divisor, or which is zero. Following this procedure we obtain:

$$
\begin{array}{r|rrrrrrrr}
 & 3x^6 & -7x^5 & +14x^4 & +3x^3 & -6x^2 & +12x & -3 & \\
\hline
x+2 & 3x^7 & -x^6 & & +31x^4 & & & +21x & +5 \\
 & 3x^7 & +6x^6 & & & & & & \\
\cline{2-9}
 & & -7x^6 & & +31x^4 & & & +21x & +5 \\
 & & -7x^6 & -14x^5 & & & & & \\
\cline{3-9}
 & & & 14x^5 & +31x^4 & & & +21x & +5 \\
 & & & 14x^5 & +28x^4 & & & & \\
\cline{4-9}
 & & & & 3x^4 & & & +21x & +5 \\
 & & & & 3x^4 & +6x^3 & & & \\
\cline{5-9}
 & & & & & -6x^3 & & +21x & +5 \\
 & & & & & -6x^3 & -12x^2 & & \\
\cline{6-9}
 & & & & & & 12x^2 & +21x & +5 \\
 & & & & & & 12x^2 & +24x & \\
\cline{7-9}
 & & & & & & & -3x & +5 \\
 & & & & & & & -3x & -6 \\
\cline{8-9}
 & & & & & & & & 11
\end{array}
$$

Thus the quotient is $3x^6 - 7x^5 + 14x^4 + 3x^3 - 6x^2 + 12x - 3$, and the remainder is 11.

# CHAPTER 9

# FUNCTIONS AND RELATIONS

**Basic Attacks and Strategies for Solving Problems in this Chapter. See pages 90 to 104 for step-by-step solutions to problems.**

A relation is any set of ordered pairs, for example, the set

$$A = \{(2, 3), (5, 2), (6, 3)\}$$

is a relation. The set of all first coordinates of set $A$ is called the domain of the relation, and the set of all second coordinates is said to be the range of the relation. Thus, for relation $A$, the sets $\{2, 5, 6\}$ and $\{3, 2\}$ are the domain and range, respectively.

There are two ways to specify the ordered pairs in a relation. One method is simply to list them. The other method is to give the rule (equation) for obtaining them.

One of the most fundamental ideas of mathematics is that of a function. There are two essentially equivalent ways of defining the concept. First, a function is a relation in which no two different ordered pairs have the same first coordinates. The domain and range of a function are the sets of first and second coordinates, respectively. For example, the above relation $A$ is a function. On the other hand, the relation

$$B = \{(1, 5), (3, 9), (1, 9)\}$$

is not a function because the ordered pairs $(1, 5)$ and $(1, 9)$ have the same first coordinates.

The second definition of a function emphasizes that a function is a rule (equation) that pairs every element $x$ in a set $D$ with a unique element $y$ in a set $R$, where set $D$ is the domain and set $R$ is the range or image set of $x$. When a function (or relation) is given in terms of a rule (equation), the domain is the set of all possible replacements for the variable $x$. If the domain of a function (or relation) is not specified, it is assumed to be all real numbers that do not yield undefined terms in the equation. That is, we cannot use values of $x$ in the domain that will produce 0 in a denominator or the square root of a negative number.

The graph of a function (or relation) is sometimes helpful in determining the domain and range. Also, the notation,

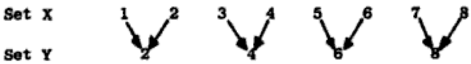
$$y = f(x),$$

defined to be the value of the function $f$ at $x$ or the value of $y$ associated with a given value of $x$, is useful in determining the domain and range of the function.

# Step-by-Step Solutions to Problems in this Chapter, "Functions and Relations"

• **PROBLEM 157**

If $X = \{1,2,3,4,5,6,7,8\}$ and $Y = \{2,4,6,8\}$, use $f(x)$ notation to indicate the image of each element of $X$ in the following mapping.




<u>Solution:</u> The mapping of an element $x$ in the set $X$ to an element $y$ in the set $Y$ may be written in $f(x)$ notation as $f(x) = y$ when $f$ is a function mapping $x$ to $y$. That is, $f: x \to y$. Therefore,

$$f(1) = 2,\ f(2) = 2,\ f(3) = 4,\ f(4) = 4,\ f(5) = 6,\ f(6) = 6,$$
$$f(7) = 8,\ f(8) = 8$$

• **PROBLEM 158**

Find the image of each element in

$$A = \{1,2,3,4,5,6,7,8,9\}$$

under the following mapping:

$$f(x) = \begin{cases} 2x, & \text{if } x < 5 \\ 8, & \text{if } x \geq 5 \end{cases}$$

<u>Solution:</u> The image of each element in $A = \{1,2,3,4,5,6,7,8,9\}$ under the mapping $f(x)$, is $f(1), f(2), f(3), f(4), f(5), f(6), f(7), f(8), f(9)$. $f(x)$ has two corresponding values, depending on the value of $x$. If $x < 5$, $f(x) = 2x$, thus for

$$x = 1,\ f(1) = 2(1) = 2$$
$$x = 2,\ f(2) = 2(2) = 4$$
$$x = 3,\ f(3) = 2(3) = 6$$
$$x = 4,\ f(4) = 2(4) = 8$$

and if $x \geq 5$, $f(x) = 8$, thus for

$$x = 5,\ f(5) = 8$$
$$x = 6,\ f(6) = 8$$
$$x = 7,\ f(7) = 8$$
$$x = 8,\ f(8) = 8$$
$$x = 9,\ f(9) = 8$$

• **PROBLEM 159**

Find the relation defined by $y^2 = 25 - x^2$, where $x$ belongs to

$D = \{0,3,4,5\}$.

Solution: $x$ takes on the values 0,3,4, and 5. Replacing $x$ by these values in the equation $y^2 = 25 - x^2$ we obtain the corresponding values of $y$:

| $x$ | $y^2 = 25 - x^2$ | $y$ |
|---|---|---|
| 0 | $y^2 = 25 - 0$<br>$y^2 = 25$<br>$y = \sqrt{25}$<br>$y = \pm 5$ | $\pm 5$ |
| 3 | $y^2 = 25 - 3^2$<br>$y^2 = 25 - 9$<br>$y^2 = 16$<br>$y = \sqrt{16}$<br>$y = \pm 4$ | $\pm 4$ |
| 4 | $y^2 = 25 - 4^2$<br>$y^2 = 25 - 16$<br>$y^2 = 9$<br>$y = \sqrt{9}$<br>$y = \pm 3$ | $\pm 3$ |
| 5 | $y^2 = 25 - 5^2$<br>$y^2 = 25 - 25$<br>$y^2 = 0$<br>$y = 0$ | 0 |

Hence the relation defined by $y^2 = 25 - x^2$ where $x$ belongs to $D = \{0,3,4,5\}$ is

$$\{(0,5),(0,-5),(3,4),(3,-4),(4,3),(4,-3),(5,0)\}.$$

**• PROBLEM 160**

Find the relation $M$ over set $S = \{1,2,3\}$ if

$$M = \{(x,r(x)) : r(x) = 2x - 1\}$$

Solution: $x$ takes on the values 1,2, and 3. Replacing $x$ by these values in the equation $r(x) = 2x - 1$ we obtain the corresponding values of $r(x)$:

| $x$ | $r(x) = 2x - 1$ | $r(x)$ |
|---|---|---|
| 1 | $r(x) = 2(1)-1$<br>$= 2 - 1$<br>$= 1$ | 1 |
| 2 | $r(x) = 2(2)-1$<br>$= 4 - 1$<br>$= 3$ | 3 |
| 3 | $r(x) = 2(3)-1$<br>$= 6 - 1$<br>$= 5$ | 5 |

Thus the rule of correspondence $r(x) = 2x - 1$ determines the set of ordered pairs

$$\{(1,1),(2,3),(3,5)\}$$

But the relation must be a subset of $S \times S$. Since (3,5) is not a subset of $S \times S$ (5 is not a member of $S$) we eliminate this pair. Hence,

$$M = \{(1,1),(2,3)\}$$

Find the relation Q over $S \times T$ if $S = \{1,2,3\}$, $T = \{4,5\}$, and the rule of correspondence is

$$r(x) = x + 2.$$

Solution: We first find the image of each element in S by substituting each element for x in the rule of correspondence $r(x) = x + 2$.

$$r(1) = 1 + 2 = 3 \qquad r(2) = 2 + 2 = 4$$

$$r(3) = 3 + 2 = 5.$$

Thus, the rule of correspondence determines the following set of ordered pairs:

$$\{(1,3), (2,4), (3,5)\}.$$

However, the relation Q must be a subset of $S \times T$, which equals $\{(1,4), (1,5), (2,4), (2,5), (3,4), (3,5)\}$. Therefore the point (1,3) won't appear in Q because (1,3) doesn't appear in $S \times T$. Therefore, the relation over $S \times T$ determined by $r(x) = x + 2$ is

$$Q = \{(2,4), (3,5)\}.$$

We can use set-builder notation to describe the relation discussed in the above example. In the example, a set of ordered pairs was determined by a rule of correspondence. The first component, x, was chosen from the domain, S. The second component, r(x), was the corresponding image from the range, T. Thus, we can describe the relation Q in the following manner:

$$Q = \{\Big(x, r(x)\Big): r(x) = x + 2\}.$$

This notation refers to all ordered pairs [x, r(x)], such that $r(x) = x + 2$.

• PROBLEM 162

Which of the following sets are functions of x?

$$A = \{(5,1),(4,2),(4,3),(6,4)\},$$

$$B = \{(x,y) \mid y = |x|\},$$

$$C = \{(x,y) \mid x = |y|\}?$$

Solution: A function is a relation having the property that each member of its domain is paired with exactly one member of its range. Thus, set A is not a function, for it contains the pairs (4,2) and (4,3) - that is, one member of its domain, 4, is paired with more than one member of its range, 2 and 3. If each x value has only one corresponding y value, any vertical line only intersects the graph of a function at one point. Thus, from

Fig. 1 $y=|x|$

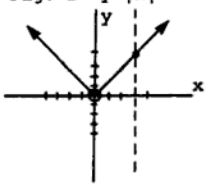



Fig. 2 $x=|y|$

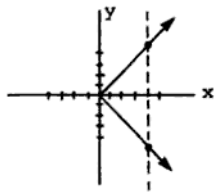



figure 1 we note that set B is a function. Notice that a function may contain two pairs with the same second member; for example, our function B contains the pairs (1,1) and (- 1,1).

From figure 2 we note that a vertical line intersects the graph of C in two places, thus there are x values of C which have more than one corresponding y value, and C is not a function.

**• PROBLEM 163**

Let the domain of $M = \{(x,y): y = x\}$ be the set of real numbers. Is $M$ a function?

Fig. A

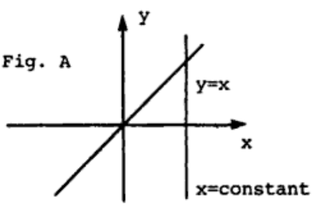
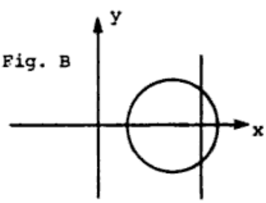



Fig. B




**Solution:** The range is also the set of real numbers since $y = \{y | y = x\}$. The graph of $y = x$ is the graph of a line ($y = mx + b$ where $m = 1$ and $b = 0$). See fig. A. If for every value of x in the domain, there corresponds only one y value then y is said to be a function of x. Since each element in the domain of M has exactly one element for its image, M is a function. Also notice that a vertical line ($x$ = constant) crosses the graph $y = x$ only once. Whenever this is true the graph defines a function. Consult figure B.
The vertical line ($x$ = constant) crosses the graph of the circle twice; i.e., for each x,y is not unique, therefore the graph does not define a function.

**• PROBLEM 164**

If $g(x) = x^2 + 5x - 3$, find $g(-7)$.

**Solution:** Substitute - 7 for x everywhere in the equation:

$$g(-7) = (-7)^2 + 5(-7) - 3$$
$$= 49 - 35 - 3 = 11.$$

**• PROBLEM 165**

> Let $f$ be a mapping with the rule of correspondence
> $$f(x) = 3x^2 - 2x + 1.$$
> Find $f(1)$, $f(-3)$, $f(-b)$.

Solution: In order to find $f(1)$, $f(-3)$, $f(-b)$, we replace $x$ by 1, (-3), and (-b) respectively in our equation for $f(x)$, $f(x) = 3x^2 - 2x + 1$. Thus

$$\begin{aligned} f(1) &= 3(1)^2 - 2(1) + 1 \\ &= 3 - 2 + 1 \\ &= 1 + 1 \\ &= 2 \\ f(-3) &= 3(-3)^2 - 2(-3) + 1 \\ &= 3(9) + 6 + 1 \\ &= 27 + 7 \\ &= 34 \\ f(-b) &= 3(-b)^2 - 2(-b) + 1 \\ &= 3(b^2) + 2b + 1 \\ &= 3b^2 + 2b + 1 \end{aligned}$$

**• PROBLEM 166**

> If $f(t) = 6t + 13$, find $f(5) - f(4)$.

Solution: To find $f(5)$ we substitute 5 for t everywhere in the equation, that is:

$$f(5) = 6(5) + 13 = 43$$

Similarly, $f(4) = 6(4) + 13 = 37$

and now subtract $f(4)$ from $f(5)$. Therefore,

$$f(5) - f(4) = 43 - 37 = 6.$$

**• PROBLEM 167**

> If $f(x) = (x - 2)/(x + 1)$, find the function values $f(2)$, $f(\frac{1}{2})$, and $f(-3/4)$.

Solution: To find $f(2)$, we replace x by 2 in the given formula for $f(x)$, $f(x) = x - 2/x + 1$; thus

$$f(2) = \frac{2 - 2}{2 + 1} = \frac{0}{3} = 0.$$

Similarly, $f(\frac{1}{2}) = \dfrac{\frac{1}{2} - 2}{\frac{1}{2} + 1}$.

Multiply numerator and denominator by 2,

$$= \frac{2(\tfrac{1}{2} - 2)}{2(\tfrac{1}{2} + 1)} \quad .$$

Distribute, $= \dfrac{2(\tfrac{1}{2}) - 2 \cdot 2}{2(\tfrac{1}{2}) + 2}$

$$= \frac{1 - 4}{1 + 2}$$

$$= -\frac{3}{3} = -1.$$

$$f(-3/4) = \frac{-3/4 - 2}{-3/4 + 1} \quad .$$

Multiply numerator and denominator by 4,

$$= \frac{4(-3/4 \;\; - 2)}{4(-3/4 \;\; + 1)} \quad .$$

Distribute, $= \dfrac{4(-3/4) - 4(2)}{4(-3/4) + 4(1)}$

$$= \frac{-3 - 8}{-3 + 4}$$

$$= \frac{-11}{1}$$

$$= -11.$$

**• PROBLEM 168**

If $g(x) = x^2 - 2x + 1$, find the given element in the range.

a) $g(-2)$ b) $g(0)$ c) $g(a + 1)$ d) $g(a - 1)$

<u>Solution:</u> a) To find $g(-2)$, substitute $-2$ for $x$ in the given equation.

$$g(x) = g(-2)$$

$$= (-2)^2 - 2(-2) + 1$$

$$= 4 + 4 + 1$$

$$= 8 + 1$$

$$= 9$$

Hence, $g(-2) = 9$

b) To find $g(0)$, substitute 0 for $x$ in the given equation.

$$g(x) = g(0)$$

$$= (0)^2 - 2(0) + 1$$

$$= 0 - 0 + 1$$

$$= 1$$

Hence, $g(0) = 1$

c) To find $g(a + 1)$, substitute $a + 1$ for $x$ in given equation.

$$g(x) = g(a + 1)$$

$$= (a + 1)^2 - 2(a + 1) + 1$$

$$= (a^2 + 2a + 1) - 2a - 2 + 1$$

$$= a^2 + \cancel{2a} + 1 - \cancel{2a} - 2 + 1$$

$$= a^2 + 1 - 2 + 1$$

$$= a^2 + 0$$

$$= a^2$$

Hence, $g(a + 1) = a^2$ .

d) To find $g(a - 1)$, substitute $a - 1$ for $x$ in given equation.

$$g(x) = g(a - 1)$$

$$= (a - 1)^2 - 2(a - 1) + 1$$

$$= (a^2 - 2a + 1) - 2a + 2 + 1$$

$$= a^2 - 2a + 1 - 2a + 2 + 1$$

$$= a^2 - 4a + 4$$

Hence, $g(a - 1) = a^2 - 4a + 4$

**• PROBLEM 169**

Let f be the function whose domain is the set of all real numbers, whose range is the set of all numbers greater than or equal to 2, and whose rule of correspondence is given by the equation $f(x) = x^2 + 2$. Find $3f(0) + f(-1)f(2)$.

<u>Solution:</u> The rule of correspondence in this example is expressed by the equation $f(x) = x^2 + 2$. To find the number in the range that is associated with any particular number in the domain, we merely replace the letter x wherever it appears in the equation $f(x) = x^2 + 2$ by the given number. Thus

$$f(0) = 0^2 + 2 = 2 \,, \quad f(-1) = (-1)^2 + 2 = 1 + 2 = 3 \,,$$

$f(2) = 2^2 + 2 = 4 + 2 = 6$ , and

$3f(0) + f(-1)f(2) = 3(2) + (3)(6) = 6 + 18 = 24.$

**• PROBLEM 170**

Show that $f(a) = f(-a)$ if $f(x) = x^2 + 3$.

Solution: If $f(a) = f(-a)$, one should obtain an identity when a and then $-a$ are substituted into the equation. For the given equation we have:

$$f(-a) = (-a)^2 + 3 = a^2 + 3 = f(a).$$

**• PROBLEM 171**

Find the domain D and the range R of the function $\left(x, \frac{x}{|x|}\right)$.

Solution: Note that the y-value of any coordinate pair $(x,y)$ is $\frac{x}{|x|}$. We can replace x in the formula $\frac{x}{|x|}$ with any number except 0, since the denominator, $|x|$ , can not equal 0, (i.e. $|x| \neq 0$) which is equivalent to $x \neq 0$. This is because division by 0 is undefined. Therefore, the domain D is the set of all real numbers except 0. If x is negative, i.e. $x < 0$, then $|x| = -x$ by definition. Hence, if x is negative, then $\frac{x}{|x|} = \frac{x}{-x} = -1$. If x is positive, i.e. $x > 0$, then $|x| = x$ by definition. Hence, if x is positive, then $\frac{x}{|x|} = \frac{x}{x} = 1$. (The case where $x = 0$ has already been found to be undefined). Thus, there are only two numbers $-1$ and 1 in the range R of the function; that is, $R = \{-1, 1\}$.

**• PROBLEM 172**

Describe the domain and range of the function $f = (x,y) | y = \sqrt{9 - x^2}\}$ if x and y are real numbers.

Solution: In determining the domain we are interested in the values of x which yield a real value for y. Since the square root of a negative number is not a real number, the domain is restricted to those values of x which make the radicand positive or zero. Therefore $x^2$ cannot exceed 9 which means that x cannot exceed 3 or be less than $-3$. A convenient way to express this is to write $-3 \leq x \leq 3$, which is read "x is greater than or equal to $-3$ and less than or equal to 3." This is the domain of the function the range is the set of values that y can assume. To determine the range of the function we note that the largest value of y occurs when $x = 0$. Then $y = \sqrt{9 - 0} = 3$. Likewise, the smallest value of y occurs when $x = 3$ or $x = -3$. Then $y = \sqrt{9 - 9} = 0$. Since this is an inclusive interval of the real axis, the range of y is $0 \leq y \leq 3$.

• **PROBLEM 173**

Find the set of ordered pairs $\{(x,y)\}$ if $y = x^2 - 2x - 3$ and $D = \{x \mid x \text{ is an integer and } 1 \le x \le 4\}$.

Solution: We first note that $D = \{1,2,3,4\}$. Substituting these values of x in the equation

$$y = x^2 - 2x - 3,$$

we find the corresponding y values. Thus,

$$\text{for } x = 1,\ y = 1^2 - 2(1) - 3 = 1 - 2 - 3 = -4$$

$$\text{for } x = 2,\ y = 2^2 - 2(2) - 3 = 4 - 4 - 3 = -3,$$

$$\text{for } x = 3,\ y = 3^2 - 2(3) - 3 = 9 - 6 - 3 = 0, \quad \text{and}$$

$$\text{for } x = 4,\ y = 4^2 - 2(4) - 3 = 16 - 8 - 3 = 5.$$

Hence $\{(x,y)\} = \{(1,-4), (2,-3), (3,0), (4,5)\}$.

• **PROBLEM 174**

If $f(x) = 3x + 4$ and $D = \{x \mid -1 \le x \le 3\}$, find the range of $f(x)$.

Solution: We first prove that the value of $3x + 4$ increases when x increases. If $X > x$, then we may multiply both sides of the inequality by a positive number to obtain an equivalent inequality. Thus, $3X > 3x$. We may also add a number to both sides of the inequality to obtain an equivalent inequality. Thus

$$3X + 4 > 3x + 4.$$

Hence, if x belongs to D, the function value $f(x) = 3x + 4$ is least when $x = -1$ and greatest when $x = 3$. Consequently, since $f(-1) = -3 + 4 = 1$ and $f(3) = 9 + 4 = 13$, the range is all y from 1 to 13; that is,

$$R = \{y \mid 1 \le y \le 13\}.$$

• **PROBLEM 175**

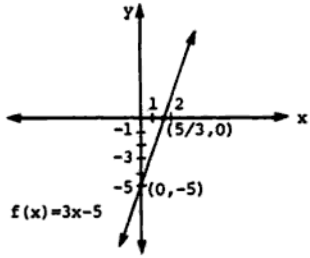
Find the zeros of the function f if $f(x) = 3x - 5$.

Solution: The zeros of the function $f(x) = 3x - 5$ are those values of x for which $3x - 5 = 0$:

$$3x - 5 = 0$$

$$3x = 5$$

$$x = \frac{5}{3}$$




Thus $x = 5/3$ is a zero of $f(x) = 3x - 5$, which means that the graph of $f(x)$ crosses the x axis at the point $(5/3, 0)$ (see figure).

**• PROBLEM 176**

Find the zeros of the function

$$\frac{2x+7}{5} + \frac{3x-5}{4} + \frac{33}{10}.$$

Solution: Let the function $f(x)$ be equal to $\frac{2x+7}{5} + \frac{3x-5}{4} + \frac{33}{10}$.
A number, a, is a zero of a function $f(x)$ if $f(a) = 0$. A zero of $f(x)$ is a root of the equation $f(x) = 0$. Thus, the zeros of the function are the roots of the equation

$$\frac{2x+7}{5} + \frac{3x-5}{4} + \frac{33}{10} = 0.$$

The least common denominator, LCD, of the denominators of 5, 4, and 10 is 20. This is a fractional equation which can be solved by multiplying both members of the equation by the LCD.

$$20\left(\frac{2x+7}{5} + \frac{3x-5}{4} + \frac{33}{10}\right) = (20)(0)$$

$$4(2x+7) + 5(3x-5) + (2\cdot 33) = 0.$$

Distributing,

$$8x + 28 + 15x - 25 + 66 = 0.$$

$$23x + 69 = 0$$

$$23x = -69$$

$$x = -3$$

Hence $x = -3$ is the zero of the given function.

**• PROBLEM 177**

For each of the following functions, with domain equal to the set of all whole numbers, find: (a) $f(0)$; (b) $f(1)$; (c) $f(-1)$; (d) $f(2)$; (e) $f(-2)$.

(1) $f(x) = 2x^3 - 3x + 4$

(2) $f(x) = x^2 + 1$.

**Solution:** In order to find each f(x) value, we replace x by the given value in each equation. Thus:

(1) $f(x) = 2x^3 - 3x + 4$

(a) $f(0) = 2(0)^3 - 3(0) + 4$
$= 2(0) - 0 + 4$
$= 0 + 4$
$= 4$

(b) $f(1) = 2(1)^3 - 3(1) + 4$
$= 2(1) - 3 + 4$
$= 2 - 3 + 4$
$= 3$

(c) $f(-1) = 2(-1)^3 - 3(-1) + 4$
$= 2(-1) + 3 + 4$
$= -2 + 3 + 4$
$= 5$

(d) $f(2) = 2(2)^3 - 3(2) + 4$
$= 2(8) - 6 + 4$
$= 16 - 6 + 4$
$= 14$

(e) $f(-2) = 2(-2)^3 - 3(-2) + 4$
$= 2(-8) + 6 + 4$
$= -16 + 10$
$= -6$

(2) $f(x) = x^2 + 1$

(a) $f(0) = (0)^2 + 1$
$= 0 + 1$
$= 1$

(b) $f(1) = (1)^2 + 1$
$= 1 + 1$
$= 2$

(c) $f(-1) = (-1)^2 + 1$
$= 1 + 1$
$= 2$

(d) $f(2) = (2)^2 + 1$
$= 4 + 1$
$= 5$

(e) $f(-2) = (-2)^2 + 1$

$$= 4 + 1$$

$$= 5$$

Notice that the range is contained in the set of whole numbers.

**• PROBLEM 178**

If $y = f(x) = (x^2 - 2)/(x^2 + 4)$ and $x = t + 1$, express y as a function of t.

Solution: y is given as a function of x, $y = f(x) = \dfrac{x^2 - 2}{x^2 + 4}$. To express y as a function of t, replace x by t + 1 (since $x = t + 1$) in the formula for y. Thus,

$$y = f(x) = f(t + 1) = \frac{(t + 1)^2 - 2}{(t + 1)^2 + 4}$$

$$= \frac{(t^2 + 2t + 1) - 2}{(t^2 + 2t + 1) + 4}$$

$$= \frac{t^2 + 2t - 1}{t^2 + 2t + 5}$$

$$= g(t).$$

Hence, $y = g(t)$; that is, y is now a function of t since y has been expressed in terms of t.

**• PROBLEM 179**

If $f(x) = x^2 - x - 3$, $g(x) = (x^2 - 1)/(x + 2)$, and $h(x) = f(x) + g(x)$, find $h(2)$.

Solution:

$h(x) = f(x) + g(x)$, and we are told that $f(x) = x^2 - x - 3$ and $g(x) = x^2 - 1/x + 2$; thus $h(x) = (x^2 - x - 3) + (x^2 - 1)/(x + 2)$.

To find $h(2)$, we replace x by 2 in the above formula for $h(x)$,

$$h(2) = \left[(2)^2 - 2 - 3\right] + \left(\frac{2^2 + 1}{2 + 2}\right)$$

$$= (4 - 2 - 3) + \left(\frac{4 - 1}{4}\right)$$

$$= (-1) + \left(\frac{3}{4}\right)$$

$$= -\frac{4}{4} + \frac{3}{4}$$

$$= -\frac{1}{4}.$$

Thus, $h(2) = -\frac{1}{4}$.

**• PROBLEM 180**

Let $f(x) = 2x^2$ with domain $D_f = R$ (or, alternatively, C) and $g(x) = x - 5$ with $D_g = R$ (or C). Find (a) $f + g$ (b) $f - g$ (c) $fg$ (d) $\frac{f}{g}$.

Solution: (a) $f + g$ has domain R (or C) and

$$(f + g)(x) = f(x) + g(x) = 2x^2 + x - 5$$

for each number x. For example, $(f + g)(1) = f(1) + g(1) = 2(1)^2 + 1 - 5 = 2 - 4 = -2$.

(b) $f - g$ has domain R (or C) and

$$(f - g)(x) = f(x) - g(x) = 2x^2 - (x - 5) = 2x^2 - x + 5$$

for each number x. For example, $(f - g)(1) = f(1) - g(1) = 2(1)^2 - 1 + 5 = 2 + 4 = 6$.

(c) $fg$ has domain R (or C) and

$$(fg)(x) = f(x) \cdot g(x) = 2x^2 \cdot (x - 5) = 2x^3 - 10x^2$$

for each number x. In particular, $(fg)(1) = 2(1)^3 - 10(1)^2 = 2 - 10 = -8$.

(d) $\frac{f}{g}$ has domain R (or C) excluding the number $x = 5$ (when $x = 5$, $g(x) = 0$ and division by zero is undefined) and

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{2x^2}{x-5}$$

for each number $x \neq 5$. In particular, $\left(\frac{f}{g}\right)(1) = \frac{2(1)^2}{1 - 5} = \frac{2}{-4} = -\frac{1}{2}$.

**• PROBLEM 181**

If $D = \{x \mid x$ is an integer and $-2 \leq x \leq 1\}$, find the function $\{(x, f(x)) \mid f(x) = x^3 - 3$ and x belongs to D$\}$.

Solution: $D = \{-2,-1,0,1\}$. Substituting these values of x in the equation $f(x) = x^3 - 3$, we find the corresponding $f(x)$ values. Thus,

$$f(-2) = (-2)^3 - 3 = -8 - 3 = -11$$

$$f(-1) = (-1)^3 - 3 = -1 - 3 = -4$$

$$f(0) = 0^3 - 3 = 0 - 3 = -3$$

and $$f(1) = 1^3 - 3 = 1 - 3 = -2.$$

Hence, $$f = \{(x, f(x)) \mid f(x) = x^3 - 3 \text{ and } x \text{ belongs to } D\}$$
$$= \{(-2, -11), (-1, -4), (0, -3), (1, -2)\}$$

• **PROBLEM 182**

Let $f$ be the linear function that is defined by the equation $f(x) = 3x + 2$. Find the equation that defines the inverse function $f^{-1}$.

Solution: To find the inverse function $f^{-1}$, the given equation must be solved for x in terms of y. Let $x = f^{-1}(y)$.

Solving the given equation for x:

$$y = 3x + 2, \text{ where } y = f(x).$$

Subtract 2 from both sides of this equation:

$$y - 2 = 3x + \cancel{2} - \cancel{2}$$

$$y - 2 = 3x.$$

Divide both sides of this equation by 3:

$$\frac{y - 2}{3} = \frac{3x}{3}$$

$$\frac{y - 2}{3} = x$$

or $$x = \frac{y}{3} - \frac{2}{3}$$

Hence, the inverse function $f^{-1}$ is given by::

$$x = f^{-1}(y) = \frac{y}{3} - \frac{2}{3}$$

$$\text{or } x = f^{-1}(y) = \frac{1}{3}y - \frac{2}{3}.$$

Of course, the letter that we use to denote a number in the domain of the inverse function is of no importance whatsoever, so this last equation can be rewritten $f^{-1}(u) = \frac{1}{3}u - \frac{2}{3}$, or $f^{-1}(s) = \frac{1}{3}s - \frac{2}{3}$, and it will still define the same function $f^{-1}$.

**• PROBLEM 183**

Show that the inverse of the function $y = x^2 + 4x - 5$ is not a function.

Solution: Given the function $f$ such that no two of its ordered pairs have the same second element, the inverse function $f^{-1}$ is the set of ordered pairs obtained from $f$ by interchanging in each ordered pair the first and second elements. Thus, the inverse of the function $y = x^2 + 4x - 5$ is $x = y^2 + 4y - 5$.

The given function has more than one first component corresponding to a given second component. For example, if $y = 0$, then $x = -5$ or 1. If the elements (-5,0) and (1,0) are reversed, we have (0,-5) and (0,1) as elements of the inverse. Since the first component 0 has more than one second component, the inverse is not a function (a function can have only one $y$ value corresponding to each $x$ value).

# CHAPTER 10

# SOLVING LINEAR EQUATIONS

**Basic Attacks and Strategies for Solving Problems in this Chapter.** **See pages 105 to 129 for step-by-step solutions to problems.**

When solving a linear equation, the object is to simplify the equation using one or more of the four fundamental operations such that the equation has as its final form

**variable = constant.**

The constant is the solution.

**TYPE I:** Linear equations with the unknown in the numerator.

This category of linear equations consists of a number of general forms and variations. Three of these forms are the equations

$ax + b = c$, $ax + b = dx + c$, and $a(x + e) = f$.

When solving an equation of form

$$ax + b = c,$$

where $a \neq 0$, the first step is to add the opposite of the constant $b$ to each side of the equation and simplify. If coefficient $a = 1$, then the final equation is in the form *variable = constant* which is the solution. On the other hand, if coefficient $a \neq 1$ and $a \neq 0$, then the next step is to remove the coefficient from the variable by multiplying each side of the equation by the reciprocal of the coefficient or dividing each side of the equation by the coefficient and simplify. If coefficient $a$ and other constant terms in the equation involve fractions, then it may be easier to multiply both sides of the equation by the LCD and simplify. In any case, the result is the form *variable = constant* which is the solution.

To solve an equation of form

$$ax + b = dx + c,$$

where $a$ and $d$ are not zero, the beginning strategy is to rewrite the equation so that only one variable term exists in the equation. This is done by adding the opposite of the variable term on the right-hand side to both sides of the equation and simplifying. With only one variable term, the next step is to add the opposite of the constant term on the left-hand side of the equation to both sides and simplify so that only one constant term remains. Finally, multiply each side of the equation by the reciprocal of the coefficient of the variable and simplify. The result is the equation *variable = constant* which is the solution.

When solving an equation involving one or more sets of parentheses, (e.g.,

$$a(x + e) = f,$$

where $a$ and $e$ are not zero), the first step in the solution procedure is to apply the distributive property to remove the parentheses. The remainder of the solution procedure is as indicated for the equation of the form

$$ax + b = c.$$

**TYPE II:** Linear equations with the unknown in the numerator and/or denominator.

A typical form of Type II linear equations is as follows:

$$\frac{ax+b}{cx+d} = \frac{e}{fx+g},$$

where $a$, $c$ and constant terms are not zero. The beginning strategy for this and other variations of the equation is to remove or eliminate the denominators by multiplying both sides of the equation by the LCD and simplify. The resulting equation will conform to one of the forms in Type I linear equations. Thus, the remainder of the solution procedure follows what is indicated for Type I equations. However, a critical final step in the procedure must be observed. It is necessary to check the final constant value of the variable by substituting it in the original equation to determine if it really is the solution.

**TYPE III:** Linear equations with the unknown under a radical sign.

The property of squaring both sides of an equation is used as the first step in the solution procedure of this type of linear equation. When the result is simplified, this step should have eliminated all radical signs from the variable in the equation. However, if all of the radical signs are not removed from the variable, then usually it is necessary to rewrite the equation by transposing appropriately and again squaring both sides of the equation and simplify. Once the radical signs are removed, then the remainder of the solution procedure follows what is given above in Type I equations. It is necessary to check the final result in the original equation to determine if it is a solution.

# Step-by-Step Solutions to Problems in this Chapter, "Solving Linear Equations"

## UNKNOWN IN NUMERATOR

**• PROBLEM 184**

Solve $3x - 5 = 4$ for x.

Solution: Since this equation is to be solved for x, place the term x on one side of the equation, $3x - 5 = 4$. Add 5 to both sides of the equation.

$$3x - 5 + 5 = 4 + 5 \tag{1}$$

Since $-5 + 5 = 0$ and $4 + 5 = 9$, Equation (1) reduces to:

$$3x = 9. \tag{2}$$

Since it is desired to get the term x on one side of the equation, divide both sides of Equation (2) by 3.

$$\frac{3x}{3} = \frac{9}{3}. \tag{3}$$

Since $\frac{3x}{3}$ reduces to 1x and since $\frac{9}{3}$ reduces to 3, Equation(3) becomes: $1x = 3$. Since $1x = x$, $x = 3$. Therefore, the equation has been solved for x.

Check: By substituting $x = 3$ into the original equation we have

$$3(3) - 5 \stackrel{?}{=} 4$$

$$9 - 5 \stackrel{?}{=} 4$$

$$4 = 4.$$

Note that, upon substitution of the solution into the original equation, the equation is reduced to the identity $4 = 4$.

**• PROBLEM 185**

Solve the equation $6x - 3 = 7 + 5x$.

Solution: To solve for x in the equation $6x - 3 = 7 + 5x$, we wish to obtain an equivalent equation in which each term in one member involves x, and each term in the other member is a constant. If we add $(-5x)$ to both members, then only one side of the equation will have an x term:

$$6x - 3 + (-5x) = 7 + 5x + (-5x)$$

$$6x + (-5x) - 3 = 7 + 0$$

$$x - 3 = 7$$

Now, adding 3 to both sides of the equation we obtain,

$$x - 3 + 3 = 7 + 3$$

$$x + 0 = 10$$

$$x = 10$$

Thus, our solution is $x = 10$. Now we check this value.

Check: Substitute 10 for x in the original equation:

$$6x - 3 = 7 + 5x$$

$$6(10) - 3 = 7 + 5(10)$$

$$60 - 3 = 7 + 50$$

$$57 = 57.$$

**• PROBLEM 186**

Solve the equation $4x - 5 = x + 7$.

<u>Solution:</u> The problem here is to list the elements in the set

$$S = \{x \mid 4x - 5 = x + 7\}$$

To find these elements, use the additive principle and the multiplicative principle. By using these principles, convert the description of $S$ to the form $S = \{x \mid x = \ldots\}$.

The computation can be arranged in the following manner:

| | |
|---|---|
| $4x - 5 = x + 7$ | Original equation |
| $4x - 5 + 5 = x + 7 + 5$ | Adding 5 to both sides |
| $4x = x + 12$ | |
| $4x + (-x) = x + 12 + (-x)$ | Adding $-x$ to both sides |
| $3x = 12$ | |
| $\frac{1}{3}(3x) = \frac{1}{3}(12)$ | Multiplying both sides by $\frac{1}{3}$ |
| $x = 4$ | |

Hence the solution set is $S = \{x \mid x = 4\} = \{4\}$.

**• PROBLEM 187**

Solve, justifying each step. $3x - 8 = 7x + 8$.

| | |
|---|---|
| <u>Solution:</u> | $3x - 8 = 7x + 8$ |
| Adding 8 to both members, | $3x - 8 + 8 = 7x + 8 + 8$ |
| Additive inverse property, | $3x + 0 = 7x + 16$ |
| Additive identity property, | $3x = 7x + 16$ |
| Adding $(-7x)$ to both members, | $3x - 7x = 7x + 16 - 7x$ |
| Commuting, | $-4x = 7x - 7x + 16$ |
| Additive inverse property, | $-4x = 0 + 16$ |
| Additive identity property, | $-4x = 16$ |

Dividing both sides by -4, $x = \frac{16}{-4}$

$x = -4$

Check: Replacing x by -4 in the original equation:

$$\begin{aligned} 3x - 8 &= 7x + 8 \\ 3(-4) - 8 &= 7(-4) + 8 \\ -12 - 8 &= -28 + 8 \\ -20 &= -20 \end{aligned}$$

• **PROBLEM 188**

Solve for x: $2x + 5 = 7 - x.$

<u>Solution:</u> Add x to both sides: $2x + 5 + x = 7 - x + x$

Combine terms: $3x + 5 = 7$

Subtract 5 from both sides: $3x + 5 - 5 = 7 - 5$

Combine terms: $3x = 2$

Divide both sides by 3: $\frac{3x}{3} = \frac{2}{3}$

$x = \frac{2}{3}$

• **PROBLEM 189**

Solve for x: $7x - 3 = 2(x + 3).$

<u>Solution:</u>

$7x - 3 = 2(x + 3)$

Distributing,

$7x - 3 = 2x + 6$

Adding (-2x) to both sides, $7x - 3 - 2x = 6$

Combining terms, $5x - 3 = 6$

Adding 3 to both sides, $5x = 6 + 3$

Combining terms, $5x = 9$

Dividing both sides by 5, $x = \frac{9}{5}$

Check: Replacing x by $\frac{9}{5}$ in the original equation,

$$\begin{aligned} 7x - 3 &= 2(x + 3) \\ 7\left(\frac{9}{5}\right) - 3 &= 2\left(\frac{9}{5} + 3\right) \\ \frac{63}{5} - \frac{15}{5} &= \frac{18}{5} + 6 \\ \frac{48}{5} &= \frac{18}{5} + \frac{30}{5} \\ \frac{48}{5} &= \frac{48}{5} \end{aligned}$$

Solve each equation (find the solution set), and check each solution.
(a) $4(6x + 5) - 3(x - 5) = 0$. (b) $8 + 3x = -4(x - 2)$.

Solution: (a) $4(6x + 5) - 3(x - 5) = 0$

distributing, $24x + 20 - 3x + 15 = 0$

combining like terms,
$$21x + 35 = 0$$

adding (-35) to both sides,
$$21x = -35$$

dividing both sides by 21,
$$x = \frac{-35}{21} = \frac{-5}{3}$$

Therefore the solution set to this equation is $\left\{\frac{-5}{3}\right\}$.

Check: Replace x by $\frac{-5}{3}$ in the equation,

$$4(6x + 5) - 3(x - 5) = 0$$
$$4\left[6\left(\frac{-5}{3}\right) + 5\right] - 3\left[\frac{-5}{3} - 5\right] = 0$$
$$4\left(\frac{-30}{3} + 5\right) - 3\left(\frac{-5}{3} - \frac{15}{3}\right) = 0$$
$$4(-10 + 5) - 3\left(\frac{-20}{3}\right) = 0$$
$$4(-5) + 20 = 0$$
$$-20 + 20 = 0$$
$$0 = 0$$

(b) $8 + 3x = -4(x - 2)$

distributing, $8 + 3x = -4x + 8$

adding 4x to both sides,
$$8 + 7x = 8$$

adding (-8) to both sides,
$$7x = 0$$

dividing both sides by 7,
$$x = 0$$

Therefore the solution set to this equation is $\{0\}$, (not to be confused with the null set $\{\ \}$).

Check: Replace x by 0 in the equation,

$$8 + 3x = -4(x - 2)$$
$$8 + 3(0) = -4(0 - 2)$$
$$8 + 0 = -4(-2)$$
$$8 = 8$$



Solve the equation $2(x + 3) = (3x + 5) - (x - 5)$.

Solution: We transform the given equation to an equivalent equation where we can easily recognize the solution set.

$$2(x + 3) = 3x + 5 - (x - 5)$$

$$\text{Distribute, } 2x + 6 = 3x + 5 - x + 5$$

$$\text{Combine terms, } 2x + 6 = 2x + 10$$

Subtract 2x from both sides, $6 = 10$

Since $6 = 10$ is not a true statement, there is no real number which will make the original equation true. The equation is inconsistent and the solution set is $\phi$, the empty set.

**• PROBLEM 192**

Solve the equation $\frac{3}{4}x + \frac{7}{8} + 1 = 0$.

Solution: There are several ways to proceed. First we observe that $\frac{3}{4}x + \frac{7}{8} + 1 = 0$ is equivalent to

$\frac{3}{4}x + \frac{7}{8} + \frac{8}{8} = 0$, where we have converted 1 into $\frac{8}{8}$. Now, combining fractions we obtain:

$$\frac{3}{4}x + \frac{15}{8} = 0$$

Subtract $\frac{15}{8}$ from both sides:

$$\frac{3}{4}x = \frac{-15}{8}$$

Multiplying both sides by $\frac{4}{3}$:

$$\left(\frac{4}{3}\right) \frac{3}{4}x = \left(\frac{4}{3}\right)\left(\frac{-15}{8}\right)$$

Cancelling like terms in numerator and denominator:

$$x = \frac{-5}{2}$$

A second method is to multiply both sides of the equation by the least common denominator, 8:

$$8\left(\frac{3}{4}x + \frac{7}{8} + 1\right) = 8(0)$$

Distributing: $8\left(\frac{3}{4}\right)x + 8\left(\frac{7}{8}\right) + 8 \cdot 1 = 0$

$$(2 \cdot 3)x + 7 + 8 = 0$$

$$6x + 15 = 0$$

Subtract 15 from both sides: $6x = -15$

Divide both sides by 6: $x = \frac{-15}{6}$

Cancelling 3 from numerator and denominator: $x = \frac{-5}{2}$

**• PROBLEM 193**

Solve the equation $2\left(\frac{2}{3}y + 5\right) + 2(y + 5) = 130.$

Solution: The procedure for solving this equation is as follows:

$\frac{4}{3}y + 10 + 2y + 10 = 130,$ Distributive property

$\frac{4}{3}y + 2y + 20 = 130,$ Combining like terms

$\frac{4}{3}y + 2y = 110,$ Subtracting 20 from both sides

$\frac{4}{3}y + \frac{6}{3}y = 110,$ Converting 2y into a fraction with denominator 3

$\frac{10}{3}y = 110,$ Combining like terms

$y = 110 \cdot \frac{3}{10} = 33,$ Dividing by $\frac{10}{3}$

Check: Replace y by 33 in our original equation,

$$2\left(\frac{2}{3}(33) + 5\right) + 2(33 + 5) = 130$$

$$2(22 + 5) + 2(38) = 130$$

$$2(27) + 76 = 130$$

$$54 + 76 = 130$$

$$130 = 130$$

Therefore the solution to the given equation is $y=33$.

**• PROBLEM 194**

Solve the equation

$$\frac{1}{2}x + \frac{2}{3} = \frac{1}{4}x - \frac{1}{6}$$

Solution: Since 2, 3, 4, and 6, the denominators of the fractions, are all factors of 12, and there is no smaller number which contains 2, 3, 4, and 6 as factors, 12 is the least common multiple (LCM). We may therefore multiply both sides of the given equation by 12 to eliminate the fractions.

$$\left(\frac{1}{2}x + \frac{2}{3}\right)12 = \left(\frac{1}{4}x - \frac{1}{6}\right)12$$

Distribute, $\left(\frac{1}{2}x\right)12 + \left(\frac{2}{3}\right)12 = \left(\frac{1}{4}x\right)12 - \left(\frac{1}{6}\right)(12)$

$$6x + 8 = 3x - 2$$

Add (- 3x) to both sides,

$$6x + 8 + (-3x) = 3x - 2 + (-3x)$$

$$6x + (-3x) + 8 = 3x + (-3x) - 2$$

$$3x + 8 = -2$$

Add (- 8) to both sides

$$3x + 8 + (-8) = -2 + (-8)$$

$$3x = -10$$

Divide both sides by 3, $x = -\frac{10}{3}$

Thus, the solution is x = - 10/3, and we have

$\left\{x \middle| \frac{1}{2}x + \frac{2}{3} = \frac{1}{4}x - \frac{1}{6}\right\} = \left\{\frac{-10}{3}\right\}$ To verify this statement we perform the following check:

Check: Replace x by - 10/3 in the original equation,

$$\frac{1}{2}x + \frac{2}{3} = \frac{1}{4}x - \frac{1}{6}$$

$$\frac{1}{2}\left(\frac{-10}{3}\right) + \frac{2}{3} = \frac{1}{4}\left(\frac{-10}{3}\right) - \frac{1}{6}$$

$$\frac{-10}{6} + \frac{2}{3} = \frac{-10}{12} - \frac{1}{6}$$

Convert each fraction into a fraction whose denominator is 12. Here we are using the fact that 12 is the least common denominator (this is an alternative method to multiplying both members by the LCM 12). Thus

$$\frac{2}{2}\left(\frac{-10}{6}\right) + \frac{4}{4}\left(\frac{2}{3}\right) = \frac{-10}{12} - \frac{2}{2}\left(\frac{1}{6}\right)$$

$$\frac{-20}{12} + \frac{8}{12} = \frac{-10}{12} - \frac{2}{12}$$

$$\frac{-12}{12} = \frac{-12}{12}$$

$$-1 = -1$$

Since substitution of x by (- 10/3) results in this equivalent equation which is always true, - 10/3 is indeed a root of the equation.

**• PROBLEM 195**

Solve for x:

$$\frac{x}{2} + \frac{x}{3} = 12.$$

**Solution:** The Least Common Denominator is 6. Multiply both members of the equation by 6: $6\left(\frac{x}{2}+\frac{x}{3}\right) = 6(12).$

Use distributive law: $3x + 2x = 72.$

Collect terms: $5x = 72.$

Divide by 5: Therefore, $x = 14\frac{2}{5}.$

Check: Substitute $14\frac{2}{5} = \frac{72}{5}$ for $x$ in the given equation:

$$\frac{\frac{72}{5}}{2} + \frac{\frac{72}{5}}{3} = 12$$

$$\left(\frac{72}{5} \cdot \frac{1}{2}\right) + \left(\frac{72}{5} \cdot \frac{1}{3}\right) = 12$$

$$\frac{36}{5} + \frac{24}{5} = 12$$

$$\frac{60}{5} = 12$$

$$12 = 12$$

**• PROBLEM 196**

Find the set indicated by
$\left\{x \mid \frac{1}{2}x - \frac{2}{3} = \frac{3}{4}x + \frac{1}{12}\right\}$

**Solution:** The set indicated by $\left\{x \mid \frac{1}{2}x - \frac{2}{3} = \frac{3}{4}x + \frac{1}{12}\right\}$ is the set of all x such that x makes the statement

$$\frac{1}{2}x - \frac{2}{3} = \frac{3}{4}x + \frac{1}{12}$$

true. Hence to obtain the required set, we must solve the equation

$$\frac{1}{2}x - \frac{2}{3} = \frac{3}{4}x + \frac{1}{12}.$$

Since 2, 3, 4, and 12, the denominators of the fractions are all factors of 12, we may multiply both sides of the equation by 12 to eliminate the fractions. Therefore, 12 is called the least common multiple (LCM). Thus,

$$12\left(\frac{1}{2}x - \frac{2}{3}\right) = 12\left(\frac{3}{4}x + \frac{1}{12}\right)$$

Distribute, $12\left(\frac{1}{2}x\right) - 12\left(\frac{2}{3}\right) = 12\left(\frac{3}{4}x\right) + 12\left(\frac{1}{12}\right)$

$$6x - 8 = 9x + 1$$

Add $(-9x)$ to both sides, $6x - 8 + (-9x) = 9x + 1 + (-9x)$

commute, $6x + (-9x) - 8 = 9x + (-9x) + 1$

$$-3x - 8 = 1$$

Add 8 to both sides, $-3x - 8 + 8 = 1 + 8$

$$-3x = 9$$

Divide both sides by - 3 to obtain,

$$x = -3.$$

Thus, our solution is $x = -3$, and the set indicated by $\left\{x \mid \frac{1}{2}x - \frac{2}{3} = \frac{3}{4}x + \frac{1}{12}\right\}$ is $\{x \mid x = -3\}$. Now, we check this solution.

Check: Substitute (- 3) for x in our original equation,

$$\frac{1}{2}x - \frac{2}{3} = \frac{3}{4}x + \frac{1}{12}$$

$$\left(\frac{1}{2}\right)(-3) - \frac{2}{3} = \frac{3}{4}(-3) + \frac{1}{12}$$

$$\frac{-3}{2} - \frac{2}{3} = \frac{-9}{4} + \frac{1}{12}$$

Convert each fraction into a fraction whose denominator is 12. Here we are using the fact that 12 is the least common denominator (we could also multiply both members by the LCM 12 as before). Thus,

$$\frac{6}{6}\left(\frac{-3}{2}\right) - \frac{4}{4}\left(\frac{2}{3}\right) = \frac{3}{3}\left(\frac{-9}{4}\right) + \frac{1}{12}$$

$$\frac{-18}{12} - \frac{8}{12} = \frac{-27}{12} + \frac{1}{12}$$

$$\frac{-26}{12} = \frac{-26}{12}$$

Since substitution for x by (- 3) results in this equivalent equation, which is always true, (- 3) is indeed a root of the equation.

**• PROBLEM 197**

Solve the equation $\frac{3}{2}x - \frac{2}{3} = 2x + 1$.

Solution: Subtract $\frac{3}{2}x$ from both sides of the given equation:

$$\cancel{\frac{3}{2}x} - \frac{2}{3} - \cancel{\frac{3}{2}x} = 2x + 1 - \frac{3}{2}x$$

$$-\frac{2}{3} = \frac{4}{2}x - \frac{3}{2}x + 1$$

$$-\frac{2}{3} = \frac{1}{2}x + 1$$

$$-\frac{2}{3} = \frac{x}{2} + 1$$

Subtract 1 from both sides of this equation:

$$-\frac{2}{3} - 1 = \frac{x}{2} + \cancel{1} - \cancel{1}$$

$$-\frac{2}{3} - \frac{3}{3} = \frac{x}{2}$$

$$-\frac{5}{3} = \frac{x}{2}$$

Multiply both sides of this equation by 2:

$$2\left(-\frac{5}{3}\right) = \cancel{2}\left(\frac{x}{\cancel{2}}\right)$$

$$-\frac{10}{3} = x$$

Thus the solution set of our given equation is the set $\left\{-\frac{10}{3}\right\}$.

• **PROBLEM 198**

Solve $\quad \frac{1}{2x} - \frac{5}{16} = \frac{1}{x}$ .

**Solution:** In order to rid an equation of fractions we multiply both sides by the least common multiple. In this case our L.C.M. is 16x:

$$16x\left(\frac{1}{2x} - \frac{5}{16}\right) = 16x\left(\frac{1}{x}\right)$$

Distributing, $\quad 16x\left(\frac{1}{2x}\right) - 16x\left(\frac{5}{16}\right) = \frac{16x}{x}$

Cancelling out like terms in numerator and denominator:

$$8 - 5x = 16$$

Subtracting 8 from both sides:

$$-5x = 16 - 8$$

$$-5x = 8$$

Dividing both sides by $-5$:

$$x = -\frac{8}{5}$$

Check:

Substitute $\frac{-8}{5}$ for x in $\frac{1}{2x} - \frac{5}{16} = \frac{1}{x}$ :

$$\frac{1}{2\left(-\frac{8}{5}\right)} - \frac{5}{16} = \frac{1}{\left(-\frac{8}{5}\right)}$$

$$\frac{1}{\left(\frac{-16}{5}\right)} - \frac{5}{16} = \frac{1}{-\frac{8}{5}}$$

Since division by a fraction is equivalent to multiplication by the reciprocal

$$\frac{1}{\left(\frac{-16}{5}\right)} = 1 \cdot \left(\frac{5}{-16}\right) = \frac{-5}{16}$$

and $$\frac{1}{\left(\frac{-8}{5}\right)} = 1 \cdot \left(\frac{5}{-8}\right) = \frac{-5}{8}$$

Hence, $$\frac{-5}{16} - \frac{5}{16} = \frac{-5}{8}$$

$$\frac{-10}{16} = \frac{-5}{8}$$

$$\frac{-5}{8} = \frac{-5}{8}$$

**• PROBLEM 199**

> Solve $A = \frac{h}{2}(b + B)$ for h.

__Solution:__ Since the given equation is to be solved for h, obtain h on one side of the equation. Multiply both sides of the equation $A = \frac{h}{2}(b + B)$ by 2. Then, we have:

$$2(A) = 2\left(\frac{h}{2}(b + B)\right).$$

Therefore: $$2(A) = \frac{\cancel{2}h}{\cancel{2}}(b + B)$$

$$2A = h(b + B). \qquad (1)$$

Since it is desired to obtain h on one side of the equation, divide both sides of equation (1) by (b + B).

$$\frac{2A}{(b + B)} = \frac{h\cancel{(b + B)}}{\cancel{(b + B)}}.$$

Therefore: $$\frac{2A}{b + B} = h.$$

Thus, the given equation, $A = \frac{h}{2}(b + B)$, is solved for h.

This is the form of the formula used to determine values of h for a set of trapezoids, if the area and lengths of the bases are known.

**• PROBLEM 200**

> Solve $\frac{1}{R} = \frac{1}{a} + \frac{1}{b}$ for a.

__Solution:__ To solve for a we must obtain a alone on one side of the equation,

$$\frac{1}{R} = \frac{1}{a} + \frac{1}{b}. \qquad (1)$$

To do this we proceed as follows: Multiply Equation (1) by Rab. Then,

$$Rab\left(\frac{1}{R}\right) = Rab\left(\frac{1}{a} + \frac{1}{b}\right).$$

Therefore: $\frac{\cancel{R}ab}{\cancel{R}} = \frac{R\cancel{a}b}{\cancel{a}} + \frac{Ra\cancel{b}}{\cancel{b}}.$

Therefore: $ab = Rb + Ra.$ (2)

Subtracting Ra from both sides of Equation (2), we obtain:

$$ab - Ra = Rb + Ra - Ra.$$

Therefore: $ab - Ra = Rb.$ (3)

We can now factor a from both terms of the left side of Equation (3), obtaining:

$$a(b - R) = Rb. \quad (4)$$

Now, we divide both sides of Equation (4) by (b - R):

$$\frac{a\cancel{(b - R)}}{\cancel{(b - R)}} = \frac{Rb}{(b - R)} .$$

Thus, we find $a = \frac{Rb}{b - R}$ .

**• PROBLEM 201**

> Solve the equation $a(x + b) = bx + c$ for $x$ if $a \neq b$.

<u>Solution:</u>

| | |
|---|---|
| $ax + ab = bx + c$ | Distributive property |
| $ax + ab + (-bx) = bx + c + (-bx)$ | adding (-bx) to both sides |
| $ax + (-bx) + ab = bx + (-bx) + c$ | commutative law of addition |
| $ax + (-bx) + ab = 0 + c$ | additive inverse property |
| $ax + (-bx) + ab = c$ | additive identity property |
| $ax + (-bx) + ab + (-ab) = c + (-ab)$ | adding (-ab) to both sides |
| $ax + (-bx) + 0 = c + (-ab)$ | additive inverse property |
| $ax - bx = c - ab$ | additive identity property |
| $(a - b)x = c - ab$ | factoring out x |
| $x = \frac{c - ab}{a - b}$ if $a \neq b$ | Dividing by (a - b) |

If $a = b$ the denominator of this fraction is zero, and thus the fraction has no meaning.

**• PROBLEM 202**

> Find a solution of the equation
> $$3x + 4y + 5z = 13 \quad (1)$$

<u>Solution:</u> The above equation is linear in x, y, and z. Any ordered triple (x, y, z) which satisfies it is a solution. If we chose x = 2, and y = 3, by substitution

$$6 + 12 + 5z = 13$$
$$z = -1$$

The one solution of Equation 1 is $x = 2$, $y = 3$, and $z = -1$.

**Obviously, the number of solutions is unlimited, since any choice of values for two of the variables will determine the value of the third variable.**

## UNKNOWN IN NUMERATOR AND / OR DENOMINATOR

**• PROBLEM 203**

Find the solutions of the equation $\frac{4x - 7}{x - 2} = 3 + \frac{1}{x - 2}$.

Solution: Assume that there is a number $x$ such that

$$\frac{4x - 7}{x - 2} = 3 + \frac{1}{x - 2}$$

In order to eliminate the fractions multiply both sides of the equation by $x - 2$ to obtain

$$(x - 2)\ \frac{4x - 7}{x - 2} = \left(3 + \frac{1}{x - 2}\right)(x - 2)$$

Thus

$$4x - 7 = 3(x - 2) + \frac{x - 2}{x - 2}$$

$$4x - 7 = 3(x - 2) + 1$$

$$4x - 7 = 3x - 6 + 1$$

$$4x - 7 = 3x - 5$$

Add $(-3x)$ to both sides, $\quad 4x - 7 + (-3x) = -5$

$$x - 7 = -5$$

Add 7 to both sides, $\quad x = -5 + 7$

and hence $x = 2$.

We have shown that if $x$ is a solution of the equation

$$\frac{4x - 7}{x - 2} = 3 + \frac{1}{x - 2},$$

then $x = 2$. But if we substitute $x = 2$ in the right-hand member of the equation we obtain

$$3 + \frac{1}{0}$$

and we know that we cannot divide by zero. Hence 2 is not a solution.

Before we analyze the process which led to the conclusions that 2 was a possible solution to our equation, let us see exactly why our equation has no solution. To do this, we note that

$$3 + \frac{1}{x - 2} = 3 \cdot \frac{x - 2}{x - 2} + \frac{1}{x - 2} = \frac{3(x - 2) + 1}{x - 2} = \frac{3x - 6 + 1}{x - 2} = \frac{3x - 5}{x - 2}$$

and hence that the original equation is equivalent to

$$\frac{4x - 7}{x - 2} = \frac{3x - 5}{x - 2} \qquad (1)$$

Now we know that two fractions, $\frac{a}{b}$ and $\frac{c}{d}$ are equal if and only if $ad = bc$. Thus (1) holds, providing that $x \neq 2$, if and only if

$$(x - 2)(4x - 7) = (x - 2)(3x - 5) \qquad (2)$$

holds. But, since $x \neq 2$, $x - 2 \neq 0$, and we can divide both sides of (2) by $x - 2$ and have

$$4x - 7 = 3x - 5$$

which gives $x = 2$, a contradiction. In other words, the only possible solution is a number which we knew in advance could not be a solution, and hence there are no solutions to our given equation.

**• PROBLEM 204**

Solve the equation

$$\frac{x}{x + 1} + \frac{5}{8} = \frac{5}{2(x + 1)} + \frac{3}{4}$$

Solution: Since $(x + 1)$, 8, $2(x + 1)$, and 4, the denominators of the fractions, are all factors of $8(x + 1)$, and there is no smaller number which contains $(x + 1)$, 8, $2(x + 1)$, and 4 as factors, $8(x + 1)$ is the least common multiple (LCM). We may therefore multiply both sides of the given equation by $8(x + 1)$ to eliminate the fractions.

$$8(x + 1)\left(\frac{x}{x + 1} + \frac{5}{8}\right) = 8(x + 1)\left(\frac{5}{2(x + 1)} + \frac{3}{4}\right).$$

Distribute,

$$8(x + 1)\left(\frac{x}{x + 1}\right) + 8(x + 1)\left(\frac{5}{8}\right)$$

$$= 8(x + 1)\left(\frac{5}{2(x + 1)}\right) + 8(x + 1)\left(\frac{3}{4}\right)$$

Cancel like terms, $8x + 5(x + 1) = 4(5) + 6(x + 1)$

Distribute, $8x + 5x + 5 = 20 + 6x + 6$

Combine terms, $13x + 5 = 26 + 6x$

Add $(-6x)$ to both sides

$$13x + 5 + (-6x) = 26 + 6x + (-6x)$$

$$7x + 5 = 26$$

Add $(-5)$ to both sides

$$7x + 5 + (-5) = 26 + (-5)$$

$$7x = 21$$

Divide both sides by 7, $x = 3$.

Thus, our solution is $x = 3$, and we have

$$\left\{x \middle| \frac{x}{x+1} + \frac{5}{8} = \frac{5}{2(x+1)} + \frac{3}{4}\right\} = \{3\}.$$

To verify this statement we perform the following check.

Check: Replace x by 3 in the original equation,

$$\frac{x}{x+1} + \frac{5}{8} = \frac{5}{2(x+1)} + \frac{3}{4}$$

$$\frac{3}{3+1} + \frac{5}{8} = \frac{5}{2(3+1)} + \frac{3}{4}$$

$$\frac{3}{4} + \frac{5}{8} = \frac{5}{2(4)} + \frac{3}{4}$$

$$\frac{3}{4} + \frac{5}{8} = \frac{5}{8} + \frac{3}{4}$$

$$\frac{3}{4} + \frac{5}{8} = \frac{3}{4} + \frac{5}{8}$$

Since substitution of x by 3 results in this equivalent equation, which is always true, 3 is indeed a root of the equation.

**• PROBLEM 205**

Solve the equation

$$\frac{5}{x-1} + \frac{1}{4-3x} = \frac{3}{6x-8} \quad .$$

Solution: By factoring out a common factor of -2 from the denominator of the term on the right side of the given equation, the given equation becomes:

$$\frac{5}{x-1} + \frac{1}{4-3x} = \frac{3}{-2(-3x+4)} = \frac{3}{-2(4-3x)} = -\frac{3}{2(4-3x)}$$

Hence,

$$\frac{5}{x-1} + \frac{1}{4-3x} = -\frac{3}{2(4-3x)}$$

Adding $\frac{3}{2(4-3x)}$ to both sides of this equation:

$$\frac{5}{x-1} + \frac{1}{4-3x} + \frac{3}{2(4-3x)} = 0. \qquad (1)$$

Now, in order to combine the fractions, the least common denominator (l.c.d.) must be found. The l.c.d. is found in the following way: list all the different factors of the denominators of the fractions. The exponent to be used for each factor in the l.c.d. is the greatest value of the exponent for each factor in any denominator. Therefore, the l.c.d. of the given fractions is:

$$2^1(x-1)^1(4-3x)^1 = 2(x-1)(4-3x)$$

Hence, equation (1) becomes:

$$\frac{(2)(4-3x)(5)}{(2)(4-3x)(x-1)} + \frac{(2)(x-1)(1)}{(2)(x-1)(4-3x)} + \frac{(x-1)(3)}{(x-1)(2)(4-3x)} = 0 \quad (2)$$

Simplifying equation (2):

$$\frac{10(4 - 3x) + 2(x - 1) + 3(x - 1)}{2(x - 1)(4 - 3x)} = 0$$

$$\frac{40 - 30x + 2x - 2 + 3x - 3}{2(x - 1)(4 - 3x)} = 0$$

$$\frac{-25x + 35}{2(x - 1)(4 - 3x)} = 0$$

Multiplying both sides of this equation by $2(x - 1)(4 - 3x)$:

$$2\cancel{(x - 1)(4 - 3x)}\frac{-25x + 35}{2\cancel{(x - 1)(4 - 3x)}} = 2(x - 1)(4 - 3x)(0)$$

$$-25x + 35 = 0$$

Adding 25x to both sides of this equation:

$$\cancel{-25x} + 35 + \cancel{25x} = 0 + 25x$$

$$35 = 25x$$

Dividing both sides of this equation by 25:

$$\frac{35}{25} = \frac{\cancel{25}x}{\cancel{25}}$$

$$\frac{7}{5} = x$$

Therefore, the solution set to the equation $\frac{5}{x-1} + \frac{1}{4-3x} = \frac{3}{6x-8}$ is: $\left\{\frac{7}{5}\right\}$.

**• PROBLEM 206**

Find $\left\{x \middle| \frac{2}{x+1} - 3 = \frac{4x+6}{x+1}\right\}$.

Solution: The required set is the set of all $x$ such that

$$\frac{2}{x+1} - 3 = \frac{4x+6}{x+1}$$

Multiplying each member by $(x + 1)$ to eliminate the fractions, we obtain

$$\left(x + 1\right)\left(\frac{2}{x+1} - 3\right) = \left(\frac{4x+6}{x+1}\right)\left(x + 1\right)$$

Distributing, $(x + 1)\left(\frac{2}{x+1}\right) - (x + 1)3 = 4x + 6$

$$2 - (3x + 3) = 4x + 6$$

$$2 - 3x - 3 = 4x + 6$$

$$-1 - 3x = 4x + 6$$

Adding $(-4x)$ to both sides,

$$-1 - 3x - 4x = 6$$

$$-1 - 7x = 6$$

Adding 1 to both sides,

$$-7x = 7$$
$$x = -1$$

If we now substitute (-1) for x in our original equation,

$$\frac{2}{x+1} - 3 = \frac{4x+6}{x+1}$$

$$\frac{2}{-1+1} - 3 = \frac{4(-1)+6}{-1+1}$$

$$\frac{2}{0} - 3 = \frac{-4+6}{0}$$

Since division by zero is impossible the above equation is not defined for $x = -1$. Hence we conclude that the equation has no roots and

$$\left\{x \,\middle|\, \frac{2}{x+1} - 3 = \frac{4x+6}{x+1}\right\} = \emptyset \text{, where } \emptyset \text{ is the empty set.}$$

**• PROBLEM 207**

Solve

$$\frac{3}{x-1} + \frac{1}{x-2} = \frac{5}{(x-1)(x-2)}.$$

Solution: First we will eliminate the fractions by finding the least common denominator, LCD. This is done by multiplying the denominators and taking the highest power of each factor which appears, only once.

$$(x-1)(x-2)(x-1)(x-2)$$

$$\text{LCD} = (x-1)(x-2)$$

Multiplying both sides of the equation by the LCD will remove the fractions and give:

$$(x-1)(x-2)\left(\frac{3}{x-1} + \frac{1}{x-2}\right)$$

$$= \left(\frac{5}{(x-1)(x-2)}\right)(x-1)(x-2)$$

$$3(x-2) + (x-1) = 5$$

$$3x - 6 + x - 1 = 5$$

$$4x - 7 = 5$$

$$4x = 12$$

$$x = 3$$

Substituting $x = 3$ into the original equation

$$\frac{3}{2} + 1 = \frac{5}{(2)(1)}$$

$$\frac{5}{2} = \frac{5}{2}$$

we find $x = 3$ satisfies the original equation.

**• PROBLEM 208**

Solve $\dfrac{3}{x - 1} + \dfrac{2}{x + 1} = \dfrac{6}{x^2 - 1}$.

Solution: First we obtain the Least Common Denominator, LCD, by multiplying the denominators,

$$(x - 1)(x + 1)(x^2 - 1), \text{ or}$$

$(x - 1)(x + 1)[(x - 1)(x + 1)]$, and taking each factor's highest power once.

$$\text{LCD} = (x - 1)(x + 1) = (x^2 - 1) \qquad (1)$$

Then multiply both sides of the equation by the LCD to remove the fractions and obtain:

$$(x - 1)(x + 1)\left(\frac{3}{x - 1} + \frac{2}{x + 1}\right)$$

$$= \left(\frac{6}{x^2 - 1}\right)(x - 1)(x + 1)$$

$$3(x + 1) + 2(x - 1) = 6 \qquad (2)$$

$$3x + 3 + 2x - 2 = 6$$

$$5x + 1 = 6$$

$$5x = 5$$

$$x = 1$$

Substituting $x = 1$ into Equation 2, we can readily see that it is a solution of that equation. However, $x = 1$ is not an admissible value of $x$ for Equation 1, because division by 0 is undefined; therefore, $x = 1$ is not a solution of Equation 1. It is an extraneous root that was introduced by the multiplication by the LCD. The original equation does not have a solution.

**• PROBLEM 209**

Solve the equation

$$\frac{2x}{3 + x} + \frac{3 + x}{3} = 2 + \frac{x^2}{3(x - 3)}.$$

Solution: In order to eliminate the fractions in this equation we multiply both members of the equation by the Least Common Denominator (the LCD). Our denominators are $(3 + x)$, $3$, and $3(x - 3)$. Thus the LCD is $3(3 + x)(x - 3)$. Therefore

$$[3(3 + x)(x - 3)]\left[\frac{2x}{3 + x} + \frac{3 + x}{3}\right] = [3(3 + x)(x - 3)]\left[2 + \frac{x^2}{3(x-3)}\right].$$

Distribute:

$$[3(3 + x)(x - 3)]\left[\frac{2x}{3 + x}\right] + [3(3 + x)(x - 3)]\left[\frac{3 + x}{3}\right]$$
$$= [3(3 + x)(x - 3)](2) + [3(3 + x)(x - 3)]\left[\frac{x^2}{3(x - 3)}\right].$$

Cancelling like terms in numerator and denominator,

$$3(x - 3)(2x) + (3 + x)(x - 3)(3 + x) = 3\cdot 2(3 + x)(x - 3) + (3 + x)x^2$$

Factoring both sides of the equation,

$$6x(x - 3) + (9 + 6x + x^2)(x - 3) = 6(3x + x^2 - 9 - 3x) + (3x^2 + x^3)$$
$$6x(x - 3) + (9 + 6x + x^2)(x - 3) = 6(x^2 - 9) + (3x^2 + x^3)$$

Distributing the two terms on the left side and the one term on the right side of this equation,

$$(6x^2 - 18x) + (9x + 6x^2 + x^3) - (27 + 18x + 3x^2) = (6x^2 - 54) + (3x^2 + x^3)$$

Grouping terms and simplifying,

$$6x^2 - 18x + 9x + 6x^2 + x^3 - 27 - 18x - 3x^2 = 6x^2 - 54 + 3x^2 + x^3$$
$$x^3 + 9x^2 - 27x - 27 = x^3 + 9x^2 - 54$$

Subtract $x^3$ from both sides of this equation:

$$x^3 + 9x^2 - 27x - 27 - x^3 = x^3 + 9x^2 - 54 - x^3$$
$$9x^2 - 27x - 27 = 9x^2 - 54$$

Subtract $9x^2$ from both sides of this equation:

$$9x^2 - 27x - 27 - 9x^2 = 9x^2 - 54 - 9x^2$$
$$-27x - 27 = -54$$

Add $27$ to both sides of this equation:

$$-27x - 27 + 27 = -54 + 27$$
$$27x = -27$$

Divide both sides of this equation by $-27$:

$$\frac{-27x}{-27} = \frac{-27}{-27}$$

Therefore, $x = 1$.

To verify that $x = 1$ is the solution to our problem, we perform the following check.

Check: Replace $x$ by $1$ in our original equation,

$$\frac{2x}{3 + x} + \frac{3 + x}{3} = 2 + \frac{x^2}{3(x - 3)}$$

$$\frac{2(1)}{3 + 1} + \frac{3 + 1}{3} = 2 + \frac{(1)^2}{3(1 - 3)}$$

$$\frac{2}{4} + \frac{4}{3} = 2 + \frac{1}{3(-2)}$$

$$\frac{2}{4} + \frac{4}{3} = 2 + \frac{1}{-6}$$

$$\frac{2}{4} + \frac{4}{3} = 2 - \frac{1}{6}$$

Multiplying both members by LCD, 12:

$$12\left(\frac{2}{4} + \frac{4}{3}\right) = 12\left(2 - \frac{1}{6}\right)$$

$$12\left(\frac{2}{4}\right) + 12\left(\frac{4}{3}\right) = 12(2) - 12\left(\frac{1}{6}\right)$$

$$6 + 16 = 24 - 2$$

$$22 = 22$$

Hence, $x = 1$ is our solution, and our solution set is $\{1\}$.

## UNKNOWN UNDER RADICAL SIGN

**• PROBLEM 210**

Solve $\sqrt{x - 3} = 4$.

**Solution:** Square both sides of the given equation to obtain:

$$\left(\sqrt{x - 3}\right)^2 = 4^2$$

Note $(\sqrt{a})^2 = \sqrt{a} \cdot \sqrt{a} = \sqrt{a \cdot a} = \sqrt{a^2} = a$; thus $\left(\sqrt{x - 3}\right)^2 = x - 3$, and we obtain:

$$x - 3 = 16$$

$$x = 19$$

Check: Substitute 19 for x in the original equation,

$$\sqrt{x - 3} = 4$$

$$\sqrt{19 - 3} = 4$$

$$\sqrt{16} = 4$$

$$4 = 4$$

**• PROBLEM 211**

Solve the equation $\sqrt{3x + 1} = 5$.

**Solution:** Square both members:

$$3x + 1 = 25.$$

Solve for x: $x = 8$.

Check: $\sqrt{3(8) + 1} = \sqrt{25} = 5$.

It should be recalled that $\sqrt{25} = +5$, and does not equal $+5$; that is, when no sign precedes the radical the positive value of the root is to be taken. If both positive and negative roots are meant, we shall write both signs before the radical.

**• PROBLEM 212**

Solve $\sqrt{5}x + \sqrt{3} = \sqrt{3}x - \sqrt{5}$ .

Solution: Add $-\sqrt{3x}$ to both sides of the given equation,

$$\sqrt{5}x + \sqrt{3} - \sqrt{3}x = \sqrt{3}x - \sqrt{5} - \sqrt{3}x \ .$$

Commute terms and add $-\sqrt{3}$ to both sides,

$$\sqrt{5}x - \sqrt{3}x + \sqrt{3} - \sqrt{3} = \sqrt{3}x - \sqrt{3}x - \sqrt{5} - \sqrt{3}$$

$$\sqrt{5}x - \sqrt{3}x = -\sqrt{5} - \sqrt{3} \ .$$

Using the distributive law,

$$x(\sqrt{5} - \sqrt{3}) = -\sqrt{5} - \sqrt{3}$$

$$x = \frac{-\sqrt{5} - \sqrt{3}}{\sqrt{5} - \sqrt{3}} \ .$$

Factor $(-1)$ from the numerator:

$$x = \frac{-(\sqrt{5} + \sqrt{3})}{\sqrt{5} - \sqrt{3}}$$

To rationalize the denominator, we multiply numerator and denominator by the conjugate $(\sqrt{5} + \sqrt{3})$ of the denominator; hence:

$$x = \frac{-(\sqrt{5} + \sqrt{3})(\sqrt{5} + \sqrt{3})}{(\sqrt{5} - \sqrt{3})(\sqrt{5} + \sqrt{3})}$$

$$x = \frac{-(\sqrt{5}\sqrt{5} + 2\sqrt{3}\sqrt{5} + \sqrt{3}\sqrt{3})}{\sqrt{5}\cdot\sqrt{5} + \sqrt{5}\sqrt{3} - \sqrt{5}\sqrt{3} - \sqrt{3}\sqrt{3}}$$

Note $\sqrt{a}\cdot\sqrt{b} = \sqrt{ab}$ , therefore:

$$x = \frac{-(\sqrt{25} + 2\sqrt{15} + \sqrt{9})}{\sqrt{25} - \sqrt{9}}$$

$$x = \frac{-(5 + 2\sqrt{15} + 3)}{5 - 3}$$

$$x = \frac{-(8 + 2\sqrt{15})}{2}$$

$$x = \frac{-2(4 + \sqrt{15})}{2}$$

$$x = -(4 + \sqrt{15})$$

$$x = -4 - \sqrt{15}$$

**• PROBLEM 213**

Solve the equation

$$\sqrt{x} = 7 + \sqrt{x - 7}$$

**Solution:** Squaring both sides of the given equation,

$$x = 49 + 14\sqrt{x-7} + x - 7$$

Simplifying

$$-42 = 14\sqrt{x-7}$$

$$-3 = \sqrt{x-7} \qquad (1)$$

Squaring both sides of equation (1),

$$9 = x - 7$$

$$x = 16$$

Checking the root by substitution in the given equation:

$$\sqrt{16} \neq 7 + \sqrt{16-7}$$

$$4 \neq 7 + 3$$

Clearly $x = 16$ does not satisfy the given equation, and therefore the equation has no roots. The fact that the given equation has no roots could have been anticipated from equation (1), $-3 = \sqrt{x-7}$, since the positive root is indicated in the original equation.

**• PROBLEM 214**

Solve $\sqrt{2}x - 2 = 2x - \sqrt{2}$.

**Solution:** Add $(-2x)$ to both sides of the given equation:

$$\sqrt{2}x - 2 - 2x = -\sqrt{2}$$

Now, add 2 to both sides:

$$\sqrt{2}x - 2x = 2 - \sqrt{2}$$

Use the distributive law:

$$x(\sqrt{2} - 2) = 2 - \sqrt{2}$$

$$x = \frac{2 - \sqrt{2}}{\sqrt{2} - 2}$$

Multiply both sides by $(-1)$:

$$-x = \frac{-(2 - \sqrt{2})}{\sqrt{2} - 2}$$

$$-x = \frac{-2 + \sqrt{2}}{-2 + \sqrt{2}}$$

$$-x = 1$$

$$x = -1$$

Solve $\sqrt{4x + 5} + 2\sqrt{x - 3} = 17$.

Solution: Transpose:

$$\sqrt{4x + 5} - 17 = -2\sqrt{x - 3}.$$

Square: $4x + 5 - 34\sqrt{4x + 5} + 289 = 4(x - 3)$

$$4x + 5 - 34\sqrt{4x + 5} + 289 = 4x - 12.$$

Transpose: $-34\sqrt{4x + 5} = 4x - 12 - 4x - 5 - 289.$

Simplify: $-34\sqrt{4x + 5} = -306$

$$\sqrt{4x + 5} = 9.$$

Square: $4x + 5 = 81.$

Solve for x: $x = 19.$

Check: $\sqrt{4(19) + 5} + 2\sqrt{(19) - 3} \stackrel{?}{=} 17$

$$\sqrt{81} + 2\sqrt{16} \stackrel{?}{=} 17$$

$$9 + 2(4) \stackrel{?}{=} 17$$

$$17 = 17.$$

Sol.: $x = 19.$



Solve $\sqrt{x - 2} - \sqrt{x + 3} = 1$.

Solution: $\sqrt{x - 2} - \sqrt{x + 3} = 1$

$\sqrt{x - 2} = \sqrt{x + 3} + 1.$ Transpose $\sqrt{x + 3}$.

$x - 2 = x + 3 + 2\sqrt{x + 3} + 1.$ Square both sides of equation.

$x - 2 - x - 3 - 1 = 2\sqrt{x + 3}.$ Transpose and combine terms.

$$-6 = 2\sqrt{x + 3}$$

$$-3 = \sqrt{x + 3}$$

$9 = x + 3.$ Square both sides of equation.

Solving gives $x = 6.$

Check: $\sqrt{6 - 2} - \sqrt{6 + 3} \stackrel{?}{=} 1$

$$\sqrt{4} - \sqrt{9} \stackrel{?}{=} 1$$

$$2 - 3 \neq 1$$
$$-1 \neq 1$$

Therefore x = 6 is not a solution. x = 6 is an extraneous root. The two expressions are not equal for any value of the unknown.

**• PROBLEM 217**

Solve the equation

$$\frac{\sqrt{x+1} + \sqrt{x-1}}{\sqrt{x+1} - \sqrt{x-1}} = \frac{4x-1}{2}.$$

Solution: We can use the following law to rewrite the given proportion: If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$. Applying this law we have:

$$\frac{\sqrt{x+1} + \sqrt{x-1} + (\sqrt{x+1} - \sqrt{x-1})}{\sqrt{x+1} + \sqrt{x-1} - (\sqrt{x+1} - \sqrt{x-1})} = \frac{4x - 1 + (2)}{4x - 1 - (2)} \quad \text{or,}$$

$$\frac{2\sqrt{x+1}}{2\sqrt{x-1}} = \frac{4x+1}{4x-3}. \text{ Eliminating } \frac{2}{2} \text{ we have:}$$

$$\frac{\sqrt{x+1}}{\sqrt{x-1}} = \frac{4x+1}{4x-3}.$$

Squaring both sides of the equation gives us,

$$\left(\frac{\sqrt{x+1}}{\sqrt{x-1}}\right)^2 = \left(\frac{4x+1}{4x-3}\right)^2 \quad \text{or} \quad \frac{(\sqrt{x+1})^2}{(\sqrt{x-1})^2} = \frac{(4x+1)^2}{(4x-3)^2}.$$

Finding the above squares we obtain:

$$\frac{x+1}{x-1} = \frac{16x^2 + 8x + 1}{16x^2 - 24x + 9}.$$

We can again rewrite this new proportion as:

$$\frac{x + 1 + (x-1)}{x + 1 - (x-1)} =$$

$$\frac{16x^2 + 8x + 1 + (16x^2 - 24x + 9)}{16x^2 + 8x + 1 - (16x^2 - 24x + 9)} \qquad \text{or,}$$

$$\frac{2x}{2} = \frac{32x^2 - 16x + 10}{32x - 8};$$

$$\text{therefore, } x = \frac{32x^2 - 16x + 10}{32x - 8} = \frac{2(16x^2 - 8x + 5)}{2(16x - 4)};$$

$$\text{thus,} \qquad x = \frac{16x^2 - 8x + 5}{16x - 4};$$

and multiplying both sides of this equation by (16x - 4) we have,

$x(16x - 4) = 16x^2 - 8x + 5$ or,

$16x^2 - 4x = 16x^2 - 8x + 5.$

Now, combining similar terms we obtain:

$16x^2 - 16x^2 - 4x + 8x = 5$ or

$$4x = 5.$$

Therefore, $x = \frac{5}{4}$ .

# CHAPTER 11

# PROPERTIES OF STRAIGHT LINES

**Basic Attacks and Strategies for Solving Problems in this Chapter. See pages 130 to 152 for step-by-step solutions to problems.**

Any equation that can be put in the standard form

$$ax + by = c,$$

where $a$, $b$, and $c$ are real numbers and $a$ and $b$ are not both 0, is defined to be a linear equation in two variables. The graph of this form is a straight line. In order to graph a linear equation in two variables, we simply graph its solution set. This means that we draw a straight line through all the points whose coordinates satisfy the equation.

Two important points on the graph of a straight line, if they exist, are the $x$- and $y$-intercepts or the points where the graph crosses the axes. The $x$-intercept of the line is the $x$-coordinate of the point that the line has in common with the $x$-axis, and the $y$-intercept is the $y$-coordinate of the point that the line has in common with the $y$-axis. Since any point on the $x$-axis has a $y$-coordinate of 0, we can find the $x$-intercept by letting $y = 0$ and solving the equation for $x$. Similarly, we can find the $y$-intercept by letting $x = 0$ and solving for $y$. For example, the $x$-intercept and $y$-intercept of the equation

$$3x + 4y = 12$$

are given by

$$3x + 4(0) = 12$$

or $x = 4$ and

$$3(0) + 4y = 12$$

or $y = 3$, respectively.

Graphing straight lines by finding the intercepts works best when the coefficients of $x$ and $y$ are factors of the constant term in the equation.

A linear equation or linear function, whose graph is a nonvertical straight line, has a slope. The slope, $m$, of a line is a measure of the steepness of the line in that it is the ratio of the vertical change (difference of ordinates) between any two points, $(x_1, y_1)$ and $(x_2, y_2)$, on the line to the horizontal change (difference of abscissas) between the two points. Thus, the most popular procedure for finding the slope of a line is given by the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1},$$

where $x_1$ is not equal to $x_2$.

When calculating the slope using this formula, it does not matter which point is designated as the "first" point and which is designated as the "second" point. The only warning is that once the points are designated then they must not be changed throughout the problem. Geometrically, a straight line that gets higher as it goes from left to right has a positive slope; a line that gets lower as it goes from left to right has a negative slope; a horizontal line has a slope of 0; and a vertical line has an undefined slope.

If a linear equation can be written in the form

$$y = mx + b,$$

then the $m$ represents the slope of the graph of the equation and the $b$ is the $y$-intercept.

Some techniques for determining the equation of a line, when given certain facts about the line, are as follows:

(1) A procedure for finding the equation of a line having a slope of $r$ and containing a point $(c, d)$ includes first using the slope formula,

$$\frac{y - y_1}{x - x_1} = m \quad \text{or} \quad y - y_1 = m(x - x_1),$$

and substituting $r$ for $m$, $d$ for $y_1$, and $c$ for $x_1$. Then write the resulting equation in the form

$$ax + by = c.$$

(2) The procedure for finding the equation of the line having a slope of $r$ and $y$-intercept of $c$ includes the use of the slope-intercept form of the equation,

$$y = mx + b.$$

Then, substitute $r$ for $m$ and $c$ for $b$ in the equation.

Of the various graphing techniques for a linear equation, the most obvious way is to calculate any two ordered pairs that satisfy the equation, plot the points, and draw a straight line through them. Depending on how the equation is arranged, there may be faster and more convenient ways to obtain the graph.

If the graph of a line is in slope-intercept form, then the graph can be obtained by marking the $y$-intercept and using the slope to move from that point to the second point on the line. One must remember that the slope is the ratio of how much the line moves up or down compared to how much the line moves right or left. If the equation of a line is in intercept form, then it is easy to mark the two intercepts and draw the straight line through them.

# Step-by-Step Solutions to Problems in this Chapter, "Properties of Straight Lines"

## SLOPES, INTERCEPTS AND POINTS ON GIVEN LINES

**• PROBLEM 218**

Find the slope of $f(x) = 3x + 4$.

Solution: Two points on the line determined by $f(x) = 3x + 4$ are $A(0,4)$ and $B(1,7)$.

$$\frac{\text{difference of ordinates}}{\text{difference of abscissas}} = \frac{7 - 4}{1 - 0} = 3$$

Note that the ordinates are the y-coordinates and the abscissas are the x-coordinates. The slope determined by points A and B is 3. Hence, the slope of $f(x) = 3x + 4$ is 3. In general, the slope of a linear function of the form $f(x) = mx + b$ is m.

**• PROBLEM 219**

Show that the slope of the segment joining (1,2) and (2,6) is equal to the slope of the segment joining (5,15) and (10,35).

Solution: The slope of the line segment, m, joining the points $(x_1,y_1)$ and $(x_2,y_2)$ is given by the formula

$$m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$

Therefore, the slope of the segment joining (1,2) and (2,6) is

$$\frac{6 - 2}{2 - 1} = \frac{4}{1} = 4.$$

The slope of the segment joining (5,15) and (10,35) is

$$\frac{35 - 15}{10 - 5} = \frac{20}{5} = 4.$$

Thus, the slopes of the two segments are equal.

**• PROBLEM 220**

Determine the constant A so that the lines $3x - 4y = 12$ and $Ax + 6y = -9$ are parallel.

Solution: If two non-vertical lines are parallel, their slopes are equal. Thus the lines $Ax + By + C = 0$ and $Ax + By + D = 0$ are parallel (since both have slope $= -A/B$). We are given two lines:

$$3x - 4y = 12 \qquad (1)$$

$$Ax + 6y = -9 \qquad (2)$$

We must make the coefficients of y the same for both equations in order to equate the coefficients of x. Multiply (1) by $-3/2$ to obtain

$$\frac{-3}{2}(3x - 4y) = -\frac{3}{2}(12)$$

$$-\frac{9}{2}x + 6y = -18 \qquad (3)$$

$$Ax + 6y = -9 \qquad (2)$$

Transpose the constant terms of (3) and (2) to the other side.

Adding 18 to both sides, $\quad \frac{-9}{2}x + 6y + 18 = 0 \qquad (4)$

Adding 9 to both sides. $\quad Ax + 6y + 9 = 0 \qquad (5)$

(4) and (5) will now be parallel if the coefficients of the x-terms are the same. Thus the constant A is $-9/2$. Then equation (5) becomes $-9/2x + 6y + 9 = 0$. We can also express (5) in its given form, $Ax + 6y = -9$ or $-9/2\,x + 6y = -9$.

We also can write it in a form that has the same coefficient of x as (1), which clearly shows that they have equal slopes.

$$3x - 4y = 12 \qquad (1)$$

$$-\frac{9}{2}x + 6y = -9$$

Multiply the second equation by $-2/3$ to obtain a coefficient of x equal to 3.

$$-\frac{2}{3}\left(\frac{-9}{2}x + 6y\right) = -\frac{2}{3}(-9)$$

$$3x - 4y = 6$$

Now equations (1), $3x - 4y = 12$, and the equation $3x - 4y = 6$ are parallel since the coefficients of x and y are identical.

**• PROBLEM 221**

Find the slope and Y-intercept of the following lines.

(a) $y = 3x - 1$ (b) $y = 1 - 4x$ (c) $2y = 4x + 7$

Solution: a) The equation of a line is: $y = mx + b$, where m is the slope of the line and b is the y-intercept of the line. Hence, the line $y = 3x - 1$ has slope $= 3$ and y-intercept $= -1$.

b) The line $y = 1 - 4x$ can be rewritten, using the commutative law, as $y = -4x + 1$. Hence, the slope of this line $= -4$ and the y-intercept $= 1$.

c) The line $2y = 4x + 7$, after dividing both sides by 2, can be rewritten as:

$$\frac{2y}{2} = \frac{4x + 7}{2}$$

$$y = 2x + \frac{7}{2}$$

Hence, the slope $= 2$ and the y-intercept $= \frac{7}{2}$.

**• PROBLEM 222**

Find the slope, the y-intercept, and the x-intercept of the equation $2x - 3y - 18 = 0$.

Solution: The equation $2x - 3y - 18 = 0$ can be written in the form of the general linear equation, $ax + by = c$.

$$2x - 3y - 18 = 0$$

$$2x - 3y = 18$$

To find the slope and y-intercept we derive them from the formula of the general linear equation $ax + by = c$. Dividing by $b$ and solving for $y$ we obtain:

$$\frac{a}{b}x + y = \frac{c}{b}$$

$$y = \frac{c}{b} - \frac{a}{b}x$$

where $-\frac{a}{b}$ = slope and $\frac{c}{b}$ = y-intercept.

To find the x-intercept, solve for $x$ and let $y = 0$:

$$x = \frac{c}{b} - \frac{b}{a}y$$

$$x = \frac{c}{a}$$

In this form we have $a = 2$, $b = -3$, and $c = 18$. Thus,

$$\text{slope} = -\frac{a}{b} = -\frac{2}{-3} = \frac{2}{3}$$

$$\text{y-intercept} = \frac{c}{b} = \frac{18}{-3} = -6$$

$$\text{x-intercept} = \frac{c}{a} = \frac{18}{2} = 9$$

**• PROBLEM 223**

The equation $F = \frac{9}{5}C + 32$ relates the Fahrenheit and centigrade temperature scales. What do the numbers $\frac{9}{5}$ and 32 represent?

Solution: An equation in the form $y = mx + b$ is a linear equation with slope $m$ and y-intercept $b$. Thus, with $F = 9/5\,C + 32$, 32 is the y-intercept and 9/5 is the slope. That is, the number 32 tells us that when the centigrade thermometer reads 0, the Fahnrenheit thermometer reads 32. The number 9/5 is the slope of the line we would obtain if we graphed our equation in an axis system in which centigrade temperatures are measured on the horizontal axis and Fahrenheit temperatures are measured on the vertical axis; that is, the number 9/5 is the number of units of Fahrenheit temperature rise per unit of centigrade temperature rise. If a body's temperature increases $1^0$C, then it increases $9/5^0$F. If a body's temperature increases $-10^0$(decreases $10^0$) C, then it increases $9/5(-10)^0 = -18^0$F.

**• PROBLEM 224**

The slope and one point of a line are given. Is the Y-intercept positive or negative?

(a) $m = \frac{22}{7}$, $(1, \pi)$ (b) $m = \sqrt{2}$, $(1, 1.414)$

Solution: a) The equation of a line is: $y = mx + b$, where $m$ is the slope and $b$ is the y-intercept. Given the slope $m$ and one point of the line, the y-intercept $b$ can be found. Thus it can be determined whether the y-intercept b is positive or negative. For the line with slope $m = 22/7$ and which contains the point $(1,\pi)$:

$$y = mx + b$$
$$\pi = \frac{22}{7}(1) + b$$
$$\pi = \frac{22}{7} + b \qquad (1)$$

Since $\pi$ is approximately $\frac{22}{7}$, equation (1) becomes:

$$\frac{22}{7} = \frac{22}{7} + b$$

Subtract 22/7 from both sides to obtain:

$$\frac{22}{7} - \frac{22}{7} = \cancel{\frac{22}{7}} + b - \cancel{\frac{22}{7}}$$
$$0 = b$$

Hence, the y-intercept b is neither positive nor negative, since the y-intercept $b = 0$.

b) For the line with slope $m = \sqrt{2}$ and which contains the point (1,1.414):

$$y = mx + b$$
$$1.414 = \sqrt{2}(1) + b$$
$$1.414 = \sqrt{2} + b \qquad (2)$$

Since $\sqrt{2}$ is approximately 1.414, equation (2) becomes:

$$1.414 = 1.414 + b$$

Subtract 1.414 from both sides to obtain:

$$1.414 - 1.414 = \cancel{1.414} + b - \cancel{1.414}$$
$$0 = b$$

Again, the y-intercept b is neither positive nor negative, since $b = 0$.

**• PROBLEM 225**

Show that the slope of the segment joining (1,2) and (3,8) is equal to the slope of the segment joining (4,11) and (8,23)

__Solution:__ The slope of the segment joining (1,2) and (3,8) is

$$\frac{8-2}{3-1} = \frac{6}{2} = 3 .$$

The slope of the segment joining (4,11) and (8,23) is

$$\frac{23-11}{8-4} = \frac{12}{4} = 3 .$$

Therefore, the slopes of the two segments are equal.

## FINDING EQUATIONS OF LINES

**• PROBLEM 226**

The two points $P_1(1,-2)$ and $P_2(4,1)$ determine a line. What is the equation of the line?

**Solution:** The slope of the line segment connecting the two points is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - (-2)}{4 - 1} = \frac{3}{3} = 1.$$

Now we know the slope and at least one point on the line. Therefore, let $P(x,y)$ be any point on the line. Then the slope between the points $P$ and $P_1$ (or, alternatively, between $P$ and $P_2$) must be 1. Therefore,

$$m = \frac{\Delta y}{\Delta x} = 1 = \frac{y - (-2)}{x - 1}$$

$$x - 1 = y + 2 \quad \text{by cross multiplying}$$

$$y = x - 3 \quad \text{by solving for } y.$$

The required equation is $y = x - 3$. Note that both of the given points satisfy this equation.

| for $P_1(1,-2)$: | for $P_2(4,1)$: |
|---|---|
| $y = x - 3$ | $y = x - 3$ |
| $-2 \stackrel{?}{=} 1 - 3$ | $1 \stackrel{?}{=} 4 - 3$ |
| $-2 = -2$ | $1 = 1.$ |

**• PROBLEM 227**

Find the equation for the line passing through (3,5) and (-1,2).

**Solution A:** We use the two-point form with $(x_1,y_1) = (3,5)$ and $(x_2,y_2) = (-1,2)$. Then

$$\frac{y - y_1}{x - x_1} = m = \frac{y_2 - y_1}{x_2 - x_1}.$$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 5}{-1 - 3} \quad \text{thus} \quad \frac{y - 5}{x - 3} = \frac{-3}{-4}.$$

Cross multiply, $-4(y - 5) = -3(x - 3)$.

Distributing, $-4y + 20 = -3x + 9$

$$3x - 4y = -11.$$

**Solution B:** Does the same equation result if we let $(x_1,y_1) = (-1,2)$ and $(x_2,y_2) = (3,5)$?

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - 2}{3 - (-1)} \quad \text{thus} \quad \frac{y - 2}{x + 1} = \frac{3}{4}.$$

Cross multiply, $4(y - 2) = 3(x + 1)$

$$3x - 4y = -11.$$

Hence, either replacement results in the same equation.

**• PROBLEM 228**

> Write the equation of the lines that contain the following points:
> (a) (1,2) and (3,4)     (b) (-2,1) and (2,3)

Solution: a) The equation of a line is: $y = mx + b$, where $m$ = slope and $b$ = y-intercept. The slope, m, of any line can be found by using the equation:

$$m = \frac{y_2 - y_1}{x_2 - x_1} ,$$

where $(x_1,y_1)$ and $(x_2,y_2)$ are two points. After the slope $m$ is found, the y-intercept $b$ can be found by substituting one point of the line and the value of the slope $m$ into the equation of the line. Hence, for the line that contains the points (1,2) and (3,4) where $(x_1,y_1) =$ (1,2) and (3,4) $= (x_2,y_2)$:

$$\text{slope}: m = \frac{4 - 2}{3 - 1}$$
$$= 2/2$$
$$= 1$$

Using $m = 1$ and the point (1,2) to find the y-intercept $b$ we have:

$$y = mx + b$$
$$2 = 1(1) + b$$
$$2 = 1 + b$$
$$1 = b$$

Therefore, the equation of the line that contains the points (1,2) and (3,4) is:

$$y = 1x + 1$$

or

$$y = x + 1 .$$

b) For the line that contains the points (-2,1) and (2,3), where $(x_1,y_1) = (-2,1)$ and $(x_2,y_2) = (2,3)$:

$$\text{slope} = m = \frac{3 - 1}{2-(-2)}$$
$$= \frac{2}{4} = \frac{1}{2}$$

Using $m = 1/2$ and the point (2,3) to find the y-intercept $b$ we have:

$$y = mx + b$$
$$3 = \frac{1}{2}(2) + b$$
$$3 = \frac{2}{2} + b$$
$$3 = 1 + b$$
$$b = 2$$

Therefore, the equation of the line that contains the points (-2,1) and (2,3) is:

$$y = \frac{1}{2} x + 2 .$$

Find the equation of the line which passes through the points (-3,1) and (7,11).

Solution: The general equation for a line is $y = mx + b$, where $m$ is the slope of the line and $b$ is the y-intercept. Replacing (-3,1) and (7,11) for $x$ and $y$ in this equation, we obtain the equations $1 = m(-3) + b$, and $11 = m(7) + b$; or:

$$1 = -3m + b \qquad (1)$$

and

$$11 = 7m + b \qquad (2)$$

Thus, we solve equations (1) and (2) for $m$ and $b$.
Subtracting equation (2) from (1):

$$\begin{array}{r} 1 = -3m + b \\ -\ \underline{(11 = 7m + b\ )} \\ -10 = -10m \\ m = 1 \end{array}$$

Replacing $m$ by $1$ in equation (1) we solve for $b$:

$$1 = (-3)(1) + b$$

$$1 = -3 + b$$
$$b = 4$$

Hence the equation of the line passing through (-3,1) and (7,11) is $y = (1)x + 4$ or $y = x + 4$.

• PROBLEM 230

(a) Find the equation of the line passing through (2,5) with slope 3.

(b) Suppose a line passes through the y-axis at (0,b). How can we write the equation if the point-slope form is used?

Solution: (a) In the point-slope form, let $x_1 = 2$, $y_1 = 5$, and $m = 3$. The point-slope form of a line is:

$$y - y_1 = m(x - x_1)$$
$$y - 5 = 3(x - 2)$$
$$y - 5 = 3x - 6 \qquad \text{Distributive property}$$
$$y = 3x - 1 \qquad \text{Transposition}$$

(b)
$$y - b = m\ (x - 0)$$
$$y = mx + b\ .$$

• PROBLEM 231

Find the equation of the line through (-1,2) and (3,1).

Solution: The equation of a line is in the form $y = mx + b$, where $m$ is the slope of the line, and $b$ is the y-intercept. Given 2 points on a line, $(x_1,y_1)$ and $(x_2,y_2)$, we can determine the slope of the line by means of the formula

$$m = \frac{(y_2 - y_1)}{(x_2 - x_1)}$$

Therefore

$$m = \frac{1 - 2}{3 - (-1)} = \frac{-1}{4}$$

[Observe that it makes no difference which point is considered as $(x_1, y_1)$.]

To determine the y-intercept, we replace (x,y) by either of the given points, and m by -1/4 in the equation $y = mx + b$. Thus, using (-1,2) and solving for b:

$$2 = -\frac{1}{4}(-1) + b$$

$$2 = \frac{1}{4} + b$$

$$b = 2 - \frac{1}{4} = \frac{8}{4} - \frac{1}{4} = \frac{7}{4}$$

Therefore, the equation of the line through (-1,2) and (3,1) is

$$y = -\frac{1}{4}x + \frac{7}{4}. \qquad (1)$$

We may multiply both members by 4 to obtain an equivalent equation to equation (1):

$$4(y) = 4\left(\frac{-1}{4}x + \frac{7}{4}\right)$$

Distributing,

$$4y = -1x + 7$$

$$4y = -x + 7$$

Adding x to both sides of this equation:

$$4y + x = -x + 7 + x \quad \text{or} \quad x + 4y = 7.$$

**• PROBLEM 232**

What is the equation of the line through the point (3,5) whose slope is 2?

Solution: Let P(x,y) be any point on this line other than (3,5). Using the definition of slope we have

$$m = \frac{\Delta y}{\Delta x} \qquad 2 = \frac{y - 5}{x - 3}$$

$$2x - 6 = y - 5 \quad \text{by cross-multiplying}$$

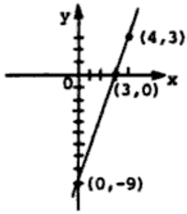
$$y = 2x - 1 \quad \text{by solving for y.}$$

This equation is satisfied by the coordinates (3,5) and represents a line with a slope of 2.

## GRAPHING TECHNIQUES

**• PROBLEM 233**

Construct the graph of the function defined by $y = 3x - 9$.




<u>Solution:</u> An equation of the form $y = mx + b$ is a linear equation; that is, the equation of a straight line.

A straight line can be determined by two points. Let us choose the intercepts. The x-intercept lies on the x-axis and the y-intercept is on the y-axis.

We find the intercepts by assigning 0 to x and solving for y and by assigning 0 to y and solving for x. It is helpful to have a third point. We find a third point by assigning 4 to x and solving for y. Thus we get the following table of corresponding numbers:

| x | $y = 3x - 9$ | y |
|---|---|---|
| 0 | $y = 3(0) - 9 = 0 - 9 =$ | -9 |
| 4 | $y = 3(4) - 9 = 12 - 9 =$ | 3 |

Solving for x to get the x-intercept:

$$y = 3x - 9$$

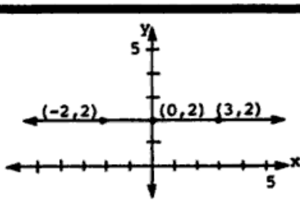
$$y + 9 = 3x$$

$$x = \frac{y + 9}{3}$$

When $y = 0$, $x = \frac{9}{3} = 3$. The three points are (0,-9), (4,3), and (3,0). Draw a line through them (see sketch).

**• PROBLEM 234**

**Graph the constant function $2y = 4$.**



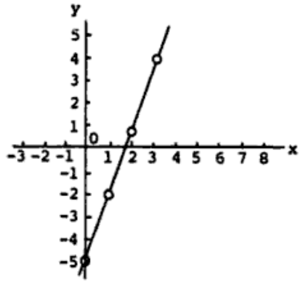

<u>Solution:</u> First rewrite $2y = 4$ in y-form. If $2y = 4$, then $y = 2$. Hence, $g = \{(x,y): y = 2\}$.

| x | -2 | 0 | 3 |
|---|---|---|---|
| y | 2 | 2 | 2 |

For all values of x, y is equal to 2. The graph of g is a straight line with slope 0 and y-intercept (0,2).

**• PROBLEM 235**

> Graph the function 3x - 5.



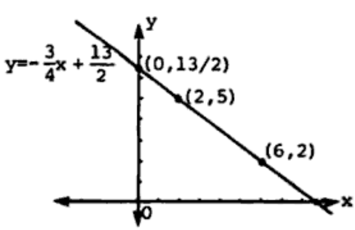

Solution: Let $y = 3x - 5$; then assign values to x and compute the corresponding values of y, the results being conveniently arranged in a table.

| x | y = 3x - 5 | y |
|---|---|---|
| 0 | $y = 3(0) - 5$ <br> $= 0 - 5$ <br> $= -5$ | -5 |
| 1 | $y = 3(1) - 5$ <br> $= 3 - 5$ <br> $= -2$ | -2 |
| 2 | $y = 3(2) - 5$ <br> $= 6 - 5$ <br> $= 1$ | 1 |
| 3 | $y = 3(3) - 5$ <br> $= 9 - 5$ <br> $= 4$ | 4 |

The various points (x,y) are then plotted and joined by a smooth curve, which turns out to be a straight line. See the accompanying figure.

**• PROBLEM 236**

> Find the equation of the line passing through the points (2,5) and (6,2). Check the results graphically.



**Solution:** The general equation for a line is $y = mx + b$, where $m$ is the slope of the line and $b$ the y-intercept. Replacing (2,5) and (6,2) for $x$ and $y$ in this equation we obtain

$$5 = m(2) + b \qquad (1)$$

and

$$2 = m(6) + b \qquad (2)$$

Thus, we solve $5 = 2m + b$ and $2 = 6m + b$ for $m$ and $b$:

subtracting equation (2) from (1):

$$\begin{array}{r} 5 = 2m + b \\ -\ \underline{(2 = 6m + b)} \\ 3 = -4m \\ m = \frac{-3}{4} \end{array}$$

Replacing $m$ by $\frac{-3}{4}$ in equation (1) we solve for $b$:

$$5 = \left(\frac{-3}{4}\right)(2) + b$$

$$5 = \frac{-6}{4} + b$$

$$b = 5 + \frac{6}{4} = \frac{20}{4} + \frac{6}{4} = \frac{26}{4} = \frac{13}{2}$$

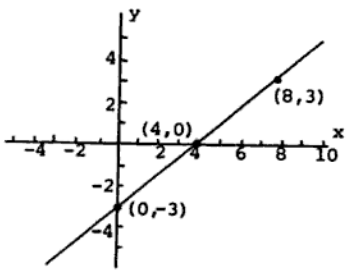
Hence, the equation of the line passing through (2,5) and (6,2) is

$$y = \frac{-3}{4}x + \frac{13}{2}.$$

This result is shown graphically in the accompanying figure.

**• PROBLEM 237**

Graph the function defined by $3x - 4y = 12$.




**Solution:** Solve for y: $3x - 4y = 12$

$$-4y = 12 - 3x$$

$$y = -3 + \frac{3}{4}x$$

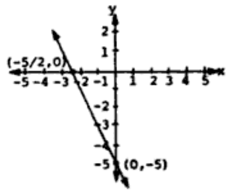
$$y = \frac{3}{4}x - 3.$$

The graph of this function is a straight line since it is of the form $y = mx + b$. The y-intercept is the point $(0, -3)$ since for $x = 0$, $y = b = -3$. The x-intercept is the point (4, 0) since for y = 0,

$x = (y + 3) \cdot \frac{4}{3} = (0 + 3) \cdot \frac{4}{3} = 4$. These two points, (0,-3) and (4,0) are sufficient to determine the graph (see the figure). A third point, (8,3), satisfying the equation of the function is plotted as a partial check of the intercepts. Note that the slope of the line is $m = \frac{3}{4}$. This means that y increases 3 units as x increases 4 units anywhere along the line.

**• PROBLEM 238**

If f(x) = -2x - 5, find the (a) slope, (b) x-intercept, and (c) y-intercept. (d) Graph the function.




**Solution:** f(x) = mx + b is called a linear function where m and b are constants. m is the slope of the line and b is the y-intercept of the line. In this case, f(x) = -2x - 5, m = -2 and b = -5. Therefore,

(a) slope: m = -2

The x-intercept is located on the x-axis where f(x) = = 0. Then we solve for x.

$$f(x) = mx + b = 0$$
$$mx = -b$$
$$x = \frac{-b}{m} = \text{x-intercept}$$

Hence,

(b) x-intercept: $\frac{-b}{m} = \frac{-(-5)}{-2} = \frac{5}{-2} = \frac{-5}{2}$

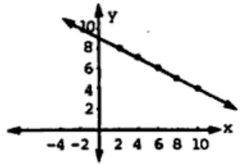
(c) y-intercept: b = -5

(d) We can graph the function by locating the points where the graph crosses the y-axis, (0, -5), and the x-axis, $\left(-\frac{5}{2}, 0\right)$. Recall again that (0,b) is the y-intercept and that $\left(\frac{-b}{m}, 0\right)$ is the x-intercept.

**• PROBLEM 239**

Graph the function $y = 9 - \frac{x}{2}$.

**Solution:** Writing the function as $y = -\frac{1}{2}x + 9$, we see that




it is a straight line since it is of the form $y = mx + b$. The y-intercept of the graph is $(0,b)$, i.e., $(0,9)$. The slope of the line is $m = -\frac{1}{2}$. This means that y decreases 1 unit as x increases 2 units anywhere along the line. To see this, choose even values of x in the interval $0 \leq x \leq 10$. Determine the ordered pairs listed in the following table using the equation $y = 9 - \frac{x}{2}$.

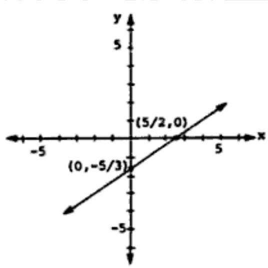
| x | 0 | 2 | 4 | 6 | 8 | 10 |
|---|---|---|---|---|---|---|
| y | 9 | 8 | 7 | 6 | 5 | 4 |

Plotting these points, as illustrated in the figure, the graph is a straight line passing through the first, second and fourth quadrants. Although we have plotted a limited number of points, the coordinates of each point on the line will satisfy the equation $y = 9 - \frac{x}{2}$ and, conversely, each ordered pair which satisfies the equation will determine a point on the line.

**• PROBLEM 240**

Write the equation in slope-intercept form; specify the slope and the y-intercept of the line. Sketch the graph of the equation.

$$2x - 3y = 5$$




<u>Solution:</u> The equation $y = mx + b$ is the slope-intercept form of a line in which m is the slope of the line and b is the y-intercept of the line. If the given equation is solved for y, then the slope-intercept form of a line will be obtained. Thus, adding 3y to both sides of the given equation:

$$2x - \cancel{3y} + \cancel{3y} = 5 + 3y$$

$$2x = 5 + 3y$$

Subtracting 5 from both sides of this equation:

$$2x - 5 = \cancel{5} + 3y - \cancel{5}$$

$$2x - 5 = 3y$$

Dividing both sides by 3:

$$\frac{2x - 5}{3} = \frac{3y}{3}$$

$$\frac{2x - 5}{3} = y$$

$$\frac{2}{3}x - \frac{5}{3} = y \qquad \text{or} \qquad y = \frac{2}{3}x - \frac{5}{3} \qquad (1)$$

Equation (1) is an equation in the slope-intercept form of a line. Comparing equation (1) with the slope-intercept form, $y = mx + b$, slope $= m = \frac{2}{3}$ and y - intercept $= b = -\frac{5}{3}$. To graph this equation it is sufficient to find one more point besides the y - intercept $\left(0, -\frac{5}{3}\right)$ which lies on the line (since two points determine a line). We can find the x - intercept by solving for x in the given equation, and use this as our second point on the line. Thus,

$$2x - 3y = 5$$

$$2x = 3y + 5$$

$$x = \frac{3}{2}y + \frac{5}{2},$$

and $\frac{5}{2}$ is the x - intercept. Thus, the point $\left(\frac{5}{2}, 0\right)$ is on the line (see figure).

• **PROBLEM 241**

The following table was constructed by reading the coordinates of selected points on a graphed line. Determine the equation of the line.

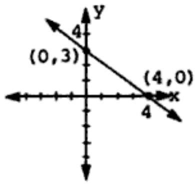
| x | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|
| y | 5 | 3 | 1 | -1 | -3 | -5 |

<u>Solution:</u> The equation of the line is of the form $y = mx + b$ where m represents the slope and b represents the y-intercept.

For each interval in the table, as x increases 1 unit, y decreases 2 units. Therefore, the slope of the line connecting these points is $m = \frac{\Delta y}{\Delta x} = \frac{-2}{1} = -2$. The y-intercept is given as (0,1) since for x = 0, y = 1 as given in the table. The required equation is one which represents a straight line with a slope of -2 and y-intercept of 1, that is, $y = -2x + 1$. It can be verified that each listed ordered pair will satisfy this equation.

**• PROBLEM 242**

Use the intercept form to graph $3x + 4y = 12$.




Solution: To find the x intercept substitute y = 0 in the expression and solve for x.

$$3x + 0 = 12$$
$$3x = 12$$
$$x = 4$$

The x intercept is (4,0).

Find the y intercept by substituting x = 0 in the expression and solve for y.

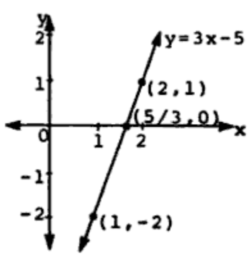
$$0 + 4y = 12$$
$$4y = 12$$
$$y = 3$$

The y intercept is (0,3).

Locate the points (4,0) and (0,3) and join them with the required straight line. See the figure.

**• PROBLEM 243**

a) Find the zeros of the function f if $f(x) = 3x - 5$.
b) Sketch the graph of the equation $y = 3x - 5$.



**Solution:** a) The zeros of a function are the numbers for which the value of the function is 0. Therefore, let $f(x) = 3x - 5 = 0$. Solving this equation:

$$3x - 5 = 0$$

Add 5 to both sides of this equation.

$$3x - 5 + 5 = 0 + 5$$

$$3x = 5$$

Divide both sides of this equation by 3.

$$\frac{3x}{3} = \frac{5}{3}$$

$$x = \frac{5}{3}$$

This number is the only zero of f (see the graph of f in the figure).

b) Note that the equation of a line is: $y = mx + b$ where m is the slope of the line and b is the y-intercept. Since the given equation is in this form, the graph is a line. It is only necessary to find two points of the graph in order to draw it. Let $x = 1$. Then $f(x) = f(1) = 3(1) - 5 = 3 - 5 = -2$. Hence, one point is $(1,-2)$. Let $x = 2$. Then $f(x) = f(2) = 3(2) - 5 = 6 - 5 = 1$. Therefore, $(2,1)$ is the other point. These two points determine the straight line shown in the figure.

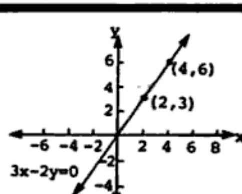
**• PROBLEM 244**

Discuss the graph of the function $y = -3x + 4$.

**Solution:** The graph is a straight line since it is of the form $y = mx + b$. The line intersects the y-axis at the point $(0,4)$. That is, when $x = 0$ then $y = 4$. The y-intercept in this example corresponds to $b = 4$. The slope of the line is $m = -3$. This means that y decreases 3 units as x increases 1 unit, anywhere along the line.

**• PROBLEM 245**

Are the following points on the graph of the equation $3x-2y=0$? (a) point $(2,3)$? (b) point $(3,2)$? (c) point $(4,6)$?




**Solution:** The point $(a,b)$ lies on the graph of the equation $3x - 2y = 0$ if replacement of x and y by a and b, respectively, in the given equation results in an equation which is true.

(a) Replacing (x,y) by (2,3):

$$3x - 2y = 0$$
$$3(2) - 2(3) = 0$$
$$6 - 6 = 0$$
$$0 = 0, \text{ which is true.}$$

Therefore (2,3) is a point on the graph.

(b) Replacing (x,y) by (3,2):

$$3x - 2y = 0$$
$$3(3) - 2(2) = 0$$
$$9 - 4 = 0$$
$$5 = 0, \text{ which is not true.}$$

Therefore (3,2) is not a point on the graph.

(c) Replacing (x,y) by (4,6):

$$3x - 2y = 0$$
$$3(4) - 2(6) = 0$$
$$12 - 12 = 0$$
$$0 = 0, \text{ which is true.}$$

Therefore (4,6) is a point on the graph.

This problem may also be solved geometrically as follows: draw the graph of the line $3x - 2y = 0$ on the coordinate axes. This can be done by solving for y:

$$3x - 2y = 0$$
$$-2y = -3x$$
$$y = \frac{-3}{-2}x = \frac{3}{2}x,$$

and plotting the points shown in the following table:

| x | $y = \frac{3}{2}x$ |
|---|---|
| 0 | 0 |
| 1 | $3/2 = 1\frac{1}{2}$ |
| 2 | 3 |
| -2 | -3 |

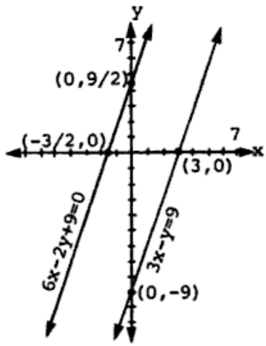
(See accompanying figure.)

Observe that we obtain the same result as in our algebraic solution. The points (2,3) and (4,6) lie on the line $3x - 2y = 0$, whereas (3,2) does not.

**• PROBLEM 246**

Show that the graphs of $3x - y = 9$ and $6x - 2y + 9 = 0$ are parallel lines.

**Solution:** If the slopes of two lines are equal, the lines




are parallel. Thus we must show that the two slopes are equal. In standard form the equation of a line is $y = mx + b$, where m is the slope.

Putting $3x - y = 9$ in standard form,

$$-y = 9 - 3x$$

$$y = -9 + 3x$$

$$y = 3x - 9.$$

Thus the slope of the first line is 3. Putting $6x - 2y + 9 = 0$ in standard form,

$$-2y + 9 = -6x$$

$$-2y = -6x - 9$$

$$y = 3x + \frac{9}{2}.$$

Thus the slope of this line is also 3. The slopes are equal. Hence, the lines are parallel.

To graph these equations pick values of x and substitute them into the equation to determine the corresponding values of y. Thus we obtain the following tables of values. Notice we need only <u>two</u> points to plot a line (2 points determine a line).

$6x - 2y + 9 = 0$ $\qquad$ $3x - y = 9$

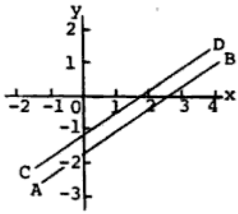
$y = 3x + \frac{9}{2}$ $\qquad$ $y = 3x - 9$

| | | |
|---|---|---|
| x | 0 | $-\frac{3}{2}$ |
| y | $\frac{9}{2}$ | 0 |

| | | |
|---|---|---|
| x | 0 | 3 |
| y | -9 | 0 |

(See accompanying figure)

Determine whether there is a point of intersection of the graphs of $2x - 3y = 5$ and $6x - 9y = 10$.




Solution: Geometric discussion. Rewriting the given linear equations in standard form, $y = mx + b$, the slope, m, can be read directly.

$$2x - 3y = 5 \qquad\qquad 6x - 9y = 10$$
$$-3y = 5 - 2x \qquad\qquad -9y = 10 - 6x$$
$$y = \frac{2}{3}x - \frac{5}{3} \qquad\qquad y = \frac{6}{9}x - \frac{10}{9}$$
$$m = \frac{2}{3} \qquad\qquad m = \frac{6}{9} = \frac{2}{3}$$

Recall that if the slope, m, of two lines are equal, the lines are parallel. This can be seen in the figure.

The graph of the first equation is the line AB through the point $(1, -1)$ with slope $\frac{2}{3}$, and the graph of the second equation is the line CD through the point $\left(\frac{2}{3}, -\frac{2}{3}\right)$, with slope $\frac{2}{3}$. The lines are parallel, hence there is no point of intersection and the equations are inconsistent.

Algebraic discussion. If the members of the first equation are multiplied by 3, and if the members of the resulting equation are subtracted from the corresponding members of the second equation, we obtain

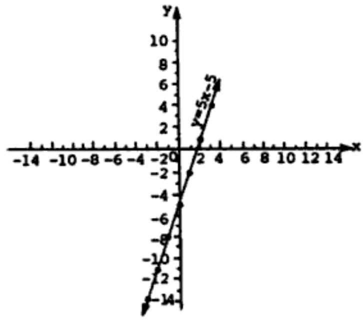
$$3(2x - 3y) = 3(5)$$
$$6x - 9y = 15$$

$$\text{Then,} \quad \begin{array}{r} 6x - 9y = 10 \\ -(6x - 9y = 15) \\ \hline 0 = -5, \end{array}$$ which is impossible.

The steps that were taken were based on the assumption that the given equations had a solution. The fact that an impossible conclusion results proves that the assumption was false. In other words, the two equations have no common solution, and are therefore inconsistent.

**• PROBLEM 248**

Find the point of intersection of the graphs of $3x - y = 5$ and $9x - 3y = 15$.




**Solution:** (1) $3x - y = 5$

(2) $9x - 3y = 15$

Divide the second equation by 3 to obtain:

(1) $3x - y = 5$

Thus any pair of values $(x,y)$ which satisfies the first equation also satisfies the second equation. Hence the same straight line is the graph of both equations. It follows that there is no unique solution, but rather that every point on the common line is a solution. The two equations are dependent.

To solve the pair of dependent equations algebraically it is sufficient to assign an arbitrary value to $x$ (or $y$), and then to solve for $y$ (or $x$) in either equation.

$$3x - y = 5$$

$$-y = 5 - 3x \qquad \text{Add } -3x \text{ to both sides.}$$

$$y = 3x - 5 \qquad \text{Multiply by } -1.$$

To graph this equation, choose values of $x$ and obtain their corresponding values of $y$ from $y = 3x - 5$. The following table is constructed.

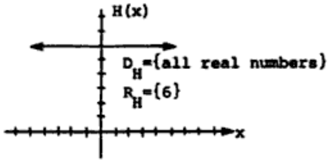
| | | | | | | | |
|---|---|---|---|---|---|---|---|
| x | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
| y | -14 | -11 | -8 | -5 | -2 | 1 | 4 |

We then plot the points found in the table and draw a smooth curve (which turns out to be a straight line) through them.

**• PROBLEM 249**

Sketch the graph of the function $H = \{(x,H(x)) | H(x) = 6\}$.

**Solution:** $H(x) = 6$ can be expressed in the form $H(x) = mx + b$, for in this particular example $m = 0$; hence $H(x) = 6$ can be written as $H(x) = 0 \cdot x + 6$. From this, regardless of the choice of a value for $x$,




the corresponding value for H(x) will be 6. When there is no domain set given, it is taken to be the largest subset of the real numbers for which the corresponding H(x) value is also real. Hence the domain of H is {all real numbers} and the range is {6}. The graph of H is a horizontal line, i.e., has slope = m = 0 and H-intercept = b = 6. The graph is sketched in the figure.

**• PROBLEM 250**

Given the function $f$ defined by the equation

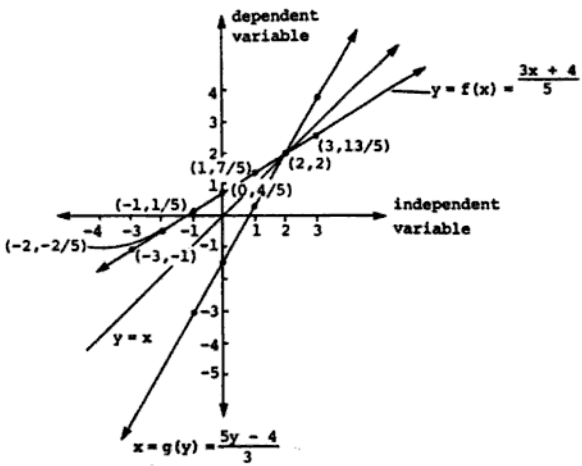
$$y = f(x) = \frac{3x + 4}{5}, \tag{1}$$

where the domain (and the range) of $f$ is the set R of all real numbers.

(a) Find the equation $x = g(y) = f^{-1}(x)$ that defines $f^{-1}$.

(b) Show that $f^{-1}(f(x)) = x$.

(c) Show that $f(f^{-1}(x)) = y$.




**Solution:** (a) The definition of a function $f$ is a set of ordered pairs $(x,y)$ where

1) $x$ is an element of a set X
2) $y$ is an element of a set Y, and
3) no two pairs in $f$ have the same first element.

By definition, $f$ is the infinite set of ordered pairs

$$\left\{\left(x, \frac{3x+4}{5}\right) \middle| x \in R\right\},$$

which includes $(0, 4/5)$, $(2,2)$, $(7,5)$, $(12,8)$, $(-3,-1)$, etc. Furthermore, no two ordered pairs have the same first element . That is, for each element of X a unique value of Y is assigned. For example, if $x = 0$, we obtain only one y value, $4/5$.

We construct the following table to calculate the x and corresponding y values. Note that x is the independent variable and y is the dependent variable.

| $x$ | $\frac{3x+4}{5}$ = | $y$ | $(x,y)$ |
|---|---|---|---|
| $-3$ | $\frac{3(-3)+4}{5} = \frac{-5}{5}$ | $-1$ | $(-3,-1)$ |
| $-2$ | $\frac{3(-2)+4}{5} = -\frac{2}{5}$ | $-\frac{2}{5}$ | $\left(-2, -\frac{2}{5}\right)$ |
| $-1$ | $\frac{3(-1)+4}{5} = \frac{1}{5}$ | $\frac{1}{5}$ | $\left(-1, \frac{1}{5}\right)$ |
| $0$ | $\frac{3(0)+4}{5} = \frac{4}{5}$ | $\frac{4}{5}$ | $\left(0, \frac{4}{5}\right)$ |
| $1$ | $\frac{3(1)+4}{5} = \frac{7}{5}$ | $\frac{7}{5}$ | $\left(1, \frac{7}{5}\right)$ |
| $2$ | $\frac{3(2)+4}{5} = \frac{10}{5}$ | $2$ | $(2,2)$ |
| $3$ | $\frac{3(3)+4}{5} = \frac{13}{5}$ | $\frac{13}{5}$ | $\left(3, \frac{13}{5}\right)$ |

See the accompanying figure, which shows the graph of the function $f$ (which is also the graph of the equation $y = \frac{3x+4}{5}$). We can say that $f$ carries (or maps) any real number $x$ into the number $\frac{3x+4}{5}$:

$$f: x \to \frac{3x+4}{5}.$$

Now to find the inverse function, we must find a function which takes each element of the original set Y and relates it to a unique value of X. There cannot be two values of X for a given value of Y in order for the inverse function to exist. That is, if this is true: $(x_1, y)$ and $(x_2, y)$, then there is no $f^{-1}$.

To find $x = g(y)$, we solve for x in terms of y.

Given: $$y = \frac{3x+4}{5}$$

Multiply both sides by 5,

$$5y = 3x + 4$$

Subtract 4 from both sides,

$$5y - 4 = 3x$$

Divide by 3 and solve for x,

$$x = \frac{5y-4}{3} = f^{-1}(x) = g(y).$$

Choose y values and find their corresponding x-values, as shown in the following table. Note that y is the independent variable and x is the dependent variable.

| $y$ | $g(y)=\frac{5y-4}{3}=$ | $x$ | $(y,x)$ |
|---|---|---|---|
| $-3$ | $\frac{5(-3)-4}{3}$ | $\frac{-19}{3}=-6\frac{1}{3}$ | $\left(-3,-6\frac{1}{3}\right)$ |
| $-2$ | $\frac{5(-2)-4}{3}$ | $\frac{-14}{3}=-4\frac{2}{3}$ | $\left(-2,-4\frac{2}{3}\right)$ |
| $-1$ | $\frac{5(-1)-4}{3}$ | $\frac{-9}{3}=-3$ | $(-1,-3)$ |
| $0$ | $\frac{5(0)-4}{3}$ | $\frac{-4}{3}=-1\frac{1}{3}$ | $\left(0,-1\frac{1}{3}\right)$ |
| $1$ | $\frac{5(1)-4}{3}$ | $\frac{1}{3}$ | $\left(1,\frac{1}{3}\right)$ |
| $2$ | $\frac{5(2)-4}{3}$ | $2$ | $(2,2)$ |
| $3$ | $\frac{5(3)-4}{3}$ | $\frac{11}{3}=3\frac{2}{3}$ | $\left(3,3\frac{2}{3}\right)$ |

See graph. Since there is only one value of $y$ for each value of $x$, this equation defines the inverse function $f^{-1}$. The graph of $f^{-1}$ is the image of the graph of $f$ in the mirror $y = x$.

(b) Given $f(x) = \frac{3x+4}{5} = y.$

Then perform the operation of $f^{-1}$ on $y = f(x)$ where $f^{-1}(x) = \frac{5y-4}{3}$. That is, substitute for $y$: $\frac{3x+4}{5}$.

$$f^{-1}\big(f(x)\big) = f^{-1}\left(\frac{3x+4}{5}\right) = \frac{5\left(\frac{3x+4}{5}\right)-4}{3} = \frac{\frac{15x+20}{5}-4}{3}$$

$$= \frac{\frac{15x+20-20}{5}}{3} = \frac{3x}{3} = x, \qquad \text{or}$$

$$f^{-1}\big(f(x)\big) = \frac{5f(x)-4}{3} = \frac{5\left(\frac{3x+4}{5}\right)-4}{3} = x.$$

(c) We now perform the operation of $f$ on $f^{-1}(x)$. Substitute for $f^{-1}(x)$: $\frac{5y-4}{3} = x$. Note $f(x) = \frac{3x+4}{5}$

$$f\big(f^{-1}(x)\big) = f\left(\frac{5y-4}{3}\right) = \frac{3\left(\frac{5y-4}{3}\right)+4}{5}$$

$$= \frac{5y-4+4}{5} = \frac{5y}{5} = y = f(x)$$

Comment. Since $f = \left\{\left(x, \frac{3x+4}{5}\right)\right\}$, this function $f$ may be thought of as a sequence of directions listing the operations that must be performed on $x$ to get $\frac{3x+4}{5}$. These operations are, in order: take any number $x$, multiply it by 3, add 4, and then divide by 5. The inverse function

$$f^{-1} = \left\{\left(y, \frac{5y-4}{3}\right)\right\}$$

tells us to multiply by 5, subtract 4, and then divide by 3. This "undoes," in reverse order, the operations performed by $f$. The function $f^{-1}$ could be called "the undoing function" because it undoes what the function $f$ has done.

# CHAPTER 12

# LINEAR INEQUALITIES

**Basic Attacks and Strategies for Solving Problems in this Chapter.** **See pages 153 to 176 for step-by-step solutions to problems.**

There is a wide variety of techniques for solving different kinds of inequalities. The most basic technique is the use of the properties of inequality. The properties can be summarized by saying that, except in two cases, if a certain inequality originally exists between two quantities and the both quantities are changed in the same way (by one of the four fundamental operations), then the same kind of inequality exists between the resulting quantities. The two exceptions are that whenever you multiply or divide both sides of an inequality by the same negative number, the direction of the inequality is reversed. An example of applying the basic technique is as follows:

$$
\begin{aligned}
-3x + 5 &> -7 \\
-3x + 5 + (-5) &> -7 + (-5) \\
-3x &> -12 \\
-3x/(-3) &< -12/(-3) \\
x &< 4
\end{aligned}
$$

The procedure for graphing inequalities involving one variable includes the following steps:

(1) Solve the inequality as shown in the above example.

(2) If the equation is of the form $x < a$, draw a number line and draw the endpoint $a$ as an open circle to show that it is not a part of the graph.

(3) Draw a solid line from the open circle to the left to show the numbers included in the graph.

If the form of the solution is $x > a$, then a solid line is drawn from the open circle to the right. If the form of the solution of the inequality is $x \le a$ or $x \ge a$, then the only change from the above procedure is a closed or solid circle in the respective graphs.

The procedure for graphing a compound inequality, that is, two inequalities connected by "and" and "or," begins by graphing each inequality separately. If the two inequalities are connected by the word "or," then we graph the union of all points on either graph on a real number line. If the two inequalities are connected by the word "and," then we graph their intersection, or the parts they have in common, on a real number line.

The procedure for graphing a linear inequality in two variables involves the following steps:

(1) Replace the inequality symbol with an equal sign. The resulting equality represents the boundary for the solution set.

(2) Graph the boundary found in Step 1 using a solid line if the boundary is included in the solution set (that is, if the original inequality symbol was either $\leq$ or $\geq$). Use a broken line to graph the boundary if it is not included in the solution set. (It is not included if the original inequality was either $<$ or $>$.)

(3) Choose any convenient point not on the boundary and substitute the coordinates into the original inequality. If the resulting statement is true, the graph (shaded) lies on the same side of the boundary as the chosen point. If the resulting statement is false, the graph (shaded) lies on the opposite side of the boundary.

To solve an inequality that involves absolute value, one first isolates the absolute value on the left side of the inequality symbol. Then, rewrites the absolute value inequality as an equivalent continued or compound inequality that does not contain absolute value symbols. In general, if $a$ is a positive number, then $|x| < a$ is equivalent to $-a < x < a$ and $|x| > a$ is equivalent to $a < -x$ or $x >$ a.

# Step-by-Step Solutions to Problems in this Chapter, "Linear Inequalities"

## SOLVING INEQUALITIES AND GRAPHING

**• PROBLEM 251**

Solve the inequality $2x + 5 > 9$.

<u>Solution:</u>

| | |
|---|---|
| $2x + 5 + (-5) > 9 + (-5)$ . | Adding $-5$ to both sides. |
| $2x + 0 > 9 + (-5)$ | Additive inverse property |
| $2x > 9 + (-5)$ | Additive identity property |
| $2x > 4$ | Combining terms |
| $\frac{1}{2}(2x) > \frac{1}{2} \cdot 4$ | Multiplying both sides by $\frac{1}{2}$. |
| $x > 2$ | |

The solution set is

$$X = \{x | 2x + 5 > 9\}$$
$$= \{x | x > 2\}$$

(that is all $x$, such that $x$ is greater than 2).

**• PROBLEM 252**

Determine the values of x for which $3x + 2 < 0$.

<u>Solution:</u> We may add $-2$ to both members, to give $3x < -2$. We may then multltply both sides of the inequality by 1/3; hence $x < -2/3$, which is the solution. In other words, the solution consists of the set of all numbers which are less than $-2/3$, and can be expressed

in solution set notation: $\{x : x < -2/3\}$ , (meaning the set of all x such that x is less than $-2/3$).

**• PROBLEM 253**

Solve the inequality $2x - 5 > 3$.

<u>Solution:</u> $2x - 5 > 3$ Given

Add 5 to both sides of the given inequality.

$2x - 5 + 5 > 3 + 5$

Therefore: $2x > 8$ (1).

Divide both sides of inequality (1) by 2. Dividing both sides of an inequality by a positive number does not change the direction of the inequality.

$$\frac{2x}{2} > \frac{8}{2}$$

Therefore: $x > 4$, and $x$ is any real number greater than 4.

**• PROBLEM 254**

Solve $4 - 5x < -3$.

<u>Solution:</u> $4 - 5x < -3 \quad (1)$

Subtract 4 from both sides of inequality (1).

$$4 - 5x - 4 < -3 - 4$$

Therefore: $-5x < -7 \quad (2)$

Divide both sides of inequality (2) by $-5$. Dividing both sides of an inequality by a negative number changes the direction of the inequality.

$$\frac{-5x}{-5} > \frac{-7}{-5}$$

Therefore: $x > \frac{7}{5}$.

Hence, $x$ is any real number greater than $\frac{7}{5}$.

**• PROBLEM 255**

Solve the inequality $\frac{1}{3}x + 6 < 2$.

<u>Solution:</u> $\frac{1}{3}x + 6 < 2 \quad (1)$

Subtract 6 from both sides of inequality (1).

$$\frac{1}{3}x + 6 - 6 < 2 - 6$$

Therefore: $\frac{1}{3}x < -4 \quad (2)$.

Multiply both sides of inequality (2) by 3. Multiplying both sides of an inequality by a positive number does not change the direction of the inequality.

$$3\left(\frac{1}{3}x\right) < 3(-4)$$

Therefore: $x < -12$ and $x$ is any real number less than $-12$.

**• PROBLEM 256**

Illustrate one (a) conditional inequality, (b) identity, and (c) inconsistent inequality.

**Solution:** (a) A conditional inequality is an inequality whose validity depends on the values of the variables in the sentence. That is, certain values of the variables will make the sentence true, and others will make it false. $3 - y > 3 + y$ is a conditional inequality for the set of real numbers, since it is true for any replacement less than zero and false for all others.

(b) $x + 5 > x + 2$ is an identity for the set of real numbers, since for any real valued x, the expression on the left is greater than the expression on the right.

(c) $5y < 2y + y$ is inconsistent for the set of non-negative real numbers. For any x greater than 0 the sentence is always false. A sentence is inconsistent if it is always false when its variables assume allowable values.

**• PROBLEM 257**

Solve the inequality $4x + 3 < 6x + 8$.

**Solution:** In order to solve the inequality $4x + 3 < 6x + 8$, we must find all values of x which make it true. Thus, we wish to obtain x alone on one side of the inequality.

Add $-3$ to both sides:

$$\begin{array}{r} 4x + 3 < 6x + 8 \\ \underline{\quad -3 \qquad -3} \\ 4x < 6x + 5 \end{array}$$

Add $-6x$ to both sides:

$$\begin{array}{r} 4x < \quad 6x + 5 \\ \underline{-6x \quad -6x \qquad} \\ -2x < 5 \qquad \end{array}$$

In order to obtain x alone we must divide both sides by $(-2)$. Recall that dividing an inequality by a negative number reverses the inequality sign, hence

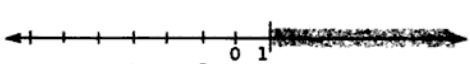
$$\frac{-2x}{-2} > \frac{5}{-2}$$

Cancelling $\frac{-2}{-2}$ we obtain, $x > -\frac{5}{2}$

Thus, our solution is $\left(x : x > -\frac{5}{2}\right)$ (the set of all x such that x is greater than $-\frac{5}{2}$).

**• PROBLEM 258**

Solve the inequality $4x - 5 \geq -6x + 5$.




$\{x|x \geq 1\} = [1, \infty)$

**Solution:** To solve this compound statement we solve for $x$ as follows:

Adding $5$ to both sides of the given inequality we have:

$$4x \geq -6x + 10$$

Adding $6x$ to both sides: $10x \geq 10$

Multiplying both sides by $\frac{1}{10}$ : $x \geq 1$

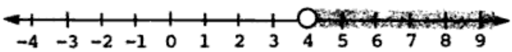
Therefore $S = \{x | x \geq 1\}$

The region representing this set on the number line is shown in the diagram.

You should note that the bracket in the graph includes the point 1.

**• PROBLEM 259**

Find the solution set of inequality $5x - 9 > 2x + 3$.




**Solution:** To find the solution set of the inequality $5x - 9 > 2x + 3$, we wish to obtain an equivalent inequality in which each term in one member involves $x$, and each term in the other member is a constant. Thus, if we add $(- 2x)$ to both members, only one side of the inequality will have an $x$ term:

$$5x - 9 + (- 2x) > 2x + 3 + (- 2x)$$

$$5x + (-2x) - 9 > 2x + (- 2x) + 3$$

$$3x - 9 > 3$$

Now, adding 9 to both sides of the inequality we obtain,

$$3x - 9 + 9 > 3 + 9$$

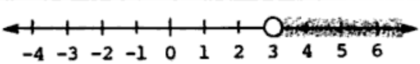
$$3x > 12$$

Dividing both sides by 3, we arrive at $x > 4$.

Hence the solution set is $\{x | x > 4\}$, and is pictured in the accompanying figure.

**• PROBLEM 260**

Solve $3(x + 2) < 5x$.




**Solution:**

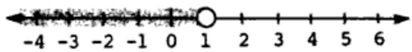
| | |
|---|---|
| $3(x + 2) < 5x$ | Given |
| $3x + 6 < 5x$ | Distributive property |
| $-2x < -6$ | Transposition (adding (-6) and (-5x) to both sides of the inequality and simplifying) |

| | |
|---|---|
| $x > 3$ | Multiplicative property (multiply both members of the previous inequality by $-\frac{1}{2}$). Note that multiplying both sides of an inequality by a negative number changes the sense of the inequality. |

The solution set is $\{x: x > 3\}$ (see figure).
The unshaded circle above 3 on the number line indicates that $x = 3$ is not included in the solution set.

**• PROBLEM 261**

> Solve $2(x + 1) < 4$.



**Solution:**

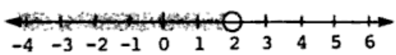
| | |
|---|---|
| $2(x + 1) < 4$ | Given |
| $2x + 2 < 4$ | Distributive property |
| $2x < 2$ | Additive property (with -2) |
| $x < 1$ | Multiplicative property (with $\frac{1}{2}$) |

The solution set of $2(x + 1) < 4$ is $\{x: x < 1\}$. The graph of the solution set can be seen in the accompanying figure.

The solution set of $x < 1$ is equal to the solution set of $2(x + 1) < 4$ because the inequalities are equivalent. We have solved an inequality when we know its solution set.

**• PROBLEM 262**

> Solve $-3(x - 5) > x + 7$.



**Solution:**

| | |
|---|---|
| $-3(x - 5) > x + 7$ | Given |
| $-3x + 15 > x + 7$ | $a(b - c) = ab - ac$, Distributive Law |
| $-4x > -8$ | Subtracting 15 and x from both members of the inequality, Transposition |
| $x < 2$ | Multiplying both members of the |

inequality by $-\frac{1}{4}$, Multiplicative Property. Note that multiplying both members of an inequality by a negative number changes the sense of the inequality.

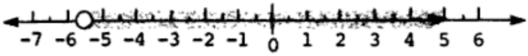
The solution set is $\{x: x < 2\}$ and the graph can be seen in the accompanying figure.

The unshaded circle above 2 on the number line indicates that $x = 2$ is not included in the solution set.

**• PROBLEM 263**

> Solve the inequality $3x - 4 < 5x + 7$.

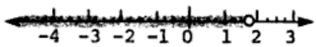
**Solution:** By subtracting 3x from both sides of the given inequality, we obtain the equivalent inequality



$-4 < 2x + 7$. Now we subtract 7 from both sides, and we obtain the equivalent inequality $-11 < 2x$. Finally, we divide by the positive number 2, and we have $-11/2 < x$. Thus, our solution set consists of all points greater than $\frac{-11}{2}$, $\left\{x : x > \frac{-11}{2}\right\}$, as pictured in the accompanying figure.

**• PROBLEM 264**

Solve the inequality $-5(x - 1) \geq 3(x - 3)$.



| | |
|---|---|
| **Solution:** $-5x + 5 \geq 3x - 9$ | Distributing |
| $-5x + 5 + (-5) \geq 3x - 9 + (-5)$ | Adding $(-5)$ to both sides |
| $-5x \geq 3x - 9 + (-5)$ | Additive inverse property |
| $-5x \geq 3x - 14$ | Combining terms |
| $-5x + (-3x) \geq 3x - 14 + (-3x)$ | Adding $(-3x)$ to both sides |
| $-5x + (-3x) \geq -14 + 3x + (-3x)$ | Commuting |
| $-5x + (-3x) \geq -14$ | Additive inverse property |
| $-8x \geq -14$ | Simplifying |
| $\left(-\frac{1}{8}\right)(-8x) \leq (-14)\left(-\frac{1}{8}\right)$ | Multiplying both sides by $-\frac{1}{8}$ |

Notice that multiplying by a negative number reverses the inequality, that is it goes from greater than and equal to, to less than and equal to.

$$x \leq \frac{14}{8}$$

Reducing to lowest terms $x \leq \frac{7}{4}$

The solution set is

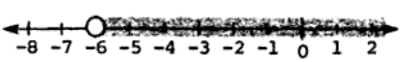
$$X = \{x \mid -5(x - 1) \geq 3(x - 3)\}$$

$$= \left\{x \mid x \leq \frac{7}{4}\right\},$$

that is, all $x$ such that $X$ is less than or equal to $\frac{7}{4}$. The solution is pictured above.

**• PROBLEM 265**

Solve $\quad \frac{1}{6}x - 3 < \frac{3}{4}x + \frac{1}{2}$.



**Solution:** We can eliminate the fractional coefficients by multiplying both members of the inequality by the least common denominator of the fractions. Multiplying by 12, we have

$$2x - 36 < 9x + 6.$$

Isolating the constant terms on the left side of the inequality sign and the x-terms on the right side by transposition, we have:

$$-36 - 6 < 9x - 2x .$$

Then simplifying:

$$-42 < 7x.$$

Dividing both members of this inequality by 7 yields:

$$-6 < x$$

Hence, the solution set is $\{x: x > -6\}$ and the graph is shown in the accompanying figure.

The unshaded circle above -6 on the number line indicates that $x = -6$ is not included in the solution set.

**• PROBLEM 266**

Solve $7\left(\frac{2}{3}x - 1\right) > 2(x - 6)$

**Solution:** $7\left(\frac{2}{3}x - 1\right) > 2(x - 6)$ (1)

$\frac{14}{3}x - 7 > 2x - 12$ (2) Distributive Property

Subtract 2x from both sides of inequality (2).

$$\frac{14}{3}x - 7 - 2x > 2x - 12 - 2x$$

Therefore: $\frac{8}{3}x - 7 > -12$ (3).

Add 7 to both sides of inequality (3).

$$\frac{8}{3}x - 7 + 7 > -12 + 7$$

Therefore: $\frac{8}{3}x > -5$ (4).

Multiply both sides of inequality (4) by $\frac{3}{8}$.

Multiplying both sides of an inequality by a positive number does not change the direction of the inequality. Therefore:

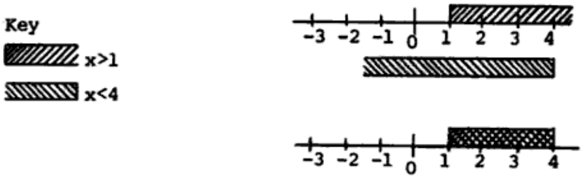
$$\frac{3}{8}\left(\frac{8}{3}x\right) > \frac{3}{8}(-5) \quad \text{and} \quad x > -\frac{15}{8}.$$

Hence, x is any real number greater than $-\frac{15}{8}$.

**• PROBLEM 267**

Solve the inequality

$$\frac{1}{x - 1} > \frac{1}{3} .$$




**Solution:** Since the fraction $\frac{1}{x-1} > \frac{1}{3}$; that is, since the fraction $\frac{1}{x-1}$ is greater than 0, $x - 1$ must be positive. Hence, $x - 1 > 0$. If both sides of the given equation are multiplied by $3(x - 1)$, then:

$$3(x - 1)\,\frac{1}{x - 1} > 3(x - 1)\left(\frac{1}{3}\right)$$

$$3 > x - 1 .$$

Note that multiplying both sides of an inequality by a positive number $\big($in this case, $3(x - 1)\big)$ does not change the sign of the inequality.

Now we have the double restrictions

$$x - 1 > 0 \quad \text{and} \quad 3 > x - 1$$

and the solution set is the intersection of the solution sets of these two inequalities. Solving each of the two inequalities, we find that

$$x > 1 \quad \text{and} \quad x < 4$$

The solution set is the intersection of the two inequalities, as can be seen on a number line (see diagrams). The intersection of these two inequalities is the set $1 < x < 4$. Hence, the solution set is:

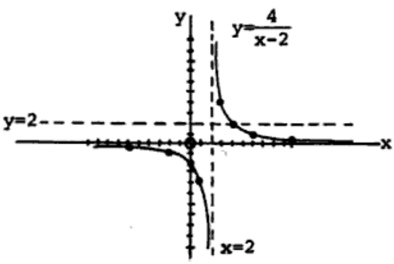
$$X = \{x \mid 1 < x < 4\}$$

(The endpoints $X = 1$ and $X = 4$ are not included in the solution set).

**• PROBLEM 268**

Solve the inequality

$$\frac{4}{x - 2} < 2.$$



Solution: The inequality is meaningless for $x = 2$ because when $x = 2$ the denominator of the left member is 0, making the fraction undefined.

If $x > 2$, $x - 2$ is positive (since $x > 2$ is equivalent to $x - 2 > 0$), and multiplication of the given inequality by $x - 2$ yields

$$\begin{aligned} 4 &< 2(x - 2) \\ 4 &< 2x - 4 \\ 8 &< 2x \\ 4 &< x \\ x &> 4. \end{aligned}$$

Thus, the solution is the intersection of $x > 2$ and $x > 4$, $x > 2 \cap x > 4$, which is $\{x | x > 4\}$.

If $x < 2$, $x - 2$ is negative (since $x < 2$ is equivalent to $x - 2 < 0$), and multiplication by $x - 2$ yields

$$4 > 2(x - 2)$$

because multiplication by a negative number reverses an inequality.

Distributing, $\quad 4 > 2x - 4$

Adding 4 to both sides,

$$8 > 2x$$

Dividing both sides by 2,

$$4 > x, \quad \text{or} \quad x < 4.$$

Thus the solution is the intersection of $x < 2$ and $x < 4$, $x < 2 \cap x < 4$, which is

$\{x | x < 2\}$. Hence

$$\frac{4}{x - 2} < 2$$

if $x < 2$ or if $x > 4$.

A graphical solution of the problem (see diagram) can be obtained by sketching the equilateral hyperbola $y = 4/(x - 2)$ and the line $y = 2$. The hyperbola may be sketched from its vertical asymptote $x = 2$, its horizontal asymptote $y = 0$, its intercepts $x = 0$, $y = -2$, and a few other points obtained by substitution and symmetry. It is then possible to observe the values of $x$ for which the hyperbola is below the line, namely,

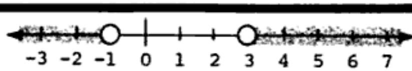
$$x < 2 \quad \text{and} \quad x > 4.$$

The same diagram also shows that $[4/(x - 2)] > 2$ for $2 < x < 4$.

## • PROBLEM 269

Find the solution set of the disjunction

$$2 - 3x > 5 \quad \text{or} \quad 2x - 1 > 5.$$




Solution: A disjunction is a compound sentence using the connective 'or'. The union of the solution sets of the two sentences comprising the compound sentence is the solution set of the disjunction. For this problem the solution set is

$$\{x: 2 - 3x > 5\} \cup \{x: 2x - 1 > 5\}$$

We solve each inequality independently and find the union of their solution set:

$$2 - 3x > 5 \quad \text{or} \quad 2x - 1 > 5$$

$$-3x > 3 \quad \text{or} \quad 2x > 6$$

Solving for $x$, we divide both members of the inequality by a negative number (-3). Therefore the direction of the inequality is reversed.

$$x < -1 \quad \text{or} \quad x > 3$$

The solution set of $2 - 3x > 5$ or $2x - 1 > 5$ is shown on the graph. The unshaded circles above $-1$ and $3$ on the number line indicate that these values are not included in the solution set.

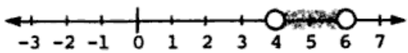
Compound sentences can also be formed by connecting two sentences with the word 'and.' A compound sentence using the connective and is called a conjunction. The solution set of a conjunction is the set of replacements that are common to the solution sets of the sentences making-up the conjunction (their intersection). We may write the solution set as:

$$\{x\colon x > a\} \cap \{x\colon x < b\} = \{x\colon a < x < b\}.$$

**• PROBLEM 270**

Find the solution set of the conjunction

$$\tfrac{1}{2}x + 1 > 3 \quad \text{and} \quad x > 2x - 6$$





__Solution:__ The solution set must be such that $x$ satisfies both inequalities simultaneously. The solution set of the conjunction is

$$\{x\colon \tfrac{1}{2}x + 1 > 3\} \cap \{x\colon x > 2x - 6\}$$

$$\tfrac{1}{2}x + 1 > 3 \quad \text{and} \quad x > 2x - 6$$

$$\tfrac{1}{2}x > 2 \quad \text{and} \quad -x > -6$$

$$x > 4 \quad \text{and} \quad x < 6$$

(multiplying by negative 1 reverses the inequality).

Note: if $x = 4$ the sentence $\tfrac{1}{2}x + 1 > 3$ becomes false, i.e.,

$$\tfrac{1}{2}(4) + 1 = 3 > 3 \quad \text{false}$$

and if $x = 6$ the sentence $x > 2x - 6$ becomes false

also $(6 > 2(6) - 6 = 6$ false).

Therefore the solution set cannot include these two points, and the values of $x$ which make both sentences true simultaneously are the inequalities $x$ greater than but not equal to $4$ and less than but not equal to $6$. That is $\{x\colon 4 < x < 6\}$. (See the number line).

**• PROBLEM 271**

If $1 < a$, show that $a < a^2$.

__Solution:__ We are given $a>1$, and we know $1>0$, thus using the transitive property (if $x>y$ and $y>z$, then $x>z$) we conclude $a>o$. Since $a$ is positive we may multiply both sides of the inequality $1<a$ by $a$ to obtain an equivalent inequality,

$$1 \cdot a < a \cdot a$$

$$a < a^2$$

Thus we have shown if,

$$1<a, a<a^2.$$

• **PROBLEM 272**

Show that if $0 < a < 1$, then $a^2 < a$.

Solution: The relation $0 < a < 1$ means that $a > 0$ and $a < 1$. Thus $a$ is positive and less than one. Since $a$ is positive we may multiply both sides of the inequality, $a < 1$, by $a$.

$$a(a) < a(1)$$

hence

$$a^2 < a .$$

(To visualize this concept pick a number between 0 and 1. If we choose $\frac{1}{2}$, then

$$0 < \tfrac{1}{2} < 1, \text{ and } (\tfrac{1}{2})^2 < \tfrac{1}{2} \text{ or } \tfrac{1}{4} < \tfrac{1}{2}).$$

• **PROBLEM 273**

Prove that if $a > b > 0$, then

$$\frac{1}{a} < \frac{1}{b}.$$

Solution: Since a and b are both positive (given), ab is positive because the product of two positive numbers is always positive. Now, since $ab > 0$, we may divide both sides of $a > b$ by ab to obtain

$$\frac{a}{ab} > \frac{b}{ab}.$$

Cancelling like terms in numerator and denominator,

$$\frac{1}{b} > \frac{1}{a},$$

which is equivalent to $\frac{1}{a} < \frac{1}{b}$. To complete the proof, the student should check that the steps are reversible, as follows:

Check. $\frac{1}{b} > \frac{1}{a}$. Multiply both sides of the inequality by the least common denominator obtained by multiplying the two denominators together.

$$\frac{ab}{b} > \frac{ab}{a} = a > b.$$

• **PROBLEM 274**

Under what conditions does the inequality $1/a \le 1/b$ imply that $b \le a$?

Solution: If $ab > 0$, we may multiply both sides of the first inequality by ab to obtain

$$ab\left(\frac{1}{a}\right) \le ab\left(\frac{1}{b}\right) \quad \text{or} \quad b \le a ,$$

whereas if $ab < 0$, such multiplication produces the reverse inequality $a \le b$. Therefore, the inequality $1/a \le 1/b$ implies that $b \le a$ provided that $ab > 0$. We don't have to consider the case $ab = 0$ because if $ab = 0$, either $a = 0$ or $b = 0$, which would mean that one side of our original inequality $1/a \le 1/b$ would be undefined (as it would be in the form $1/0$).

**• PROBLEM 275**

Solve the inequality $\sqrt{x-3} \le 2 - \sqrt{x+1}$.

<u>Solution:</u>

| | |
|---|---|
| $\sqrt{x-3} \le 2 - \sqrt{x+1}$ | Given |
| $x - 3 \le 4 - 4\sqrt{x+1} + x + 1$ | Squaring |
| $-8 \le -4\sqrt{x+1}$ | Transposing and simplifying |
| $2 \ge \sqrt{x+1}$ | Dividing by $-4$. |

Note that dividing both sides of an inequality by a negative number changes the sense of the inequality.

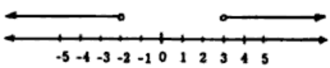
| | |
|---|---|
| $4 \ge x + 1$ | Squaring |
| $3 \ge x$ or $x \le 3$ | Solving for $x$ |

Check: If $x = 3$, $\sqrt{3-3} = 2 - \sqrt{3+1}$.
If $x > 3$, the left member, $\sqrt{x-3}$, is positive, but the right member, $2 - \sqrt{x+1}$, is negative. Hence, the inequality is not satisfied. Nor is the inequality satisfied if $x < 3$, for in this case the left member is not a real number. Hence, the solution set is $\{3\}$.

If the inequality were a strict inequality, the solution set would be the null set. If the left member is to be a real number, it must be positive and $x$ must be greater than 3, but in this case the right member must be negative. However, it is impossible for a positive number to be less than a negative number.

**• PROBLEM 276**

What is the set $\{x < -2\} \cap \{x > 3\}$?




<u>Solution:</u> An element belongs to the intersection of two sets, if, and only if, it belongs to both of them. Thus, in order for a number to belong to our intersection, it would have to be both less than $-2$ and greater than 3. There is no such number, so the intersection is the empty set; that is,

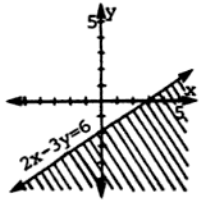
$$\{x < -2\} \cap \{x > 3\} = \emptyset .$$

This can be seen from the accompanying number line representation, which illustrates that the two graphs have no points in common.

## INEQUALITIES WITH TWO VARIABLES

**• PROBLEM 277**

Solve $2x - 3y \ge 6$




Solution: The statement $2x - 3y \geq 6$ means $2x - 3y$ is greater than or equal to 6. Symbolically, we have $2x - 3y > 6$ or $2x - 3y = 6$. Consider the corresponding equality and graph $2x - 3y = 6$. To find the x-intercept, set $y = 0$

$$2x - 3y = 6$$
$$2x - 3(0) = 6$$
$$2x = 6$$
$$x = 3$$

{3,0} is the x-intercept.

To find the y-intercept, set $x=0$

$$2x - 3y = 6$$
$$2(0) - 3y = 6$$
$$-3y = 6$$
$$y = -2$$

{0,-2} is the y-intercept.

A line is determined by two points. Therefore draw a straight line through the two intercepts {3,0} and {0,-2}. Since the inequality is mixed, a solid line is drawn through the intercepts. This line represents the part of the statement $2x - 3y = 6$.

We must now determine the region for which the inequality $2x - 3y > 6$ holds.

Choose two points to decide on which side of the line the region $x - 3y > 6$ lies. We shall try the points (0,0) and (5,1).

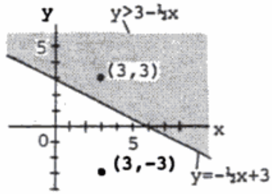
| For (0,0) | For (5,1) |
|---|---|
| $2x - 3y > 6$ | $2x - 3y > 6$ |
| $2(0) - 3(0) > 6$ | $2(5) - 3(1) > 6$ |
| $0 - 0 > 6$ | $10 - 3 > 6$ |
| $0 > 6$ | $7 > 6$ |
| False | True |

The inequality, $2x - 3y > 6$, holds true for the point (5,1). We shade this region of the xy-plane. That is, the area lying below the line $2x - 3y = 6$ and containing (5,1).

Therefore, the solution contains the solid line, $2x - 3y = 6$, and the part of the plane below this line for which the statement $2x - 3y > 6$ holds.

**• PROBLEM 278**

Solve the inequality $x + 2y \geq 6$ for y in terms of x and draw its graph.




Solution: To solve for y in terms of x, obtain y alone on one side of the inequality and x and any constants on the other. Subtracting x from both sides of

$x + 2y \geq 6$ gives $2y \geq 6 - x$

Divide the equation by 2

$y \geq 3 - \frac{1}{2}x$

The points in the x - y plane which will satisfy this equation are those satisfying

$y > 3 - \frac{1}{2}x$ and $y = 3 - \frac{1}{2}x$.

Consider the case,

$y = 3 - \frac{1}{2}x$

which is a graph of the solid straight line with y -intercept 3 and slope $-\frac{1}{2}$. (See diagram.)

We must also find those points which satisfy

$y > 3 - \frac{1}{2}x$.

Choose two points which lie on either side of the line

$y = 3 - \frac{1}{2}x$ to find the region where

$y > 3 - \frac{1}{2}x$.

We shall choose (3, 3) and (3, - 3) (see diagram).

For (3, 3)

$y > 3 - \frac{1}{2}x$

For (3, - 3)

$y > 3 - \frac{1}{2}x$

$$3 > 3 - \frac{1}{2}(3) \qquad\qquad -3 > 3 - \frac{1}{2}(3)$$

$$3 > \frac{3}{2} \qquad\qquad -3 \not> \frac{3}{2}$$

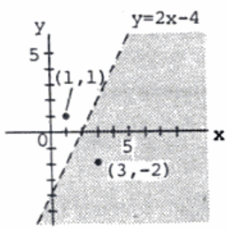
(3, 3) satisfies the inequality. (3, - 3) does not satisfy the inequality.

Thus, all the points in the region where (3, 3) lies satisfy $y > 3 - \frac{1}{2}x$. That is all those points above the line $y = 3 - \frac{1}{2}x$ satisfy $y > 3 - \frac{1}{2}x$ and lie in the shaded area.

Consequently, the graphical solution of $y \geq -\frac{1}{2}x + 3$ are those points which lie on the solid line $y = -\frac{1}{2} + 3$ and those points in the shaded area $y > -\frac{1}{2}x + 3$.

**• PROBLEM 279**

Solve the inequality $2x - y > 4$ for y in terms of x, and draw its graph.




Solution: To solve for y in terms of x obtain y on one side of the inequality and x on the other. Given

$$2x - y > 4 \qquad (1)$$

Add $-2x$ to both sides of

$$2x - y > 4 \qquad (1)$$

We obtain $\quad -y > 4 - 2x \qquad (2)$

Multiply (2) by $-1$ and reverse the inequality sign since we are multiplying by a negative number. We obtain y in terms of x

$$y < -4 + 2x \qquad (3)$$

Rewriting (3)

$$y < 2x - 4 \qquad (3)$$

Graphing (3) consider the equation first as an equality

$$y = 2x - 4 \qquad (4)$$

We draw the graph of (4) as a dotted line since the points of the given inequality $y < 2x - 4$ do not satisfy $y = 2x - 4$. To draw $y = 2x - 4$, we note the slope is 2 and the y-intercept is $-4$.

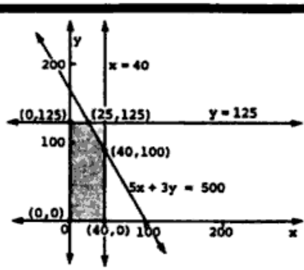
To determine what region of the x - y plane satisfies $y < 2x - 4$ choose a point on either side of the dotted line. Let us take the points $(3, -2)$ and $(1, 1)$ (see diagram). Substitute these points into the given inequality and see which point will satisfy it.

| For (3, - 2) | For (1, 1) |
|---|---|
| $y < 2x - 4$ | $y < 2x - 4$ |
| $-2 < 2(3) - 4$ | $1 < 2 - 4$ |
| $-2 < 2$ | $1 \nless -2$ |

Now hatch in that portion of the plane containing $(3, -2)$. All the points to the right of the dotted line $y = 2x - 4$ will satisfy the given inequality.

**• PROBLEM 280**

A livestock farmer has 500 acres to devote to grazing. He estimates that cattle require 5 acres per head and sheep require 3 acres per head. He has winter shelter facilities for 40 head of cattle and for 125 sheep. What constraints are imposed on the number of cattle and sheep he can raise?




__Solution:__ Let $x$ represent the number of cattle raised and $y$ the number of sheep. Since he cannot raise a negative number of either cattle or sheep, we have the constraints

$$x \geq 0 \qquad (1)$$
$$y \geq 0 \qquad (2)$$

Since $5x$ acres are required for the cattle and $3y$ acres for the sheep and there are only 500 acres available, we have

$$5x + 3y \leq 500 \qquad (3)$$

Since he can winter only 40 cattle,

$$x \leq 40 \qquad (4)$$

Since he can winter only 125 sheep,

$$y \leq 125 \qquad (5)$$

Relations (1) through (5) are the constraints.

The graph of the constraints in this example is a convex set of points. The corner points of the shaded polygon are (0,0), (40,0), (40,100), (25,125), and (0,125).

## INEQUALITIES COMBINED WITH ABSOLUTE VALUES

**• PROBLEM 281**

Express the inequality $|x| < 3$ without using absolute value signs.

Solution: According to the law of absolute values which states that $|a| < b$ is equivalent to $-b < a < b$, where b is any positive number, $|x| < 3$ is equivalent to $-3 < x < 3$.

**• PROBLEM 282**

Solve the inequality $|5 - 2x| > 3$.

Solution: The property of absolute values states that $|a| = +a$ or $|a| = -a$. Therefore: $|5 - 2x| = 5 - 2x$ or $-(5 - 2x)$. Thus, the given inequality becomes two new inequalities:

$$5 - 2x > 3, \quad -(5 - 2x) > 3.$$

Now, we must solve for x in both inequalities. For the first, we subtract 5 from both sides of the inequality, and then divide by -2. We must keep in mind that division or multiplication by a negative number reverses the inequality sign. Thus, for $5 - 2x > 3$ we have:

$$5 - 5 - 2x > 3 - 5$$

$$-2x > -2$$

$$\frac{-2x}{-2} > \frac{-2}{-2}$$

$$x < 1.$$

For the second inequality, we first take the negative of all the terms inside the parentheses. Thus, for $-(5 - 2x) > 3$ we have:

$$-5 + 2x > 3.$$

Now, we add 5 to both sides of the inequality, and then divide by 2. Thus, we obtain:

$$-5 + 5 + 2x > 3 + 5$$

$$2x > 8$$

$$\frac{2x}{2} > \frac{8}{2}$$

$$x > 4.$$

Therefore, the above inequality holds when $x < 1$, and when $x > 4$.

**• PROBLEM 283**

Solve

$$\left|\frac{4x}{5} - 1\right| > 3$$


<u>Solution:</u> We note the following about absolute values. If b is a nonnegative real number, then a is a real number for which $|a| > b$ if and only if $a > b$ or $a < -b$. See the figure.

In the given example, this inequality is satisfied if either

$$\frac{4x}{5} - 1 > 3 \quad \text{or} \quad \frac{4x}{5} - 1 < -3$$

is satisfied. By adding 1 to each member of these inequalities, we get

$$\frac{4x}{5} > 4 \qquad \text{and} \qquad \frac{4x}{5} < -2$$

Hence, by multiplying by 5/4 in each case, we note that the original inequality is satisfied by values of x that are greater than 5 and by values of x that are less than $-5/2$, that is, by $x > 5$ and by $x < -5/2$.

The solution set is therefore

$$\{x|\ x > 5\} \cup \{x|x < -5/2\}.$$

**• PROBLEM 284**

Find the solution set of the inequality

$$|-2x+6| > 8$$

<u>Solution:</u> By a property of inequalities involving absolute values, the solution set of the given inequality is the union of the solution sets of

$$-2x + 6 > 8 \qquad \text{and} \qquad -2x + 6 < -8$$

$$-2x > 2 \qquad\qquad -2x < -14$$

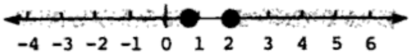
$$x < -1 \qquad\qquad x > 7$$

Hence the solution set is

$$\{x|x < -1\} \cup \{x|x > 7\}.$$

**• PROBLEM 285**

Graph $\{x: |3x - 4| \geq 2\}$.




Solution: In general, the required graph of $\{x: |ax + b| \geq c\}$ is the union of two sets: $\{x: ax + b \geq c\} \cup \{x: ax + b \leq -c\}$. Therefore, the required graph of $\{x: |3x - 4| \geq 2\}$ is the union of two sets:

$$\{x: 3x - 4 \geq 2\} \cup \{x: 3x - 4 \leq -2\}$$

$$3x - 4 \geq 2 \quad \text{or} \quad 3x - 4 \leq -2$$

$$3x \geq 6 \quad \text{or} \quad 3x \leq 2$$

$$x \geq 2 \qquad\qquad x \leq 2/3$$

The solution set of $\{x: |3x - 4| \geq 2\}$ is

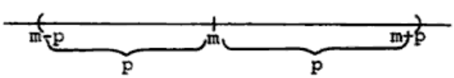
$$\{x: x \geq 2\} \cup \{x: x \leq 2/3\}$$

The graph of the solution set is the union of two rays. Notice that the shaded circles above 2/3 and 2 on the number line indicate that 2/3 and 2 are included in the solution set.

**• PROBLEM 286**

Express the inequality $|2x - 1| < 5$ without using absolute value signs.

Solution: If x is a number such that $|x - m| < p$, then x must lie in the interval between the points $m - p$ and $m + p$; that is, if $|x - m| < p$, then $-p < x - m < p$. Then add $+m$ to all the members of the inequality to obtain $m - p < x < m + p$. Observe that the point m is the midpoint of this interval and that the length of the interval is 2p. Note the number line.




length = p + p = 2p

Therefore, the inequalities $m - p < x < m + p$ and $|x - m| < p$ are equivalent. Hence, the given inequality $|2x - 1| < 5$ reduces to:

$1 - 5 < 2x < 1 + 5$ where x is replaced by 2x, m is replaced by 1, and p is replaced by 5.

Therefore,

$$-4 < 2x < 6$$

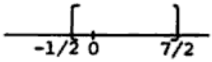
Now, divide each term of these inequalities by 2:

$$\frac{-4}{2} < \frac{2x}{2} < \frac{6}{2}$$

$$-2 < x < 3.$$

**• PROBLEM 287**

Solve the inequality $|2x - 3| \leq 4$.




$x = [-1/2, 7/2]$

**Solution:** In general, $|ax + b| \leq c$ is equivalent to $-c \leq ax + b \leq c$. Therefore $|2x - 3| \leq 4$ implies that $-4 \leq 2x - 3 \leq 4$. This statement is the conjunction of the statement $-4 \leq 2x - 3$ and the statement $2x - 3 \leq 4$. Hence the solution set of the conjunction is the intersection of the solution sets of the two component propositions.

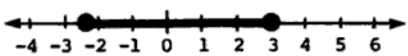
The computation may be arranged in the following manner:

| | |
|---|---|
| $-4 \leq 2x - 3 \leq 4$ | |
| $-1 \leq 2x \leq 7$ | Adding 3 to both sides of both inequalities |
| $-\frac{1}{2} \leq x \leq \frac{7}{2}$ | Multiplying both sides of both inequalities by $\frac{1}{2}$ |

The solution set is represented on the number line as in the figure.

**• PROBLEM 288**

Solve $|3x - 1| \leq 8$.




**Solution:** Since $|a| = a$ if $a \geq 0$ and $|a| = -a$ if $a \leq 0$. We must solve two equations

$$3x - 1 \leq 8$$

$$-(3x - 1) \leq 8 \text{ or } 3x - 1 \geq -8.$$

(Note that multiplying an inequality by a negative number, i.e., -1, reverses the inequality.)

The solution set will be the conjunction of the solution sets of each equation; that is,

$$\{x: 3x - 1 \leq 8\} \text{ and } \{x: 3x - 1 \geq -8\}.$$

We must find

$$\{x: 3x - 1 \leq 8\} \cap \{x: 3x - 1 \geq -8\}$$

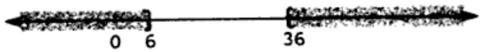
$$3x - 1 \leq 8 \text{ and } 3x - 1 \geq -8$$

$$3x \leq 9 \text{ and } \quad 3x \geq -7$$

$$x \leq 3 \text{ and } \quad x \geq -2\tfrac{1}{3}.$$

The solution set is $\left\{x: -2\frac{1}{3} \leq x \leq 3\right\}$. See the figure.

Find the values of x satisfying the statement $\left|\frac{x}{3} - 7\right| \geq 5.$




$(-\infty,6] \cup [36,\infty)$

**Solution:** In general, $|ax + b| \geq c$ implies that $ax + b \geq c$ or $ax + b \leq -c$. Therefore, for

$$\left|\frac{x}{3} - 7\right| \geq 5 :$$

(1) $\frac{x}{3} - 7 \geq 5$ or (2) $\frac{x}{3} - 7 \leq -5$

Case (1) $\frac{x}{3} - 7 \geq 5$

$$\frac{x}{3} \geq 12$$

$$x \geq 36$$

Case (2) $\frac{x}{3} - 7 \leq -5$

$$\frac{x}{3} \leq 2$$

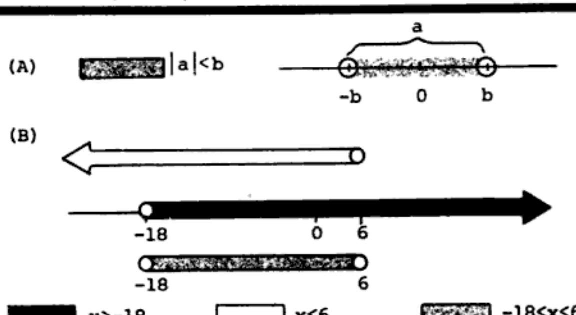
$$x \leq 6$$

We may consider the solution set of the inequality in this example as the union of the two disjoint sets $\{x \mid x \geq 36\}$ and $\{x \mid x \leq 6\}$. This union may be represented on the number line as in the accompanying figure.

• PROBLEM 290

Solve

$$\left|\frac{x}{3} + 2\right| < 4$$




**Solution:** Now, if b is a nonnegative real number, then a is a real number for which $|a| < b$ if and only if $-b < a < b$. See number line (A).

We apply this rule to the given problem. Therefore, $\left|\frac{x}{3} + 2\right| < 4$ is equivalent to

$-4 < \frac{x}{3} + 2 < 4$. In other words, this inequality is satisfied if and only if both

$$\frac{x}{3} + 2 < 4 \qquad \text{and} \qquad \frac{x}{3} + 2 > -4$$

are satisfied. By adding $-2$ to each member of these inequalities, we get

$$\frac{x}{3} < 2 \qquad \text{and} \qquad \frac{x}{3} > -6$$

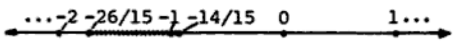
Hence, multiplying by 3 in each case, we note that the original inequality is satisfied by values of x that satisfy both $x < 6$ and $x > -18$. We can observe the solution from diagram (B). Therefore, the solution set is

$$\{x| \ -18 < x < 6\} \quad \text{or} \quad \{x|x < 6\} \cap \{x|x > -18\}.$$

**• PROBLEM 291**

Find all x for which

$$\left|\frac{4}{3} + x\right| \leq \frac{2}{5} .$$




Solution: Note the following rule for absolute values: for $|a + b| \leq c$ where a,b,c are any real numbers, $-c \leq a + b \leq c$. Therefore, the given inequality, involving the absolute value, reduces to:

$$-\frac{2}{5} \leq \frac{4}{3} + x \leq \frac{2}{5} .$$

Subtract $\frac{4}{3}$ from the three parts of the inequality above,

$$-\frac{2}{5} - \frac{4}{3} \leq \frac{4}{3} + x - \frac{4}{3} \leq \frac{2}{5} - \frac{4}{3}$$

$$-\frac{2}{5} - \frac{4}{3} \leq x \leq \frac{2}{5} - \frac{4}{3} .$$

Getting a common denominator of 15 for the fractions involved in the above inequality,

$$-\frac{3(2)}{3(5)} - \frac{5(4)}{5(3)} \leq x \leq \frac{3(2)}{3(5)} - \frac{5(4)}{5(3)}$$

$$-\frac{6}{15} - \frac{20}{15} \leq x \leq \frac{6}{15} - \frac{20}{15}$$

$$-\frac{26}{15} \leq x \leq -\frac{14}{15}$$

Thus, all rational numbers between $\frac{-26}{15}$ and $\frac{-14}{15}$ are solutions. The figure gives the solution set on the rational line.

**• PROBLEM 292**

Find the solution set of $|2x + 5| \leq x + 3$.

Fig. A

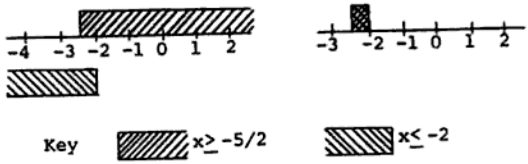
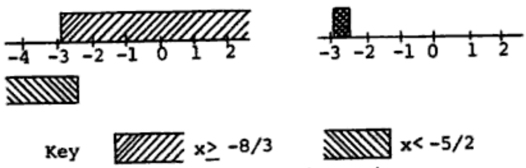



Fig. B




Solution: Case 1. If $2x + 5 \geq 0$, then $|2x + 5| = 2x + 5$ and the inequality becomes

$$2x + 5 \leq x + 3$$
$$x \leq -2$$

For Case 1 we have the simultaneous restrictions

$$2x + 5 \geq 0 \quad \text{and} \quad x \leq -2$$

or

$$x \geq -\frac{5}{2} \quad \text{and} \quad x \leq -2$$

The solution set for Case 1 is

$$X_1 = \left(x \,|\, x \geq -\frac{5}{2} \text{ and } x \leq -2\right)$$
$$= \left(x \,|\, -\frac{5}{2} \leq x \leq -2\right)$$

This inequality holds by noting Figure A. The solution set for Case 1 is the intersection of the two inequalities on the number line. This intersection is the set

$$-2\tfrac{1}{2} \leq x \leq -2.$$

Case 2. If $2x + 5 < 0$, then $|2x + 5| = -(2x + 5)$ and the inequality becomes

$$-(2x + 5) \leq x + 3$$

Multiplying an inequality by -1 reverses the direction of the inequality.

$$2x + 5 \geq -(x + 3) = -x - 3$$
$$3x \geq -8$$
$$x \geq -\frac{8}{3}$$

For Case 2 we have

$$2x + 5 < 0 \quad \text{and} \quad x \geq -\frac{8}{3}$$

or

$$x < -\frac{5}{2} \quad \text{and} \quad x \geq -\frac{8}{3}$$

The solution set is

$$X_2 = \left(x \,|\, x < -\frac{5}{2} \text{ and } x \geq -\frac{8}{3}\right)$$
$$= \left(x \,|\, -\frac{8}{3} \leq x < -\frac{5}{2}\right)$$

This inequality holds by noting Figure B. The solution set for Case 2 is the intersection of the two inequalities on the number line. This intersection is the set

$$-\frac{8}{3} \le x < -\frac{5}{2}.$$

Finally, the solution set, X, of the given inequality is the union of $X_1$ and $X_2$.

$$\begin{aligned} X &= X_1 \cup X_2 \\ &= \left\{x \middle| -\frac{8}{3} \le x < -\frac{5}{2}\right\} \cup \left\{x \middle| -\frac{5}{2} \le x \le -2\right\} \\ &= \left\{x \middle| -\frac{8}{3} \le x \le -2\right\} \end{aligned}$$

**• PROBLEM 293**

Replace the inequality $1 < x < 3$ by a single inequality involving an absolute value.

Solution: Recall:

$$a - b < x < a + b$$

$$= -b < x - a < b, \quad \text{subtracting } a$$

$$= |x - a| < b, \quad \text{definition of absolute value}$$

Replacing a by 2 and b by 1 we obtain:

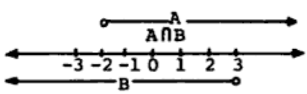
$$2 - 1 < x < 2 + 1$$

$$-1 < x - 2 < 1$$

$$|x - 2| < 1.$$

**• PROBLEM 294**

Let $A = \{x|x > -2\}$ and $B = \{x|x < 3\}$. Describe these sets as collections of points of the number scale. What is $A \cap B$? $A \cup B$?




Solution: The set A consists of all numbers that are greater than -2, so a point belongs to A if, and only if, it lies to the right of the point -2 of the number scale. Similarly, we think of B graphically as the set of points to the left of 3. The intersection $A \cap B$, that is, the set of points in both A and B, is illustrated in the diagram. It consists of the points between -2 and 3, so we have the set equation

$$A \cap B = \{x|x > -2\} \cap \{x|x < 3\} = \{x|-2 < x < 3\}.$$

Every real number belongs to at least one of the sets A or B, since every real number is either greater than -2 or less than 3 (and some numbers are both). In other words, the union $A \cup B = R$, the set of all real numbers.

# CHAPTER 13

# SYSTEMS OF LINEAR EQUATIONS AND INEQUALITIES

**Basic Attacks and Strategies for Solving Problems in this Chapter. See pages 177 to 198 for step-by-step solutions to problems.**

Algebraic methods of solving systems of linear equations in two variables involve applying certain laws of algebra to individual equations or combinations of equations. One of the algebraic solution procedures for a 2 by 2 system of linear equations is called the substitution method. The procedure is:

(1) Choose one of the variables and one of the equations. Rearrange the chosen equation so that it is solved for the chosen variable in terms of the other variable.

(2) Substitute the expression for the chosen variable obtained in Step 1 into the remaining equation. The result will be an equation that involves only one variable.

(3) Solve the equation obtained in Step 2. The number obtained is one of the coordinates of the solution for the system.

(4) Substitute the number obtained in Step 3 into the expression for the variable chosen in Step 1 and simplify. The result is the other coordinate of the ordered pair solution.

When the system of equations is inconsistent, the substitution method produces a false mathematical statement which means that the system has no solution. On the other hand, when the system of equations is dependent, the substitution method produces a true statement in which no variable appears. An infinite number of ordered pairs represents the solution set.

A second algebraic solution procedure for a 2 by 2 system of linear equations is called the addition-subtraction or elimination method. Its strategy is to add or subtract multiples of the two given equations so that one of the variables is eliminated. The resulting equation involves only one variable whose value can be easily solved for. Substitute the value of this variable in either of the original equations and solve for the other variable. The values of the variables represent the coordinates of the solution. As with the previous solution method, the addition-subtraction method can be used to alert the problem solver to inconsistent and dependent systems of equations.

An algebraic solution of a system of three linear equations in three variables can be achieved by using the substitution and/or addition-subtraction method(s). The primary step is to use one of the methods to reduce the original system to a 2 by 2 system and then solve the reduced system using an algebraic method. Once a solution is found for the reduced system, choose one of the original equations, substitute the values already obtained, and solve for the remaining unknown. The result is the third coordinate of the ordered triplet which is the solution for the original system.

## GRAPHING METHOD

The graphing method of solving a $2 \times 2$ system of linear equations involves graphing each equation and determining a point of intersection of the graphs. If the graphs of the equations are non-parallel, non-coincident lines, there is exactly one point of intersection which is the solution. Parallel lines indicate no solution and coincident lines indicate an infinite set of solutions.

## SOLVING SYSTEMS OF INEQUALITIES AND GRAPHING

When solving by graphing a system of inequalities in two variables (where the inequality symbols are all strictly > and/or <), the first step is to rewrite each inequality in the system in the form of

$$y > mx + b \quad \text{or} \quad y < mx + b.$$

The second step is to graph the linear equation,

$$y = mx + b,$$

for each inequality as a straight dotted line. Determine in what region of the $x$-$y$ plane each inequality holds true by selecting points on both sides of the corresponding dotted line and substitute them into the variable statement of the inequality. Shade in the side of the line whose points make the inequality a true statement. Represent each shaded area with a unique pattern (e.g., diagonal shading, vertical shading, etc.). The solution is the intersection of all the shaded areas representing points whose ordered pairs satisfy all conditions in the original system of inequalities.

If the system of inequalities contains the $\geq$ and/or $\leq$ symbols, then the only change in the graphing solution procedure above is to graph the linear equation,

$$y = mx + b,$$

for each inequality as a straight non-dotted (solid) line which will be a part of the solution. If unique pairs of these equations are formed into 2 by 2 linear systems, then they can be solved algebraically as indicated above. The result of the solution of each system yields the coordinates of one of the vertices of the shaded area that represents the solution of the original system.

# Step-by-Step Solutions to Problems in this Chapter, "Systems of Linear Equations and Inequalities"

## SOLVING EQUATIONS IN TWO VARIABLES AND GRAPHING

**• PROBLEM 295**

Solve the simultaneous equations $2x + 4y = 11$, $-5x + 3y = 5$ by the method of substitution and by the method of elimination by addition.

Solution: The method of substitution involves solving for one variable in terms of the other and then substituting the obtained value into the second equation. Thus, we solve the first equation for $x$ and substitute in the second:

$$2x + 4y = 11$$
$$2x = 11 - 4y$$
$$x = \frac{11 - 4y}{2}$$

Replacing $x$ by $\left(\frac{11 - 4y}{2}\right)$ in the second equation,

$$-5\left(\frac{11 - 4y}{2}\right) + 3y = 5$$
$$\frac{-55 + 20y}{2} + 3y = 5$$
$$\frac{-55}{2} + 10y + 3y = 5$$

Multiply both sides by 2,

$$-55 + 20y + 6y = 10$$
$$26y = 65$$
$$y = \frac{65}{26} = \frac{5}{2}\,.$$

Substituting this value for $y$ into the first equation:

$$2x + 4\left(\frac{5}{2}\right) = 11$$
$$2x + 10 = 11$$
$$2x = 1$$
$$x = \frac{1}{2}\,.$$

We obtain the same result by the method of elimination by addition.

$$2x + 4y = 11 \qquad (1)$$
$$-5x + 3y = 5 \qquad (2)$$

Multiplying equation (1) by 5 and equation (2) by 2 and adding the result we obtain:

$$10x + 20y = 55$$
$$-\ \underline{10x + 6y = 10}$$
$$26y = 65$$
$$y = \frac{65}{26} = \frac{5}{2}$$

Once again, replacing $y$ by $\frac{5}{2}$ in equation (1):

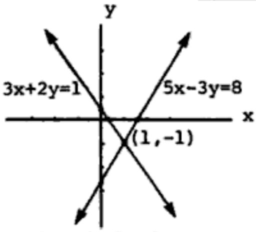
$$2x + 4\left(\frac{5}{2}\right) = 11$$

$$2x + 10 = 11$$
$$2x = 1$$
$$x = \frac{1}{2}$$

Thus $\left\{\left(\frac{1}{2}, \frac{5}{2}\right)\right\}$ is the solution to the given system of equations.

**• PROBLEM 296**

Solve the equations $3x + 2y = 1$ and $5x - 3y = 8$ simultaneously.




Solution: We have 2 equations in 2 unknowns,

$$3x + 2y = 1 \qquad (1)$$

and

$$5x - 3y = 8 \qquad (2)$$

There are several methods to solve this problem. We have chosen to multiply each equation by a different number so that when the two equations are added, one of the variables drops out. Thus

multiplying the first by 3: $\quad 9x + 6y = 3$

and the second by 2: $\quad 10x - 6y = 16$

and adding: $\quad 19x = 19$

$$x = 1$$

Substituting $x = 1$ in the first equation:

$$3(1) + 2y = 3 + 2y = 1$$
$$2y = -2$$
$$y = -1$$

(Alternatively, $y$ might have been found by multiplying the first equation by 5, the second by -3, and adding.)

In this case, then, there is a unique solution: $x = 1$, $y = -1$. This may be checked by replacing $x$ by 1 and $y$ by (-1) in each equation. In equation (1):

$$3x + 2y = 1$$
$$3(1) + 2(-1) = 1$$
$$3 - 2 = 1$$
$$1 = 1$$

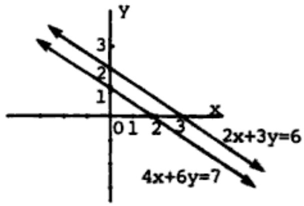
In equation (2):

$$5x - 3y = 8$$
$$5(1) - 3(-1) = 8$$
$$5 - (-3) = 8$$
$$5 + 3 = 8$$
$$8 = 8$$

In other words, the lines whose equations are $3x+2y=1$ and $5x-3y=8$ meet in one and only one point: $(1,-1)$. This, again, may be checked graphically, as seen in the diagram.

## • PROBLEM 297

Solve the equations $2x + 3y = 6$ and $4x + 6y = 7$ simultaneously.




Solution: We have 2 equations in 2 unknowns,

$$2x + 3y = 6 \qquad (1)$$

and

$$4x + 6y = 7 \qquad (2)$$

There are several methods to solve this problem. We have chosen to multiply each equation by a different number so that when the two equations are added, one of the variables drops out. Thus

multiplying equation (1) by 2: $4x + 6y = 12$ (3)

multiplying equation (2) by -1: $\underline{-4x - 6y = -7}$ (4)

adding equations (3) and (4): $0 = 5$

We obtain a peculiar result!

Actually, what we have shown in this case is that if there were a simultaneous solution to the given equations, then 0 would equal 5. But the conclusion is impossible; therefore there can be no simultaneous solution to these two equations, hence no point satisfying both.

The straight lines which are the graphs of these equations must be parallel if they never intersect, but not identical, which can be seen from the graph of these equations (see the accompanying diagram).

## • PROBLEM 298

Solve for $x$ and $y$.

$$x + 2y = 8 \qquad (1)$$

$$3x + 4y = 20 \qquad (2)$$

Solution: Solve equation (1) for $x$ in terms of $y$:

$$x = 8 - 2y \qquad (3)$$

Substitute $(8 - 2y)$ for $x$ in (2):

$$3(8 - 2y) + 4y = 20 \qquad (4)$$

Solve (4) for $y$ as follows:

Distribute: $24 - 6y + 4y = 20$

Combine like terms and then subtract 24 from both sides:

$$24 - 2y = 20$$

$$24 - 24 - 2y = 20 - 24$$

$$-2y = -4 \qquad \text{Divide both sides by } -2: \quad y = 2$$

Substitute 2 for y in equation (1):

$$x + 2(2) = 8$$

$$x = 4$$

Thus, our solution is $x = 4$, $y = 2$.

Check: Substitute $x = 4$, $y = 2$ in equations (1) and (2):

$$4 + 2(2) = 8$$

$$8 = 8$$

$$3(4) + 4(2) = 20$$

$$20 = 20$$

**• PROBLEM 299**

Solve the equations $2x + 3y = 6$ and $y = -(2x/3) + 2$ simultaneously.

Solution: We have 2 equations in 2 unknowns,

$$2x + 3y = 6 \qquad (1)$$

$$y = -(2x/3) + 2 \qquad (2)$$

There are several methods of solution for this problem. Since equation (2) already gives us an expression for y, we use the method of substitution. Substituting $-(2x/3) + 2$ for y in the first equation:

$$2x + 3\left(-\frac{2x}{3} + 2\right) = 6$$

Distributing,

$$2x - 2x + 6 = 6$$

$$6 = 6$$

Apparently we have gotten nowhere! The result $6 = 6$ is true, but indicates no solution. Actually, our work shows that no matter what real number x is, if y is determined by the second equation, then the first equation will always be satisfied.

The reason for this peculiarity may be seen if we take a closer look at the equation $y = -(2x/3) + 2$. It is equivalent to $3y = -2x + 6$, or $2x + 3y = 6$.

In other words, the two equations are equivalent. Any pair of values of x and y which satisfies one satisfies the other.

It is hardly necessary to verify that in this case the graphs of the given equations are identical lines, and that there are an infinite number of simultaneous solutions of these equations.

**• PROBLEM 300**

Find the solution set for the system:

$$3x + 5y = -9$$

$$x - 5y = 17$$

Solution: Upon examination we see that if the left members of the two equations are added and the right members of the two equations are added, we obtain the equation $4x = 8$. (This is justified by the additive principle of equations; we are simply adding equal quantities to both sides of an equation.) Hence the $y$ terms have been eliminated, since the coefficients were additive inverses. This new equation, $4x = 8$, can be easily seen to have $\{2\}$ for its solution set. Now, if we use this value for $x$, we see that upon substituting it into either of the two equations in our system, for example, $3x + 5y = -9$, we obtain $3(2) + 5y = -9$. Upon simplifying, this becomes $y = -3$. (You should convince yourself that had the other equation in the system been selected, the value of $y$ would have been found to be $-3$.)

Therefore the single solution for our system is the ordered pair $(2,-3)$.

**• PROBLEM 301**

Solve algebraically: $\begin{cases} 4x + 2y = -1 & (1) \\ 5x - 3y = 7 & (2) \end{cases}$

Solution: We arbitrarily choose to eliminate x first.

Multiply (1) by 5: $20x + 10y = -5$ (3)

Multiply (2) by 4: $20x - 12y = 28$ (4)

Subtract, (3) - (4): $22y = -33$ (5)

Divide (5) by 22: $y = -\frac{33}{22} = -\frac{3}{2}$.

To find x, substitute $y = -\frac{3}{2}$ in either of the original equations. If we use Eq. (1), we obtain $4x + 2(-3/2) = -1$, $4x - 3 = -1$, $4x = 2$, $x = \frac{1}{2}$.

The solution $\left(\frac{1}{2},-\frac{3}{2}\right)$ should be checked in both equations of the given system.

Replacing $\left(\frac{1}{2},-\frac{3}{2}\right)$ in Eq. (1):

$$\begin{aligned} 4x + 2y &= -1 \\ 4\left(\frac{1}{2}\right)+2\left(-\frac{3}{2}\right) &= -1 \\ \frac{4}{2} - 3 &= -1 \\ 2 - 3 &= -1 \\ -1 &= -1 \end{aligned}$$

Replacing $\left(\frac{1}{2},-\frac{3}{2}\right)$ in Eq. (2):

$$\begin{aligned} 5x - 3y &= 7 \\ 5\left(\frac{1}{2}\right)-3\left(-\frac{3}{2}\right) &= 7 \\ \frac{5}{2} + \frac{9}{2} &= 7 \\ \frac{14}{2} &= 7 \\ 7 &= 7 \end{aligned}$$

(Instead of eliminating x from the two given equations, we could have eliminated y by multiplying Eq. (1) by 3, multiplying Eq. (2) by 2, and then adding the two derived equations.)

Solve one equation for one unknown and substitute in the other equation to find the solutions of the following systems.

$$xy = 1 \quad (1)$$

$$x + 2y = 3 \quad (2)$$

Solution: We can solve equation (2) for x by adding (-2y) to both sides:

$$x = 3 - 2y \quad (3)$$

Substituting (3 - 2y) for x in equation (1):

$$(3-2y)y = 1$$

Distributing,

$$3y - 2y^2 = 1$$

$$3y - 2y^2 - 1 = 0$$

$$-2y^2 + 3y - 1 = 0$$

Factoring,

$$(-2y + 1)(y-1) = 0$$

Whenever the product of two numbers ab = 0, either a = 0 or b = 0. Thus, either

$$-2y + 1 = 0 \quad \text{or} \quad y - 1 = 0$$

$$-2y = -1$$

and

$$y = \frac{-1}{-2} = \frac{1}{2} \text{ or } \quad y = 1$$

Replacing y by $\frac{1}{2}$ in equation (3), we obtain the corresponding x value:

$$x = 3 - 2\left(\frac{1}{2}\right)$$

$$x = 3 - 1$$

$$x = 2$$

Replacing y by 1 in equation (3), we obtain

$$x = 3 - 2(1)$$

$$x = 3 - 2$$

$$x = 1$$

Thus, the two solutions to this system appear to be $\left(2,\frac{1}{2}\right)$ and (1,1). These can be verified by the following check: Replace (x,y) by (2,1/2) in equations (1) and (2):

$$xy = 1 \quad (1)$$

$$(2)\left(\frac{1}{2}\right) = 1$$

$$\frac{2}{2} = 1$$

$$1 = 1$$

$$x + 2y = 3 \quad (2)$$

$$2 + 2\left(\frac{1}{2}\right) = 3$$

$$2 + \frac{2}{2} = 3$$

$$2 + 1 = 3$$
$$3 = 3$$

Replace (x,y) by (1,1) in equations (1) and (2):

$$xy = 1 \qquad (1)$$
$$1 \cdot 1 = 1$$
$$1 = 1$$

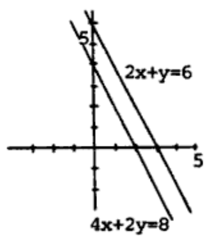
$$x + 2y = 3 \qquad (2)$$
$$1 + 2(1) = 3$$
$$1 + 2 = 3$$
$$3 = 3$$

Thus, the solutions to this system are indeed $\left(2, \frac{1}{2}\right)$ and (1,1).

**• PROBLEM 303**

Determine the nature of the system of linear equations

$$2x + y = 6 \qquad (1)$$
$$4x + 2y = 8 \qquad (2)$$




Solution: These linear equations may be written in the standard form y = mx + b:

$$y = -2x + 6 \qquad (3-1)$$
$$\text{and } y = -2x + 4 \qquad (4-2)$$

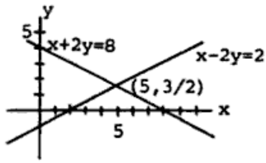
Observe that the slope of each line is m = -2, but the y-intercepts are different, that is, b = 6 for equation (3-1) and b = 4 for equation (4-2). The lines are therefore parallel and distinct. The graph below also indicates that the lines are parallel. The system is therefore inconsistent, and there is no solution.

**• PROBLEM 304**

Determine the nature of the system of linear equations

$$x + 2y = 8$$

$$x - 2y = 2.$$




Solution: Add the two equations, eliminating the y-terms, to obtain a single equation in terms of x. Values of x satisfying this equation will yield solutions of the system.

$$\begin{array}{rcll} x + 2y & = & 8 & \quad (1) \\ + \; x - 2y & = & 2 & \quad (2) \\ \hline 2x & = & 10 & \\ x & = & 5 & \end{array}$$

Substituting $x = 5$ into Equation (1) yields $y = (8 - x)/2 = (8 - 5)/2 = \frac{3}{2}$ or into Equation (2) yields $y = (2 - x)/(-2) = (2 - 5)/(-2) = \frac{3}{2}$. Thus we have $x = 5$, $y = \frac{3}{2}$ as the only solution of the system. Alternately, the figure indicates that the lines intersect in the point $\left(5, \frac{3}{2}\right)$. The system is therefore consistent and independent. Substitution of $x = 5$ and $y = \frac{3}{2}$ in both equations yields

$$5 + 2\left(\frac{3}{2}\right) = 8, \text{ or } 8 = 8$$

$$5 - 2\left(\frac{3}{2}\right) = 2, \text{ or } 2 = 2$$

so that $x = 5$, $y = 3/2$, is a solution, and the only solution of the system.

**• PROBLEM 305**

Show that the following pair of equations is dependent by showing that the two equations are equivalent.

$$3x - 2y = 9$$

$$4y - 6x = -18$$

Solution: We can derive $4y - 6x = -18$ from $3x - 2y = 9$ by applying field properties.

$$3x - 2y = 9$$

Multiplying both sides by $(-2)$, $(-2)(3x - 2y) = (-2)9$

Distributing, $-6x + 4y = -18$

Commuting, $4y - 6x = -18$

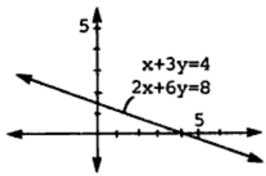
Thus, since $3x - 2y = 9$ is equivalent to $4y - 6x = -18$ the two equations are dependent.

**• PROBLEM 306**

Determine the nature of the system of linear equations

$$x + 3y = 4 \qquad (1)$$

$$2x + 6y = 8. \qquad (2)$$




Solution: If the first equation is multiplied by 2, the solution of the system will not be altered. Note, however, that the two equations are then identical. The graph too, indicates that the lines coincide, and therefore the system is consistent and dependent. It can be verified by substitution that three of the solutions are $x = 1$, $y = 1$; $x = 7$, $y = -1$; and $x = -5$, $y = 3$.

**• PROBLEM 307**

Solve for $x$ and $y$: $\begin{cases} 3x + 5y = 9, & (1) \\ 7x - 10y = 8 & (2) \end{cases}$

Solution: Multiply each member of the first equation by 2; thus $6x + 10y = 18$. Now add each member of the resulting equation to the corresponding member of the second equation.

$$\begin{array}{r} 6x + 10y = 18 \\ \underline{7x - 10y = 18} \\ 13x = 26 \\ x = 2 \end{array}$$

To solve for $y$, replace $x$ by 2 in either equation. Using equation (1),

$$\begin{aligned} 3x + 5y &= 9 \\ 3(2) + 5y &= 9 \\ 6 + 5y &= 9 \\ 5y &= 3 \\ y &= 3/5 \end{aligned}$$

Therefore our solution is $x = 2$, $y = 3/5$.

Check: To verify our solutions, we substitute the values 2 and 3/5 for $x$ and $y$ in equations (1) and (2):

$$3x + 5y = 9$$
$$3(2) + 5(3/5) = 9$$
$$6 + 3 = 9$$
$$9 = 9$$

$$7x - 10y = 8$$
$$7(2) - 10(3/5) = 8$$
$$14 - 6 = 8$$
$$8 = 8$$

## • PROBLEM 308

Solve for $x$ and $y$.

$$3x + 2y = 23 \qquad (1)$$
$$x + y = 9 \qquad (2)$$

Solution: Multiply equation (2) by $-3$:

$$-3x - 3y = -27 \qquad (3)$$

Add equations (1) and (3):

$$\begin{array}{r} 3x + 2y = 23 \\ \underline{-3x - 3y = -27} \\ -y = -4 \\ y = 4 \end{array}$$

Substitute 4 for $y$ in equation (1):

$$3x + 2(4) = 23$$
$$3x + 8 = 23$$

Subtract 8 from both sides: $3x = 15$

Divide each side by 3: $x = 5$

Hence our solution is, $x = 5$ and $y = 4$.

Check: Substitute 5 for $x$ and 4 for $y$ in equation (1):

$$3(5) + 2(4) = 23$$
$$23 = 23$$

Substitute 5 for $x$ and 4 for $y$ in equation (2):

$$5 + 4 = 9$$
$$9 = 9 .$$

## • PROBLEM 309

Solve for $x$ and $y$.

$$4x + 3y = 23 \qquad (1)$$
$$2x - 5y = 31 \qquad (2)$$

Solution: Multiply equation (2) by $-2$:

$$-4x + 10y = -62 \qquad (3)$$

Add equation (3) to equation (1):

$$\begin{array}{r} 4x + 3y = 23 \\ \underline{+ (-4x + 10y = -62)} \\ 13y = -39 \end{array}$$

$$y = -3$$

Substitute $-3$ for $y$ in equation (1):

$$4x + 3(-3) = 23$$
$$4x - 9 = 23$$
$$4x = 32$$
$$x = 8$$

Hence the solution is $x = 8, y = -3$.

Check: Substitute 8 for $x$ and $-3$ for $y$ in (1),

$$4(8) + 3(-3) = 23$$
$$32 - 9 = 23$$
$$23 = 23.$$

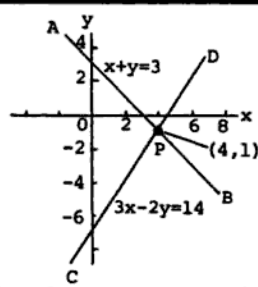
Substitute 8 for $x$ and $-3$ for $y$ in (2),

$$2(8) - 5(-3) = 31$$
$$16 + 15 = 31$$
$$31 = 31\ .$$

**• PROBLEM 310**

Find the point of intersection of the graphs of the equations:

$$\begin{cases} x + y = 3, \\ 3x - 2y = 14. \end{cases}$$




Solution: To solve these linear equations, solve for $y$ in terms of $x$. The equations will be in the form $y = mx + b$, where $m$ is the slope and $b$ is the intercept on the y-axis.

$$x + y = 3$$
$$y = 3 - x \quad \text{subtract } x \text{ from both sides}$$

$$3x - 2y = 14 \quad \text{subtract } 3x \text{ from both sides}$$

$$-2y = 14 - 3x \quad \text{divide by } -2.$$

$$y = -7 + \frac{3}{2}x$$

The graphs of the linear functions, $y = 3 - x$ and $y = -7 + \frac{3}{2}x$, can be determined by plotting only two points. For example, for $y = 3 - x$, let $x = 0$, then $y = 3$. Let $x = 1$, then $y = 2$. The two points on this first line are $(0,3)$ and $(1,2)$. For $y = -7 + 3/2\ x$, let $x = 0$, then

$y = -7$. Let $x = 1$, then $y = -5\frac{1}{2}$. The two points on this second line are $(0,-7)$ and $(1, -5\frac{1}{2})$.
To find the point of intersection P of

$$x + y = 3$$

and

$$3x - 2y = 14,$$

solve them algebraically. Multiply the first equation by 2. Add these two equations to eliminate the variable y.

$$\begin{array}{r} 2x + 2y = 6 \\ \underline{3x - 2y = 14} \\ 5x \qquad = 20 \end{array}$$

Solve for x to obtain $x = 4$. Substitute this into $y = 3 - x$ to get $y = 3 - 4 = -1$. P is (4,1). AB is the graph of the first equation, and CD is the graph of the second equation. The point of intersection P of the two graphs is the only point on both lines. The coordinates of P satisfy both equations and represent the desired solution of the problem. From the graph, P seems to be the point $(4,-1)$. These coordinates satisfy both equations, and hence are the exact coordinates of the point of intersection of the two lines.

To show that $(4,-1)$ satisfies both equations, substitute this point into both equations.

$$\begin{array}{rcl} x + y &=& 3 \\ 4 + (-1) &=& 3 \\ 4 - 1 &=& 3 \\ 3 &=& 3 \end{array} \qquad\qquad \begin{array}{rcl} 3x - 2y &=& 14 \\ 3(4) - 2(-1) &=& 14 \\ 12 + 2 &=& 14 \\ 14 &=& 14 \end{array}$$

## • PROBLEM 311

Find the solution set of the system

$$2x - 12y = 3 \qquad (1)$$

$$3x + 9y = 4 \qquad (2)$$

Solution: This system can be solved using the multiplication-addition method: The least common multiple of the x coefficients is 6. Multiply equation (1) by 3 and equation (2) by $-2$ to obtain

$$3(2x - 12y) = 3 \cdot 3$$

$$6x - 36y = 9 \qquad (3)$$

$$-2(3x + 9y) = 4.(-2)$$

$$-6x - 18y = -8 \qquad (4)$$

Adding equations (3) and (4),

$$\begin{array}{r} 6x - 36y = 9 \\ \underline{-6x - 18y = -8} \\ -54y = 1 \\ y = -\frac{1}{54} \end{array}$$

To solve for x, we substitute this value of y in either of our given equations. Substituting $\left(\frac{-1}{54}\right)$ for y in (2):

$$3x + 9y = 4$$

$$3x + 9\left(-\frac{1}{54}\right) = 4$$

$$3x - \frac{1}{6} = 4$$

$$3x = 4 + \frac{1}{6}$$

$$3x = \frac{24}{6} + \frac{1}{6}$$

$$3x = \frac{25}{6}$$

$$x = \frac{25}{18}$$

Thus our solution set is $\left\{\left(\frac{25}{18}, -\frac{1}{54}\right)\right\}$, and we perform the following check to verify this result.

Check: Replace x and y by $\left(\frac{25}{18}, -\frac{1}{54}\right)$ respectively in (1) and (2):

(1) $$2x - 12y = 3$$

$$2\left(\frac{25}{18}\right) - 12\left(\frac{-1}{54}\right) = 3$$

$$\frac{50}{18} + \frac{12}{54} = 3$$

$$\frac{50}{18} + \frac{4}{18} = 3$$

$$\frac{54}{18} = 3$$

$$3 = 3$$

(2) $$3x + 9y = 4$$

$$3\left(\frac{25}{18}\right) + 9\left(\frac{-1}{54}\right) = 4$$

$$\frac{75}{18} - \frac{9}{54} = 4$$

$$\frac{75}{18} - \frac{3}{18} = 4$$

$$\frac{72}{18} = 4$$

$$4 = 4$$

**• PROBLEM 312**

Obtain the simultaneous solution set of the equations

$$3x + 4y = -6 \qquad (1)$$

$$5x + 6y = -8 \qquad (2)$$

Solution: We eliminate one of the unknowns to obtain an equation in one unknown whose root is one of the numbers in a simultaneous solution pair. We arbitrarily choose to eliminate y. Notice that the lowest common multiple, LCM, of the coefficients of y in (1) and (2) is 12. This is because the coefficients of y are $4 = 2 \cdot 2$ and $6 = 2 \cdot 3$. Thus their LCM will be $2 \cdot 2 \cdot 3 = 12$.

To obtain a coefficient of y equal to 12 in equation (1) we multiply the equation by 3,

$$3(3x + 4y) = 3(-6)$$

$$9x + 12y = -18 \qquad (3)$$

To obtain a coefficient of $y$ equal to 12 in equation (2) we multiply equation (2) by 2,

$$2(5x + 6y) = 2(-8)$$

$$10x + 12y = -16 \qquad (4)$$

Now subtract equation (4) from equation (3),

$$\begin{array}{r} 9x + 12y = -18 \\ \underline{10x + 12y = -16} \\ -x \qquad = -2 \\ x = 2 \end{array}$$

Consequently the first number in $(x,y)$ is 2. We obtain the second number by replacing $x$ by 2 in either (1) or (2) and solving for $y$. We shall choose (1) and get

$$3(2) + 4y = -6$$

$$6 + 4y = -6$$

$$4y = -12$$

$$y = -3$$

Hence $(x,y) = (2,-3)$.

Check: Replacing $x$ by 2 and $y$ by $-3$ in (1) and (2), we get

from equation (1): $3x + 4y = -6$

$$3(2) + 4(-3) = -6$$

$$6 + (-12) = -6$$

$$-6 = -6$$

from equation (2): $5x + 6y = -8$

$$5(2) + 6(-3) = -8$$

$$10 + (-18) = -8$$

$$-8 = -8$$

Therefore the simultaneous solution set is $\{(2,-3)\}$.

## SOLVING EQUATIONS IN THREE VARIABLES

**• PROBLEM 313**

Solve the system of equations,

$$2x - y - 4z = 3 \qquad (1)$$

$$-x + 3y + z = -10 \qquad (2)$$

$$3x + 2y - 2z = -2 \qquad (3)$$

Solution: To solve a system of 3 equations in 3 unknowns, we first reduce it to a system of 2 equations in 2 unknowns, a process which can often be done many ways. Although various other algebraic manipulations may be used to arrive at the same result, we will employ the following method: Multiplying equation 1 by (-1) we obtain,

$$-2x + y + 4z = -3 \qquad (4)$$

Adding equations (4), (2), and (3) we obtain,

$$\begin{array}{r} -2x + y + 4z = -3 \\ -x + 3y + z = -10 \\ \underline{3x + 2y - 2z = -2} \\ 6y + 3z = -15 \end{array} \qquad (5)$$

Multiplying equation (2) by 3 we obtain,

$$-3x + 9y + 3z = -30 \qquad (6)$$

Adding equations (6) and (3) we obtain,

$$\begin{array}{r} -3x + 9y + 3z = -30 \\ \underline{3x + 2y - 2z = -2} \\ 11y + z = -32 \end{array} \qquad (7)$$

Multiplying equation (7) by (-3) we obtain,

$$-33y - 3z = 96 \qquad (8)$$

Adding equations (8) and (5) we obtain,

$$\begin{array}{r} -33y - 3z = 96 \\ \underline{6y + 3z = -15} \\ -27y = 81 \\ y = -3 \end{array}$$

Solving for z, we replace, y by (-3) in equation (5):

$$\begin{array}{r} 6y + 3z = -15 \\ 6(-3) + 3z = -15 \\ -18 + 3z = -15 \\ 3z = 3 \\ z = 1 \end{array}$$

Solving for x, we replace y by (-3) and z by 1 in equation (1):

$$\begin{array}{r} 2x - y - 4z = 3 \\ 2x - (-3) - 4(1) = 3 \\ 2x + 3 - 4 = 3 \\ 2x - 1 = 3 \\ 2x = 4 \\ x = 2 \end{array}$$

Thus the solution to this system is $x = 2$, $y = -3$, and $z = 1$.

Check: Replace x,y, and z by 2, -3, and 1 in each equation.

$$2x - y - 4z = 3 \qquad (1)$$

$$2(2) - (-3) - 4(1) = 3$$

$$4 + 3 - 4 = 3$$

$$3 = 3$$

$$-x + 3y + z = -10 \qquad (2)$$

$$-(2) + 3(-3) + 1 = -10$$

$$-2 - 9 + 1 = -10$$

$$-10 = -10$$

$$3x + 2y - 2z = -2 \qquad (3)$$

$$3(2) + 2(-3) - 2(1) = -2$$

$$6 - 6 - 2 = -2$$

$$-2 = -2$$

**• PROBLEM 314**

Solve the system

$$2x - y + 4z = 1 \qquad (1)$$

$$x - y + z = 0 \qquad (2)$$

$$x + y + z = 1 \qquad (3)$$

Solution: It is easiest to eliminate the variable y since the equations (1), (2), (3) differ only by a factor of +1 or - 1 for the variable y. (For the other variables x and z, the equations differ by factors of $\pm$ 2 for x and $\pm$ 4 for z).

Multiplying equation (1) by - 1 we obtain:

$$-2x + y - 4z = -1 \qquad (4)$$

$$x - y + z = 0 \qquad (2)$$

$$x + y + z = 1 \qquad (3)$$

Add equations (4) and (2) to eliminate the variable y and we obtain a new equation (5) in x and z.

$$-2x + y - 4z = -1 \qquad (4)$$

$$\underline{x - y + z = 0} \qquad (2)$$

$$-x - 3z = -1 \qquad (5)$$

Add (2) and (3) to obtain another equation (6) in the variables x and z.

$$x - y + z = 0 \qquad (2)$$

$$\underline{x + y + z = 1} \qquad (3)$$

$$2x + 2z = 1 \qquad (6)$$

Now we have a new system of 2 equations in 2 unknowns x and z:

$$- x - 3z = - 1 \qquad (5)$$

$$2x + 2z = 1 \qquad (6)$$

We must solve for one variable. The simplest way is to eliminate x. Multiply equation (5) by 2 and we obtain:

$$- 2x - 6z = - 2 \qquad (7)$$

$$2x + 2z = 1 \qquad (6)$$

Add equations (7) and (6) to obtain:

$$- 4z = - 1 \qquad (8)$$

Divide equation (8) by - 4 to solve for z.

$$z = \frac{1}{4}$$

Substitute z into either (5) or (6) to find x. For equation (5) then we have:

$$- x - 3z = - 1 \qquad (5)$$

$$z = \frac{1}{4}$$

$$- x - 3\left(\frac{1}{4}\right) = - 1$$

$$- x - \frac{3}{4} = - 1$$

$$x = \frac{1}{4}$$

Given x and z we can now solve for y by substituting x and z into any of the three original equations (1), (2) or (3). For equation (2)

$$x - y + z = 0 \qquad (2)$$

$$x = \frac{1}{4}$$

$$z = \frac{1}{4}$$

$$\frac{1}{4} - y + \frac{1}{4} = 0$$

$$- y + \frac{2}{4} = 0$$

$$y = \frac{1}{2}$$

The solution of the original system is then:
$x = \frac{1}{4}$; $y = \frac{1}{2}$ ; $z = \frac{1}{4}$ .

To check, substitute the solution into each of the three original equations (1), (2),and (3).

For (1): $$2x - y + 4z = 1 \qquad (1)$$

$$2\left(\frac{1}{4}\right) - \frac{1}{2} + 4\left(\frac{1}{4}\right) = 1$$

$$\frac{1}{2} - \frac{1}{2} + 1 = 1$$

$$1 = 1$$

For (2): $$x - y + z = 0 \qquad (2)$$

$$\frac{1}{4} - \frac{1}{2} + \frac{1}{4} = 0$$

$$\frac{2}{4} - \frac{1}{2} = 0$$

$$\frac{2}{4} - \frac{2}{4} = 0$$

$$0 = 0$$

For (3): $$x + y + z = 1 \qquad (3)$$

$$\frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$$

$$\frac{2}{4} + \frac{2}{4} = 1$$

$$1 = 1$$

**• PROBLEM 315**

Solve the system

$$2x + 3y - 4z = -8 \qquad (1)$$

$$x + y - 2z = -5 \qquad (2)$$

$$7x - 2y + 5z = 4 \qquad (3)$$

Solution: We cannot eliminate any variable from two pairs of equations by a single multiplication. However, both x and z may be eliminated from equations 1 and 2 by multiplying Equation 2 by $-2$. Then

$$2x + 3y - 4z = -8 \qquad (1)$$

$$-2x - 2y + 4z = 10 \qquad (4)$$

By addition, we have $y = 2$. Although, we may now eliminate either x or z from another pair of equations, we can more conveniently substitute $y = 2$ in Equations 2 and 3 to get two equations in two variables. Thus, making the substitution $y = 2$ in Equations 2 and 3, we have

$$x - 2z = -7 \qquad (5)$$

$$7x + 5z = 8 \qquad (6)$$

Multiply (5) by 5 and multiply (6) by 2. Then add the two new equations. Then $x = -1$. Substitute x in either (5) or (6) to find z.

The solution of the system is $x = -1$, $y = 2$, and $z = 3$. Check by substitution.

**• PROBLEM 316**

Find the solution set for the system:

$$\begin{aligned} 3x + 4y - z &= -2 \\ 2x - 3y + z &= 4 \\ x - 6y + 2z &= 5 \end{aligned}$$

**Solution:** Adding the first and second equations, we obtain another equation without a term involving z:

$$\begin{array}{r} 3x + 4y - z = -2 \\ \underline{2x - 3y + z = 4} \\ 5x + y \quad = 2 \end{array}$$

Similarly, after multiplying through by $-2$ in the second equation, we can use this new equation and the third one to obtain another equation without a term involving z:

$$\begin{array}{r} -4x + 6y - 2z = -8 \\ \underline{x - 6y + 2z = 5} \\ -3x \qquad = -3 \end{array}$$

Our problem has been somewhat simplified in that not only have we obtained an equation without a term involving z, but we have obtained one without a y term.

The solution set of $-3x = -3$ is $\{1\}$. Upon substituting this into the equation $5x + y = 2$, we find that $y = -3$. Finally, upon substituting these values for x and y in either of the three equations of the system, we can obtain a value for z. If we use the first equation, $3x + 4y - z = -2$, we find that $z = -7$.

Hence the solution set for this system is $\{(1, -3, -7)\}$.

**• PROBLEM 317**

Solve for x, y and z:

$$5x + y - z = 9, \qquad (1)$$

$$3x + y + 2z = 17, \qquad (2)$$

$$x + 2y + 3z = 20. \qquad (3)$$

**Solution:** Subtract (2) from (1):

$$\begin{array}{r} 5x + y - z = 9 \\ \underline{-(3x + y + 2z = 17)} \\ 2x - 3z = -8 \end{array} \qquad (4)$$

Multiply (2) by 2: $6x + 2y + 4z = 34.$ (5)

Subtract (3) from (5):

$$\begin{array}{r} 6x + 2y + 4z = 34 \\ \underline{-\ (x + 2y + 3z = 20)} \\ 5x + z = 14 \end{array} \qquad (6)$$

Subtract $5x$ from both sides: $z = 14 - 5x$ (7)

Substitute $(14 - 5x)$ for $z$ in equation (4):

$$2x - 3(14 - 5x) = -8 \ .$$

Distribute: $2x - 42 + 15x = -8$

$$17x - 42 = -8$$

Add 42 to both sides: $17x = 34$

$$x = 2$$

Substitute 2 for $x$ in equation (7)

$$z = 14 - 5(2) = 14 - 10 = 4 \ .$$

Therefore, $x = 2$, and $z = 4$.

Substitute in (1): $5(2) + y - 4 = 9$

$$10 + y - 4 = 9$$

$$6 + y = 9$$

Subtract 6 from both sides: $y = 3$

Thus, $x = 2$, $y = 3$, $z = 4$.

Check:

$$5(2) + 3 - 4 = 9,$$
$$9 = 9.$$

$$3(2) + 3 + 2(4) = 17,$$
$$17 = 17.$$

$$2 + 2(3) + 3(4) = 20,$$
$$20 = 20.$$

• **PROBLEM 318**

Solve the system:

$$2a - 3b + c = 2 \qquad (1)$$
$$3a + 2b - c = 4 \qquad (2)$$
$$2a - 3b + c = 5 \qquad (3)$$

<u>Solution:</u> Observe that equations (1) and (3) are inconsistent.

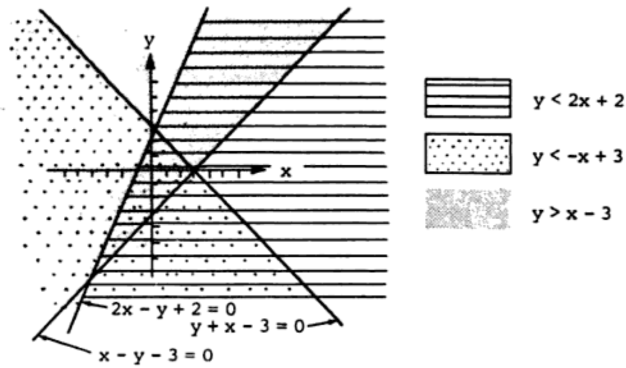
$$2a - 3b + c = 2 \neq 5 = 2a - 3b + c$$

This is a contradiction; that is, $2a - 3b + c$ cannot equal both 2 and 5 at the same time. This implies that there are no values of $a, b$, and $c$ which will solve this set of simultaneous equations, hence this system has no solution.

**• PROBLEM 319**

Solve the following system graphically.

$$y - x > -3$$
$$y - 2x < 2$$
$$x + y - 3 < 0$$




Solution: We may rewrite the system:

$$y > x - 3$$
$$y < 2x + 2$$
$$y < -x + 3$$

Graph the linear equation, $y = mx + b$, for each inequality as a straight dotted line. Thus, we graph

$$y = x - 3$$
$$y = 2x + 2$$
$$y = -x + 3$$

To determine in what region of the x - y plane the inequality holds, select points on both sides of the corresponding dotted line and substitute them into the variable statement. Shade in the side of the line whose point makes the inequality a true statement.

The graphs of the variable sentences are represented in the accompanying figure by diagonal, horizontal, and vertical shading, respectively.

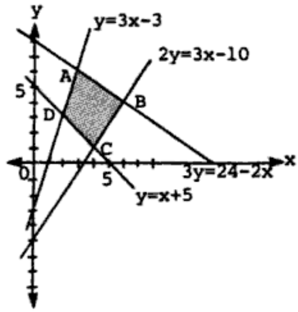
The triple-shaded triangular region is the set of all points whose coordinate pairs satisfy all three conditions as defined by the three inequalities in the system.

**• PROBLEM 320**

Draw the graph of the given system of inequalities, and determine the coordinates of the vertices of the polygon which forms the boundary.

$$y \leq 3x - 3 \qquad (1)$$
$$3y \leq 24 - 2x \qquad (2)$$

$$2y \geq 3x - 10 \tag{3}$$
$$y \geq -x + 5 \tag{4}$$




Solution: y is expressed in terms of x. For each inequality draw the corresponding equality. Choose a point on each side of each solid line to determine the area where the particular inequality holds. Shade in that region. The graph of the given system of inequalities consists of the hatched area and the four lines which form the boundary, that is, the polygon ABCD. The vertex A is found by solving the system obtained by writing Equations 1 and 2 as

$$y = 3x - 3 \tag{5}$$

$$3y = 24 - 2x \tag{6}$$

Solving the system of equations 5 and 6, we have

$$x = 3, \qquad y = 6$$

The coordinates of the vertex A are therefore (3, 6). In a similar manner, the coordinates of B, C, and D are found to be (6, 4)(4, 1) and (2, 3), respectively.

# CHAPTER 14

# DETERMINANTS AND MATRICES

**Basic Attacks and Strategies for Solving Problems in this Chapter. See pages 199 to 231 for step-by-step solutions to problems.**

Any rectangular array of numbers, given by $m$ rows and $n$ columns, is a matrix. Associated with a square matrix $A$ is a real number called a determinant of $A$, denoted by $|A|$. The procedure for finding the value of a second order determinant, that is, a determinant of a $2 \times 2$ matrix, is given by

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - bc,$$

where $a$, $b$, $c$, and $d$ are the elements in the matrix.

Second order determinants are used in finding the solution of any system of two linear equations in two variables by using the well-known Cramer's Rule. The procedure for using this rule is clearly explained in Problem #324 in this chapter. Notice that the system of equations must be in standard form before Cramer's Rule can be applied.

Another method for finding the solution of a system of two linear equations in two variables (in standard form) involves writing the system in matrix format and solving for the variables. The procedure includes finding the multiplicative inverse of the matrix involving the coefficients of the variables in the system and then multiplying this matrix throughout the matrix equation. For example, given the system

$$5x - y = 7$$

$$2x + 3y = -1$$

we can write it in matrix form as follows:

$$\begin{bmatrix} 5 & -1 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}.$$

Then, the inverse matrix $C^{-1}$ for the coefficients of the variables matrix

$$C = \begin{bmatrix} 5 & -1 \\ 2 & 3 \end{bmatrix}$$

is given by

$$C^{-1} = \left(\frac{1}{\det C}\right)\begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix} = \left(\frac{1}{17}\right)\begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix}.$$

Multiplying throughout the equation by $C^{-1}$ we obtain

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{17}\cdot\begin{bmatrix} 3 & 1 \\ -2 & 5 \end{bmatrix}\cdot\begin{bmatrix} 7 \\ -1 \end{bmatrix} \quad \text{or}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{17}\cdot\begin{bmatrix} 20 \\ -19 \end{bmatrix}.$$

Thus, $x = \dfrac{20}{17}$ and $y = \dfrac{19}{17}$.

A determinant of a $3 \times 3$ matrix can be found in more than one way. Two popular procedures are highlighted. The first procedure is the so-called "crossing pattern" which starts with rewriting the first two columns in the matrix to the right of the original matrix as follows:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \rightarrow \begin{array}{|ccc|cc} a & b & c & a & b \\ d & e & f & d & e \\ g & h & i & g & h \end{array}$$

$$\qquad\qquad\qquad\qquad - \; - \; - \qquad + \; + \; +$$

Then, lightly draw in six arrows (as shown above) and form products among the numbers along each arrow. Each arrow pointing upward yields a negative product, while each pointing downward yields a positive product. Finally, the sum of the products is the value of the determinant.

The second approach to finding the value of a determinant of a $3 \times 3$ matrix is to use the concept of cofactor. The procedure for this concept is formed by multiplying each of the three elements of the first row of a $3 \times 3$ matrix by its corresponding cofactor and then adding the three results to obtain the determinant. Problem #339 in this chapter clearly illustrates this procedure.

# Step-by-Step Solutions to Problems in this Chapter, "Determinants and Matrices"

## DETERMINANTS OF SECOND ORDER

**• PROBLEM 321**

Find the value of the determinant $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}$.

Solution: The value of a 2X2 determinant can be found by the following equation:

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc .$$

Hence, the value of $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}$ is :

$$\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = (1)(3) - (2)(2) = 3 - 4 = -1.$$

**• PROBLEM 322**

Evaluate the determinant

$$\begin{vmatrix} 3 & 5 \\ -2 & 3 \end{vmatrix}$$

Solution: The determinant of any $2 \times 2$ matrix is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Apply this rule to the given $2 \times 2$ matrix.

$$\begin{vmatrix} 3 & 5 \\ -2 & 3 \end{vmatrix} = (3)(3) - (-2)(5) = 9 + 10 = 19$$

**• PROBLEM 323**

Evaluate, or expand, the determinant

$$\begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix}$$

Solution: The determinant of a 2 by 2 matrix is defined to be

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc. \text{ Thus the solution is:}$$

$$\begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} = (2)(-1) - (3)(3) = -2 - 9 = -11.$$

Solve $\quad x + y = 3$

$\quad\quad\quad 2x + 3y = 1$

Solution: The values for x and y can be determined by use of Cramer's Rule and determinants. The value for x is the quotient of two determinants. The determinant in the denominator consists of vertical columns in which the numbers are the coefficients of the variables. The determinant in the numerator is the same as the determinant in the denominator, except that the first vertical column is replaced by the constant terms. Note that the first vertical column in the two given equations corresponds to the x term.

$$x + y = 3$$
$$2x + 3y = 1$$

(Illustration)

1st vertical column, 2nd vertical column, 3rd vertical column

The third vertical column consists of the constant terms. Hence,

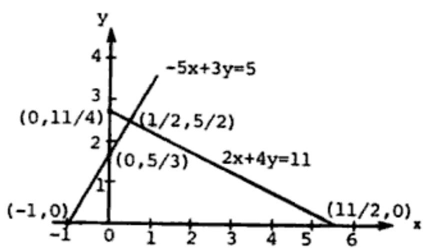
$$x = \frac{\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}} = \frac{(3)(3) - (1)(1)}{(1)(3) - (2)(1)} = \frac{9 - 1}{3 - 2} = \frac{8}{1} = 8.$$

The value for y is also the quotient of two determinants. The determinant in the denominator is the same as the determinant in the denominator used for finding x. The determinant in the numerator is the same as the determinant in the denominator, except that the second vertical column is replaced by the constant terms. Note that the second vertical column in the two given equations corresponds to the y term. (See the Illustration). Hence,

$$y = \frac{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}} = \frac{(1)(1) - (2)(3)}{(1)(3) - (2)(1)} = \frac{1 - 6}{3 - 2} = \frac{-5}{1} = -5 .$$



Solve the equations $2x + 4y = 11$, $-5x + 3y = 5$ graphically and by means of determinants.



Solution A: To solve a set of linear equations graphically we find their point of intersection (which satisfies both equations simultaneously). Draw both lines by determining their y- and x-intercepts by setting $x = 0$ and $y = 0$ respectively. See the following tables. Note: We solve for $y$ before finding the y-intercept and solve for $x$ to find the x-intercept.

$$2x + 4y = 11 \; ; \qquad 2x + 4y = 11$$
$$4y = 11 - 2x \qquad 2x = 11 - 4y$$
$$y = \frac{11 - 2x}{4} \qquad x = \frac{11 - 4y}{2}$$

$$-5x + 3y = 5 \; ; \qquad -5x + 3y = 5$$
$$3y = 5 + 5x \; ; \qquad 3y = 5 + 5x$$
$$y = \frac{5 + 5x}{3} \qquad 3y - 5 = 5x$$
$$\frac{3y - 5}{5} = x$$

$2x + 4y = 11$

| | | |
|---|---|---|
| $x$ | $\frac{11 - 2x}{4} = y$ | $y$ |
| 0 | $\frac{11 - 2(0)}{4}$ | $\frac{11}{4}$ |
| $y$ | $\frac{11 - 4y}{2} = x$ | $x$ |
| 0 | $\frac{11 - 4(0)}{2}$ | $\frac{11}{2}$ |

$-5x + 3y = 5$

| | | |
|---|---|---|
| $x$ | $\frac{5 + 5x}{3} = y$ | $y$ |
| 0 | $\frac{5 + 5(0)}{3}$ | $\frac{5}{3}$ |
| $y$ | $\frac{3y - 5}{5} = x$ | $x$ |
| 0 | $\frac{3(0) - 5}{5}$ | $-1$ |

Now each line can be plotted from two points. We see from the graph that the point of intersection is $\left(\frac{1}{2}, \frac{5}{2}\right)$.

Solution B: To solve a system of two linear equations in two unknowns by determinants, we set up the following solutions in determinant form, derived from the linear equations in standard form.

$$a_1x + b_1y = c_1$$
$$a_2x + b_2y = c_2$$

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

The denominator for both variables is formed by writing the coefficients of $x$ and $y$ in the linear equations. The numerators are formed from the denominator by replacing the column of coefficients of that unknown by the column of constants.
In this case the linear equations in standard form are:

$$2x + 4y = 11$$
$$-5x + 3y = 5$$

Then, the solution by determinants is

$$x = \frac{\begin{vmatrix} 11 & 4 \\ 5 & 3 \end{vmatrix}}{\begin{vmatrix} 2 & 4 \\ -5 & 3 \end{vmatrix}}.$$ The value of a 2X2 determinant is defined to be:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc .$$ Therefore,

$$x = \frac{(11)(3) - (4)(5)}{(2)(3) - (4)(-5)} = \frac{33 - 20}{6 - (-20)} = \frac{13}{26} = \frac{1}{2};$$

$$y = \frac{\begin{vmatrix} 2 & 11 \\ -5 & 5 \end{vmatrix}}{26} = \frac{(2)(5) - (-5)(11)}{26} = \frac{10 - (-55)}{26} = \frac{65}{26} = \frac{5}{2}.$$

This agrees with the geometrical solution.

**• PROBLEM 326**

Find the simultaneous solution set of the equations

$$3x - 6y - 2 = 0 \qquad (9)$$
$$4x + 7y + 3 = 0 \qquad (10)$$

by use of Cramer's rule.

Solution: We first add 2 to each members of (9) and -3 to each member of (10) and get

$$3x - 6y = 2$$
$$4x + 7y = -3$$

We now obtain the solution set by the following steps:

1. We form the determinant D whose elements are the coefficients of the unknowns in the order in which they appear, and get

$$D = \begin{vmatrix} 3 & -6 \\ 4 & 7 \end{vmatrix}$$

Then recalling that a 2X2 determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ may be evaluated as $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$, we get

$$(3)(7) - (-6)(4) = 45$$

2. Replace the column of coefficients of $x$ in D by the constant terms and get

$$D_x = \begin{vmatrix} 2 & -6 \\ -3 & 7 \end{vmatrix} = (2)(7) - (-6)(-3) = -4$$

3. Replace the column of coefficients of $y$ in D by the constant terms and get

$$D_y = \begin{vmatrix} 3 & 2 \\ 4 & -3 \end{vmatrix} = (3)(-3) - (2)(4) = -17$$

4. By Cramer's rule

$$x = \frac{D_x}{D} = \frac{-4}{45} = -\frac{4}{45}$$

$$y = \frac{D_y}{D} = \frac{-17}{45} = -\frac{17}{45}$$

Hence the simultaneous solution set is $\left\{\left(-\frac{4}{45}, -\frac{17}{45}\right)\right\}$.

Check: Replacing $x$ and $y$ in the given equations by the appropriate elements of the solution set, we have

$$3\left(-\frac{4}{45}\right) - 6\left(-\frac{17}{45}\right) - 2 = \frac{-12 + 102 - 90}{45} = 0 \quad \text{from (9)}$$

$$4\left(-\frac{4}{45}\right) + 7\left(-\frac{17}{45}\right) + 3 = \frac{-16-119+135}{45} = 0 \quad \text{from (10)}$$

**• PROBLEM 327**

Solve by determinants:

$$3x - 5y = 4$$
$$7x + 4y = 25.$$

Solution: The equations, as given, are in standard form for applying Cramer's rule. Therefore,

$$x = \frac{\begin{vmatrix} 4 & -5 \\ 25 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & -5 \\ 7 & 4 \end{vmatrix}} = \frac{4 \cdot 4 - 25(-5)}{3 \cdot 4 - 7(-5)} = \frac{16 + 125}{12 + 35} = \frac{141}{47} = 3$$

$$y = \frac{\begin{vmatrix} 3 & 4 \\ 7 & 25 \end{vmatrix}}{\begin{vmatrix} 3 & -5 \\ 7 & 4 \end{vmatrix}} = \frac{3 \cdot 25 - 7 \cdot 4}{47} = \frac{75 - 28}{47} = \frac{47}{47} = 1$$

This process always yields a unique solution unless the denominator determinant is equal to zero.

## • PROBLEM 328

Solve the system

$$2x + 3y = 4$$
$$3x - 2y = -2$$

Solution: This system can be solved by Cramer's rule.

The equations are in the standard form to apply the rule. That is, the constants are on one side of the equation while the unknowns are on the opposite side.

Each unknown is the quotient of two determinants. The denominator is the determinant of the coefficients. The numerator is derived from the denominator by substituting the constant terms for the coefficients of the unknown. Thus:

$$x = \frac{\begin{vmatrix} 4 & 3 \\ -2 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix}} = \frac{-8 + 6}{-4 - 9} = \frac{2}{13}$$

$$y = \frac{\begin{vmatrix} 2 & 4 \\ 3 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix}} = \frac{-4 - 12}{-4 - 9} = \frac{16}{13}$$

Check by substitution into the original equations:

$$x = \frac{2}{13} \qquad y = \frac{16}{13}$$

$$2x + 3y = 4 \qquad 3x - 2y = -2$$

$$2\left(\frac{2}{13}\right) + 3\left(\frac{16}{13}\right) = 4 \qquad 3\left(\frac{2}{13}\right) - 2\left(\frac{16}{13}\right) = -2$$

$$\frac{52}{13} = 4 \qquad\qquad -2 = -2$$

$$4 = 4$$

**• PROBLEM 329**

Solve the system

$$3x + 2y = 12$$

$$4x - 3y = -1$$

**Solution:** This system can be solved algebraically by eliminating one variable from the pair of equations or by Cramer's rule of determinants. The equations are in the proper form to use determinants to find x and y.
Hence:

$$x = \frac{\begin{vmatrix} 12 & 2 \\ -1 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 4 & -3 \end{vmatrix}} = \frac{12(-3) - 2(-1)}{3(-3) - 4(2)} = \frac{-34}{-17} = 2$$

$$y = \frac{\begin{vmatrix} 3 & 12 \\ 4 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 4 & -3 \end{vmatrix}} = \frac{3(-1) - 12(4)}{3(-3) - 4(2)} = \frac{-51}{-17} = 3$$

To verify if the solution x = 2 and y = 3 is correct, substitute both values into the original equations.

**• PROBLEM 330**

Solve the system

$$2x + 3y - 6 = 0$$
$$2y = 3x$$

**Solution:** We can solve this system of two equations in two unknowns by determinants or by adding and subtracting the equations to eliminate one variable.

For purposes of illustration, we shall use determinants to solve this system. Use Cramer's rule. We must have all the unknowns on one side of the equal sign and the constant terms on the other to apply this rule.

$$2x + 3y = 6$$
$$3x - 2y = 0$$

Each unknown, x and y, is the quotient of two determinants. Let D = denominator which is the determinant of the coefficients.

$$D = \begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} = 2(-2) - 3(3) = -4 - 9 = -13$$

The numerator, for each unknown, is obtained from the denominator by substituting the constant terms for the coefficients of the unknown. Thus the numerator for x is:

$$\begin{vmatrix} 6 & 3 \\ 0 & -2 \end{vmatrix} = 6(-2) - 3(0) = -12$$

The numerator for y is:

$$\begin{vmatrix} 2 & 6 \\ 3 & 0 \end{vmatrix} = 2(0) - 6(3) = -18$$

$$\text{and } x = \frac{\begin{vmatrix} 6 & 3 \\ 0 & -2 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix}} = \frac{12}{13}$$

$$y = \frac{\begin{vmatrix} 2 & 6 \\ 3 & 0 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix}} = \frac{18}{13}$$

To check the solution $x = \frac{12}{13}$ and $y = \frac{18}{13}$,

substitute these values into the original equations.

$$2x + 3y - 6 = 0$$

$$2\left(\frac{12}{13}\right) + 3\left(\frac{18}{13}\right) - 6 = 0$$

$$\frac{24}{13} + \frac{54}{13} - 6 = 0$$

$$\frac{78}{13} - \frac{78}{13} = 0$$

$$0 = 0$$

$$2y = 3x$$

$$2\left(\frac{18}{13}\right) = 3\left(\frac{12}{13}\right)$$

$$\frac{36}{13} = \frac{36}{13}$$

**• PROBLEM 331**

Use determinants to show that the following system is inconsistent.

$$x + y = 3(x - 2y) + 5 \qquad (1)$$

$$14y - 4x = 11 \qquad (2)$$

<u>Solution:</u> The method of solving a system of equations by determinants is based upon Cramer's rule. Cramer's rule is stated:

In a system of n linear equations in n variables, if the determinant of the coefficients is not zero, the system has a unique solution. The value of each variable is a fraction whose denominator is the determinant of the coefficients and whose numerator is the same determinant, with the coefficients of that variable replaced by the corresponding constants.

Thus, if we have two equations arranged in standard form $ax + by = c$ and $dx + ey = f$, then

$$x = \frac{\begin{vmatrix} c & b \\ f & e \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}} = \frac{ce - fb}{ae - db}$$

$$y = \frac{\begin{vmatrix} a & c \\ d & f \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}} = \frac{af - dc}{ae - db}$$

If the numerator is not zero and the denominator is zero, the system is inconsistent. We can approach this particular problem as follows: The determinant can be obtained more readily if the terms are arranged in the standard form, $ax + by = c$. Equation (1) becomes $2x - 7y = -5$ and equation (2) becomes $-4 + 14y = 11$. Hence,

$$x = \frac{\begin{vmatrix} -5 & -7 \\ 11 & 14 \end{vmatrix}}{\begin{vmatrix} 2 & -7 \\ -4 & 14 \end{vmatrix}} = \frac{-5(14) - 11(-7)}{2(14) - (-4)(-7)} = \frac{-70 + 77}{28 - 28} = \frac{7}{0}$$

$$y = \frac{\begin{vmatrix} 2 & -5 \\ -4 & 11 \end{vmatrix}}{\begin{vmatrix} 2 & -7 \\ -4 & 14 \end{vmatrix}} = \frac{2(11) - (-4)(-5)}{0} = \frac{22 - 20}{0} = \frac{2}{0}$$

Since both $x$ and $y$ are of the form $\frac{a}{0}$, $a \neq 0$, the solution set is the empty set, and the system is inconsistant.

If both numerator and denominator are zero, the values of $x$ and $y$ are indeterminate; any $(x,y)$ pair that satisfies one equation will satisfy the other also. Since the two equations have the same solution set, they are dependent equations.

**• PROBLEM 332**

Obtain the simultaneous solution set of the system of equations

$$3x - 4y = -6$$

$$2x + 5y = 19$$

by use of the multiplicative inverse of a matrix.

**Solution:** We first express the system in matrix notation as

$$\begin{bmatrix} 3 & -4 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6 \\ 19 \end{bmatrix} \qquad (1)$$

The determinant of the matrix $M = \begin{bmatrix} 3 & -4 \\ 2 & 5 \end{bmatrix}$ is $3 \cdot 5 - [(2)(-4)] = 15 + 8 = 23$. The multiplicative inverse of $M$ is

$$m^{-1} = \frac{1}{\det(m)} \times \text{(matrix of the cofactors of each element of the original matrix)}$$

$$= \frac{1}{23}\begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix}$$

since the cofactor of 3 is 5, the cofactor of $-4$ is $-2$, the cofactor of 2 is $-(-4) = 4$, and the cofactor of 5 is 3.
We now multiply each member of (1) by the inverse of $M$ and get

$$M^{-1}M\begin{bmatrix} x \\ y \end{bmatrix} = M^{-1}\begin{bmatrix} -6 \\ 19 \end{bmatrix}$$

or

$$\frac{1}{23}\begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{23}\begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} -6 \\ 19 \end{bmatrix}$$

We complete the process as follows:

$$\frac{1}{23}\begin{bmatrix} 15+8 & -20+20 \\ -6+6 & 8+15 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{23}\begin{bmatrix} -30+76 \\ 12+57 \end{bmatrix}$$

$$\frac{1}{23}\begin{bmatrix} 23 & 0 \\ 0 & 23 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{23}\begin{bmatrix} 46 \\ 69 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Therefore, the simultaneous solution set is $\{(2,3)\}$.

**• PROBLEM 333**

If $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$ and $B = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

find $A \times B$.

Solution: A matrix is a set of numbers in a rectangular arrangement. The numbers which make up a matrix are its elements. Matrix A has 2 rows and 3 columns; it is called a $2\times3$ matrix, the number of rows being written first. Matrix B has 3 rows and 1 column; it is called a $3\times1$ matrix. The product of an $m\times n$ matrix A by an $n\times p$ matrix B is an $m\times p$ matrix whose element in the ith row and jth column is the single element in the product of the ith row vector of A by the jth column vector of B. An ith row vector is a $1\times n$ matrix of the form

$$[a_{i1}\ a_{i2}\ \dots\ a_{in}] .$$

A jth column vector is $n\times1$ like $\begin{bmatrix} b_{1j} \\ b_{2j} \\ \cdot \\ \cdot \\ \cdot \\ b_{nj} \end{bmatrix}$.

$$i\times j = [a_{i1}\ a_{i2}\ \dots\ a_{in}]\begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = [a_{i1}\, b_{1j} \dots a_{in}\, b_{nj}] .$$

Since A is $2\times3$, and B is $3\times1$, $A \times B$ is $2\times1$.

$$A \times B = \begin{bmatrix} a_1x + a_2y + a_3z \\ b_1x + b_2y + b_3z \end{bmatrix}$$

The first row is $a_1x + a_2y + a_3z$ . The second row is $b_1x + b_2y + b_3z$. The one column is

$$\begin{matrix} a_1x + a_2y + a_3z \\ b_1x + b_2y + b_3z \end{matrix} .$$

If the number of columns in A is not equal to the number of rows in B, the product $A \times B$ is not defined. Furthermore, if A and B are square matrices, (matrices which have the same number of rows as columns), $A \times B$ is usually not equal to $B \times A$.

**• PROBLEM 334**

> Obtain the product of
>
> $$A = \begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5 & 1 \\ -2 & 3 \end{bmatrix}$$

<u>Solution:</u> A is a 2×2 matrix. B is a 2×2 matrix. Therefore $A \times B$ is a 2×2 matrix. (A, B, and $A \times B$ are square matrices.)

$$A \times B = \begin{bmatrix} (3)(5) + (-2)(-2) & (3)(1) + (-2)(3) \\ (1)(5) + (4)(-2) & (1)(1) + (4)(3) \end{bmatrix} = \begin{bmatrix} 19 & -3 \\ -3 & 13 \end{bmatrix}$$

Note that

$$B \times A = \begin{bmatrix} 5 & 1 \\ -2 & 3 \end{bmatrix} \times \begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} (5)(3)+(1)(1) & (5)(-2)+(1)(4) \\ (-2)(3)+(3)(1) & (-2)(-2)+(3)(4) \end{bmatrix}$$

$$= \begin{bmatrix} 16 & -6 \\ -3 & 16 \end{bmatrix} \neq \begin{bmatrix} 19 & -3 \\ -3 & 13 \end{bmatrix} = A \times B$$

**• PROBLEM 335**

> If $A = \begin{bmatrix} 3 & -5 \\ 7 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 \\ -8 & 9 \end{bmatrix}$, find AB and BA.

<u>Solution:</u> The product of two 2×2 matrices is the 2×2 matrix given by the following formula:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{pmatrix}$$

We consider the first row of the first matrix and the first column of the second matrix. (See the dotted line between the two matrices). Multiply the number in the first row and first column of the first matrix by the number in the second matrix which is in the same position. Then we multiply the number in the first row and second column of the first matrix by the number in the first column and second row of the second matrix. Adding the two products, we obtain the term in the first row, first column of the product matrix. We perform the multiplication in a similar manner on the second row of the first matrix and the first column of the second matrix, to obtain the second row, first column of the product matrix.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

We do the same for the second column of the second matrix. Therefore,

$$AB = \begin{bmatrix} 3 & -5 \\ 7 & 0 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ -8 & 9 \end{bmatrix} = \begin{bmatrix} 3(2)+(-5)(-8) & 3(4)+(-5)9 \\ 7(2)+(0)(-8) & 7(4)+(0)9 \end{bmatrix} = \begin{bmatrix} 46 & -33 \\ 14 & 28 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 4 \\ -8 & 9 \end{bmatrix}\begin{bmatrix} 3 & -5 \\ 7 & 0 \end{bmatrix} = \begin{bmatrix} 2(3)+4(7) & 2(-5)+4(0) \\ -8(3)+9(7) & (-8)(-5)+9(0) \end{bmatrix} = \begin{bmatrix} 34 & -10 \\ 39 & 40 \end{bmatrix}$$

In this case $AB \neq BA$. Hence, matrix multiplication is not commutative.

**• PROBLEM 336**

Write the solution of the system

$$2x + \pi y = \sqrt{17}$$

$$ix - 23y = 89$$

in terms of determinants.

**Solution:** The variables $x$ and $y$ can both be written as the quotient of two determinants, using Cramer's Rule. For the variable $x$, the determinant in the denominator has the coefficients of the x-terms as its first vertical column and the coefficients of the y-terms as its second vertical column. Also for the variable $x$, the determinant in the numerator is the same as the determinant in the denominator except that the first vertical column is replaced by the constant terms. Hence,

$$x = \frac{\begin{vmatrix} \sqrt{17} & \pi \\ 89 & -23 \end{vmatrix}}{\begin{vmatrix} 2 & \pi \\ i & -23 \end{vmatrix}}$$

For the variable $y$, the determinant in the denominator is the same as the determinant in the denominator used for the variable $x$. Also for the variable $y$, the determinant in the numerator is the same as the determinant in the denominator except that the second vertical column is replaced by the constant terms. Hence,

$$y = \frac{\begin{vmatrix} 2 & \sqrt{17} \\ i & 89 \end{vmatrix}}{\begin{vmatrix} 2 & \pi \\ i & -23 \end{vmatrix}}$$

Note that the column which is replaced by the constant terms in the numerator, is the column which contains the coefficients of the variable we are solving for.

**• PROBLEM 337**

Show, using determinants, that the equations of the following system are dependent.

$$5x - 3y + 7 = 0 \qquad (1)$$

$$x - 2y = 4\left(4x - \frac{11}{4}y + 5\right) + 1 \qquad (2)$$

**Solution:** Rewrite each equation in standard form.

(1) $5x - 3y + 7 = 0$

$$5x - 3y = -7$$

(2) $x - 2y = 4\left(4x - \frac{11}{4}y + 5\right) + 1$

$$x - 2y = 16x - 11y + 20 + 1$$
$$15x - 9y = -21$$

Hence, we have

$$x = \frac{\begin{vmatrix} -7 & -3 \\ -21 & -9 \end{vmatrix}}{\begin{vmatrix} 5 & -3 \\ 15 & -9 \end{vmatrix}} = \frac{(-7)(-9)-(-21)(-3)}{5(-9) - 15(-3)} = \frac{63 - 63}{-45 + 45} = \frac{0}{0}$$

$$y = \frac{\begin{vmatrix} 5 & -7 \\ 15 & -21 \end{vmatrix}}{\begin{vmatrix} 5 & -3 \\ 15 & -9 \end{vmatrix}} = \frac{5(-21)-15(-7)}{0} = \frac{-105 + 105}{0} = \frac{0}{0}$$

Since both $x$ and $y$ are in the indeterminate form $\frac{0}{0}$, any value may be assigned to one of the variables, and the same corresponding value of the other will satisfy equations.

## DETERMINANTS AND MATRICES OF THIRD AND HIGHER ORDERS

**• PROBLEM 338**

Evaluate

$$\begin{vmatrix} 2 & 3 & 5 \\ 1 & 4 & 6 \\ 7 & 2 & 8 \end{vmatrix}$$

**Solution:** We will use minors to evaluate this determinant. Choose the first row, and call its elements $a_1$, $b_1$, $c_1$. Then their corresponding minors are $A_1$, $B_1$, $C_1$. We form the products $a_1A_1$, $b_1B_1$, $c_1C_1$. Since $a_1$ is in the first row and the first column, and $1 + 1 = 2$, which is even, the sign of $a_1A_1$ is positive. Similarly, the sign of $b_1B_1$ is negative, and that of $c_3C_3$ is positive. Thus, we have:

$$a_1A_1 - b_1B_1 + c_1C_1 ,$$

and substituting we obtain:

$$2A_1 - 3B_1 + 5C_1 .$$

We find the minors $A_1$, $B_1$, and $C_1$ by eliminating from the determinant the row and column that $a_1$, $b_1$, and $c_1$ are found in. Thus,

$$A_1 = \begin{vmatrix} 4 & 6 \\ 2 & 8 \end{vmatrix} , \quad B_1 = \begin{vmatrix} 1 & 6 \\ 7 & 8 \end{vmatrix} , \quad C_1 = \begin{vmatrix} 1 & 4 \\ 7 & 2 \end{vmatrix} \text{ ; we obtain:}$$

$$2\begin{vmatrix} 4 & 6 \\ 2 & 8 \end{vmatrix} - 3\begin{vmatrix} 1 & 6 \\ 7 & 8 \end{vmatrix} + 5\begin{vmatrix} 1 & 4 \\ 7 & 2 \end{vmatrix} .$$

Now, since a determinant of the form $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ can be equivalently written as: $ad - bc$, we have:

$$2[(4)(8) - (6)(2)] - 3[(1)(8) - (6)(7)] + 5[(1)(2) - (4)(7)]$$

$= 2(32 - 12) - 3(8 - 42) + 5(2 - 28)$

$= 2(20) - 3(-34) + 5(-26)$

$= 40 + 102 - 130$

$= 142 - 130$

$= 12$

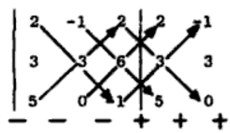
Therefore, the value of our given determinant is 12.

**• PROBLEM 339**

Evaluate the determinant:

$$D = \begin{vmatrix} 2 & -1 & 2 \\ 3 & 3 & 6 \\ 5 & 0 & -1 \end{vmatrix}$$

Solution: D is a determinant of order three, which may be evaluated by making use of six lines each of which joins the three elements whose product is to be formed.




Here the first two columns are rewritten to the right of the determinant. The products formed by following the lines running down from left to right have a plus sign attached, and those formed by following the lines running up from left to right have a negative sign attached. The algebraic sum of the products thus formed is the value of the determinant:

$$D = (2)(3)(-1) + (-1)(6)(5) + (2)(3)(0) - (5)(3)(2) - (0)(6)(2) - (-1)(3)(-1)$$

$$D = -6 - 30 + 0 - 30 - 0 -3$$

$$D = -69$$

**• PROBLEM 340**

Expand the determinant

$$D = \begin{vmatrix} 3 & 2 & 4 \\ 0 & 2 & 0 \\ 1 & 3 & 2 \end{vmatrix}$$

Solution: Here the second row contains two zeros. Hence we shall use this row to get the expansion:

$$D = -0\begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} + 2\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} -0\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= -0[(2)(2) - (4)(3)] + 2[(3)(2) - (4)(1)] -0[(3)(3) - (2)(1)]$$

$$= 0 + 2(6 - 4) -0 = 0 + 4 - 0 = 4$$

Obtain the value of

$$\begin{vmatrix} 2 & 4 & 1 \\ 3 & 6 & 2 \\ 5 & 2 & 4 \end{vmatrix}$$

Solution: If to the elements of any row (any column) of a determinant there is added m times the corresponding elements of another row (another column), the value of the determinant is unchanged. Therefore since the first two elements of the second column are twice the corresponding elements of the first column, if we multiply each element of the first column by -2 and add the product to the corresponding element in the second column, we get

$$\begin{vmatrix} 2 & 4 & 1 \\ 3 & 6 & 2 \\ 5 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 4+(-2)(2) & 1 \\ 3 & 6+(-2)(3) & 2 \\ 5 & 2+(-2)5 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 0 & 1 \\ 3 & 0 & 2 \\ 5 & -8 & 4 \end{vmatrix}$$

Now if we expand this determinant in terms of the elements of the second column, using the signs -, +, - that appear in the second column of the sign diagram, we get

$$-0 + 0 - (-8)\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 8(4 - 3) = 8$$



Find the value of
$$\begin{vmatrix} 29 & 26 & 22 \\ 25 & 31 & 27 \\ 63 & 54 & 46 \end{vmatrix}$$

Solution: We wish to rewrite the given determinant in a simpler form so as to make evaluation less complicated. Adding a multiple of each element in one column to the corresponding element in another column does not change the value of the determinant. Therefore, adding - 1 times the elements of column two to the corresponding elements of column one gives us:

$$\begin{vmatrix} 29 & 26 & 22 \\ 25 & 31 & 27 \\ 63 & 54 & 46 \end{vmatrix} = \begin{vmatrix} 29 + (-1)(26) & 26 & 22 \\ 25 + (-1)(31) & 31 & 27 \\ 63 + (-1)(54) & 54 & 46 \end{vmatrix} = \begin{vmatrix} 3 & 26 & 22 \\ -6 & 31 & 27 \\ 9 & 54 & 46 \end{vmatrix} .$$

Now, adding - 1 times the elements of column two to the corresponding elements of column three, we obtain:

$$\begin{vmatrix} 3 & 26 & 22 + (-1)(26) \\ -6 & 31 & 27 + (-1)(31) \\ 9 & 54 & 46 + (-1)(54) \end{vmatrix} = \begin{vmatrix} 3 & 26 & -4 \\ -6 & 31 & -4 \\ 9 & 54 & -8 \end{vmatrix} .$$

We can again rewrite this determinant as

$$\begin{vmatrix} 3(\ 1) & 26 & -4 \\ 3(-2) & 31 & -4 \\ 3(\ 3) & 54 & -8 \end{vmatrix}$$ , and since multiplying each element of a column of a determinant by a number is equivalent to multiplying the determinant by that number, we can write:

$$\begin{vmatrix} 3(\ 1) & 26 & -4 \\ 3(-2) & 31 & -4 \\ 3(\ 3) & 54 & -8 \end{vmatrix} = 3 \begin{vmatrix} 1 & 26 & -4 \\ -2 & 31 & -4 \\ 3 & 54 & -8 \end{vmatrix}$$ . Now,

since each element in the last column of the determinant can be written as a multiple of - 4, we obtain:

$$(3)(-4) \begin{vmatrix} 1 & 26 & 1 \\ -2 & 31 & 1 \\ 3 & 54 & 2 \end{vmatrix}$$ . We now add

- 1 times the first row to the second row. This gives us:

$$-12 \begin{vmatrix} 1 & 26 & 1 \\ -2-1 & 31-26 & 1-1 \\ 3 & 54 & 2 \end{vmatrix} = -12 \begin{vmatrix} 1 & 26 & 1 \\ -3 & 5 & 0 \\ 2 & 54 & 2 \end{vmatrix} .$$

Again, adding - 2 times the first row to the third, we obtain: $-12 \begin{vmatrix} 1 & 26 & 1 \\ -3 & 5 & 0 \\ 1 & 2 & 0 \end{vmatrix}$ .

We can now use minors to determine the value of the determinant. Let us choose column three, and call its elements $c_1$, $c_2$, $c_3$. Then their corresponding minors are $C_1$, $C_2$, $C_3$. We form the products $c_1C_1$, $c_2C_2$, $c_3C_3$. Since $c_1$ is in the first row and the third column, and $1 + 3 = 4$, which is even, the sign of $c_1C_1$ is positive. Similarly, that of $c_2C_2$ is negative, and that of $c_3C_3$ is positive. Thus, we have: $c_1C_1 - c_2C_2 + c_3C_3$. Substituting we obtain: $1C_1 - 0C_2 + 0C_3$. The last two terms vanish. We find the minor $C_1$ by eliminating from the determinant the row and column that $c_1$ is found in.

Thus, $C_1 = \begin{vmatrix} -3 & 5 \\ 1 & 2 \end{vmatrix}$ , and since $c_1 = 1$,

$c_1C_1$ also = $\begin{vmatrix} -3 & 5 \\ 1 & 2 \end{vmatrix}$. Thus, our given determinant

equals: $-12 \begin{vmatrix} -3 & 5 \\ 1 & 2 \end{vmatrix} = -12[(-3)(2) - (5)(1)]$

$$= -12(-6-5)$$

$$= -12(-11) = 132.$$

**• PROBLEM 343**

Find the value of $\begin{vmatrix} 67 & 19 & 21 \\ 39 & 13 & 14 \\ 81 & 24 & 26 \end{vmatrix}$

Solution: Our aim in this problem is to break down the given determinant into one that is easier to evaluate. We can therefore rewrite our determinant as:

$$\begin{vmatrix} 67 & 19 & 21 \\ 39 & 13 & 14 \\ 81 & 24 & 26 \end{vmatrix} = \begin{vmatrix} 10+57 & 19 & 21 \\ 0+39 & 13 & 14 \\ 9+72 & 24 & 26 \end{vmatrix}.$$

Now we can make use of one of the well-known properties of determinants; that is, if each element of a column of a determinant is expressed as the sum of two terms, the determinant can be expressed as the sum of two determinants. Thus,

$$\begin{vmatrix} 10+57 & 19 & 21 \\ 0+39 & 13 & 14 \\ 9+72 & 24 & 26 \end{vmatrix} = \begin{vmatrix} 10 & 19 & 21 \\ 0 & 13 & 14 \\ 9 & 24 & 26 \end{vmatrix} + \begin{vmatrix} 57 & 19 & 21 \\ 39 & 13 & 14 \\ 72 & 24 & 26 \end{vmatrix}.$$

The determinant can again be simplified further. Let us examine the second determinant in the above sum. Remember that multiplying each element in a column of a determinant by a number and adding that product to the corresponding elements in another column does not change the value of the determinant. Therefore, we can perform this on the determinant using $-3$ as the number, and adding the product of $-3$ and the elements of column two to the corresponding elements of column one. Thus, we obtain:

$$\begin{vmatrix} 57 & 19 & 21 \\ 39 & 13 & 14 \\ 72 & 24 & 26 \end{vmatrix} = \begin{vmatrix} 57+(-3)(19) & 19 & 21 \\ 39+(-3)(13) & 13 & 14 \\ 72+(-3)(24) & 24 & 26 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 19 & 21 \\ 0 & 13 & 14 \\ 0 & 24 & 26 \end{vmatrix}.$$

Now, since each element in a column of a determinant is zero, the value of the determinant is zero. Thus, the value of the second determinant in the above sum is zero, and we have:

$$\begin{vmatrix} 67 & 19 & 21 \\ 39 & 13 & 14 \\ 81 & 24 & 26 \end{vmatrix} = \begin{vmatrix} 10 & 19 & 21 \\ 0 & 13 & 14 \\ 9 & 24 & 26 \end{vmatrix}.$$

But, this can be rewritten as:

$$\begin{vmatrix} 10 & 19 & 19+2 \\ 0 & 13 & 13+1 \\ 9 & 24 & 24+2 \end{vmatrix} = \begin{vmatrix} 10 & 19 & 19 \\ 0 & 13 & 13 \\ 9 & 24 & 24 \end{vmatrix} + \begin{vmatrix} 10 & 19 & 2 \\ 0 & 13 & 1 \\ 9 & 24 & 2 \end{vmatrix}.$$

If two columns of a determinant have the same elements, then its value is zero. Thus the first determinant in the above sum is zero, and we are left with:

$$\begin{vmatrix} 10 & 19 & 2 \\ 0 & 13 & 1 \\ 9 & 24 & 2 \end{vmatrix}.$$

We now use minors to determine the value of the determinant. Let us choose column one, and call its elements $a_1$, $a_2$, $a_3$. Then their corresponding minors are $A_1$, $A_2$, $A_3$. We form the products $a_1A_1$, $a_2A_2$, $a_3A_3$. Since $a_1$ is in the first row and the first column, and $1 + 1 = 2$, which is even, the sign of $a_1A_1$ is positive. Similarly, the sign of $a_2A_2$ is negative, and that of $a_3A_3$ is positive. Thus, we have:

$a_1A_1 - a_2A_2 + a_3A_3$, and substituting we obtain:

$10A_1 - 0A_2 + 9A_3$. The second term vanishes. We find the minors $A_1$ and $A_3$ by eliminating from the determinant the row and column that $a_1$ and $a_3$ are found in. Thus,

$$A_1 = \begin{vmatrix} 13 & 1 \\ 24 & 2 \end{vmatrix}, \qquad A_3 = \begin{vmatrix} 19 & 2 \\ 13 & 1 \end{vmatrix}$$

and $\begin{vmatrix} 67 & 19 & 21 \\ 39 & 13 & 14 \\ 81 & 24 & 26 \end{vmatrix} = \begin{vmatrix} 10 & 19 & 2 \\ 0 & 13 & 1 \\ 9 & 24 & 2 \end{vmatrix}$

$$= 10 \begin{vmatrix} 13 & 1 \\ 24 & 2 \end{vmatrix} + 9 \begin{vmatrix} 19 & 2 \\ 13 & 1 \end{vmatrix} .$$

Now, these two determinants are easily evaluated. The first,

$\begin{vmatrix} 13 & 1 \\ 24 & 2 \end{vmatrix} = (13)(2) - (1)(24) = 26 - 24 = 2$, and the second

$\begin{vmatrix} 19 & 2 \\ 13 & 1 \end{vmatrix} = (19)(1) - (2)(13) = 19 - 26 = - 7.$

Thus, $\begin{vmatrix} 67 & 19 & 21 \\ 39 & 13 & 14 \\ 81 & 24 & 26 \end{vmatrix} = 10(2) + 9(- 7) = 20 - 63 = - 43.$

Another way to approach this problem is to use the expansion scheme for determinants of third order. Using this method we rewrite the given determinant as follows:

$$\left|\begin{array}{ccc} 67 & 19 & 21 \\ 39 & 13 & 14 \\ 81 & 24 & 26 \end{array}\right| \begin{array}{cc} 67 & 19 \\ 39 & 13 \\ 81 & 24 \end{array}$$

We multiply the elements falling on the same diagonal, thus obtaining six terms. The three terms on the lines sloping downward from left to right have a positive value, and the three on the lines sloping downward from right to left have a negative value. Upon expanding we obtain:

$$(67)(13)(26) + (19)(14)(81) + (21)(39)(24) - (21)(13)(81)$$
$$- (67)(14)(24) - (19)(39)(26).$$

Performing the indicated operations should give us the same value obtained using our previous method, that is - 43.

The advantage of the first method is that it does not involve a long multiplication process.

Find the value of

$$\begin{vmatrix} 30 & 11 & 20 & 38 \\ 6 & 3 & 0 & 9 \\ 11 & -2 & 36 & 3 \\ 19 & 6 & 17 & 22 \end{vmatrix}$$

Solution: We can simplify this determinant by multiplying each element of the second column by - 2, and adding this value to the corresponding element in the first column. This does not change the value of the determinant. Doing this we obtain:

$$\begin{vmatrix} 30 + (-2)(11) & 11 & 20 & 38 \\ 6 + (-2)(\ 3) & 3 & 0 & 9 \\ 11 + (-2)(-2) & -2 & 36 & 3 \\ 19 + (-2)(\ 6) & 6 & 17 & 22 \end{vmatrix} \quad \text{or,}$$

$$\begin{vmatrix} 8 & 11 & 20 & 38 \\ 0 & 3 & 0 & 9 \\ 15 & -2 & 36 & 3 \\ 7 & 6 & 17 & 22 \end{vmatrix} .$$

Now, we multiply each element of the second column by - 3, and add this value to the corresponding element in the fourth column. We thus obtain:

$$\begin{vmatrix} 8 & 11 & 20 & 38 + (-3)(11) \\ 0 & 3 & 0 & 9 + (-3)(\ 3) \\ 15 & -2 & 36 & 3 + (-3)(-2) \\ 7 & 6 & 17 & 22 + (-3)(\ 6) \end{vmatrix} \quad \text{or,}$$

$$\begin{vmatrix} 8 & 11 & 20 & 5 \\ 0 & 3 & 0 & 0 \\ 15 & -2 & 36 & 9 \\ 7 & 6 & 17 & 4 \end{vmatrix}$$

We now find the value of the determinant in terms of minors. Let us choose the second row of the determinant. We will call the elements of this row $a_2$, $b_2$, $c_2$, $d_2$, respectively. Their corresponding minors are $A_2$, $B_2$, $C_2$, $D_2$. We obtain the value of the determinant by multiplying each element in the chosen row by its corresponding minor as follows:

$a_2A_2$, $b_2B_2$, $c_2C_2$, $d_2D_2$. We now add each of these.

The signs are determined by the row and column of each element. Since $a_2$ is the term in the second row and the first column, and $2 + 1 = 3$, an odd number, the sign is negative. Since $b_2$ is the term in the second row, second column, and $2 + 2 = 4$, an even number, the sign is positive. Similarly, the sign for $c_2C_2$ is negative, and that for $d_2D_2$ is positive. Thus the value of the determinant is:

$- a_2A_2 + b_2B_2 - c_2C_2 + d_2D_2$. Substituting we obtain:

$- 0A_2 + 3B_2 - 0C_2 + 0D_2$. All terms vanish except the second, $3B_2$. We must now find the minor, $B_2$. This is done by eliminating the row and column in the determinant which contains $b_2 = 3$. Thus,

$$B_2 = \begin{vmatrix} 8 & 20 & 5 \\ 15 & 36 & 9 \\ 7 & 17 & 4 \end{vmatrix},$$ and

the value of the given determinant up to this points is:

$$3\begin{vmatrix} 8 & 20 & 5 \\ 15 & 36 & 9 \\ 7 & 17 & 4 \end{vmatrix}.$$

We can simplify the above determinant in the following manner; multiply each element of row three by $-1$ and add this value to the corresponding element in row two. We obtain:

$$3\begin{vmatrix} 8 & 20 & 5 \\ 15 + (-1)(7) & 36 + (-1)(17) & 9 + (-1)(4) \\ 7 & 17 & 4 \end{vmatrix} =$$

$$3\begin{vmatrix} 8 & 20 & 5 \\ 8 & 19 & 5 \\ 7 & 17 & 4 \end{vmatrix}$$

Now, multiplying each element of row 2 by $-1$ and adding this value to the corresponding element in row 1 we have:

$$3\begin{vmatrix} 8+(-1)(8) & 20+(-1)(19) & 5+(-1)(5) \\ 8 & 19 & 5 \\ 7 & 17 & 4 \end{vmatrix} =$$

$$3\begin{vmatrix} 0 & 1 & 0 \\ 8 & 19 & 5 \\ 7 & 17 & 4 \end{vmatrix} .$$

We can now obtain the value of this determinant in terms of minors. Choose row 1. Then,

$$\begin{vmatrix} 0 & 1 & 0 \\ 8 & 19 & 5 \\ 7 & 17 & 4 \end{vmatrix} = 0A_1 - 1B_1 + 0C_1. \quad \text{But,}$$

$B_1 = \begin{vmatrix} 8 & 5 \\ 7 & 4 \end{vmatrix}$. Thus the value of our given determinant is

$$(3)\ (-1)\begin{vmatrix} 8 & 5 \\ 7 & 4 \end{vmatrix} = -3\begin{vmatrix} 8 & 5 \\ 7 & 4 \end{vmatrix} .$$

Now, $\begin{vmatrix} 8 & 5 \\ 7 & 4 \end{vmatrix} = (8)(4) - (5)(7) = 32 - 35 = -3,$

and $(-3)(-3) = 9$. Therefore,

$$\begin{vmatrix} 30 & 11 & 20 & 38 \\ 6 & 3 & 0 & 9 \\ 11 & -2 & 36 & 3 \\ 19 & 6 & 17 & 22 \end{vmatrix} = 9.$$

**• PROBLEM 345**

Find the inverse $M^{-1}$ of the matrix

$$M = \begin{bmatrix} 1 & 3 & 4 \\ -2 & 4 & 5 \\ 3 & 1 & 6 \end{bmatrix}$$

and verify that $MM^{-1} = I$.

<u>Solution:</u> We find the determinant of the matrix $M$ using the method of minors and cofactors, and expanding in the first row, with the following scheme giving the sign of each minor (in this case each element of the first row).

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$D = 1\begin{vmatrix} 4 & 5 \\ 1 & 6 \end{vmatrix} -3\begin{vmatrix} -2 & 5 \\ 3 & 6 \end{vmatrix} +4\begin{vmatrix} -2 & 4 \\ 3 & 1 \end{vmatrix}$$

$$= 1(24-5) -3(-12-15) +4(-2-12) = 19 + 81 - 56 = 44$$

Furthermore, each entry $A_{ij}$ consists of the cofactor (the determinant of the two by two matrix of entries resulting when any combination of row $i$ and column $j$ is excluded) of each element of the original matrix.

$$A_{11} = \begin{vmatrix} 4 & 5 \\ 1 & 6 \end{vmatrix} = 19 \quad A_{12} = -\begin{vmatrix} -2 & 5 \\ 3 & 6 \end{vmatrix} = 27 \text{ and } A_{13} = \begin{vmatrix} -2 & 4 \\ 3 & 1 \end{vmatrix} = -14$$

Similarly, $A_{21} = -14$, $A_{22} = -6$, $A_{23} = 8$, $A_{31} = -1$, $A_{32} = -13$, and $A_{33} = 10$.

Hence, by the definition of the multiplicative inverse, the inverse matrix $M^{-1}$ becomes

$$\frac{1}{\text{determinant of } M}[\text{matrix } A]$$

where $A$ is the matrix of cofactors.

$$M^{-1} = \frac{1}{44}\begin{bmatrix} 19 & -14 & -1 \\ 27 & -6 & -13 \\ -14 & 8 & 10 \end{bmatrix}$$

Therefore,

$$M^{-1}M = \frac{1}{44}\begin{bmatrix} 19 & -14 & -1 \\ 27 & -6 & -13 \\ -14 & 8 & 10 \end{bmatrix}\begin{bmatrix} 1 & 3 & 4 \\ -2 & 4 & 5 \\ 3 & 1 & 6 \end{bmatrix}$$

$$= \frac{1}{44}\begin{bmatrix} 19+28-3 & 57-56-1 & 76-70-6 \\ 27+12-39 & 81-24-13 & 108-30-78 \\ -14-16+30 & -42+32+10 & -56+40+60 \end{bmatrix}$$

$$= \frac{1}{44}\begin{bmatrix} 44 & 0 & 0 \\ 0 & 44 & 0 \\ 0 & 0 & 44 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**• PROBLEM 346**

Expand the determinant

$$D = \begin{vmatrix} 3 & 2 & 4 \\ 1 & 5 & 2 \\ 4 & 7 & 6 \end{vmatrix}$$

in terms of the cofactors of the elements in the first row.

Solution: The cofactors of the elements in the first row of $D$ are the minors of the elements preceded respectively by the signs $+, -, +$.

The proper sign is $(-1)^{i+j}$ where $i$ is the number of the row and $j$ is the number of the column in which the element stands. The minor of a given element of a determinant is the determinant of the elements which remain after deleting the row and the column in which the given element is found. The minor of an element $a$ may be denoted by $m(a)$. Then since $a = 3$, $b = 2$, and $c = 4$, we have the corresponding cofactors

$$A = +m(3) = \begin{vmatrix} 5 & 2 \\ 7 & 6 \end{vmatrix} \quad B = -m(2) = -\begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix} \quad C = +m(4) = \begin{vmatrix} 1 & 5 \\ 4 & 7 \end{vmatrix}$$

Since a determinant may be expressed as the sum of the products formed by multiplying each element of any chosen row (column) by its cofactor, the expansion of D in terms of the cofactors of the elements in the first row is

$$D = 3A - 2B + 4c .$$

Hence we have

$$D = 3\begin{vmatrix} 5 & 2 \\ 7 & 6 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 4 & 6 \end{vmatrix} + 4\begin{vmatrix} 1 & 5 \\ 4 & 7 \end{vmatrix}$$

Evaluating a 2X2 determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ as $ad - bc$, D becomes

$$\begin{aligned} D &= 3[(5)(6) - (2)(7)] - 2[(1)(6) - (2)(4)] \\ &\qquad + 4[(1)(7) - (5)(4)] \\ &= 3(30 - 14) - 2(6 - 8) + 4(7 - 20) \\ &= 3(16) - 2(-2) + 4(-13) = 48 + 4 - 52 = 0 \end{aligned}$$

**• PROBLEM 347**

Expand the determinant

$$D = \begin{vmatrix} -2 & 4 & 3 \\ 1 & -5 & -6 \\ 3 & 1 & -2 \end{vmatrix}$$

in terms of the cofactors of the elements of the third column and check the result by expanding in terms of the cofactors of the second row.

Solution: Since in the third column $c_1 = 3$, $c_2 = -6$, and $c_3 = -2$ and the signs in the third column of the sign diagram for a 3X3 determinant,

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix},$$

are +, -, +, we have the corresponding cofactors

$$C_1 = +m(3) = \begin{vmatrix} 1 & -5 \\ 3 & 1 \end{vmatrix}, \quad C_2 = -m(-6) = -\begin{vmatrix} -2 & 4 \\ 3 & 1 \end{vmatrix},$$

$$C_3 = +m(-2) = \begin{vmatrix} -2 & 4 \\ 1 & -5 \end{vmatrix}$$

$$D = 3C_1 - (-6)C_2 + (-2)C_3 .$$

$$D = 3\begin{vmatrix} 1 & -5 \\ 3 & 1 \end{vmatrix} - (-6)\begin{vmatrix} -2 & 4 \\ 3 & 1 \end{vmatrix} + (-2)\begin{vmatrix} -2 & 4 \\ 1 & -5 \end{vmatrix}$$

$$\begin{aligned} &= 3[(1)(1) - (-5)(3)] + 6[(-2)(1) - (4)(3)] \\ &\qquad - 2[(-2)(-5) - (4)(1)] \\ &= 3(1 + 15) + 6(-2-12) - 2(10 - 4) = 48 - 84 - 12 = -48 \end{aligned}$$

The signs in the second row of the sign diagram are -, +, -, and the elements are 1, -5, and -6. Hence

$$D = -1\begin{vmatrix} 4 & 3 \\ 1 & -2 \end{vmatrix} + (-5)\begin{vmatrix} -2 & 3 \\ 3 & -2 \end{vmatrix} - (-6)\begin{vmatrix} -2 & 4 \\ 3 & 1 \end{vmatrix}$$

$$= -1[(4)(-2) - (3)(1)] - 5[(-2)(-2) - (3)(3)]$$
$$+ 6[(-2)(1) - (4)(3)]$$
$$= -1(-8-3) + (-5)(4-9) + 6(-2-12)$$
$$= 11 + 25 - 84 = -48$$

If one or more of the elements of a determinant are zero, it is advisable to expand the determinant in terms of the cofactors of the elements of the row or column that contains the greatest number of zeros.

**• PROBLEM 348**

Use Cramer's rule to solve the system of equations

$$3x + y - 2z = -3$$
$$2x + 7y + 3z = 9$$
$$4x - 3y - z = 7$$

Solution: The terms in the left members are arranged in the proper order and only the constant terms appear in the right members. Hence, we proceed as follows:

$$Ax = B \text{ where } A = \begin{bmatrix} 3 & 1 & -2 \\ 2 & 7 & 3 \\ 1 & -3 & -1 \end{bmatrix};\ x = \begin{bmatrix} x \\ y \\ z \end{bmatrix};\ B = \begin{bmatrix} -3 \\ 9 \\ 7 \end{bmatrix}$$

Step 1:
The determinant of $A$ is:

$$D = \begin{vmatrix} 3 & 1 & -2 \\ 2 & 7 & 3 \\ 4 & -3 & -1 \end{vmatrix}$$

Compute the determinant $D$ by the method of minors and cofactors. That is, compute the sum of the products of a number, the minor, and the determinant of the two by two matrix resulting when the row and column containing the minor is crossed out. The summation can be along any row or column and the following scheme provides the sign for each minor.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

The minors may be the elements along any chosen row or column of the matrix. The cofactor is the determinant of the four terms remaining when the row and column of the chosen minor is eliminated.

$$= 3\begin{vmatrix} 7 & 3 \\ -3 & -1 \end{vmatrix} -1\begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix} -2\begin{vmatrix} 2 & 7 \\ 4 & -3 \end{vmatrix}$$

The determinant of a two by two matrix is computed as follows

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

$$= 3(-7+9) - 1(-2-12) - 2(-6-28)$$
$$D = 6 + 14 + 68 = 88$$

Step 2: Now compute $D_x$, $D_y$, $D_z$ i.e., the determinants of the matrices resulting when you replace the column containing the coefficients of the variable under consideration by the constant terms in matrix B while keeping the other two columns the same.

$$D_x = \begin{vmatrix} -3 & 1 & -2 \\ 9 & 7 & 3 \\ 7 & -3 & -1 \end{vmatrix}$$

$$= -3 \begin{vmatrix} 7 & 3 \\ -3 & -1 \end{vmatrix} -1 \begin{vmatrix} 9 & 3 \\ 7 & -1 \end{vmatrix} -2 \begin{vmatrix} 9 & 7 \\ 7 & -3 \end{vmatrix}$$

$$= -3(-7+9) -1(-9-21) -2(-27-49)$$
$$= -6 + 30 + 152 = 176$$

Step 3:

$$D_y = \begin{vmatrix} 3 & -3 & -2 \\ 2 & 9 & 3 \\ 4 & 7 & -1 \end{vmatrix}$$

$$= 3 \begin{vmatrix} 9 & 3 \\ 7 & -1 \end{vmatrix} +3 \begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix} -2 \begin{vmatrix} 2 & 9 \\ 4 & 7 \end{vmatrix}$$

$$=3(-9-21) +3(-2-12) -2(14-36)$$
$$= -90 - 42 + 44 = -88$$

Step 4:

$$D_z = \begin{vmatrix} 3 & 1 & -3 \\ 2 & 7 & 9 \\ 4 & -3 & 7 \end{vmatrix}$$

$$= 3 \begin{vmatrix} 7 & 9 \\ -3 & 7 \end{vmatrix} -1 \begin{vmatrix} 2 & 9 \\ 4 & 7 \end{vmatrix} -3 \begin{vmatrix} 2 & 7 \\ 4 & -3 \end{vmatrix}$$

$$= 3(49+27) -1(14-36) -3(-6-28)$$
$$= 228 + 22 + 102 = 352$$

Step 5:

$$x = \frac{D_x}{D} = \frac{176}{88} = 2$$
$$y = \frac{D_y}{D} = \frac{-88}{88} = -1$$
$$z = \frac{D_z}{D} = \frac{352}{88} = 4$$

by Cramer's rule. Hence the solution set is $\{(2,-1,4)\}$; it can be checked by the method of substitution.

**• PROBLEM 349**

> Using determinants, solve the system
>
> $$2x - y - 2z = 4,$$
> $$x + 3y - z = -1,$$
> $$x + 2y + 3z = 5.$$

Solution: Use Cramer's rule $\frac{\Delta_{13}}{\Delta_3}$ to solve for the variables $x,y,z$, $i = x,y,z$. $\Delta_{13}$ is the determinant of the system, $\Delta_3$, with the ith column replaced by the elements to the right of the equal signs in the system's equations. The determinant of the system is

$$\Delta_3 = \begin{vmatrix} 2 & -1 & -2 \\ 1 & 3 & -1 \\ 1 & 2 & 3 \end{vmatrix}$$

Compute the determinant by expansion in minors and cofactors. For simplicity allow the minors to be the elements of the first column. The following scheme gives the means for determining the sign of the minors.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Each minor is multiplied by its cofactor, i.e., the determinant of the two by two matrix resulting when the row and column containing the minor are crossed out.

$$\Delta_3 = 2[3(3)-2(-1)] -1[(-1)(3) -2(-2)] +1[(-1)(-1) -3(-2)] = 2(11)-1(1) + 1(7) = 28$$

Since $\Delta_3 \neq 0$, the system possesses a unique solution. Also allow the minors of $\Delta_{i3}$ to be the elements of the first column. Then we get, by application of the rule stated above,

$$\Delta_3 x = \begin{vmatrix} 4 & -1 & -2 \\ -1 & 3 & -1 \\ 5 & 2 & 3 \end{vmatrix} =$$

$$\Delta_{x3} = 4[3(3)-2(-1)] - (-1)[-1(3)-2(-2)]+5[-1(-1)-3(-2)]$$
$$= 4(11)+1(1)+5(7) = 80,$$

$$\Delta_3 y = \begin{vmatrix} 2 & 4 & -2 \\ 1 & -1 & -1 \\ 1 & 5 & 3 \end{vmatrix} =$$

$$\Delta_{y3} = 2[-1(3)-5(-1)]-1[4(3) - 5(-2)] +1[4(-1)-(-1)(-2)]$$
$$= 2(2)-1(22)+1(-6) = -24,$$

$$\Delta_3 z = \begin{vmatrix} 2 & -1 & 4 \\ 1 & 3 & -1 \\ 1 & 2 & 5 \end{vmatrix} =$$

$$\Delta_{z3} = 2[3(5)-2(-1)]-1[-1(5)-2(4)]+1[-1(-1)-3(4)] = 2(17)-1(-13)+1(-11)=36,$$

whence, since

$$x = \frac{\Delta_{x3}}{\Delta_3} \quad \text{or} \quad \Delta_3 X = \Delta_{x3}$$
$$y = \frac{\Delta_{y3}}{\Delta_3} \quad \text{or} \quad \Delta_3 Y = \Delta_{y3}$$
$$z = \frac{\Delta_{z3}}{\Delta_3} \quad \text{or} \quad \Delta_3 Z = \Delta_{z3},$$

then

$$x = \frac{80}{28} = \frac{20}{7},\ y = -\frac{24}{28} = -\frac{6}{7},\ z = \frac{36}{28} = \frac{9}{7}.$$

That this solution satisfies the given system of equations is readily verified.

**• PROBLEM 350**

Show that the following equations are not independent:

$$\begin{aligned} 5x + 4y + 11z &= 3 \\ 6x - 4y + 2z &= 1 \\ x + 3y + 5z &= 2 \end{aligned}$$

<u>Solution:</u> To show that a system of linear equations is independent first compute the determinant of the matrix of the coefficients. If this result is equal to zero the equations are dependent and if

the result is not equal to zero the equations are independent or consistent. The matrix of coefficients is:

$$m = \begin{bmatrix} 5 & 4 & 11 \\ 6 & -4 & 2 \\ 1 & 3 & 5 \end{bmatrix}$$

Call the determinant of (m), D. Thus

$$D = \begin{vmatrix} 5 & 4 & 11 \\ 6 & -4 & 2 \\ 1 & 3 & 5 \end{vmatrix} = 5\begin{vmatrix} -4 & 2 \\ 3 & 5 \end{vmatrix} - 4\begin{vmatrix} 6 & 2 \\ 1 & 5 \end{vmatrix} + 11\begin{vmatrix} 6 & -4 \\ 1 & 3 \end{vmatrix}$$

$$= 5(-20 - 6) - 4(30 - 2) + 11(18 + 4)$$

$$D = -130 - 112 + 242 = 0$$

Hence, since $D = 0$, the equations are not independent, and no unique solution set exists.

**• PROBLEM 351**

Obtain the simultaneous solution set of

$$x + 3y + 4z = 15$$
$$-2x + 4y + 5z = 12$$
$$3x + y + 6z = 29$$

Solution: The matrix notation for the system of equations is

$$MX = B \text{ where } M = \begin{bmatrix} 1 & 3 & 4 \\ -2 & 4 & 5 \\ 3 & 1 & 6 \end{bmatrix}; \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; \quad B = \begin{bmatrix} 15 \\ 12 \\ 29 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & 3 & 4 \\ -2 & 4 & 5 \\ 3 & 1 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ 12 \\ 29 \end{bmatrix} \qquad (2)$$

The matrix of the coefficients is the matrix M discussed in the previous problem, and its inverse $M^{-1}$ was found to be

$$\frac{1}{44}\begin{bmatrix} 19 & -14 & -1 \\ 27 & -6 & -13 \\ -14 & 8 & 10 \end{bmatrix}$$

Therefore, if we multiply each member of (2) by $M^{-1}$, we get $M^{-1}MX = M^{-1}B$ or

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{44}\begin{bmatrix} 19 & -14 & -1 \\ 27 & -6 & -13 \\ -14 & 8 & 10 \end{bmatrix}\begin{bmatrix} 15 \\ 12 \\ 29 \end{bmatrix}$$

which is $IX = M^{-1}B$. Then, performing the indicated operations, we have $X = M^{-1}B$ or

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{44}\begin{bmatrix} 285 - 168 - 29 \\ 405 - 72 - 377 \\ -210 + 96 + 290 \end{bmatrix} = \frac{1}{44}\begin{bmatrix} 88 \\ -44 \\ 176 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

Therefore, the simultaneous solution set is $\{(2,-1,4)\}$.

**• PROBLEM 352**

Solve by determinants

$$2x - y - z = -3$$

$$x + y + z = 6$$

$$x - 2y + 3z = 6$$

Solution: Three equations in three unknowns can be solved algebraically by eliminating the same variable from two different pairs of equations.

We can also solve this system using Cramer's rule as in the system with only two unknowns.

The given equations are in the appropriate form to set up the needed determinants. Thus:

$$x = \frac{\begin{vmatrix} -3 & -1 & -1 \\ 6 & 1 & 1 \\ 6 & -2 & 3 \end{vmatrix}}{\begin{vmatrix} 2 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix}} = \frac{15}{15} = 1$$

$$y = \frac{\begin{vmatrix} 2 & -3 & -1 \\ 1 & 6 & 1 \\ 1 & 6 & 3 \end{vmatrix}}{\begin{vmatrix} 2 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix}} = \frac{30}{15} = 2$$

$$z = \frac{\begin{vmatrix} 2 & -1 & -3 \\ 1 & 1 & 6 \\ 1 & -2 & 6 \end{vmatrix}}{\begin{vmatrix} 2 & -1 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix}} = \frac{45}{15} = 3$$

If 1, 2, and 3, are substituted for x, y, and z, respectively, in each of the three equations, it will be found that they satisfy each equation, and therefore $x = 1$, $y = 2$, and $z = 3$, is the correct solution of the given system of equations.

Check:

$$2x - y - z = -3 \qquad x + y + z = 6 \qquad x - 2y + 3z = 6$$

$$2(1) - 2 - 3 = -3 \qquad 1 + 2 + 3 = 6 \qquad 1 - 2(2) + 3(3) = 6$$

$$-3 = -3 \qquad 6 = 6 \qquad 6 = 6$$

**• PROBLEM 353**

Without expanding, prove that

$$D = \begin{vmatrix} x & y & 2x \\ z & w & 2z \\ u & v & 2u \end{vmatrix} = 0$$

Solution: If the elements of a column (row) of a determinant are multiplied by any number m, the determinant is multiplied by m. Therefore,

$$D = \begin{vmatrix} x & y & 2x \\ z & w & 2z \\ u & v & 2u \end{vmatrix} = 2\begin{vmatrix} x & y & x \\ z & w & z \\ u & v & u \end{vmatrix}$$

since the elements of the third column of D are multiples of 2. Since two rows (two columns) of this determinant are identical, the value of

$$D = \begin{vmatrix} x & y & 2x \\ z & w & 2z \\ u & v & 2u \end{vmatrix} = 2\begin{vmatrix} x & y & x \\ z & w & z \\ u & v & u \end{vmatrix} = 0$$

(the first and third columns are identical)

**• PROBLEM 354**

If $D_1 = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & k \end{vmatrix}$, $D_2 = \begin{vmatrix} a & g & x \\ b & h & y \\ c & k & z \end{vmatrix}$

and d = tx, e = ty, f = tz, prove without expanding that $D_1 = -tD_2$.

Solution: In $D_1$ if we replace d by tx, e by ty and f by tz, we have

$$D_1 = \begin{vmatrix} a & b & c \\ tx & ty & tz \\ g & h & k \end{vmatrix}$$

If the elements of a row (a column) of a determinant are multiplied by any number t, the determinant is multiplied by t. Therefore

$$D_1 = \begin{vmatrix} a & b & c \\ tx & ty & tz \\ g & h & k \end{vmatrix} = \begin{vmatrix} a & b & c \\ t(x) & t(y) & t(z) \\ g & h & k \end{vmatrix}$$

$$= t\begin{vmatrix} a & b & c \\ x & y & z \\ g & h & k \end{vmatrix}$$

$$= t\begin{vmatrix} a & x & g \\ b & y & h \\ c & z & k \end{vmatrix}$$ since the rows of this determinant are the columns of the one just above.

Now, since interchanging two columns (rows) of a determinant changes the sign of the determinant, interchanging columns 2 and 3 gives us:

$$= -t\begin{vmatrix} a & g & x \\ b & h & y \\ c & k & z \end{vmatrix} \quad \text{But} \quad \begin{vmatrix} a & g & x \\ b & h & y \\ c & k & z \end{vmatrix} = D_2;$$

hence, $D_1 = -tD_2$.

**• PROBLEM 355**

Show that $\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} = 3abc - a^3 - b^3 - c^3.$

Solution: The following is a known property of determinants: $\begin{vmatrix} a_1+\bar{a}_1 & b_1 & c_1 \\ a_2+\bar{a}_2 & b_2 & c_2 \\ a_3+\bar{a}_3 & b_3 & c_3 \end{vmatrix} =$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \bar{a}_1 & b_1 & c_1 \\ \bar{a}_2 & b_2 & b_2 \\ \bar{a}_3 & b_3 & c_3 \end{vmatrix}.$$ Notice that in our given determinant there are two columns with their elements expressed as the sum of two terms. We will first apply the above property to the first column. We thus obtain:

$$\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} = \begin{vmatrix} b & a-b & a \\ c & b-c & b \\ a & c-a & c \end{vmatrix} + \begin{vmatrix} c & a-b & a \\ a & b-c & b \\ b & c-a & c \end{vmatrix}.$$

Now, applying this property to the second column of both determinants on the right side of the equal sign we obtain:

$$\begin{vmatrix} b & a & a \\ c & b & b \\ a & c & c \end{vmatrix} + \begin{vmatrix} b & -b & a \\ c & -c & b \\ a & -a & c \end{vmatrix} + \begin{vmatrix} c & a & a \\ a & b & b \\ b & c & c \end{vmatrix} + \begin{vmatrix} c & -b & a \\ a & -c & b \\ b & -a & c \end{vmatrix}.$$

But, if each element in a column of a determinant is multiplied by a number p, in this case p = - 1, then the value of the determinant is multiplied by p. That is,

$$\begin{vmatrix} a_1 & pb_1 & c_1 \\ a_2 & pb_2 & c_2 \\ a_3 & pb_3 & c_3 \end{vmatrix} = p \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} .$$

Thus, our above determinants become,

$$= \begin{vmatrix} b & a & a \\ c & b & b \\ a & c & c \end{vmatrix} - \begin{vmatrix} b & b & a \\ c & c & b \\ a & a & c \end{vmatrix} + \begin{vmatrix} c & a & a \\ a & b & b \\ b & c & c \end{vmatrix} - \begin{vmatrix} c & b & a \\ a & c & b \\ b & a & c \end{vmatrix} .$$

Recall that when two columns of a determinant are identical, the value of the determinant is zero. Thus, the first three determinants vanish and we are left with $- \begin{vmatrix} c & b & a \\ a & c & b \\ b & a & c \end{vmatrix} .$

To evaluate this third order determinant we employ the following method: rewrite the first two columns of the determinant next to the third column, obtaining:

$$- \left|\begin{array}{ccc|cc} c & b & a & c & b \\ a & c & b & a & c \\ b & a & c & b & a \end{array}\right. .$$

Draw three diagonal lines sloping downward from left to right, each of which encompasses three elements of the determinant. Do this also from right to left.

The diagram now looks like: - $\left|\begin{array}{ccc|cc} c & b & a & c & b \\ a & c & b & a & c \\ b & a & c & b & a \end{array}\right. .$

We now form the products of the elements in each of the six diagonals, preceding each of the terms in the left to right diagonals by a positive sign, and each of the terms in the right to left diagonals by a negative sign. The sum of the six products is the required expansion of the determinant. Thus, we obtain:

$$- (c \cdot c \cdot c + b \cdot b \cdot b + a \cdot a \cdot a - acb - cba - bac) =$$

$$- c^3 - b^3 - a^3 + 3abc = 3abc - a^3 - b^3 - c^3.$$

Show that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$.

Solution: We can rewrite the given determinant by adding each element in the third row to its corresponding element in the second row, and then adding each sum to the corresponding element in the first row. Thus,

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} =$$

$$\begin{vmatrix} a-b-c+(2b+2c) & 2a+(b-c-a+2c) & 2a+(2b+c-a-b) \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}.$$

Now, since multiplying each element of a row of a determinant by a number is equivalent to multiplying the determinant by that number, we can factor out $(a + b + c)$ from the first row obtaining:

$$(a+b+c) \times \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}.$$

We now leave the first column unaltered, and subtract each element of the first column from the corresponding element in the second and third columns. Doing this we obtain:

$$(a+b+c) \times \begin{vmatrix} 1 & 1-1 & 1-1 \\ 2b & b-c-a-2b & 2b-2b \\ 2c & 2c-2c & c-a-b-2c \end{vmatrix} =$$

$$(a+b+c) \times \begin{vmatrix} 1 & 0 & 0 \\ 2b & -b-c-a & 0 \\ 2c & 0 & -c-a-b \end{vmatrix}.$$

We can now use minors to determine the value of the determinant. Let us choose row one, and call its elements $a_1$, $b_1$, $c_1$. Then their corresponding minors are $A_1$, $B_1$, $C_1$. We form the products $a_1A_1$, $b_1B_1$, $c_1C_1$. Since $a_1$ is in the first row and the first column, and $1 + 1 = 2$, which is even, the sign of $a_1A_1$ is positive. Similarly, that of $b_1B$ is negative, and that of $c_1 C_1$ is positive. Thus, we have $a_1A_1 - b_1B_1 + c_1C_1$. Substituting we obtain: $1A_1 - 0B_1 + 0C_1$. The last two terms vanish. We find the minor $A_1$ by eliminating from the determinant the row and column that $a_1$ is found in. Thus,

$$A_1 = \begin{vmatrix} -b-c-a & 0 \\ 0 & -c-a-b \end{vmatrix}, \text{ and since}$$

$a_1 = 1$, $a_1A_1$ also equals $\begin{vmatrix} -b-c-a & 0 \\ 0 & -c-a-b \end{vmatrix}$.

Thus, our given determinant equals:

$$= (a+b+c) \times \begin{vmatrix} -b-c-a & 0 \\ 0 & -c-a-b \end{vmatrix}.$$

But $\begin{vmatrix} -b-c-a & 0 \\ 0 & -c-a-b \end{vmatrix} = (-b-c-a)(-c-a-b) - (0)(0)$

$$= bc + c^2 + ac + ab + ac + a^2 + b^2 + bc + ab$$

$$= (a+b+c)^2,$$

and $(a+b+c) \times \begin{vmatrix} -b-c-a & 0 \\ 0 & -c-a-b \end{vmatrix} = (a+b+c)(a+b+c)^2$

$$= (a+b+c)^3.$$

# CHAPTER 15

# FACTORING EXPRESSIONS AND FUNCTIONS

**Basic Attacks and Strategies for Solving Problems in this Chapter.** **See pages 232 to 264 for step-by-step solutions to problems.**

In carrying out the factorization of a non-fractional polynomial expression, the following general steps are applicable:

(1) Determine if there is a greatest common factor in the polynomial and then factor it out. For example, 3 is the greatest common monomial factor of $12x - 3$ since it can be factored as $3(4x - 1)$.

(2) If the polynomial has two terms (a binomial), then see if it is the difference of two squares, or the sum or difference of two cubes, and then factor accordingly. Remember, if it is the sum of two squares then it will not be factorable.

(3) If the polynomial has three terms (a trinomial), then it is either a perfect square trinomial which will factor into the square of a binomial, or it is not a perfect square trinomial, in which case you use a trial and error method. For example, the factorization of

$$6x^2 - x - 2 \quad \text{is} \quad (2x + 1)(3x - 2),$$

found by trial and error after finding the possible factors of the first and last terms in the polynomial.

(4) If the polynomial has more than three terms, then try to factor it by grouping. For example, the factorization of the expression

$$ax + bx + 2a + 2b$$

is found by grouping and factoring as follows:

$$x(a + b) + 2(a + b) = (x + 2)(a + b).$$

(5) As a final check, see if any of the factors you have written can be factored further. If you have overlooked a common factor, you can catch it as this point.

Factoring a fractional or rational expression is done by first factoring the numerator and denominator and then dividing the numerator and denominator by any factors they have in common. This process is called reducing the expression to the lowest terms.

The procedure for combining two or more rational expressions depends on the operation(s) involved and whether the denominators are alike or different. When adding or subtracting fractional expressions with different denominators, find the LCD for all denominators and change each rational expression to an equivalent expression that has the LCD. Then, add or subtract the numerators of the expressions and apply the common denominator. If the denominators are alike, simply add or subtract the numerators and apply the common denominator.

To multiply two or more fractional expressions, multiply numerators and multiply denominators to obtain the product. Reduce this product to the lowest terms.

Finally, to divide one fractional expression by another fractional expression, simply multiply the first expression by the reciprocal of the second expression. Then, reduce the result (quotient) to the lowest terms.

# Step-by-Step Solutions to Problems in this Chapter, "Factoring Expressions and Functions"

NONFRACTIONAL

**• PROBLEM 357**

Factor $x(y + z) + u(y + z)$.

Solution: By the distributive law,

$$x(y + z) + u(y + z) = (x + u)(y + z) \qquad (1)$$

To check if this factoring is correct, distribute the two products on the left side of equation (1).

$$x(y + z) + u(y + z) = (xy + xz) + (uy + uz)$$

$$\text{or } x(y + z) + u(y + z) = xy + xz + uy + uz \qquad (2)$$

Calculating the product on the right side of equation (1):

$$(x + u)(y + z) = xy + uy + xz + uz$$

$$\text{or } (x + u)(y + z) = xy + xz + uy + uz \qquad (3)$$

Since the right side of equations (2) and (3) are equal, equation (1) is true.

**• PROBLEM 358**

Factor A) $4a^2b - 2ab$

B) $9ab^2c^3 - 6a^2c + 12ac$

C) $ac + bc + ad + bd$

Solution: Find the highest common factor of each polynomial.

A) $4a^2b = 2 \cdot 2 \cdot a \cdot a \cdot b$

$2ab = 2 \cdot a \cdot b$

The highest common factor of the two terms is therefore $2ab$. Hence,

$$4a^2b - 2ab = 2ab(2a - 1)$$

B) $9ab^2c^3 = 3 \cdot 3 \cdot a \cdot b \cdot b \cdot c \cdot c \cdot c$

$6a^2c = 3 \cdot 2 \cdot a \cdot a \cdot c$

$12ac = 3 \cdot 2 \cdot 2 \cdot a \cdot c$

The highest common factor of the three terms is $3ac$. Then,

$$9ab^2c^3 - 6a^2c + 12ac = 3ac(3b^2c^2 - 2a + 4)$$

C) An expression may sometimes be factored by grouping terms having a common factor and thus getting new terms containing a common factor. The type form for this case is $ac+bc+ad+bd$, because the terms $ac$ and $bc$ have the common factor $c$, and $ad$ and $bd$ have the common factor $d$. Then,

$$ac + bc + ad + bd = c(a + b) + d(a + b)$$

Factoring out $(a + b)$, we obtain:

$$= (a + b)(c + d).$$

**• PROBLEM 359**

> Factor the following polynomials:
>
> (a) $15\,ac + 6bc - 10ad - 4bd$
>
> (b) $3a^2c + 3a^2d^2 + 2b^2c + 2b^2d^2$ .

Solution: (a) Group terms which have a common factor. Here, they are already grouped. Then factor.

$$15\,ac + 6\,bc - 10ad - 4bd = 3c(5a + 2b) - 2d(5a + 2b)$$

Factoring out $(5a + 2b)$

$$3c(5a + 2b) - 2d(5a + 2b) = (5a + 2b)(3c - 2d),$$

(b) Apply the same method as in (a).

$$3a^2c + 3a^2d^2 + 2b^2c + 2b^2d^2 = 3a^2\left(c + d^2\right) + 2b^2\left(c + d^2\right)$$

$$= \left(c + d^2\right)\left(3a^2 + 2b^2\right).$$

**• PROBLEM 360**

> Factor $ax + by + ay + bx$ completely.

Solution: Group the terms which have a common factor. The first and last terms have the factor $x$ in common, while the second and third terms have the factor $y$ in common. Hence we may rewrite the expression, and we have

$$ax + bx + ay + by = x(a + b) + y(a + b)$$

Now factor out $(a + b)$

$$= (a + b)(x + y).$$

**• PROBLEM 361**

> Factor $x^2 + 7x + 12$.

Solution: Since $7x = 4x + 3x$, $x^2 + 7x + 12 = x^2 + 4x + 3x + 12$. Factor out the common factor of $x$ from the first two terms. Also, factor out the common factor of 3 from the last two terms. Therefore,

$$x^2 + 7x + 12 = x(x + 4) + 3(x + 4).$$

Now factor out the common factor of $(x + 4)$ from the right side to obtain:

$$x^2 + 7x + 12 = (x + 4)(x + 3).$$

Factor $xy - 3y + y^2 - 3x$ completely.

Solution: Note that the first and last terms have a common factor of $x$. Also note that the second and third factors have a common factor of $y$. Hence, group the $x$ and $y$ terms together and factor out the $x$ and $y$ from their respective two terms. Therefore,

$$xy - 3y + y^2 - 3x = (xy - 3x) + \left(-3y + y^2\right)$$

Since $\left(-3y + y^2\right) = \left(y^2 - 3y\right)$,

$$xy - 3y + y^2 - 3x = (xy - 3x) + \left(y^2 - 3y\right)$$
$$= x(y - 3) + y(y - 3)$$

Now factor out the common factor $(y - 3)$ from both terms:

$$xy - 3y + y^2 - 3x = (x + y)(y - 3).$$

• PROBLEM 363

Factor $2ax - 3by - 2ay + 3bx$ completely.

Solution: Note that the first and the last terms have a common factor of $x$. Also note that the second and third factors have a common factor of $y$. Hence, group the $x$ and $y$ terms and factor out the $x$ and $y$ from their respective two terms. Therefore:

$$2ax - 3by - 2ay + 3bx = (2ax + 3bx) + (-3by - 2ay)$$
$$= x(2a + 3b) + y(-3b - 2a)$$

Since $-3b - 2a = -2a - 3b$,

$$2ax - 3by -2ay + 3bx = x(2a + 3b) + y(-2a - 3b)$$

Factor out $-1$ from the term $(-2a -3b)$:

$$(-2a - 3b) = -1(2a + 3b).$$

Therefore,

$$2ax - 3by - 2ay + 3bx = x(2a + 3b) + y\Big((-1)(2a + 3b)\Big)$$
$$= x(2a + 3b) - y(2a + 3b)$$

Factoring out the common factor $(2a + 3b)$ from both terms:

$$2ax - 3by - 2ay + 3bx = (x - y)(2a + 3b).$$

• PROBLEM 364

Factor $x^2 - x - 12$ over the integers.

Solution: A quadratic equation whose roots are a and b may be written in the form: $(x - a)(x - b) = x^2 - (a + b)x + ab = 0$. Considering $x^2 - x - 12$, the coefficient of $x$ is $-1$ and the constant is $-12$; thus, we want to find the values for a and b such that

$$a + b = 1 \quad \text{and} \quad a \cdot b = -12$$

One of the numbers must be negative and the other one positive, since only a negative multiplied by a positive will give us a negative quantity $(-12)$ for $a \cdot b$.

After examining the possible factors of -12, we find that 4 and -3 are the desired ones since $4 + (-3) = 1$. Thus, let $a = 4$, $b = -3$, and

$$x^2 - x - 12 = x^2 - (4 - 3)x + (4)(-3)$$
$$= (x - 4)(x + 3).$$

**• PROBLEM 365**

Factor $x^2 + 7x + 10$.

Solution: We are given $x^2 + 7x + 10$. We may use the formula

$$x^2 + (b+c)x + bc = (x+b)(x+c) \qquad (1)$$

to factor this polynomial. That is, we set

$$x^2 + 7x + 10 = x^2 + (b+c)x + bc \; .$$

Thus the coefficient of the $x$ term, $7 = b + c$ and $10 = bc$. We now must find the two numbers $b$ and $c$ whose sum is seven and whose product is 10. We first check all pairs of numbers whose product is ten:

(a) $1 \times 10 = 10$; hence $b = 1$, $c = 10$.

We reject these values because $b + c$ must equal 7, but $1 + 10 = 11$.

(b) $2 \times 5 = 10$; hence $b = 2$, $c = 5$.

We check the sum of these values, and note $b + c = 2 + 5 = 7$.

Thus $b = 2$ and $c = 5$ are the correct values. Now we go back to equation (1)

$x^2 + (b+c)x + bc = (x+b)(x+c)$, and substituting our values of $b$ and $c$ we obtain:

$$x^2 + (2+5)x + (2 \cdot 5) = (x+2)(x+5)$$
$$x^2 + 7x + 10 = (x+2)(x+5) \; .$$

**• PROBLEM 366**

Factor $(z + 1)^2 - b^2$.

Solution: Since $(z + 1)^2 - b^2$ is the difference of two squares we apply the formula for the difference of two squares, $x^2 - y^2 = (x + y)(x - y)$, replacing x by $(z + 1)$ and y by b to obtain:

$$(z + 1)^2 - b^2 = [(z + 1) + b][(z + 1) - b].$$

**• PROBLEM 367**

Factor $2x^2 - 3y^2$.

Solution: Since $a = (\sqrt{a})^2$, $2 = (\sqrt{2})^2$ and $3 = (\sqrt{3})^2$.

Therefore, $2x^2 - 3y^2 = (\sqrt{2}x)^2 - (\sqrt{3}y)^2$

$(\sqrt{2}x)^2 - (\sqrt{3}y)^2$ is the difference of two squares, hence we apply the formula for the difference of two squares,

$u^2 - v^2 = (u + v)\ (u - v)$, letting $u = \sqrt{2}x$ and $v = \sqrt{3}y$.

Thus, we obtain: $(\sqrt{2}x + \sqrt{3}y)(\sqrt{2}x - \sqrt{3}y)$

**• PROBLEM 368**

Factor the expression $16a^2 - 4(b - c)^2$.

Solution: $16 = 4^2$, thus $16a^2 = 4^2a^2$. Since $a^xb^x = (ab)^x$, $16a^2 = 4^2a^2 = (4a)^2$. Similarly $4 = 2^2$, thus $4(b-c)^2 = 2^2(b-c)^2 = [2(b-c)]^2$. Hence $16a^2 - 4(b-c)^2 = (4a)^2 - [2(b-c)]^2$. We are now dealing with the difference of two squares. Applying the formula for the difference of two squares, $x^2 - y^2 = (x+y)(x-y)$, and replacing x by 4a and y by 2(b-c) we obtain:

$$(4a)^2 - [2(b-c)]^2 = [4a + 2(b-c)][4a - 2(b-c)]$$
$$= (4a + 2b - 2c)(4a - 2b + 2c).$$

Therefore, $16a^2 - 4(b-c)^2 = (4a + 2b - 2c)(4a - 2b + 2c)$.

**• PROBLEM 369**

Factor $4x^2y^2 - 36x^2z^2$ completely.

Solution: We observe,

$$4x^2y^2 = 2^2x^2y^2 \qquad \text{and} \qquad 36x^2z^2 = 6^2x^2z^2.$$

Since,

$$x^ay^az^a = (xyz)^a,\ 2^2x^2y^2 = (2xy)^2$$

and,

$$6^2x^2z^2 = (6xz)^2.$$

Thus,

$$4x^2y^2 - 36x^2z^2 = (2xy)^2 - (6xz)^2,$$

the difference of two squares. We apply the formula for the difference of two squares,

$$a^2 - b^2 = (a+b)(a-b),$$

replacing a by 2 xy and b by 6xz:

$$4x^2y^2 - 36x^2z^2 = (2xy + 6xz)(2xy - 6xz).$$

We may now factor 2x from each of the above factors since,

$$2xy + 6xz = 2x(y + 3z)$$
and,
$$2xy - 6xz = 2x(y - 3z).$$
Thus,
$$\begin{aligned} 4x^2y^2 - 36x^2z^2 &= (2x)(y + 3z)(2x)(y - 3z) \\ &= (2x)(2x)(y + 3z)(y - 3z) \\ &= 4x^2(y + 3z)(y - 3z) \end{aligned}$$

**• PROBLEM 370**

Factor $x^2 - y^2$.

Solution: Add and subtract xy from the given expression. Note that this procedure doesn't change the value of the expression because $x^2 - y^2 + xy - xy = x^2 - y^2 + (xy - xy) = x^2 - y^2 + 0 = x^2 - y^2$.

Therefore,
$$x^2 - y^2 = x^2 - y^2 + xy - xy.$$

Also, by the commutative law of addition:
$$\begin{aligned} x^2 - y^2 &= x^2 - y^2 + xy - xy \\ &= x^2 - y^2 + (xy - xy) \\ &= x^2 + (xy - xy) - y^2 \end{aligned}$$
$$\text{or } x^2 - y^2 = x^2 + xy - xy - y^2 \qquad (1)$$

Again, applying the commutative law of addition to the second and third terms of equation (1):
$$x^2 - y^2 = x^2 - xy + xy - y^2 \qquad (2)$$

Factoring x from the first two terms of equation (2) and also factoring y from the last two terms of equation (2):
$$x^2 - y^2 = x(x - y) + y(x - y)$$

By the distributive property:
$$\begin{aligned} x^2 - y^2 &= x(x - y) + y(x - y) \\ &= (x + y)(x - y) \end{aligned}$$

**• PROBLEM 371**

Simplify $(x + 2y)(x - 2y)(x^2 + 4y^2)$.

Solution: Here we use the factoring formula $a^2 - b^2 = (a - b)(a + b)$ to rewrite the product $(x + 2y)(x - 2y)$:
$$\begin{aligned} (x + 2y)(x - 2y) &= (x)^2 - (2y)^2 \quad \text{difference of two squares.} \\ &= x^2 - 4y^2 . \end{aligned}$$
Hence, $(x + 2y)(x - 2y)(x^2 + 4y^2) = (x^2 - 4y^2)(x^2 + 4y^2) \qquad (1)$

Now, again use the factoring formula given above to rewrite the right side of equation (1) in which $x^2 = a$ and $4y^2 = b$. Hence,

$$\begin{aligned}(x^2 - 4y^2)(x^2 + 4y^2) &= (x^2)^2 - (4y^2)^2 \\ &= x^4 - 4^2y^4 \quad \text{since } (a^x)^y = a^{x \cdot y} \\ &= x^4 - 16y^4\end{aligned}$$

Hence, equation (1) becomes:

$$(x + 2y)(x - 2y)(x^2 + 4y^2) = x^4 - 16y^4 .$$

**• PROBLEM 372**

Factor $25a^2 + 30ab + 9b^2$.

**Solution:** Note that $25a^2 = 5^2\ a^2 = (5a)^2$, $9b^2 = 3^2b^2 = (3b)^2$ and $30ab = 2(5a)(3b)$. The given expression can be rewritten as:

$$25a^2 + 30ab + 9b^2 = (5a)^2 + 2(5a)(3b) + (3b)^2. \qquad (1)$$

Also, the formula for the square of a binomial sum is:

$$\begin{aligned}(x + y)^2 = (x + y)(x + y) &= x^2 + xy + xy + y^2 \\ &= x^2 + 2xy + y^2\end{aligned}$$

thus,

$$(x + y)^2 = x^2 + 2xy + y^2 \qquad (2)$$

The right side of equation (1) corresponds to the right side of equation (2), where $x = 5a$ and $y = 3b$. Hence, using the left side of equation (2), equation (1) can be rewritten as:

$$\begin{aligned}25a^2 + 30ab + 9b^2 &= (5a)^2 + 2(5a)(3b) + (3b)^2 \\ &= (5a + 3b)^2\end{aligned}$$

**• PROBLEM 373**

Factor $x^3 - 8$.

**Solution:** Since $8 = 2^3$, $x^3 - 8 = (x)^3 - (2)^3$. Therefore, $x^3 - 8$ is the difference of two cubes. The formula for the difference of two cubes is:

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Replacing a by x and b by 2;

$$\begin{aligned}x^3 - 8 &= (x)^3 - (2)^3 \\ &= (x - 2)\ [x^2 + (x)(2) + (2)^2] \\ x^3 - 8 &= (x - 2)(x^2 + 2x + 4)\end{aligned}$$

Factor $8a^3 - 27$.

Solution: Note that $8 = (2)^3$, hence $8a^3 = (2^3)(a^3) = (2a)^3$

$$\text{and } 27 = (3)^3$$

So, $$8a^3 - 27 = (2a)^3 - (3)^3.$$

Recall the formula for the difference of two cubes:

$$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$$

Substituting 2a for x, and 3 for y:

$$(2a)^3 - (3)^3 = (2a - 3)\left[(2a)^2 + 3(2a) + 3^2\right]$$
$$= (2a - 3)(4a^2 + 6a + 9)$$

Hence, $8a^3 - 27 = (2a - 3)(4a^2 + 6a + 9)$.

• PROBLEM 375

Factor $27x^3 - 8y^3$.

Solution: Note that $27 = 3^3$; thus $27x^3 = (3^3)(x^3)$.

Since $a^x b^x = (ab)^x$, $(3^3)(x^3) = (3x)^3$.

Similarly, $8 = 2^3$; thus $8y^3 = (2^3)(y^3) = (2y)^3$.

Therefore $27x^3 - 8y^3 = (3x)^3 - (2y)^3$, the difference of two cubes. We apply the formula for the difference of two cubes, $(a^3 - b^3) = (a - b)(a^2 + ab + b^2)$, replacing a by 3x and b by 2y. Thus, we obtain:

$$27x^3 - 8y^3 = (3x)^3 - (2y)^3$$
$$= (3x - 2y)\left[(3x)^2 + (3x)(2y) + (2y)^2\right]$$
$$= (3x - 2y)(9x^2 + 6xy + 4y^2).$$

• PROBLEM 376

Find the factors of $125m^3n^6 - 8a^3$.

Solution: Note that $125 = 5 \cdot 5 \cdot 5 = 5^3$. Also since $a^{xy} = (a^x)^y$, $n^{6} = n^{2\ 3} = (n^2)^3$. Thus $125m^3n^6 = 5^3m^3(n^2)^3$. Since $a^x b^x c^x = (abc)^x$, $5^3m^3(n^2)^3 = (5mn^2)^3$. $8 = 2 \cdot 2 \cdot 2 = 2^3$, thus $8a^3 = 2^3a^3 = (2a)^3$. Now $125m^3n^6 - 8a^3 = (5mn^2)^3 - (2a)^3$, which is the difference of two cubes. Apply the formula for the difference of two cubes $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$, replacing x by $5mn^2$ and y by 2a. Hence,

$$125m^3n^6 - 8a^3 = (5mn^2 - 2a)\left[(5mn^2)^2 + 5mn^2(2a) + (2a)^2\right].$$
$$= (5mn^2 - 2a)(25m^2n^4 + 10amn^2 + 4a^2)$$

Factor $(x + y)^3 + z^3$.

Solution: The given expression is the sum of two cubes. The formula for the sum of two cubes can be used to factor the given expression. This formula is:

$$a^3 + b^3 = (a + b)\left(a^2 - ab + b^2\right).$$

Using this formula and replacing a by x + y and b by z:

$$(x + y)^3 + z^3 = \left[(x+y) + z\right]\left[(x+y)^2 - (x+y)z + z^2\right]$$

• PROBLEM 378

Factor $125x^3 + 64y^3$.

Solution: Noting that $125x^3 = 5^3x^3 = (5x)^3$ and $64y^3 = 4^3y^3 = (4y)^3$, we obtain

$$125x^3 + 64y^3 = (5x)^3 + (4y)^3.$$

Thus we have the sum of two cubes. Applying the formula for the sum of two cubes,

$$a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right),$$

and replacing a by (5x) and b by (4y) we obtain $(5x)^3 + (4y)^3$

$$= (5x + 4y)\left[(5x)^2 - (5x)(4y) + (4y)^2\right]$$

$$= (5x + 4y)\left(5^2x^2 - 20xy + 4^2y^2\right)$$

$$= (5x + 4y)\left(25x^2 - 20xy + 16y^2\right).$$

• PROBLEM 379

Factor $5x^3 + 8y^3$.

Solution: Recall the formula for the sum of two cubes:

$$a^3 + b^3 = (a + b)\left(a^2 - ab + b^2\right).$$

To obtain $5x^3 + 8y^3$ as the sum of two cubes, we must express $5x^3$ and $8y^3$ as perfect cubes: Note that

$$\left(\sqrt[3]{a}\right)^3 = \left(a^{\frac{1}{3}}\right)^3 = a^1 = a;$$

thus, $\left(\sqrt[3]{5}\right)^3 = \left(5^{\frac{1}{3}}\right)^3 = 5^1 = 5$.

So, we can write $5x^3 = \left(\sqrt[3]{5}\right)^3x^3 = \left[\sqrt[3]{5}x\right]^3$;

and $8 = 2^3$, so $8y^3 = 2^3y^3 = (2y)^3$.

Thus, $5x^3 + 8y^3 = \left(\sqrt[3]{5}x\right)^3 + (2y)^3$.

Substituting $\sqrt[3]{5}x$ for a and 2y for b in the above general formula we obtain:

$$\left(\sqrt[3]{5x}\right)^3 + (2y)^3 = \left(\sqrt[3]{5x} + 2y\right)\left[\left(\sqrt[3]{5x}\right)^2 - \sqrt[3]{5x}\cdot 2y + (2y)^2\right],$$

and since $\left(\sqrt[3]{5x}\right)^2 = \left(5^{\frac{1}{3}}\right)^2 x^2 = 5^{\frac{2}{3}} x^2 = \sqrt[3]{5^2}x^2 = \sqrt[3]{25}x^2$ ,

$$= \left(\sqrt[3]{5x} + 2y\right)\left(\sqrt[3]{25x^2} - 2\sqrt[3]{5}xy + 4y^2\right).$$

**• PROBLEM 380**

Factor: (a) $2x^2 + 2y^2$ (b) $a^3x + b^3x$.

Solution: (a) First we factor out a 2 from this expression. Thus $2x^2 + 2y^2 = 2\left(x^2 + y^2\right)$. $x^2 + y^2$, the sum of two like even powers, cannot be factored. Thus, it is a prime expression. Hence $2x^2 + 2y^2 = 2\left(x^2 + y^2\right)$.

(b) First we factor out an x from this expression. Thus, $a^3x + b^3x = \left(a^3 + b^3\right)x$. $\left(a^3 + b^3\right)$ is the sum of two cubes. Applying the formula for the sum of two cubes: $c^3 + d^3 = (c + d)\left(c^2 - cd + d^2\right)$, replacing c by a and d by b we obtain

$$a^3 + b^3 = (a + b)\left(a^2 - ab + b^2\right).$$

Hence, $a^3x + b^3x = (a + b)\left(a^2 - ab + b^2\right)x$.

**• PROBLEM 381**

Find the following special products:

(a) $3xy^2\left(2x^2 - 3x + 6\right)$

(b) $\left(2x^2 - 3y\right)\left(2x^2 + 3y\right)$

(c) $(3a + 2)(4a - 5)$

(d) $\left(9x^2 + 3xy^2 + y^4\right)\left(3x - y^2\right)$ .

Solution:

(a) Use the distributive property.

$$\begin{aligned} 3xy^2\left(2x^2 - 3x + 6\right) &= 3xy^2\left(2x^2\right) - 3xy^2(3x) + 3xy^2(6) \\ &= (3\cdot 2)\left(x\cdot x^2\right)y^2 - (3\cdot 3)(x\cdot x)y^2 + (3\cdot 6)xy^2 \\ &= 6x^3y^2 - 9x^2y^2 + 18xy^2 \end{aligned}$$

(b) Use the difference of two squares, $(a-b)(a+b) = a^2 - b^2$, replacing a by $2x^2$ and b by $3y$,

$$\begin{aligned} \left(2x^2-3y\right)\left(2x^2+3y\right) &= \left(2x^2\right)^2 - (3y)^2 = 2^2\left(x^2\right)^2 - 3^2y^2 \\ &= 4x^{2\cdot 2} - 9y^2 \\ &= 4x^4 - 9y^2 \end{aligned}$$

(c) Use the distributive property.

$$\begin{aligned} (3a+2)(4a-5) &= 3a(4a-5) + 2(4a-5) \\ &= 12a^2 - 15a + 8a - 10 \\ &= 12a^2 - 7a - 10 \end{aligned}$$

(d) Use the difference of two cubes, $\left(a^2+ab+b^2\right)(a-b) = a^3 - b^3$, replacing a by $3x$ and b by $y^2$. Thus

$$\left(9x^2+3xy^2+y^4\right)\left(3x-y^2\right) = \left[(3x)^2+3xy^2+\left(y^2\right)^2\right]\left(3x-y^2\right) = (3x)^3 - \left(y^2\right)^3$$
$$= 27x^3 - y^6$$

**• PROBLEM 382**

Factor $a^4 - b^4$.

**Solution:** Note that $a^4 = (a^2)^2$ and $b^4 = (b^2)^2$; thus

$a^4 - b^4 = (a^2)^2 - (b^2)^2$, the difference of two squares. Thus we apply the formula for the difference of two squares, $x^2 - y^2 = (x + y)(x - y)$, replacing x by $a^2$ and y by $b^2$ to obtain:

$$a^4 - b^4 = (a^2)^2 - (b^2)^2 = (a^2 + b^2)(a^2 - b^2).$$

Since $a^2 - b^2$ is also the difference of two squares, we once again apply the above formula to obtain:

$$a^2 - b^2 = (a + b)(a - b).$$

Therefore, $a^4 - b^4 = (a^2 + b^2)(a + b)(a - b)$.

**• PROBLEM 383**

Evaluate $2x^2y - 8y^3$.

**Solution:** First note that there is a common factor of $y$ in both terms. Thus

$$2x^2y - 8y^3 = y\left(2x^2 - 8y^2\right)$$

There is also a common factor of 2 which can be factored out,

$$= 2y\left(x^2 - 4y^2\right)$$

Observe $4y^2 = 2^2y^2 = (2y)^2$, thus $x^2 - 4y^2 = x^2 - (2y)^2$, the difference of two squares. Applying the formula for the difference of two squares, $a^2 - b^2 = (a+b)(a-b)$. Replacing $a$ by $x$ and $b$ by $2y$,

$$x^2 - (2y)^2 = (x + 2y)(x - 2y).$$

Thus

$$2y\left(x^2 - 4y^2\right) = 2y(x + 2y)(x - 2y)$$

and

$$2x^2 - 8y^3 = 2y(x + 2y)(x - 2y) .$$

**• PROBLEM 384**

Factor $16x^4y^2 - 250xy^5$.

**Solution:** Factoring out the common monomial factor $2xy^2$, we have

$$16x^4y^2 - 250xy^5 = 2xy^2\left(8x^3 - 125y^3\right).$$

$8 - 2 \cdot 2 \cdot 2 = 2^3$, thus $8x^3 = 2^3x^3 = (2x)^3$, and
$125 = 5 \cdot 5 \cdot 5 = 5^3$, thus $125y^3 = 5^3y^3 = (5y)^3$.
Hence, $2xy^2\left(8x^3 - 125y^3\right) = 2xy^2\left[(2x)^3 - (5y)^3\right]$. Since $(2x)^3 - (5y)^3$ is the difference of two cubes, we apply the formula for the difference of two cubes.

$$a^3 - b^3 = (a - b)\left(a^2 + ab + b^2\right),$$

replacing a by 2x and b by 5y. Thus

$$2xy^2\left(8x^3 - 125y^3\right) = 2xy^2(2x - 5y)\left[(2x)^2 + (2x)(5y) + (5y)^2\right]$$

$$= 2xy^2(2x - 5y)\left(4x^2 + 10xy + 25y^2\right).$$

Therefore $16x^4y^2 - 250xy^5 = 2xy^2(2x - 5y)\left(4x^2 + 10xy + 25y^2\right)$.

**• PROBLEM 385**

Factor A) $a^4 + 4a^2 + 4$

B) $9a^2 - 6ab^2 + b^4$ .

Solution: The first example is a trinomial which is a perfect square, in the form:

$$x^2 + 2xy + y^2 = x^2 + xy + xy + y^2 = (x+y)(x+y) = (x+y)^2 .$$

For example A), replace $x$ by $a^2$ and $y$ by $2$ to obtain

$$a^2 + 4a^2 + 4 = \left(a^2\right)^2 + 2\cdot a^2\cdot 2 + 2^2 = \left(a^2 + 2\right)^2 ,$$

The second example is a trinomial perfect square whose form is:

$$x^2 - 2xy + y^2 = x^2 - xy - xy + y^2 = (x-y)(x-y) = (x-y)^2 .$$

For example B) replace $x$ by $3a$ and $y$ by $b^2$ to obtain:

$$9a^2 - 6ab^2 + b^4 = (3a)^2 - 2(3a)\left(b^2\right) + \left(b^2\right)^2$$

$$= \left(3a - b^2\right)\left(3a - b^2\right)$$

$$= \left(3a - b^2\right)^2 .$$

**• PROBLEM 386**

Factor $a^4 - a^2 - 12$ completely.

Solution: We factor a trinomial of degree two in this manner:

$$x^2 + (c + d)x + cd = (x + c)(x + d).$$

In this case $x = \left(a^2\right)$

$$a^4 - a^2 - 12 = \left(a^2\right)^2 - \left(a^2\right) - 12.$$

$$c + d = -1 \qquad c \cdot d = -12.$$

We must find two numbers whose sum is -1 and whose product is -12. The two numbers which satisfy these two conditions are -4 and 3. Thus,

$$a^4 - a^2 - 12 = \left(a^2\right)^2 - a^2 - 12$$

$$= \left(a^2\right)^2 + (-4 + 3)\left(a^2\right) + (-4)(3)$$

$$= a^4 - a^2 - 12.$$

Therefore,

$$a^4 - a^2 - 12 = (a^2 + 3)(a^2 - 4).$$

The first factor on the right does not factor further, but the second factor; $a^2 - 4$, is a difference of two squares. Completion of the factorization gives

$$a^4 - a^2 - 12 = (a^2 + 3)(a + 2)(a - 2).$$

• **PROBLEM 387**

Factor the expression $49x^6 - 25y^6$.

Solution: Noting that $49x^6 = 7^2(x^3)^2 = (7x^3)^2$ and $25y^6 = 5^2(y^3)^2 = (5y^3)^2$, we obtain

$$49x^6 - 25y^6 = (7x^3)^2 - (5y^3)^2.$$

Thus we have the difference of two squares. Applying the formula for the difference of two squares $a^2 - b^2 = (a+b)(a-b)$, and replacing a by $(7x^3)$ and b by $(5y^3)$, we obtain $(7x^3)^2 - (5y^3)^2 = [(7x^3) + (5y^3)][(7x^3) - (5y^3)] = (7x^3 + 5y^3)(7x^3 - 5y^3)$.

• **PROBLEM 388**

Factor $128x^6 - 2y^6$.

Solution: We first observe that 2 may be factored from this expression. Thus

$$128x^6 - 2y^6 = 2(64x^6 - y^6).$$

Now, since $a^{b\cdot c} = (a^b)^c$, $x^6 = x^{3\cdot 2} = (x^3)^2$ and

$$y^6 = y^{3\cdot 2} = (y^3)^2.$$

$$\begin{aligned}\text{Therefore } 2(64x^6 - y^6) &= 2[64(x^3)^2 - (y^3)^2] \\ &= 2[8^2(x^3)^2 - (y^3)^2] \\ &= [(8x^3)^2 - (y^3)^2].\end{aligned}$$

Thus, we have the difference of two squares. Applying the formula for the difference of two squares, $a^2 - b^2 = (a+b)(a-b)$, and, replacing a by $8x^3$ and b by $y^3$, we obtain

$$2[(8x^3)^2 - (y^3)^2] = 2\left[\left(8x^3 + y^3\right)\left(8x^3 - y^3\right)\right]$$
$$= 2\left[\left(2^3x^3 + y^3\right)\left(2^3x^3 - y^3\right)\right]$$
$$= 2\left[(2x)^3 + y^3][(2x)^3 - y^3\right].$$

Now since the expressions in brackets are the sum and difference of two cubes, respectively, we apply the formulas for the sum and difference of two cubes:

$$a^3 + b^3 = (a+b)\left(a^2 - ab + b^2\right)$$
$$a^3 - b^3 = (a-b)\left(a^2 + ab + b^2\right).$$

Replacing a by 2x and b by y we have

$$2\left[(2x)^3 + y^3\right]\left[(2x)^3 - y^3\right] = 2(2x+y)\left(4x^2-2xy+y^2\right)(2x-y)\left(4x^2+2xy+y^2\right)$$

Therefore, $128x^6 - 2y^6 = 2(2x+y)\left(4x^2-2xy+y^2\right)(2x-y)\left(4x^2+2xy+y^2\right)$.

**• PROBLEM 389**

Factor: (a) $4a^4 - b^6$
(b) $a^2 - b^2 + 2bc - c^2$

Solution: (a) First note that $4a^4 - b^6$ may be expressed as the difference of two squares: $4a^4 - b^6 = \left(2a^2\right)^2 - \left(b^3\right)^2$. Recall the formula for the difference of two squares: $x^2 - y^2 = (x-y)(x+y)$. Replacing $x$ by $2a^2$ and $y$ by $b^3$ in this formula we obtain:

$$4a^4 - b^6 = \left(2a^2\right)^2 - \left(b^3\right)^2 = \left(2a^2 - b^3\right)\left(2a^2 + b^3\right)$$

(b) Observe that $b^2 - 2bc + c^2$ is a perfect square, since the first and last terms are perfect squares and positive, and the middle term is twice the product of the square roots of the end terms. Then, $b^2 - 2bc + c^2 = (b - c)^2$.

Thus, express the given algebraic expression as the difference of two squares.

$$a^2 - b^2 + 2bc - c^2 = a^2 - \left(b^2 - 2bc + c^2\right) = a^2 - (b - c)^2$$

Then apply the formula for the difference of two squares

$$\left(x^2 - y^2\right) = (x-y)(x+y),$$

replacing $x$ by $a$ and $y$ by $(b-c)$:

$$a^2 - (b-c)^2 = [a - (b-c)][a + (b-c)]$$
$$= (a - b + c)(a + b - c)$$

Thus,
$$a^2 - b^2 + 2bc - c^2 = (a - b + c)(a + b - c)$$

**• PROBLEM 390**

Factor $x^6 - 64$ completely.

**Solution:** We observe that,

$$x^6 = x^{3 \cdot 2} = (x^3)^2 \quad \text{and} \quad 64 = 8^2.$$

Thus,

$$x^6 - 64 = (x^3)^2 - 8^2,$$

the difference of two squares. Applying the formula for the difference of two squares,

$$a^2 - b^2 = (a + b)(a - b)$$

and replacing a by $x^3$ and b by 8, we obtain,

$$x^6 - 64 = (x^3 + 8)(x^3 - 8).$$

Since ,

$$8 = 2^3,\ x^6 - 64 = (x^3 + 8)(x^3 - 8)$$

$$= (x^3 + 2^3)(x^3 - 2^3).$$

Thus each resulting factor, factors further, as a sum and as a difference of two cubes, respectively. Applying the formulas for the sum and difference of two cubes,

$$(a^3 + b^3) = (a + b)(a^2 - ab + b^2)$$

$$(a^3 - b^3) = (a - b)(a^2 + ab + b^2),$$

and replacing a by x and b by 2, we obtain,

$$x^6 - 64 = (x + 2)(x^2 - 2x + 4)(x - 2)(x^2 + 2x + 4).$$

It is of interest to see what results on factoring $x^6 - 64$ as a difference of two cubes. We get, in this case,

$$x^6 - 64 = (x^2)^3 - 4^3$$

$$= (x^2 - 4)(x^4 + 4x^2 + 16),$$

in which

$$x^2 - 4 = x^2 - 2^2$$

factors as $(x + 2)(x - 2)$. A comparison of the two results shows that $x^4 + 4x^2 + 16$ must be factorable as,

$$x^4 + 4x^2 + 16 = (x^2 - 2x + 4)(x^2 + 2x + 4).$$

This may be verified by multiplication, but it is not easy to see directly. Treating,

$$x^6 - 64$$

as the difference of two squares is much simpler than thinking of it as the difference of two cubes.

Find the LCM of: $x^2 + 2x + 1$, $x^2 - 1$, and $x^2 - 3x + 2$.

Solution: Factor each term.

$$x^2 + 2x + 1 = (x + 1)^2, \quad x^2 - 1 = (x - 1)(x + 1),$$

$$x^2 - 3x + 2 = (x - 1)(x - 2)$$

As with integers, the LCM must contain all of the prime factors as a product, and each prime factor must contain the largest exponent it has in any of the factored forms. Each factor is used only once, regardless of the number of times it appears. Therefore,

$$\text{LCM} = (x + 1)^2 (x - 1)(x - 2)$$

• PROBLEM 392

Find the LCM of: $6x^2 + 24x + 24$, $4x^2 - 8x - 12$, and $3x^2 + 9x + 6$.

Solution: Factor each expression completely. Constant factors should be written as a product of prime numbers.

$$6x^2 + 24x + 24 = 6(x^2 + 4x + 4) = 6(x + 2)^2 = (3)(2)(x + 2)^2$$

$$4x^2 - 8x - 12 = 4(x^2 - 2x - 3) = 4(x + 1)(x - 3)$$

$$= (2)^2(x + 1)(x - 3)$$

$$3x^2 + 9x + 6 = 3(x^2 + 3x + 2) = 3(x + 1)(x + 2).$$

Each of the factors of these expressions appears in the product known as the LCM. Each factor is raised to the highest power to which it appears in any one of the expressions. Therefore,

$$\text{LCM} = (2)^2(3)(x + 2)^2(x + 1)(x - 3).$$

• PROBLEM 393

Find the LCM of: $(x - 1)^2$, $(1 - x)^3$, $1 - x^3$.

Solution: Factor each polynomial completely. Notice in the factoring of the second and the third polynomials that $-1$ may be factored from the expressions first so that the terms of highest degree in the factors will have positive coefficients.

$$(x - 1)^2 = (x - 1)^2$$

$$(1 - x)^3 = [-1(x - 1)]^3 = (-1)^3(x - 1)^3 = -(x - 1)^3$$

$$1 - x^3 = (-1)(x^3 - 1) = -(x - 1)(x^2 + x + 1).$$

($x^3 - 1$ is the difference of two cubes.)

Each of the factors of these expressions appears in the product known as the LCM. Each factor is raised to the highest power to which it appears in any one of the expressions. Therefore the

$$\text{LCM} = (x - 1)^3(x^2 + x + 1).$$

## FRACTIONAL

**• PROBLEM 394**

Simplify $\dfrac{x^2 - y^2}{x + y}$.

Solution: $x^2 - y^2$ is the difference of two squares. Applying the formula for the difference of two squares, $a^2-b^2 = (a + b)(a - b)$, with $x = a$ and $y = b$,

$$\frac{x^2 - y^2}{x + y} = \frac{(x + y)(x - y)}{x + y} = x - y .$$

**• PROBLEM 395**

Perform the indicated operation

$$\frac{x^2 - y^2}{2x} \cdot \frac{4x^2}{x + y}$$

Solution: $x^2 - y^2$ is the difference of two squares. Applying the formula for the difference of two squares

$$a^2 - b^2 = (a+b)(a-b)$$

$$x^2 - y^2 = (x+y)(x-y)$$

Thus $$\frac{x^2 - y^2}{2x} \cdot \frac{4x^2}{x + y} = \frac{(x + y)(x - y)}{2x} \cdot \frac{4x^2}{x + y}$$

$$= \frac{(x+y)(x-y)(4x^2)}{(2x)(x+y)}$$

Since $4x^2 = (2x)(2x)$,

$$= \frac{(\cancel{2x})(2x)\cancel{(x+y)}(x-y)}{\cancel{(2x)}\cancel{(x+y)}}$$

$$= 2x(x - y)$$

**• PROBLEM 396**

Combine $\dfrac{3x + y}{x^2 - y^2} - \dfrac{2y}{x(x - y)} - \dfrac{1}{x + y}$ into a single fraction.

**Solution:** Fractions which have unlike denominators must be transformed into fractions with the same denominator before they may be combined. This identical denominator is the least common denominator (L.C.D.), the least common multiple of the denominators of the fractions to be added. In the process of transforming the fractions to a common denominator we make use of the fact that the numerator and denominator of a fraction may be multiplied by the same non-zero number without changing the value of the fraction. In our case the denominators are $x^2 - y^2 = (x + y)(x - y)$, $x(x - y)$, and $x + y$. Therefore the LCD is $x(x + y)(x - y)$, and we proceed as follows:

$$\frac{3x + y}{x^2 - y^2} - \frac{2y}{x(x-y)} - \frac{1}{x+y} = \frac{3x + y}{(x+y)(x-y)} - \frac{2y}{x(x-y)} - \frac{1}{x+y}$$

$$= \frac{x(3x + y)}{x(x+y)(x-y)} - \frac{(x + y)2y}{(x+y)(x)(x-y)} - \frac{x(x - y)}{x(x-y)(x+y)}$$

$$= \frac{3x^2 + xy}{x(x+y)(x-y)} - \frac{2xy + 2y^2}{x(x+y)(x-y)} - \frac{x^2 - xy}{x(x+y)(x-y)}$$

$$= \frac{3x^2 + xy - (2xy + 2y^2) - (x^2 - xy)}{x(x+y)(x-y)}$$

$$= \frac{3x^2 + xy - 2xy - 2y^2 - x^2 + xy}{x(x+y)(x-y)}$$

$$= \frac{3x^2 - x^2 + xy + xy - 2xy - 2y^2}{x(x+y)(x-y)}$$

$$= \frac{2x^2 - 2y^2}{x(x+y)(x-y)}$$

$$= \frac{2(x^2 - y^2)}{x(x+y)(x-y)}$$

$$= \frac{2\cancel{(x+y)}\cancel{(x-y)}}{x\cancel{(x+y)}\cancel{(x-y)}}$$

$$= \frac{2}{x}$$

**• PROBLEM 397**

Perform the following subtraction: $\dfrac{3x}{x^2 - 4} - \dfrac{4}{2 - x}$ .

**Solution:** Note that $x^2 - 4 = x^2 - 2^2$, the difference of two squares. Using the formula for the difference of two squares, $a^2 - b^2 = (a-b)(a+b)$, we have: $x^2 - 4 = x^2 - 2^2 = (x-2)(x+2)$.

Thus,

$$\frac{3x}{x^2 - 4} - \frac{4}{2 - x} = \frac{3x}{(x-2)(x+2)} - \frac{4}{2 - x}$$

$$= \frac{3x}{(x-2)(x+2)} + \frac{-4}{2 - x} \quad .$$

Multiplying numerator and denominator of $\frac{-4}{2 - x}$ by $(-1)$:

$$\frac{-4(-1)}{(2-x)(-1)} = \frac{4}{-2 + x} = \frac{4}{x - 2} \quad .$$

Thus,

$$\frac{3x}{x^2 - 4} - \frac{4}{2 - x} = \frac{3x}{(x-2)(x+2)} + \frac{4}{x - 2}$$

$$= \frac{3x}{(x-2)(x+2)} + \frac{4(x+2)}{(x-2)(x+2)}$$

(Notice that $\frac{x+2}{x+2} = 1$, therefore multiplication by this fraction does not alter the value of the term)

$$= \frac{3x + 4(x+2)}{(x-2)(x+2)} = \frac{3x + 4x + 8}{(x-2)(x+2)}$$

$$= \frac{7x + 8}{(x-2)(x+2)} \quad .$$

• **PROBLEM 398**

Simplify: $\dfrac{\frac{1}{a - b} + \frac{1}{a + b}}{1 + \frac{b^2}{a^2 - b^2}}$

**Solution:** This is a complex fraction, a fraction whose numerator and denominator both contain fractions. To simplify it, multiply the numerator and denominator by the least common denominator, LCD. To find the LCD of several fractions, first factor each denominator into its prime factors.

$$a - b = (a - b)$$
$$a + b = (a + b)$$
$$a^2 - b^2 = (a - b)(a + b)$$

The LCD of the fractions is the product of the highest power of the different prime factors, with each prime factor being used only once. Hence $(a - b)(a + b)$ is our LCD. Multiplying, we obtain:

$$\frac{(a-b)(a+b)\left[\frac{1}{a - b} + \frac{1}{a + b}\right]}{(a-b)(a+b)\left[1 + \frac{b^2}{a^2 - b^2}\right]}$$

Distributing in the numerator and denominator, and recalling that

$a^2 - b^2 = (a - b)(a + b)$ we have:

$$\frac{\frac{\cancel{(a-b)}(a+b)}{\cancel{(a-b)}} + \frac{(a-b)\cancel{(a+b)}}{\cancel{(a+b)}}}{(a-b)(a+b) + \frac{b^2\cancel{(a-b)}\cancel{(a+b)}}{\cancel{(a-b)}\cancel{(a+b)}}} = \frac{(a+b) + (a-b)}{(a-b)(a+b) + b^2}$$

$$= \frac{a + b + a - b}{a^2 - b^2 + b^2} = \frac{2a}{a^2} = \frac{2}{a} \; .$$

**• PROBLEM 399**

Combine the following fractions $\dfrac{x}{x^2 - y^2} + \dfrac{2}{y - x} - 5$ .

Solution: Note that the denominator $x^2 - y^2$ is the difference of two squares. Using the formula for the difference of two squares

$$a^2 - b^2 = (a + b)(a - b) ,$$

factor $x^2 - y^2$ into $(x + y)(x - y)$. Next observe that $-(y-x) = -y+x = x-y$. Then multiplying numerator and denominator of

$$\left(\frac{2}{y - x}\right) \text{ by } -1,$$

$$\frac{-1}{-1}\left(\frac{2}{y - x}\right) = \frac{-2}{x - y} \; .$$

Therefore,

$$\frac{x}{x^2 - y^2} + \frac{2}{y - x} - 5 = \frac{x}{(x+y)(x-y)} + \frac{-2}{x - y} - \frac{5}{1}$$

Thus the terms $(x+y)$,$(x-y)$, and $1$ appear in our denominators. In order to combine fractions, transform them into equivalent fractions with a common denominator. Using $(x+y)(x-y)$ as our least common denominator, LCD, multiply each term by the necessary factor to yield a denominator of $(x+y)(x-y)$:

$$\frac{x}{x^2 - y^2} + \frac{2}{y - x} - 5 = \frac{x}{(x+y)(x-y)} + \left(\frac{x+y}{x+y}\right)\left(\frac{-2}{x-y}\right) + \frac{(x+y)(x-y)}{(x+y)(x-y)}\left(\frac{-5}{1}\right)$$

$$= \frac{x}{(x+y)(x-y)} + \frac{-2x - 2y}{(x+y)(x-y)} + \frac{-5(x^2 - y^2)}{(x+y)(x-y)}$$

$$= \frac{x - 2x - 2y - 5x^2 + 5y^2}{(x+y)(x-y)}$$

$$= \frac{-x - 2y - 5x^2 + 5y^2}{(x+y)(x-y)}$$

$$= \frac{-x - 2y - 5x^2 + 5y^2}{x^2 - y^2}$$

**• PROBLEM 400**

Reduce $\dfrac{3x - 6}{x^2 - 4}$ to lowest terms.

Solution: Factor the expression in both the numerator and denominator. In the numerator we factor 3 from both terms, and observing that the denominator is the difference of two squares, $x^2 - 2^2$, we obtain:

$$\frac{3x - 6}{x^2 - 4} = \frac{3(x - 2)}{(x - 2)(x + 2)}$$

$$= \frac{3}{x + 2}$$

The numerator and denominator were divided by $x - 2$.

**• PROBLEM 401**

Combine the following fractions:

$$\frac{12b - 16a}{3a^2 - 3b^2} + \frac{5}{a + b} - \frac{1}{b - a} .$$

Solution: In order to add fractions with unlike denominators, we must find a least common denominator, L.C.D., which is the least common multiple of the denominators of the fractions to be added. First, factor the different denominators

$$3a^2 - 3b^2 = 3\left(a^2 - b^2\right) = 3(a+b)(a-b)$$

$$a + b = (a + b)$$

$$b - a = -1(a - b)$$

Notice that the last denominator differs from the 3rd factor $(a - b)$ of the first denominator by a factor of $-1$. Factor out this minus one. Hence

$$\frac{12b - 16a}{3a^2 - 3b^2} + \frac{5}{a + b} - \frac{1}{(-1)(a-b)} = \frac{12b - 16a}{3(a+b)(a-b)} + \frac{5}{a + b} + \frac{1}{a - b} .$$

Now, the least common denominator is $3(a-b)(a+b)$. Transform the fractions to equivalent fractions all having the same denominator. Then, add the numerators.

$$\frac{12b - 16a}{3(a+b)(a-b)} + \frac{5}{(a + b)} + \frac{1}{(a - b)} = \frac{12b - 16a}{3(a+b)(a-b)}$$

$$+ \frac{5(3)(a-b)}{3(a+b)(a-b)} + \frac{(1)3(a+b)}{3(a-b)(a+b)} =$$

$$\frac{12b - 16a + 15a - 15b + 3a + 3b}{3(a+b)(a-b)} = \frac{2a}{3\left(a^2 - b^2\right)} .$$

**• PROBLEM 402**

Simplify $\dfrac{4x + 10}{4x^2 + 20x + 25}$.

Solution: First we factor a 2 from the numerator, thus

$$\frac{4x + 10}{4x^2 + 20x + 25} \quad \frac{2(2x + 5)}{4x^2 + 20x + 25}.$$

Factoring the denominator, $= \frac{2(2x + 5)}{(2x + 5)(2x + 5)} = \frac{2}{2x + 5}$. Since the factor $(2x + 5)$ appears in the denominator it may not equal zero, as division by zero is not defined. Thus

$$2x + 5 \neq 0$$

$$2x \neq -5$$

$$x \neq \frac{-5}{2}.$$

Therefore, $\frac{4x + 10}{4x^2 + 20x + 25} = \frac{2}{2x + 5}$, $x \neq \frac{-5}{2}$.

**• PROBLEM 403**

> Reduce $\frac{4x - 20}{50 - 2x^2}$ to lowest terms.

Solution: Factor the numerator and the denominator:

$$\frac{4x - 20}{50 - 2x^2} = \frac{4(x - 5)}{2(25 - x^2)} = \frac{4(x - 5)}{2(5 - x)(5 + x)}$$

Multiply the numerator and denominator by $(-1)$ to reverse the sign of the factor $(5 - x)$ in the denominator. Then divide both the numerator and denominator by $2(x - 5)$.

$$\frac{(-1)[4(x - 5)]}{(-1)[2(5 - x)(5 + x)]} = \frac{-4(x - 5)}{2(x - 5)(5 + x)}$$

Dividing, we obtain:

$$-\frac{2}{x + 5} \quad .$$

**• PROBLEM 404**

> Simplify $\frac{a^2 - 3ab + 2b^2}{2b^2 + ab - a^2}$ .

Solution: Factoring the numerator and denominator of the given fraction we obtain:

$$\frac{a^2 - 3ab + 2b^2}{2b^2 + ab - a^2} = \frac{(a - 2b)(a - b)}{(2b - a)(b + a)}$$

If we negate both factors in the numerator we do not change the fraction's value( because negating both factors gives us -1 multiplied by -1, which equals 1; and multiplication by 1 does not change the expression's value). Thus, we have:

$$\frac{[-(a-2b)][-(a-b)]}{(2b-a)(b+a)} = \frac{(2b-a)(b-a)}{(2b-a)(b+a)} = \frac{b-a}{b+a},$$

since $\frac{2b-a}{2b-a} = 1.$

• **PROBLEM 405**

> Perform the following addition: $\dfrac{2x}{x^2-4} + \dfrac{3}{x^2-5x+6}$.

__Solution:__ Factor the denominators into polynomial factors. Hence,

$$x^2 - 4 = (x+2)(x-2)$$
$$x^2 - 5x + 6 = (x-3)(x-2).$$

Therefore:

$$\frac{2x}{x^2-4} + \frac{3}{x^2-5x+6} = \frac{2x}{(x+2)(x-2)} + \frac{3}{(x-3)(x-2)} \quad (1)$$

Now, find the least common denominator ($\ell$.c.d) of the two fractions on the right side of equation (1). This is done by writing down all the different factors that appear in the two denominators. The exponent to be used for each factor is the smallest number of times that the factor appears in either denominator. Hence,

$$\ell\text{.c.d.} = (x+2)^1(x-2)^1(x-3)^1 = (x+2)(x-2)(x-3).$$

Multiplying each fraction by a fraction of the appropriate form, and with unit value, produces an equivalent fraction whose denominator is the $\ell$.c.d. Therefore:

$$\frac{2x}{x^2-4} + \frac{3}{x^2-5x+6} = \frac{(x-3)(2x)}{(x-3)(x+2)(x-2)} + \frac{(x+2)(3)}{(x+2)(x-3)(x-2)}$$

$$= \frac{(x-3)(2x) + 3(x+2)}{(x+2)(x-2)(x-3)}$$

distribute,
$$= \frac{2x^2 - 6x + 3x + 6}{(x+2)(x-2)(x-3)}$$

combine terms
$$= \frac{2x^2 - 3x + 6}{(x+2)(x-2)(x-3)}.$$

• **PROBLEM 406**

> Divide $\dfrac{2x-8}{x+1}$ by $\dfrac{3x^2-12x}{x^2-1}$.

__Solution:__ The problem can be written as:

$$\frac{\dfrac{2x-8}{x+1}}{\dfrac{3x^2-12x}{x^2-1}}$$

To divide fractions invert the denominator and multiply the inverted fraction by the numerator. Thus,

$$\frac{\frac{2x - 8}{x + 1}}{\frac{3x^2 - 12x}{x^2 - 1}} = \frac{2x - 8}{x + 1} \cdot \frac{x^2 - 1}{3x^2 - 12x}$$

$$= \frac{2(x - 4)}{x + 1} \cdot \frac{(x + 1)(x - 1)}{3x(x - 4)}$$

factoring and dividing out common factors

$$= \frac{2(x - 1)}{3x} \quad .$$

**• PROBLEM 407**

Subtract $\dfrac{2x - 3}{x^2 - 3x + 2}$ from $\dfrac{2 - x}{x^2 - 2x + 1}$.

Solution: To subtract $\dfrac{2x - 3}{x^2 - 3x + 2}$ from $\dfrac{2 - x}{x^2 - 2x + 1}$ we must change these fractions to equivalent fractions with a common denominator.

$$\frac{2 - x}{x^2 - 2x + 1} - \frac{2x - 3}{x^2 - 3x + 2} = \frac{2 - x}{(x - 1)(x - 1)} - \frac{2x - 3}{(x - 1)(x - 2)}$$

Denominators were factored for convenience. Multiplying the numerator and denominator of $\dfrac{2 - x}{(x - 1)(x - 1)}$ by the denominator of $\dfrac{2x - 3}{(x - 1)(x - 2)}$ does not change the value of the original fraction. Likewise, we can multiply the numerator and denominator of $\dfrac{2x - 3}{(x - 1)(x - 2)}$ by the denominator of $\dfrac{2 - x}{(x - 1)(x - 1)}$

$$\frac{2 - x}{x^2 - 2x + 1} - \frac{2x - 3}{x^2 - 3x + 2} = \frac{2 - x}{(x - 1)(x - 1)} - \frac{2x - 3}{(x - 1)(x - 2)}$$

$$= \frac{(2x- x)(x - 1)(x - 2)}{(x - 1)(x - 2)(x - 1)(x - 1)}$$

$$- \frac{(2x - 3)(x - 1)(x - 1)}{(x - 1)(x - 2)(x - 1)(x - 1)}$$

$$= \frac{(2 - x)(x - 1)(x - 2) - (2x - 3)(x - 1)(x - 1)}{(x - 1)(x - 2)(x - 1)(x - 1)}$$

Rule for subtracting fractions with common denominators

$$= \frac{(x - 1)[(2 - x)(x - 2) - (2x - 3)(x - 1)]}{(x - 1)[(x - 2)(x - 1)(x - 1)]}$$

$$= \frac{(2 - x)(x - 2) - (2x - 3)(x - 1)}{(x - 2)(x - 1)(x - 1)}$$

factor of $(x - 1)$ cancels

$$= \frac{-x^2 + 4x - 4 - (2x^2 - 5x + 3)}{(x - 1)^2(x - 2)}$$

$$= \frac{-3x^2 + 9x - 7}{(x - 1)^2(x - 2)}$$

$$= - \frac{3x^2 - 9x + 7}{(x - 1)^2(x - 2)} .$$

The numerator is not factorable so the fraction can not be reduced. Either of the last two fractions could be given as the answer.

**• PROBLEM 408**

Combine

$$\frac{1}{x^2 + x} - \frac{4}{x^2 - 1} + \frac{1}{x^2 - x}, \quad (x \neq 0,1,-1).$$

Solution: Note that the denominators are $x^2 + x = x(x+1)$, $x^2 - 1$, the difference of two squares, $= (x+1)(x-1)$, and $x^2 - x = x(x-1)$. Before fractions can be combined they must be transformed so that all will have the same denominator. This identical denominator is called the Least Common Denominator (L.C.D.). The L.C.D. is the least common multiple of the denominators of the fractions to be added. Thus, in our example the least common denominator is $x(x-1)(x+1)$. Multiply the numerator and denominator of the first fraction by $x-1$, those of the second fraction by $x$, and those of the third by $(x+1)$. Then the sum is given by

$$\left(\frac{x-1}{x-1}\right)\left(\frac{1}{x(x+1)}\right) - \left(\frac{x}{x}\right)\left(\frac{4}{(x+1)(x-1)}\right) + \left(\frac{x+1}{x+1}\right)\left(\frac{1}{x(x-1)}\right)$$

$$= \frac{x-1}{x(x-1)(x+1)} - \frac{4x}{x(x-1)(x+1)} + \frac{x+1}{x(x-1)(x+1)}$$

$$= \frac{(x-1) - 4x + x+1}{x(x-1)(x+1)} = \frac{-2x}{x(x-1)(x+1)} = \frac{-2}{(x-1)(x+1)} .$$

Thus,

$$\frac{1}{x^2 + x} - \frac{4}{x^2 - 1} + \frac{1}{x^2 - x} = \frac{-2}{(x-1)(x+1)} .$$

To check that these equations are equivalent, substitute any value of $x$ (we use $x = 2$) in both sides of the equation. From the left side we obtain:

$$\frac{1}{2^2 + 2} - \frac{4}{2^2 - 1} + \frac{1}{2^2 - 2} = \frac{1}{4 + 2} - \frac{4}{4 - 1} + \frac{1}{4 - 2}$$

$$= \frac{1}{6} - \frac{4}{3} + \frac{1}{2}$$

$$= \frac{1}{6} - \frac{8}{6} + \frac{3}{6}$$

$$= \frac{-4}{6}$$

$$= \frac{-2}{3}$$

From the right side we obtain:

$$\frac{-2}{(2-1)(2+1)} = \frac{-2}{(1)(3)} = \frac{-2}{3}$$

Thus, both members yield the same result.

**• PROBLEM 409**

Find the product of

$$\frac{x^2 - x}{x^2 - x - 2} \quad \text{and} \quad \frac{x - 2}{x^2}.$$

Solution:

$$\frac{x^2 - x}{x^2 - x - 2} \cdot \frac{x - 2}{x^2} = \frac{x(x - 1)(x - 2)}{(x - 2)(x + 1)x^2} \quad \text{by}$$

factoring

$$= \frac{x - 1}{(x + 1)x} \quad \text{by dividing}$$

out common factors x and x - 2

$$= \frac{x - 1}{x^2 + x}$$

**• PROBLEM 410**

Combine into a single fraction

$$\frac{2x}{x^2-6x+9} - \frac{8}{x^2-2x-3} - \frac{1}{x+1}.$$

Solution: Fractions which have unlike denominators cannot be combined directly. First they must be transformed into fractions with the same denominator. This identical denominator is called the Least Common Denominator (L.C.D.).

Before we can obtain the L.C.D., we factor each individual denominator.

$$x^2 - 6x + 9 = (x-3)(x-3) = (x-3)^2$$

$$x^2 - 2x - 3 = (x-3)(x+1)$$

$$x + 1 = (x + 1)$$

To find the L.C.D., we consider all the different factors. Take the highest value of the exponent of each factor. Thus, factoring the denominators, we obtain

$$\frac{2x}{(x-3)^2} - \frac{8}{(x-3)(x+1)} - \frac{1}{x+1}$$

and the L.C.D. is $(x-3)^2(x+1)$. We shall now rewrite the three given fractions as equivalent fractions, each having the denominator $(x-3)^2(x+1)$. To this end, multiply numerator and denominator of the first fraction by $x + 1$, of the second fraction by $x - 3$, and of the third fraction by $(x-3)^2$. This gives

$$\frac{2x(x+1)}{(x-3)^2(x+1)} - \frac{8(x-3)}{(x-3)^2(x+1)} - \frac{(x-3)^2}{(x+1)(x-3)^2}$$

$$= \frac{2x^2+2x-(8x-24)-(x^2-6x+9)}{(x-3)^2(x+1)}$$

$$= \frac{2x^2+2x-8x+24-x^2+6x-9}{(x-3)^2(x+1)}$$

$$= \frac{2x^2-x^2+2x-8x+6x+24-9}{(x-3)^2(x+1)}$$

$$= \frac{x^2+15}{(x-3)^2(x+1)}$$

Check. The given expression should be equal to the resulting fraction for all permissible values of $x$, that is, for all values of $x$ except 3 and -1 ($x = 3$ and $x = -1$ give us zero in the denominator of the fraction, which is undefined). Replacing $x$ arbitrarily by 2,

$$\left.\frac{2x}{x^2-6x+9} - \frac{8}{x^2-2x-3} - \frac{1}{x+1}\right]_{x=2}$$

$$= \frac{2(2)}{(2)^2-6(2)+9} - \frac{8}{(2)^2-2(2)-3} - \frac{1}{2+1}$$

$$= \frac{4}{4-12+9} - \frac{8}{4-4-3} - \frac{1}{3}$$

$$= \frac{4}{1} - \frac{8}{-3} - \frac{1}{3}$$

$$= \frac{12}{3} + \frac{8}{3} - \frac{1}{3} = \frac{19}{3}$$

and

$$\left.\frac{x^2+15}{(x-3)^2(x+1)}\right]_{x=2} = \frac{(2)^2+15}{(2-3)^2(2+1)} = \frac{4+15}{(-1)^2(3)} = \frac{19}{1\cdot 3} = \frac{19}{3}$$

Hence, we have shown that the given expression holds true for $x = 2$ in the uncombined and combined forms.

**• PROBLEM 411**

Reduce $\frac{x^2 - 5x + 4}{x^2 - 7x + 12}$ to lowest terms.

Solution: Factor the expressions in both the numerator and denominator and cancel like terms.

$$\frac{x^2 - 5x + 4}{x^2 - 7x + 12} = \frac{(x-1)(x-4)}{(x-3)(x-4)}$$

$$= \frac{x-1}{x-3}$$

The numerator and the denominator were both divided by $x - 4$.

**• PROBLEM 412**

Reduce to lowest form: $\frac{a^3 - 8b^3}{2a^2 - 8b^2}$.

<u>Solution:</u> Factor the numerator and denominator as completely as possible.
The numerator, $a^3 - 8b^3 = a^3 - (2b)^3$, is the difference of two cubes. Apply the following formula:

$$\left(x^3 - y^3\right) = (x - y)\left(x^2 + xy + y^3\right)$$

Replacing $x$ by $a$ and $y$ by $2b$, we obtain:

$$\left(a^3 - 8b^3\right) = \left[a^3 - (2b)^3\right] = \left(a - (2b)\right)\left(a^2 + 2ab + 4b^2\right)$$

For the denominator, factor out the highest common factor 2.

$$2a^2 - 8b^2 = 2\left(a^2 - 4b^2\right)$$

where $a^2 - 4b^2$ is the difference of two squares. Recall the formula for the difference of two squares: $x^2 - y^2 = (x-y)(x+y)$. Substitute $a$ for $x$ and $2b$ for $y$ to obtain:

$$\left(2a^2 - 8b^2\right) = 2\left(a^2 - 4b^2\right) = 2(a - 2b)(a + 2b)$$

Then, writing the factored forms and cancelling:

$$\frac{a^3 - 8b^3}{2a^2 - 8b^2} = \frac{\cancel{(a - 2b)}\left(a^2 + 2ab + 4b^2\right)}{2(a + 2b)\cancel{(a - 2b)}} = \frac{a^2 + 2ab + 4b^2}{2(a + 2b)}$$

**• PROBLEM 413**

Divide $\dfrac{y^2 + y - 20}{y - 3}$ by $\dfrac{y^2 - 16}{y^2 + y - 12}$ .

<u>Solution:</u> Dividing by a nonzero polynomial is the same as multiplying by its reciprocal. That is,

$$\frac{y^2 + y - 20}{y - 3} \div \frac{y^2 - 16}{y^2 + y - 12} = \frac{y^2 + y - 20}{y - 3} \cdot \frac{y^2 + y - 12}{y^2 - 16}$$

Factor each numerator and denominator, where possible. Note that $y^2 + y - 20 = (y + 5)(y - 4)$

$$y^2 + y - 12 = (y + 4)(y - 3),$$

and $y^2 - 16 = y^2 - 4^2$, the difference of two squares. Using the formula for the difference of two squares, $\left(a^2 - b^2\right) = (a - b)(a + b)$, replace a by $y$ and b by 4 to obtain, $\left(y^2 - 16\right) = (y - 4)(y + 4)$.

Thus, $\dfrac{y^2 + y - 20}{y - 3} \cdot \dfrac{y^2 + y - 12}{y^2 - 16}$

$$= \frac{(y + 5)(y - 4)}{y - 3} \cdot \frac{(y + 4)(y - 3)}{(y - 4)(y + 4)} \qquad (1)$$

$$= \frac{(y + 5)(y - 4)(y + 4)(y - 3)}{(y - 3)(y - 4)(y + 4)} \qquad (2)$$

$$= \frac{(y + 5)(y - 4)(y + 4)(y - 3)}{(y - 4)(y + 4)(y - 3)} \qquad (3)$$

$$= y + 5.$$

Note that in equation (2) we are dividing by $(y - 3)(y - 4)(y + 4)$. If any of these factors equal 0, then we are dividing by zero, making our fraction in-valid. Thus, in order to be certain we are proceeding correctly, we must establish the following restrictions:

$$(y - 3) \neq 0, \quad (y - 4) \neq 0, \quad (y + 4) \neq 0;$$

thus, $\quad y \neq 3, \quad y \neq 4, \quad y \neq -4.$

Therefore, $\dfrac{y^2 + y - 20}{y - 3} \div \dfrac{y^2 - 16}{y^2 + y - 12} = y + 5,$

and $y \neq 3, 4, -4$.

• PROBLEM 414

Combine the following fractions:

$$\frac{3}{x^2 - 3x + 2} + \frac{1}{x^2 - 5x + 6} - \frac{2}{x^2 - 4x + 3}$$

Solution: Factor each denominator. Multiply together the highest power of each factor only once regardless of the number of times each factor appears to obtain the Least Common Denominator, LCD.

$$x^2 - 3x + 2 = (x - 1)(x - 2),$$

$$x^2 - 5x + 6 = (x - 2)(x - 3),$$

$$x^2 - 4x + 3 = (x - 1)(x - 3);$$

the LCD can be seen to be $(x - 1)(x - 2)(x - 3)$.

In factored form we have:

$$\frac{3}{(x - 1)(x - 2)} + \frac{1}{(x - 2)(x - 3)} - \frac{2}{(x - 1)(x - 3)}$$

Now multiply each fraction by the LCD to obtain:

$$\frac{3(x - 1)(x - 2)(x - 3)}{(x - 1)(x - 2)} + \frac{1(x - 1)(x - 2)(x - 3)}{(x - 2)(x - 3)}$$

$$- \frac{2(x - 1)(x - 2)(x - 3)}{(x - 1)(x - 3)};$$

and cancel like terms to obtain:

$$3(x - 3) + (x - 1) - 2(x - 2)$$

$$= 3x - 9 + (x - 1) - (2x - 4)$$

$$= 3x - 9 + x - 1 - 2x + 4 = 2x - 6.$$

Now, divide by the LCD, so as not to change the value of the given fraction. Thus, we have:

$$\frac{2x - 6}{(x - 1)(x - 2)(x - 3)}$$

and simplified to give:

$$\frac{2(x - 3)}{(x - 1)(x - 2)(x - 3)} = \frac{2}{(x - 1)(x - 2)} .$$

$$= \frac{2x - 6}{LCD}$$

$$= \frac{2(x - 3)}{(x - 1)(x - 2)(x - 3)}$$

$$= \frac{2}{(x - 1)(x - 2)} .$$

**• PROBLEM 415**

Multiply

$$\frac{y^2 + 3y + 2}{y - 3} \text{ by } \frac{y^2 - 7y + 12}{y^2 + y - 2} .$$

Solution: Factor the terms in the numerators and denominators of both fractions and cancel like terms.

$$\left(\frac{y^2 + 3y + 2}{y - 3}\right)\left(\frac{y^2 - 7y + 12}{y^2 + y - 2}\right)$$

$$= \frac{(y + 1)(y + 2)(y - 3)(y - 4)}{(y - 3)(y + 2)(y - 1)}$$

$$= \frac{(y +1)(y - 4)}{y - 1}$$ dividing out common factors $(y + 2)$ and $(y - 3)$

$$= \frac{y^2 - 3y - 4}{y - 1}$$

Either of the last two fractions may be accepted as correct results.

**• PROBLEM 416**

Divide $\frac{x - 3}{x^2 - 7x + 12}$ by $\frac{x^2 - 6x + 9}{x^2 - 8x + 16}$ .

Solution: Dividing by a fraction is equivalent to multiplying by its reciprocal; hence:

$$\frac{x - 3}{x^2 - 7x + 12} \div \frac{x^2 - 6x + 9}{x^2 - 8x + 16} = \frac{x - 3}{x^2 - 7x + 12} \times \frac{x^2 - 8x + 16}{x^2 - 6x + 9}$$

Factoring the numerators and denominators where possible we obtain:

$$\frac{x-3}{x^2-7x+12} \times \frac{x^2-8x+16}{x^2-6x+9} = \frac{x-3}{(x-4)(x-3)} \cdot \frac{(x-4)(x-4)}{(x-3)(x-3)}$$

$$= \frac{(x-3)(x-4)(x-4)}{(x-4)(x-3)(x-3)(x-3)}$$

Cancelling out the like terms which appear in both the numerators and denominators (since $\frac{x-3}{x-3} = 1$, and $\frac{x-4}{x-4} = 1$) gives us:

$$\frac{(x-3)(x-4)(x-4)}{(x-4)(x-3)(x-3)(x-3)} = \frac{x-4}{(x-3)(x-3)}$$

Therefore,

$$\frac{x-3}{x^2-7x+12} \div \frac{x^2-6x+9}{x^2-8x+16} = \frac{x-4}{(x-3)(x-3)} .$$

**• PROBLEM 417**

Divide

$$\frac{2y^2 - 11y + 12}{6y^2 - 6y - 12} \quad \text{by} \quad \frac{3y^2 - 14y + 8}{2y^2 - 6y + 4} .$$

Solution: Divide the two fractions by inverting the fraction to the right of the division sign and multiplying the inverted fraction by the fraction on the left of the division sign. Thus,

$$\frac{2y^2 - 11y + 12}{6y^2 - 6y - 12} \div \frac{3y^2 - 14y + 8}{2y^2 - 6y + 4}$$

$$= \frac{2y^2 - 11y + 12}{6y^2 - 6y - 12} \cdot \frac{2y^2 - 6y + 4}{3y^2 - 14y + 8}$$

Now factor the expressions in the numerator and denominator of each fraction:

$$\frac{(2y-3)(y-4)\ 2(y-2)(y-1)}{6(y-2)(y+1)(3y-2)(y-4)}$$

Cancel common terms in the numerator and denominator to obtain:

$$\frac{(2y-3)(y-1)}{3(y+1)(3y-2)} .$$

**• PROBLEM 418**

Find $\frac{4a^2 + 4ab + b^2}{2a^3 + 16b^3} \div \frac{4a^2 - b^2}{6a + 12b}$

__Solution:__ Division by a fraction is equivalent to multiplication by its reciprocal, hence:

$$\frac{4a^2 + 4ab + b^2}{2a^3 + 16b^3} \div \frac{4a^2 - b^2}{6a + 12b} = \frac{4a^2 + 4ab + b^2}{2a^3 + 16b^3} \times \frac{6a + 12b}{4a^2 - b^2}$$

Factor the numerators and denominators as completely as possible.

$4a^2 + 4ab + b^2$ is called a trinomial perfect square for it is in the form $(2a)^2 + 2(2ab) + b^2$. The formula for factoring a trinomial perfect square is given by $x^2 + 2xy + y^2 = (x+y)(x+y) = (x+y)^2$. Replacing $x$ by $2a$ and $y$ by $b$ we obtain:

$$4a^2 + 4ab + b^2 = (2a + b)(2a + b) = (2a + b)^2$$

$$2a^3 + 16b^3 = 2\left(a^3 + 8b^3\right) = 2\left(a^3 + (2b)^3\right)$$ where $a^3 + (2b)^3$ us the sum of two cubes. The formula for the sum of two cubes is

$$x^3 + y^3 = (x + y)\left(x^2 - xy + y^2\right).$$

Replacing $x$ by $a$ and $y$ by $2b$ we obtain:

$$a^3 + (2b)^3 = (a + 2b)\left(a^2 - 2ab + (2b)^2\right)$$
$$= (a + 2b)\left(a^2 - 2ab + 4b^2\right)$$

thus, $$2a^3 + 16b^3 = 2(a + 2b)\left(a^2 - 2ab + 4b^2\right)$$

Remove the highest common factor from $6a + 12b$ which is $6$. Hence,

$$6a + 12b = 6(a + 2b)$$

$4a^2 - b^2$ is the difference of two squares, $(2a)^2 - b^2$. Applying the formula for the difference of two squares $x^2 - y^2 = (x-y)(x+y)$:

$$4a^2 - b^2 = (2a)^2 - b^2 = (2a - b)(2a + b).$$

Now, express all the denominators and numerators in their factored form, and cancel:

$$\frac{4a^2 + 4ab + b^2}{2a^3 + 16b^3} \times \frac{6a + 12b}{4a^2 - b^2} = \frac{\cancel{(2a + b)}(2a + b)}{\overset{}{\underset{1}{\cancel{2}}}\cancel{(a+2b)}(a^2 - 2ab + 4b^2)} \times \frac{\overset{3}{\cancel{6}}\cancel{(a + 2b)}}{(2a - b)\cancel{(2a + b)}}$$
$$= \frac{3(2a + b)}{\left(a^2 - 2ab + 4b^2\right)(2a - b)}$$

**• PROBLEM 419**

> Perform the indicated operation,
>
> $$\frac{x^3 - y^3}{x^2 - 5x + 6} \cdot \frac{x^2 - 4}{x^2 - 2xy + y^2}$$

__Solution:__ We factor numerators and denominators to enable us to cancel terms.

$$x^3 - y^3$$

is the difference of two cubes. Thus we factor it applying the formula for the difference of two cubes,

$$a^3 - b^3 = (a - b)\left(a^2 + ab + b^2\right),$$

replacing a by x and b by y. Thus,

$$x^3 - y^3 = (x - y)\left(x^2 + xy + y^2\right).$$

$x^2 - 5x+6$ is factored as $(x-2)(x-3)$.

$$x^2 - 4 = x^2 - 2^2,$$

the difference of two squares. Applying the formula for the difference of two squares,

$$a^2 - b^2 = (a + b)(a - b),$$

and replacing a by x and b by 2 we obtain,

$$x^2 - 4 = (x + 2)(x - 2).$$

$$x^2 - 2xy + y^2 = (x - y)(x - y).$$

Thus,

$$\frac{x^3-y^3}{x^2 - 5x + 6} \cdot \frac{x^2 - 4}{x^2 - 2xy + y^2} = \frac{(x - y)\left(x^2 + xy + y^2\right)}{(x - 2)(x - 3)}$$

$$\cdot \frac{(x + 2)(x - 2)}{(x - y)(x - y)}$$

$$= \frac{\left(x^2 + xy + y^2\right)\cancel{(x - y)}\cancel{(x - 2)}(x + 2)}{(x - 3)\cancel{(x - 2)}\cancel{(x - y)}(x - y)}$$

$$= \frac{\left(x^2 + xy + y^2\right)(x + 2)}{(x - 3)(x - y)}$$

# CHAPTER 16

# SOLVING QUADRATIC EQUATIONS BY FACTORING

**Basic Attacks and Strategies for Solving Problems in this Chapter. See pages 265 to 310 for step-by-step solutions to problems.**

A quadratic equation in standard form is written as

$$ax^2 + bx + c = 0,$$

where $a$, $b$, and $c$ are constants and $a \neq 0$. To solve a quadratic equation by factoring involves the following steps:

(1) Eliminate any fractions from the equation.

(2) Write the equation in standard form.

(3) Factor the left side or non-zero side of the equation.

(4) Use the zero-property to set each factor equal to 0 which yields linear equations. (The zero-property states that for real number $x$ and $y$, $xy = 0$ if and only if $x = 0$ or $y = 0$ or both equal 0).

(5) Solve the resulting linear equations.

For example, the solution of the quadratic equation

$$x^2 - 8x + 15 = 0$$

by factoring is given as follows:

$$(x - 3)(x - 5) = 0$$

$$x - 3 = 0 \quad \text{and} \quad x - 5 = 0$$

$$x = 3 \qquad\qquad x = 5$$

Thus, the solution set is $\{3, 5\}$.

If the quadratic equation contains one or more radicals then the first step in the procedure for solving the equation is to eliminate the radical(s) from the equation. The standard procedure for doing this is to use the squaring property of equality, that is, square both sides of the original equation and simplify. Then, the solution procedure proceeds with the steps outlined above. Notice that if a radical expression in the equation equals a negative real number, then the solution is an extraneous solution. This means that it satisfies the equation obtained after removing the radical(s), but does not satisfy the original equation.

A quadratic equation that is not factorable in standard form can be solved by completing the square. This technique involves the following steps:

(1) Write the given standard quadratic equation in the form

$$x^2 + (b/a)x = -c/a,$$

where $a \neq 0$.

(2) Then, take $^1/_2$ of the coefficient of $x$, square it, and add the result to both sides of the equation in Step 1 to obtain a perfect square trinomial on the left-hand side of the equation as follows:

$$x^2 + (b/a)x + (b/2a)^2 = -c/a + (b/2a)^2$$

(3) Factor the left-hand side of the equation in Step 2 to get:

$$(x + b/2a)^2 = -c/a + (b/2a)^2.$$

(4) Find the values of $x$ by taking the square root of both sides of the equation in Step 3.

For example, the solution of

$$x^2 + 4x - 9 = 0$$

by completing the square as follows:

$$x^2 + 4x = 9 \quad \text{or} \quad x^2 + 4x + (2)^2 = 9 + (2)^2$$

$$(x + 2)^2 = 13$$

$$x + 2 = \pm\sqrt{13}$$

$$x = -2 \pm\sqrt{13}.$$

# Step-by-Step Solutions to Problems in this Chapter, "Solving Quadratic Equations by Factoring"

## EQUATIONS WITHOUT RADICALS

• **PROBLEM 420**

Show that $x^2 + 2x + 5 = 20$ is a conditional equation.

Solution: A conditional equation is an equation for which there exists at least one value which may be substituted for the variable that makes the equation false, but is true for other values. It is sufficient to exhibit one replacement for x that makes the equation true and one that makes it false.

$$\text{Let } x = 3: \quad (3)^2 + 2(3) + 5 \stackrel{?}{=} 20$$
$$9 + 6 + 5 \stackrel{?}{=} 20$$
$$20 = 20$$
$$\text{Let } x = -4: \quad (-4)^2 + (-4) + 5 \stackrel{?}{=} 20$$
$$16 - 8 + 5 \stackrel{?}{=} 20$$
$$13 \neq 20$$

When $x = -4$, this value of x makes the equation false. For $x = 3$, the equation is true. Therefore, $x^2 + 2x + 5 = 20$ is a conditional equation.

Notice that we have not solved the equation in this example . An equation is solved when its solution set is completely known.

• **PROBLEM 421**

Solve the equation $(3x - 7)(x + 2) = 0$.

Solution: When a given product of two numbers that are equal to zero, $ab = 0$, either a must equal zero or b must equal zero (or both equal zero). So if $(3x - 7)(x + 2) = 0$, then $(3x - 7) = 0$ or $(x + 2) = 0$.

| | |
|---|---|
| $3x - 7 = 0$ | $x + 2 = 0$ |
| Add 7 to both sides: | Subtract 2 from both sides: |
| $3x = 7$ | |
| Divide both sides by 3: | |
| $x = \frac{7}{3}$ | $x = -2$ |

Hence $x = \frac{7}{3}$ or $x = -2$, and our solution set is $\left\{\frac{7}{3}, -2\right\}$.

**• PROBLEM 422**

Solve the equation $3x^2 + 5x = 0$.

**Solution:** Because division by zero is impossible, we must not divide by x, since x might be equal to zero. Instead of dividing by x we factor x from the left side of the equation to obtain:

$$x(3x + 5) = 0.$$

Whenever we have a situation where $ab = 0$ (the product of two or more numbers equal to zero) either $a = 0$ or $b = 0$. Therefore $x = 0$, or $3x + 5 = 0$. Subtract 5 from each side of the second equation to obtain:

$$3x = -5$$

Divide both sides by 3 to obtain $x = -\frac{5}{3}$. The two solutions of the given equation are $x = 0$ and $x = -\frac{5}{3}$.

To check the validity of the two solutions we substitute them into the given equation. Thus,

when $x = 0$

$$3x^2 + 5x = 0$$

$$3(0)^2 + 5(0) = 0$$

$$0 = 0$$

when $x = -\frac{5}{3}$

$$3x^2 + 5x = 0$$

$$3\left(-\frac{5}{3}\right)^2 + 5\left(-\frac{5}{3}\right) = 0$$

$$3\left(\frac{25}{9}\right) + 5\left(-\frac{5}{3}\right) = 0$$

$$\frac{25}{3} - \frac{25}{3} = 0$$

$$0 = 0$$

**• PROBLEM 423**

Solve the equation $x^2 + 8x + 15 = 0$.

**Solution:** Since $(x + a)(x + b) = x^2 + bx + ax + ab$

$= x^2 + (a + b)\,x + ab$, we may factor the given equation, $0 = x^2 + 8x + 15$, replacing $a + b$ by 8 and $ab$ by 15. Thus,

$$a + b = 8, \quad \text{and}$$

$$ab = 15.$$

We want the two numbers a and b whose sum is 8 and whose product is 15. We check all pairs of numbers whose product is 15:

(a) $1 \cdot 15 = 15$; thus $a = 1$, $b = 15$ and $ab = 15$.

$1 + 15 = 16$, therefore we reject these values because $a + b \neq 8$.

(b) $3 \cdot 5 = 15$; thus $a = 3$, $b = 5$, and $ab = 15$.

$3 + 5 = 8$. Therefore $a + b = 8$, and we accept these values.

Hence $\quad x^2 + 8x + 15 = 0 \qquad$ is equivalent to

$$0 = x^2 + (3 + 5)x + 3 \cdot 5 = (x + 3)(x + 5)$$

Hence, $\quad x + 5 = 0 \quad$ or $x + 3 = 0$

since the product of these two numbers is zero, one of the numbers must be zero. Hence, $x = -5$, or $x = -3$, and the solution set is $X = \{-5, -3\}$.

The student should note that $x = -5$ or $x = -3$. We are certainly not making the statement, that $x = -5$, and $x = -3$. Also, the student should check that both these numbers do actually satisfy the given equations and hence are solutions.

Check: Replacing x by $(-5)$ in the original equation:

$$x^2 + 8x + 15 = 0$$

$$(-5)^2 + 8(-5) + 15 = 0$$

$$25 - 40 + 15 = 0$$

$$-15 + 15 = 0$$

$$0 = 0$$

Replacing x by $(-3)$ in the original equation:

$$x^2 + 8x + 15 = 0$$

$$(-3)^2 + 8(-3) + 15 = 0$$

$$9 - 24 + 15 = 0$$

$$-15 + 15 = 0$$

$$0 = 0.$$

**Find the roots of the equation $x^2 + 6x + 8 = 0$.**

Solution: In order to obtain the roots of this equation we must factor it, that is, put it in the form $(x + a)(x + b) = 0$. Using our method for multiplying polynomials we find $(x + a)(x + b) = x^2 + ax + bx + ab = x^2 + (a + b)x + ab$. We are given $x^2 + 6x + 8 = 0$. Thus in our case

$$a + b = 6 \quad (1)$$

$$\text{and } ab = 8 \quad (2)$$

That is, we want to find 2 numbers a and b whose sum is 6 and whose product is 8.

Checking all pairs of numbers whose product is 8:

(a) $8 \times 1 = 8$; hence 8 and 1 satisfy equation 2, however

$8 + 1 = 9 \neq 6$; thus we reject these values, for they fail to satisfy equation (1)

(b) $4 \times 2 = 8$; hence 4 and 2 satisfy equation 2, and

$4 + 2 = 6$ satisfying equation (1).

Thus, we conclude $a = 4$, $b = 2$, and we may write

$x^2 + 6x + 8 = \left(x^2 + (4 + 2)x + 4 \cdot 2\right)$ as $(x + 4)(x + 2)$

Since $x^2 + 6x + 8 = 0$ and $x^2 + 6x + 8 = (x + 4)(x + 2)$,

$$(x + 4)(x + 2) = 0$$

Recall that if the product of 2 numbers, $ab = 0$, either $a = 0$, or $b = 0$.

Hence $x + 4 = 0$ or $x + 2 = 0$

and $x = -4$ or $x = -2$

Check: Replace x by (-4) in our given equation,

$$(-4)^2 + 6(-4) + 8 = 0$$

$$16 - 24 + 8 = 0$$

$$-8 + 8 = 0$$

$$0 = 0$$

Now, replace x by (-2) in our given equation,

$$(-2)^2 + 6(-2) + 8 = 0$$

$$4 - 12 + 8 = 0$$

$$-8 + 8 = 0$$

$$0 = 0$$

Therefore, the roots of the given equation are $x = -4$, $x = -2$.

**• PROBLEM 425**

Solve: $x^2 - 5x - 14 = 0$.

Solution: To find the roots of this quadratic, we factor it (put it in the form $(x + a)(x + b) = 0$).

Note that $(x + a)(x + b) = x^2 + (a + b)x + ab$

Thus, in our quadratic, $x^2 + (-5)x + (-14)$,

$$a + b = -5 \tag{1}$$

and

$$ab = -14 \tag{2}$$

That is, we want the two numbers, a and b, whose sum is $(-5)$, and whose product is $(-14)$.

To find these numbers, we can check the set of numbers whose product is $(-14)$:

(a) $(-14) \times (1) = -14$, therefore equation (2) is satisfied, now check these values in equation (1):

$(-14) + (1) = -13 \neq -5$ therefore we reject these values.

(b) $(-7) \times (2) = -14$, therefore equation (2) is satisfied, now check these values in equation (1):

$(-7) + 2 = -5$ hence both equations are satisfied and we conclude

$$a = -7 \quad \text{and } b = 2.$$

Thus

$$\begin{aligned} x^2 - 5x - 14 &= x^2 + (-7 + 2)x + (-7)(2) \\ &= [x + (-7)][x + 2] \\ &= (x - 7)(x + 2) = 0 \end{aligned}$$

By the fundamental principle, if the product of two numbers $yz = 0$, then either $y = 0$ or $z = 0$; hence if

$$(x - 7)(x + 2) = 0$$

| | | |
|---|---|---|
| either $x - 7 = 0$ | or | $x + 2 = 0$ |
| add 7 to both sides | | subtract 2 from both sides |
| $x = 7$ | or | $x = -2$ |

This proves that if the equation has roots, they must be either 7 or $-2$. We check these values by substituting in the given equation:

$$\text{If } x = 7, \text{ then } x^2 - 5x - 14 = (7)^2 - 5(7) - 14$$
$$= 49 - 35 - 14$$
$$= 49 - 49$$
$$= 0$$

$$\text{If } x = -2, \text{ then } x^2 - 5x - 14 = (-2)^2 - 5(-2) - 14$$
$$= 4 + 10 - 14$$
$$= 14 - 14$$
$$= 0$$

We may now conclude that the solution to our equation is $x = 7$ or $x = -2$.

**• PROBLEM 426**

Find the roots of $x^2 - 3x - 10 = 0$.

Solution: To find the roots of this quadratic, we factor it (put it in the form $(x + a)(x + b) = 0$).

Note that $(x + a)(x + b) = x^2 + (a + b)x + ab$

Thus in our quadratic, $x^2 + (-3)x + (-10)$,

$$a + b = -3 \qquad (1)$$

and $$ab = -10. \qquad (2)$$

That is, we want the two numbers a and b whose sum is $(-3)$, and whose product is $(-10)$.

To find these numbers, we can check the set of numbers whose product is $(-10)$:

(a) $(-10) \times (1) = -10$, therefore equation (2) is satisfied, now check these values in equation (1):
$(-10) + (1) = -9 \neq -3$ therefore we reject these values.

(b) $(-5) \times (2) = -10$, therefore equation (2) is satisfied, now checking these values in equation (1): $(-5) + 2 = -3$.

Hence both equations are satisfied and we conclude

$$a = -5 \quad \text{and} \quad b = 2.$$

$$\text{Thus, } x^2 - 3x - 10 = x^2 + (-5 + 2)x + (-5)(2)$$
$$= [x + (-5)][x + 2]$$
$$= (x - 5)(x + 2) = 0.$$

Hence, by the fundamental property which states that if $ab = 0$, either $a = 0$ or $b = 0$, $x - 5 = 0$ or $x + 2 = 0$ and

$$x = 5 \qquad \text{or} \qquad x = -2.$$

This proves that if the equation has roots, they must be either 5 or $-2$. So far we have not proved that these are roots. We can check this by substituting in the given equation. If $x = 5$, then

$$\begin{aligned} x^2 - 3x - 10 &= (5)^2 - 3(5) - 10 \\ &= 25 - 15 - 10 \\ &= 25 - 25 \\ &= 0 \end{aligned}$$

Thus 5 is indeed a root of the equation.

If $x = -2$, then

$$\begin{aligned} x^2 - 3x - 10 &= (-2)^2 - 3(-2) - 10 \\ &= 4 + 6 - 10 \\ &= 10 - 10 \\ &= 0 \end{aligned}$$

Thus $-2$ is also a root.

Such a check not only has a logical purpose, but it also assures us that we have not made a mistake in arithmetic. We may now conclude that the solution to our equation is $x = 5$ or $x = -2$.

**• PROBLEM 427**

Find the roots of the function $G$ defined by the rule

$$G(x) = x^2 + 5x + 6.$$

**Solution:** To find the roots of the function $G$, find those values of $x$ which satisfy $G(x) = 0$. Let $x^2 + 5x + 6 = 0$. In factored form this may be written $(x + 3)(x + 2) = 0$. Values of $x$ which make this product $= 0$ satisfy $x + 3 = 0$ or $x + 2 = 0$. Hence $x = -3$ or $x = 2$.

Check:

for $x = -3$      for $x = -2$

$(-3)^2 + 5(-3) + 6 = 9 - 15 + 6 = 0$      $(-2)^2 + 5(-2) + 6 = 4 - 10 + 6 = 0$

**• PROBLEM 428**

Solve and check the roots of the equation

$$2x^2 - 3x + 1 = 0$$

**Solution:** Factor the given equation into a product of two polynomials: therefore: $(2x - 1)(x - 1) = 0$.

When $ab = 0$, either $a = 0$ or $b = 0$ where a and b are real numbers. Therefore, either $2x - 1 = 0$ or $x - 1 = 0$.

Therefore, either $x = \frac{1}{2}$ or $x = 1$.

Hence, the roots to the given equation are:

$x = \frac{1}{2}$ and $x = 1$.

To check these roots, substitute $x = \frac{1}{2}$ in the given equation:

$$2\left(\frac{1}{2}\right)^2 - 3\left(\frac{1}{2}\right) + 1 = 0$$

$$2\left(\frac{1}{4}\right) - \frac{3}{2} + 1 = 0$$

$$\frac{1}{2} - \frac{3}{2} + 1 = 0$$

$$-1 + 1 = 0$$

$$0 = 0$$

Now, substitute $x = 1$ in the given equation:

$$2(1)^2 - 3(1) + 1 = 0$$

$$2 - 3 + 1 = 0$$

$$-1 + 1 = 0$$

$$0 = 0$$

**• PROBLEM 429**

Solve the equation $2x^2 - 7x + 6 = 0$.

Solution: To solve this equation we must find the values of x. We find these values by factoring the given equation into a product of two polynomials. To do this, factors of 6 must be found which give a coefficient of $-7$ for x when the two polynomials are multiplied together. The factors $(2x - 3)(x - 2)$ accomplish this. When $ab = 0$ where a and b are any numbers, either $a = 0$ or $b = 0$. Therefore, either $2x - 3 = 0$ or $x - 2 = 0$. To solve for x in the first equation we add 3 to both sides of the equation, $2x - 3 = 0$, and then divide by 2. Thus, we have:

$$2x - 3 + 3 = 0 + 3$$

$$2x = 3$$

$$\frac{2x}{2} = \frac{3}{2}$$

$$x = \frac{3}{2}.$$

To solve for x in the second equation we add 2 to both sides of the equation, $x - 2 = 0$. Thus, we have:

$$x - 2 + 2 = 0 + 2$$

$$x = 2.$$

Therefore, the roots of the given equation are:

$$x = \frac{3}{2},\ x = 2.$$

To check these roots, do the following: Substituting $x = \frac{3}{2}$ in the given equation, we find:

$$2\left(\frac{3}{2}\right)^2 - 7\left(\frac{3}{2}\right) + 6 = 0$$

$$\frac{9}{2} - \frac{21}{2} + \frac{12}{2} = 0$$

$$0 = 0.$$

Substituting $x = 2$ in the given equation, we find:

$$2(2)^2 - 7(2) + 6 = 0$$

$$8 - 14 + 6 = 0$$

$$0 = 0.$$

Thus, the two roots are valid.

**• PROBLEM 430**

Solve the following for x:

(a) $x^2 - 3x = 0$

(b) $6x^2 + 5x - 4 = 0$

Solution: (a) Factor the common factor x from the left side of the given equation:

$$x^2 - 3x = x(x - 3)$$

Since $x^2 - 3x = 0$,

$$x(x - 3) = 0. \qquad (1)$$

Whenever a product $ab = 0$, where a and b are any two numbers, either $a = 0$ or $b = 0$. Then, equation (1) becomes,

$$x = 0 \text{ or } x - 3 = 0$$

$$x = 3$$

Hence, the solution set is: $\{0, 3\}$.

(b) Factor the left side of the given equation into a product of two polynomials:

$$6x^2 + 5x - 4 = (3x + 4)(2x - 1)$$

Since $6x^2 + 5x - 4 = 0$,

$$(3x + 4)(2x - 1) = 0$$

Thus,

$$3x + 4 = 0 \quad \text{or} \quad 2x - 1 = 0$$

$$3x = -4 \quad \text{or} \quad 2x = 1$$

$$x = \frac{-4}{3} \quad \text{or} \quad x = \frac{1}{2}$$

Hence, the solution set is: $\left\{\frac{4}{3}, \frac{1}{2}\right\}$.

**• PROBLEM 431**

Solve the equation $5y^2 = 6y$ by the factoring method.

Solution: Add $(-6y)$ to both members of the given equation

$$5y^2 - 6y = 0$$

Factor y from the left member, $y(5y - 6) = 0$.

When the product of two numbers $ab = 0$ either $a = 0$, or $b = 0$. Thus, either $y = 0$ or $(5y - 6) = 0$.

Solving for y in the second equation, $5y - 6 = 0$:

add $-6$ to both sides, $5y = 6$

divide by 5, $y = 6/5$.

Therefore, the solution set is $\{0, 6/5\}$.

Check: To check these values we replace y by 0 and then by 6/5 in the original equation:

(a) when $y = 0$

$$5y^2 = 6y$$

$$5(0)^2 = 6(0)$$

$$0 = 0$$

(b) when $y = \frac{6}{5}$

$$5\left(\frac{6}{5}\right)^2 = 6\left(\frac{6}{5}\right)$$

$$5\left(\frac{36}{25}\right) = \frac{36}{5}$$

$$\frac{36}{5} = \frac{36}{5}$$

Thus, the solution set of the given equation is indeed $\{0, 6/5\}$.

Solve $4x^2 = 8x$.

Solution: The temptation to divide both sides by $4x$ to arrive at: $x = 2$, should be avoided, for if $x = 0$ we are performing an operation which is undefined. Although 2 actually is a root, there happens to be another root, which is lost in this process.

When solving equations, avoid multiplying or dividing by anything but nonzero numbers. In this case, there is no harm in dividing both sides by the number 4:

$$x^2 = 2x$$

We then add $-2x$ to both sides, to arrive at:

$$x^2 - 2x = 0$$

Factoring: $$x(x-2) = 0$$

Whenever the product of two numbers $ab = 0$, either $a = 0$ or $b = 0$. Therefore,

$$x = 0 \text{ or } x - 2 = 0,$$

and

$$x = 0 \text{ or } x = 2.$$

Check: To verify that the roots of this equation are $x = 0$ and $x = 2$, we replace $x$ by each value in the original equation. Replacing $x$ by 0 in $4x^2 = 8x$:

$$4(0)^2 = 8(0)$$
$$0 = 0$$

Replacing $x$ by 2:

$$4(2)^2 = 8(2)$$
$$4(4) = 16$$
$$16 = 16$$

Thus, the roots of the equation are 0 and 2, and the solution set is

$$\{0,2\}.$$

• PROBLEM 433

Solve the equation $4x^2 = 9x$.

Solution: To solve the given equation we must find the values of x which satisfy the equation. To do this we proceed as follows: Subtract 9x from both sides of the given equation. Thus, we have:

$$4x^2 - 9x = 0. \qquad (1)$$

Now, factor x from the terms on the left side of Equation (1):

$$x(4x - 9) = 0.$$

When $ab = 0$ where a and b are any numbers, then either $a = 0$ or $b = 0$. Therefore: either $x = 0$, or $4x - 9 = 0$.

To solve for x in the second equation, add 9 to both sides of $4x - 9 = 0$, and then divide both sides of the equation by 4. Thus,

$$4x - 9 + 9 = 0 + 9$$
$$4x = 9$$
$$\frac{4x}{4} = \frac{9}{4}$$

$$x = \frac{9}{4}.$$

Therefore, the two roots to the given equation are $x = 0$ and $x = \frac{9}{4}$. To check these two roots, do the following: Substitute $x = 0$ in the equation, $4x^2 = 9x$. Then:

$$4(0)^2 = 9(0)$$
$$4(0) = 0$$
$$0 = 0.$$

Now, substitute $x = \frac{9}{4}$ in the equation, $4x^2 = 9x$. Then:

$$4\left(\frac{9}{4}\right)^2 = 9\left(\frac{9}{4}\right)$$

$$4\left(\frac{81}{16}\right) = \frac{81}{4}$$

$$\frac{81}{4} = \frac{81}{4}.$$

Therefore, the two values $x = 0$, $x = \frac{9}{4}$ are valid.

**• PROBLEM 434**

Solve the equation $6x^2 = 2 - x$.

__Solution:__ Write the equation in standard quadratic form by adding $x - 2$ to both sides of the equation. Then we have $6x^2 + x - 2 = 0$. In factored form this becomes $(3x + 2)(2x - 1) = 0$. The values of $x$ that make this product $= 0$ satisfy

$$3x + 2 = 0 \quad \text{or} \quad 2x - 1 = 0$$
$$3x = -2 \quad \text{or} \quad 2x = 1$$
$$x = -\tfrac{2}{3} \quad \text{or} \quad x = \tfrac{1}{2}$$

Check:

for $x = -\frac{2}{3}$

$$6\left(-\tfrac{2}{3}\right)^2 \stackrel{?}{=} 2 - \left(-\tfrac{2}{3}\right)$$

$$6\left(\frac{4}{9}\right) \stackrel{?}{=} \frac{18}{9} - \left(-\frac{6}{9}\right)$$

$$\frac{24}{9} = \frac{24}{9}$$

for $x = \frac{1}{2}$

$$6\left(\tfrac{1}{2}\right)^2 \stackrel{?}{=} 2 - \left(\tfrac{1}{2}\right)$$

$$6\left(\tfrac{1}{4}\right) \stackrel{?}{=} \frac{8}{4} - \left(\frac{2}{4}\right)$$

$$\frac{6}{4} = \frac{6}{4}$$

Therefore the solution set is $\{-\frac{2}{3}, \frac{1}{2}\}$.

**• PROBLEM 435**

Solve the equation $2x^2 = x + 6$ by the factoring method.

__Solution:__

| | |
|---|---|
| $2x^2 = x + 6$ | given equation |
| $2x^2 - x - 6 = 0$ | adding $-x -6$ to each member |
| $(2x + 3)(x - 2) = 0$ | factoring left member |

Whenever the product of 2 numbers $ab = 0$ either $a = 0$ or b = 0. Thus either $2x + 3 = 0$ or $x - 2 = 0$

| | |
|---|---|
| $2x + 3 = 0$ | setting the first factor equal to 0 |
| $2x = -3$ | |
| $x = -\frac{3}{2}$ | solving for x |
| $x - 2 = 0$ | setting second factor equal to 0 |
| $x = 2$ | solving for x |

Consequently the solution set is $\left\{-\frac{3}{2},\ 2\right\}$.

The solution set can be verified by replacing x in the given equation by each element in the set.

Check:

| | |
|---|---|
| $x = -\frac{3}{2}$ | $x = 2$ |
| $2x^2 = x + 6$ | $2x^2 = x + 6$ |
| $2\left(\frac{-3}{2}\right)^2 = \frac{-3}{2} + 6$ | $2(2)^2 = 2 + 6$ |
| $2\left(\frac{9}{4}\right) = \frac{-3}{2} + \frac{12}{2}$ | $2 \cdot 4 = 2 + 6$ |
| $\frac{9}{2} = \frac{9}{2}$ ✓ | $8 = 8$ ✓ |

**• PROBLEM 436**

Solve the equation $4x^2 = 4x - 1$. (1)

Solution: Subtract 4x from both sides of equation (1).

$$4x^2 - 4x = 4x - 1 - 4x$$

Therefore: $4x^2 - 4x = -1$ (2).

Add 1 to both sides of equation (2).

$$4x^2 - 4x + 1 = -1 + 1$$

$$4x^2 - 4x + 1 = 0 \quad (3)$$

Factor the left side of equation (3) as a product of two polynomials. Therefore,

$$(2x - 1)(2x - 1) = 0.$$

When $ab = 0$, either $a = 0$ or $b = 0$, where a and b are real numbers. Therefore, $(2x - 1) = 0$, $(2x - 1) = 0$.

$$x = \frac{1}{2},\ x = \frac{1}{2} \text{ (Both roots are equal to } \tfrac{1}{2}\text{).}$$

In order to check that $\frac{1}{2}$ is a solution to the given equation, substitute $x = \frac{1}{2}$ in the given equation:

$$4\left(\frac{1}{2}\right)^2 = 4\left(\frac{1}{2}\right) - 1$$

$$4\left(\frac{1}{4}\right) = 2 - 1$$

$$1 = 1$$

**• PROBLEM 437**

Solve the equation $4x^2 = 100$.

__Solution:__ To solve this equation we must find the values for x which satisfy the equation. To do this we proceed as follows: Subtract 100 from both sides of the given equation. Thus,

$$4x^2 - 100 = 100 - 100$$

$$4x^2 - 100 = 0. \qquad (1)$$

Now, factor 4 from the left side of Equation (1):

$$4(x^2 - 25) = 0.$$

Next, factor $x^2 - 25$ into a product of two polynomials. To do this, notice that $x^2 - 25$ is the difference between two squares, that is, $x^2 - 5^2$. Thus, the two factors are $(x - 5)(x + 5)$. Thus, we have:

$$4(x - 5)(x + 5) = 0.$$

Dividing both sides of this equation by 4, we obtain:

$$\frac{4(x - 5)(x + 5)}{4} = \frac{0}{4}.$$

Therefore, $(x - 5)(x + 5) = 0$.

When $ab = 0$, where a and b are any numbers, either $a = 0$ or $b = 0$. Therefore, either $x - 5 = 0$, or $x + 5 = 0$.

To solve for x in the first equation add 5 to both sides of $x - 5 = 0$. Thus,

$$x - 5 + 5 = 0 + 5$$

$$x = 5.$$

To solve for x in the second equation subtract 5 from both sides of $x + 5 = 0$. Thus,

$$x + 5 - 5 = 0 - 5$$

$$x = -5.$$

Therefore, the solution of the given equation is $x = 5$, $x = -5$.

To check these two solutions, do the following:

Substituting $x = 5$ in the given equation, we find:

$$4(5)^2 = 100$$

$$100 = 100.$$

Substituting $x = -5$ in the given equation, we find:

$$4(-5)^2 = 100$$

$$100 = 100.$$

Thus, the obtained values of x are valid.

**• PROBLEM 438**

Solve the following equations by factoring.

(a) $2x^2 + 3x = 0$ (c) $z^2 - 2z - 3 = 0$

(b) $y^2 - 2y - 3 = y - 3$ (d) $2m^2 - 11m - 6 = 0$

Solution: (a) $2x^2 + 3x = 0$. Factoring out the common factor of x from the left side of the given equation,

$$x(2x + 3) = 0.$$

Whenever a product $ab = 0$, where a and b are any two numbers, either $a = 0$ or $b = 0$. Then, either

$$x = 0 \quad \text{or} \quad 2x + 3 = 0$$

$$2x = -3$$

$$x = \frac{-3}{2}$$

Hence, the solution set to the original equation $2x^2 + 3x = 0$ is: $\left\{-\frac{3}{2}, 0\right\}$

(b) $y^2 - 2y - 3 = y - 3$. Subtract $(y - 3)$ from both sides of the given equation:

$$y^2 - 2y - 3 - (y - 3) = y - 3 - (y - 3)$$

$$y^2 - 2y - \cancel{3} - y + \cancel{3} = \cancel{y} - \cancel{3} - \cancel{y} + \cancel{3}$$

$$y^2 - 3y = 0.$$

Factor out a common factor of y from the left side of this equation:

$$y(y - 3) = 0.$$

Thus, $y = 0$ or $y - 3 = 0$

$$y = 3$$

Therefore, the solution set to the original equation $y^2 - 2y - 3 = y - 3$ is: $\{0,3\}$.

(c) $z^2 - 2z - 3 = 0$. Factor the original equation into a product of two polynomials:

$$z^2 - 2z - 3 = (z - 3)(z + 1) = 0$$

Hence,

$(z - 3)(z + 1) = 0$; and $z - 3 = 0$ or $z + 1 = 0$
$$z = 3 \qquad z = -1$$

Therefore, the solution set to the original equation $z^2 - 2z - 3 = 0$ is: $\{-1,3\}$.

(d) $2m^2 - 11m - 6 = 0$. Factor the original equation into a product of two polynomials:

$$2m^2 - 11m - 6 = (2m + 1)(m - 6) = 0$$

Thus,
$$2m + 1 = 0 \quad \text{or} \quad m - 6 = 0$$
$$2m = -1 \qquad m = 6$$
$$m = \frac{-1}{2}$$

Therefore, the solution set to the original equation $2m^2 - 11m - 6 = 0$ is: $\left\{-\frac{1}{2},\ 6\right\}$.

**• PROBLEM 439**

Solve the equation $a^2x + 4c^2x - 10c = 5a - 4acx$ for $x$.

Solution: Add $4acx$ and $10c$ to both sides.

$$a^2x + 4acx + 4c^2x = 5a + 10c,$$

Factor out $x$ from the left-hand side and 5 from the right-hand side.

$$\left(a^2 + 4ac + 4c^2\right)x = 5(a + 2c),$$

Factor the left-hand side which is a trinomial perfect square. Take the square roots of the first and last terms, and join them by the sign of the middle term.

$$(a + 2c)^2x = 5(a + 2c),$$

Solve for $x$.

$$x = \frac{5(a + 2c)}{(a + 2c)^2} = \frac{5}{a + 2c} .$$

The solution just obtained is also the zero of the function

$$a^2x + 4c^2x - 10c - (5a - 4acx) = 0 .$$

**• PROBLEM 440**

Solve $\dfrac{x}{x - 2} + \dfrac{x - 1}{2} = x + 1.$

Solution: First eliminate the fractions to facilitate solution. This is done by multiplying both sides of the equation by the Least Common Denominator, LCD. The LCD is obtained by multiplying the denominators of every fraction: LCD = $2(x - 2)$; and multiplying each side by this, the equation becomes:

$$2(x - 2)\left(\frac{x}{x - 2} + \frac{x - 1}{2}\right) = (x + 1)2(x - 2)$$

$$2x + (x - 1)(x - 2) = 2(x + 1)(x - 2)$$

$$2x + x^2 - 3x + 2 = 2x^2 - 2x - 4$$

$$x^2 - x - 6 = 0$$

This can be solved by factoring and setting each factor equal to zero.

$$(x - 3)(x + 2) = 0$$

| | |
|---|---|
| $x - 3 = 0$ | $x + 2 = 0$ |
| $x = 3$ | $x = -2$ |

Since both of these solutions are admissible values of x, they both should satisfy the original equation.

Check for $x = 3$:

$$\frac{3}{1} + \frac{2}{2} = 3 + 1$$

$$3 + 1 = 3 + 1$$

Check for $x = -2$:

$$\frac{-2}{-4} + \frac{-3}{2} = -2 + 1$$

$$\frac{1}{2} + \frac{-3}{2} = -1$$

$$-1 = -1$$

**• PROBLEM 441**

Solve the equation $\frac{4}{x + 1} + \frac{3}{x} = 2$.

Solution: Our two denominators are $x$ and $x + 1$. They have no common factors, thus our least common multiple, LCM, is $x(x+1)$. We multiply both members by $x(x+1)$ to obtain

$$x(x+1)\left[\frac{4}{x + 1} + \frac{3}{x}\right] = 2[x(x+1)]$$

$$\frac{x(x+1)4}{x + 1} + \frac{x(x+1)3}{x} = 2x(x+1)$$

$$4x + 3(x + 1) = 2x(x + 1)$$

and then

$$4x + 3x + 3 = 2x^2 + 2x$$

$$7x + 3 = 2x^2 + 2x$$

We add $-(7x + 3)$ to both sides

$$0 = 2x^2 + 2x - (7x + 3)$$
$$2x^2 + 2x - 7x - 3 = 0$$

We thus have to solve the quadratic equation

$$2x^2 - 5x - 3 = 0$$

Factoring, we have

$$(2x + 1)(x - 3) = 0 ;$$

and whenever a product of two numbers $ab = 0$ either $a = 0$ or $b = 0$. Therefore either

$$2x + 1 = 0 \quad \text{or} \quad x - 3 = 0$$

and $$x = -\tfrac{1}{2} \quad \text{or} \quad x = 3$$

Thus the only possible roots are $(-\frac{1}{2})$ and 3. We must check if these values are indeed roots. Replace $x$ by $(-\frac{1}{2})$ in the original equation,

$$\frac{4}{x + 1} + \frac{3}{x} = 2$$

$$\frac{4}{(-\frac{1}{2}) + 1} + \frac{3}{(-\frac{1}{2})} = 2$$

$$\frac{4}{(\frac{1}{2})} + \frac{3}{(-\frac{1}{2})} = 2$$

Since division by a fraction is equivalent to multiplication by its reciprocal,

$$4.2 + 3\cdot(-2) = 2$$
$$8 + (-6) = 2$$
$$2 = 2$$

Now replace $x$ by 3 in the original equation,

$$\frac{4}{x + 1} + \frac{3}{x} = 2$$

$$\frac{4}{3 + 1} + \frac{3}{3} = 2$$

$$\frac{4}{4} + \frac{3}{3} = 2$$

$$1 + 1 = 2$$

$$2 = 2$$

Thus $-\frac{1}{2}$ and 3 are both roots and our solution set is $\{-\frac{1}{2}, 3\}$.

**• PROBLEM 442**

Solve

$$\frac{x + 1}{x^2 - 5x + 6} + \frac{x + 2}{x^2 - 7x + 12} = \frac{6}{x^2 - 6x + 8}.$$

Solution: Factor the denominator of each fraction to obtain

$$\frac{x + 1}{(x - 2)(x - 3)} + \frac{x + 2}{(x - 3)(x - 4)} = \frac{6}{(x - 2)(x - 4)}.$$

Obtain the Least Common Denominator, LCD, by multiplying the denominators of each fraction together and using the highest power of each factor only once, that is,

$$(x - 2)(x - 3)(x - 4)(x - 3)(x - 2)(x - 4)$$

$$LCD = (x - 2)(x - 3)(x - 4).$$

Multiply both sides of the equation by the LCD to remove all fractions and obtain:

$$(x - 2)(x - 3)(x - 4)\left[\frac{x + 1}{(x - 2)(x - 3)} + \frac{x + 2}{(x - 3)(x - 4)}\right]$$

$$= \left[\frac{6}{(x - 2)(x - 4)}\right](x - 2)(x - 3)(x - 4)$$

$$(x + 1)(x - 4) + (x + 2)(x - 2) = 6(x - 3)$$

$$(x + 1)(x - 4) + (x + 2)(x - 2) = 6x - 18$$

$$(x^2 - 3x - 4) + (x^2 - 4) = 6x - 18$$

$$2x^2 - 3x - 8 = 6x - 18$$

$$2x^2 - 9x + 10 = 0$$

$$(2x - 5)(x - 2) = 0$$

| | |
|---|---|
| $2x - 5 = 0$ | $x - 2 = 0$ |
| $2x = 5$ | $x = 2$ |
| $x = \frac{5}{2}$ | |

Substituting $x = 2$ into the original equation shows that $x = 2$ is an extraneous root, since it is not an admissible value for the original equation. (It makes two of the denominators $(x - 2)(x - 3)$ and $(x - 2)(x - 4)$, equal to zero.)

$x = \frac{5}{2}$ is an admissible value of x for the original equation and is a solution if it will satisfy the original equation.

Check: $x = \frac{5}{2}$

$$\frac{\frac{7}{2}}{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)} + \frac{\frac{9}{2}}{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)} = \frac{6}{\left(\frac{1}{2}\right)\left(-\frac{3}{2}\right)}$$

$$\frac{14}{-1} + \frac{18}{3} = \frac{24}{-3}$$

$$-14 + 6 = -8$$

$$-8 = -8$$

## EQUATIONS WITH RADICALS

**• PROBLEM 443**

Solve the equation $\sqrt{2x^2 - 9} = x$.

__Solution:__ Squaring both sides, we have

$$2x^2 - 9 = x^2$$

$$x^2 = 9$$

$$x = 3 \quad \text{or} \quad x = -3$$

Both 3 and -3 will satisfy the equation $2x^2 - 9 = x^2$ since $2(3)^2 - 9 = 9 = (3)^2$ and $2(-3)^2 - 9 = 9 = (-3)^2$. However, -3 does not satisfy the original equation since $\sqrt{2(-3)^2 - 9} = \sqrt{9} = 3 \neq -3$. An extraneous root was introduced by squaring. Thus the solution set is $\{3\}$.

**• PROBLEM 444**

Solve $3\sqrt{x} + 4 = x$.

__Solution:__ Adding (-4) to both sides of the given equation,

$$3\sqrt{x} = x - 4.$$

Squaring both sides

$$(3\sqrt{x})^2 = (x - 4)^2$$

$$3^2(\sqrt{x})^2 = (x - 4)(x - 4)$$

Since $(\sqrt{a})^2 = \sqrt{a}\cdot\sqrt{a} = \sqrt{a\cdot a} = \sqrt{a^2} = a$, $(\sqrt{x})^2 = x$, and we obtain:

$$9x = (x - 4)(x - 4)$$

$$9x = x^2 - 8x + 16$$

Adding (-9x) to both sides,

$$x^2 - 17x + 16 = 0$$

Factoring, $\quad (x - 1)(x - 16) = 0$

Whenever the product of two numbers $ab = 0$, either $a = 0$ or $b = 0$. Thus

$$x - 1 = 0 \quad \text{or} \quad x - 16 = 0$$

$$x = 1 \quad \text{or} \quad x = 16$$

Hence, the possible roots are 1 and 16.

Check, replacing $x$ by 1:

$$3\sqrt{1} + 4 = 1,$$

$3(1) + 4 = 7 = 1$, which is false.

Hence 1 is an extraneous root.

Check, replacing $x$ by 16:

$$3\sqrt{16} + 4 = 16,$$

$$3(4) + 4 = 12 + 4 = 16, \text{ which is true.}$$

Therefore, the only root of the given equation is 16.

**• PROBLEM 445**

Solve the equation $2y = \sqrt{2y + 5} + 1$.

**Solution:** $2y = \sqrt{2y + 5} + 1$

Isolate the radical term by adding $-1$ to both sides:

$$2y - 1 = \sqrt{2y + 5}$$

Eliminate the radical by squaring both sides,

$$4y^2 - 4y + 1 = 2y + 5$$

Put in standard quadratic form,

$$4y^2 - 6y - 4 = 0$$

Dividing both sides by 2, this reduces to

$$2y^2 - 3y - 2 = 0$$

Solve by factoring:

$$(2y + 1)(y - 2) = 0$$

and set each factor $= 0$ to find all values of $y$ which make this product $= 0$.

$$\begin{array}{r|l} (2y + 1) = 0 & (y - 2) = 0 \\ y = -\frac{1}{2} & y = 2 \end{array}$$

Check: for $y = -\frac{1}{2}$ | for $y = 2$

$$2(-\tfrac{1}{2}) \stackrel{?}{=} \sqrt{2(-\tfrac{1}{2}) + 5} + 1 \qquad 2(2) \stackrel{?}{=} \sqrt{2(2) + 5} + 1$$

$$-1 \stackrel{?}{=} \sqrt{4} + 1 \qquad 4 \stackrel{?}{=} \sqrt{9} + 1$$

$$-1 \neq 3 \qquad 4 = 4$$

Therefore $y = -\frac{1}{2}$ is an extraneous root, and the solution set is $\{2\}$.

**• PROBLEM 446**

Solve the equation $\sqrt{x^2 - 3x} = 2x - 6$.

**Solution:** Remove the radical by squaring both sides of the equation, and obtain:

$$\sqrt{x^2 - 3x})^2 = (2x - 6)^2 \qquad \text{or}$$

$$x^2 - 3x = 4x^2 - 24x + 36$$

Writing in standard form, move every term to one side of the equation.

$$3x^2 - 21x + 36 = 0$$

Dividing all terms by 3, and factoring,

$$\frac{3x^2}{3} - \frac{21x}{3} + \frac{36}{3} = \frac{0}{3}$$

$$x^2 - 7x + 12 = 0$$

$$(x - 3)(x - 4) = 0$$

The roots are: $x = 3$, $x = 4$.

Check: Substituting $x = 3$ in the original equation

$$\sqrt{9 - 9} = 6 - 6$$

$$0 = 0$$

Substituting $x = 4$ in the original equation

$$\sqrt{4} = 8 - 6$$

$$2 = 2.$$

Observe that both $x = 3$ and $x = 4$ satisfy the original equation, and there are no extraneous roots.

**• PROBLEM 447**

Find the solution set of $\sqrt{x + 2} + 4 = x$.

Solution: Subtract 4 from both sides of the given equation.

$$\sqrt{x + 2} + \cancel{4} - \cancel{4} = x - 4$$

$$\sqrt{x + 2} = x - 4$$

Square both sides of this equation.

$$(\sqrt{x + 2})^2 = (x - 4)^2$$

Since

$$(\sqrt{x + 2})^2 = \sqrt{x + 2}\ \sqrt{x + 2} = \sqrt{(x + 2)(x + 2)} =$$

$$\sqrt{(x + 2)^2} = x + 2$$

$$x + 2 = (x - 4)^2$$

$$x + 2 = x^2 - 8x + 16$$

Subtract $(x + 2)$ from both sides of this equation

$$x + 2 - (x + 2) = x^2 - 8x + 16 - (x + 2)$$

$$\cancel{x} + \cancel{2} - \cancel{x} - \cancel{2} = x^2 - 8x + 16 - x - 2$$

$$0 = x^2 - 9x + 14$$

Factor the right side of this equation into a product of two polynomials.

$$0 = (x - 7)(x - 2)$$

Whenever a product ab = 0, where a and b are any two numbers, either a = 0 or b = 0. Hence, either

$$x - 7 = 0 \qquad \text{or} \qquad x - 2 = 0$$

$$x = 7 \qquad \text{or} \qquad x = 2$$

To check whether the solutions, 2 and 7, are indeed solutions, x will be replaced by both values in the original equation. Check: Replacing x by 2 in the original equation,

$$\sqrt{x + 2} + 4 = x$$

$$\sqrt{2 + 2} + 4 = 2$$

$$\sqrt{4} + 4 = 2$$

$$2 + 4 = 2$$

$$6 = 2,$$

Which is false. Therefore, 2 is not a solution. Replacing x by 7 in the original equation,

$$\sqrt{x + 2} + 4 = x$$

$$\sqrt{7 + 2} + 4 = 7$$

$$\sqrt{9} + 4 = 7$$

$$3 + 4 = 7$$

$$7 = 7$$

Which is true. Therefore, 7 is a solution. As a result, the solution set to the original equation is {7}.

• **PROBLEM 448**

Find the solution of the equation

$$\sqrt{3 - 2x} = 3 - \sqrt{2x + 2}\ .$$

<u>Solution:</u> Assume that there is a number x such that $\sqrt{3 - 2x} = 3 - \sqrt{2x + 2}$. Squaring both sides, we have

$$(\sqrt{3 - 2x})^2 = (3 - \sqrt{2x + 2})^2$$

$$(\sqrt{3 - 2x})^2 = 9 - 6\sqrt{2x + 2} + (\sqrt{2x + 2})^2$$

Since $(\sqrt{a})^2 = (a^{\frac{1}{2}})^2 = a^{2/2} = a^1 = a$

$$(\sqrt{3 - 2x})^2 = 3 - 2x \quad \text{and} \quad (\sqrt{2x + 2})^2 = 2x + 2$$

Thus we obtain

$$3 - 2x = 9 - 6\sqrt{2x + 2} + 2x + 2$$

Adding $6\sqrt{2x + 2}$ to both sides,

$$(3 - 2x\ ) + 6\sqrt{2x + 2} = 9 + 2x + 2$$

$$(3 - 2x) + 6\sqrt{2x + 2} = 11 + 2x$$

Adding $-(3 - 2x)$ to both sides,

$$6\sqrt{2x + 2} = 11 + 2x - (3 - 2x)$$

$$6\sqrt{2x + 2} = 11 + 2x - 3 + 2x$$

$$6\sqrt{2x + 2} = 4x + 8$$

Dividing both sides by 2,

$$3\sqrt{2x + 2} = 2x + 4$$

Squaring both sides of this new equation, we have

$$(3\sqrt{2x + 2})^2 = (2x + 4)^2$$

$$3^2(\sqrt{2x + 2})^2 = (2x + 4)(2x + 4)$$

$$9(\sqrt{2x + 2})^2 = 4x^2 + 16x + 16$$

Recall $(\sqrt{2x + 2})^2 = 2x + 2$, thus

$$9(2x + 2) = 4x^2 + 16x + 16$$

$$18x + 18 = 4x^2 + 16x + 16$$

Dividing both sides by 2,

$$9x + 9 = 2x^2 + 8x + 8$$

Adding $-(9x + 9)$ to both sides,

$$0 = 2x^2 + 8x + 8 - (9x + 9)$$

$$2x^2 + 8x + 8 - 9x - 9 = 0$$

$$2x^2 - x - 1 = 0$$

Factoring, $(2x + 1)(x - 1) = 0$

Whenever a product of two numbers $ab = 0$ either $a = 0$ or $b = 0$. Thus, either $2x + 1 = 0$ or $x - 1 = 0$

$$x = -\tfrac{1}{2} \text{ or } x = 1$$

so that the only possible roots are $-\frac{1}{2}$ and 1. We must check if these values are indeed roots. Replace $x$ by $(-\frac{1}{2})$ in our original equation

$$\sqrt{3 - 2x} = 3 - \sqrt{2x + 2}$$

$$\sqrt{3 - 2(-\tfrac{1}{2})} = 3 - \sqrt{2(-\tfrac{1}{2}) + 2}$$

$$\sqrt{3 + \frac{2}{2}} = 3 - \sqrt{-\frac{2}{2} + 2}$$

$$\sqrt{3 + 1} = 3 - \sqrt{-1 + 2}$$

$$\sqrt{4} = 3 - \sqrt{1}$$

$$2 = 3 - 1$$

$$2 = 2$$

Thus $(-\frac{1}{2})$ is a root.

Now replace x by 1 in our original equation

$$\sqrt{3 - 2x} = 3 - \sqrt{2x + 2}$$
$$\sqrt{3 - 2(1)} = 3 - \sqrt{2(1) + 2}$$
$$\sqrt{3 - 2} = 3 - \sqrt{2 + 2}$$
$$\sqrt{1} = 3 - \sqrt{4}$$
$$1 = 3 - 2$$
$$1 = 1$$

Therefore 1 is also a root, and the solution set is $\{-\frac{1}{2}, 1\}$.

**• PROBLEM 449**

Solve the equation $\sqrt{x + 7} + x = 13$.

Solution: Subtract x from both sides of the equation which gives $\sqrt{x + 7} = 13 - x$. Then square both sides, obtaining

$$x + 7 = (13 - x)^2 = 169 - 26x + x^2,$$

Since we have just shown $169 - 26x + x^2 = x + 7$, we may subtract $x + 7$ from both members to obtain:

$$169 - 7 - 26x - x + x^2 = (x + 7) - (x + 7)$$

Thus, $x^2 - 27x + 162 = 0$

Factor to obtain, $(x - 9)(x - 18) = 0$.

When we have a product, $ab = 0$, either $a = 0$ or $b = 0$; thus with $(x - 9)(x - 18) = 0$, either $x - 9 = 0$ or $x - 18 = 0$.

Thus, $x = 9$ or $x = 18$.

Checking the value $x = 9$ in the original equation, we find

$$\sqrt{9 + 7} + 9 = 13$$
$$\sqrt{16} + 9 = 13$$
$$4 + 9 = 13$$
$$13 = 13$$

and $x = 9$ is seen to be a root. However, if we try to check $x = 18$, we find

$$\sqrt{18 + 7} + 18 \neq 13$$

Since $\sqrt{25} + 18 \neq 13$

$$5 + 18 \neq 13$$
$$23 \neq 13;$$

so that $x = 18$ is not a root of the original equation. Hence, there is only one solution of the problem: $x = 9$.

Solve the equation $\sqrt{11 - x} - \sqrt{x + 6} = 3$.

Solution: The process of eliminating the radical is simpler if the equation is rewritten with one radical on each side of the equal sign before squaring. Thus,

$$\sqrt{11 - x} = 3 + \sqrt{x + 6}$$

Squaring both sides:

$$(\sqrt{11 - x})^2 = (3 + \sqrt{x + 6})^2$$

$$11 - x = 9 + 6\sqrt{x + 6} + x + 6 \qquad \text{or}$$

$$6\sqrt{x + 6} = -4 - 2x$$

Dividing both sides by 2 and squaring:

$$3\sqrt{x + 6} = -2 - x$$

$$(3\sqrt{x + 6})^2 = (-2 - x)^2$$

$$9(x + 6) = x^2 + 4x + 4$$

$$9x + 54 = x^2 + 4x + 4$$

Writing in standard form, collect all the terms on one side of the equal sign:

$$x^2 - 5x - 50 = 0.$$

In factored form $(x - 10)(x + 5) = 0$

$$x - 10 = 0, \qquad x + 5 = 0.$$

Therefore, the roots are $x = 10$ and $x = -5$.

Check: Substituting $x = -5$ in the original equation

$$\sqrt{16} - \sqrt{1} \stackrel{?}{=} 3$$

$$4 - 1 = 3$$

$$3 = 3$$

Substituting $x = 10$ in the original equation

$$\sqrt{1} - \sqrt{16} \stackrel{?}{=} 3$$

$$1 - 4 \neq 3$$

$$-3 \neq 3$$

Observe that $x = -5$ satisfies the original equation, but $x = 10$ does not and is therefore an extraneous root.

Solve the equation

$$\sqrt{x^2 + 24x + 3} = 2x + 3.$$

<u>Solution:</u> Squaring both members,

$$\left(\sqrt{x^2 + 24x + 3}\right)^2 = (2x + 3)^2$$

Since $\left(\sqrt{a}\right)^2 = \sqrt{a}\,\sqrt{a} = \sqrt{a \cdot a} = \sqrt{a^2} = a$, $\left(\sqrt{x^2 + 24x + 3}\right)^2 = x^2 + 24x + 3$.

Thus, $x^2 + 24x + 3 = (2x + 3)(2x + 3)$

$$x^2 + 24x + 3 = 4x^2 + 12x + 9$$

Adding $-(x^2 + 24x + 3)$ to both sides,

$$3x^2 - 12x + 6 = 0$$

Dividing both members by 3,

$$x^2 - 4x + 2 = 0$$

Adding 2 to both members,

$$x^2 - 4x + 4 = 2$$

$$(x - 2)^2 = 2$$

Taking the square root of both sides,

$$\sqrt{(x - 2)^2} = \pm\sqrt{2}$$

$$x - 2 = \pm\sqrt{2}$$

Adding 2 to both sides, $\quad x = 2 \pm \sqrt{2}$.

To check that the roots of this equation are indeed $2 + \sqrt{2}$ and $2 - \sqrt{2}$ we replace x by these values in our original equation. Replacing x by $2 + \sqrt{2}$,

$$\sqrt{x^2 + 24x + 3} = 2x + 3$$

$$\sqrt{(2 + \sqrt{2})^2 + 24(2 + \sqrt{2}) + 3} = 2(2 + \sqrt{2}) + 3$$

$$\sqrt{4 + 4\sqrt{2} + 2 + 48 + 24\sqrt{2} + 3} = 4 + 2\sqrt{2} + 3$$

$$\sqrt{28\sqrt{2} + 57} = 7 + 2\sqrt{2}$$

$$28\sqrt{2} + 57 = (7 + 2\sqrt{2})^2$$

$$28\sqrt{2} + 57 = 49 + 28\sqrt{2} + 4(2)$$

$$28\sqrt{2} + 57 = 28\sqrt{2} + 49 + 8$$

$$28\sqrt{2} + 57 = 28\sqrt{2} + 57$$

Replacing x by $2 - \sqrt{2}$,

$$\sqrt{x^2 + 24x + 3} = 2x + 3$$

$$\sqrt{(2 - \sqrt{2})^2 + 24(2 - \sqrt{2}) + 3} = 2(2 - \sqrt{2}) + 3$$

$$\sqrt{4 - 4\sqrt{2} + 2 + 48 - 24\sqrt{2} + 3} = 4 - 2\sqrt{2} + 3$$

$$\sqrt{-28\sqrt{2} + 57} = 7 - 2\sqrt{2}$$

$$-28\sqrt{2} + 57 = \left(7 - 2\sqrt{2}\right)^2$$

$$-28\sqrt{2} + 57 = 49 - 28\sqrt{2} + 4(2)$$

$$-28\sqrt{2} + 57 = -28\sqrt{2} + 49 + 8$$

$$-28\sqrt{2} + 57 = -28\sqrt{2} + 57$$

Thus our solution set is $\{2 + \sqrt{2},\ 2 - \sqrt{2}\}$.

Find the solution set of the equation $\sqrt{x+7} = 2x - 1$.

Solution: Assume that there is a number $x$ such that $\sqrt{x+7} = 2x - 1$. Squaring both sides, we have

$$(\sqrt{x+7})^2 = (2x-1)^2$$

Note $(\sqrt{a})^2 = \left(a^{\frac{1}{2}}\right)^2 = a^{2/2} = a^1 = a$ thus $(\sqrt{x+7})^2 = x + 7$.
Replacing $(\sqrt{x+7})^2$ by $x + 7$ we obtain

$$x + 7 = (2x-1)(2x-1)$$
$$x + 7 = 4x^2 - 4x + 1$$

Adding $-(x + 7)$ to both members,

$$0 = 4x^2 - 4x + 1 - (x+7)$$
$$4x^2 - 4x + 1 - x - 7 = 0$$
$$4x^2 - 5x - 6 = 0$$

Thus factoring

$$4x^2 - 5x - 6 = (4x+3)(x-2) = 0$$

Whenever a product of two numbers $ab = 0$ either $a = 0$ or $b = 0$, thus either $4x + 3 = 0$ or $x - 2 = 0$ and

$$x = \frac{-3}{4} \quad \text{or} \quad x = 2$$

Note that at this point we have not proved that either $x = 2$ or $x = -\frac{3}{4}$ is a solution of our equation, but simply that if there is any solution it must be either $2$ or $-\frac{3}{4}$. Thus we must check our values by substituting them into our original equation. Replacing $x$ by $2$ in $\sqrt{x+7} = 2x - 1$ we obtain

$$\sqrt{2+7} = 2(2) - 1$$
$$\sqrt{9} = 4 - 1$$
$$3 = 3.$$

So $2$ is indeed a solution of our equation. On the other hand, $-\frac{3}{4}$ is not a solution since

$$\sqrt{-\tfrac{3}{4} + 7} = 2(-\tfrac{3}{4}) - 1$$
$$\sqrt{6\tfrac{1}{4}} = \frac{-6}{4} - 1$$
$$\sqrt{\frac{25}{4}} = \frac{-6}{4} - \frac{4}{4}$$
$$\frac{\sqrt{25}}{\sqrt{4}} = \frac{-10}{4}$$
$$\frac{5}{2} \neq \frac{-10}{4}$$

Thus the solution set is $\{2\}$.

A number such as $-\frac{3}{4}$ obtained in this way is sometimes called an extraneous root — a term we prefer not to use since it implies that we do have a root of some kind or another.

Note also that if the equation had been given in the form

$$\sqrt{x+7} + 1 = 2x$$

and we had squared both sides, we would have obtained

$$x + 7 + 2\sqrt{x + 7} + 1 = 4x^2$$

and would not have eliminated the radical. For this reason we always "isolate" a radical on one side of the equation before squaring.

**• PROBLEM 453**

> Solve the equation
>
> $$\sqrt{x^2 - 3x + 27} = 2x + 3 .$$

Solution: Squaring both members,

$$\left(\sqrt{x^2 - 3x + 27}\right)^2 = (2x + 3)^2 \qquad (1)$$

Since $\left(\sqrt{a}\right)^2 = \sqrt{a}\,\sqrt{a} = \sqrt{a \cdot a} = \sqrt{a^2} = a,$

$$\left(\sqrt{x^2 - 3x + 27}\right)^2 = x^2 - 3x + 27.$$

Thus equation (1) becomes:

$$x^2 - 3x + 27 = (2x + 3)(2x + 3)$$

$$x^2 - 3x + 27 = 4x^2 + 12x + 9$$

Adding $-\left(x^2 - 3x + 27\right)$ to both members,

$$x^2 - 3x + 27 - \left(x^2 - 3x + 27\right) = 4x^2 + 12x + 9 - \left(x^2 - 3x + 27\right)$$

$$0 = 4x^2 + 12x + 9 - x^2 + 3x - 27$$

$$4x^2 - x^2 + 12x + 3x + 9 - 27 = 0$$

$$3x^2 + 15x - 18 = 0$$

Dividing both members by 3, $x^2 + 5x - 6 = 0$

$$(x + 6)(x - 1) = 0;$$

Whenever the product of two numbers $ab = 0$, either $a = 0$, or $b = 0$. Thus

$$x + 6 = 0 \quad \text{or} \quad x - 1 = 0$$

and

$$x = -6 \quad \text{or} \quad x = 1$$

Before we can conclude that the roots to this equation are $-6$ and $1$ we must perform the following check: Replacing $x$ by $-6$ in the original equation,

$$\sqrt{x^2 - 3x + 27} = 2x + 3$$

$$\sqrt{(-6)^2 - 3(-6) + 27} \stackrel{?}{=} 2(-6) + 3$$

$$\sqrt{36 + 18 + 27} \stackrel{?}{=} -12 + 3$$

$$\sqrt{81} \neq -9$$

Since substitution of $(-6)$ for $x$ results in a statement which isn't true, $\sqrt{81} = +9$ not $-9$ (unless the negative square root is indicated), $(-6)$ is not part of our solution. $(-6)$ is an extraneous root. Replacing $x$ by $1$ in the original equation,

$$\sqrt{x^2 - 3x + 27} = 2x + 3$$

$$\sqrt{1^2 - 3(1) + 27} = 2\,(1) + 3$$

$$\sqrt{1 - 3 + 27} = 2 + 3$$

$$\sqrt{25} = 5$$

$$5 = 5$$

Thus the solution set is $\{1\}$.

Solve the equation $\sqrt{5x - 11} - \sqrt{x - 3} = 4$.

Solution:

Add $\sqrt{x - 3}$ to both sides, $\sqrt{5x - 11} = \sqrt{x - 3} + 4$.

Square both members to eliminate one of the radicals:

$$(\sqrt{5x - 11})^2 = (\sqrt{x - 3} + 4)^2$$

$$(\sqrt{5x - 11})^2 = (\sqrt{x - 3} + 4)(\sqrt{x - 3} + 4)$$

$$(\sqrt{5x - 11})^2 = (\sqrt{x - 3})^2 + 4\sqrt{x - 3} + 4\sqrt{x - 3} + 16.$$

Since $(\sqrt{a})^2 = \sqrt{a} \cdot \sqrt{a} = \sqrt{a^2} = a$,

$$(\sqrt{5x - 11})^2 = 5x - 11$$

and $(\sqrt{x - 3})^2 = x - 3$.

Therefore, $5x - 11 = x - 3 + 8\sqrt{x - 3} + 16$.

Combine terms, $5x - 11 = x + 8\sqrt{x - 3} + 13$

Add (-13) to both sides, $5x - 11 - 13 = x + 8\sqrt{x - 3}$

Add (-x) to both sides, $5x - 24 - x = 8\sqrt{x - 3}$

$$4x - 24 = 8\sqrt{x - 3}$$

Factoring 4 from each member,

$$4(x - 6) = 4(2\sqrt{x - 3}).$$

Divide both sides by 4, $x - 6 = 2\sqrt{x - 3}$.

Square both sides, $(x - 6)^2 = (2\sqrt{x - 3})^2$

Since $(ab)^2 = a^2b^2$,

$$(x - 6)^2 = (2)^2 (\sqrt{x - 3})^2$$

$$(x - 6)(x - 6) = 4(\sqrt{x - 3})^2$$

Recall: $(\sqrt{a})^2 = \sqrt{a}\sqrt{a} = \sqrt{a^2} = a$;

Thus, $(\sqrt{x - 3})^2 = x - 3$.

Replacing $(\sqrt{x - 3})^2$ by $x - 3$ we obtain,

$$x^2 - 12x + 36 = 4(x - 3).$$

Distribute, $x^2 - 12x + 36 = 4x - 12$.

Add (-4x) to both sides,

$$x^2 - 12x + 36 - 4x = -12$$

Combine terms, $x^2 - 16x + 36 = -12$

Add 12 to both sides,

$$x^2 - 16x + 36 + 12 = 0$$

Combine terms, $$x^2 - 16x + 48 = 0$$

Factor, $$(x - 12)(x - 4) = 0$$

Whenever a product ab = 0, either a = 0, or b = 0; thus either $(x - 12) = 0$, or $(x - 4) = 0$, therefore $x = 12$ or $x = 4$.

Now we must check to verify that 12 and 4 are indeed roots of the given equation.

Check: To check if 12 is a root, we replace x by 12 in the original equation,

$$\sqrt{5x - 11} - \sqrt{x - 3} = 4$$

$$\sqrt{5(12) - 11} - \sqrt{12 - 3} = 4$$

$$\sqrt{60 - 11} - \sqrt{9} = 4$$

$$\sqrt{49} - \sqrt{9} = 4$$

$$7 - 3 = 4$$

$$4 = 4$$

Thus 12 is a root of the equation.

Now to check if 4 is a root, we replace x by 4 in the original equation,

$$\sqrt{5x - 11} - \sqrt{x - 3} = 4$$

$$\sqrt{5(4) - 11} - \sqrt{4 - 3} = 4$$

$$\sqrt{20 - 11} - \sqrt{1} = 4$$

$$\sqrt{9} - \sqrt{1} = 4$$

$$3 - 1 = 2 \neq 4.$$

Since substitution of x by 4 does not result in a valid equation, 4 is not a root, and our solution set is $\{12\}$.

• **PROBLEM 455**

Solve and check: $\sqrt{x + 10} + \sqrt[4]{x + 10} = 2.$

**Solution:** Let $y = \sqrt[4]{x + 10}$ then $y^2 = \sqrt{x + 10}$.

Substituting, the original equation may be written

$$y^2 + y - 2 = 0.$$

Factor $\qquad (y + 2)(y - 1) = 0.$

Set each factor = 0 to find all values of x which can make the product = 0

$$y + 2 = 0 \quad\Big|\quad y - 1 = 0$$

$$y = -2 \quad\Big|\quad y = 1$$

for $y = -2$ $\qquad\qquad$ for $y = 1$

$$y = \sqrt[4]{x + 10} = -2 \qquad\qquad y = \sqrt[4]{x + 10} = 1$$

$$x + 10 = 16 \qquad\qquad x + 10 = 1$$

$$x = 6 \qquad\qquad x = -9$$

Check: for $x = 6$: $\sqrt{6 + 10} + \sqrt[4]{6 + 10} \stackrel{?}{=} 2$

$$\sqrt{16} + \sqrt[4]{16} = 2$$

$$4 + 2 \neq 2$$

This root does not check.

for $x = -9$: $\sqrt{-9 + 10} + \sqrt[4]{-9 + 10} = 2$

$$1 + \sqrt[4]{1} = 2$$

$$1 + 1 = 2$$

$$2 = 2.$$

This root checks.

**• PROBLEM 456**

Solve the equation: $x^2 - x + 2\sqrt{x^2 - x - 5} = 8.$

Solution: We notice that the first two terms of the left member, $x^2 - x$, appear under the radical also. Hence we add $-5$ to both members to get

$$x^2 - x - 5 + 2\sqrt{x^2 - x - 5} = 3.$$

$$x^2 - x - 5 + 2\sqrt{x^2 - x - 5} - 3 = 0.$$

Then setting

$$y = \sqrt{x^2 - x - 5}$$

for brevity, we have

$$y^2 + 2y - 3 = 0.$$

Factoring:

$$(y + 3)(y - 1) = 0.$$

Setting each factor equal to zero we obtain:

$$y = -3, \quad y = 1.$$

The positively signed radical denoted by y cannot have the negative real value, -3. But y = 1 is permissible; we obtain:

$$\sqrt{x^2 - x - 5} = 1.$$

Squaring both sides:

$$x^2 - x - 5 = 1.$$

Subtracting 1 from both sides:

$$x^2 - x - 6 = 0.$$

Factoring and setting each factor equal to zero:

$$(x - 3)(x + 2) = 0$$

$$x - 3 = 0 \quad x + 2 = 0$$

$$x = 3 \qquad x = -2$$

Thus there are two solutions of the given equation.

Check: To verify this result we replace x by 3 in our original equation,

$$x^2 - x + 2\sqrt{x^2 - x - 5} = 8$$

$$3^2 - 3 + 2\sqrt{3^2 - 3 - 5} = 8$$

$$9 - 3 + 2\sqrt{9 - 3 - 5} = 8$$

$$6 + 2\sqrt{1} = 8$$

$$6 + 2(1) = 8$$

$$6 + 2 = 8$$

$$8 = 8.$$

Now we replace x by -2 in our original equation

$$x^2 - x + 2\sqrt{x^2 - x - 5} = 8$$

$$(-2)^2 - (-2) + 2\sqrt{(-2)^2 - (-2) - 5} = 8$$

$$4 + 2 + 2\sqrt{4 + 2 - 5} = 8$$

$$6 + 2\sqrt{1} = 8$$

$$6 + 2 = 8$$

$$8 = 8.$$

Solve $2\sqrt{\frac{x}{a}} + 3\sqrt{\frac{a}{x}} = \frac{b}{a} + \frac{6a}{b}$ .

Solution: Let $\sqrt{\frac{x}{a}} = y$; then $\sqrt{\frac{a}{x}} = \frac{1}{y}$ ;

Hence, $2y + \frac{3}{y} = \frac{b}{a} + \frac{6a}{b}$ ;

$$yab\left(2y + \frac{3}{y}\right) = \left(\frac{b}{a} + \frac{6a}{b}\right)yab$$

$$2y^2ab + 3ab = b^2y + 6a^2y$$

$$2aby^2 - 6a^2y - b^2y + 3ab = 0,$$

$$(2ay - b)(by - 3a) = 0; \qquad by - 3a = 0$$

$$2ay - b = 0$$

$$2ay = b \qquad by = 3a$$

$$y = \frac{b}{2a}, \quad \text{or} \quad \frac{3a}{b};$$

Substitute these two values of y:

$$\sqrt{\frac{x}{a}} = y$$

$$\sqrt{\frac{x}{a}} = \frac{b}{2a} \qquad \sqrt{\frac{x}{a}} = \frac{3a}{b}$$

square both sides. | square both sides

$$\frac{x}{a} = \frac{b^2}{4a^2} \qquad \frac{x}{a} = \frac{9a^2}{b^2}$$

multiply both sides by a. | multiply both sides by a.

$$x = \frac{b^2a}{4a^2} = \frac{b^2}{4a} \qquad x = \frac{9a^2 \cdot a}{b^2} = \frac{9a^3}{b^2}$$

The solution is:

$$x = \left\{\frac{b^2}{4a}, \frac{9a^3}{b^2}\right\}$$



Find the real solutions of the equation

$$\sqrt{3x + 1} + \sqrt{x - 4} = 3.$$

Solution: Adding $(-\sqrt{x-4})$ to both sides,

$$\sqrt{3x + 1} = 3 - \sqrt{x - 4} .$$

Squaring both members,

$$(\sqrt{3x + 1})^2 = \left(3 - \sqrt{x - 4}\right)^2.$$

Since $\left(\sqrt{a}\right)^2 = \sqrt{a} \cdot \sqrt{a} = \sqrt{a \cdot a} = \sqrt{a^2} = a$,

$$\left(\sqrt{3x + 1}\right)^2 = 3x + 1 .$$

Thus,

$$3x + 1 = (3 - \sqrt{x - 4})(3 - \sqrt{x - 4})$$

$$3x + 1 = 9 - 6\sqrt{x - 4} + x - 4.$$

Adding $-(3x + 1)$ to both sides we obtain,

$$3x + 1 - (3x + 1) = 9 - 6\sqrt{x - 4} + x - 4 - (3x + 1)$$

$$0 = 9 - 6\sqrt{x - 4} + x - 4 - 3x - 1$$

$$9 - 4 - 1 + x - 3x - 6\sqrt{x - 4} = 0$$

$$4 - 2x - 6\sqrt{x - 4} = 0$$

$$4 - 2x = 6\sqrt{x - 4}$$

At this point it may be observed that $x$ must be $\geqq 4$ if the right member is to be real, and that $x \leqq 2$ if the left member is to be positive.

Squaring both members,

$$(4 - 2x)^2 = (6\sqrt{x - 4})^2$$

$$(4-2x)(4-2x) = (6)^2(\sqrt{x - 4})^2$$

$$16 - 16x + 4x^2 = 36(x - 4)$$

$$16 - 16x + 4x^2 = 36x - 144$$

Subtract $(36x - 144)$ from both sides:

$$16 - 16x + 4x^2 - (36x - 144) = 36x - 144 - (36x - 144)$$

$$16 - 16x + 4x^2 - 36x + 144 = 0$$

$$4x^2 - 52x + 160 = 0$$

Dividing by 4, $\quad x^2 - 13x + 40 = 0$

Factoring, $\quad (x - 5)(x - 8) = 0$

Whenever the product of two numbers $ab = 0$, either $a = 0$ or $b = 0$; thus either

$$x - 5 = 0 \quad \text{or} \quad x - 8 = 0;$$

hence

$$x = 5 \quad \text{or} \quad x = 8 .$$

These are the solutions of this last quadratic equation, but they may not be solutions of the original equation. Therefore, we must check these roots.

Check: Replace $x$ by 5 in the original equation.

$$\sqrt{3x + 1} + \sqrt{x - 4} = 3$$

$$\sqrt{3(5) + 1} + \sqrt{5 - 4} = 3$$

$$\sqrt{15 + 1} + \sqrt{1} = 3$$

$$\sqrt{16} + 1 = 3$$

$$4 + 1 = 3$$

$$5 = 3$$

Since substitution of $x$ by 5 results in a statement which isn't true $(5 = 3)$, $x = 5$ is not a root of the original equation.

Now replace $x$ by 8 in the original equation.

$$\sqrt{3x + 1} + \sqrt{x - 4} = 3$$

$$\sqrt{3(8) + 1} + \sqrt{8 - 4} = 3$$

$$\sqrt{24 + 1} + \sqrt{4} = 3$$

$$\sqrt{25} + 2 = 3$$

$$5 + 2 = 3$$

$$7 = 3$$

Substitution of x by 8 also results in a statement which isn't true (7 = 3), therefore $x = 8$ is not a root of the original equation.

Thus, there are no real solutions to the given equation.

• PROBLEM 459

Solve $2(x + 2)^{\frac{1}{2}} = (x + 1)^{\frac{1}{2}} - 2$.

Solution: Squaring gives

$$4(x + 2) = x + 1 - 4(x + 1)^{\frac{1}{2}} + 4$$

$$4x + 8 = x + 5 - 4(x + 1)^{\frac{1}{2}}.$$

Transposing $\quad -4(x + 1)^{\frac{1}{2}} = 3x + 3.$

Squaring again $\quad 16(x + 1) = 9x^2 + 18x + 9.$

$$16x + 16 = 9x^2 + 18x + 9.$$

Transposing again $\quad 9x^2 + 2x - 7 = 0.$

Factoring $\quad (9x - 7)(x + 1) = 0.$

Set each factor = 0 to find all values of x for which the product = 0

$$9x - 7 = 0 \qquad x + 1 = 0$$

$$x = \frac{7}{9} \qquad x = -1.$$

Check: for $x = \frac{7}{9}$

$$2\left(\frac{7}{9} + 2\right)^{\frac{1}{2}} \stackrel{?}{=} \left(\frac{7}{9} + 1\right)^{\frac{1}{2}} - 2$$

$$2\left(\frac{7}{9} + \frac{18}{9}\right)^{\frac{1}{2}} \stackrel{?}{=} \left(\frac{7}{9} + \frac{9}{9}\right)^{\frac{1}{2}} - 2$$

$$2\sqrt{\frac{25}{9}} \stackrel{?}{=} \sqrt{\frac{16}{9}} - 2$$

$$2\left(\frac{5}{3}\right) \stackrel{?}{=} \frac{4}{3} - 2$$

$$\frac{10}{3} \neq -\frac{2}{3}$$

for $x = -1$

$$2(-1 + 2)^{\frac{1}{2}} \stackrel{?}{=} (-1 + 1)^{\frac{1}{2}} - 2$$

$$2\sqrt{1} \stackrel{?}{=} \sqrt{0} - 2$$

$$2 \neq -2.$$

Neither of these values is a root of the given equation.

The above example illustrates that:

1. Two expressions involving radicals may not be equal for any value of the unknown.
2. Extraneous roots may be introduced by squaring.
3. Results must always be checked. There is no other way to determine whether or not a result is a root of the given equation.

**• PROBLEM 460**

| |
|---|
| Solve $(5x - 4)^{\frac{1}{2}} = (2x + 1)^{\frac{1}{2}} + 1.$ |

<u>Solution:</u> Squaring gives $5x - 4 = 2x + 1 + 2(2x + 1)^{\frac{1}{2}} + 1.$

Transposing $3x - 6 = 2(2x + 1)^{\frac{1}{2}}$

Squaring again $9x^2 - 36x + 36 = 4(2x + 1)$

$$9x^2 - 36x + 36 = 8x + 4.$$

Transposing again $9x^2 - 44x + 32 = 0.$

Factoring $(9x - 8)(x - 4) = 0.$

Set each factor = 0 to find all values of x for which the product = 0.

$$9x - 8 = 0 \qquad x - 4 = 0$$

$$x = \frac{8}{9} \qquad x = 4$$

Check: for $x = \frac{8}{9}$

$$\left[5\left(\frac{8}{9}\right) - 4\right]^{\frac{1}{2}} \stackrel{?}{=} \left[2\left(\frac{8}{9}\right) + 1\right]^{\frac{1}{2}} + 1$$

$$\left[\frac{40}{9} - \frac{36}{9}\right]^{\frac{1}{2}} \stackrel{?}{=} \left[\frac{16}{9} + \frac{9}{9}\right]^{\frac{1}{2}} + 1$$

$$\sqrt{\frac{4}{9}} \stackrel{?}{=} \sqrt{\frac{25}{9}} + 1$$

$$\frac{2}{3} \stackrel{?}{=} \frac{5}{3} + 1$$

$$\frac{2}{3} \neq \frac{8}{3}.$$

for $x = 4$

$$[5(4) - 4]^{\frac{1}{2}} \stackrel{?}{=} [2(4) + 1]^{\frac{1}{2}} + 1$$

$$\sqrt{16} = \sqrt{9} + 1$$

$$4 \stackrel{?}{=} 3 + 1$$

$$4 = 4.$$

The value 4 satisfies the given equation, but $\frac{8}{9}$ does not.

**• PROBLEM 461**

Solve the equation $2x^{2/5} + 5x^{1/5} - 3 = 0$.

Solution: This equation may be solved as a quadratic equation if we let $P(x) = y = x^{1/5}$. Then $x^{2/5} = y^2$ and, by substituting these expressions in the equation to be solved, we have $2y^2 + 5y - 3 = 0$. We can solve this equation for $y$ by factoring:

$$2y^2 + 5y - 3 = 0$$
$$(2y - 1)(y + 3) = 0$$
$$(2y - 1) = 0 \quad \text{or} \quad (y + 3) = 0$$

therefore, $\quad y = \frac{1}{2} \quad$ or $\quad y = -3$

Now recall that $y = x^{1/5}$. Hence $x^{1/5} = \frac{1}{2}$ or $x^{1/5} = -3$.

Hence, by raising both sides of each of these equations to the fifth power, we have $x = 1/32$ or $x = -243$.

Therefore the solution set is $\{1/32, -243\}$.

**• PROBLEM 462**

If $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$, prove that

$$\frac{x^2 + a^2}{x+a} + \frac{y^2 + b^2}{y+b} + \frac{z^2 + c^2}{z+c} = \frac{(x+y+z)^2 + (a+b+c)^2}{x+y+z+a+b+c}.$$

Solution: Let $\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = k$, so that $x = ak$, $y = bk$, $z = ck$; then, substituting for x we obtain:

$$\frac{x^2 + a^2}{x + a} = \frac{(ak)^2 + a^2}{ak + a} = \frac{a^2k^2 + a^2}{ak + a} = \frac{a^2 (k^2 + 1)}{a(k + 1)}$$
$$= \frac{a(k^2 + 1)}{k + 1}.$$

Similarly, by substituting for y and z we obtain:

$$\frac{y^2 + b^2}{y + b} = \frac{b(k^2 + 1)}{k + 1}, \quad \frac{z^2 + c^2}{z + c} = \frac{c(k^2 + 1)}{k + 1}.$$

Therefore, $\frac{x^2 + a^2}{x + a} + \frac{y^2 + b^2}{y + b} + \frac{z^2 + c^2}{z + c} =$

$$\frac{a(k^2 + 1)}{k + 1} + \frac{b(k^2 + 1)}{k + 1} + \frac{c(k^2 + 1)}{k + 1} =$$

$$\frac{a(k^2 + 1) + b(k^2 + 1) + c(k^2 + 1)}{k + 1}.$$ Now,

factoring $(k^2 + 1)$ from the numerator we obtain:

$$\frac{(k^2 + 1)(a + b + c)}{k + 1} .$$

Now, performing the multiplication in the numerator we have:

$$\frac{ak^2 + a + bk^2 + b + ck^2 + c}{k + 1} =$$

$$\frac{k^2(a + b + c) + (a + b + c)}{k + 1} .$$

Multiplying both numerator and denominator by $\frac{a + b + c}{a + b + c}$, which equals 1 and does not change the value of the fraction, we then obtain:

$$\frac{k^2(a + b + c)^2 + (a + b + c)^2}{k(a + b + c) + a + b + c} .$$

Since the first term in the numerator can be rewritten as

$$k^2\left(a^2 + ab + ac + ab + b^2 + bc + ac + bc + c^2\right) =$$

$$k^2a^2 + k^2ab + k^2ac + k^2ab + k^2b^2 + k^2bc + k^2ac + k^2bc + k^2c^2 =$$

$(ka + kb + kc)^2$, we can rewrite the fraction as

$$\frac{(ka + kb + kc)^2 + (a + b + c)^2}{(ka + kb + kc) + a + b + c} ,$$

and substituting for ka, kb, kc, we obtain:

$$\frac{(x + y + z)^2 + (a + b + c)^2}{x + y + z + a + b + c} .$$

## SOLVING BY COMPLETING THE SQUARE

**• PROBLEM 463**

**Complete the square in $x^2 + x - 1$.**

**Solution:** We proceed adding the square of half the coefficient of x and, also subtracting it. That is, we write

$$x^2 + x - 1 = x^2 + x - 1 + \tfrac{1}{4} - \tfrac{1}{4}$$

$$= x^2 + x - 1 + \left(\tfrac{1}{2}\right)^2 - \left(\tfrac{1}{2}\right)^2$$

Associating, $$= \left[x^2 + x + \left(\tfrac{1}{2}\right)^2\right] - 1 - \left(\tfrac{1}{2}\right)^2$$

$$= [x + \tfrac{1}{2}]^2 - 1 - \tfrac{1}{4}$$

$$= \left(x + \tfrac{1}{2}\right)^2 - \tfrac{4}{4} - \tfrac{1}{4}$$

$$= \left(x + \tfrac{1}{2}\right)^2 - \tfrac{5}{4}$$

Solve $x^2 - 6x + 8 = 0$.

Solution: This problem may be solved by the method of completing the square: Arrange the equation with the constant term in the right member

$$x^2 - 6x = -8.$$

Take $\frac{1}{2}$ of the coefficient of x, square this, and add the result to both members. Thus, $\frac{1}{2}$ of -6 is -3, and $(-3)^2 = 9$. Add 9 to both members:

$$x^2 - 6x + 9 = -8 + 9 = 1.$$

This procedure makes the left member a perfect square. Factor,

$$(x - 3)^2 = 1.$$

Extract the square root of both members,

$$x - 3 = \pm 1.$$

When $x - 3 = +1$, then $x = 4$ and when $x - 3 = -1$, then $x = 2$.

Check: for $x = 4$:

$$4^2 - 6(4) + 8 = 0$$
$$16 - 24 + 8 = 0$$
$$0 = 0$$

for $x = 2$:

$$2^2 - 6(2) + 8 = 0$$
$$4 - 12 + 8 = 0$$
$$0 = 0.$$

Sol: $x = \{4,2\}$.



Solve $2x^2 + 8x + 4 = 0$ by completing the square.

Solution: $2x^2 + 8x + 4 = 0$

Divide both members by 2, the coefficient of $x^2$.

$$x^2 + 4x + 2 = 0$$

Subtract the constant term, 2, from both members.

$$x^2 + 4x = -2$$

Add to each member the square of one-half the coefficient of the term in x.

$$x^2 + 4x + 4 = -2 + 4$$

Factor

$$(x + 2)^2 = 2$$

Set the square root of the left member (a perfect square) equal to $\pm$ the square root of the right member and solve for $x$.

$$x + 2 = \sqrt{2} \quad \text{or} \quad x + 2 = -\sqrt{2}$$

The roots are $\sqrt{2} - 2$ and $-\sqrt{2} - 2$. Check each solution.

$$\begin{aligned} 2(\sqrt{2} - 2)^2 + 8(\sqrt{2} - 2) + 4 &= 2(2 - 4\sqrt{2} + 4) + 8\sqrt{2} - 16 + 4 \\ &= 4 - 8\sqrt{2} + 8 + 8\sqrt{2} - 16 + 4 \\ &= 0 \end{aligned}$$

$$\begin{aligned} 2(-\sqrt{2} - 2)^2 + 8(-\sqrt{2} - 2) + 4 &= 2(2 + 4\sqrt{2} + 4) - 8\sqrt{2} - 16 + 4 \\ &= 4 + 8\sqrt{2} + 8 - 8\sqrt{2} - 16 + 4 \\ &= 0 \end{aligned}$$

**• PROBLEM 466**

Solve the equation $3x^2 + 6x - 7 = 0$.

Solution: This quadratic equation cannot be solved by factoring, but may be solved by the method of completing the square. Adding 7 to both sides, we have $3x^2 + 6x = 7$. Multiplying both sides now by $\frac{1}{3}$, we have $x^2 + 2x = 7/3$. We are now in a position to complete the square. The computation can be arranged in the following manner: Add the square of $\frac{1}{2}$ the coefficient of $x$ to both sides, i.e., 1. Then rewrite as the equality of two squares.

$$x^2 + 2x + 1 = \frac{7}{3} + 1$$

$$(x + 1)^2 = \left(\sqrt{\frac{7}{3} + 1}\right)^2$$

$$(x + 1)^2 = \left(\sqrt{\frac{10}{3}}\right)^2$$

Adding $-\left(\sqrt{\frac{10}{3}}\right)^2$ to both sides,

$$(x + 1)^2 - \left(\sqrt{\frac{10}{3}}\right)^2 = 0$$

Factoring:

$$\left[(x + 1) + \sqrt{\frac{10}{3}}\right]\left[(x + 1) - \sqrt{\frac{10}{3}}\right] = 0$$

Hence

$$x + 1 + \sqrt{\frac{10}{3}} = 0 \quad \text{or} \quad x + 1 - \sqrt{\frac{10}{3}} = 0$$

$$x = -1 - \sqrt{\frac{10}{3}} \quad \text{or} \quad x = -1 + \sqrt{\frac{10}{3}}$$

Therefore the solution set is

$$\left\{-1 + \sqrt{\frac{10}{3}}, -1 - \sqrt{\frac{10}{3}}\right\}.$$

**• PROBLEM 467**

Solve by completing the square: $-2x^2 + 3x + 5 = 0$.

Solution: $-2x^2 + 3x + 5 = 0$. Divide by $-2$, the coefficient of $x^2$,

$$x^2 - \frac{3}{2}x - \frac{5}{2} = 0$$

Add $+\frac{5}{2}$ to both sides of the equation

$$x^2 - \frac{3}{2}x - \frac{5}{2} + \frac{5}{2} = 0 + \frac{5}{2}$$

$$x^2 - \frac{3}{2}x = \frac{5}{2}$$

Take ½ the coefficient of $x$ and square it to make $x^2 - \frac{3}{2}x$ a perfect square trinomial.

$$x^2 - \frac{3}{2}x + \left[\tfrac{1}{2}(-\tfrac{3}{2})\right]^2 = \frac{5}{2} + \left[\tfrac{1}{2}(-\tfrac{3}{2})\right]^2 \quad \text{or} \quad x^2 - \frac{3}{2}x + \frac{9}{16} = \frac{5}{2} + \frac{9}{16}$$

$$x^2 - \frac{3}{2}x + \frac{9}{16} = \frac{40}{16} + \frac{9}{16} = \frac{49}{16}.$$

To factor the trinomial perfect square, $x^2 - \frac{3}{2}x + \frac{9}{16}$, take the square roots of the first and last terms, and join these square roots by the sign of the middle term. Therefore,

$$x^2 - \frac{3}{2}x + \frac{9}{16} = \left(x - \frac{3}{4}\right)^2$$

Then,

$$\left(x - \frac{3}{4}\right)^2 = \frac{49}{16}.$$

Taking square roots we obtain:

$$x - \frac{3}{4} = \pm \frac{7}{4}$$

Solving for $x$:

$$x = +\frac{7}{4} + \frac{3}{4} = \frac{10}{4} = \frac{5}{2}$$

$$x = -\frac{7}{4} + \frac{3}{4} = -\frac{4}{4} = -1$$

The two solutions are

$$x = \frac{5}{2} \quad \text{and} \quad x = -1.$$

**• PROBLEM 468**

Solve $3x^2 + 5x - 2 = 0$.

<u>Solution:</u> This problem may be solved by the method of completing the square: First, divide both members by 3, the coefficient of the $x^2$ term,

$$x^2 + \frac{5}{3}x - \frac{2}{3} = 0.$$

Arrange the equation with the constant term in the right member. Thus,

$$x^2 + \frac{5}{3}x = \frac{2}{3}.$$

Take $\frac{1}{2}$ the coefficient of $x$, square this, and add the result to both members. Thus, $\frac{1}{2}$ of $\frac{5}{3}$ is $\frac{5}{6}$ and $\left(\frac{5}{6}\right)^2 = \frac{25}{36}$.

Add $\frac{25}{36}$ to both members,

$$x^2 + \frac{5}{3}x + \frac{25}{36} = \frac{2}{3} + \frac{25}{36} = \frac{49}{36}.$$

This makes the left member a perfect square. Factor:

$$\left[x + \frac{5}{6}\right]^2 = \frac{49}{36}.$$

Extract the square root of both members

$$x + \frac{5}{6} = \pm \frac{7}{6}.$$

When $x + \frac{5}{6} = \frac{7}{6}$, then $x = \frac{7}{6} - \frac{5}{6} = \frac{2}{6} = \frac{1}{3}$.

When $x + \frac{5}{6} = -\frac{7}{6}$, then $x = -\frac{7}{6} - \frac{5}{6} = -\frac{12}{6} = -2$.

Check: for $x = \frac{1}{3}$:

$$3\left(\frac{1}{3}\right)^2 + 5\left(\frac{1}{3}\right) - 2 = 0$$

$$3\left(\frac{1}{9}\right) + 5\left(\frac{1}{3}\right) - 2 = 0$$

$$\frac{3}{9} + \frac{15}{9} - \frac{18}{9} = 0$$

$$0 = 0.$$

For $x = -2$:

$$3(-2)^2 + 5(-2) - 2 = 0$$

$$3(4) + 5(-2) - 2 = 0$$

$$12 - 10 - 2 = 0$$

$$0 = 0.$$

Sol: $x = \left\{\frac{1}{3}, -2\right\}$.

**• PROBLEM 469**

Complete the square in both x and y in $x^2+2x+y^2-3y$.

Solution: To complete the square in x, take half the coefficient of x and square it. Add and subtract this value from the given expression. Therefore:

$$\left[\frac{1}{2}(2)\right]^2 = [1]^2 = 1, \text{ and } x^2+2x+y^2-3y = x^2+2x+y^2-3y+1-1.$$

Commuting, $x^2+2x+y^2-3y = x^2+2x+1+y^2-3y-1 = (x+1)^2+y^2-3y-1$. (1)

Now, take half the coefficient of y and square it. Add and subtract this value from equation (1).

$$\left[\frac{1}{2}(-3)\right]^2 = \left[-\frac{3}{2}\right]^2 = \frac{9}{4}, \text{ and}$$

$x^2 + 2x + y^2 - 3y = (x+1)^2 + y^2 - 3y - 1 + \frac{9}{4} - \frac{9}{4}$. Commuting,

$x^2 + 2x + y^2 - 3y = (x+1)^2 + y^2 - 3y + \frac{9}{4} - 1 - \frac{9}{4}$

$= (x+1)^2 + \left(y - \frac{3}{2}\right)^2 - 1 - \frac{9}{4} = (x+1)^2 + \left(y - \frac{3}{2}\right)^2 - \frac{4}{4} - \frac{9}{4}$. Hence, $x^2 + 2x + y^2 - 3y$ $= (x+1)^2 + \left(y - \frac{3}{2}\right)^2 - \frac{13}{4}$.

**• PROBLEM 470**

Solve the following equations by completing the square.

(a) $x^2 + 2x - 1 = 0$ (c) $3t^2 - 2t + 1 = 0$

(b) $x^2 - 8x + 20 = 0$

Solution: To complete the square of any equation, take half the coefficient of the variable term (i.e., the term in which the variable is raised to the first power) and square it. Then, add and subtract the resulting value from both sides of the original equation.

(a) $x^2 + 2x - 1 = 0$. In this case, the variable term is 2x and the coefficient of this term is 2. Then, completing the square:

$\left[\frac{1}{2}(2)\right]^2 = (1)^2 = 1$. Now, the original equation becomes:

$$x^2 + 2x - 1 + 1 - 1 = 0 + 1 - 1$$
$$\left(x^2 + 2x + 1\right) - 1 - 1 = 0$$
$$(x + 1)^2 - 2 = 0$$

Adding 2 to both sides:

$$(x + 1)^2 - \cancel{2} + \cancel{2} = 0 + 2$$
$$(x + 1)^2 = 2$$

Taking the square root of both sides:

$$\sqrt{(x + 1)^2} = \pm\sqrt{2}$$
$$x + 1 = \pm\sqrt{2}$$

Subtracting 1 from both sides:

$$x + \cancel{1} - \cancel{1} = \pm\sqrt{2} - 1$$
$$x = \pm\sqrt{2} - 1$$

Therefore, the solution set to the original equation, $x^2 + 2x - 1 = 0$, is: $\{\sqrt{2} - 1, -\sqrt{2} - 1\}$.

(b) $x^2 - 8x + 20 = 0$. In this case, the variable term is $-8x$ and the coefficient of this term is $-8$. Then, completing the square:

$\left[\frac{1}{2}(-8)\right]^2 = (-4)^2 = 16$. Now, the original equation becomes:

$$x^2 - 8x + 20 + 16 - 16 = 0 + 16 - 16.$$
$$(x^2 - 8x + 16) + 20 - 16 = 0$$
$$(x - 4)^2 + 4 = 0$$

Subtracting 4 from both sides:

$$(x - 4)^2 + \cancel{4} - \cancel{4} = 0 - 4$$
$$(x - 4)^2 = -4$$

Taking the square root of both sides:

$$\sqrt{(x - 4)^2} = \pm\sqrt{-4}$$
$$x - 4 = \pm\sqrt{-4} = \pm\sqrt{(-1)(4)} = \pm\sqrt{-1}\sqrt{4}$$
$$= \pm\, i(2)$$

or $\quad x - 4 = \pm\, 2i$

Adding 4 to both sides:

$$x - \cancel{4} + \cancel{4} = \pm\, 2i + 4$$
$$x = \pm\, 2i + 4$$

Therefore, the solution set to the original equation, $x^2 - 8x + 20 = 0$, is: $\{2i + 4, -2i + 4\}$.

(c) $3t^2 - 2t + 1 = 0$. In this case, the variable term is $-2t$ and the coefficient is $-2$. Then, completing the square:

$\left[\frac{1}{2}(-2)\right]^2 = (-1)^2 = 1$. Now, the original equation becomes:

$$3t^2 - 2t + 1 + 1 - 1 = 0 + 1 - 1$$
$$(3t^2 - 2t + 1) + 1 - 1 = 0$$
$$3t^2 - 2t + 1 = 0 \qquad (1)$$

The roots of equation (1) can be found by using the quadratic formula, $x = \dfrac{-b\pm\sqrt{b^2 - 4ac}}{2a}$, since equation (1) is a quadratic equation with $a = 3$, $b = -2$, and $c = 1$. therefore,

$$t = \frac{-(-2)\pm\sqrt{(-2)^2 - 4(3)(1)}}{2(3)}$$
$$= \frac{2\pm\sqrt{4 - 12}}{6}$$
$$= \frac{2\pm\sqrt{-8}}{6}$$
$$= \frac{2\pm\sqrt{(-1)(8)}}{6}$$
$$= \frac{2\pm\sqrt{(-1)(4)(2)}}{6}$$
$$= \frac{2\pm\sqrt{-1}\sqrt{4}\sqrt{2}}{6}$$

$$= \frac{2 \pm (i)(2)(\sqrt{2})}{6}$$

$$= \frac{2 \pm 2\sqrt{2}i}{6}$$

$$t = \frac{1}{3} \pm \frac{\sqrt{2}i}{3}$$

Therefore, the solution set to the original equation, $3t^2 - 2t + 1 = 0$, is: $\left\{ \frac{1}{3} + \frac{\sqrt{2}}{3}i, \frac{1}{3} - \frac{\sqrt{2}}{3}i \right\}$.

# CHAPTER 17

# SOLUTIONS BY QUADRATIC FORMULA

**Basic Attacks and Strategies for Solving Problems in this Chapter. See pages 311 to 358 for step-by-step solutions to problems.**

The quadratic formula is a very useful tool in mathematics because it allows one to solve all types of quadratic equations. The procedure for using the quadratic formula, given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

is as follows:

(1) Eliminate any fraction from the equation.

(2) Write the equation in standard form,

$$ax^2 + bx + c = 0.$$

(3) Substitute the coefficients $a$, $b$, and $c$ of the quadratic equation in standard form into the formula.

(4) Calculate the results to obtain the solution set.

There is an interrelationship of the roots of a quadratic equation which allows one to have a rapid method for verifying the roots of the equation. In particular, the sum of the roots of any quadratic equation is $-b/a$ and the product of the roots is $c/a$. For example, consider the equation

$$x^2 - 2x - 8 = 0.$$

The sum of the roots is

$$-b/a = -(-2)/1 = 2,$$

and the product of the roots is

$$c/a = -8/1 = -8.$$

The roots of the equation are 4 and – 2. Thus, the sum and product of the roots are consistent with the above results.

The expression $b^2 - 4ac$ in the quadratic formula is called the discriminant. It provides a procedure for determining the character of the roots of the equation. In particular, if the discriminant is negative, then the roots are two complex numbers; if the discriminant is a positive number that is also a perfect square, then the two roots are rational numbers; if the discriminant is a positive number that is *not* a perfect square, then the two roots are irrational numbers; and if the discriminant is zero, then there is only one rational solution.

# Step-by-Step Solutions to Problems in this Chapter, "Solutions by Quadratic Formula"

## COEFFICIENTS WITH INTEGERS, FRACTIONS, RADICALS, AND VARIABLES

**• PROBLEM 471**

Solve for $x$: $4x^2 - 7 = 0$.

Solution: This quadratic equation can be solved for $x$ using the quadratic formula, which applies to equations in the form $ax^2 + bx + c = 0$ (in our equation $b = 0$). There is, however, an easier method that we can use:

Adding 7 to both members, $4x^2 = 7$

dividing both sides by 4, $x^2 = \frac{7}{4}$

taking the square root of both sides, $x = \pm\sqrt{\frac{7}{4}} = \pm\frac{\sqrt{7}}{2}$.

The double sign $\pm$ (read "plus or minus") indicates that the two roots of the equation are

$$+\frac{\sqrt{7}}{2} \text{ and } -\frac{\sqrt{7}}{2}.$$

**• PROBLEM 472**

Obtain the quadratic equation in standard form that is equivalent to $4x - 3 = 5x^2$.

Solution: The standard form of a quadratic equation is $ax^2 + bx + c = 0$. Starting with our given equation $4x - 3 = 5x^2$, we add $(-5x^2)$ to both members,

$$(4x - 3) + (-5x^2) = 5x^2 + (-5x^2)$$

$$(4x - 3) + (-5x^2) = 0$$

commuting we obtain $-5x^2 + 4x - 3 = 0$

This is the required equation with $a = -5$, $b = 4$, and $c = -3$.

**• PROBLEM 473**

Find the roots of the equation $x^2 + 12x - 85 = 0$.

Solution: The roots of this equation may be found using the quadratic formula

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

In this equation $A = 1$, $B = 12$, and $C = -85$. Hence, by the quadratic formula,

$$x = \frac{-12 + \sqrt{144 + 340}}{2} \quad \text{or} \quad x = \frac{-12 - \sqrt{144 + 340}}{2}$$

$$x = \frac{-12 + 22}{2} \quad \text{or} \quad x = \frac{-12 - 22}{2}$$

Therefore $x = 5$ or $x = -17$. This is equivalent to the statement that the solution set is $\{-17, 5\}$.

**• PROBLEM 474**

Use the quadratic formula to solve for x in the equation

$x^2 - 5x + 6 = 0.$

Solution: The quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, is used to solve equations in the form $ax^2 + bx + c = 0$. Here $a = 1$, $b = -5$, and $c = 6$. Hence

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4\cdot1\cdot6}}{2\ 1} = \frac{5 \pm \sqrt{25 - 24}}{2}$$

$$= \frac{5 \pm \sqrt{1}}{2}$$

$$= \frac{5 \pm 1}{2}$$

$$= \frac{5 + 1}{2} \quad \text{or} \quad \frac{5 - 1}{2}$$

$$= \frac{6}{2} \quad \text{or} \quad \frac{4}{2}$$

$$= 3 \quad \text{or} \quad 2$$

Thus the roots of the equation $x^2 - 5x + 6 = 0$ are $x = 3$ and $x = 2$.

• PROBLEM 475

Solve the equation $x^2 + 5x + 6 = 0$ by the quadratic formula.

Solution: We use the quadratic formula, which states

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \text{ for cases}$$

where $ax^2 + bx + c = 0$. For this equation, $a = 1$, $b = 5$, $c = 6$. Therefore the solutions are

$$x = \frac{-5 \pm \sqrt{25 - 4(1)(6)}}{2 \cdot 1}$$

$$= \frac{-5 \pm \sqrt{25 - 24}}{2} = \frac{-5 \pm \sqrt{1}}{2}$$

or $\qquad x_1 = \frac{-5 + 1}{2} = -2, \; x_2 = \frac{-5 - 1}{2} = -3.$

• PROBLEM 476

Solve $6x^2 - 7x - 20 = 0.$

Solution: $6x^2 - 7x - 20 = 0$ is not factorable. Therefore, find the roots of the quadratic equation $ax^2 + bx + c$ using:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

where $a = 6$, $b = -7$, $c = -20$.

$$x = \frac{7 \pm \sqrt{49 - 4(6)(-20)}}{12}$$

$$x = \frac{7 \pm \sqrt{529}}{12} = \frac{7 \pm 23}{12}.$$

Therefore, $$x_1 = \frac{7 + 23}{12} = \frac{30}{12} = \frac{5}{2}$$

$$x_2 = \frac{7 - 23}{12} = -\frac{16}{12} = -\frac{4}{3}.$$

**• PROBLEM 477**

Solve the equation $2x^2 - 5x + 3 = 0$.

Solution:

(1) $$2x^2 - 5x + 3 = 0$$

Equation (1) is a quadratic equation of the form $ax^2 + bx + c = 0$ in which $a = 2$, $b = -5$, and $c = 3$.

Therefore, the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

may be used to find the solutions of equation (1). Substituting the values for a, b, and c in the quadratic formula:

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(2)(3)}}{2(2)}$$

$$x = \frac{5 \pm \sqrt{1}}{4}$$

$$x = \frac{5 + 1}{4} = \frac{3}{2}, \quad \text{and } x = \frac{5 - 1}{4} = 1$$

Check: Substituting $x = \frac{3}{2}$ in the given equation,

$$2\left(\frac{3}{2}\right)^2 - 5\left(\frac{3}{2}\right) + 3 = 0$$

$$0 = 0$$

Substituting $x = 1$ in the given equation,

$$2(1)^2 - 5(1) + 3 = 0$$

$$0 = 0$$

**• PROBLEM 478**

Solve $x^2 - 7x + 10 = 0$.

**Solution:** $x^2 - 7x + 10 = 0$ is a quadratic equation of the form $ax^2 + bx + c = 0$ with $a = 1$, $b = -7$, $c = 10$. The roots of the equation may be found using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Substituting values for a,b, and c

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 + 4(1)(10)}}{2(1)}.$$

$$x = \frac{7 \pm \sqrt{49 - 4(1)(10)}}{2}$$

$$x = \frac{7 \pm \sqrt{9}}{2} = \frac{7 \pm 3}{2}$$

$$x = \frac{7+3}{2} = 5; \; x = \frac{7-3}{2} = 2.$$

Check: for $x = 5$,

$$\begin{aligned} (5)^2 - 7(5) + 10 &= 0 \\ 25 - 35 + 10 &= 0 \\ 0 &= 0 \end{aligned}$$

for $x = 2$,

$$\begin{aligned} (2)^2 - 7(2) + 10 &= 0 \\ 4 - 14 + 10 &= 0 \\ 0 &= 0. \end{aligned}$$

More simply, the problem could have been solved by factoring:

$$x^2 - 7x + 10 = 0$$

$$(x - 5)(x - 2) = 0.$$

Set each factor equal to zero to find all values of x which make the product = 0.

| $x - 5 = 0$ | $x - 2 = 0$ |
|---|---|
| $x = 5$ | $x = 2$ |

**• PROBLEM 479**

> Solve the equation $3x^2 - 5x + 2 = 0$ by means of the quadratic formula.

**Solution:** The quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, applies to equations of the form $ax^2 + bx + c = 0$. The equation $3x^2 - 5x + 2 = 0$ is in this form with $a = 3$, $b = -5$, and $c = 2$. Substituting these values into our quadratic formula we obtain

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(2)}}{2(3)}$$

$$= \frac{5 \pm \sqrt{25 - 24}}{6} = \frac{5 \pm 1}{6}$$

$$= \frac{6}{6} \text{ and } \frac{4}{6}$$

$$x = 1 \text{ and } \frac{2}{3}$$

Hence the solution set is $\left\{1, \frac{2}{3}\right\}$. We can verify that the elements of $\left\{1, \frac{2}{3}\right\}$ are the roots of the given equation by means of the following check: We replace $x$ by $1$ in our original equation

$$3(1)^2 - 5(1) + 2 = 0$$
$$3 - 5 + 2 = 0$$
$$-2 + 2 = 0$$
$$0 = 0$$

Now we replace $x$ by $\frac{2}{3}$ in the original equation

$$3\left(\frac{2}{3}\right)^2 - 5\left(\frac{2}{3}\right) + 2 = 0$$
$$3\left(\frac{4}{9}\right) - \frac{10}{3} + 2 = 0$$
$$\frac{12}{9} - \frac{10}{3} \cdot \frac{3}{3} + 2 = 0$$
$$\frac{12}{9} - \frac{30}{9} + 2 = 0$$
$$\frac{-18}{9} + 2 = 0$$
$$-2 + 2 = 0$$
$$0 = 0$$

Thus $\left\{1, \frac{2}{3}\right\}$ are indeed the roots of the given equation.

**• PROBLEM 480**

Solve for the roots of the equation $6x^2 + 5x - 2 = 0$ and for the roots of the equation $3x^2 + 4x - 4 = 0$.

Solution: To solve for the roots of an equation in the form $ax^2 + bx + c = 0$ we use the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In our first case $a = 6$, $b = 5$, and $c = -2$. Substituting these values into the quadratic formula we obtain,

$$x = \frac{-5 \pm \sqrt{25 - 4(6)(-2)}}{12} = \frac{-5 \pm \sqrt{73}}{12}.$$

In our second case $a = 3$, $b = 4$, and $c = -4$. Substituting these values into the quadratic formula we obtain

$$x = \frac{-4 \pm \sqrt{4^2 - 4(3)(-4)}}{2(3)} = \frac{-4 \pm \sqrt{16 + 48}}{6} = \frac{-4 \pm \sqrt{64}}{6} = \frac{-4 \pm 8}{6}$$

$$= \frac{-4 + 8}{6} = \frac{4}{6} = \frac{2}{3}$$

or

$$= \frac{-4 - 8}{6} = \frac{-12}{6} = -2$$

Thus $x = -2$ or $2/3$.

## • PROBLEM 481

> Solve the quadratic equation
>
> $$6x^2 - x - 35 = 0 .$$

Solution: Here we have a quadratic equation of the form $ax^2 + bx + c = 0$ with $a = 6$, $b = -1$, $c = -35$. Substituting in the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

we find

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 6(-35)}}{2 \cdot 6} = \frac{1 \pm \sqrt{1 + 840}}{12} = \frac{1 \pm 29}{12}$$

Hence the roots are

$$x = \frac{1 + 29}{12} = \frac{30}{12} = \frac{5}{2}\, , \quad x = \frac{1 - 29}{12} = -\frac{28}{12} = -\frac{7}{3}\, .$$

The quadratic equation of this example could also be solved by factoring. We find that

$$6x^2 - x - 35 \equiv (2x - 5)(3x + 7),$$

and since the equation will be satisfied if either of the two linear factors is set equal to zero, we get the two solutions found above. That is,

$$\begin{aligned} 2x - 5 &= 0 \\ 2x &= 5 \\ x &= 5/2 \end{aligned} \qquad\qquad \begin{aligned} 3x + 7 &= 0 \\ 3x &= -7 \\ x &= -\,7/3 \end{aligned}$$

To verify these results perform the following check.

Check: Replace $x$ by $5/2$ in the original equation

$$\begin{aligned} 6x^2 - x - 35 &= 0 \\ 6\left(\frac{5}{2}\right)^2 - \frac{5}{2} - 35 &= 0 \\ 6\left(\frac{25}{4}\right) - \frac{5}{2} - 35 &= 0 \\ \frac{150}{4} - \frac{2}{2}\left(\frac{5}{2}\right) - 35 &= 0 \\ \frac{150}{4} - \frac{10}{4} - 35 &= 0 \\ \frac{140}{4} - 35 &= 0 \\ 35 - 35 &= 0 \\ 0 &= 0 \end{aligned}$$

Now replace $x$ by $-\frac{7}{3}$ in the original equations

$$6x^2 - x - 35 = 0$$
$$6\left(-\frac{7}{2}\right)^2 - \left(-\frac{7}{3}\right) - 35 = 0$$
$$6\left(\frac{49}{9}\right) + \frac{7}{3} - 35 = 0$$
$$\frac{294}{9} + \frac{3\cdot 7}{3\cdot 3} - 35 = 0$$
$$\frac{294}{9} + \frac{21}{9} - 35 = 0$$
$$\frac{315}{9} - 35 = 0$$
$$35 - 35 = 0$$
$$0 = 0$$

**• PROBLEM 482**

Solve $x^2 + 2x - 5 = 0$.

Solution: $x^2 + 2x - 5 = 0$ is a nonfactorable quadratic equation of the form $ax^2 + bx + c = 0$. Therefore, to find the roots of the equation use the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with $a = 1$, $b = 2$, $c = -5$.

$$x = \frac{-2 \pm \sqrt{4 - 4(1)(-5)}}{2}$$

$$x = \frac{-2 \pm \sqrt{24}}{2} = \frac{-2 \pm \sqrt{4\cdot 6}}{2} = \frac{-2 \pm \sqrt{4}\cdot\sqrt{6}}{2}.$$

This may be simplified as follows:

$$x = \frac{-2 \pm 2\sqrt{6}}{2} = \frac{2(-1 \pm \sqrt{6})}{2} = -1 \pm \sqrt{6}.$$

**• PROBLEM 483**

Use the Quadratic Formula to solve the following equation: $x^2 - 7x - 7 = 0$.

Solution: The quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, is used to solve equations in the form $ax^2 + bx + c = 0$. $x^2 - 7x - 7 = 0$ is in this form, with $a = 1$, $b = -7$, and $c = -7$. Thus,

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(-7)}}{2(1)}$$

$$x = \frac{7 \pm \sqrt{49 + 28}}{2}$$

$$x = \frac{7 \pm \sqrt{77}}{2}$$

Thus, the solution to the given equation is $x = \frac{7 + \sqrt{77}}{2}$, $x = \frac{7 - \sqrt{77}}{2}$.

**• PROBLEM 484**

Solve the equation $3x^2 + 5x - 7 = 0$.

**Solution:** In order to solve a quadratic of the form $ax^2 + bx + c = 0$, we employ the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} .$$

In our example $a = 3$, $b = 5$, $c = -7$. Substituting these values in our formula we obtain,

$$x = \frac{-5 \pm \sqrt{5^2 - 4(-7)(3)}}{2(3)}$$

$$= \frac{-5 \pm \sqrt{25 + 84}}{6}$$

$$= \frac{-5 \pm \sqrt{109}}{6}$$

Thus the two solutions to the equation $3x^2 + 5x - 7 = 0$ are $\frac{-5 + \sqrt{109}}{6}$ and $\frac{-5 - \sqrt{109}}{6}$ which can be verified by direct substitution in the original equation.

**• PROBLEM 485**

Use the quadratic formula to solve the equation

$$3x^2 + 4x - 5 = 0.$$

**Solution:** The quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad \text{applies to the situation}$$

where $ax^2 + bx + c = 0$. A comparison of the given equation $3x^2 + 4x - 5 = 0$ with the equation $ax^2 + bx + c = 0$ shows that $a = 3$, $b = 4$, and $c = -5$. Substituting these values in the quadratic formula we obtain:

$$x = \frac{-4 \pm \sqrt{16 - 4 \cdot 3(-5)}}{2 \cdot 3}$$

$$= \frac{-4 \pm \sqrt{16 + 60}}{6}$$

$$= \frac{-4 \pm \sqrt{76}}{6}$$

Since $76 = 4 \cdot 19$

$$= \frac{-4 \pm \sqrt{4 \cdot 19}}{6}$$

Recall $\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$

$$= \frac{-4 \pm \sqrt{4} \cdot \sqrt{19}}{6}$$

$$= \frac{-4 \pm 2\sqrt{19}}{6}$$

Factoring 2 out of the numerator and denominator

$$= \frac{\cancel{2}(-2 \pm \sqrt{19})}{\cancel{2}(3)}$$

Cancelling 2 from numerator and denominator we conclude

$$x = \frac{-2 \pm \sqrt{19}}{3}$$

**• PROBLEM 486**

Solve $t^2 - 8t + 3 = 0$ by the quadratic formula.

**Solution:** Recall the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

which applies to the situation where $ax^2 + bx + c = 0$. In our case, $a = 1$, $b = -8$, $c = 3$, and

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - (4 \cdot 1 \cdot 3)}}{2 \cdot 1}$$

$$= \frac{8 \pm \sqrt{64 - 12}}{2 \cdot 1} = \frac{8 \pm \sqrt{52}}{2}$$

$= \frac{8}{2} \pm \frac{\sqrt{52}}{2}$ by the definition of addition of fractions

$= 4 \pm \frac{\sqrt{13 \cdot 4}}{2}$ because $\frac{8}{2} = 4$, and $52 = 13 \cdot 4$

$= 4 \pm \frac{\sqrt{13} \cdot \sqrt{4}}{2}$ Recall $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$

$= 4 \pm \frac{2\sqrt{13}}{2}$ because $\sqrt{4} = 2$

$= 4 \pm \sqrt{13}$

**• PROBLEM 487**

Solve the equation $x^2 = 4x - 1$.

**Solution:** Subtract 4x from both sides of the given equation:

$$x^2 - 4x = 4x - 1 - 4x$$

$$x^2 - 4x = -1 \qquad (1)$$

Add 1 to both sides of equation (1).

$$x^2 - 4x + 1 = -1 + 1$$

$$x^2 - 4x + 1 = 0 \qquad (2)$$

An equation of the form $ax^2 + bx + c = 0$ where a, b, and c are real numbers, and $a \neq 0$, is called a second degree or quadratic equation over the real numbers. The following formula, called the quadratic formula, may be used to find the roots or solutions to quadratic equations:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Therefore, equation (2) is a quadratic equation where $a = 1$, $b = -4$, and $c = 1$. Substituting these values into the quadratic formula, then:

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)}. \qquad (3)$$

Thus:

$$x = \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm \sqrt{4}\sqrt{3}}{2} = \frac{4 \pm 2\sqrt{3}}{2}$$

$$= \frac{\cancel{2}(2 \pm \sqrt{3})}{\cancel{2}} = 2 \pm \sqrt{3}.$$

$$x = 2 + \sqrt{3} \text{ and } x = 2 - \sqrt{3} \qquad (4)$$

If these values of x are substituted in the original equation it will be found that they satisfy the given equation, so that $x = 2 + \sqrt{3}$ and $x = 2 - \sqrt{3}$ are roots of the equation.

**• PROBLEM 488**

> Use the quadratic formula to solve the equation
>
> $8z(z + 1) = 1$ for $z$.

Solution: Distributing, $8z(z) + 8z(1) = 1$

$8z^2 + 8z = 1$

Adding (-1) to both sides, $8z^2 + 8z - 1 = 0$

We use the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

to solve equations in the form $ax^2 + bx + c = 0$. In our case $a = 8$, $b = 8$, and $c = -1$. Applying the quadratic formula to solve for $z$ we obtain

$$z = \frac{-8 \pm \sqrt{8^2 - 4(8)(-1)}}{2(8)}$$

$$= \frac{-8 \pm \sqrt{64 + 32}}{16}$$

$$= \frac{-8 \pm \sqrt{96}}{16}$$

$$= \frac{-8 \pm \sqrt{16 \cdot 6}}{16}$$

$$= \frac{-8 \pm \sqrt{16}\sqrt{6}}{16}$$

$$= \frac{-8 \pm 4\sqrt{6}}{16}$$

$$= \frac{4(-2 \pm 1\sqrt{6})}{4(4)}$$

$$= \frac{-2 \pm \sqrt{6}}{4}$$

$$= \frac{-2}{4} \pm \frac{\sqrt{6}}{4}$$

$$= -\frac{1}{2} \pm \frac{\sqrt{6}}{4}$$

Hence, $z = -\frac{1}{2} + \frac{\sqrt{6}}{4}, \ -\frac{1}{2} - \frac{\sqrt{6}}{4}$.

**• PROBLEM 489**

> Solve $\frac{1}{x} + \frac{1}{x+2} = 2$ .

**Solution:** In order to eliminate the fractions in this equation, we multiply both sides of the equation by the lowest common multiple (L.C.M), the expression of lowest degree into which each of the original expressions can be divided without a remainder. The L.C.M. is the product obtained by taking each factor to the highest degree. Thus in our case the L.C.M. is $(x^1)(x + 2)^1$ and we multiply each member by $x(x + 2)$.

$$x(x + 2)\left[\frac{1}{x} + \frac{1}{x+2}\right] = 2\left[x(x + 2)\right]$$

Distributing, $\quad x(x + 2)\left(\frac{1}{x}\right) + x(x + 2)\left(\frac{1}{x+2}\right) = 2x \cdot (x + 2)$

Cancelling, $\quad x + 2 + x = 2x^2 + 4x$

Combining, $\quad 2x + 2 = 2x^2 + 4x$

Dividing both sides by 2, $x + 1 = x^2 + 2x$

Adding $-(x + 1)$ to both sides, $\quad 0 = x^2 + 2x - (x + 1)$

$$x^2 + 2x - x - 1 = 0$$

$$x^2 + x - 1 = 0$$

Since this is an expression in the form $ax^2 + bx + c = 0$ we may use the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

to find its roots. In our case $a = 1$, $b = 1$, and $c = -1$. Hence

$$x = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{-1 \pm \sqrt{1 + 4}}{2}$$

$$= \frac{-1 \pm \sqrt{5}}{2}$$

Thus, $$x = \frac{-1 + \sqrt{5}}{2} \text{ or } \frac{-1 - \sqrt{5}}{2} .$$

Check: In order to verify these solutions, we substitute them for $x$ in our original equation.

(a) Replace $x$ by $\frac{-1 + \sqrt{5}}{2}$ :

$$\frac{1}{x} + \frac{1}{x + 2} = 2$$

$$\frac{1}{\frac{-1 + \sqrt{5}}{2}} + \frac{1}{\frac{-1 + \sqrt{5}}{2} + 2} = 2$$

Since $\sqrt{5} \approx 2.24$ replace $\sqrt{5}$ by 2.24

$$\frac{1}{\frac{-1 + 2.24}{2}} + \frac{1}{\frac{-1 + 2.24}{2} + 2} \approx 2$$

$$\frac{1}{\frac{1.24}{2}} + \frac{1}{\frac{1.24}{2} + 2} \approx 2$$

$$\frac{1}{.62} + \frac{1}{.62 + 2} \approx 2$$

$$\frac{1}{.62} + \frac{1}{2.62} \approx 2$$

$$1.61 + .38 \approx 2$$

$$1.99 \approx 2$$

(b) Replace $x$ by $\frac{-1 - \sqrt{5}}{2}$

$$\frac{1}{x} + \frac{1}{x + 2} = 2$$

$$\frac{1}{\frac{-1-\sqrt{5}}{2}} + \frac{1}{\frac{-1-\sqrt{5}}{2} + 2} = 2$$

Again replace $\sqrt{5}$ by 2.24

$$\frac{1}{\frac{-1-2.24}{2}} + \frac{1}{\frac{-1-2.24}{2} + 2} \approx 2$$

$$\frac{1}{\frac{-3.24}{2}} + \frac{1}{\frac{-3.24}{2} + 2} \approx 2$$

$$\frac{1}{-1.62} + \frac{1}{-1.62+2} \approx 2$$

$$\frac{1}{-1.62} + \frac{1}{.38} \approx 2$$

$$-.62 + 2.63 \approx 2$$

$$2.01 \approx 2$$

Therefore $x = \frac{-1 \pm \sqrt{5}}{2}$ are indeed solutions to our equation, and our solution set is $\left\{\frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}\right\}$.

**• PROBLEM 490**

Solve the equation $x^2 - x + 1 = 0$.

Solution: (1) $x^2 - x + 1 = 0$

Equation (1) is a quadratic equation of the form $ax^2 + bx + c = 0$ in which $a = 1$, $b = -1$, and $c = 1$. Therefore, the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ may be used to find solutions of equation (1). Substituting the values for a, b, and c in the quadratic formula:

(2) $x = \frac{-(-1) \pm \sqrt{(-1)^2 - (4)(1)(1)}}{2(1)}$

(3) $x = \frac{1 \pm \sqrt{-3}}{2}$

(4) $x = \frac{1 + \sqrt{-3}}{2}$ and $x = \frac{1 - \sqrt{-3}}{2}$

Substitution of each of these roots in the original Equation 1 will show that they satisfy the equation.

**• PROBLEM 491**

Solve the equation $\sqrt{x+1} + \sqrt{2x+3} - \sqrt{8x+1} = 0$.

Solution: Add $\sqrt{8x+1}$ to both sides,

$$\sqrt{x+1} + \sqrt{2x+3} = \sqrt{8x+1}.$$

Square both sides of the equation,

$$(\sqrt{x+1} + \sqrt{2x+3})(\sqrt{x+1} + \sqrt{2x+3}) = (\sqrt{8x+1})^2$$

$$(\sqrt{x+1})^2 + 2\sqrt{x+1}\,\sqrt{2x+3} + (\sqrt{2x+3})^2 = (\sqrt{8x+1})^2$$

Since $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$,

$$(\sqrt{x+1})^2 + 2\sqrt{(x+1)(2x+3)} + (\sqrt{2x+3})^2 = (\sqrt{8x+1})^2.$$

Recall: $(\sqrt{a})^2 = \sqrt{a} \cdot \sqrt{a} = \sqrt{a^2} = a$.

Thus, $(\sqrt{x+1})^2 = x + 1$

$(\sqrt{2x+3})^2 = 2x + 3$

and $\quad (\sqrt{8x + 1})^2 = 8x + 1$

Substituting these values we obtain,

$$x + 1 + 2\sqrt{(x + 1)(2x + 3)} + 2x + 3 = 8x + 1.$$

Combine terms, $3x + 4 + 2\sqrt{(x + 1)(2x + 3)} = 8x + 1$

Add (-4) to both sides, $3x + 2\sqrt{(x + 1)(2x + 3)} = 8x - 3$

Add (-3x) to both sides $\quad 2\sqrt{(x + 1)(2x + 3)} = 5x - 3$

Multiply the terms within the radical,

$$2\sqrt{2x^2 + 5x + 3} = 5x - 3$$

Square both members,

$$(2\sqrt{2x^2 + 5x + 3})^2 = (5x - 3)(5x - 3).$$

Since $(ab)^2 = a^2b^2$,

$$(2)^2 (\sqrt{2x^2 + 5x + 3})^2 = (5x - 3)(5x - 3)$$

$$4(\sqrt{2x^2 + 5x + 3})^2 = 25x^2 - 30x + 9.$$

Once again recall: $(\sqrt{2x^2 + 5x + 3}) = (2x^2 + 5x + 3)$.
Substituting this value, we obtain

$$4(2x^2 + 5x + 3) = 25x^2 - 30x + 9$$

Distribute,

$$8x^2 + 20x + 12 = 25x^2 - 30x + 9$$

Add $(-8x^2)$ to both sides,

$$20x + 12 = 25x^2 - 30x + 9 - 8x^2$$

$$20x + 12 = 17x^2 - 30x + 9$$

Add (- 20x) to both sides,

$$12 = 17x^2 - 50x + 9$$

Add (- 12) to both sides,

$$17x^2 - 50x - 3 = 0$$

We can find the roots of this equation using the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, which applies to the situation $ax^2 + bx + c = 0$. In our case $a = 17$, $b = -50$, and $c = -3$. Thus

$$x = \frac{50 \pm \sqrt{2500 + 204}}{34}$$

$$= \frac{50 \pm \sqrt{2704}}{34}$$

$$= \frac{50 \pm 52}{34} = \frac{50}{34} \pm \frac{52}{34}$$

$$= \frac{102}{34} \text{ and } -\frac{2}{34} \;;$$

Thus, $x = 3$,

and $\quad x = -\frac{1}{17}$.

Check: To verify that 3 and $-1/17$ are indeed roots of the given equation we replace x by these values in the original equation,

$$\sqrt{x+1} + \sqrt{2x+3} - \sqrt{8x+1} = 0$$

(a) Substituting 3 for x:

$$\sqrt{3+1} + \sqrt{2(3)+3} - \sqrt{8(3)+1} = 0$$
$$\sqrt{4} + \sqrt{9} - \sqrt{25} = 0$$
$$2 + 3 - 5 = 0$$
$$0 = 0$$

Thus, 3 is a root of the equation.

(b) Substitute $-\frac{1}{17}$ for x:

$$\sqrt{-\frac{1}{17}+1} + \sqrt{2\left(-\frac{1}{17}\right)+3} - \sqrt{8\left(-\frac{1}{17}\right)+1} = 0$$

$$\sqrt{-\frac{1}{17}+\frac{17}{17}} + \sqrt{-\frac{2}{17}+\frac{51}{17}} - \sqrt{-\frac{8}{17}+\frac{17}{17}} = 0$$

$$\sqrt{\frac{16}{17}} + \sqrt{\frac{49}{17}} - \sqrt{\frac{9}{17}} = 0$$

Since $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$,

$$\frac{4}{\sqrt{17}} + \frac{7}{\sqrt{17}} - \frac{3}{\sqrt{17}} = 0$$

$$\frac{8}{\sqrt{17}} \neq 0$$

Since substitution of x by $-1/17$ does not result in a valid equation, $-1/17$ is not a root, and our solution set is $\{3\}$.

**• PROBLEM 492**

| |
|---|
| Solve for $y$ if $6x^2 + 9y^2 + x - 6y = 0$. |

Solution: Note: The standard form of a quadratic equation is $az^2 + bz + c = 0$ where $a \neq 0$. This type of equation can be solved by using the quadratic formula:

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Thus, we first put the equation in standard form,

$$9y^2 - 6y + (6x^2 + x) = 0$$

where $a = 9$, $b = -6$, and $c = 6x^2 + x$. Therefore the solutions are

$$y = \frac{6 \pm \sqrt{36 - 36(6x^2 + x)}}{18}$$

$$= \frac{6 \pm \sqrt{36[1 - (6x^2 + x)]}}{18}$$

$$= \frac{6 \pm 6\sqrt{1 - 6x^2 - x}}{18} = \frac{1 \pm \sqrt{1 - 6x^2 - x}}{3}$$

or

$$y_1 = \frac{1 + \sqrt{1 - 6x^2 - x}}{3}, \quad y_2 = \frac{1 - \sqrt{1 - 6x^2 - x}}{3}$$

**• PROBLEM 493**

Solve for $y$ if $2x^2 + y^2 + 2xy - 2x = 0$.

Solution: In standard form, the equation becomes

$$y^2 + (2x)y + (2x^2 - 2x) = 0 \qquad (1)$$

Equation (1) is now in the standard form of a quadratic equation, $az^2 + bz + c = 0$, where $a \neq 0$. This type of equation can be solved by using the quadratic formula:

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Then, $a = 1$, $b = 2x$, and $c = 2x^2 - 2x$. Therefore the solutions are

$$y = \frac{-2x \pm \sqrt{4x^2 - 4(2x^2 - 2x)}}{2}$$

$$= \frac{-2x \pm \sqrt{4[x^2 - (2x^2 - 2x)]}}{2}$$

$$= \frac{-2x \pm 2\sqrt{x^2 - 2x^2 + 2x}}{2} = -x \pm \sqrt{2x - x^2}$$

or

$$y_1 = -x + \sqrt{2x - x^2}, \quad y_2 = -x - \sqrt{2x - x^2}$$

**• PROBLEM 494**

Solve for $x$ by using the quadratic formula.

(a) $3x^2 = x + 6$ (b) $5x^2 - 6x + 7 = 0$.

Solution: The quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

is employed to solve equations in the form $ax^2 + bx + c = 0$.

(a) $3x^2 = x + 6$

In order to transform this equation into the desired form, we add $-(x + 6)$ to both sides,

$$3x^2 - (x + 6) = (x + 6) - (x + 6)$$

$$3x^2 - x - 6 = 0$$

Thus $a = 3$, $b = -1$, and $c = -6$. Substituting these values into the quadratic formula, we obtain

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(3)(-6)}}{2(3)}$$

$$x = \frac{1 \pm \sqrt{1 + 72}}{6}$$

$$x = \frac{1 \pm \sqrt{73}}{6}$$

(b) $5x^2 - 6x + 7 = 0$

This equation is already in the form $ax^2 + bx + c = 0$ with $a = 5$, $b = -6$, and $c = 7$. Therefore

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(5)(7)}}{2(5)}$$

$$x = \frac{6 \pm \sqrt{36 - 140}}{10}$$

$$x = \frac{6 \pm \sqrt{-104}}{10}$$

$$x = \frac{6 \pm \sqrt{(-1)(4)(26)}}{10}$$

$$x = \frac{6 \pm \sqrt{(-1)} \cdot \sqrt{4} \cdot \sqrt{26}}{10}$$

$$x = \frac{6 \pm i(2)\sqrt{26}}{10}$$

Factoring out a 2 from each term:

$$x = \frac{2(3 \pm i\sqrt{26})}{2(5)}$$

$$x = \frac{3 \pm i\sqrt{26}}{5}$$

• PROBLEM 495

Solve the equation

$$4\sqrt{\frac{3 - x}{3 + x}} - \sqrt{\frac{3 + x}{3 - x}} = \sqrt{2}\ .$$

Solution: Although this equation is not in quadratic form, the fact that the two radicals in the left member are reciprocal to each other suggests that the equation may be reduced to a tractable form. For brevity, let

$$y = \sqrt{\frac{3 - x}{3 + x}}$$

Then the equation becomes

$$4y - \frac{1}{y} = \sqrt{2},$$

Multiplying by $y$ and transferring terms to one side, we obtain:

$$4y^2 - 1 = \sqrt{2}\ y$$

$$4y^2 - \sqrt{2}y - 1 = 0, \text{ which is}$$

a quadratic in $y$. The solutions of this equation are found from the quadratic formula,

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $a = 4$, $b = -\sqrt{2}$ and $c = -1$. Solving, we find

$$y = \frac{\sqrt{2} \pm \sqrt{2} + 16}{8}$$

$$= \frac{\sqrt{2} \pm \sqrt{18}}{8} = \frac{\sqrt{2} \pm 3\sqrt{2}}{8}$$

$$= \frac{\sqrt{2}(1 \pm 3)}{8}$$

Thus,

$$y = \frac{\sqrt{2}}{2}, \quad \text{or} \quad y = -\frac{\sqrt{2}}{4}.$$

Now, since $y = \sqrt{\frac{3-x}{3+x}}$, denotes a square root with a prefixed positive sign understood. Hence $y$ may be either real and positive or complex, but not real and negative. Consequently the value $y = -\sqrt{2}/4$ can lead only to extraneous solutions and may therefore be discarded. Therefore, we have:

$$\sqrt{\frac{3 - x}{3 + x}} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

Squaring both sides:

$$\frac{3 - x}{3 + x} = \frac{1}{2},$$

Cross-multiplying, we have:

$$6 - 2x = 3 + x,$$

Collecting terms so that the variable $x$ is on one side and the numerical quantities on the other side:

$$3x = 3,$$

Dividing by three:

$$x = 1.$$

This is the only solution of the given equation; its correctness may readily be checked as follows: Substituting $x = 1$,

$$4\sqrt{\frac{3 - 1}{3 + 1}} - \sqrt{\frac{3 + 1}{3 - 1}} = \sqrt{2}$$

$$4\frac{\sqrt{2}}{\sqrt{4}} - \frac{\sqrt{4}}{\sqrt{2}} = \sqrt{2}$$

$$4\frac{\sqrt{2}}{2} - \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$2\sqrt{2} - \frac{2}{\sqrt{2}} = \sqrt{2}$$

Multiply by $\sqrt{2}$: $2\sqrt{2}\sqrt{2} - 2 = \sqrt{2}\sqrt{2}$

$$4 - 2 = 2$$

$$2 = 2$$

**• PROBLEM 496**

Solve for x: $3x^2 + 5 = 0$.

Solution: This quadratic equation can be solved for x using the quadratic formula, which applies to equations in the form $ax^2 + bx + c = 0$ (in our equation $b = 0$). There is, however, an easier method that we can use: adding $-5$ to both sides,

$$3x^2 = -5$$

dividing both sides by 3,

$$x^2 = -\frac{5}{3}$$

Taking the square root of both sides,

$$x = \pm\sqrt{-\frac{5}{3}} = \pm\sqrt{(-1)\left(\frac{5}{3}\right)} = \pm\sqrt{-1}\sqrt{\frac{5}{3}}$$

By definition $\sqrt{-1} = i$. Thus,

$$\pm\sqrt{-1}\sqrt{\frac{5}{3}} = \pm i\sqrt{\frac{5}{3}} = \pm i\sqrt{\frac{5}{3}}\cdot\frac{3}{3} = \pm i\sqrt{\frac{15}{9}} = \pm i\frac{\sqrt{15}}{\sqrt{9}} = \pm\frac{i\sqrt{15}}{3}.$$

Thus, $$x = \frac{i\sqrt{15}}{3}, -\frac{i\sqrt{15}}{3}.$$

**497**

Use the quadratic formula to solve $2x^2 - 5x + 8 = 0$.

Solution: Recall the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

which applies to the situation where $ax^2 + bx + c = 0$. In our case $a = 2$, $b = -5$, and $c = 8$. Substituting these values into the quadratic formula we obtain:

$$x = \frac{5 \pm \sqrt{25 - 4 \cdot 2 \cdot 8}}{2 \cdot 2}$$

$$= \frac{5 \pm \sqrt{25 - 64}}{4} = \frac{5 \pm \sqrt{-39}}{4}$$

Note that the $\sqrt{-39}$, the square root of a negative number, is not defined for real numbers, hence we must use the imaginary number system.

Since $\sqrt{ab} = \sqrt{a}\cdot\sqrt{b}$, $\sqrt{-39} = \sqrt{(-1)\cdot(39)} = \sqrt{-1}\cdot\sqrt{39}$

By definition $i = \sqrt{-1}$, so $\sqrt{-39} = i\sqrt{39}$

Therefore $$x = \frac{5 \pm i\sqrt{39}}{4}$$

**• PROBLEM 498**

Use the quadratic formula to solve the following equation for x:

$$x^2 + 2x + 4 = 0$$

Solution: The quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

is used to solve equations in the form $ax^2 + bx + c = 0$. Consider the equation $x^2 + 2x + 4 = 0$. Here $a = 1$, $b = 2$, $c = 4$. Hence the roots are

$$\frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 4}}{2 \cdot 1} = \frac{-2 \pm \sqrt{4 - 16}}{2} = \frac{-2 \pm \sqrt{-12}}{2}$$

$$= \frac{-2 \pm \sqrt{(-3)4}}{2} = \frac{-2 \pm \sqrt{-3}\ \sqrt{4}}{2} = \frac{-2 \pm \sqrt{4}\ \sqrt{-3}}{2} = \frac{-2 \pm 2\sqrt{-3}}{2}$$

$$= \frac{2(-1 \pm \sqrt{-3})}{2} = \begin{cases} -1 + \sqrt{-3} = -1 + \sqrt{-1 \cdot 3} = -1 + \sqrt{-1}\ \sqrt{3} \\ -1 - \sqrt{-3} = -1 - \sqrt{-1 \cdot 3} = -1 - \sqrt{-1}\ \sqrt{3} \end{cases}$$

$$= -1 + i\sqrt{3}$$
$$= -1 - i\sqrt{3}$$

Notice that the roots of a quadratic equation may be imaginary, even though the coefficients are real.

• PROBLEM 499

Solve $x^2 + 2x + 5 = 0$.

Solution: $x^2 + 2x + 5 = 0$ is a nonfactorable quadratic equation of the form $ax^2 + bx + c = 0$. Therefore, to find the roots of the equation use the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ with } a = 1,\ b = 2,\ c = 5.$$

$$x = \frac{-2 \pm \sqrt{4 - 4(1)(5)}}{2}$$

$$x = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm \sqrt{-1} \cdot \sqrt{16}}{2}.$$

In this case the roots involve imaginary numbers. The result can be simplified by using $i = \sqrt{-1}$ to give

$$x = \frac{-2 \pm 4i}{2} = -1 \pm 2i.$$

• PROBLEM 500

Solve the equation $2x^2 + 5x + 8 = 0$.

**Solution:** Letting A = 2, B = 5, and C = 8, we substitute these values in the quadratic formula in the following manner:

$$x = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \quad \text{or} \quad x = \frac{-B - \sqrt{B^2 - 4AC}}{2A}$$

$$x = \frac{-(5) + \sqrt{(5)^2 - 4(2)(8)}}{2(2)} \quad \text{or} \quad x = \frac{-(5) - \sqrt{(5)^2 - 4(2)(8)}}{2(2)}$$

$$x = \frac{-5 + \sqrt{-39}}{4} \quad \text{or} \quad x = \frac{-5 - \sqrt{-39}}{4}$$

Therefore the solution set is

$$\left\{\frac{-5 + \sqrt{-39}}{4}, \frac{-5 - \sqrt{-39}}{4}\right\}$$

Since $\sqrt{-39}$ is not a real number, we recognize that the members of the solution set of the equation are mixed imaginary numbers. Furthermore, since $\sqrt{-39} = i\sqrt{39}$, the solution set may be rewritten as

$$\left\{\frac{-5 + i\sqrt{39}}{4}, \frac{-5 - i\sqrt{39}}{4}\right\}$$

This emphasizes a need for the extension of the set of real numbers into the set of complex numbers.

**• PROBLEM 501**

Solve $x^2 + 2x + 5 = 0$, by the quadratic formula.

**Solution:** Recall the quadratic formula:

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, which applies to the situation

where $ax^2 + bx + c = 0$. In our case, $x^2 + 2x + 5 = 0$,

$a = 1$, $b = 2$, and $c = 5$; hence $x = \dfrac{-2 \pm \sqrt{(2)^2 - 4(1)(5)}}{2 \cdot 1}$

$$= \frac{-2 \pm \sqrt{4 - 20}}{2} = \frac{-2 \pm \sqrt{-16}}{2}$$

Note $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$, therefore $\sqrt{-16} = \sqrt{16 \cdot (-1)}$

$$= \sqrt{16} \cdot \sqrt{(-1)}$$

$$= 4\sqrt{(-1)}$$

By definition $i = \sqrt{(-1)}$,

$$= 4i$$

hence, $= \dfrac{-2 \pm 4i}{2}$. Therefore,

$$x = -1 \pm 2i.$$

As a check we substitute $x = -1 \pm 2i$ into $x^2 + 2x + 5$:

$$(-1 \pm 2i)^2 + 2(-1 \pm 2i) + 5$$

$$= 1 \mp 4i + 4i^2 - 2 \pm 4i + 5$$

$$= \mp 4i \pm 4i = 0 \text{ by the additive inverse}$$

property and since $i^2 = -1$ by definition, $4i^2 = 4(-1) = -4$ hence,

$$= 1 - 4 - 2 + 5$$

$$= 6 - 6$$

$$= 0$$

• **PROBLEM 502**

Solve the equation $4x^2 = 8x - 7$ by means of the quadratic formula.

Solution: The quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, applies to equations of the form $ax^2 + bx + c = 0$. If we add $(-8x + 7)$ to both sides of our given equation we obtain $4x^2 - 8x + 7 = 8x - 7 = 0$ which is an equation in the form $ax^2 + bx + c = 0$ with $a = 4$, $b = -8$, and $c = 7$. Substituting these values into the quadratic formula we obtain

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(4)(7)}}{2(4)}$$

$$= \frac{8 \pm \sqrt{64 - 112}}{8}$$

$$= \frac{8 \pm \sqrt{-48}}{8}$$

Since $\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$, $\sqrt{-48} = \sqrt{-1 \cdot 48} = \sqrt{-1}\,\sqrt{48}$. Recall $\sqrt{-1} = i$.

Thus $\sqrt{-48} = \sqrt{-1}\,\sqrt{48} = i\sqrt{48}$ and

$$x = \frac{8 \pm i\sqrt{48}}{8}.$$

We can further break down this radical by noting

$$\sqrt{48} = \sqrt{16 \cdot 3} = \sqrt{16} \cdot \sqrt{3} = 4\sqrt{3}.$$

Thus,

$$x = \frac{8 \pm 4i\sqrt{3}}{8}$$

$$x = \frac{8}{8} \pm \frac{4i\sqrt{3}}{8}$$

$$x = 1 \pm \frac{i\sqrt{3}}{2}$$

Hence the solution set is $\left\{1 + \frac{i\sqrt{3}}{2},\ 1 - \frac{i\sqrt{3}}{2}\right\}$.

We can verify that these two complex numbers are the roots of the given equation by means of the following check: We replace $x$ by $1 + \frac{i\sqrt{3}}{2}$ in the original equation:

$$4\left(1 + \frac{i\sqrt{3}}{2}\right)^2 = 8\left(1 + \frac{i\sqrt{3}}{2}\right) - 7$$

$$4\left[1 + \frac{2i\sqrt{3}}{2} + \left(\frac{i\sqrt{3}}{2}\right)^2\right] = 8 + 8\,\frac{i\sqrt{3}}{2} - 7$$

$$4\left[1 + \frac{2i\sqrt{3}}{2} + i^2\left(\frac{\sqrt{3}}{2}\right)^2\right] = 1 + 4i\sqrt{3}$$

$$4\left[1 + i\sqrt{3} - \left(\frac{\sqrt{3}}{2}\frac{\sqrt{3}}{2}\right)\right] = 1 + 4i\sqrt{3}$$

$$4\left[1 + i\sqrt{3} - \frac{3}{4}\right] = 1 + 4i\sqrt{3}$$

$$4 + 4i\sqrt{3} - 3 = 1 + 4i\sqrt{3}$$

$$1 + 4i\sqrt{3} = 1 + 4i\sqrt{3}$$

Now we replace $x$ by $1 - \frac{i\sqrt{3}}{2}$ in the original equation:

$$4\left(1 - \frac{i\sqrt{3}}{2}\right)^2 = 8\left(1 - \frac{i\sqrt{3}}{2}\right) - 7$$

$$4\left[1 - \frac{2i\sqrt{3}}{2} + \left(\frac{i\sqrt{3}}{2}\right)^2\right] = 8 - \frac{8i\sqrt{3}}{2} - 7$$

$$4\left[1 - i\sqrt{3} + i^2\left(\frac{\sqrt{3}}{2}\right)^2\right] = 1 - 4i\sqrt{3}$$

$$4\left[1 - i\sqrt{3} - 1\left(\frac{\sqrt{3}}{2}\right)^2\right] = 1 - 4i\sqrt{3}$$

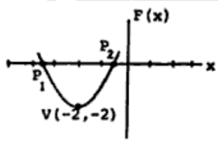
$$4\left(1 - i\sqrt{3} - \frac{3}{4}\right) = 1 - 4i\sqrt{3}$$

$$4 - 4i\sqrt{3} - 3 = 1 - 4i\sqrt{3}$$

$$1 - 4i\sqrt{3} = 1 - 4i\sqrt{3}$$

**• PROBLEM 503**

Find the roots of the function $F$ whose rule of correspondence is $F(x) = 2x^2 + 8x + 4$.




<u>Solution:</u> The roots of the function $F$ are those values of $x$ which satisfy $F(x) = 0$. Therefore we seek the solution set of $2x^2 + 8x + 4 = 0$, or $x^2 + 4x + 2 = 0$ (dividing both sides of the equation by 2). Using the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with $a = 1$, $b = 4$, and $c = 2$, we have:

$$x = \frac{-4 \pm \sqrt{(4)^2 - 4(1)(2)}}{2(1)}$$

$$x = \frac{-4 \pm \sqrt{16 - 8}}{2}$$

$$x = \frac{-4 \pm \sqrt{4 \cdot 2}}{2}$$

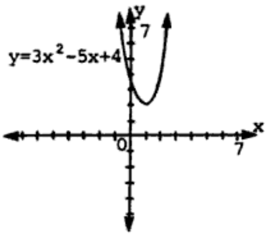
$$x = \frac{-4 \pm 2\sqrt{2}}{2} = -2 \pm \sqrt{2}$$

Hence the roots are $x_1 = -2 + \sqrt{2}$ and $x_2 = -2 - \sqrt{2}$.

The graph of the quadratic function F is a parabola and intersects the domain axis at the points $P_1(-2-\sqrt{2},0)$ and $P_2(-2+\sqrt{2},0)$.

**• PROBLEM 504**

Solve $3x^2 - 5x + 4 = 0$ by the quadratic formula.




<u>Solution:</u> Recall the quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, which applies to equations in the form $ax^2 + bx + c = 0$. In our case

$a = 3 \qquad b = -5 \qquad c = 4$.

Substituting these values into the quadratic formula we obtain,

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(4)}}{2 \cdot 3}$$

$$x = \frac{5 \pm \sqrt{25 - 48}}{6} = \frac{5 \pm \sqrt{-23}}{6}$$

Since $\sqrt{-23}$ is not a real number, x is not a real number, and, consequently, the equation has no real roots. The graph of $y = 3x^2 - 5x + 4$ is shown in the accompanying figure. Notice that the graph does not cross the x-axis. This is because on the x-axis $y = 0$. Hence $y = 0 = 3x^2 - 5x + 4$, which is the equation we have just shown to have no real roots.

## INTERRELATIONSHIPS OF ROOTS: SUMS; PRODUCTS

**• PROBLEM 505**

Determine the quadratic equation whose roots are

$\frac{1}{2}$ and $-\left(\frac{2}{3}\right)$.

<u>Solution:</u> $x = \frac{1}{2}$ (1) $\qquad x = -\frac{2}{3}$ (2)

Subtract $\frac{1}{2}$ from both sides of equation (1).

$$x - \frac{1}{2} = \frac{1}{2} - \frac{1}{2}$$

Therefore: $x - \frac{1}{2} = 0$ (3).

Add $\frac{2}{3}$ to both sides of equation (2).

$$x + \frac{2}{3} = - \frac{2}{3} + \frac{2}{3}$$

Therefore: $x + \frac{2}{3} = 0$ (4).

Hence, from equations (3) and (4):

$$\frac{2x - 1}{2} = 0 \quad (5) \quad \text{and} \quad \frac{3x + 2}{3} = 0 \quad (6).$$

Multiply both sides of equation (5) by 2 and multiply both sides of equation (6) by 3.

Therefore: $2x - 1 = 0$ and $3x + 2 = 0$
Hence, $(2x - 1)(3x + 2) = (0)(0) = 0$
or $6x^2 - 3x + 4x - 2 = 0$
or $6x^2 + x - 2 = 0$.

**• PROBLEM 506**

Show that the roots of the quadratic equation $x^2 - x - 3 = 0$ are

$$x_1 = \frac{1 + \sqrt{13}}{2} \quad \text{and} \quad x_2 = \frac{1 - \sqrt{13}}{2}$$

Solution: We use the quadratic formula derived from the quadratic equation, $ax^2 + bc + c = 0$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $x^2 - x - 3 = 0$, $a = 1$, $b = -1$, and $c = -3$. Replacing these values in the quadratic formula,

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-3)}}{2(1)}$$

$$x = \frac{1 \pm \sqrt{13}}{2}$$

$$x_1 = \frac{1 + \sqrt{13}}{2} \qquad x_2 = \frac{1 - \sqrt{13}}{2}$$

According to the Factor Theorem: If $r$ is a root of the equation $f(x) = 0$, i.e., if $f(r) = 0$, then $(x - r)$ is a factor of $f(x)$.

$x_1$ and $x_2$ are roots of $x^2 - x - 3 = 0$. Thus,

$$\left(x - \frac{1 + \sqrt{13}}{2}\right)\left(x - \frac{1 - \sqrt{13}}{2}\right)$$

are factors, and

$$x^2 - x - 3 = \left(x - \frac{1 - \sqrt{13}}{2}\right)\left(x - \frac{1 - \sqrt{13}}{2}\right)$$

Find the sum and product of the roots of the equation

$$3x^2 - 2x + 1 = 0.$$

Solution: The given equation is a quadratic equation in which a = 3, b = - 2, and c = 1. Using the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to find the roots of the given equation:

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(1)}}{2(3)} = \frac{2 \pm \sqrt{-8}}{6}$$

$$= \frac{1}{3} \pm \frac{\sqrt{4}\sqrt{-2}}{6} = \frac{1}{3} \pm \frac{2\sqrt{-2}}{6}$$

$$= \frac{1 \pm \sqrt{-2}}{3}$$

The sum of the roots is:

$$\frac{1 + \sqrt{-2}}{3} + \frac{1 - \sqrt{-2}}{3} = \frac{1 + \sqrt{-2} + 1 - \sqrt{-2}}{3} = \frac{2}{3}.$$

The product of the root is:

$$\left(\frac{1 + \sqrt{-2}}{3}\right)\left(\frac{1 - \sqrt{-2}}{3}\right) = \frac{1 + \sqrt{-2} - \sqrt{-2} - (-2)}{9}$$

$$= \frac{1}{3}.$$



Find the sum and product of the roots of the equation $3x^2 + 13x - 10 = 0$.

Solution: The roots of an equation of the form $ax^2 + bx + c = 0$ can be found using the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

For the given equation, a = 3, b = 13, c = -10. Therefore, the roots of the equation $3x^2 + 13x - 10 = 0$ are

$$x = \frac{-13 \pm \sqrt{169 + 120}}{6} = \frac{-13 \pm 17}{6} = -5 \text{ or } \frac{2}{3}.$$

The sum of the roots is $-5 + \frac{2}{3} = -\frac{13}{3}$. The product of the roots is $(-5)\left(\frac{2}{3}\right) = -\frac{10}{3}$.

Check. The sum of the roots of a quadratic equation is $r_1 + r_2 = \frac{-b}{a}$ and the product of the roots of a quadratic equation is $r_1 \cdot r_2 = \frac{c}{a}$. From the equation, $-\frac{b}{a} = -\frac{13}{3}$ and $\frac{c}{a} = -\frac{10}{3}$.

**• PROBLEM 509**

Without solving, find the sum and product of the roots of $8x^2 = 2x + 3$.

Solution: Given a quadratic equation in standard form, $ax^2 + bx + c = 0$, the sum of the roots is given by $\frac{-b}{a}$ and the product of the roots by $\frac{c}{a}$. Adding $-(2x + 3)$ to both sides of the given equation, we obtain $8x^2 - 2x - 3 = 0$, a quadratic equation in standard form with $a = 8$, $b = -2$, and $c = -3$. Thus:

$$\text{Sum of roots} = -\frac{b}{a} = \left(\frac{-2}{8}\right) = \frac{1}{4}.$$

$$\text{Product of roots} = \frac{c}{a} = \frac{-3}{8}.$$

**• PROBLEM 510**

Find the sum and the product of the roots in each of the following equations: $x^2 - 3x + 2 = 0$, $2x^2 + 8x - 5 = 0$, and $\sqrt{2}x^2 + 5x - \sqrt{8} = 0$

Solution: There are two relations between the roots and coefficients of a quadratic equation. When we want to find the roots of the quadratic function, $f(x) = ax^2 + +bx + c$, we set $f(x) = 0$. Then $ax^2 + bx + c = 0$. By the quadratic formula, the roots are

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Adding $r_1 + r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}$

$$= \frac{-2b}{2a} = \frac{-b}{a}$$

Multiplying $r_1 \cdot r_2 = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$

$$= \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}$$

Therefore, the sum of the roots is $-b/a$ and the product of the roots is $c/a$. Thus in the following tabulation $-b/a$ is the sum of the roots, and $c/a$ is the product of the roots.

| Equation | Sum of roots | Product of roots |
|---|---|---|
| $x^2 - 3x + 2 = 0$ Thus $a = 1$, $b = -3$, $c = 2$ | $-\frac{b}{a} = \frac{-(-3)}{1} = 3$ | $\frac{c}{a} = \frac{2}{1} = 2$ |
| $2x^2 + 8x - 5 = 0$ Thus $a = 2$, $b = 8$, $c = -5$ | $-\frac{b}{a} = \frac{-8}{2} = -4$ | $\frac{c}{a} = \frac{-5}{2}$ |
| $\sqrt{2}x^2 + 5x - \sqrt{8} = 0$ Thus $a = \sqrt{2}$, $b = 5$, $c = -58$ | $-\frac{b}{a} = \frac{-5}{\sqrt{2}}$ | $\frac{c}{a} = \frac{-\sqrt{8}}{\sqrt{2}} = -\sqrt{4} = -2$ |

These two relations provide a rapid method for verifying the roots of a quadratic equation. For the first equation, $x^2 - 3x + 2 = 0$, we can solve for the roots by factoring.

$$x^2 - 3x + 2 = 0$$
$$(x - 2)(x - 1) = 0$$
$$x_1 = 2 \qquad x_2 = 1$$

The sum was found by the formula to be 3.

$$x_1 + x_2 = 2 + 1 = 3$$

The product was found to be 2.

$$x_1 \cdot x_2 = 2 \cdot 1 = 2$$

Similarly for the last two equations.

$2x^2 + 8x - 5 = 0$; $-b/a = -4$; $c/a = -5/2$

$$x_1 = \frac{-8 + \sqrt{8^2 - 4(2)(-5)}}{2(2)} \qquad x_2 = \frac{-8 - \sqrt{8^2 - 4(2)(-5)}}{2(2)}$$

$$= \frac{-8 + \sqrt{104}}{4} \qquad = \frac{-8 - \sqrt{104}}{4}$$

$$x_1 + x_2 = \frac{-8 + \sqrt{104}}{4} + \frac{-8 - \sqrt{104}}{4} = \frac{-16}{4} = -4 = -b/a$$

$$x_1 \cdot x_2 = \left(\frac{-8 + \sqrt{104}}{4}\right)\left(\frac{-8 - \sqrt{104}}{4}\right) = \frac{64 - 104}{16} = \frac{-40}{16} = \frac{-5}{2} = c/a$$

$$\sqrt{2}\, x^2 + 5x - \sqrt{8} = 0$$

$$x_1 = \frac{-5 + \sqrt{25 - 4(-\sqrt{8})\sqrt{2}}}{2\sqrt{2}} \qquad x_2 = \frac{-5 - \sqrt{25 - 4(-\sqrt{8})\sqrt{2}}}{2\sqrt{2}}$$

$$= \frac{-5 + \sqrt{25 - 4(-4)}}{2\sqrt{2}} \qquad = \frac{-5 - \sqrt{25 - 4(-4)}}{2\sqrt{2}}$$

$$= \frac{-5 + \sqrt{41}}{2\sqrt{2}} \qquad = \frac{-5 - \sqrt{41}}{2\sqrt{2}}$$

$$x_1 + x_2 = \left(\frac{-5 + \sqrt{41}}{2\sqrt{2}}\right) + \left(\frac{-5 - \sqrt{41}}{2\sqrt{2}}\right) = \frac{-10}{2\sqrt{2}} = \frac{-5}{\sqrt{2}} = \frac{-b}{a}$$

$$x_1 \cdot x_2 = \left(\frac{-5 + \sqrt{41}}{2\sqrt{2}}\right)\left(\frac{-5 - \sqrt{41}}{2\sqrt{2}}\right) = \frac{25 - 41}{2 \cdot 2 \cdot 2} = \frac{-16}{8} = -2 =$$

$$= \frac{c}{a}$$

**• PROBLEM 511**

Determine the quadratic equation whose roots are $x = 2 + \sqrt{3}$ and $x = 2 - \sqrt{3}$.

Solution: We can determine the quadratic equation from the sum and the product of the roots. A quadratic equation whose roots are $x_1$ and $x_2$ may be written in the form

$$x^2 - (x_1 + x_2)x + x_1 \cdot x_2 = 0$$

where the sum of the roots is $x_1 + x_2 = -\frac{b}{a}$ and the product of the roots is $x_1 \cdot x_2 = \frac{c}{a}$. Here,

$$x_1 = 2 + \sqrt{3} \text{ and } x_2 = 2 - \sqrt{3}.$$

Then, $x_1 + x_2 = 2 + \sqrt{3} + 2 - \sqrt{3} = 4$ and

and $x_1 \cdot x_2 = (2 + \sqrt{3})(2 - \sqrt{3}) = 4 - 3 = 1.$

Hence, the equation is:

$$x^2 - 4x + 1 = 0.$$

**• PROBLEM 512**

Find the equation whose roots are $3 + \sqrt{2}$ and $3 - \sqrt{2}$.

Solution: The roots of a quadratic equation $ax^2 + bx + c$ can be characterized by the following:

the sum of the roots

$$r_1 + r_2 = \frac{-b}{a} \text{ and}$$

the product of the roots

$$r_1 \cdot r_2 = \frac{c}{a}.$$

The sum of the roots is $(3 + \sqrt{2}) + (3 - \sqrt{2}) = 6$. Hence,

$$-\frac{b}{a} = 6 \text{ or } \frac{b}{a} = -6.$$

The product of the roots is

$$(3 + \sqrt{2})(3 - \sqrt{2}) = 3^2 - (\sqrt{2})^2 = 9 - 2 = 7.$$

This is the constant term of the required equation. We obtain this from the quadratic function $ax^2 + bx + c = 0$. Divide by a.

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0.$$

Then $\frac{b}{a}$ is the coefficient of x and $\frac{c}{a}$ is the constant term. Thus, here $\frac{b}{a} = -6$ and $\frac{c}{a} = 7$. Hence, the equation is

$$x^2 - 6x + 7 = 0.$$

Check: $x^2 - 6x + 7 = 0$ with $a = 1$, $b = -6$, $c = 7$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(7)}}{2(1)} = \frac{6 \pm \sqrt{36 - 28}}{2}$$

$$= \frac{6 \pm \sqrt{8}}{2} = \frac{6 \quad 2\sqrt{2}}{2} = 3 + \sqrt{2} \text{ and } 3 - \sqrt{2}.$$

**• PROBLEM 513**

Find a quadratic equation whose roots are $3 + 2\sqrt{3}$ and $3 - 2\sqrt{3}$.

Solution: A quadratic equation is an equation of the form $ax^2 + bx + c = 0$, where a,b, and c are constants and $a \neq 0$. If both sides of this quadratic equation are divided by a, then:

$$\frac{ax^2 + bx + c}{a} = \frac{0}{a}$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad (1)$$

Note that this last result is valid since $a \neq 0$. If $r_1$ and $r_2$ are the roots of a quadratic equation, then the sum of these roots, S, is,

$S = r_1 + r_2 = -\frac{b}{a}$ and the product of these roots, P, is: $P = r_1 \cdot r_2 = c/a$.

Note that the coefficient of the x-term in equation (1) is $\frac{b}{a}$. In relation to the sum of the roots, S, this coefficient $= \frac{b}{a} = -\left(-\frac{b}{a}\right) = -(S) = -S$. Hence, equation (1) can be rewritten as,

$$x^2 + (-S)x + \frac{c}{a} = 0$$

or

$$x^2 - Sx + \frac{c}{a} = 0 \qquad (2)$$

Also, note that the constant term on the left side of equation (1), or $\frac{c}{a}$, is also the product, P, of the roots. Hence, equation (2) can be rewritten as:

$$x^2 - Sx + P = 0 \qquad (3)$$

The sum of the roots is:

$S = r_1 + r_2$, and here $r_1$ and $r_2$ are $3 + 2\sqrt{3}$, $3 - 2\sqrt{3}$

Thus,

$$S = (3 + 2\sqrt{3}) + (3 - 2\sqrt{3})$$
$$= 3 + \cancel{2\sqrt{3}} + 3 - \cancel{2\sqrt{3}}$$
$$= 3 + 3$$
$$= 6$$

The product of the roots is:

$$P = r_1 \cdot r_2$$
$$= (3 + 2\sqrt{3})(3 - 2\sqrt{3})$$
$$= 9 + \cancel{6\sqrt{3}} - \cancel{6\sqrt{3}} - 4(3)$$
$$= 9 - 12$$
$$= -3$$

Then, replacing S and P by 6 and -3 respectively in equation (3):

$$x^2 - Sx + P = x^2 - 6x + (-3) = 0$$

or

$$x^2 - 6x - 3 = 0,$$

which is in the form $ax^2 + bx + c = 0$ of a quadratic equation.

**• PROBLEM 514**

Form the equation whose roots are $2 + \sqrt{3}$ and $2 - \sqrt{3}$.

Solution: The roots are $2 + \sqrt{3}$ and $2 - \sqrt{3}$. Hence, $x = 2 + \sqrt{3}$ and $x = 2 - \sqrt{3}$. Subtract $(2 + \sqrt{3})$ from the first equation:

$$x - (2 + \sqrt{3}) = (2 + \sqrt{3}) - (2 + \sqrt{3}) = 0 ,$$

or

$$x - (2 + \sqrt{3}) = 0.$$

Subtract $(2 - \sqrt{3})$ from the second equation:

$$x - (2 - \sqrt{3}) = (2 - \sqrt{3}) - (2 - \sqrt{3}) = 0 ,$$

or

$$x - (2 - \sqrt{3}) = 0 .$$

Therefore,

$$[x - (2 + \sqrt{3})][x - (2 - \sqrt{3})] = (0)(0) = 0 ,$$

or

$$[x - (2 + \sqrt{3})][x - (2 - \sqrt{3})] = 0 . \qquad (1)$$

Equation (1) is in the form $(x - a)(x - b) = 0$ where $a$ corresponds to $(2 + \sqrt{3})$ and $b$ corresponds to $(2 - \sqrt{3})$. Also:

$$\begin{aligned}(x - a)(x - b) &= x^2 - ax - bx + ab \\ &= x^2 - (a + b)x + ab . \qquad (2)\end{aligned}$$

Notice that $a$ and $b$ are the roots; that is, $2 + \sqrt{3}$ and $2 - \sqrt{3}$. The sum of the roots is:

$$\begin{aligned}a + b = (2 + \sqrt{3}) + (2 - \sqrt{3}) &= 2 + \cancel{\sqrt{3}} + 2 - \cancel{\sqrt{3}} \\ &= 4 .\end{aligned}$$

The product of the roots is:

$$\begin{aligned}a \cdot b = (2 + \sqrt{3})(2 - \sqrt{3}) &= 4 + \cancel{2\sqrt{3}} - \cancel{2\sqrt{3}} - 3 \\ &= 4 - 3 \\ &= 1 .\end{aligned}$$

Hence, using the form of equation (2):

$$[x - (2 + \sqrt{3})][x - (2 - \sqrt{3})] = x^2 - (4)x + 1 = 0$$

or

$$x^2 - 4x + 1 = 0 ,$$

which is the equation whose roots are

$$2 + \sqrt{3} \text{ and } 2 - \sqrt{3} .$$

**• PROBLEM 515**

Find the equation whose roots are $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ .

Solution: The roots of the equation are $x = \frac{\alpha}{\beta}$ and $x = \frac{\beta}{\alpha}$. Subtract $\frac{\alpha}{\beta}$ from both sides of the first equation:

$$x - \frac{\alpha}{\beta} = \frac{\alpha}{\beta} - \frac{\alpha}{\beta} = 0 ,$$

or

$$x - \frac{\alpha}{\beta} = 0 .$$

Subtract $\frac{\beta}{\alpha}$ from both sides of the second equation:

$$x - \frac{\beta}{\alpha} = \frac{\beta}{\alpha} - \frac{\beta}{\alpha} = 0 ,$$

or

$$x - \frac{\beta}{\alpha} = 0 .$$

Therefore:

$$\left(x - \frac{\alpha}{\beta}\right)\left(x - \frac{\beta}{\alpha}\right) = (0)(0) = 0 ,$$

or

$$\left(x - \frac{\alpha}{\beta}\right)\left(x - \frac{\beta}{\alpha}\right) = 0 . \qquad (1)$$

Equation (1) is of the form: $(x - c)(x - d) = 0$, or

$$x^2 - cx - dx + cd = 0 , \text{ or}$$

$$x^2 - (c + d)x + cd = 0 . \qquad (2)$$

Note that $c$ corresponds to the root $\frac{\alpha}{\beta}$ and $d$ corresponds to the root $\frac{\beta}{\alpha}$. The sum of the roots is:

$$\begin{aligned}c + d = \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha(\alpha)}{\alpha(\beta)} + \frac{\beta(\beta)}{\beta(\alpha)} &= \frac{\alpha^2}{\alpha\beta} + \frac{\beta^2}{\alpha\beta} \\ &= \frac{\alpha^2 + \beta^2}{\alpha\beta}\end{aligned}$$

The product of the roots is:

$$c \cdot d = \frac{\alpha}{\beta} \cdot \frac{\beta}{\alpha} = \frac{\alpha\beta}{\beta\alpha} = \frac{\alpha\beta}{\alpha\beta} = 1 .$$

Using the form of equation (2):

$$\left(x - \frac{\alpha}{\beta}\right)\left(x - \frac{\beta}{\alpha}\right) = x^2 - \left(\frac{\alpha^2 + \beta^2}{\alpha\beta}\right)x + 1 = 0 .$$

Hence,

$$x^2 - \left(\frac{\alpha^2 + \beta^2}{\alpha\beta}\right)x + 1 = 0 , \qquad (3)$$

Multiply both sides of equation (3) by $\alpha\beta$ .

$$\alpha\beta\left[x^2 - \left(\frac{\alpha^2 + \beta^2}{\alpha\beta}\right)x + 1\right] = \alpha\beta(0)$$

Distributing,

$$\alpha\beta x^2 - (\alpha^2 + \beta^2)x + \alpha\beta = 0 ,$$

which is the equation whose roots are $\frac{\alpha}{\beta}$ , $\frac{\beta}{\alpha}$ .

**• PROBLEM 516**

Find the equation whose roots are the negatives of the roots of $x^2 + 7x - 2 = 0$.

Solution: The roots of a quadratic equation $ax^2 + bx + c = 0$ are given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} .$$

Therefore:

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

By adding $r_1$ and $r_2$:

$$r_1 + r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-2b}{2a} = \frac{-b}{a}.$$

We see that the sum of the roots is:

$$r_1 + r_2 = \frac{-b}{a}.$$

Then multiply: $r_1 \cdot r_2 = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$

$$= \frac{\left(-b + \sqrt{b^2 - 4ac}\right)\left(-b - \sqrt{b^2 - 4ac}\right)}{4a^2}$$

$$= \frac{b^2 + \cancel{b\sqrt{b^2 - 4ac}} \cancel{-b\sqrt{b^2 - 4ac}} - (b^2-4ac)}{4a^2}$$

$$= \frac{4ac}{4a^2} = \frac{c}{a}.$$

For the given equation $a = 1$, $b = 7$, $c = -2$. If the roots of the given equation are $r_1$ and $r_2$, we seek an equation whose roots are $-r_1$ and $-r_2$. From the given equation, we have $r_1 + r_2 = -\frac{b}{a} = -\frac{7}{1} = -7$. Thus

$$-r_1 + (-r_2) = -(r_1 + r_2) = 7$$

and the coefficient of the first-degree term in the required equation is -7. The product of the roots of the given equation is $\frac{c}{a} = \frac{-2}{1} = -2$. Since $r_1 \cdot r_2 = (-r_1)(-r_2)$, the product of the roots of the required equation is also -2. Hence, the constant term of the required equation is -2. The required equation can be written

$$x^2 - 7x - 2 = 0.$$

**• PROBLEM 517**

Find the value of $k$ if one root is twice the other.

$$x^2 - kx + 18 = 0.$$

Solution: If $x_1$ and $x_2$ are the roots of the quadratic equation $ax^2 + bx + c$, then $x_1 + x_2 = -b/a$ and $x_1 \cdot x_2 = c/a$. For $x^2 - kx + 18 = 0$, $a = 1$, $b = -k$, and $c = 18$. Since for this quadratic, one root is twice the other, let the roots be $r$ and $2r$. Their sum is $r + 2r = 3r$. The sum of the roots is $-b/a = k$. Hence $k = 3r$. The product of the roots $r \cdot 2r = 2r^2$ is equal to $c/a = 18$.

Thus, $\quad 2r^2 = 18;\ r^2 = 9;\ r = \pm 3$

Therefore, $\quad k = \pm 3 \cdot 3 = \pm 9$

Check: The roots of $x^2 - 9x + 18 = 0$ are 3, and $2 \cdot 3 = 6$.
The roots of $x^2 + 9x + 18 = 0$ are $-3$, and $2(-3) = -6$.

**• PROBLEM 518**

Find the values of the constant k in the equation

$$2x^2 - kx + 3k = 0$$

if the difference of the roots is $\frac{5}{2}$.

Solution: The given equation is a quadratic equation since it is in the form $ax^2 + bx + c = 0$. If both sides of the given equation are divided by 2, then:

$$\frac{2x^2 - kx + 3k}{2} = \frac{0}{2}$$

$$x^2 - \frac{k}{2}x + \frac{3}{2}k = 0 \tag{1}$$

Equation (1) is in the form $x^2 - Sx + P = 0$, where S = sum of the roots = $\frac{k}{2}$ and P = product of the roots = $\frac{3}{2}k$.

Let the roots be $r_1$ and $r_2$. Since the difference of the roots is $\frac{5}{2}$ and the sum of the roots is $\frac{k}{2}$, the following equations result:

$$r_1 - r_2 = \frac{5}{2} \quad (2) \qquad \text{and} \qquad r_1 + r_2 = \frac{k}{2} \qquad (3)$$

Solving for $r_1$ by adding equations (2) and (3):

$$\begin{aligned} r_1 - r_2 &= \frac{5}{2} \\ r_1 + r_2 &= \frac{k}{2} \\ \hline 2r_1 &= \frac{5}{2} + \frac{k}{2} \\ 2r_1 &= \frac{5 + k}{2}. \end{aligned}$$

Multiplying both sides by $\frac{1}{2}$:

$$\frac{1}{\cancel{2}}\left(\cancel{2}r_1\right) = \frac{1}{2}\left[\frac{5 + k}{2}\right]$$

$$r_1 = \frac{5 + k}{4}$$

$$r_1 = \frac{k + 5}{4} \qquad (4)$$

Solving for $r_2$ by subtracting equation (3) from equation (2):

$$\begin{aligned} r_1 - r_2 &= \frac{5}{2} \\ \left(- r_1 + r_2\right. &= \left.\frac{k}{2}\right) \\ \hline -2r_2 &= \frac{5}{2} - \frac{k}{2} \\ -2r_2 &= \frac{5 - k}{2} \qquad (5) \end{aligned}$$

Multiply both sides of equation (5) by $-\frac{1}{2}$:

$$\left(-\frac{1}{2}\right)\left(-2r_2\right) = \left(-\frac{1}{2}\right)\left(\frac{5 - k}{2}\right)$$

$$r_2 = \frac{(-1)(5 - k)}{(2)(2)}$$

$$= \frac{-5 + k}{4}$$

$$r_2 = \frac{k - 5}{4} \tag{6}$$

Multiplying equations (4) and (6):

$$r_1 r_2 = \left(\frac{k + 5}{4}\right)\left(\frac{k - 5}{4}\right)$$

$$= \frac{k^2 + \cancel{5k} - \cancel{5k} - 25}{16}$$

$$r_1 r_2 = \frac{k^2 - 25}{16}.$$

However, it was found earlier that the product, P, of the roots was $\frac{3}{2}k$; that is, $r_1 r_2 = \frac{3}{2}k$. Then, setting these two expressions for $r_1 r_2$ equal:

$$\frac{k^2 - 25}{16} = \frac{3}{2}k$$

$$\frac{k^2 - 25}{16} = \frac{3k}{2}.$$

Subtract $\frac{3k}{2}$ from both sides of this equation:

$$\frac{k^2 - 25}{16} - \frac{3k}{2} = \frac{3k}{2} - \frac{3k}{2}$$

$$\frac{k^2 - 25}{16} - \frac{3k}{2} = 0.$$

Obtaining a common denominator of 16 for the two fractions on the left side of the equation:

$$\frac{k^2 - 25}{16} - \frac{8(3k)}{8(2)} = 0$$

$$\frac{k^2 - 25}{16} - \frac{24k}{16} = 0$$

$$\frac{k^2 - 25 - 24k}{16} = 0.$$

$$\frac{k^2 - 24k - 25}{16} = 0$$

Multiply both sides of this equation by 16:

$$16\left(\frac{k^2 - 24k - 25}{16}\right) = 16(0)$$

$$k^2 - 24k - 25 = 0.$$

Factor the left side of this equation into a product of two polynomials:

$$(k - 25)(k + 1) = 0 \tag{7}$$

Whenever a product $ab = 0$, where $a$ and $b$ are any two numbers, either $a = 0$ or $b = 0$. Then, equation (7) becomes:

$$k - 25 = 0 \quad \text{or} \quad k + 1 = 0$$

$$k = 25 \quad \text{or} \quad k = -1.$$

Hence, the two solutions are $k = -1$ and $k = 25$.

**• PROBLEM 519**

Find the value of $k$ if, in the equation $2x^2 - kx^2 + 4x + 5k = 0$, one root is the reciprocal of the other.

Solution: By using the distributive property in relation to the $x^2$ terms, the given equation becomes:

$$2x^2 - kx^2 + 4x + 5k = (2-k)x^2 + 4x + 5k = 0 \quad \text{or}$$

$$(2-k)x^2 + 4x + 5k = 0.$$

Divide both sides of this equation by $(2-k)$:

$$\frac{(2-k)x^2 + 4x + 5k}{(2-k)} = \frac{0}{(2-k)}$$

$$x^2 + \frac{4}{2-k}x + \frac{5k}{2-k} = 0. \tag{1}$$

Equation (1) is in the form $x^2 - Sx + P = 0$, where $S$ is the sum of the roots of this equation, and $P$ is the product of the roots of this equation. If $r_1$ and $r_2$ are the roots of equation (1), then

$$P = r_1r_2 = \frac{5k}{2-k}$$

$$r_1r_2 = \frac{5k}{2-k}. \tag{2}$$

It was also given in this problem that one root is the reciprocal of the other. Then,

$$r_1 = \frac{1}{r_2}.$$

Multiply both sides of this equation by $r_2$:

$$r_2(r_1) = \cancel{r_2}\left(\frac{1}{\cancel{r_2}}\right)$$

$$r_2r_1 = 1 \qquad \text{or}$$

$$r_1r_2 = 1 \qquad (3)$$

Since equations (2) and (3) are two expressions for $r_1r_2$, the right sides of these equations can be set equal to each other:

$$\frac{5k}{2-k} = 1.$$

Multiply both sides of this equation by (2-k):

$$\cancel{(2-k)}\left[\frac{5k}{\cancel{2-k}}\right] = (2-k)(1)$$

$$5k = 2-k.$$

Add k to both sides of this equation:

$$5k + k = 2 \cancel{-k} + \cancel{k}.$$

$$6k = 2.$$

Divide both sides of this equation by 6:

$$\frac{6k}{6} = \frac{2}{6}$$

$$k = \frac{2}{6}$$

$$\text{or } k = \frac{1}{3}.$$

Therefore, the value of k in the equation $2x^2 - kx^2 + 4x + 5k = 0$ is $\frac{1}{3}$.

**• PROBLEM 520**

> If $\alpha$ and $\beta$ are the roots of $x^2 - px + q = 0$, find the value of (1) $\alpha^2 + \beta^2$, (2) $\alpha^3 + \beta^3$.

Solution: The roots of the given equation are $\alpha$ and $\beta$. Hence, $x = \alpha$ and $x = \beta$. Subtract $\alpha$ from both sides of the first equation:

$$x - \alpha = \alpha - \alpha = 0,$$

or

$$x - \alpha = 0.$$

Subtract $\beta$ from both sides of the second equation:

$$x - \beta = \beta - \beta = 0,$$

or

$$x - \beta = 0.$$

Hence,

$$(x - \alpha)(x - \beta) = (0)(0) = 0,$$

or

$$(x - \alpha)(x - \beta) = 0.$$

Also,

$$(x - \alpha)(x - \beta) = x^2 - \alpha x - \beta x + \alpha\beta = 0$$

or

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0 \qquad (1)$$

Comparing the given equation with equation (1):

$$\alpha + \beta = p \quad (\text{eq.}2), \text{ and } \quad \alpha\beta = q \quad (\text{eq. } 3)$$

Therefore, squaring both sides of equation (2):

$$(\alpha + \beta)^2 = p^2$$

$$(\alpha + \beta)(\alpha + \beta) = p^2$$

$$\alpha^2 + \alpha\beta + \alpha\beta + \beta^2 = p^2$$

$$\alpha^2 + 2\alpha\beta + \beta^2 = p^2 \qquad (4)$$

Subtract $2\alpha\beta$ from both sides of equation (4):

$$\alpha^2 + \cancel{2\alpha\beta} + \beta^2 - \cancel{2\alpha\beta} = p^2 - 2\alpha\beta$$

$$\alpha^2 + \beta^2 = p^2 - 2\alpha\beta$$

$$\alpha^2 + \beta^2 = p^2 - 2q\ ,$$

since $\alpha\beta = q$.

To obtain an expression for $\alpha^3 + \beta^3$, cube both sides of equation (2).

$$(\alpha + \beta)^3 = p^3$$

$$(\alpha + \beta)(\alpha + \beta)^2 = p^3$$

$$(\alpha + \beta)\left(\alpha^2 + 2\alpha\beta + \beta^2\right) = p^3$$

Distributing the left side of this equation:

$$\left(\alpha^3 + 2\alpha^2\beta + \alpha\beta^2\right) + \left(\alpha^2\beta + 2\alpha\beta^2 + \beta^3\right) = p^3$$

Combining terms and simplifying the left side of this equation:

$$\alpha^3 + 2\alpha^2\beta + \alpha\beta^2 + \alpha^2\beta + 2\alpha\beta^2 + \beta^3 = p^3$$

$$\alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3 = p^3$$

$$\left(\alpha^3 + \beta^3\right) + \left(3\alpha^2\beta + 3\alpha\beta^2\right) = p^3$$

Factor out $3\alpha\beta$ from the second term on the left side of this equation:

$$\left(\alpha^3 + \beta^3\right) + 3\alpha\beta(\alpha + \beta) = p^3$$

$$\left(\alpha^3 + \beta^3\right) + 3q(p) = p^3\ ,$$

since $\alpha\beta = q$ and $\alpha + \beta = p$ . Hence,

$$\left(\alpha^3 + \beta^3\right) + 3pq = p^3$$

Subtract $3pq$ from both sides of this equation.

$$\left(\alpha^3 + \beta^3\right) + 3pq - 3pq = p^3 - 3pq$$

$$\left(\alpha^3 + \beta^3\right) = p^3 - 3pq$$

Factor out $p$ from the right side of this equation.

$$\alpha^3 + \beta^3 = p\left(p^2 - 3q\right)$$

Therefore, $\alpha^2 + \beta^2 = p^2 - 2q$, and $\alpha^3 + \beta^3 = p\left(p^2 - 3q\right)$.

**• PROBLEM 521**

Solve $\left(1 - a^2\right)(x + a) - 2a\left(1 - x^2\right) = 0.$

**Solution:** The given equation may be rewritten by finding the product of the first two terms on the left side and distributing the remaining terms on the left side:

$$\left(1 - a^2\right)(x + a) - 2a\left(1 - x^2\right) = 0$$

$$\left(x - a^2x + a - a^3\right) - \left(2a - 2ax^2\right) = 0$$

$$x - a^2x + a - a^3 - 2a + 2ax^2 = 0$$

$$2ax^2 + x - a^2x - a^3 - 2a + a = 0$$

$$2ax^2 + \left(1 - a^2\right)x - a - a^3 = 0$$

$$2ax^2 + \left(1 - a^2\right)x - a\left(1 + a^2\right) = 0 \qquad (1)$$

Equation (1) is in the form $cz^2 + dz + e = 0$, which is a quadratic equation. Therefore, equation (1) is a quadratic equation, and $2a$ corresponds to $c$, $(1 - a^2)$ corresponds to $d$ and $-a(1 + a^2)$ corresponds to $e$. One of the roots is clearly $a$ because, when $x = a$, the given equation is satisfied; that is,

$$(1-a^2)(x+a)-2a(1-x^2) = (1-a^2)(a+a)-2a(1-a^2) = (1-a^2)(2a)-2a(1-a^2)$$

Using the commutative property:

$$\begin{aligned}(1-a^2)(a+a) - 2a(1-a^2) &= (1-a^2)(2a)-2a(1-a^2)\\ &= 2a(1-a^2) - 2a(1-a^2)\\ &= 0 .\end{aligned}$$

$$\begin{aligned}\text{The product of the roots} &= x_1 \cdot x_2\\ &= \frac{e}{c}\\ &= \frac{-a(1+a^2)}{2a}\\ &= -\frac{(1+a^2)}{2}\end{aligned}$$

Since one of the roots is $a$, let this root be $x_1$. Hence,

$$x_1 \cdot x_2 = a \cdot x_2 = -\frac{(1+a^2)}{2}$$

or

$$ax_2 = -\frac{(1+a^2)}{2}$$

or $x_2$, the second root,

$$\begin{aligned}&= -\frac{(1+a^2)}{2}\left(\frac{1}{a}\right)\\ &= -\frac{(1+a^2)}{2a}\end{aligned}$$

Thus, the roots are $a$, and $\dfrac{-(1+a^2)}{2a}$.

**• PROBLEM 522**

Find the condition that the roots of $ax^2 + bx + c = 0$ may be (1) both positive, (2) opposite in sign, but the greater of them negative.

Solution: The given equation, $ax^2 + bx + c = 0$, is a quadratic equation. Let $\alpha$ and $\beta$ be the roots of this equation. The product of the roots of the quadratic equation is:

$$\alpha\beta = \frac{c}{a} .$$

The sum of the roots of the quadratic equation is:

$$\alpha + \beta = \frac{-b}{a} .$$

(1) If the roots are both positive, $\alpha\beta$ is positive, and therefore $c$ and $a$ have like signs.

Also, since $\alpha + \beta$ is positive, $\frac{b}{a}$ is negative; therefore $b$ and $a$ have unlike signs.

Hence, the required condition is that the signs of $a$ and $c$ should be like, and $b$ and $a$ have unlike signs.

(2) If the roots are of opposite signs, $\alpha\beta$ is negative, and therefore $c$ and $a$ have unlike signs.

Also, since $\alpha + \beta$ has the sign of the greater root, it is negative, and therefore $\frac{b}{a}$ is positive; therefore $b$ and $a$ have like signs.

Hence, the required condition is that the signs of $a$ and $b$ should be like, and $c$ and $a$ have unlike signs.

**• PROBLEM 523**

Find the discriminant of $3x^2 - 7x + 5 = 0$. Then solve.

Solution: Recall that the discriminant is $b^2 - 4ac$, which appears under the radical in the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

applying to equations in the form $ax^2 + bx + c = 0$. In our case, $a = 3$, $b = -7$, $c = 5$ and the discriminant is

$$b^2 - 4ac = (-7)^2 - (4\cdot3\cdot5) = 49 - 60 = -11.$$

A negative discriminant means there is a negative under the radical, which results in imaginary roots. Hence, $3x^2 - 7x + 5 = 0$ has two imaginary roots. To find these roots, we use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{7 \pm \sqrt{-11}}{6}.$$

Since $\sqrt{ab} = \sqrt{a}\sqrt{b}$, $\sqrt{-11} = \sqrt{-1(11)} = \sqrt{(-1)}\sqrt{11}$. By definition $i = \sqrt{(-1)}$, thus $\sqrt{-11} = i\sqrt{11}$ and the roots are

$$\frac{7 \pm i\sqrt{11}}{6}.$$

**• PROBLEM 524**

Compute the value of the discriminant and then determine the nature of the roots of each of the following four equations:

$$4x^2 - 12x + 9 = 0,$$

$$3x^2 - 7x - 6 = 0$$

$$5x^2 + 2x - 9 = 0$$

and $x^2 + 3x + 5 = 0$.

Solution: The discriminant, the term of the quadratic formula which appears under the radical, is $b^2 - 4ac$. It can be used to determine the nature of the roots of equations in the form $ax^2 + bx + c = 0$. Assuming a,b,c are real numbers, then,

(1) if $b^2 - 4ac > 0$, the roots are real and unequal

(2) if $b^2 - 4ac = 0$, the roots are real and equal

(3) if $b^2 - 4ac < 0$, the roots are imaginary

Assuming a,b,c are real and rational numbers then,

(1) if $b^2 - 4ac$ is a perfect square $\neq 0$, the roots are real, rational and unequal,

(2) if $b^2 - 4ac = 0$, the roots are real, rational, and equal,

(3) if $b^2 - 4ac > 0$, but not a perfect square, the roots are real, irrational and unequal,

(4) if $b^2 - 4ac < 0$, the roots are imaginary.

(a) $4x^2 - 12x + 9 = 0$

Here a,b,c are rational numbers,

$$a = 4,\ b = -12 \text{ and } c = 9.$$

Therefore,

$$b^2 - 4ac = (-12)^2 - 4(4)(9) = 144 - 144 = 0$$

Since the discriminant is 0, the roots are rational and equal.

(b) $3x^2 - 7x - 6 = 0$

Here a,b,c are rational numbers,

$$a = 3,\ b = -7, \text{ and } c = -6.$$

Therefore,

$$b^2 - 4ac = (-7)^2 - 4(3)(-6) = 49 + 72 = 121 = 11^2$$

Since the discriminant is a perfect square, the roots are rational and unequal.

(c) $5x^2 + 2x - 9 = 0$

Here a,b,c are rational numbers,

$$a = 5,\ b = 2 \text{ and } c = -9$$

Therefore,

$$b^2 - 4ac = 2^2 - 4(5)(-9) = 4 + 180 = 184$$

Since the discriminant is greater than zero, but not a perfect square, the roots are irrational and unequal.

(d) $x^2 + 3x + 5 = 0$

Here a,b,c are rational numbers,

$a = 1$, $b = 3$, and $c = 5$

Therefore,

$$b^2 - 4ac = 3^2 - 4(1)(5) = 9 - 20 = -11$$

Since the discriminant is negative the roots are imaginary.

**• PROBLEM 525**

Discuss the nature of the roots of

(a) $3x^2-7x + 3 = 0$ (b) $5x^2 + 3x + 1 = 0$

Solution: Equations (a) and (b) are of the form,

$$ax^2 + bx + c = 0.$$

In (a): $a = 3$, $b = -7$, $c = 3$

In (b): $a = 5$, $b = 3$, $c = 1$.

First we find the value of the discriminant, $b^2 - 4ac$, in each case.

If $b^2 - 4ac > 0$ the roots are real and unequal.

If $b^2 - 4ac = 0$ the roots are real and equal.

If $b^2 - 4ac < 0$ the roots are imaginary.

(a)
$$\begin{aligned} b^2 - 4ac &= (-7)^2 - (4 \cdot 3 \cdot 3) \\ &= 49 - 36 \\ &= 13. \end{aligned}$$

Thus, the roots are real and unequal.

(b)
$$\begin{aligned} b^2 - 4ac &= (3)^2 - (4 \cdot 5 \cdot 1) \\ &= 9 - 20 \\ &= -11. \end{aligned}$$

Thus, there are no real roots, i.e., the roots are imaginary.

**• PROBLEM 526**

Determine the character of the roots of the equation $2x^2 - x + 5 = 0$.

Solution: The given equation is a quadratic equation where $a = 2$, $b = -1$, and $c = 5$. The discriminant of a quadratic equation is: $b^2 - 4ac$. Therefore, the discriminant of the equation is $1 - 40 = -39$.

By the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$,

the roots of the given equation are:

$$x = \frac{-(-1) \pm \sqrt{-39}}{2(2)} = \frac{1 \pm \sqrt{-39}}{4}$$

$$= \frac{1}{4} \pm \frac{\sqrt{-39}}{4} = \frac{1}{4} \pm \frac{\sqrt{39}}{4}\sqrt{-1}$$

Therefore, $x = \dfrac{1}{4} \pm \dfrac{\sqrt{39}}{4}i$. Considering that the discriminant is less than zero, and the roots obtained for the given equation, the roots are conjugate complex numbers.

**• PROBLEM 527**

Determine the character of the roots of the equation $4x^2 - 12x + 9 = 0$.

Solution: The given equation is a quadratic equation where $a = 4$, $b = -12$, and $c = 9$. The discriminant of this equation,

$b^2 - 4ac$, is $144 - 144 = 0$.

By the quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$,

the only root of the given equation is:

$$x = \frac{-(-12) \pm \sqrt{0}}{2(4)} = \frac{12}{8} = \frac{3}{2}.$$

Since the discriminant is equal to zero and by the root obtained for the given equation, there is only one real rational root.

**• PROBLEM 528**

Determine the character of the roots of the equation $x^2 - 5x + 6 = 0$.

Solution: The given equation is a quadratic equation where $a = 1$, $b = -5$, and $c = 6$. The discriminant of this equation, $b^2 - 4ac$, is $25 - 24 = 1$.

By the quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the roots of the given equation are:

$$x = \frac{-(-5) \pm \sqrt{1}}{2(1)} = \frac{5 \pm 1}{2}.$$

Therefore, $x = \frac{5+1}{2} = 3$ and $x = \frac{5-1}{2} = 2$.

Hence, the roots of the given equation are real, unequal, and rational. $\left[\text{Note: The roots are rational since } x = 3 = \frac{3}{1} \text{ and } x = 2 = \frac{2}{1}\right]$.

**• PROBLEM 529**

> Without solving the equation
>
> $$2x^2 - 3x + 5 = 0,$$
>
> determine the nature of its roots.

Solution: To determine the nature of the roots of a quadratic equation we look at the discriminant $b^2 - 4ac$ $\left(\text{this is the term that appears under the radical in the quadratic formula } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)$, which is used for equations in the form of $ax^2 + bx + c = 0$. In our example $a = 2$, $b = -3$, $c = 5$. Thus, the discriminant, $b^2 - 4ac = 9 - 4(2)(5) = -31 < 0$.

Our discriminant is negative. This means that we have a negative number appearing under the radical in our quadratic formula, which indicates that the roots are imaginary.

**• PROBLEM 530**

> Compute the value of the discriminant and determine the nature of the roots in each of the following three equations: $4x^2 - 4\sqrt{5}x + 5 = 0$, $\sqrt{3}x^2 - 6x + \sqrt{12} = 0$, and $\sqrt{2}x^2 + 3x + \sqrt{5} = 0$.

Solution: The discriminant, the term of the quadratic formula which appears under the radical, is $b^2 - 4ac$. It can be used to determine the nature of the roots of equations in the form $ax^2 + bx + c = 0$.

Assuming a, b, c are real numbers, then

(1) if $b^2 - 4ac > 0$, the roots are real and unequal

(2) if $b^2 - 4ac = 0$, the roots are real and equal

(3) if $b^2 - 4ac < 0$, the roots are imaginary.

Assuming a, b, c are real and rational numbers, then

(1) if $b^2 - 4ac$ is a perfect square $\neq 0$, the roots are real, rational and unequal.

(2) if $b^2 - 4ac = 0$, the roots are real, rational and equal

(3) if $b^2 - 4ac > 0$, but not a perfect square, the roots are real, irrational and unequal,

(4) if $b^2 - 4ac < 0$ the roots are imaginary.

(a) $4x^2 - 4\sqrt{5}x + 5 = 0$. Here a, b, c are real, but not all rational, with $a = 4$, $b = -4\sqrt{5}$, and $c = 5$. Therefore

$$b^2 - 4ac = (-4\sqrt{5})^2 - 4(4)(5) = (4^2)(\sqrt{5})^2 - 80 = (16\cdot5) - 80$$

$$= 80 - 80 = 0.$$

Since the discriminant equals zero, the roots are real and equal.

(b) $\sqrt{3}x^2 - 6x + \sqrt{12} = 0$. Here a, b, c are real, but not all rational, with $a = \sqrt{3}$, $b = -6$, and $c = \sqrt{12}$. Therefore

$$b^2 - 4ac = (-6)^2 - 4\sqrt{3}\sqrt{12} = 36 - 4\sqrt{3\cdot12} = 36 - 4\sqrt{36}$$

$$= 36 - 4(6) = 36 - 24 = 12.$$

Since the discriminant is greater than zero, the roots are real and unequal.

(c) $\sqrt{2}x^2 + 3x + \sqrt{5} = 0$. Here a, b, c are real, but not all rational, with $a = \sqrt{2}$, $b = 3$, and $c = \sqrt{5}$. Therefore

$$b^2 - 4ac = (3)^2 - 4(\sqrt{2})(\sqrt{5}) = 9 - 4\sqrt{2\cdot5} = 9 - 4\sqrt{10}$$

$$\approx 9 - 4(3\ 2) = 9 - 12.8 < 0.$$

Since the discriminant is less than zero, the roots are imaginary.

**• PROBLEM 531**

Can the expression $16x^2 - 76x + 21$ be factored into rational factors?

Solution: To determine if this quadratic polynomial has rational factors we look at its discriminant, $b^2 - 4ac$ (this is the term that appears under the radical in the quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, used for equations in the form of $ax^2 + bx + c = 0$).

In our example, $a = 16$, $b = -76$, $c = 21$ and our discriminant $b^2 - 4ac = (-76)^2 - 4 \cdot 16 \cdot 21 =$

$$5776 - 1344 = 4432.$$

Now, recall what the discriminant tells us about the nature of the roots:

If the discriminant ($b^2-4ac$) is positive or zero, roots are real

If the discriminant ($b^2-4ac$) is negative roots are complex

If the discriminant ($b^2-4ac$) is a perfect square roots are rational

If the discriminant ($b^2-4ac$) is zero roots are equal and rational.

Hence, roots are rational only if the discriminant is zero or a perfect square.

Looking at the column of perfect squares in a table of square roots, we note that 4,432 is not a perfect square, hence the expression $16x^2 - 76x + 21$ cannot be factored into rational factors.

**• PROBLEM 532**

Show that the graph of $y = -x^2 + x - 1$ has no real zeros.

Solution: The real zeros of $y = -x^2 + x - 1$ are those real values of x for which the curve crosses the x-axis, i.e., satisfying:

$$0 = -x^2 + x - 1. \tag{1}$$

The discriminant, $b^2 - 4ac$, of Equation (1) is equal to $(-1)^2 - 4(-1)(-1) = 1 - 4 = -3$. Since the discriminant is negative, Equation (1) has no real roots, and the given function has no real zeros.

**• PROBLEM 533**

Show that the equation $2x^2 - 6x + 7 = 0$ cannot be satisfied by any real values of x.

Solution: We want to show that the values of x, that is the roots, are not real. The discriminant, $b^2 - 4ac$, of an equation of the form $ax^2 + bx + c = 0$, is useful in describing the nature of the roots. If the discriminant is negative, the roots are not real. They are complex in the form $a \pm bi$. Here

$$a = 2,\ b = -6,\ c = 7; \text{ so that}$$

$$b^2 - 4ac = (-6)^2 - 4(2)(7) = -20.$$

Therefore the roots are imaginary.

To show that this is true, we apply the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(2)(7)}}{2(2)} = \frac{6 \pm \sqrt{36 - 56}}{4}$$

$$x = \frac{6 \pm \sqrt{-20}}{4} = \frac{6 \pm \sqrt{4}\sqrt{5}\sqrt{-1}}{4} = \frac{6 \pm 2i\sqrt{5}}{4}$$

$$x = \frac{3 \pm i\sqrt{5}}{2},$$

$$x = \left\{\frac{3 + i\sqrt{5}}{2}, \quad \frac{3 - i\sqrt{5}}{2}\right\}$$

**• PROBLEM 534**

If the equation $x^2 + 2(k+2)x + 9k = 0$ has equal roots, find $k$.

Solution: The given equation is a quadratic equation of the form $ax^2 + bx + c = 0$. In the given equation, $a = 1$, $b = 2(k+2)$, and $c = 9k$. A quadratic equation has equal roots if the discriminant, $b^2 - 4ac$, is zero.

$$b^2 - 4ac = [2(k+2)]^2 - 4(1)(9k) = 0$$

$$4(k+2)^2 - 36k = 0$$

$$4(k+2)(k+2) - 36k = 0$$

$$4(k^2+4k+4) - 36k = 0$$

Distributing, $\quad 4k^2 + 16k + 16 - 36k = 0$

$$4k^2 - 20k + 16 = 0 .$$

Divide both sides of this equation by 4:

$$\frac{4k^2 - 20k + 16}{4} = \frac{0}{4}$$

or

$$k^2 - 5k + 4 = 0 .$$

Factoring the left side of this equation into a product of two polynomials:

$$(k-4)(k-1) = 0.$$

When the product $ab = 0$, where $a$ and $b$ are any two numbers, either $a = 0$ or $b = 0$. Hence, in the case of this problem, either

$$k - 4 = 0 \quad \text{or} \quad k - 1 = 0.$$

Therefore, $\quad k = 4 \quad \text{or} \quad k = 1.$

# CHAPTER 18

# SOLVING QUADRATIC INEQUALITIES

**Basic Attacks and Strategies for Solving Problems in this Chapter.** **See pages 359 to 373 for step-by-step solutions to problems.**

A quadratic inequality is a mathematical statement that can be put in one of the forms

$$ax^2 + bx + c > 0,$$

$$ax^2 + bx + c < 0,$$

$$ax^2 + bx + c \geq 0, \text{ or}$$

$$ax^2 + bx + c \leq 0,$$

where $a \neq 0$. One general method for solving quadratic inequalities is by factoring, if indeed

$$ax^2 + bx + c$$

is factorable. The procedure is as follows:

(1) Arrange the inequality such that only 0 is on one side.

(2) Factor $ax^2 + bx + c$.

(3) Use the factors from Step 2 and the inequalities that satisfy the cases below and find a solution set for each.

Case I: The product of the factors is positive.

Case II: The product of the factors is negative.

(4) Obtain a complete solution set of the original inequality by combining all of the solutions found in the two cases into one set. This is done by taking the union of the solution sets of the two cases.

Graphing each of the various solution sets on a number line is useful in finding the complete solution set of the original quadratic inequality.

In the event that

$$ax^2 + bx + c$$

in a quadratic inequality is not factorable, then completing the square using one of the quadratic formulas is an appropriate solution procedure. Once the roots for the corresponding equation

$$ax^2 + bx + c = 0$$

have been determined, then in order to find the complete solution set we must determine the region into which the roots divide the $x$-axis and satisfy the original inequality.

# Step-by-Step Solutions to Problems in this Chapter, "Solving Quadratic Inequalities"

**• PROBLEM 535**

Solve the inequality $(2x - 1)(x + 2) < 0$.

Solution: Since the two factors must be of opposite sign for their product to be negative, we have the two tentative possibilities:

$$2x - 1 < 0, \quad x + 2 > 0,$$

or $$2x - 1 > 0, \quad x + 2 < 0.$$

Solving the first pair of inequalities:

$2x - 1 < 0$ and $x + 2 > 0$

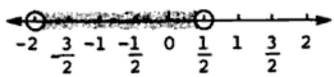
add 1 to both sides: | subtract 2 from both sides:

$2x < 1$ |

divide both sides by 2: |

$x < \frac{1}{2}$ and $x > -2$

Thus, the first pair implies that $x < \frac{1}{2}$ and $x > -2$, or $-2 < x < \frac{1}{2}$; the graph is as follows:




Solving the second pair of inequalities:

$2x - 1 > 0$ and $x + 2 < 0$

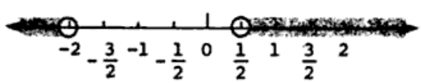
Adding 1 to both sides: |

$2x > 1$ | Subtracting 2 from both sides:

Dividing both sides by 2: |

$x > \frac{1}{2}$ and $x < -2$

Thus, the second pair implies that $x > \frac{1}{2}$ and $x < -2$; the graph is as follows:



Since there is no x such that $x > \frac{1}{2}$ and $x < -2$ we reject this solution.

The complete solution is thus the solution to the first pair of inequalities, $\{x : -2 < x < \frac{1}{2}\}$.

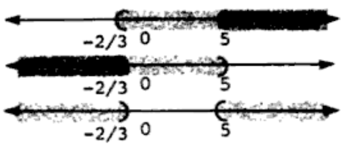
**• PROBLEM 536**

> Solve the quadratic inequality $3x^2 - 13x - 10 > 0$.

Fig. A

Fig. B

Fig. C




<u>Solution:</u> This statement may be written in factored form as

$$(3x + 2)(x - 5) > 0$$

Hence we know that either both factors are positive or both are negative, since the product of a positive factor and a negative factor would be less than zero. This leads us to the following cases:

Case (1) $\quad 3x + 2 > 0$ and $x - 5 > 0$

$3x > -2$ and $x > 5$

$x > -\frac{2}{3}$ and $x > 5$

On the number line this part of the solution set may be represented as the intersection of the intervals $(-\frac{2}{3},\infty)$ and $(5,\infty)$, as pictured in Fig. A. Hence $(-\frac{2}{3},\infty) \cap (5,\infty) = (5,\infty)$, which is represented by the double-shaded region in diagram A.

Case (2) $\quad 3x + 2 < 0$ and $x - 5 < 0$

$x < -\frac{2}{3}$ and $x < 5$

From Fig. B it can be seen that this appears as

$$(-\infty,-\tfrac{2}{3}) \cap (-\infty,5) = (-\infty,-\tfrac{2}{3})$$

Finally, since Case 1 and Case 2 represent the disjunction of two propositions, we conclude that the solution set of our inequality is the union of the sets identified in these two cases. That is,

$$(5,\infty) \cup (-\infty,-\tfrac{2}{3})$$

is the solution set of the inequality, as pictured in Fig. C.

**• PROBLEM 537**

> Obtain the solution set of $2x^2 > x + 6$.

<u>Solution:</u>

$2x^2 > x + 6$ given

$2x^2 - x - 6 > 0$ adding $-x -6$ to each member

$(2x + 3)(x - 2) > 0$ factoring the left member

Now the product of the two factors on the left is greater than zero, or positive, if both factors are positive or if both are negative. Hence we seek the simultaneous solution set of the two inequalities

$$2x + 3 > 0$$

and
$$x - 2 > 0$$

and also the simultaneous set of

$$2x + 3 < 0$$

and
$$x - 2 < 0$$

The first two inequalities form Case I.

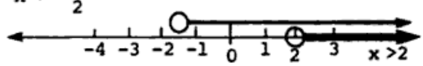
$$2x + 3 > 0 \qquad\qquad x - 2 > 0$$

$$2x > -3$$

$$x > \frac{-3}{2} \qquad \text{and} \qquad x > 2$$

Therefore, in order for $x > -3/2$ and $x > 2$, x must be greater than two. That is, $x > -3/2 \cap x > 2 = x > 2$.

Case I $\quad x > -\frac{3}{2}$




Case II concerns the second pair of inequalities.

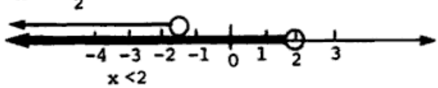
$$2x + 3 < 0 \qquad \text{and} \qquad x - 2 < 0$$

$$2x < -3$$

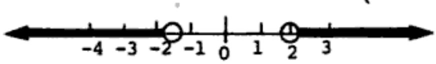
$$x < -\frac{3}{2} \qquad\qquad x < 2$$

The total solution for Case II is $x < -\frac{3}{2} \cap x < 2 = x < -\frac{3}{2}$

Case II $\quad x < -\frac{3}{2}$




The solution set for $2x^2 > x + 6$ is the union of Case I and Case II; that is; $\{x|x > 2\} \cup \left\{x|x < -\frac{3}{2}\right\}$




**• PROBLEM 538**

Find the set $S = \{x|x^2 + 2x - 8 < 0\}$.

<u>Solution:</u> To get the required set, we find the solution set of

$$x^2 + 2x - 8 < 0$$

by the following procedure:

$(x + 4)(x - 2) < 0$ factoring left member of given inequality

In order for a product to be negative (less than zero), one factor must be positive and the other must be negative. Thus there are two cases to be considered here:

Case I $\quad x + 4 > 0$

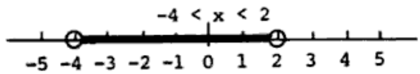
$\qquad\qquad x - 2 < 0$

or

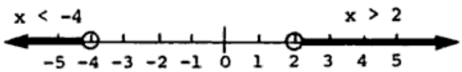
Case II $\quad x + 4 < 0$

$\qquad\qquad x - 2 > 0$

The solution to Case I is $x > -4$ and $x < 2$. We can see this from diagram (A). Note that the solution includes all those numbers between -4 and 2, but not the endpoints themselves.

(A) 


For the second case, the solutions are $x < -4$ and $x > 2$. See the accompanying number line ((B)).

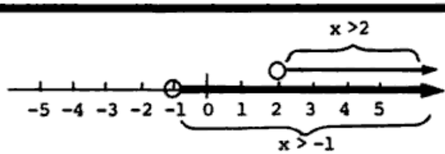
(B) 

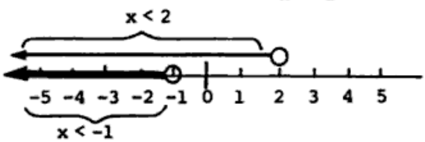

However, we can see from diagram (B) that x cannot at the same time be less than -4 and greater than 2. Hence, there is no solution for this case. That is $x < -4 \cap x > 2 = \emptyset$

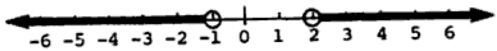
Therefore, the solution set is $S = \{x \mid -4 < x < 2\}$.

**• PROBLEM 539**

Find the solution set of $(x + 1)/(x - 2) > 0$.

(A) 


(B) 

(C) 

Solution: The fraction $(x + 1)/(x - 2)$ is positive, i.e., greater than zero, if the numerator and denominator are both positive or both negative. Hence, we seek the set of numbers that satisfies the two inequalities

$$x + 1 > 0 \qquad (1)$$

$$x - 2 > 0 \qquad (2)$$

simultaneously, and also those which satisfy

$$x + 1 < 0 \qquad (3)$$

$$x - 2 < 0 \qquad (4)$$

simultaneously.

Thus, we have two cases to consider in order to find the solution set of $(x + 1)/(x - 2) > 0$.

Case I

$x + 1 > 0$ and $x - 2 > 0$

$x > -1$ and $x > 2$

We can see the solution from diagram (A).

Hence, for Case I the solution set is the solution of $x > -1 \cap x > 2$, $\{x > 2\}$.

Case II

$x + 1 < 0$ and $x - 2 < 0$

$x < -1$ and $x < 2$

Thus the solution is the solution to $x < -1 \cap x < 2$ or $\{x \mid x < -1\}$. See (B).

Note that none of these solutions include the endpoints.

The solution set of $(x + 1)/(x - 2) > 0$ is thus the union of Case I and Case II; that is $\{x \mid x > 2\} \cup \{x| < -1\}$ (see diagram (C)).

• **PROBLEM 540**

| |
|---|
| Solve $x^2 - 5x + 4 \leq 0$. |

Solution: We factor $x^2 - 5x + 4$ and have

$$(x - 1)(x - 4) \leq 0$$

Since the product is negative, one of the factors is positive and the other is negative. Therefore, either

$$x - 1 \geq 0 \text{ and } x - 4 \leq 0 \qquad \text{Case 1}$$

or $\qquad x - 1 \le 0$ and $x - 4 \ge 0 \qquad$ Case 2

Solving for $x$ in each inequality, we obtain

$x \ge 1$ and $x \le 4 \qquad$ Case 1

or $\qquad x \le 1$ and $x \ge 4 \qquad$ Case 2

The solution to the equation is the set of all $x$, such that $x \ge 1$ and $x \le 4$, or

$x \le 1$ and $x \ge 4$.

**• PROBLEM 541**

Find the solution set of $2x^2 - 3x - 5 > 0$.

Solution: Factoring, we have

$$(2x - 5)(x + 1) > 0.$$

If the product of two factors is positive, both factors must be positive or both factors must be negative. We must consider two cases.

Case 1 $\qquad\qquad$ Case 2

$$(2x - 5 > 0 \text{ and } x + 1 > 0) \text{ or } (2x - 5 < 0 \text{ and } x + 1 < 0)$$

$$\left(x > \tfrac{5}{2} \text{ and } x > -1\right) \text{ or } \left(x < \tfrac{5}{2} \text{ and } x < -1\right)$$

$$\left\{x:\ x > \tfrac{5}{2}\right\} \cup \{x:\ x < -1\}$$

Since $\frac{5}{2} > -1$, then $x > \frac{5}{2}$ implies $x > -1$. Therefore the solution set of Case 1 is $\left\{x:\ x > \frac{5}{2}\right\}$. Similarly, $-1 < \frac{5}{2}$ so that $x < -1$ implies $x < \frac{5}{2}$. The solution set of Case 2 is $\{x: x < -1\}$. The solution set is the union of the solution sets of Case 1 and Case 2.

**• PROBLEM 542**

Solve the inequality $x^2 - x - 2 \le 0$.

Solution: Factoring the left side of the given inequality,

$$(x - 2)(x + 1) \le 0.$$

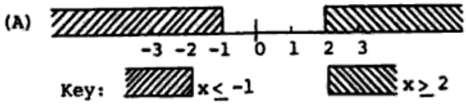
If the product of two numbers is negative, one of the numbers is positive and the other is negative. Hence, there are two cases:

Case 1: $\quad x - 2 \ge 0,\ x + 1 \le 0$

Solving these two inequalities,

$$x \ge 2, \quad x \le -1$$

Graph these new inequalities on number line (A).




Note that there is no value of x which satisfies both inequalities at the same time since these two inequalities do not intersect anywhere on the number line (A).

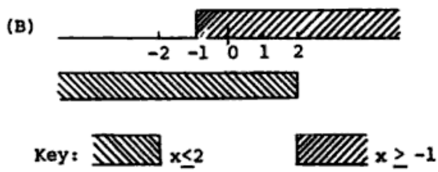
Thus $x \le -1 \cap x \ge 2 = \emptyset$

Case 2: $x - 2 \le 0,\ x + 1 \ge 0.$

Solving these two inequalities,

$$x \le 2, \quad x \ge -1$$

Graph these inequalities on number line (B).




The interval of x which satisfies both inequalities at the same time is $-1 \le x \le 2$. Note that the two inequalities intersect in this interval on number line (B), that is

$$x \ge -1 \cap x \le 2 = -1 \le x \le 2.$$

Hence, the solution to the inequality $x^2 - x - 2 \le 0$ is the set:

$$\{x \mid -1 \le x \le 2\}$$

**• PROBLEM 543**

Find the solution set of $x^2 - 5x + 4 \le 0$.

Solution: First we find the solution for the equality.

$$x^2 - 5x + 4 = 0$$

$$(x - 4)(x - 1) = 0$$

$$x = 4 \text{ or } x = 1$$

For the product $(x - 4)(x - 1)$ to be less than zero, either of the two expressions must be less than zero, but not both simultaneously. $(x - 4)(x - 1)$ will equal zero only when

$x = 4$ or $x = 1$ so that the endpoints 1 and 4 must be included in the solution set. $(x - 4)(x - 1)$ is less than zero when x is greater than one but less than four. The solution set is the union of the set whose elements are the included endpoints and the set whose elements are values of x between 1 and 4. Therefore the solution is the closed interval $\{x: 1 \le x \le 4\}$.

**• PROBLEM 544**

Solve the inequality $x^2 > 4$.

Solution: The given relation may be replaced by an equivalent one, by subtracting 4 from both sides of the inequality. Thus,

$$x^2 - 4 > 0 \qquad (1)$$

Now $x^2 - 4$ may be factored into two linear expressions, and we have

$$(x + 2)(x - 2) > 0 \qquad (2)$$

This expression is not true, when either of the two factors is zero; that is, when

$$(x + 2) = 0 \quad \text{or} \quad (x - 2) = 0,$$

or when

$$x = -2 \qquad \text{or} \qquad x = 2.$$

Hence we have the product of two factors, one of which vanishes for $x = -2$ and the other for $x = 2$. These are therefore the critical values: as x increases from values less than $-2$ to values greater than $-2$, the expression $x + 2$ changes sign, from negative to positive; likewise, $x - 2$ changes from negative to positive as x passes through the critical value 2.

We now make use of the fact that the product of two quantities of like sign (both positive or both negative) is positive, whereas the product of two quantities of unlike sign is negative. Hence x must be such that either both factors in (2) are positive or both are negative. This yields the desired ranges

$$x > 2 \quad \text{or} \quad x < -2 \qquad (3)$$

It will often be found helpful to plot the critical values on a line representation of the real numbers, and then to consider in turn those values of x less than the left most critical point, those between each adjacent pair of critical points, and finally those greater than the rightmost critical point. It is possible also to plot the function $y = f(x)$ and thus determine the values of x for which $f(x) > 0$. Thus, if we plot $y = x^2 - 4$, we find that the graph lies above the X-axis when either of the inequalities (3) is obeyed.

**• PROBLEM 545**

Solve the inequality $2x^2 + 3x + 2 < 0$.

Solution: Divide the left side of the given inequality by 2:

$$\frac{2x^2 + 3x + 2}{2} < \frac{0}{2}$$

$$x^2 + \frac{3}{2}x + 1 < 0 \qquad (1)$$

To factor the left side of inequality (1), complete the square in x. This is done by taking half the coefficient of the x term and squaring this value. The result is then added to and subtracted from both sides of the inequality. Then,

$$\left[\frac{1}{2}\left(\frac{3}{2}\right)\right]^2 = \left[\frac{3}{4}\right]^2 = \frac{3}{4}\cdot\frac{3}{4} = \frac{9}{16}$$

$$x^2 + \frac{3}{2}x + 1 + \frac{9}{16} - \frac{9}{16} < 0 + \frac{9}{16} - \frac{9}{16}$$

or

$$\left(x^2 + \frac{3}{2}x + \frac{9}{16}\right) + 1 - \frac{9}{16} < 0 + \frac{9}{16} - \frac{9}{16}$$

$$\left(x + \frac{3}{4}\right)^2 + 1 - \frac{9}{16} < 0 + \frac{9}{16} - \frac{9}{16}$$

$$\left(x + \frac{3}{4}\right)^2 + \frac{7}{16} < 0 + \frac{9}{16} - \frac{9}{16}.$$

Subtract $\frac{7}{16}$ from both sides of this inequality:

$$\left(x + \frac{3}{4}\right)^2 + \cancel{\frac{7}{16}} - \cancel{\frac{7}{16}} < 0 + \cancel{\frac{9}{16}} - \cancel{\frac{9}{16}} - \frac{7}{16}$$

$$\left(x + \frac{3}{4}\right)^2 < -\frac{7}{16} \qquad (2)$$

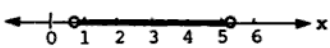
(Note that the constant term in the squared polynomial, namely $\frac{3}{4}$, is just half the coefficient of the x-term in inequality (1)).

In reference to inequality (2), the square of any real number is always greater than or equal to zero; that is, the square of any real number cannot be negative. Therefore, there is no solution to inequality (2) and hence there is no solution to the given inequality.

**• PROBLEM 546**

Solve the inequalities

$x^2 - 6x + 4 > 0$ and $x^2 - 6x + 4 < 0$ .




<u>Solution:</u> The function which we are considering here is $x^2 - 6x + 4$; that is, $f(x) = x^2 - 6x + 4$. We must find where this function is positive or greater than zero and where it is negative or less than zero. We set $f(x) = 0$ and find the roots of this equation, $x^2 - 6x + 4 = 0$. Apply the quadratic formula. In this case $a = 1$, $b = -6$, and $c = 4$.

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(4)}}{2(1)} = \frac{6 \pm \sqrt{36 - 16}}{2} = \frac{6 \pm \sqrt{20}}{2}$$

$$= \frac{6 \pm \sqrt{4}\sqrt{5}}{2} = \frac{6 \pm 2\sqrt{5}}{2} = 3 \pm \sqrt{5}$$

Thus, the roots are:

$$x_1 = 3 + \sqrt{5} \approx 3 + 2.2 = 5.2$$
$$x_2 = 3 - \sqrt{5} \approx 3 - 2.2 = 0.8$$

Mark the roots on the x-axis and consider the regions into which the roots divide the x-axis (see Figure). They are $x < 0.8$, $0.8 < x < 5.2$, $x > 5.2$. For each of these regions choose a value of $x$ and see if $f(x) < 0$ or if $f(x) > 0$ holds. For the first region, we select $x = 0$. Substitute this value into $f(x)$:

$$0^2 - 6(0) + 4 = 0 - 0 + 4 = 4 > 0$$

Therefore

$$f(x) > 0 \quad \text{for} \quad x < 0.8$$

or more precisely

$$x < 3 - \sqrt{5} .$$

For the second region, $0.8 < x < 5.2$, we choose $x = 3$.

$$(3)^2 - 6(3) + 4 = 9 - 18 + 4 = -9 + 4 = -5 < 0$$

Therefore

$$f(x) < 0 \quad \text{for} \quad 0.8 < x < 5.2 \quad \text{or more exactly}$$
$$3 - \sqrt{5} < x < 3 + \sqrt{5} .$$

For the third region $x > 5.2$, we try $x = 6$.

$$6^2 - 6(6) + 4 = 36 - 36 + 4 = 0 + 4 = 4 > 0$$

Therefore

$$f(x) > 0 \quad \text{when} \quad x > 5.2;$$

that is,

$$f(x) > 0 \quad \text{when} \quad x > 3 + \sqrt{5} .$$

In conclusion, recalling that $f(x) = x^2 - 6x + 4$, we have found that $x^2 - 6x + 4 > 0$ for $x < 3 - \sqrt{5}$ and for $x > 3 + \sqrt{5}$. Furthermore, $x^2 - 6x + 4 < 0$ for $3 - \sqrt{5} < x < 3 + \sqrt{5}$. Note that the function is zero at the points $3 - \sqrt{5}$ and $3 + \sqrt{5}$.

**• PROBLEM 547**

Solve the inequality $x^2 + 9x - 7 \leq 0$.

Fig A

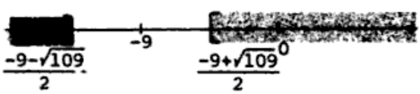
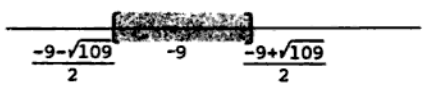



Fig. B




Solution: We solve the given inequality by the method of completing the square. To complete the square, take one half of the coefficient of $x$ and square it. Add this quantity to both sides of the inequality. Here it is $\frac{1}{2}(9) = 9/2$ and $(9/2)^2 = 81/4$

$$x^2 + 9x + \frac{81}{4} \leq 7 + \frac{81}{4} \qquad (1)$$

Write this quadratic expression as a binomial squared.

$$\left(x + \frac{9}{2}\right)^2 \leq \frac{109}{4} \qquad (2)$$

Subtracting $\frac{109}{4}$ from both sides and expressing it as $\left(\sqrt{\frac{109}{4}}\right)^2$ on the left side of equation (2)

$$\left(x + \frac{9}{2}\right)^2 - \left(\sqrt{\frac{109}{4}}\right)^2 \le 0$$

The expression $a^2 - b^2$ can be factored into $(a - b)(a + b)$. Similarly, for this example:

$$\left[\left(x + \frac{9}{2}\right) - \frac{\sqrt{109}}{2}\right]\left[\left(x + \frac{9}{2}\right) + \frac{\sqrt{109}}{2}\right] \le 0$$

Hence we know that, since the product here is nonpositive, either

Case (1) $\left(x + \frac{9}{2}\right) - \frac{\sqrt{109}}{2} \ge 0$ and $\left(x + \frac{9}{2}\right) + \frac{\sqrt{109}}{2} \le 0$

or

Case (2) $\left(x + \frac{9}{2}\right) - \frac{\sqrt{109}}{2} \le 0$ and $\left(x + \frac{9}{2}\right) + \frac{\sqrt{109}}{2} \ge 0$

Case (1) $x + \frac{9}{2} - \frac{\sqrt{109}}{2} \ge 0$ and $x + \frac{9}{2} + \frac{\sqrt{109}}{2} \le 0$

$x \ge \frac{-9 + \sqrt{109}}{2}$ and $x \le \frac{-9 - \sqrt{109}}{2}$

But this conjunction is logically false since no number can be larger than or equal to

$$\frac{-9 + \sqrt{109}}{2}$$

and at the same time be less than or equal to the smaller number

$$\frac{-9 - \sqrt{109}}{2}$$

(See Figure A). Thus, x cannot be a value in both sets at the same time. Therefore this case leads to the null or empty set.

Case (2) $x + \frac{9}{2} - \frac{\sqrt{109}}{2} \le 0$ and $x + \frac{9}{2} + \frac{\sqrt{109}}{2} \ge 0$

$x \le \frac{-9 + \sqrt{109}}{2}$ and $x \ge \frac{-9 - \sqrt{109}}{2}$

Diagrammatically, the solution set is given in Figure B. Thus, the solution set for this inequality is

$$\left\{x \,\middle|\, \frac{-9 - \sqrt{109}}{2} \le x \le \frac{-9 + \sqrt{109}}{2}\right\} = \left[\frac{-9 - \sqrt{109}}{2}, \frac{-9 + \sqrt{109}}{2}\right]$$

**• PROBLEM 548**

Solve the inequality

$$x - 6 > \frac{18 - 15x}{x^2 + 2x - 3} \quad .$$

**Solution:** We first subtract $\frac{18 - 15x}{x^2 + 2x - 3}$ from both sides of the inequality, obtaining

$$x - 6 - \frac{18 - 15x}{x^2 + 2x - 3} > 0.$$

In order to combine terms, we convert $x - 6$ into a fraction with $x^2 + 2x - 3$ as its denominator. Thus

$$\frac{(x^2 + 2x - 3)}{(x^2 + 2x - 3)} \cdot (x - 6) - \frac{18 - 15x}{x^2 + 2x - 3} > 0$$

Note that since $\frac{x^2 + 2x - 3}{x^2 + 2x - 3} = 1$, multiplication of $(x - 6)$ by this fraction does not alter the value of the inequality.

$$\frac{(x^2 + 2x - 3)(x - 6)}{x^2 + 2x - 3} - \frac{18 - 15x}{x^2 + 2x - 3} > 0$$

$$\frac{x^3 + 2x^2 - 3x - 6x^2 - 12x + 18}{x^2 + 2x - 3} - \frac{18 - 15x}{x^2 + 2x - 3} > 0$$

$$\frac{x^3 - 4x^2 - 15x + 18}{x^2 + 2x - 3} - \frac{18 - 15x}{x^2 + 2x - 3} > 0$$

$$\frac{x^3 - 4x^2 - 15x + 18 - 18 + 15x}{x^2 + 2x - 3} > 0$$

$$\frac{x^3 - 4x^2}{x^2 + 2x - 3} > 0$$

Now we factor numerator and denominator. Thus

$$\frac{x^2(x - 4)}{(x - 1)(x + 3)} > 0.$$

We now want all values of x which make $\frac{x^2(x - 4)}{(x - 1)(x + 3)}$ greater than zero. If $(x - 1) = 0$ or $(x + 3) = 0$ this fraction is undefined, thus we must place the restrictions

$$x - 1 \neq 0 \quad \text{and} \quad x + 3 \neq 0$$

or

$$x \neq 1 \quad \text{and} \quad x \neq -3.$$

Next we must eliminate all values of x which make $\frac{x^2(x - 4)}{(x - 1)(x + 3)}$ equal to zero (for we only want it to be greater than zero). The numerator will be zero if $x^2 = 0$ or $x - 4 = 0$, thus $x \neq 0$ and $x \neq 4$. We now have critical values $x = -3$, $x = 0$, $x = 1$, $x = 4$.

We must test values of x in all ranges bordering on these critical values: (a) $x < -3$, (b) $-3 < x < 0$, (c) $0 < x < 1$, (d) $1 < x < 4$, (e) $x > 4$, to find the ranges in which the inequality holds:

(a) To test if the inequality holds for $x < -3$, choose any value of $x < -3$, we will use $-4$, and replace x by this value in the given inequality:

$$\frac{x^2(x - 4)}{(x - 1)(x + 3)} > 0$$

$$\frac{(-4)^2\ (-4 - 4)}{(-4 - 1)(-4 + 3)} > 0$$

$$\frac{16\ (-8)}{(-5)(-1)} > 0$$

$$\frac{-128}{5} > 0$$

Since a negative number is not greater than zero, the range $x < -3$ is not part of the solution.

(b) To test if the inequality holds for $-3 < x < 0$, choose a value of x between 0 and $-3$, we will use $-1$, and replace x by this value in the inequality:

$$\frac{x^2(x-4)}{(x-1)(x+3)} > 0$$

$$\frac{(-1)^2(-1-4)}{(-1-1)(-1+3)} > 0$$

$$\frac{1(-5)}{(-2)(2)} > 0$$

$$\frac{-5}{-4} > 0$$

$$\frac{5}{4} > 0$$

Since 5/4 is indeed greater than zero, the range $-3 < x < 0$ is part of the solution.

(c) Testing if $0 < x < 1$ is part of the solution, we choose a value of x between 0 and 1, we will use $\frac{1}{2}$, and replace x by this value in the inequality:

$$\frac{x^2(x-4)}{(x-1)(x+3)} > 0$$

$$\frac{(\frac{1}{2})^2(\frac{1}{2}-4)}{(\frac{1}{2}-1)(\frac{1}{2}+3)} > 0$$

$$\frac{(1/4)(-3\frac{1}{2})}{(-\frac{1}{2})(3\frac{1}{2})} > 0$$

$$\frac{(1/4)(-7/2)}{(-\frac{1}{2})(7/2)} > 0$$

$$\frac{-7/8}{-7/4} > 0$$

$$-\frac{7}{8} \cdot -\frac{4}{7} > 0$$

$$\frac{1}{2} > 0$$

Since $\frac{1}{2}$ is indeed greater than zero, the range $0 < x < 1$ is part of the solution.

(d) Testing if $1 < x < 4$ is part of the solution, we choose a value of x between 1 and 4. We will use 2, and replace x by this value in the inequality:

$$\frac{x^2(x-4)}{(x-1)(x+3)} > 0$$

$$\frac{(2)^2(2-4)}{(2-1)(2+3)} > 0$$

$$\frac{4(-2)}{(1)(5)} > 0$$

$$\frac{-8}{5} > 0$$

Since a negative number is not greater than zero, the range $1 < x < 4$ is not part of the solution.

(e) Testing if $x > 4$ is part of the solution, we choose any value of x greater than 5, we will use 5, and replace x by this value in the inequality:

$$\frac{x^2(x-4)}{(x-1)(x+3)} > 0$$

$$\frac{(5)^2(5-4)}{(5-1)(5+3)} > 0$$

$$\frac{25\ (1)}{(4)(8)} > 0$$

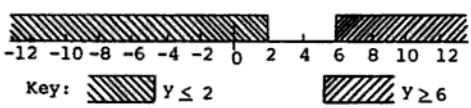
$$\frac{25}{32} > 0$$

Since 25/32 is indeed greater than zero, the range $x > 4$ is part of the solution.

Thus, the permissible ranges for which the inequality $x - 6 > \frac{18 - 15x}{x^2 + 2x - 3}$ holds are $-3 < x < 0$, $0 < x < 1$, $x > 4$.

**• PROBLEM 549**

> If $x$ is a real quantity, prove that the expression $\frac{x^2+2x-11}{2(x-3)}$ can have all numerical values except such as lie between 2 and 6.




**Solution:** Let the given expression be represented by y, so that

$$\frac{x^2+2x-11}{2(x-3)} = y\ ;$$

then cross-multiplying and transposing, we have

$$x^2+2x-11 = 2y(x-3)$$

$$x^2+2x-11 = 2xy - 6y$$
$$x^2+2x-11-2xy+6y = 0$$
$$x^2+2x-2xy+6y-11 = 0$$
$$x^2+2x(1-y)+6y-11 = 0,$$

or

$$x^2+2(1-y)x + (6y-11) = 0 \qquad (1)$$

Equation (1) is in the form $az^2 + bz + c = 0$, which is a quadratic equation. Hence, equation (1) is a quadratic equation, with $a = 1$, $b = 2(1-y)$, and $c = 6y-11$. In order that $x$ may have real values, the discriminant, $b^2-4ac$, must be positive; that is, in order that $x$ may have real values, $[2(1-y)]^2 - 4(1)(6y-11)$ must be positive; or $4(1-y)^2 - (24y-44)$ must be positive, i.e., $4(1-y)^2 - (24y-44) \geq 0$. Dividing by 4 and simplifying:

$$\frac{4(1-y)^2 - (24y-44)}{4} \geq \frac{0}{4}$$

$$\frac{4(1-y)^2 - 24y+44}{4} \geq 0$$

$$(1-y)^2 - 6y + 11 \geq 0$$
$$\left(1 - 2y + y^2\right) - 6y + 11 \geq 0$$
$$y^2 - 8y + 12 \geq 0 .$$

Factoring the left side of the inequality into a product of two polynomials:

$$(y - 6)(y - 2) \geq 0 .$$

Hence, the factors of this product must both be positive or both negative, since the entire product is positive.

Case 1: Both factors positive:

$$y - 6 \geq 0 \quad \text{and} \quad y - 2 \geq 0$$
$$y \geq 6 \quad \text{and} \quad y \geq 2$$

The two inequalities, $y \geq 6$ and $y \geq 2$, mean $y \geq 6 \cap y \geq 2$, and thus yield the single inequality $y \geq 6$, since this single inequality satisfies the two inequalities.

Case 2: Both factors negative:

$$y - 6 \leq 0 \quad \text{and} \quad y - 2 \leq 0$$
$$y \leq 6 \quad \text{and} \quad y \leq 2$$

The two inequalities, $y \leq 6$ and $y \leq 2$, mean $y \leq 6 \cap y \leq 2$, and thus yield the single inequality $y \leq 2$, since this single inequality satisfies the two inequalities. Therefore, $y$ may have real values only when $y \geq 6$ (Case 1) and $y \leq 2$ (Case 2). The real values of $y$ are indicated on the accompanying number line. Therefore, $y$ cannot lie between 2 and 6, but $y$ may have any other value.

# CHAPTER 19

# GRAPHING QUADRATIC EQUATIONS/ CONICS AND INEQUALITIES

**Basic Attacks and Strategies for Solving Problems in this Chapter.** **See pages 374 to 403 for step-by-step solutions to problems.**

Parabolas, circles, ellipses, and hyperbolas are conic sections. Procedures for using formulas and obtaining graphical representation of each of these conics is given below.

## PARABOLA

Equations of the standard form

$$y = ax^2 + bx + c,$$

where $a \neq 0$, have parabolas for graphs. The common method of graphing the solution set of any given quadratic equation is to first put the equation in standard form and then find a number of ordered pairs in a table that satisfy the equation and graph them.

The points associated with the intercepts and the vertex of a parabola are very useful in the procedure for graphing

$$y = ax^2 + bx + c.$$

The $y$-intercept is given by $y = c$ when $x = 0$ is substituted in the equation. The point associated with the intercept is $(0, c)$. The $x$-intercepts (if they exist) are found as a result of substituting $y = 0$ in the original equation and solving for $x$ using the quadratic formula. The points associated with these intercepts are of the form $(x, 0)$ and $(-x, 0)$.

The vertex is the highest or lowest point on a parabola and it always occurs on the graph when the $x$-coordinate is $x = -b/2a$, which we use to solve for $y$ in the equation for the parabola.

Using the points associated with the intercepts and the vertex one can sketch the graph of

$$y = ax^2 + bx + c,$$

the parabola. Note that if the coefficient of the $x^2$ term is positive, then the parabola is concave upward. On the other hand, if it is negative, then the parabola is concave downward.

Another way to locate the vertex of a parabola is by completing the square of the expression

$$ax^2 + bx,$$

where $a$ must be 1 or $-1$ in the equation

$$y = ax^2 + bx + c.$$

For example, one can complete the square of

$$y = -x^2 - 6x + 1$$

as follows:

$$y = -(x^2 + 6x + 9) + 1 + 9 = -(x + 3)^2 + 10.$$

Hence, the $x$-component of the vertex is $-3$. The $y$-component is found by substituting $x = -3$ in the original equation and solve to obtain $y = 10$. The vertex is $(-3, 10)$. The parabola will be concave downward. If the equation has the form

$$x = ay^2 + b,$$

then the parabola opens to the right if $a$ is positive and to the left if $a$ is negative. If

$$f(x) = -f(x)$$

then the axis of symmetry is the $x$-axis which is a horizontal line through the vertex. Notice that the graph is not a function.

## CIRCLE

The equation of a circle with center $(a, b)$ and radius $r$ is given by

$$(x - a)^2 + (y - b)^2 = r^2.$$

To graph the equation of a circle, the first step is to write the given equation in the above form such that the center and radius are easily determined. The next step is to plot the center and find and plot points that are $r$ distance in horizontal and vertical positions from the center. Finally, connect the plotted points with a smooth curve to form a circle.

## ELLIPSE

The ellipse is given by the equation

$$x^2/a^2 + y^2/b^2 = 1,$$

where $a \neq 0$, $b \neq 0$, in standard form. The ellipse will cross the $x$-axis at $(a, 0)$ and $(-a, 0)$ and cross the $y$-axis at $(0, b)$ and $(0, -b)$. If $a = b$ then the ellipse will be a circle. A reasonably accurate sketch of the graph of an ellipse can be obtained by plotting the above four points and connecting them with a smooth curve.

## HYPERBOLA

The graph of the equation

$$x^2/a^2 - y^2/b^2 = 1$$

is the standard form of a hyperbola centered at the origin with a horizontal transverse axis. The graph will have $x$-intercepts at $-a$ and $a$. The equation of the form

$$y^2/a^2 - x^2/b^2 = 1$$

with a vertical transverse axis has a graph that is a hyperbola centered at the origin with $y$-intercepts at $-a$ and $a$. To draw a graph, we can plot the points associated with the intercepts and construct a rectangle whose sides pass through these points. An important aid in sketching either of the preceding equations is to draw the asymptotes of the graph. The asymptotes can be found by drawing a line through curves of the rectangle whose sides pass through $(-a, 0)$, $(a, 0)$, $(0, -b)$, and $(0, b)$ on the axes. These points are the vertices of hyperbola.

The graphical method for the solution set of the equation

$$ax^2 + bx + c > 0 \quad \text{or} \quad ax^2 + bx + c < 0$$

entails plotting the graph of

$$ax^2 + bx + c = 0$$

and determining what set of points satisfy the original inequality. Once this is determined, the graph is completed by shading the area that represents the solution set.

# Step-by-Step Solutions to Problems in this Chapter, "Graphing Quadratic Equations/ Conics and Inequalities"

## PARABOLAS

**• PROBLEM 550**

Draw the graphs of $f(x) = x^2$, $g(x) = 3x^2$, and also $h(x) = \frac{1}{2}x^2$ on one set of coordinate axes.

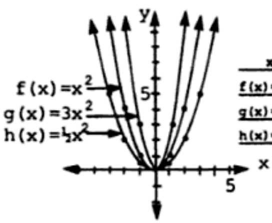



Fig. A

| x | 0 | 1 | -1 | 2 | -2 | 3 | -3 |
|---|---|---|---|---|---|---|---|
| $f(x)=x^2$ | 0 | 1 | 1 | 4 | 4 | 9 | 9 |
| $g(x)=3x^2$ | 0 | 3 | 3 | 12 | 12 | 27 | 27 |
| $h(x)=\frac{1}{2}x^2$ | 0 | ½ | ½ | 2 | 2 | 4½ | 4½ |

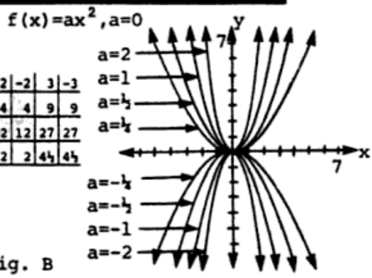



Fig. B

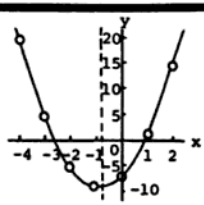
**Solution:** We construct a composite table showing the values of each function corresponding to selected values for x.

In the example, we graphed three instances of the function $f(x) = ax^2$, $a > 0$. For different values of a, how do the graphs compare? (Fig. A). Assigning a given value to a has very little effect upon the main characteristics of the graph. The coefficient a serves as a "stretching factor" relative to the y-axis. As a increases, the two branches of the curve approach the y-axis. The curve becomes "thinner". As a decreases, the curve becomes "flatter" and approaches the x-axis.

The graph of $f(x)^2 = ax$, $a \neq 0$, is called a parabola. (Fig. B). The point (0,0) is the vertex, or turning point, of the curve; the y-axis is the axis of symmetry. The value of a determines the shape of the curve. For $a > 0$ the parabola opens upward and for $a < 0$ the parabola opens downward.

**• PROBLEM 551**

Graph the function $3x^2 + 5x - 7$.



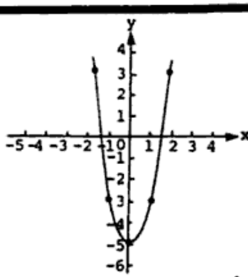
Solution: Let $y = 3x^2 + 5x - 7$. Substitute values for x and then find the corresponding values of y. This is done in the following table.

| x | $y = 3x^2 + 5x - 7$ |
|---|---|
| -4 | 21 |
| -3 | 5 |
| -2 | -5 |
| -1 | -9 |
| 0 | -7 |
| 1 | 1 |
| 2 | 15 |

These points are plotted and joined by a smooth curve in the figure.

• PROBLEM 552

Graph $y = 2x^2 - 5$.



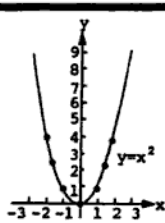

Solution: Graphs of the form $y = kx^2 + c$ are parabolas that stretch upward from a minimum point on the y-axis. From the table

| x | -2 | -1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| y | 3 | -3 | -5 | -3 | 3 |

we obtain the graph shown in the accompanying figure.

• PROBLEM 553

Sketch the graph of the function $\{(x, x^2)\}$.



**Solution:** Substitute values of x and obtain the corresponding values of $f(x) = x^2$. This was done in the following table:

| $x$ | $f(x) = x^2$ | $f(x)$ |
|---|---|---|
| -3 | $f(-3) = (-3)^2$ $= 9$ | 9 |
| -2 | $f(-2) = (-2)^2$ $= 4$ | 4 |
| -1 | $f(-1) = (-1)^2$ $= 1$ | 1 |
| 0 | $f(0) = 0^2$ $= 0$ | 0 |
| 1 | $f(1) = 1^2$ $= 1$ | 1 |
| 2 | $f(2) = 2^2$ $= 4$ | 4 |
| 3 | $f(3) = 3^2$ $= 9$ | 9 |

We plotted these points and then joined them to obtain the part of the graph shown in the figure.

**• PROBLEM 554**

Find the graph of $y = x^2 - 3x + 2$.

| $x$ | $y$ |
|---|---|
| -2 | 12 |
| -1 | 6 |
| 0 | 2 |
| 1 | 0 |
| 2 | 0 |
| 3 | 2 |
| 4 | 6 |
| 5 | 12 |

Fig. 1

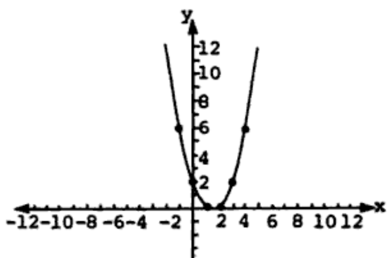



Fig. 2

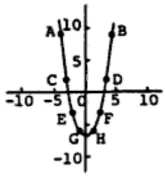
**Solution:** First, construct the chart given in Fig. 1, by choosing values of x and finding the corresponding value for y by substituting into the equation,

$$y = x^2 - 3x + 2.$$

Next, plot the corresponding ordered pairs on the coordinate system. Having done this, connect the points with a smooth curve as in Fig. 2.

If we had recognized the function as a parabola whose general shape is that of the given curve, then we could have done almost as well with three or four points.

Draw a graph of the set of ordered pairs which satisfy the function $f(x) = x^2 - 7$.



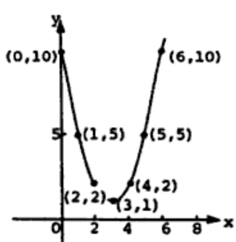

Solution: The following table lists a sufficient sequence of ordered pairs to determine the general nature of the curve:

| x | 0 | 1 | 2 | 3 | 4 | -1 | -2 | -3 | -4 |
|---|---|---|---|---|---|---|---|---|---|
| y = f(x) | -7 | -6 | -3 | 2 | 9 | -6 | -3 | 2 | 9 |

Plotting these points, we obtain the curve illustrated in the figure. Note also that the function is given by a quadratic equation with the coefficient of the $x^2$-term positive. This implies that the graph will be a parabola opening upward. Since for this function, $f(x) = f(-x)$, the graph will be symmetric with respect to the y-axis. The point with x-coordinate 0 will be the vertex and minimum point of the parabola. The graph should confirm what the table suggests; the minimum point is (0,7). The range of the function is then limited to real numbers equal to or greater than -7. The domain of the parabola is the set of real numbers. The points A(-4,9) and B(4,9) in the figure are said to be symmetric with respect to the y-axis. Similarly, C and D, E and F, and G and H are symmetric with respect to the y-axis.

• PROBLEM 556

Construct the graph of the function defined by $y = x^2 - 6x + 10$.



Solution: We are given the function $y = x^2 - 6x + 10$. The most general form of the quadratic function is

$y = ax^2 + bx + c$ where a, b, and c are constants. If a is positive, the curve opens upward and it is U-shaped. If a is negative, the curve opens downward and it is inverted U-shaped.

Since $a = 1 > 0$ in the given equation, the graph is a parabola that opens upward. To determine the pairs of values (x,y) which satisfy this equation, we express the quadratic function in terms of the square of a linear function of x.

$$y = x^2 - 6x + 10 = x^2 - 6x + 9 + 1$$

$$= (x - 3)^2 + 1$$

y is least when $x - 3 = 0$ This is true because the square of any number, be it positive or negative, is a positive number. Therefore y would always be greater than or equal to one. Thus the minimum value of y is one when $x - 3 = 0$ or $x = 3$.

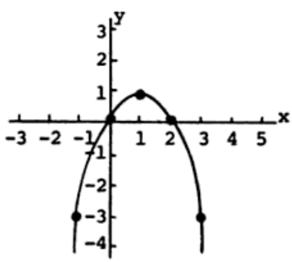
In order to plot the curve, we select values for x and calculate the corresponding y values. (See the table.)

| x | $x^2 - 6x + 10 =$ | y |
|---|---|---|
| 0 | $(0)^2 - 6(0) + 10$ | 10 |
| 1 | $(1)^2 - 6(1) + 10$ | 5 |
| 2 | $(2)^2 - 6(2) + 10$ | 2 |
| 3 | $(3)^2 - 6(3) + 10$ | 1 |
| 4 | $(4)^2 - 6(4) + 10$ | 2 |
| 5 | $(5)^2 - 6(5) + 10$ | 5 |
| 6 | $(6)^2 - 6(6) + 10$ | 10 |

The points and graphs determined by the table are shown in the accompanying figure.

**• PROBLEM 557**

Graph $x^2 + y - 2x = 0$.



**Solution:** First we solve for $y$. Subtract $x^2$ from both sides of the given equation.

$$\cancel{x^2} + y - 2x - \cancel{x^2} = 0 - x^2$$
$$y - 2x = -x^2$$

Add $2x$ to both sides of this equation.

$$y - \cancel{2x + 2x} = -x^2 + 2x$$
$$y = -x^2 + 2x$$

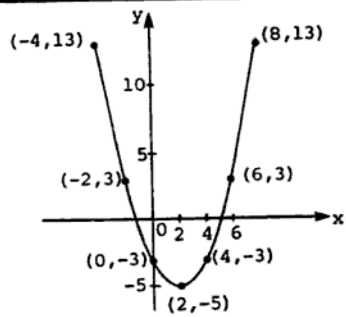
Then we construct the table by substituting values of $x$ into this derived equation to find corresponding values of $y$,

| x | -2 | -1 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|
| y | -8 | -3 | 0 | 1 | 0 | -3 | -8 |

from which we obtain the Figure.

**• PROBLEM 558**

Construct the graph of $\{(x,y) \mid y = \frac{1}{2}x^2 - 2x - 3\}$.




**Solution:** In general, the graph of the quadratic function $y = ax^2 + bx + c$ is a parabola. Some of its properties are:

1. If $a > 0$, the graph opens upward.
2. If $a < 0$, the graph opens downward.

We note that $a = \frac{1}{2}$ in the equation $y = \frac{1}{2}x^2 - 2x - 3$. Hence the graph is a parabola opening upward since $a > 0$. In order to determine the vertex $(x_v, y_v)$ of the parabola, we complete the square by the following procedure:

Factor out $\frac{1}{2}$ from $y = \frac{1}{2}x^2 - 2x - 3$. Then $\frac{1}{2}x^2 - 2x - 3 = \frac{1}{2}(x^2 - 4x - 6)$ Complete the square of $x^2 - 4x$. This is done by taking $\frac{1}{2}$ of the coefficient of $x$ and squaring it. Thus, the constant term is

$$[\tfrac{1}{2}(-4)]^2 = (-2)^2 = 4.$$

To keep the same equation we must retain $-6$. Thus, we express it as $4 - 10$. Then, $\frac{1}{2}(x^2 - 4x - 6) = \frac{1}{2}(x^2 - 4x + 4 - 10)$.

Factor $(x^2 - 4x + 4)$ into $(x-2)(x-2) = (x-2)^2$.

Then, $\frac{1}{2}(x^2 - 4x + 4 - 10) = \frac{1}{2}\left[(x-2)^2 - 10\right]$.

Now, $y = \frac{1}{2}\left[(x-2)^2 - 10\right]$. Since the parabola opens upward, the vertex is the value where $y$ is minimum. $(x-2)^2$ will always be positive since the square of a positive or of a negative number is always positive. Therefore, the minimum value of $y$ will occur when $(x-2) = 0$. Consequently $y$ is least when $x-2 = 0$ or when $x = 2$. To find $y$, substitute $x = 2$ into $y = \frac{1}{2}[(x-2)^2 - 10]$.

$$\begin{aligned} y &= \tfrac{1}{2}[(2-2)^2 - 10] \\ &= \tfrac{1}{2}[0^2 - 10] \\ &= \tfrac{1}{2}[-10] \\ &= -5 . \end{aligned}$$

Hence, the vertex is (2,-5).

We now assign numbers to $x$ and calculate the corresponding $y$-values to obtain points on the parabola. This is done in the following table:

| $x$ | $\frac{1}{2}x^2-2x-3$ | $y$ |
|---|---|---|
| -4 | $\frac{1}{2}(-4)^2-2(-4)-3$ | 13 |
| -2 | $\frac{1}{2}(-2)^2-2(-2)-3$ | 3 |
| 0 | $\frac{1}{2}(0)^2-2(0)-3$ | -3 |
| 2 | $\frac{1}{2}(2)^2-2(2)-3$ | -5 |
| 4 | $\frac{1}{2}(4)^2-2(4)-3$ | -3 |
| 6 | $\frac{1}{2}(6)^2-2(6)-3$ | 3 |
| 8 | $\frac{1}{2}(8)^2-2(8)-3$ | 13 |

The graph is shown in the accompanying figure.

**• PROBLEM 559**

Show that the quadratic equation $y = 2x^2 - 20x + 25$ is the equation of a parabola.

Solution: If we can write $y = 2x^2 - 20x + 25$ in the standard form $y = a(x-h)^2 + k$, we will show that its graph is a parabola. The axis of symmetry will be the line $x = h$. The vertex will be at $(h,k)$. Notice that the standard form has a term with a perfect square in it. We use a method known as completing the square to rewrite

$$y = 2x^2 - 20x + 25$$

(a) Subtract the constant term, 25, from both members.

$$y - 25 = 2x^2 - 20x$$

(b) Factor 2 from the right member.

$$y - 25 = 2(x^2 - 10x)$$

(c) We need to add 25 within the parentheses since 25 is the square of one-half the coefficient of the term in x. If we do, then we must add 50 to the left member to maintain equality, since we have in fact added two times the quantity added in the parentheses ($2 \times 25 = 50$) to the right member.

$$y + 25 = 2(x^2 - 10x + 25)$$

(d) The expression within the parentheses is now a perfect square.

$$y + 25 = 2(x - 5)^2$$

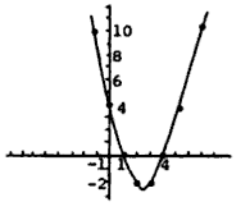
(e) Next we subtract 25 from each member.

$$y = 2(x - 5)^2 - 25$$

This is now in standard form $y = a(x - h)^2 + k$. Here $a = 2$, $h = 5$, and $k = -25$. Thus, the graph of $y = 2x^2 - 20x + 25$ is a parabola. Its axis of symmetry is the line $x = 5$; its vertex is at $(5,-25)$.

**• PROBLEM 560**

Sketch the graph of the quadratic function $y = x^2 - 5x + 4$.




Solution: Since the coefficient of the $x^2$-term of the quadratic function is positive, its graph will be a parabola opening upward. Therefore the minimum point occurs at the vertex. The x-coordinate of the vertex is given by $-\dfrac{\text{coefficient of } x}{2 \cdot \text{coefficient of } x^2} = -\dfrac{(-5)}{2(1)} = \dfrac{5}{2}$. The y-coordinate is then $y = \left(\dfrac{5}{2}\right)^2 - 5\left(\dfrac{5}{2}\right) + 4 = -\dfrac{9}{4}$. The minimum point is $(5/2\ ,\ -9/4)$. The axis of symmetry is the vertical line through the point $(5/2, -9/4)$. The y-intercept, found by setting $x = 0$, is $y = 0^2 - 5(0) + 4 = 4$, i.e., $(0,4)$. The point symmetric to $(0,4)$ is a point with the same y-coordinate, 4, but 5/2 units to the right of the axis of symmetry, line $x = \dfrac{5}{2}$. The point is therefore $(5,4)$. The x-intercepts satisfy

$$0 = x^2 - 5x + 4$$

$$0 = (x - 4)(x - 1)$$

$$\begin{array}{r|r} x - 4 = 0 & x - 1 = 0 \\ x = 4 & x = 1 \end{array}$$

The x-intercepts are therefore $(1,0)$ and $(4,0)$. These points are also symmetric with respect to the axis of symmetry. Each point has y-coordinate 0 and is either $\frac{3}{2}$ units to the left or right of the axis line $x = \frac{5}{2}$.

**• PROBLEM 561**

The graph of the quadratic function $y = ax^2 + bx + c$ is a parabola. Find the equation of a parabola passing through the points $(-1, 11)$, $(1, 3)$, and $(2, 5)$, by determining the values of a, b, and c from the given data.

**Solution:** Each of the three points given lies on the parabola and therefore each one must satisfy the quadratic function for a parabola,

$$ax^2 + bx + c = y.$$

For each point we substitute the coordinates of x and y into the quadratic function,

For (- 1, 11) $a(-1)^2 + b(-1) + c = 11$

$$a - b + c = 11 \qquad (1)$$

For (1,3) $a(1)^2 + b(1) + c = 3$

$$a + b + c = 3 \qquad (2)$$

For (2,5) $a(2)^2 + b(2) + c = 5$

$$4a + 2b + c = 5 \qquad (3)$$

We have obtained a system of three equations with three unknowns.

$$a - b + c = 11 \qquad (1)$$

$$a + b + c = 3 \qquad (2)$$

$$4a + 2b + c = 5 \qquad (3)$$

We can eliminate by by adding (1) and (2). We obtain a new equation in a and c.

$$a - b + c = 11 \qquad (1)$$

$$\underline{a + b + c = 3} \qquad (2)$$

$$2a + 2c = 14 \qquad (4)$$

We have one equation in two unknowns. We need another equation in a and c before we can solve for a or c.

Eliminate b from two other equations. Let us choose (2) and (3).

$$a + b + c = 3 \qquad (2)$$

$$4a + 2b + c = 5 \qquad (3)$$

Multiply (2) by - 2 in order to eliminate the variable b. Then add (5) and (3)

$$-2a - 2b - 2c = -6 \qquad (5)$$

$$\underline{4a + 2b + c = 5} \qquad (3)$$

$$2a - c = -1 \qquad (6)$$

Now we have two equations (4) and (6) in two unknowns, a and c.

$$2a + 2c = 14 \qquad (4)$$

$$2a - c = -1 \qquad (6)$$

Subtract equation (6) from (4) to eliminate a.

$$2a + 2c = 14 \qquad (4)$$

$$-\ \underline{2a - c = -1} \qquad (6)$$

$$3c = 15 \qquad (7)$$

Divide (7) by 3

$$c = 5$$

Substitute c into either (4) or (6) to find the value of a. Choose (4)

$$2a + 2c = 14 \qquad (4)$$

$$2a + 2(5) = 14$$

$$2a + 10 = 14$$

$$2a = 4 \qquad (8)$$

Divide (8) by 2

$$a = 2$$

To find b substitute a and c into any of the three original equations (1), (2), or (3). Let us choose (2).

$a = 2;\ c = 5$

$$a + b + c = 3 \qquad (2)$$
$$2 + b + 5 = 3$$

$$b + 7 = 3$$

$$b = -4$$

Therefore the solution of the original system is

$a = 2$

$b = -4$

$c = 5$

The equation of a parabola is

$y = ax^2 + bx + c$

For this particular parabola the equation is

$y = 2x^2 - 4x + 5,$

by substituting the values of a, b, and c.

Each of the three given points satisfy this equation.

Check

For $(-1, 11)$

$$11 = 2(-1)^2 - 4(-1) + 5$$

$$11 = 2 + 4 + 5$$

$$11 = 11$$

For (1, 3)

$$3 = 2(1)^2 - 4(1) + 5$$

$$3 = 2 - 4 + 5$$

$$3 = 3$$

For (2, 5)

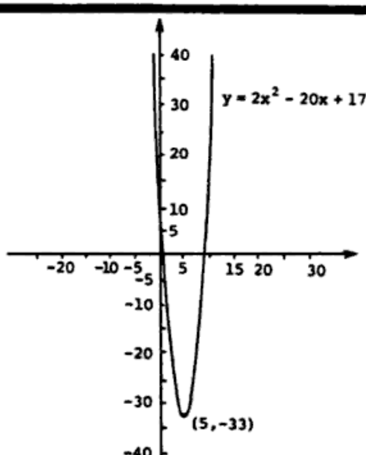
$$5 = 2(2)^2 - 4(2) + 5$$

$$5 = 8 - 8 + 5$$

$$5 = 5$$

**• PROBLEM 562**

What is the minimum value of the expression $2x^2 - 20x + 17$?




Solution: Consider the function $y = 2x^2 - 20x + 17$. This function is defined by a second degree equation. The coefficient of its $x^2$ term is positive. Hence the curve is a parabola opening upward. Thus, the minimum point of this curve occurs at the vertex. The x-coordinate is equal to

$$-\frac{\text{coefficient of } x \text{ term}}{2 \cdot \text{coefficient of } x^2 \text{ term}} = -\frac{b}{2a} = -\frac{(-20)}{2(2)} = \frac{20}{4} = 5.$$

For $x = 5$, $y = 2(5)^2 - 20(5) + 17 = -33$. Therefore the minimum value of the expression $2x^2 - 20x + 17$ for any value of x is $-33$. This minimum value is assumed only when $x = 5$.

**• PROBLEM 563**

Find the coordinates of the maximum point of the curve

$y = -3x^2 - 12x + 5$, and locate the axis of symmetry.

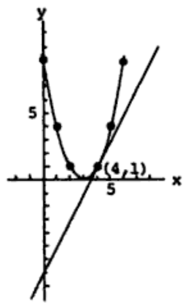
Solution: The curve is defined by a second degree equation. The coefficient of the $x^2$ term is negative. Hence, the graph of this curve is a parabola opening downward. The maximum point of the curve occurs at the vertex and has the x-coordinate:

$$-\frac{\text{coefficient of x term}}{2(\text{coefficient of } x^2 \text{ term})} = -\frac{b}{2a} = -\frac{-12}{2(-3)} = \frac{12}{-6} = -2.$$

For $x = -2$, $y = -3(-2)^2 - 12(-2) + 5 = 17$. Hence the coordinates of the vertex are $(-2,17)$. The curve is symmetric with respect to the vertical line through its vertex. The axis of symmetry of this curve is the vertical line through the point $(-2,17)$, i.e., the line $x = -2$.

**• PROBLEM 564**

Find the equation of the tangent to the parabola $y = x^2 - 6x + 9$, if the slope of the tangent equals 2.




Solution: The equation of a straight line is $y = mx + k$ where m is the slope and k is the y-intercept. The equation

$$y = 2x + k \qquad (1)$$

represents a family of parallel lines with slope 2, some of which intersect the parabola in two points, others which have no point of intersection with the parabola, and just one which intersects the parabola in only one point. The problem is to find the value of k so that the graph of Equation 1 intersects the parabola in just one point. If we solve the system

$$y = 2x + k \qquad (1)$$

$$y = x^2 - 6x + 9 \qquad (2)$$

by substitution, we get for the first step

$$2x + k = x^2 - 6x + 9 \qquad \text{or}$$

$$x^2 - 8x + 9 - k = 0 \qquad (3)$$

This is a quadratic equation of the form $ax^2 + bx + c = 0$. The discriminant determines the nature of the roots when $ax^2 + bx + c = 0$. The condition that Equation 3 has but one solution is that the discriminant, $b^2 - 4ac$ equals 0. Therefore, if $a = 1$, $b = -8$, $c = 9 - k$, then $b^2 - 4ac = 64 - 4(9 - k) = 0$ or $k = -7$.

Substituting this value of k in equation 1, we have $y = 2x - 7$ which is the equation of the tangent to the given parabola when the slope of the tangent is equal to 2. The figure is the graph of the parabola and the tangent. The student may verify that the point of contact is (4, 1). This is shown by substituting (4, 1) into

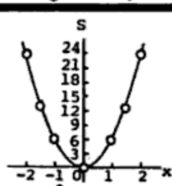
$$y = 2x - 7 = x^2 - 6x + 9$$

$$1 = 2(4) - 7 = 4^2 - 6(4) + 9$$

$$1 = \qquad 1 = 1$$

**• PROBLEM 565**

The surface S of a cube is given by the formula $S = 6x^2$, where x represents the length of an edge. Graph S as a function of x.



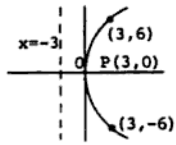

Solution: In the formula $S = 6x^2$, negative values of x may be used as well as positive ones. The points determined by these negative values of x belong to the graph of $S = 6x^2$, although they have no meaning in relation to the cube. The table of values from which the graph in the figure was constructed follows:

| x | $S = 6x^2$ | S |
|---|---|---|
| -2 | $S = 6(-2)^2 = 6(4) = 24$ | 24 |
| $-\frac{3}{2}$ | $S = 6\left(-\frac{3}{2}\right)^2 = 6\left(\frac{9}{4}\right) = \frac{27}{2}$ | $\frac{27}{2}$ |
| -1 | $S = 6(-1)^2 = 6(1) = 6$ | 6 |
| 0 | $S = 6(0)^2 = 6(0) = 0$ | 0 |
| 1 | $S = 6(1)^2 = 6(1) = 6$ | 6 |
| $\frac{3}{2}$ | $S = 6\left(\frac{3}{2}\right)^2 = 6\left(\frac{9}{4}\right) = \frac{27}{2}$ | $\frac{27}{2}$ |
| 2 | $S = 6(2)^2 = 6(4) = 24$ | 24 |

Note that the axis of ordinates is labeled S. Moreover, units of different size are used on the two axes; this may be done if convenient, but should be avoided where possible.

**• PROBLEM 566**

Discuss the graph of the equation $y^2 = 12x$.



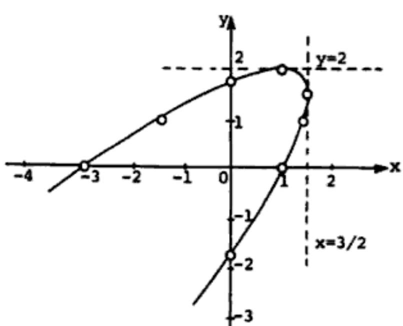

Solution: The equation written as $x = \frac{1}{12}y^2$ is a quadratic equation with the coefficient of the $y^2$ term positive. Therefore the graph is a parabola opening to the right. Since $f(x) = -f(x)$ the parabola is symmetric with respect to the x-axis. Point (0,0) satisfies the equation and lies on the axis of symmetry. Hence the vertex of the parabola is at (0,0), (see figure). The focus of the parabola lies on the axis of symmetry, $y = 0$, at the point $(p,0)$ where $4p$ = coefficient of x in the original equation: $4p = 12$, $p = 3$. Therefore the focus is at (3,0). The directrix is the vertical line $x = -p = -3$. When $x = 3$, $y = 12x = 12(3) = \pm 6$. Therefore the points (3,6) and (3,-6) are points on the graph. The graph of this parabola is not the graph of a function since for any given value of x there is more than one corresponding value of y.

**• PROBLEM 567**

Discuss the rational integral equation

$$x^2 - 2xy + y^2 + 2x - 3 = 0,$$

and plot its graph.




Solution: We solve this equation for y in terms of x.

$$x^2 - 2xy + y^2 + 2x - 3 = 0 \qquad (1)$$

Subtract $x^2$ from both sides of equation (1):

$$\cancel{x^2} - 2xy + y^2 + 2x - 3 - \cancel{x^2} = 0 - x^2$$

$$-2xy + y^2 + 2x - 3 = -x^2 \qquad (2)$$

Subtract (2x-3) from both sides of equation (2):

$$-2xy + y^2 + 2x - 3 -(2x - 3) = -x^2 - (2x - 3)$$

$$-2xy + y^2 + \cancel{2x} - \cancel{3} - \cancel{2x} + \cancel{3} = -x^2 - 2x + 3$$

$$y^2 - 2xy = -x^2 - 2x + 3 \qquad (3)$$

To complete the square on the left side of equation (3), take half the coefficient of $y$ and square it.

$$[\tfrac{1}{2}(-2x)]^2 = [-x]^2 = x^2 .$$

Add this value to both sides of equation (3):

$$y^2 - 2xy + x^2 = -\cancel{x^2} - 2x + 3 + \cancel{x^2}$$

$$y^2 - 2xy + x^2 = -2x + 3$$

$$y^2 - 2xy + x^2 = 3 - 2x$$

or

$$(y - x)^2 = 3 - 2x .$$

Take the square root of each side:

$$\sqrt{(y-x)^2} = \pm\sqrt{3 - 2x}$$

$$y - x = \pm\sqrt{3 - 2x}$$

Add $x$ to both sides:

$$y - \cancel{x} + \cancel{x} = x \pm \sqrt{3 - 2x}$$

$$y = x \pm \sqrt{3 - 2x} \qquad (4)$$

This explicit relation shows immediately that: (a) for every real value of $x$ less than 3/2, there are two distinct real values of $y$; (b) for $x = 3/2$, there is only one y-value; namely, $y = 3/2$. (c) for $x > 3/2$, $y$ is complex. Hence, we know that the graph of equation (1) will not extend to the right of the line $x = 3/2$. In addition, relation (4) enables us to compute the two values of $y$ corresponding to each permissible value of $x$; thus, when $x = 0$, $y = \pm\sqrt{3}$, and when $x = 1$, we get $y = 2$ and $y = 0$. Substitute values for $x$ and find the corresponding value of $y$. This is done in the following table:

| $x$ | $x \pm \sqrt{3-2x} = y$ |
|---|---|
| $-3$ | $-3 \pm \sqrt{3-2(-3)} = -3 \pm \sqrt{3+6} = -3 \pm \sqrt{9} = -3 \pm 3 = 0, -6$ |
| $-\frac{3}{2}$ | $-\frac{3}{2} \pm \sqrt{3-2(-\frac{3}{2})} = -\frac{3}{2} \pm \sqrt{3+3} = -\frac{3}{2} \pm \sqrt{6} = -\frac{3}{2} \pm 2.45 = 0.95, -3.95$ |
| $0$ | $0 \pm \sqrt{3-2(0)} = 0 \pm \sqrt{3-0} = 0 \pm \sqrt{3} = \pm\sqrt{3} = \pm 1.73$ |
| $1$ | $1 \pm \sqrt{3-2(1)} = 1 \pm \sqrt{3-2} = 1 \pm \sqrt{1} = 1 \pm 1 = 2, 0$ |
| $\frac{3}{2}$ | $\frac{3}{2} \pm \sqrt{3-2(\frac{3}{2})} = \frac{3}{2} \pm \sqrt{3-3} = \frac{3}{2} \pm \sqrt{0} = \frac{3}{2} \pm 0 = \frac{3}{2}$ |

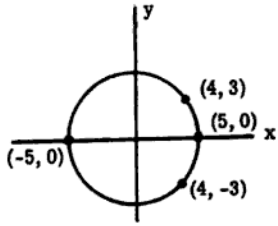
These points may be plotted and joined by a smooth curve.

The graph obtained with the help of the above discussion is shown in the figure. By the methods of analytic geometry, it may be shown that this curve is a parabola, and additional characteristics of the curve may be determined.

## CIRCLES, ELLIPSES AND HYPERBOLAS

**• PROBLEM 568**

Discuss the graph of the equation $x^2 + y^2 = 25$.




Solution: This is an equation of the form $x^2 + y^2 = r^2$, and therefore its graph is a circle with radius $r = 5$ and center at the origin (see figure). Note that the graph does not represent a function since, except for $x = -5$ or $x = 5$, each permissible value of x is associated with two values of y. For example, for $x = 4$, we have the ordered pairs (4,3) and (4,-3). The domain of this function is $\{x \mid -5 \le x \le 5\}$. The range of this function is $\{y \mid -5 \le y \le 5\}$.

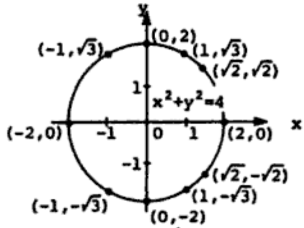
**• PROBLEM 569**

Sketch the graph of the equation $x^2 + y^2 = 4$.

Solution: Substitute values for x and then find the corresponding values of y. This is done in the following table:

| $x$ | $y = \pm\sqrt{4 - x^2}$ |
|---|---|
| $-1$ | $\pm\sqrt{3} = \pm 1.73$ |
| $0$ | $\pm 2$ |
| $1$ | $\pm\sqrt{3} = \pm 1.73$ |
| $\sqrt{2} = 1.41$ | $\pm\sqrt{2} = \pm 1.41$ |

These points have been plotted and a smooth curve has been drawn through them in the figure. We could plot more points and then "fill in" the rest of the graph, but instead we will use some of our knowledge of the coordinate plane. Our given equation is equivalent to the equation $\sqrt{x^2 + y^2} = 2$. In the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$, where d is the distance be-




tween the points $(x_1, y_1)$ and $(x_2, y_2)$, let the origin or $(0,0) = (x_1, y_1)$ and let $(x,y) = (x_2, y_2)$. Therefore,

$$d = \sqrt{(x - 0)^2 + (y - 0)^2}$$

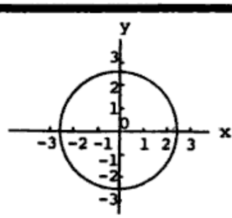
$$d = \sqrt{x^2 + y^2}\ .$$

Hence, the number $\sqrt{x^2 + y^2}$ is the distance between the point $(x,y)$ and the origin. Thus, in words, our equation says, "The distance between the point $(x,y)$ and the origin is 2." Clearly, the set of points that are two units from the origin is the circle whose center is the origin and whose radius is 2. This circle is therefore the graph of our given equation, and we have drawn it in the figure.

Most of the graphs of equations that we deal with in mathematics are "one-dimensional" figures, such as the circle we just drew. In these cases, we say that the graph is a curve. Not all simple looking equations in x and y, however, have graphs that are simple curves.

**• PROBLEM 570**

Graph the equation $2x^2 + 2y^2 - 13 = 0$.




**Solution:** In order to verify that this is the equation of a circle, we put the equation in the standard form: add 13 to both sides of the given equation.

$$2x^2 + 2y^2 - 13 + 13 = 0 + 13$$

$$2x^2 + 2y^2 = 13$$

Divide both sides of this equation by 2.

$$\frac{2x^2 + 2y^2}{2} = \frac{13}{2}$$

or

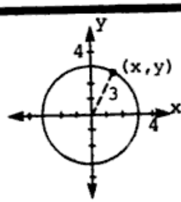
$x^2 + y^2 = \frac{13}{2}$, which is the standard form for the equation of a circle with its center at the origin (0,0) and

$$\text{radius} = r = \sqrt{\frac{13}{2}} = \frac{\sqrt{13}}{\sqrt{2}} = \frac{\sqrt{13}}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{26}}{2}.$$

Therefore, the radius of the circle is approximately $\frac{5.1}{2}$ or 2.55. The graph is represented in the Figure.

• PROBLEM 571

Find the equation for the circle whose center is at the origin and whose radius is 3.




Solution: A circle is the set of all points in a plane at a given distance from a fixed point. The fixed point is called the center of the circle and the measure of the given distance is called the radius of the circle. Thus to find the equation for the circle whose center is at the origin and whose radius is 3, we wish to find the equation of all points at a distance of 3 from the origin, (0,0).

The distance formula for the distance between two points $(x_1,y_1)$ and $(x_2,y_2)$ is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

In our case $d = 3, (x_1,y_1) = (0,0)$. Thus

$$3 = \sqrt{(x - 0)^2 + (y - 0)^2}$$

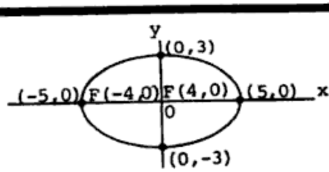
$$3 = \sqrt{x^2 + y^2}$$

Squaring both sides, $x^2 + y^2 = 9$.

Hence the equation for the circle whose center is at the origin, with radius 3, is $x^2 + y^2 = 9$.

• PROBLEM 572

Discuss the graph of $\frac{x^2}{25} + \frac{y^2}{9} = 1$.




Solution: Since this is an equation of the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, with $a = 5$ and $b = 3$, it represents an ellipse. The simplest way to sketch the curve is to find its intercepts.

If we set x = 0, then

$$y = \sqrt{\left(1 - \frac{x^2}{25}\right)9} = \sqrt{\left(1 - \frac{0^2}{25}\right)9} = \pm 3$$

so that the y-intercepts are at (0,3) and (0,-3). Similarly, the x-intercepts are found for y = 0:

$$x = \sqrt{\left(1 - \frac{y^2}{9}\right)25}$$

$$= \sqrt{\left(1 - \frac{0^2}{9}\right)25}$$

$$= \pm 5$$

to be at (5,0) and (-5,0) (see figure). To locate the foci we note that

$$c^2 = a^2 - b^2 = 5^2 - 3^2$$

$$c^2 = 25 - 9 = 16$$

$$c = \pm 4.$$

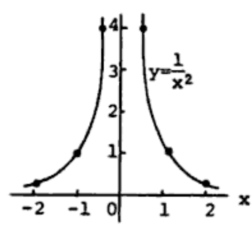
The foci lie on the major axis of the ellipse. In this case it is the x-axis since a = 5 is greater than b = 3. Therefore, the foci are ($\pm c$,0), that is, at (-4,0) and (4,0). Therefore, the foci are at (-4,0) and (4,0). The sum of the distances from any point on the curve to the foci is 2a = 2(5) = 10.

**• PROBLEM 573**

Sketch the graph of the function $\{(x, 1/x^2)\}$.

**Solution:** Substitute different values of x and find the corresponding value of y or $1/x^2$.

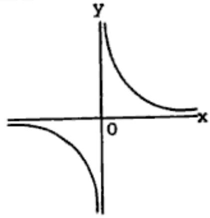
| x | $y = 1/x^2$ |
|---|---|
| -2 | $\frac{1}{(-2)^2} = \frac{1}{4}$ |
| -1 | $\frac{1}{(-1)^2} = \frac{1}{1} = 1$ |
| $-\frac{1}{2}$ | $\frac{1}{(-\frac{1}{2})^2} = \frac{1}{\frac{1}{4}} = 4$ |
| $\frac{1}{2}$ | $\frac{1}{(\frac{1}{2})^2} = \frac{1}{\frac{1}{4}} = 4$ |
| 1 | $\frac{1}{1^2} = \frac{1}{1} = 1$ |
| 2 | $\frac{1}{2^2} = \frac{1}{4}$ |




The graph is shown in the figure.

• PROBLEM 574

Draw the graph of $xy = 6$.



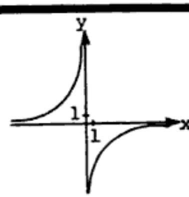

Solution: Since the product is positive the values of $x$ and $y$ must have the same sign, that is, when $x$ is positive $y$ must also be positive and when $x$ is negative then $y$ is also negative. Moreover, neither $x$ nor $y$ can be zero(or their product would be zero not 6), so that the graph never touches the coordinate axes. Solve for $y$ and we obtain $y = 6/x$. Substituting values of $x$ into this equation we construct the following chart:

| x: | -6 | -3 | -2 | -1 | 1 | 2 | 3 | 6 |
|---|---|---|---|---|---|---|---|---|
| y: | -1 | -2 | -3 | -6 | 6 | 3 | 2 | 1 |

The graph is obtained by plotting the above points and then joining them with a smooth curve, remembering that the curve can never cross a coordinate axis. The graph of the equation, $xy = k$, is a hyperbola for all nonzero real values of $k$. If $k$ is negative, then $x$ and $y$ must have opposite signs, and the graph is in the second and fourth quadrants as opposed to the first and third.

• PROBLEM 575

Graph the equation $xy = -4$.




Solution: The graph of the equation, $xy = c$, is a hyperbola for all non-zero real values of $c$. In this case, $c = -4$ and thus the particular equation is $xy = -4$. Since the product is negative, the values of $x$ and $y$ must have different signs. If $x$ is positive, then $y$ is negative and so part of the graph lies in quadrant IV. On the other hand, if $x$ is negative, then $y$ is positive and the other part of the hyperbola is located in quadrant II. If we solve for $y$, then $y = -4/x$ and $x \neq 0$. Then the graph never touches the $y$-axis. Thus the line $x = 0$ is an asymptote of the graph. On the other hand, solving for $x$, we have $x = -4/y$ and $y \neq 0$. Thus the graph never crosses the x-axis and the line $y = 0$ is an asymptote. We now prepare a table of values by selecting values for $x$ and finding the corresponding values for $y$. See table and graph.

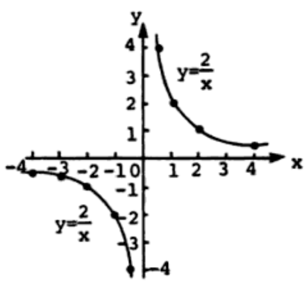
| x | $\frac{-4}{x}$ = | y |
|---|---|---|
| -3 | $\frac{-4}{-3}$ | $\frac{4}{3} = 1\frac{1}{3}$ |
| -2 | $\frac{-4}{-2}$ | 2 |
| -1 | $\frac{-4}{-1}$ | 4 |
| 1 | $\frac{-4}{1}$ | -4 |
| 2 | $\frac{-4}{2}$ | -2 |
| 3 | $\frac{-4}{3}$ | $\frac{-4}{3} = -1\frac{1}{3}$ |

**• PROBLEM 576**

**Sketch the graph of the equation y = 2/x.**

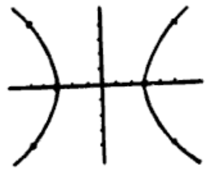
Solution: Substitute values for x and then find the corresponding values for y. This is done in the following table.

| x | $y = \frac{2}{x}$ |
|---|---|
| -4 | $-\frac{1}{2}$ |
| -3 | $-\frac{2}{3}$ |
| -2 | -1 |
| -1 | -2 |
| $-\frac{1}{2}$ | -4 |
| $\frac{1}{2}$ | 4 |
| 1 | 2 |
| 2 | 1 |
| 3 | $\frac{2}{3}$ |
| 4 | $\frac{1}{2}$ |




The graph is shown in the figure. This graph is an example of an equilateral hyperbola. Notice in the graph that, when x takes on larger and larger positive values, y gets closer and closer to 0. When x takes on larger and larger negative values, y also gets closer and closer to 0. Also, when x gets closer and closer to 0, y either takes on larger and larger positive values or larger and larger negative values. Note also that x cannot be 0, since $y = \frac{2}{x} = \frac{2}{0}$ is not defined.

Discuss the graph of $\frac{x^2}{9} - \frac{y^2}{9} = 1$.



Solution: $\frac{x^2}{9} - \frac{y^2}{9} = 1$ is an equation of the form

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with $a = 3$ and $b = 3$. Therefore the graph is a hyperbola. The x-intercepts are found by setting $y = 0$:

$$\frac{x^2}{9} - \frac{0^2}{9} = 1$$

$$x^2 = 9$$

$$x = \pm 3.$$

Thus, the x-intercepts are at $(-3,0)$ and $(3,0)$. There are no y-intercepts since for $x = 0$ there are no real values of y satisfying the equation, i.e., no real value of y satisfies

$$\frac{0^2}{9} - \frac{y^2}{9} = 1$$

$$y^2 = -9, \quad y = \sqrt{-9}.$$

Solving the original equation for y:

$$y = \sqrt{\left(1 - \frac{x^2}{9}\right)(-9)} \quad \text{or} \quad y = \sqrt{x^2 - 9}$$

shows that there will be no permissible values of x in the interval $-3 < x < 3$. Such values of x do not yield real values for y. For $x = 5$ and $x = -5$ use the equation for y to obtain the ordered pairs $(5,4)$, $(5,-4)$, $(-5,4)$, and $(-5,-4)$ as indicated in the figure. The foci of the hyperbola are located at $(\pm c,0)$, where

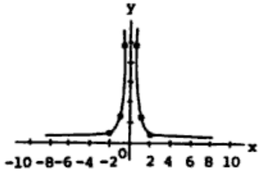
$$c^2 = a^2 + b^2$$

$$c^2 = 3^2 + 3^2 = 9 + 9 = 18$$

$$c = \pm\sqrt{18} = \pm 3\sqrt{2}.$$

Therefore, the foci are at $(-3\sqrt{2},0)$ and $(3\sqrt{2},0)$.

Discuss the graph of the function $y = \frac{12}{x^2}$.




Solution: Intercepts: Since division by zero is not defined, x cannot = 0, hence there is no y-intercept. Similarly, there is no x-intercept since y cannot = 0, because no value of x allowed in the given equation yields a value of y = 0. Symmetry: The curve is symmetric with respect to the y-axis since the x-term appears squared in the given function and hence $f(x) = f(-x)$. Domain: There is no limitation on x, except that $x \neq 0$. Range: Since $y = \frac{12}{x^2}$ is a positive number divided by a positive number, y must be positive. Therefore, the curve exists only in the first and second quadrants. Plotting: We note that, in the first quadrant, as x increases y decreases. Several points to illustrate this are listed in the following table.

| x | 0.5 | 1 | 2 | 3 | ... | 10 |
|---|---|---|---|---|---|---|
| y | 48 | 12 | 3 | 1.3 | ... | 0.12 |

After plotting these points, and tracing the curve in the first quadrant, the second branch is drawn in quadrant II, using the principle of symmetry. The curve is illustrated in the figure.

• PROBLEM 579

Graph the equation $x^2 - 2y^2 = 4$.

Fig. A

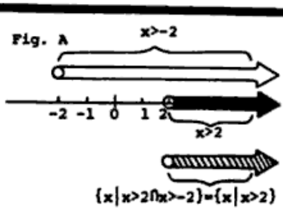
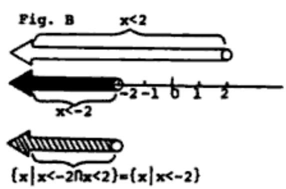
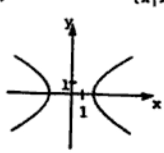



Fig. B



**Solution:** First, we shall determine the intercepts. Set $x = 0$, and find $-2y^2 = 4$, $y^2 = -2$, $y = \pm i\sqrt{2}$. This means that the curve does not meet the y-axis. Set $y = 0$, and find $x = \pm 2$. Thus, the graph only crosses the x-axis. To avoid fractions and negative radicands, we solve for $x$ in terms of $y$ rather than for $y$ in terms of $x$: $x = \pm\sqrt{2y^2 + 4}$. Assigning values to $y$ and computing corresponding values of $x$, we get the accompanying table.

| $y$ | $\pm\sqrt{2y^2 + 4}$ | $x$ |
|---|---|---|
| -3 | $\pm\sqrt{2(-3)^2 + 4}$ | $\pm\sqrt{22} = \pm 4.7$ |
| -2 | $\pm\sqrt{2(-2)^2 + 4}$ | $\pm\sqrt{12} = \pm 3.5$ |
| -1 | $\pm\sqrt{2(-1)^2 + 4}$ | $\pm\sqrt{6} = \pm 2.4$ |
| 0 | $\pm\sqrt{2(0)^2 + 4}$ | $\pm\sqrt{4} = \pm 2$ |
| 1 | $\pm\sqrt{2(1)^2 + 4}$ | $\pm\sqrt{6} = \pm 2.4$ |
| 2 | $\pm\sqrt{2(2)^2 + 4}$ | $\pm\sqrt{12} = \pm 3.5$ |
| 3 | $\pm\sqrt{2(3)^2 + 4}$ | $\pm\sqrt{22} = \pm 4.7$ |

Now plot the points. Note for each value of $y$, there are two values of $x$. For example, if $y = -3$, $x = +4.7$ and $x = -4.7$. Thus the two points are $(4.7,-3)$ and $(-4.7,-3)$. The curve of $x^2 - 2y^2 = 4$ is called a hyperbola and consists of two parts, or branches. See Figure. If we solve the equation for $y$, we obtain:

$$\begin{aligned} x^2 - 2y^2 &= 4 \\ -2y^2 &= 4 - x^2 \\ y^2 &= \frac{4 - x^2}{-2} \\ &= \frac{-4 + x^2}{2} = \frac{x^2 - 4}{2} \\ y &= \pm\sqrt{\frac{x^2 - 4}{2}} \end{aligned}$$

Now to obtain real values for $y$, the radicand cannot be negative. Therefore, $x^2 - 4 \geq 0$, and factoring $(x - 2)(x + 2) \geq 0$. There are two statements $(x - 2)(x + 2) = 0$ and $(x - 2)(x + 2) > 0$ to be considered. The roots are $x = 2$ and $x = -2$ when the factors are set equal to zero.

If we consider the inequality $(x - 2)(x + 2) > 0$, the two factors must be both positive or both negative.

Case I: both are positive

$$\begin{aligned} x - 2 > 0 \quad &\text{and} \quad x + 2 > 0 \\ x > 2 \quad &\text{and} \quad x > -2 \end{aligned}$$

Therefore, $x > 2$ (see number line A)

Case II: both are negative

$$\begin{aligned} x - 2 < 0 \quad &\text{and} \quad x + 2 < 0 \\ x < 2 \quad &\text{and} \quad x < -2 \end{aligned}$$

Therefore, $x < -2$ (see number line B)

$$\{x|x < -2 \cap x < 2\} = \{x|x < -2\}$$

To summarize $x = -2$ and $x = 2$, and $x < -2$ or $x > 2$. More simply, the domain of $x$ is $\{x|x \le -2 \text{ or } x \ge 2\}$. From

$$y = \pm\sqrt{\frac{x^2 - 4}{2}},$$

the range is the set of all real numbers.

**• PROBLEM 580**

> Find the inverse function of $f$ if $y = f(x) = 2\sqrt{9 - x^2}$ and $f$ has domain $\{x|-3 \le x \le 0\}$ and range $\{y|0 \le y \le 6\}$.

**Solution:** The equation $y = f(x)$ determines the number $y$ in the range of $f$, $\{y|0 \le y \le 6\}$, from a given number $x$ of the domain, $\{x|-3 \le x \le 0\}$. Now we want to know if this equation also determines $x$ when $y$ is given. If a function $f$ is given by a simple formula $y = f(x)$, we can often obtain $f^{-1}$ by solving this for $x$ so that $x = f^{-1}(y)$. Therefore,

$$y = 2\sqrt{9 - x^2} = 2\left(9 - x^2\right)^{\frac{1}{2}}$$

Squaring both sides, $$y^2 = 2^2\left[\left(9 - x^2\right)^{\frac{1}{2}}\right]^2 = 4\left(9 - x^2\right)$$

Distributing, $$y^2 = 36 - 4x^2$$

$$y^2 - 36 = -4x^2$$

$$x^2 = \frac{y^2 - 36}{-4} = \frac{36 - y^2}{4}$$

$$x = \frac{\pm\sqrt{36 - y^2}}{2}$$

In order to obtain a function of $y$, we must have for each value of $y$ one and only one value of $x$. Thus, we choose only one sign. Now the domain of the inverse function is the range of $f$, and the range of the inverse function is the domain of $f$. Therefore, the domain of $f^{-1}(y)$ is to be $0 \le y \le 6$ and the range is $-3 \le x \le 0$. Now since $x$ can assume negative values then we must consider the negative values of $\pm\sqrt{36 - y^2}$. Hence, the inverse function is

$$x = \frac{-\sqrt{36 - y^2}}{2}.$$

Plot each function by selecting values in the domain and finding the corresponding values in the range.

For $y = 2\sqrt{9 - x^2}$

Domain $\{x|-3 \le x \le 0\}$

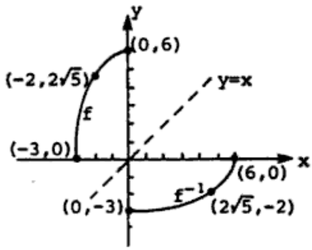
| $x$ | $2\sqrt{9 - x^2}$ | $y$ |
|---|---|---|
| $-3$ | $2\sqrt{9 - (-3)^2}$ | $0$ |
| $-2$ | $2\sqrt{9 - (-2)^2}$ | $2\sqrt{5}$ |
| $-1$ | $2\sqrt{9 - (-1)^2}$ | $4\sqrt{2}$ |
| $0$ | $2\sqrt{9 - (0)^2}$ | $6$ |

For $x = -\sqrt{36 - y^2}/2$

Domain $\{y|0 \le y \le 6\}$

| $y$ | $-\sqrt{36 - y^2}/2$ | $x$ |
|---|---|---|
| $0$ | $-\sqrt{36 - 0^2}/2$ | $-3$ |
| $1$ | $-\sqrt{36 - 1^2}/2$ | $-\sqrt{35}/2$ |
| $2$ | $\sqrt{36 - 2^2}/2$ | $-2\sqrt{2}$ |
| $3$ | $-\sqrt{36 - 3^2}/2$ | $-3\sqrt{3}/2$ |
| $4$ | $-\sqrt{36 - 4^2}/2$ | $-\sqrt{5}$ |
| $5$ | $-\sqrt{36 - 5^2}/2$ | $-\sqrt{11}/2$ |
| $6$ | $-\sqrt{36 - 6^2}/2$ | $0$ |

See graph




The graphs of $f$ and $f^{-1}$ show that if $(a,b)$ is a point of the graph of $f$, then $(b,a)$ is a point of the graph of $f^{-1}$. Furthermore, this is true by the definition of an inverse function which states that, given the function $f$ such that no two of its ordered pairs have the same second element, the inverse function $f^{-1}$ is the set of ordered pairs obtained from $f$ by interchanging in each ordered pair the first and second elements. The graphs of $f$ and $f^{-1}$ are symmetric with respect to the line $y = x$; that is, each is the image of the other in the mirror $y = x$.

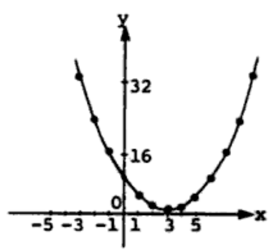
## INEQUALITIES

**• PROBLEM 581**

Find the solution set of $x^2 - 6x + 10 > 0$ by the graphical method.

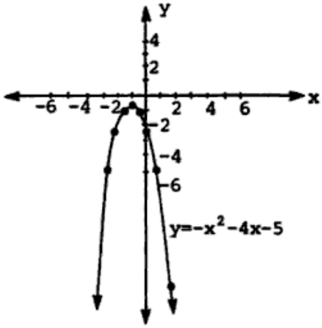
Solution: First we graph the function $y = x^2 - 6x + 10$. Assign values to x and then calculate y-values.

| x | $x^2 - 6x + 10$ | y |
|---|---|---|
| $-3$ | $(-3)^2 - 6(-3) + 10$ | 37 |
| $-2$ | $(-2)^2 - 6(-2) + 10$ | 26 |
| $-1$ | $(-1)^2 - 6(-1) + 10$ | 17 |
| 0 | $(0)^2 - 6(0) + 10$ | 10 |
| 1 | $(1)^2 - 6(1) + 10$ | 5 |
| 2 | $(2)^2 - 6(2) + 10$ | 2 |
| 3 | $(3)^2 - 6(3) + 10$ | 1 |
| 4 | $(4)^2 - 6(4) + 10$ | 2 |




See graph. The curve is the graph of $y = x^2 - 6x + 10$. Since the graph is entirely above the X axis, the solution set of $x^2 - 6x + 10 > 0$ is the set of all real numbers.

Find the solution set of $-x^2 - 4x - 5 > 0$.




**Solution:** To find the graphical solution, select values of x and find the corresponding y values. See the table. Note that $y = -x^2 - 4x - 5 = -[x^2 + 4x + 5]$.

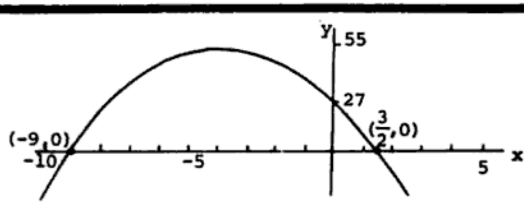
| x | $-[x^2 + 4x + 5] =$ | y |
|---|---|---|
| -3 | $-[(-3)^2 + 4(-3) + 5] = -(9 - 12 + 5) =$ | -2 |
| -2 | $-[(-2)^2 + 4(-2) + 5] = -(4 - 8 + 5) =$ | -1 |
| -1 | $-[(-1)^2 + 4(-1) + 5] = -(1 - 4 + 5) =$ | -2 |
| 0 | $-[(0)^2 + 4(0) + 5] = -(0 + 0 + 5) =$ | -5 |
| 1 | $-[(1)^2 + 4(1) + 5] = -(1 + 4 + 5) =$ | -10 |
| 2 | $-[(2)^2 + 4(2) + 5] = -(4 + 8 + 5) =$ | -17 |
| 3 | $-[(3)^2 + 4(3) + 5] = -(9 + 12 + 5) =$ | -26 |

The graph of $y = -x^2 - 4x - 5$, as shown in the figure, lies entirely below the x-axis. Consequently

$$\{x \mid -x^2 - 4x - 5 > 0\} = \emptyset, \text{ the empty set.}$$



Solve the inequality $-2x^2 - 15x + 27 > 0$.



**Solution:** The graph of the corresponding quadratic function $y = -2x^2 - 15x + 27$ is a parabola which is concave down since the coefficient of the $x^2$-term is negative. Its zeros are the solution set of the quadratic equation $-2x^2 - 15x + 27 = 0$.

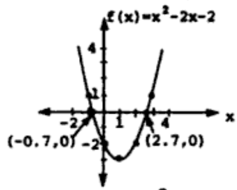
$(2x - 3)(-x - 9) = 0$ factor

$$\begin{array}{r|r} 2x - 3 = 0 & -x - 9 = 0 \\ x = \frac{3}{2} & x = -9 \end{array}$$ set each factor = 0 to find all x's which make the product 0.

Therefore, the solution set of the given inequality is the set of all x's such that the graph of $-2x^2 - 15x + 27$ lies above the x-axis, i.e., $\{x | y = -2x^2 - 15 + 27 > 0\}$. This set is $\{x | -9 < x < \frac{3}{2}\}$. (see Figure).

**• PROBLEM 584**

Solve: $x^2 < 2x + 2$.




**Solution:** To solve the inequality $x^2 < 2x + 2$, transfer all the terms to one side. Subtracting $(2x+2)$ from both sides,

$$x^2 - 2x - 2 < 0 .$$

We shall solve this inequality graphically. The function that we are dealing with is $f(x) = x^2 - 2x - 2$. We must find the region where it is less than zero. Thus, we graph the left side as a function of $x$. Choose values of $x$ and calculate the corresponding $f(x)$ values, as shown in the following table:

| $x$ | $x^2-2x-2$ | $f(x)$ |
|---|---|---|
| -3 | $(-3)^2-2(-3)-2 = 9+6-2$ | 13 |
| -2 | $(-2)^2-2(-2)-2 = 8-2$ | 6 |
| -1 | $(-1)^2-2(-1)-2 = 1+2-2$ | 1 |
| 0 | $(0)^2-2(0)-2 = 0-0-2$ | -2 |
| 1 | $(1)^2-2(1)-2 = 1-2-2$ | -3 |
| 2 | $(2)^2-2(2)-2 = 4-4-2$ | -2 |
| 3 | $(3)^2-2(3)-2 = 9-6-2$ | 1 |

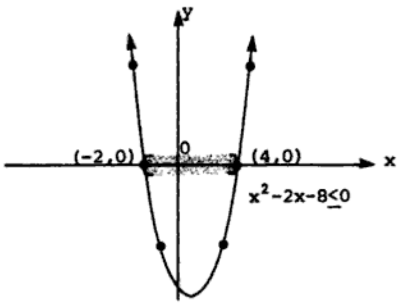
The graph of $f(x) = x^2-2x-2$ is shown in the accompanying figure. $f(x)<0$ when the curve lies below the x-axis, that is, for values of x between -0.7 and 2.7. (see figure) Hence, $-0.7<x<2.7$.

Since these results were read from the curve, they are only approximations. If the student should read 2.6 or 2.8 instead of 2.7, his result would be acceptable.

In case more accuracy is desired, we can solve the corresponding equation $x^2 - 2x - 2 = 0$ and find $x = 1 \pm \sqrt{3}$. Thus the curve crosses the x-axis when $x = 1 - \sqrt{3}$ and $x = 1 + \sqrt{3}$, and the inequality is true for $1 - \sqrt{3} < x < 1 + \sqrt{3}$. Note $x = 1 + \sqrt{3} \approx 2.7$ and $x = 1 - \sqrt{3} \approx -0.7$.

**• PROBLEM 585**

Construct a graphical representation of the inequality $x^2 - 2x - 8 \le 0$ and identify the solution set.




Solution: The graph of the relation $\{(x, y) | y = x^2 - 2x - 8\}$ is sketched in the figure. A table of values can be constructed.

| x | $x^2 - 2x - 8$ | y |
|---|---|---|
| - 3 | $(-3)^2 - 2(-3) - 8$ | 7 |
| - 2 | $(-2)^2 - 2(-2) - 8$ | 0 |
| - 1 | $(-1)^2 - 2(-1) - 8$ | - 5 |
| 0 | $(0)^2 - 2(0) - 8$ | - 8 |
| 1 | $(1)^2 - 2(1) - 8$ | - 9 |
| 2 | $(2)^2 - 2(2) - 8$ | - 8 |
| 3 | $(3)^2 - 2(3) - 8$ | - 5 |
| 4 | $(4)^2 - 2(4) - 8$ | 0 |
| 5 | $(5)^2 - 2(5) - 8$ | 7 |

We have to find the values of x for which $y \le 0$ where $y = x^2 - 2x - 8$. First we consider the case $y = 0$. Factor y into $(x - 4)(x + 2)$. Set $y = 0$ and find the roots of this equation.

$x - 4 = 0 \qquad\qquad x + 2 = 0$

$x_1 = 4 \qquad\qquad x_2 = -2$

Now, we must find where $x^2 - 2x - 8 < 0$.

We mark the roots on the x-axis and consider the regions into which the roots divide the x-axis. They are $x < -2$, $-2 < x < 4$, and $x > 4$. For each region choose an x value and investigate the algebraic signs of the factors of f(x) and also their product sign, f(x). See the following table.

| Regions | $x < -2$ | $-2 < x < 4$ | $x > 4$ |
|---|---|---|---|
| Factors of f(x) | $(x-4)(x+2)$ | $(x-4)(x+2)$ | $(x-4)(x+2)$ |
| x-value | $-3$ | 0 | 5 |
| Signs of factors | (−) (−) | (−) (+) | (+) (+) |
| $\therefore$ | $y > 0$ | $y < 0$ | $y > 0$ |

Thus, $y < 0$ for $-2 < x < 4$.

Furthermore, $y \leq 0$ when $-2 \leq x \leq 4$. See Graph.

The darkened portion is the part which represents the inequality. The solution set is the interval $[-2, 4]$.

# CHAPTER 20

# SYSTEMS OF QUADRATIC EQUATIONS

**Basic Attacks and Strategies for Solving Problems in this Chapter. See pages 404 to 463 for step-by-step solutions to problems.**

A system of two equations in two variables, in which at least one equation is quadratic, is considered to be a quadratic system of equations. The solution set for such a system consists of all ordered pairs that satisfy both equations. In most cases, the solution procedure used for quadratic systems is the substitution method; however, the addition method can also be used if the like variables are raised to the same power in both equation. In a case where the resulting equation is quadratic, then the usual procedure for solving a quadratic equation is used. The final step is to check the original system of equations to ensure that the obtained solution set is really a true solution set. These methods are the same ones that are well-known in the solution of systems of linear equations in two variables.

As a precursor to the above solution procedure, it is helpful to graph each equation in the system on the same $x - y$ coordinate system in order to anticipate the number, approximate position, and value of the solution(s). Sometimes it is absolutely necessary to construct a graphical solution of the system. This is especially true if the system contains quadratic inequalities. Such a system requires that the solution set, if it exists, is a shaded region in the plane.

If both of the equations in a system of two equations in two variables are of the second degree in both variables, then the use of linear combinations of the equations often provides a simpler means of solution than does substitution. For example, solve the following system of equations:

$$3x^2 - 7y^2 + 15 = 0 \qquad (1)$$

$$3x^2 - 4y^2 - 12 = 0 \qquad (2)$$

By forming the linear combination of – 1 times equation (1) and 1 times equation (2) we obtain

$$3y^2 - 27 = 0$$

$$y^2 = 9$$

$$y = \pm 3$$

Substituting 3 for y in equation (1) or (2), we have

$$x^2 = 16 \quad \text{or} \quad x = \pm 4.$$

Hence, (4, 3) and (– 4, 3) are solutions of the system. Now substitute – 3 for $y$ to obtain

$$x^2 = 16 \quad \text{or} \quad x = \pm 4.$$

Hence, (4, – 3) and (– 4, – 3) are also solutions.

The solutions of some systems require the application of both linear combinations and substitution. Also, it may be necessary, after substitution does not yield a simplified equation, to replace an expression in the equation with a new variable $u$ and solve the system with it included. Once the solution has been found, the replacement variable can be removed by substituting back into the original expression and simplifying to obtain the solution for the original variable.

## Step-by-Step Solutions to Problems in this Chapter, "Systems of Quadratic Equations"

### QUADRATIC/LINEAR COMBINATIONS

**• PROBLEM 586**

Solve the system

$$xy = 24, \quad (1)$$

$$y - 2x + 2 = 0. \quad (2)$$

Solution: This system is most easily solved by the method of substitution. Solve (2) for y in terms of x:

$$y = 2x - 2.$$

Substitute 2x - 2 for y in (1):

$$x(2x - 2) = 24,$$

$$2x^2 - 2x = 24,$$

$$2x^2 - 2x - 24 = 0,$$

$$x^2 - x - 12 = 0,$$

or factoring $(x+3)(x-4) = 0,$

Set each factor = 0 to find all values of x for which the product = 0.

$$x + 3 = 0 \quad\Big|\quad x - 4 = 0$$

$$x = -3 \quad\Big|\quad x = 4.$$

In Equation (1): for $x = -3$, $(-3)y = 24$ or $y = -8$; for $x = 4$, $(4)y = 24$ or $y = 6$.

In Equation (2): for $x = -3$, $y - 2(-3) + 2 = 0$ or $y = -8$; for $x = 4$, $y - 2(4) + 2 = 0$ or $y = 6$.

**• PROBLEM 587**

Solve the system

$$2x^2 - 3xy - 4y^2 + x + y - 1 = 0,$$

$$2x - y = 3.$$

Solution: A system of equations consisting of one linear and one quadratic is solved by expressing one of the unknowns in the linear equation in terms of the other, and substituting the result in the quadratic equation. From the second equation, $y = 2x - 3$. Replacing $y$ by this linear function of $x$ in the first equation, we find

$$2x^2 - 3x(2x - 3) - 4(2x - 3)^2 + x + 2x - 3 - 1 = 0.$$

$$2x^2 - 3x(2x - 3) - 4\left(4x^2 - 12x + 9\right) + x + 2x - 3 - 1 = 0$$

Distribute, $2x^2 - 6x^2 + 9x - 16x^2 + 48x - 36 + x + 2x - 3 - 1 = 0$

Combine terms, $\quad -20x^2 + 60x - 40 = 0$

Divide both sides by $-20$,

$$\frac{-20x^2}{-20} + \frac{60x}{-20} - \frac{40}{-20} = \frac{0}{-20}$$

$$x^2 - 3x + 2 = 0$$

Factoring, $\quad (x - 2)(x - 1) = 0$

Setting each factor equal to zero, we obtain:

$$x - 2 = 0 \qquad\qquad x - 1 = 0$$

$$x = 2 \qquad\qquad x = 1$$

To find the corresponding y-values, substitute the x-values in $y = 2x - 3$:

when $x = 1$, $\qquad$ when $x = 2$,

$$y = 2(1) - 3 \qquad\qquad y = 2(2) - 3$$

$$= 2 - 3 \qquad\qquad = 4 - 3$$

$$y = -1 \qquad\qquad y = 1$$

Therefore the two solutions of the system are

$$(1,-1), \qquad\qquad (2,1),$$

and the solution set is $\left\{(1,-1),(2,1)\right\}$.

**• PROBLEM 588**

Solve the following system:

$$x^2 + y^2 = 25 \qquad (1)$$

$$2x + y = 10 \qquad (2)$$

Solution: We solve equation (2) for y by adding $-2x$ to both sides:

$$y = 10 - 2x \qquad (3)$$

Replacing y by $10 - 2x$ in equation (1):

$$x^2 + (10 - 2x)^2 = 25$$

$$x^2 + 100 - 40x + 4x^2 = 25$$

$$5x^2 - 40x + 100 = 25$$

$$5x^2 - 40x + 75 = 0.$$ Factoring out 5,

$$5\left(x^2 - 8x + 15\right) = 0$$

Dividing both sides by 5,

$$x^2 - 8x + 15 = 0$$

Factoring,

$$(x - 5)(x - 3) = 0$$

Whenever the product of two numbers $ab = 0$, either $a = 0$ or $b = 0$; therefore

$$x - 5 = 0 \quad \text{or} \quad x - 3 = 0$$

$$x = 5 \quad \text{or} \quad x = 3$$

Replacing x by 5 in equation (3):

$$y = 10 - 2(5)$$

$$y = 10 - 10 = 0$$

Replacing x by 3 in equation (3):

$$y = 10 - 2(3)$$

$$y = 10 - 6$$

$$y = 4$$

Thus our solutions are $(5,0)$ and $(3,4)$. Check: Replacing $(x,y)$ by $(5,0)$ in each equation:

$$x^2 + y^2 = 25 \qquad (1)$$

$$5^2 + 0^2 = 25$$

$$25 + 0 = 25$$

$$25 = 25$$

$$2x + y = 10 \qquad (2)$$

$$2(5) + 0 = 10$$

$$10 + 0 = 10$$

$$10 = 10$$

Replacing (x,y) by (3,4) in each equation:

$$x^2 + y^2 = 25 \qquad (1)$$

$$3^2 + 4^2 = 25$$

$$9 + 16 = 25$$

$$25 = 25$$

$$2x + y = 10 \qquad (2)$$

$$2(3) + 4 = 10$$

$$6 + 4 = 10$$

$$10 = 10$$

Therefore the solution set is $\{(5,0), (3,4)\}$.

**• PROBLEM 589**

Obtain the simultaneous solution set of

$$x^2 + 2y^2 = 54 \qquad (1)$$

$$2x - y = -9. \qquad (2)$$

Solution: Equation (2) is readily solvable for y in terms of x, and so we proceed as follows:

$$-y = -9 - 2x$$

$$y = 2x + 9. \qquad (3)$$

$x^2 + 2(2x+9)^2 = 54$ replacing y by (2x+9) in equation (1)

$x^2 + 2(4x^2+36x+81) = 54$ squaring (2x+9)

$x^2 + 8x^2 + 72x + 162 = 54$ by the distributive law

$9x^2 + 72x + 108 = 0$ adding $-54$ to each member and combining terms

$x^2 + 8x + 12 = 0$ dividing by 9

$(x+6)(x+2) = 0$ factoring.

Whenever a product of two numbers $ab = 0$, either $a = 0$ or $b = 0$; thus either

$$x + 6 = 0 \quad \text{or} \quad x + 2 = 0 \quad \text{and}$$

$$x = -6 \quad \text{or} \quad x = -2.$$

To find y, we proceed as follows:

$y = 2(-6) + 9 = -12 + 9 = -3$ replacing x by -6 in equation (3)

$y = 2(-2) + 9 = -4 + 9 = 5$ replacing x by -2 in equation (3)

Therefore the simultaneous solution set is $\{(-6,-3), (-2,5)\}$.

Check. Replacing x and y by (-6) and (-3) in equations (1) and (2),

$$x^2 + 2y^2 = 54 \qquad (1)$$

$$(-6)^2 + 2(-3)^2 = 54$$

$$36 + 2(9) = 54$$

$$36 + 18 = 54$$

$$54 = 54$$

$$2x - y = -9 \qquad (2)$$

$$2(-6) - (-3) = -9$$

$$-12 + 3 = -9$$

$$-9 = -9.$$

Now replacing x and y by (-2) and (5) in equations (1) and (2),

$$x^2 + 2y^2 = 54 \qquad (1)$$

$$(-2)^2 + 2(5)^2 = 54$$

$$4 + 2(25) = 54$$

$$4 + 50 = 54$$

$$54 = 54$$

$$2x - y = -9 \qquad (2)$$

$$2(-2) - (5) = -9$$

$$-4 - 5 = -9$$

$$-9 = -9.$$

• **PROBLEM 590**

Solve the system

$$2x + y = 1$$
$$3x^2 - xy - y^2 = -2.$$

Solution: Solving the first equation for y in terms of x:

$$2x + y = 1$$

Subtracting $2x$ from both sides:

$$\cancel{2x} + y - \cancel{2x} = 1 - 2x$$
$$y = 1 - 2x$$

When we substitute this value for $y$ in the second equation we obtain:

$$3x^2 - x(1 - 2x) - (1 - 2x)^2 = -2$$
$$3x^2 - (x - 2x^2) - (1 - 4x + 4x^2) = -2$$
$$3x^2 - x + 2x^2 - 1 + 4x - 4x^2 = -2$$
$$3x^2 + 2x^2 - 4x^2 - x + 4x - 1 = -2$$
$$x^2 + 3x - 1 = -2$$

Adding 2 to both sides:

$$x^2 + 3x - 1 + 2 = -2 + 2$$
$$x^2 + 3x + 1 = 0 \qquad (1)$$

Equation (1) is a quadratic equation since it is in the form $ax^2 + bx + c = 0$. This equation can be solved by using the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad .$$

In this case, $a = 1$, $b = 3$, and $c = 1$. Then,

$$x = \frac{-3 \pm \sqrt{(3)^2 - 4(1)(1)}}{2(1)}$$
$$= \frac{-3 \pm \sqrt{9 - 4}}{2}$$
$$= \frac{-3 \pm \sqrt{5}}{2}$$
$$x = \frac{-3}{2} \pm \frac{1}{2}\sqrt{5}$$

To obtain the corresponding y-values we use the equation $y = 1 - 2x$. When $x = -3/2 + 1/2\sqrt{5}$ ,

$$y = 1 - 2\left(-\frac{3}{2} + \frac{1}{2}\sqrt{5}\right)$$
$$= 1 - 2\left(-\frac{3}{2}\right) - 2\left(\frac{1}{2}\sqrt{5}\right)$$
$$= 1 + 3 - \sqrt{5}$$
$$y = 4 - \sqrt{5}$$

When $x = -\frac{3}{2} - \frac{1}{2}\sqrt{5}$,

$$y = 1 - 2\left(-\frac{3}{2} - \frac{1}{2}\sqrt{5}\right)$$
$$= 1 - 2\left(-\frac{3}{2}\right) - 2\left(-\frac{1}{2}\sqrt{5}\right)$$
$$= 1 + 3 + \sqrt{5}$$
$$y = 4 + \sqrt{5}$$

Therefore, the solution set to the original system of equations is:

$$\left\{\left(-\frac{3}{2}+\frac{1}{2}\sqrt{5},\ 4-\sqrt{5}\right),\left(-\frac{3}{2}-\frac{1}{2}\sqrt{5},\ 4+\sqrt{5}\right)\right\}.$$

**• PROBLEM 591**

Find the solution set for the system:

$$3x - 5y = 13 \qquad (1)$$
$$y^2 = 4x \qquad (2)$$

**Solution:** Use the method of substitution to solve the system. From equation (2) we obtain:

$$x = \frac{y^2}{4}$$

Then upon substitution for $x$, equation (1) becomes:

$$3\left(\frac{y^2}{4}\right) - 5y = 13$$

Multiplying both sides of the equation by 4:

$$3y^2 - 20y = 52$$

The equation in standard quadratic form is:

$$3y^2 - 20y - 52 = 0$$
$$(3y - 26)(y + 2) = 0$$
$$3y - 26 = 0 \quad \text{or} \quad y + 2 = 0$$
$$y = \frac{26}{3} \quad \text{or} \quad y = -2$$

Returning to $x = \frac{y^2}{4}$, we see that

$$y = \frac{26}{3} \text{ implies that } x = \frac{(26/3)^2}{4} = \frac{169}{9} \text{ and}$$

$$y = 2 \text{ implies that } x = \frac{(-2)^2}{4} = 1$$

Check to see if equation (1) is satisfied.

for $x = \frac{169}{9}$, $y = \frac{26}{3}$ :

$$3\left(\frac{169}{9}\right) - 5\left(\frac{26}{3}\right) = \frac{169}{3} - \frac{130}{3} = \frac{39}{3} = 13$$

for $x = 1, y = -2$ :

$$3(1) - 5(-2) = 3 + 10 = 13$$

Hence the solution set is

$$\left\{\left(\frac{169}{9}, \frac{26}{3}\right), (1,-2)\right\} .$$

**• PROBLEM 592**

Solve the following linear-quadratic system:

$$\begin{cases} 2x + 3y = 1, & (1) \\ x^2 - 5xy - 8y^2 + 6y = 0. & (2) \end{cases}$$

**Solution:** To solve a system of equations consisting of one linear and one quadratic, express one of the unknowns in the linear equation in terms of the other variable. Substitute the result in the quadratic equation. We solve for $x$ in terms of $y$.

$$2x + 3y = 1 \quad (1) \text{ (adding } -3y \text{ to both sides)}$$
$$2x = 1 - 3y \text{ (dividing both sides by 2)}$$
$$x = \frac{1 - 3y}{2} \qquad (3)$$

Substitute in the quadratic equation:

$$\left(\frac{1 - 3y}{2}\right)^2 - 5\left(\frac{1 - 3y}{2}\right)y - 8y^2 + 6y = 0.$$

$$\left(\frac{1 - 3y}{2}\right)\left(\frac{1 - 3y}{2}\right) - 5\left(\frac{1 - 3y}{2}\right)y - 8y^2 + 6y = 0$$

$$\frac{1 - 6y + 9y^2}{4} - \frac{5y - 15y^2}{2} - 8y^2 + 6y = 0.$$

Multiplying the equation by 4.

$$4\left[\frac{1 - 6y + 9y^2}{4} - \frac{5y - 15y^2}{2} - 8y^2 + 6y\right] = [0]\ 4$$

$$1 - 6y + 9y^2 - \overset{2}{\cancel{4}}\left(\frac{5y - 15y^2}{\underset{1}{\cancel{2}}}\right) - 4 \cdot 8y^2 + 4 \cdot 6y = 0$$

$$1 - 6y + 9y^2 - 10y + 30y^2 - 32y^2 + 24y = 0$$

$$9y^2 + 30y^2 - 32y^2 - 6y - 10y + 24y + 1 = 0$$

$$7y^2 + 8y + 1 = 0$$

Factoring: $(7y + 1)(y + 1) = 0$ .
Set each factor equal to zero. Solve for $y$.

$$7y + 1 = 0 \qquad\qquad y + 1 = 0$$
$$7y = -1$$
$$y = -\frac{1}{7} \qquad\qquad y = -1$$

Substitute in the linear equation (3). For $y = -\frac{1}{7}$ ;

$$x = \frac{1 - 3y}{2} = \frac{1 - 3\left(-\frac{1}{7}\right)}{2} = \frac{1 + \frac{3}{7}}{2} = \frac{\frac{10}{7}}{2} = \frac{\overset{5}{\cancel{10}}}{7} \cdot \frac{1}{\underset{1}{\cancel{2}}} = \frac{5}{7}$$

For $y = -1$;

$$x = \frac{1 - 3y}{2} = \frac{1 - 3(-1)}{2} = \frac{1 + 3}{2} = \frac{4}{2} = 2 \ .$$

The solutions are (2,-1) and (5/7, -1/7). Each solution should be checked by substitution in both of the given equations.

Check: For (2,-1)

$$2x + 3y = 1$$
$$2(2) + 3(-1) = 1$$
$$4 - 3 = 1$$
$$1 = 1$$

For $\left(\frac{5}{7}, -\frac{1}{7}\right)$

$$2x + 3y = 1$$
$$2(5/7) + 3(-1/7) = 1$$
$$10/7 - 3/7 = 1$$
$$7/7 = 1$$
$$1 = 1$$

$$x^2 - 5xy - 8y^2 + 6y = 0$$
$$(2)^2 - 5(2)(-1) - 8(-1)^2 + 6(-1) = 0$$
$$4 + 10 - 8 - 6 = 0$$
$$6 - 6 = 0$$
$$0 = 0$$

$$x^2 - 5xy - 8y^2 + 6y = 0$$
$$\left(\frac{5}{7}\right)^2 - 5\left(\frac{5}{7}\right)\left(-\frac{1}{7}\right) - 8\left(-\frac{1}{7}\right)^2 + 6\left(-\frac{1}{7}\right) = 0$$
$$\frac{25}{49} + \frac{25}{49} - 8\left(\frac{1}{49}\right) - \frac{6}{7} = 0$$

$$\frac{25}{49} + \frac{25}{49} - \frac{8}{49} - \frac{7(6)}{7(7)} = 0$$

$$\frac{25}{49} + \frac{25}{49} - \frac{8}{49} - \frac{42}{49} = 0$$

$$\frac{25 + 25 - 8 - 42}{49} = 0$$

$$\frac{50 - 8 - 42}{49} = 0$$

$$\frac{42 - 42}{49} = 0$$

$$\frac{0}{49} = 0$$

$$0 = 0$$

In choosing the variable to be eliminated, it is advisable to avoid fractions if possible. For example, the linear equation $6x - y = 7$ should be solved for $y$ in terms of $x$: $y = 6x - 7$, rather than for $x$ in terms of $y$:

$$x = \frac{y + 7}{6} .$$

**• PROBLEM 593**

> Solve for $x$ and $y$: $\begin{cases} x^2 + 4y^2 - 8y + 2x - 3 = 0 , & (1) \\ 3y - 2x = 12. & (2) \end{cases}$

Solution: From the second equation, $3y - 2x = 12$, add $2x$ to both sides,

$$3y = 12 + 2x$$

divide both sides by 3 to obtain,

$$y = 4 + \tfrac{2}{3}x.$$

Substitution of this in the first equation gives

$$x^2 + 4\left(4 + \tfrac{2}{3}x\right)^2 - 8\left(4 + \tfrac{2}{3}x\right) + 2x - 3 = 0,$$

$$x^2 + 4\left(4 + \tfrac{2}{3}x\right)\left(4 + \tfrac{2}{3}x\right) - 32 - \frac{16}{3}x + 2x - 3 = 0$$

$$x^2 + 4\left(16 + 2\left(\frac{8}{3}\right)x + \frac{4}{9}x^2\right) - 32 - \frac{16}{3}x + 2x - 3 = 0$$

$$x^2 + 64 + \frac{4.16}{3}x + \frac{16}{9}x^2 - 32 - \frac{16}{3}x + 2x - 3 = 0$$

Group terms raised to the same exponent of $x$,

$$x^2 + \frac{16}{9}x^2 + \frac{64}{3}x - \frac{16}{3}x + 2x + 64 - 32 - 3 = 0$$

$$\frac{9}{9}x^2 + \frac{16}{9}x^2 + \frac{48}{3}x + 2x + 29 = 0$$

$$\frac{25}{9}x^2 + 16x + 2x + 29 = 0$$

$$\frac{25}{9}x^2 + 18x + 29 = 0$$

Multiply both members by 9,

$$9\left(\frac{25}{9}x^2 + 18x + 29\right) = 9 \cdot 0$$

$$25x^2 + 162x + 261 = 0$$

Factor $$(25x + 87)(x + 3) = 0.$$

When the product of two numbers $ab = 0$ either $a = 0$ or $b = 0$, thus with $(25x + 87)(x + 3) = 0$ either

$$25x + 87 = 0 \quad \text{or} \quad x + 3 = 0$$

$$25x = -87$$

$$x = \frac{-87}{25} \quad \text{or} \quad x = -3$$

To obtain the corresponding $y$ values to each $x$ value, replace the $x$ values in equation (2) and solve for $y$,

for $x = \frac{-87}{25}$: $$3y - 2\left(\frac{-87}{25}\right) = 12$$

$$3y + \frac{174}{25} = 12$$

multiply both members by 25,

$$25\left(3y + \frac{174}{25}\right) = 25(12)$$

$$75y + 174 = 300$$

$$75y = 126$$

$$y = \frac{126}{75} = \frac{42}{25}$$

for $x = -3$: $$3y - 2(-3) = 12$$

$$3y + 6 = 12$$

$$3y = 6$$

$$y = 2$$

Hence, our two pairs of solutions are $(-3,2)$ and $\left(\frac{-87}{25}, \frac{42}{25}\right)$.

Check: A) Replace $x$ and $y$ in (1) and (2) by $(-3,2)$,

(1) $$x^2 + 4y^2 - 8y + 2x - 3 = 0$$

$$(-3)^2 + 4\left(2^2\right) - 8(2) + 2(-3) - 3 = 0$$

$$9 + 16 - 16 - 6 - 3 = 0$$

$$3 - 3 = 0$$

$$0 = 0$$

(2) $$3y - 2x = 12$$

$$3(2) - 2(-3) = 12$$

$$6 + 6 = 12$$

$$12 = 12$$

B) Now replace $x$ and $y$ in (1) and (2) by $\left(\frac{-87}{25}, \frac{42}{25}\right)$,

(1) $$x^2 + 4y^2 - 8y + 2x - 3 = 0$$

$$\left(\frac{-87}{25}\right)^2 + 4\left(\frac{42}{25}\right)^2 - 8\left(\frac{42}{25}\right) + 2\left(\frac{-87}{25}\right) - 3 = 0$$

$$12.1104 + 4(2.8224) - 8(1.68) + 2(-3.48) - 3 = 0$$

$$12.1104 + 11.2896 - 13.4400 - 6.96 - 3 = 0$$

$$23.4 - 6.96 - 16.44 = 0$$

$$23.4 - 23.4 = 0$$

$$0 = 0$$

(2) $3y - 2x = 12$

$$3\left(\frac{42}{25}\right) - 2\left(\frac{-87}{25}\right) = 12$$

$$\frac{126}{25} + \frac{174}{25} = 12$$

$$\frac{300}{25} = 12$$

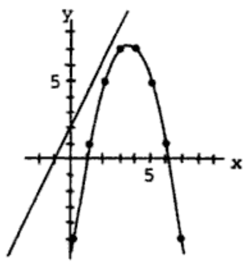
$$12 = 12$$

**• PROBLEM 594**

Solve the system

$$y = -x^2 + 7x - 5 \qquad (1)$$

$$y - 2x = 2 \qquad (2)$$




Solution: Solving Equation (2) for y yields an expression for y in terms of x. Substituting this expression in Equation (1),

$$2x + 2 = -x^2 + 7x - 5 \qquad (3)$$

We have a single equation, in terms of a single variable, to be solved. Writing Equation (3) in standard quadratic form,

$$x^2 - 5x + 7 = 0 \qquad (4)$$

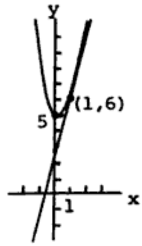
Since the equation is not factorable the roots are not found in this manner. Evaluating the discriminant will indicate whether Equation (4) has real roots. The discriminant, $b^2 - 4ac$, of Equation (4) equals $(-5)^2 - 4(1)(7) = 25 - 28 = -3$. Since the discriminant is negative, Equation (4) has no real roots, and therefore the system has no real solution. In terms of the graph, the figure shows that the parabola and the straight line have no point in common.

**• PROBLEM 595**

Solve the system

$$y = 3x^2 - 2x + 5 \tag{1}$$

$$y = 4x + 2. \tag{2}$$




Solution: To obtain a single equation with one unknown variable, x, substitute the value of y from Equation (2) in Equation (1),

$$4x + 2 = 3x^2 - 2x + 5. \tag{3}$$

Writing Equation (3) in standard quadratic form,

$$3x^2 - 6x + 3 = 0. \tag{4}$$

We may simplify Equation (4) by dividing both members by 3, which is a factor common to each term:

$$x^2 - 2x + 1 = 0. \tag{5}$$

To find the roots, factor and set each factor = 0. This may be done since a product = 0 implies one or all of the factors must = 0.

$$(x - 1)(x - 1) = 0 \tag{6}$$

$$\begin{array}{r|r} x - 1 = 0 & x - 1 = 0 \\ x = 1 & x = 1 \end{array}$$

Equation (5) has two equal roots, each equal to 1. For $x = 1$, from Equation (2) we have $y = 4(1) + 2 = 6$. Therefore the system has but one common solution:

$$x = 1,\ y = 6.$$

The figure indicates that our solution is probably correct. We may also check to see if our values satisfy Equation (1) as well:

Substituting in: $y = 3x^2 - 2x + 5$

$$6 \stackrel{?}{=} 3(1)^2 - 2(1) + 5$$

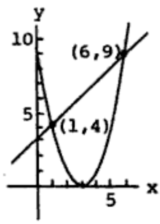
$$6 \stackrel{?}{=} 3 - 2 + 5$$

$$6 = 6.$$

Solve the system of equations

$$y = x^2 - 6x + 9 \qquad (1)$$

$$y = x + 3. \qquad (2)$$




Solution: A single equation in terms of the one variable x may be obtained by the method of substitution. Substitute the value of y, x + 3, from Equation (2) for y in Equation (1).

$$x + 3 = x^2 - 6x + 9. \qquad (3)$$

Writing Equation (3) in the standard form of a quadratic equation,

$$x^2 - 7x + 6 = 0. \qquad (4)$$

Use the usual method of solving quadratic equations. Factor the equation. Then find the values of x for which each factor may = 0.

$$(x - 1)(x - 6) = 0 \qquad (5)$$

| | |
|---|---|
| $x - 1 = 0$ | $x - 6 = 0$ |
| $x = 1$ | $x = 6$ |

The roots of Equation (4) are $x = 1$ and $x = 6$. Since $y = x + 3$, the solution of the given system is

$$x = 1,\ y = 4 \text{ and } x = 6,\ y = 9.$$

In the figure, the graph of the system, indicates that our solution is probably correct. We may prove that the solution is correct by substituting each solution in both equations of the given system, as usual. The values of y were obtained by satisfying equation (2). Now for Equation (1):

| check for $x = 1$, $y = 4$ | check for $x = 6$, $y = 9$ |
|---|---|
| $y = x^2 - 6x + 9$ | $y = x^2 - 6x + 9$ |

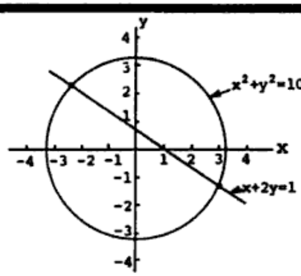
| | |
|---|---|
| $4 \stackrel{?}{=} (1)^2 - 6(1) + 9$ | $9 \stackrel{?}{=} (6)^2 - 6(6) + 9$ |
| $4 \stackrel{?}{=} 1 - 6 + 9$ | $9 \stackrel{?}{=} 36 - 36 + 9$ |
| $4 = 4$ | $9 = 9$ |

**• PROBLEM 597**

Solve the system

$$x^2 + y^2 = 10 \qquad (1)$$

$$x + 2y = 1 \qquad (2)$$




Solution: We solve the linear equation for $x$ in terms of $y$ by adding $-2y$ to both sides to obtain $x = 1 - 2y$. We substitute the result, $1 - 2y$, for $x$ in the quadratic equation to obtain

$$(1 - 2y)^2 + y^2 = 10$$

Then we have

$$1 - 4y + 4y^2 + y^2 = 10$$

We add (-10) to both sides and combine terms:

$$5y^2 - 4y - 9 = 0$$

We factor,

$$(5y - 9)(y + 1) = 0$$

Whenever the product of two numbers $ab = 0$ either $a = 0$ or $b = 0$. Thus $(5y - 9)(y + 1) = 0$ implies either $5y - 9 = 0$ or $y + 1 = 0$

$$5y = 9 \quad \text{or} \quad y = -1$$

Thus, $y = \frac{9}{5}$ or $y = -1$.

Substituting these values in turn in the linear equation, we find the corresponding values for $x$: $x + 2y = 1$, for $y = \frac{9}{5}$

$$x + 2\left(\frac{9}{5}\right) = 1,\ x = 1 - \frac{18}{5},\ x = \frac{-13}{5}$$

and for $y = -1$

$$x + 2(-1) = 1,\ x - 2 = 1,\ x = 3.$$

The solutions of the system are therefore

$$\left(x = -\frac{13}{5},\ y = \frac{9}{5}\right) \text{ and } (x = 3,\ y = -1).$$

To consider the corresponding graphs of this system, we notice that $x^2 + y^2 = 10$ represents a circle with radius $\sqrt{10}$ and $x + 2y = 1$ is the line passing through the points $(1,0)$ and $(0,\frac{1}{2})$.

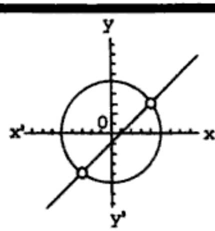
The two points where the circle and line intersect are the solution to our problem, $\left(\frac{-13}{5}, \frac{9}{5}\right)$ and $(3,-1)$, the points where $x^2 + y^2 = 10$ and $x + 2y = 1$ simultaneously.

**• PROBLEM 598**

Solve the system

$$x^2 + y^2 = 25, \qquad (1)$$

$$x - y = 1. \qquad (2)$$




Solution: Solve (2) for y (the problem can be done similarly for x instead): The method of substitution is most easily employed in this example to solve the system.

$$y = x - 1. \qquad (3)$$

Substitute $x - 1$ for y in (1):

$$x^2 + (x - 1)^2 = 25. \qquad (4)$$

$$x^2 + x^2 - 2x + 1 = 25.$$

From (4) $\qquad 2x^2 - 2x - 24 = 0,$

or $\qquad x^2 - x - 12 = 0. \qquad (5)$

Solve (5) by factoring: $\quad (x - 4)(x + 3) = 0$

$$x - 4 = 0 \quad x + 3 = 0$$

$$x = 4 \text{ or } -3.$$

Substituting 4 for x in (2), we obtain

$$4 - y = 1 \text{ or } y = 3.$$

Substituting -3 for x in (2), we obtain

$$-3 - y = 1 \text{ or } y = -4.$$

This gives $\begin{matrix} x = 4 \\ y = 3 \end{matrix}$ and $\begin{matrix} x = -3 \\ y = -4 \end{matrix}$ for the solutions.

Check:

for $x = 4$, $y = 3$

in Eq. (1): $(4)^2 + (3)^2 = 25$

$$16 + 9 = 25$$

$$25 = 25$$

in Eq. (2): $(4) - (3) = 1$

$$1 = 1$$

for $x = -3$, $y = -4$

in Eq. (1): $(-3)^2 + (-4)^2 = 25$

$$9 + 16 = 25$$

$$25 = 25$$

in Eq. (2): $(-3) - (-4) = 1$

$$-3 + 4 = 1$$

$$1 = 1.$$

Graphical meaning of the two solutions. We may plot the graph for each of the equations (1) and (2). The graph of

$$x - y = 1$$

is the straight line shown in the figure, and the graph of

$$x^2 + y^2 = 25$$

is the circle there shown. To draw the graph of (1), the student may give various values to x and calculate the corresponding values for y from $y = \pm\sqrt{25 - x^2}$.

Any point on the straight line (2) has coordinates that satisfy Equation (2). Any point on the circle (1) has coordinates that satisfy Equation (1). The points (4,3) and (-3,-4) lie on both graphs and satisfy both Equations (1) and (2). That is to say, each point of intersection of the graph of (1) with the graph of (2) gives a pair of numbers that is a solution of the system.

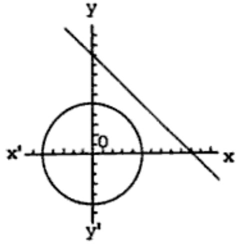
**• PROBLEM 599**

Solve the system

$$x^2 + y^2 = 25, \qquad (1)$$

$$x + y = 10, \qquad (2)$$

and draw the graph to explain the fact that the solutions are not real.




<u>Solution:</u> Use the method of substitution to obtain a single equation in terms of either one of the variables x or y. Solve Equation (2) for y (we could have chosen to solve for x). We get

$$y = 10 - x. \qquad (3)$$

Substitute this expression for y in Equation (1)

$x^2 + (10 - x)^2 = 25$

$x^2 + 100 - 20x + x^2 = 25$ Expand the equation

$2x^2 - 20x + 100 = 25$ Combine like terms

$2x^2 - 20x + 75 = 0$ Put in standard quadratic form.

This is a nonfactorable quadratic equation of the form $ax^2 + bx + c$ with $a = 2$, $b = -20$, $c = 75$. To find the roots of this equation use the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

substituting our values of a,b, and c we get

$$x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4(2)(75)}}{2(2)}$$

$$x = \frac{20 \pm \sqrt{400 - 600}}{4} = \frac{20 \pm \sqrt{-200}}{4}$$

$$x = \frac{20 \pm \sqrt{(100)(2)(-1)}}{4} = \frac{20 \pm \sqrt{100} \cdot \sqrt{2} \cdot \sqrt{-1}}{4}$$

square roots of a product.

Recall $i = \sqrt{-1}$ then $x = \dfrac{20 \pm 10i\sqrt{2}}{4}$ reduces to $x = 5 \pm \frac{5i}{2}\sqrt{2}$.

Using Equation (3):

for $x = 5 + \frac{5i}{2}\sqrt{2}$, $y = 10 - x = 10 - \left(5 + \frac{5i}{2}\sqrt{2}\right)$

$$= 10 - 5 - \frac{5i}{2}\sqrt{2} = 5 - \frac{5i}{2}\sqrt{2}$$

for $x = 5 - \frac{5i}{2}\sqrt{2}$, $y = 10 - x = 10 - \left(5 - \frac{5i}{2}\sqrt{2}\right)$

$$= 10 - 5 + \frac{5i}{2}\sqrt{2} = 5 + \frac{5i}{2}\sqrt{2}$$

Check:

for $x = 5 + \frac{5i}{2}\sqrt{2}$, $y = 5 - \frac{5i}{2}\sqrt{2}$

Eq. (1): $x^2 + y^2 = 25$

$$\left(5 + \frac{5i}{2}\sqrt{2}\right)^2 + \left(5 - \frac{5i}{2}\sqrt{2}\right)^2 \stackrel{?}{=} 25$$

remember $i^2 = -1$

$$\left(25 + \frac{50i}{2}\sqrt{2} - \frac{50}{4}\right) + \left(25 - \frac{50i}{2}\sqrt{2} - \frac{50}{4}\right) \stackrel{?}{=} 25$$

$$50 \neq 25$$

These roots do not check. The roots are extraneous.

for $x = 5 - \frac{5i}{2}\sqrt{2}$, $y = 5 + \frac{5i}{2}\sqrt{2}$

Eq. (1): $x^2 + y^2 = 25$

$$\left(5 - \frac{5i}{2}\sqrt{2}\right)^2 + \left(5 + \frac{5i}{2}\sqrt{2}\right)^2 \stackrel{?}{=} 25$$

remember $i^2 = -1$

$$\left(25 - \frac{50i}{2}\sqrt{2} + \frac{50}{4}\right) + \left(25 + \frac{50i}{2}\sqrt{2} - \frac{50}{4}\right) \stackrel{?}{=} 25$$

$$50 \neq 25.$$

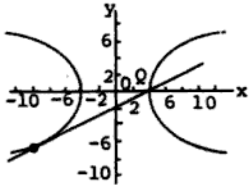
These roots do not check.

**• PROBLEM 600**

Solve for x and y: $\begin{cases} 9x^2 - 16y^2 = 144 & (1) \\ x - 2y = 4. & (2) \end{cases}$

Algebraic solution: We solve equation (2) for x,

$$x = 4 + 2y \qquad (3)$$




and substitute this expression for x in equation (1). This gives

$$9(4 + 2y)^2 - 16y^2 = 144,$$

$$9(16 + 16y + 4y^2) - 16y^2 = 144$$

Dividing both sides by 4, $9(4 + 4y + y^2) - 4y^2 = 36.$

Distributing, $36 + 36y + 9y^2 - 4y^2 = 36$

Combining terms, $36 + 36y + 5y^2 = 36$

Subtracting 36 from both sides, $5y^2 + 36y = 0$

Factoring, $y(5y + 36) = 0$

Whenever the product of two numbers $ab = 0$, either $a = 0$ or $b = 0$; thus either

$$y = 0 \quad \text{or} \quad 5y + 36 = 0$$

$$5y = -36$$

$$y = \frac{-36}{5}$$

Thus, $y = 0, \quad -\frac{36}{5}$.

Placing these values in linear equation (3):

when $y = 0$,

$$x = 4 + 2(0) = 4 + 0 = 4$$

when $y = -\frac{36}{5}$,

$$x = 4 + 2\left(-\frac{36}{5}\right) = 4 - \frac{72}{5} = \frac{20}{5} - \frac{72}{5} = -\frac{52}{5}$$

Thus the two solutions of the equations are seen to be (4, 0) and (- 52/5, - 36/5), which are then the actual coordinates of the points of intersection of the line and the hyperbola to be discussed.

Geometric solution: Construct the graph of each equation and note where the two graphs intersect. The graph of the first equation cuts the x-axis at $x = \pm 4$, and y is imaginary for any value of x between - 4 and 4. The graph consists of the two curved branches in the diagram, and is a hyperbola

The graph of the second equation is a straight line through the points (4, 0) and (0, - 2). This line intersects the hyperbola at the points P and Q, whose coordinates are approximately (4, 0) and (- 10, - 7).

**• PROBLEM 601**

Use the graphical method to find the simultaneous solution set of

$$x^2 + x - 2 > 0$$

and $$\frac{3}{4}x + \frac{3}{2} < 0$$

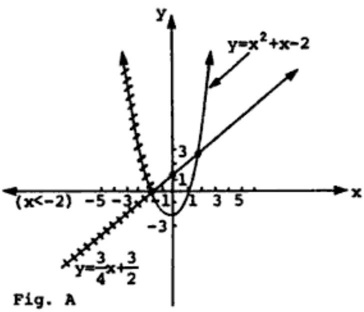
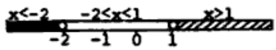



Fig. A




The slashed lines represent the simultaneous solution set.

Fig. B

Solution: We construct the graphs of

$$y = x^2 + x - 2 \text{ and } y = \frac{3}{4}x + \frac{3}{2}$$

Set up tables for both equations to calculate y values.

For $y = x^2 + x - 2$

| x | $x^2 + x - 2$ = | y |
|---|---|---|
| - 4 | $(-4)^2 + (-4) - 2$ | 10 |
| - 3 | $(-3)^2 + (-3) - 2$ | 4 |
| - 2 | $(-2)^2 + (-2) - 2$ | 0 |
| - 1 | $(-1)^2 + (-1) - 2$ | -2 |
| 0 | $(0)^2 + (0) - 2$ | -2 |
| 1 | $(1)^2 + (1) - 2$ | 0 |
| 2 | $(2)^2 + 2 - 2$ | 4 |
| 3 | $(3)^2 + 3 - 2$ | 10 |

For $y = \frac{3}{4}x + \frac{3}{2}$

| x | $\frac{3}{4}x + \frac{3}{2}$ = | y |
|---|---|---|
| -2 | $\frac{3}{4}(-2) + \frac{3}{2}$ | 0 |
| 0 | $\frac{3}{4}(0) + \frac{3}{2}$ | $\frac{3}{2}$ |
| 2 | $\frac{3}{4}(2) + \frac{3}{2}$ | 3 |

See Figure A for graphs.

Now, find the region where the inequality $x^2 + x - 2 > 0$ holds. The function $f(x) = x^2 + x - 2$ can be factored into $(x + 2)(x - 1)$. Set $f(x) = 0$ and find the roots of this equation. Here $x = -2$ and $x = 1$. Mark the roots on the x-axis and consider the regions into which the roots divide the x-axis. They are $x < -2$, $-2 < x < 1$, and $x > 1$. See Figure B.

For each of these regions, choose a value of x and investigate the algebraic signs of the factors of the function f(x). Then look at the sign of their product, f(x). The table summarizes the process.

| | $f(x) = (x + 2)(x - 1)$ | | |
|---|---|---|---|
| Regions | $x < -2$ | $-2 < x < 1$ | $x > 1$ |
| x-value | $x = -3$ | $x = 0$ | $x = 2$ |
| Factors of f(x) | $(x+2)(x-1)$ | $(x+2)(x - 1)$ | $(x+2)(x-1)$ |
| Signs of factors | (−) (−) | (+) (−) | (+) (+) |
| $\therefore$ | $f(x) > 0$ | $f(x) < 0$ | $f(x) > 0$ |

For our problem we now know that the graph of $x^2 + x - 2$ is greater than zero, that is, above the x-axis, for $x < -2$ or $x > 1$.

Call the second function g(x). Thus, $g(x) = \frac{3}{4}x + \frac{3}{2}$. We are interested in finding the values of x for which the function g(x) is negative or when it is below the x-axis. Therefore,

$$\frac{3}{4}x + \frac{3}{2} < 0 \Leftrightarrow \frac{3}{4}x < -\frac{3}{2} \Leftrightarrow 3x < -6 \Leftrightarrow x < -2$$

Hence, the solution set is $\{x \mid x < -2\}$. That is, the graph of $y = \frac{3}{4}x + \frac{3}{2}$ lies below the x- axis when $x<-2$.

Simultaneously we solve for x and we obtain $\{x \mid x < -2\}$. See Figure A.

## QUADRATIC/QUADRATIC(CONIC) COMBINATIONS

**• PROBLEM 602**

**Determine whether or not, the given number pair (x,y) is a solution of the accompanying system of equations.**

**(a) (3,−2); $2x - y = 8$, $x^2 - y^2 = 17$**

**(b) (4,−1); $\sqrt{x} - y = -1$, $xy^2 = 4$**

**Solution:** A given number pair (x,y) is a solution of a system of equations if substitution of (x,y) into each equation of the system results in a valid equation.

(a) Replacing $(3,-2)$ in $2x - y = 8$, with $x = 3$, $y = -2$:

$$2(3) - (-2) = 8$$
$$6 + 2 = 8$$
$$8 = 8 \text{ , which is true.}$$

Replacing $(3,-2)$ in $x^2 - y^3 = 17$:

$$3^2 - (-2)^3 = 17$$
$$9 - (-8) = 17$$
$$9 + 8 = 17$$
$$17 = 17 \text{ , which is true.}$$

Therefore $(3,-2)$ is a solution of this system of equations.

(b) Replacing $(4,-1)$ in $\sqrt{x} - y = -1$, with $x = 4$, $y = -1$:

$$\sqrt{4} - (-1) = -1$$
$$2 + 1 = -1$$
$$3 = -1 \text{ , which is not true;}$$

thus, even though replacing $(4,-1)$ in $xy^2 = 4$ yields $4 = 4$ which is true, $(4,-1)$ is not a solution of this system for it fails to satisfy the first equation.

**• PROBLEM 603**

Obtain the solution set of the following system of equations by substitution:

$$xy = 2 \qquad (1)$$
$$15x^2 + 4y^2 = 64 \qquad (2)$$

Solution: Equation (1) can be solved for y in terms of x by dividing both sides by x,

$$y = \frac{2}{x} \qquad (3)$$

We now replace y by $2/x$ in equation (2) and complete the process of solving as follows:

$$15x^2 + 4\left(\frac{2}{x}\right)^2 = 64$$

$$15x^2 + 4\left(\frac{4}{x^2}\right) = 64 \text{ , squaring } 2/x$$

$$15x^4 + 16 = 64x^2 \text{ , multiplying each member by } x^2$$

$$15x^4 - 64x^2 + 16 = 0, \text{ adding } -64x^2 \text{ to each member and arranging terms}$$

$$\left(x^2-4\right)\left(15x^2-4\right) = 0, \text{ factoring}$$

Whenever the product of two numbers $ab = 0$, either $a = 0$ or $b = 0$. Thus, either

$$x^2 - 4 = 0 \quad \text{or} \quad 15x^2 - 4 = 0$$
$$x^2 = 4 \qquad 15x^2 = 4$$
$$x = \pm\sqrt{4} \qquad x^2 = \frac{4}{15}$$
$$x = \pm 2 \qquad x = \pm\sqrt{\frac{4}{15}}$$

$$x = \pm \frac{2}{\sqrt{15}} = \pm \frac{2}{\sqrt{15}} \cdot \frac{\sqrt{15}}{\sqrt{15}}$$
$$= \pm \frac{2\sqrt{15}}{15}$$

To obtain y, we replace x by all four of its values in eq. (3)

$$y = \frac{2}{\pm 2}, \quad \text{replacing } x = \pm 2 \text{ in (3)}$$
$$= \pm 1$$

$$y = \frac{2}{\pm 2\sqrt{15}/15}, \quad \text{replacing } x \text{ by } \pm 2\sqrt{15}/15 \text{ in (3)}$$
$$= \pm \frac{2(15)}{2\sqrt{15}} = \pm \frac{15}{\sqrt{15}} \cdot \frac{\sqrt{15}}{\sqrt{15}} = \pm \frac{15\sqrt{15}}{15} = \pm \sqrt{15}$$

Therefore the simultaneous solution set is

$$\left\{(2, 1), (-2, -1), (2\sqrt{15}/15, \sqrt{15}), (-2\sqrt{15}/15, -\sqrt{15})\right\}$$

The solution set can be checked by the usual method.

**• PROBLEM 604**

Obtain the simultaneous solution set of

$$4x^2 - 2xy - y^2 = -5 \qquad (1)$$
$$y + 1 = -x^2 - x \qquad (2)$$

Solution: Solving equation (2) for y,

$$y = -x^2 - x - 1 \qquad (3)$$

Replacing y by $\left(-x^2 - x - 1\right)$ in equation (1),

$$4x^2 - 2x\left(-x^2 - x - 1\right) - \left(-x^2 - x - 1\right)^2 = -5$$
$$4x^2 + 2x^3 + 2x^2 + 2x - \left(-x^2 - x - 1\right)\left(-x^2 - x - 1\right) = -5$$
$$4x^2 + 2x^3 + 2x^2 + 2x - \left(x^4 + x^3 + x^2 + x^3 + x^2 + x + x^2 + x + 1\right) = -5$$
$$2x^3 + 6x^2 + 2x - \left(x^4 + 2x^3 + 3x^2 + 2x + 1\right) = -5$$
$$2x^3 + 6x^2 + 2x - x^4 - 2x^3 - 3x^2 - 2x - 1 = -5$$
$$-x^4 + 3x^2 - 1 = -5$$
$$-x^4 + 3x^2 + 4 = 0 \qquad (4)$$

Equation (4) is in quadratic form. This can be seen more easily by the following: divide both sides by -1; thus $x^4 - 3x^2 - 4 = 0$, and this can be rewritten as $\left(x^2\right)^2 - 3x^2 - 4 = 0$, which is in quadratic form. Now, factoring, we have:

$$\left(x^2 - 4\right)\left(x^2 + 1\right) = 0$$

Whenever the product of two numbers $ab = 0$, either $a = 0$ or $b = 0$. Thus, either

$$\begin{array}{llcll} x^2 - 4 = 0 & & \text{or} & x^2 + 1 = 0 & \\ x^2 & = 4 & & x^2 & = -1 \\ x & = \pm\sqrt{4} & & x & = \pm\sqrt{-1} \\ x & = \pm 2 & & x & = \pm i \end{array}$$

Now we obtain the value of y corresponding to each of the four values of x: replacing x by 2 in equation (3),

$$\begin{aligned} y &= -x^2 - x - 1 \\ &= -(2)^2 - 2 - 1 \\ &= -4 - 2 - 1 \\ &= -7 \end{aligned}$$

Replacing x by - 2 in eq. (3),

$$\begin{aligned} y &= -(-2)^2 - (-2) - 1 \\ &= -(4) + 2 - 1 \\ &= -3 \end{aligned}$$

Replacing x by i in eq. (3),

$$\begin{aligned} y &= -(i)^2 - i - 1 \\ &= -(-1) - i - 1 \\ &= -i \end{aligned}$$

Replacing x by -i in eq. (3),

$$\begin{aligned} y &= -(-i)^2 - (-i) - 1 \\ &= -(-1) + i - 1 \\ &= i \end{aligned}$$

Therefore our solution set is

$$\left\{(2,-7),\ (-2,-3),\ (i,-i),\ (-i,i)\right\}$$

which can be verified by checking.

**• PROBLEM 605**

Solve the system: $x^2 + y^2 = 1$ (1)

$x^2 - 1 = y.$ (2)

**Solution:** Although the value of y in equation (2) can be substituted in equation (1), the result will be a fourth-degree equation. Hence, we substitute the value of $x^2$ from equation (2).

$$x^2 - 1 = y$$

$$x^2 = y + 1.$$

Replacing $x^2$ by $y + 1$ in equation (1),

$$y + 1 + y^2 = 1.$$

Add (-1) to both sides, $y^2 + y = 0$.
Factor out y, $y(y + 1) = 0$.

Whenever a product of two numbers $ab = 0$, either $a = 0$ or $b = 0$; hence

$$y = 0, \quad y + 1 = 0 \quad \text{or}$$

$$y = 0, \quad y = -1.$$

Substituting $y = 0$ in equation (1),

$$x^2 + 0 = 1$$

$$x^2 = 1$$

$$x = \pm 1.$$

Two solutions are (1,0) and (-1,0).
Substituting $y = -1$ in equation (1),

$$x^2 + (-1)^2 = 1$$

$$x^2 + 1 = 1$$

$$x^2 = 0$$

$$x = 0.$$

The solution (0,-1) is considered a double root in the sense that if $y = -1$, $x = \sqrt{0}$ or $x = -\sqrt{0}$. The solution set for the system is $\{(1,0), (-1,0), (0,-1)\}$.

Check: To verify that (1,0) is a root, replace x by 1 and y by 0 in each equation.

$$x^2 + y^2 = 1 \tag{1}$$

$$(1)^2 + 0^2 = 1$$

$$1 = 1$$

$$x^2 - 1 = y \tag{2}$$

$$(1)^2 - 1 = 0$$

$$1 - 1 = 0$$

$$0 = 0.$$

To verify that (-1,0) is a root, replace x by -1 and y by 0 in each equation

$$x^2 + y^2 = 1 \qquad (1)$$

$$(1)^2 + 0 = 1$$

$$1 + 0 = 1$$

$$1 = 1$$

$$x^2 - 1 = y \qquad (2)$$

$$(-1)^2 - 1 = 0$$

$$1 - 1 = 0$$

$$0 = 0.$$

To verify that (0,-1) is a root, replace x by 0 and y by -1 in each equation,

$$x^2 + y^2 = 1 \qquad (1)$$

$$0 + (-1)^2 = 1$$

$$0 + 1 = 1$$

$$1 = 1$$

$$x^2 - 1 = y \qquad (2)$$

$$(0)^2 - 1 = -1$$

$$0 - 1 = -1$$

$$-1 = -1.$$

**• PROBLEM 606**

Obtain the solution set of

$$x^2 + 4xy - 7x = 12 \qquad (1)$$

$$3x^2 - 4xy + 4x = 15 \qquad (2)$$

Solution: Since the sum of the xy terms in the left members of equations (1) and (2) is zero, we proceed as follows: Adding equations (1) and (2),

$$4x^2 - 3x = 27$$

$$4x^2 - 3x - 27 = 0$$

Equations in the form $ax^2 + bx + c = 0$ can be solved using the quadratic formula, $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$. In our case, $a = 4$, $b = -3$, and $c = -27$, thus

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(4)(-27)}}{2(4)}$$

$$x = \frac{3 \pm \sqrt{9 + 432}}{8} = \frac{3 \pm \sqrt{441}}{8} = \frac{3 \pm 21}{8}$$

$$x = \frac{3 + 21}{8} = \frac{24}{8} = 3 \quad \text{or} \quad x = \frac{3 - 21}{8} = \frac{-18}{8} = -\frac{9}{4}$$

Thus $x = 3, -\frac{9}{4}$.

We find the second numbers in the solution pairs as follows: Replacing x by 3 in equation (1),

$$x^2 + 4xy - 7x = 12$$

$$3^2 + 4(3)y - 7(3) = 12$$

$$9 + 12y - 21 = 12$$

$$12y - 12 = 12$$

$$12y = 24$$

$$y = 2$$

Hence one simultaneous solution pair is (3,2). Then: Replacing x by -9/4 in equation (1),

$$x^2 + 4xy - 7x = 12$$

$$\left(\frac{-9}{4}\right)^2 + 4\left(\frac{-9}{4}\right)y - 7\left(\frac{-9}{4}\right) = 12$$

$$\frac{81}{16} - 9y + \frac{63}{4} = 12$$

Multiplying both sides by 16,

$$81 - 144y + 252 = 192$$

$$-144y + 333 = 192$$

$$-144y = -141$$

$$y = \frac{-141}{-144} = \frac{47}{48}$$

Therefore a second simultaneous solution pair is $\left(-\frac{9}{4}, \frac{47}{48}\right)$, and the simultaneous solution set is

$$\left\{(3,2),\ \left(-\frac{9}{4}, \frac{47}{48}\right)\right\}$$

Check: Using (3,2), we have:

From equation (1):

$$(3)^2 + 4(3)(2) - 7(3) = 9 + 24 - 21 = 33 - 21 = 12$$

From equation (2):

$$3(3)^2 - 4(3)(2) + 4(3) = 27 - 24 + 12 = 3 + 12 = 15$$

Using $\left(-\frac{9}{4}, \frac{47}{48}\right)$, we obtain:

From equation (1):

$$\left(-\frac{9}{4}\right)^2 + 4\left(\frac{-9}{4}\right)\left(\frac{47}{48}\right) - 7\left(\frac{-9}{4}\right) = \frac{81}{16} - \frac{141}{16} + \frac{63}{4} = \frac{81-141+252}{16}$$

$$= \frac{192}{16} = 12$$

From equation (2):

$$3\left(\frac{-9}{4}\right)^2 - 4\left(\frac{-9}{4}\right)\left(\frac{47}{48}\right) + 4\left(\frac{-9}{4}\right) = \frac{243}{16} + \frac{141}{16} - 9 = \frac{243+141-144}{16}$$

$$= \frac{240}{16} = 15$$

**Thus, the two pairs of solutions obtained are valid.**

**• PROBLEM 607**

Find the simultaneous solution set of the equations

$$3x^2 - 2y^2 - 6x = -23 \qquad (1)$$

$$x^2 + y^2 - 4x = 13 \qquad (2)$$

Solution: Since each of the given equations contains one term in $y^2$ and no other term involving y, we eliminate $y^2$ and then complete the process of solving as follows:

| | | |
|---|---|---|
| $3x^2 - 2y^2 - 6x = -23$ | (3) | Eq.(1) recopied |
| $2x^2 + 2y^2 - 8x = 26$ | (4) | multiplying Eq.(2) by 2 |
| $5x^2 \quad - 14x = 3$ | | adding Equations (3) and (4) |
| $5x^2 - 14x - 3 = 0$ | | adding $-3$ to each member |

We can solve equations in the form $ax^2 + bx + c = 0$ by the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. In our case, $a = 5$, $b = -14$, and $c = -3$, thus,

$$x = \frac{-(-14) \pm \sqrt{(-14)^2 - 4(5)(-3)}}{2(5)}$$

$$x = \frac{14 \pm \sqrt{196 + 60}}{10}$$

$$= \frac{14 \pm \sqrt{256}}{10}$$

$$= \frac{14 \pm 16}{10}$$

$$= \frac{14 + 16}{10} \qquad \text{or} \qquad \frac{14 - 16}{10}$$

$$= \frac{30}{10} \qquad \text{or} \qquad \frac{-2}{10}$$

Thus, $x = 3$ or $-\frac{1}{5}$.

Now, solve for y by substituting the two values for x in eq. (2):

$$(3)^2 + y^2 - 4(3) = 13 \qquad \text{replacing x by 3 in eq.(2)}$$

$$9 + y^2 - 12 = 13$$

$$-3 + y^2 = 13$$

$$y^2 = 16 \qquad \text{adding 3 to each number}$$

$$y = \pm 4$$

Hence, when $x = 3$, $y = \pm 4$. We continue by replacing x by $-\frac{1}{5}$.

$$\left(-\frac{1}{5}\right)^2 + y^2 - 4\left(-\frac{1}{5}\right) = 13 \qquad \text{replacing x by } -\frac{1}{5} \text{ in eq.(2)}$$

$$\frac{1}{25} + y^2 + \frac{4}{5} = 13$$

$$1 + 25y^2 + 20 = 325 \qquad \text{multiplying each member by 25}$$

$$25y^2 = 304 \qquad \text{adding } -21 \text{ to each member}$$

$$y^2 = \frac{304}{25}$$

$$y = \pm\sqrt{\frac{304}{25}}$$

$$y = \pm\sqrt{\frac{(16)(19)}{25}} = \pm\frac{\sqrt{16}\ \sqrt{19}}{\sqrt{25}} = \pm\frac{4\ \sqrt{19}}{5}$$

Consequently, if $x = -\frac{1}{5}$, $y = \pm 4\sqrt{19}/5$. Therefore the simultaneous solution set is

$$\left\{(3,4),(3,-4),\left(-\frac{1}{5}, 4\sqrt{19}/5\right),\left(-\frac{1}{5}, -4\sqrt{19}/5\right)\right\}.$$

Each solution pair can be checked by replacing x and y in the given equations by the appropriate number from the solution pair.

**• PROBLEM 608**

Solve the system

$$2x^2 - 3xy + 4y^2 = 3 \qquad (1)$$

$$x^2 + xy - 8y^2 = -6 \qquad (2)$$

Solution: Multiply both sides of the first equation by 2.

$$2\left(2x^2 - 3xy + 4y^2\right) = 2(3)$$

$$4x^2 - 6xy + 8y^2 = 6 \tag{3}$$

Add equation (3) to equation (2):

$$\begin{array}{r} x^2 + xy - 8y^2 = -6 \\ 4x^2 - 6xy + 8y^2 = 6 \\ \hline 5x^2 - 5xy \quad = 0 \end{array} \tag{4}$$

Factoring out the common factor, 5x, from the left side of equation (4):

$$5x(x - y) = 0$$

Whenever a product $ab = 0$, where $a,b$ are any two numbers, either $a = 0$ or $b = 0$. Hence, either

$$\begin{array}{lcl} 5x = 0 & \text{or} & x - y = 0 \\ x = 0/5 & & x = y \\ x = 0 & & \end{array}$$

Substituting $x = 0$ in equation (1):

$$\begin{aligned} 2(0)^2 - 3(0)y + 4y^2 &= 3 \\ 0 - 0 + 4y^2 &= 3 \\ 4y^2 &= 3 \\ y^2 &= 3/4 \\ y &= \pm\sqrt{3/4} \\ &= \pm\sqrt{3}/\sqrt{4} \\ &= \pm\frac{\sqrt{3}}{2} \end{aligned}$$

Hence, two solutions are: $\left(0, +\frac{\sqrt{3}}{2}\right)$, $\left(0, -\frac{\sqrt{3}}{2}\right)$.

Substituting $x$ for $y$ $(x = y)$ in equation (1):

$$\begin{aligned} 2x^2 - 3x(x) + 4(x)^2 &= 3 \\ 2x^2 - 3x^2 + 4x^2 &= 3 \\ -x^2 + 4x^2 &= 3 \\ 3x^2 &= 3 \\ x^2 &= 3/3 \\ x^2 &= 1 \\ x = \pm\sqrt{1} &= \pm 1 . \end{aligned}$$

Therefore, when $x = 1$, $y = x = 1$. Also, when $x = -1$, $y = x = -1$. Hence, two other solutions are: $(1,1)$ and $(-1,-1)$. Thus the four solutions of the system are

$$\left(0, \frac{\sqrt{3}}{2}\right), \left(0, -\frac{\sqrt{3}}{2}\right), (1,1) \text{ and } (-1,-1).$$

**• PROBLEM 609**

Obtain the simultaneous solution set of the equations

$$2x^2 + 3y^2 = 21 \tag{1}$$

$$3x^2 - 4y^2 = 23 \tag{2}$$

<u>Solution:</u> We arbitrarily select $y$ as the unknown to be eliminated and proceed: Multiplying equation (1) by 4,

$$8x^2 + 12y^2 = 84 \qquad (3)$$

Multiplying equation (2) by 3,

$$\underline{9x^2 - 12y^2 = 69} \qquad (4)$$

Adding equations (3) and (4),

$$17x^2 = 153$$

Dividing both sides by 17,

$$x^2 = 9$$

Taking the square root of both members,

$$x = \pm 3$$

We now replace $x$ by $3$ and $-3$ in equation (1) and solve for $y$. We get

$$2(3)^2 + 3y^2 = 21 \quad \text{or} \quad 2(-3)^2 + 3y^2 = 21$$

$$2(9) + 3y^2 = 21 \qquad 2(9) + 3y^2 = 21$$

Thus in either case we obtain

$$18 + 3y^2 = 21$$

Adding -18 to each member,

$$3y^2 = 3$$

$$y^2 = 1$$

$$y = \pm 1$$

Consequently, the solution set is $\{(3,1),(3,-1),(-3,1),(-3,-1)\}$.

Check: To verify these solutions we replace $x$ and $y$ by each pair in equations (1) and (2). Checking (3,1) in equation (1):

$$2x^2 + 3y^2 = 21$$

$$2(3)^2 + 3(1)^2 = 21$$

$$2(9) + 3(1)^2 = 21$$

$$2(9) + 3(1) = 21$$

$$18 + 3 = 21$$

$$21 = 21$$

in equation (2):

$$3x^2 - 4y^2 = 23$$

$$3(3)^2 - 4(1)^2 = 23$$

$$3(9) - 4(1) = 23$$

$$27 - 4 = 23$$

$$23 = 23$$

Checking (3,-1) in equation (1):

$$2x^2 + 3y^2 = 21$$

$$2(3)^2 + 3(-1)^2 = 21$$

$$2(9) + 3(1) = 21$$

$$18 + 3 = 21$$

$$21 = 21$$

in equation (2):

$$3x^2 - 4y^2 = 23$$

$$3(3)^2 - 4(-1)^2 = 23$$
$$3(9) - 4(1) = 23$$
$$27 - 4 = 23$$
$$23 = 23$$

Checking (-3,1) in equation (1):

$$2x^2 + 3y^2 = 21$$
$$2(-3)^2 + 3(1)^2 = 21$$
$$2(9) + 3(1) = 21$$
$$18 + 3 = 21$$
$$21 = 21$$

in equation (2):

$$3x^2 - 4y^2 = 23$$
$$3(-3)^2 - 4(1)^2 = 23$$
$$3(9) - 4(1) = 23$$
$$27 - 4 = 23$$
$$23 = 23$$

Checking (-3,-1) in equation (1):

$$2x^2 + 3y^2 = 21$$
$$2(-3)^2 + 3(-1)^2 = 21$$
$$2(9) + 3(1) = 21$$
$$18 + 3 = 21$$
$$21 = 21$$

in equation (2):

$$3x^2 - 4y^2 = 23$$
$$3(-3)^2 - 4(-1)^2 = 23$$
$$3(9) - 4(1) = 23$$
$$27 - 4 = 23$$
$$23 = 23$$

**• PROBLEM 610**

Obtain the simultaneous solution set of

$$3x^2 + 3y^2 + x - 2y = 20 \qquad (1)$$

$$2x^2 + 2y^2 + 5x + 3y = 9 \qquad (2)$$

Solution: Multiplying equation (1) by 2,

$$6x^2 + 6y^2 + 2x - 4y = 40 \qquad (3)$$

Multiplying equation (2) by 3,

$$6x^2 + 6y^2 + 15x + 9y = 27 \qquad (4)$$

Subtracting equation (4) from (3),

$$-13x - 13y = 13 \qquad (5)$$

Now we solve Equation (5) simultaneously with Equation (2) and complete the process of solving as indicated below.

Dividing equation (5) by 13, $y = -x - 1$ (6)

$2x^2 + 2(-x-1)^2 + 5x + 3(-x-1) = 9$, replacing $y$ by $-x-1$ in Eq. (2)

$2x^2 + 2(x^2 + 2x + 1) + 5x - 3x - 3 = 9$

$2x^2 + 2x^2 + 4x + 2 + 5x - 3x - 3 - 9 = 0$, distributing and adding $-9$ to each member

$4x^2 + 6x - 10 = 0$, combining similar terms

$2x^2 + 3x - 5 = 0$, dividing by 2

$(x-1)(2x+5) = 0$, factoring

Whenever the product of two numbers $ab = 0$, either $a = 0$ or $b = 0$; thus either

$x - 1 = 0$ or $2x + 5 = 0$,

and $x = 1$ or $x = -\frac{5}{2}$

We find the corresponding values of y as follows:

$y = -1 - 1 = -2$, repacing x by 1 in Eq. (6)

$y = \frac{5}{2} - 1 = \frac{3}{2}$, replacing x by $-\frac{5}{2}$ in Eq. (6).

Consequently the solution set is$\{(1, -2), (-5/2, 3/2)\}$.

Check: Replacing x and y by $(1, -2)$ in equations (1) and (2);

$$3x^2 + 3y^2 + x - 2y = 20 \qquad (1)$$

$$3(1)^2 + 3(-2)^2 + 1 - 2(-2) \doteq 20$$

$$3(1) + 3(4) + 1 + 4 = 20$$

$$3 + 12 + 5 = 20$$

$$20 = 20$$

$$2x^2 + 2y^2 + 5x + 3y = 9 \qquad (2)$$

$$2(1)^2 + 2(-2)^2 + 5(1) + 3(-2) = 9$$

$$2(1) + 2(4) + 5 - 6 = 9$$

$$2 + 8 + 5 - 6 = 9$$

$$15 - 6 = 9$$

$$9 = 9$$

Replacing x and y by (- 5/2, 3/2) in equations (1) and (2):

$$3x^2 + 3y^2 + x - 2y = 20 \qquad (1)$$

$$3\left(-\frac{5}{2}\right)^2 + 3\left(\frac{3}{2}\right)^2 + \left(-\frac{5}{2}\right) - 2\left(\frac{3}{2}\right) = 20$$

$$3\left(\frac{25}{4}\right) + 3\left(\frac{9}{4}\right) + \frac{-5}{2} - \frac{6}{2} = 20$$

$$\frac{75}{4} + \frac{27}{4} - \frac{10}{4} - \frac{12}{4} = 20$$

$$\frac{80}{4} = 20$$

$$20 = 20$$

$$2x^2 + 2y^2 + 5x + 3y = 9 \qquad (2)$$

$$2\left(-\frac{5}{2}\right)^2 + 2\left(\frac{3}{2}\right)^2 + 5\left(-\frac{5}{2}\right) + 3\left(\frac{3}{2}\right) = 9$$

$$2\left(\frac{25}{4}\right) + 2\left(\frac{9}{4}\right) - \frac{25}{2} + \frac{9}{2} = 9$$

$$\frac{50}{4} + \frac{18}{4} - \frac{50}{4} + \frac{18}{4} = 9$$

$$\frac{36}{4} = 9$$

$$9 = 9$$

**• PROBLEM 611**

Solve $\qquad x^2 + y^2 = 5 \qquad (1)$

$\qquad x^2 - xy + y^2 = 7 \qquad (2)$

<u>Solution:</u> Subtracting the second equation from the first, we have

$$x^2 + y^2 = 5$$

$$\frac{-(x^2 - xy + y^2 = 7)}{xy = -2} \qquad (3)$$

Thus let us consider the system

$$x^2 + y^2 = 5$$

$$xy = -2.$$

Solving the second equation for y, we obtain

$$y = -\frac{2}{x}$$

Substituting this result in equation (1), we have,

$$x^2 + \left(-\frac{2}{x}\right)^2 = 5$$

$$x^2 + \frac{(-2)^2}{(x)^2} = 5$$

$$x^2 + \frac{4}{x^2} = 5.$$

Then we multiply both sides by $x^2$ to obtain

$$x^2\left(x^2 + \frac{4}{x^2}\right) = 5(x^2)$$

$$x^2 \cdot x^2 + x^2\left(\frac{4}{x^2}\right) = 5x^2$$

$$x^4 + 4 = 5x^2$$

or $$x^4 - 5x^2 + 4 = 0.$$

Hence $(x^2 - 1)(x^2 - 4) = 0$. Whenever the product of two numbers $ab = 0$, either $a = 0$ or $b = 0$; therefore $x^2 = 1$ or $x^2 = 4$. Thus $x = \sqrt{1} = \pm 1$, or $x = \sqrt{4} = \pm 2$. Substituting these values in turn in the equation $xy = -2$ we obtain the solutions

For $x = 1$, $1(y) = -2$, $y = -2$

For $x = -1$, $-1(y) = -2$, $y = 2$

For $x = 2$, $2y = -2$, $y = -1$

For $x = -2$, $-2y = -2$, $y = 1$.

Therefore the solution to this system of equations is

$$(1,-2),\ (-1,2),\ (2,-1),\ (-2,1).$$

Check: To verify that these four pairs are indeed solutions we replace x and y by each pair in equations (1) and (2). Thus checking (1,-2) in (1),

$$x^2 + y^2 = 5$$

$$1^2 + (-2)^2 = 5$$

$$1 + 4 = 5$$

$$5 = 5$$

and in (2), $x^2 - xy + y^2 = 7$

$$(1)^2 - 1(-2) + (-2)^2 = 7$$

$$1 + 2 + 4 = 7$$

$$7 = 7.$$

Checking $(-1,2)$ in (1),

$$x^2 + y^2 = 5$$

$$(-1)^2 + (2)^2 = 5$$

$$1 + 4 = 5$$

$$5 = 5$$

and in (2), $$x^2 - xy + y^2 = 7$$

$$(-1)^2 - (-1)(2) + (2)^2 = 7$$

$$1 + 2 + 4 = 7$$

$$7 = 7.$$

Checking $(2,-1)$ in (1),

$$x^2 + y^2 = 5$$

$$(2)^2 + (-1)^2 = 5$$

$$4 + 1 = 5$$

$$5 = 5$$

and in (2),

$$x^2 - xy + y^2 = 7$$

$$(2)^2 - (2)(-1) + (-1)^2 = 7$$

$$4 + 2 + 1 = 7$$

$$7 = 7.$$

Checking $(-2,1)$ in (1),

$$x^2 + y^2 = 5$$

$$(-2)^2 + (1)^2 = 5$$

$$4 + 1 = 5$$

$$5 = 5$$

and in (2),

$$x^2 - xy + y^2 = 7$$

$$(-2)^2 - (-2)(1) + (1)^2 = 7$$

$$4 + 2 + 1 = 7$$

$$7 = 7.$$

Thus our 4 pairs are all valid solutions and the solution set is $\{(1,-2), (-1,2), (2,-1)(-2,1)\}$.

**• PROBLEM 612**

Solve the following system of equations completely:

$$\begin{cases} x^2 - 5xy + 6y^2 = 0, \\ xy - y^2 = 2. \end{cases}$$

Solution: Upon factoring the left member of the first equation, we obtain $(x - 3y)(x - 2y) = o$. Whenever the product $ab = 0$ where a and b are any two numbers, either $a = 0$ or $b = 0$. Hence, either $x - 3y = 0$ or $x - 2y = 0$.

Each of these is taken with the second given equation, and we obtain the following systems of equations:

System 1: $\begin{cases} xy - y^2 = 2, \\ x - 3y = 0, \end{cases}$ and System 2: $\begin{cases} xy - y^2 = 2, \\ x - 2y = 0. \end{cases}$

Any solution of either system will be a solution of the given system. For system 1:

$$\begin{cases} xy - y^2 = 2 & \text{(a)} \\ x - 3y = 0 & \text{(b)}. \end{cases}$$

Multiply both sides of equation (b) by y:

$$y(x - 3y) = y(0)$$

$yx - 3y^2 = 0$ or, by the commutative property of multiplication,

$$xy - 3y^2 = 0 \qquad \text{(c)}$$

Subtract equation (c) from equation (a):

$$\begin{array}{r} xy - y^2 = 2 \\ -\left(xy - 3y^2\right) = -0 \\ \hline 2y^2 = 2 \end{array}$$

Divide both sides by 2:

$$\frac{2y^2}{2} = \frac{2}{2}$$

$$\text{or } y^2 = 1$$

Take the square root of both sides

$$y = \pm\sqrt{1}$$

$$y = \pm 1$$

Substitute $y = -1$ in equation (a):

$$x(-1) - (-1)^2 = 2$$
$$-x - 1 = 2$$

Add x to both sides:

$$-x - 1 + x = 2 + x$$
$$-1 = 2 + x$$

Subtract 2 from both sides:

$$-1 - 2 = 2 + x - 2$$
$$-3 = x.$$

Hence, one solution is $(-3, -1)$.

Substitute $y = 1$ in equation (a):

$$x(1) - (1)^2 = 2$$
$$x - 1 = 2$$

Add 1 to both sides:

$$x - 1 + 1 = 2 + 1$$
$$x = 3$$

Hence, another solution is $(3,1)$.

For System 2:

$$\begin{cases} xy - y^2 = 2 & \text{(d)} \\ x - 2y = 0 & \text{(e)} \end{cases}$$

Multiply both sides of equation (e) by y:

$$y(x - 2y) = y(0)$$

$yx - 2y^2 = 0$ or, by the commutative property of multiplication,

$$xy - 2y^2 = 0 \text{(f)}$$

Subtract equation (f) from equation (d):

$$\begin{array}{r} xy - y^2 = 2 \\ -(xy - 2y^2) = -0 \\ \hline y^2 = 2 \end{array}$$

Take the square root of both sides:

$$y = \pm\sqrt{2}.$$

Substitute $y = -\sqrt{2}$ in equation (d):

$$x(-\sqrt{2}) - (-\sqrt{2})^2 = 2$$

$x(-\sqrt{2}) - (2) = 2$ or, by the commutative property of multiplication,

$$-\sqrt{2}\,x - 2 = 2$$

Add 2 to both sides:

$$-\sqrt{2}\,x - 2 + 2 = 2 + 2$$

$$-\sqrt{2}\,x = 4$$

Divide both sides by $(-\sqrt{2})$:

$$\frac{-\sqrt{2}\,x}{-\sqrt{2}} = \frac{4}{-\sqrt{2}}$$

$$x = \frac{4}{-\sqrt{2}}$$

Multiply the numerator and denominator of the fraction on the right side by $\sqrt{2}$ in order to remove the radical sign in the denominator. (Note: This process is called "rationalizing the denominator")

$$x = \frac{4}{-\sqrt{2}} = \frac{\sqrt{2}(4)}{\sqrt{2}(-\sqrt{2})} = \frac{4\sqrt{2}}{-2} = -2\sqrt{2}$$

Hence, another solution is $(-2\sqrt{2}, -\sqrt{2})$.

Substitute $y = +\sqrt{2} = \sqrt{2}$ in equation (d):

$$x(\sqrt{2}) - (\sqrt{2})^2 = 2$$

$x(\sqrt{2}) - 2 = 2$ or, by the commutative property of multiplication,

$$\sqrt{2}x - 2 = 2.$$

Add 2 to both sides.

$$\sqrt{2}x - 2 + 2 = 2 + 2$$

$$\sqrt{2}x = 4$$

Divide both sides by $\sqrt{2}$:

$$\frac{\sqrt{2}x}{\sqrt{2}} = \frac{4}{\sqrt{2}}$$

$$x = \frac{4}{\sqrt{2}}$$

Rationalize the denominator on the right side:

$$x = \frac{4}{\sqrt{2}} = \frac{\sqrt{2}(4)}{\sqrt{2}(\sqrt{2})} = \frac{4\sqrt{2}}{2} = 2\sqrt{2}.$$

Therefore, another solution is $(2\sqrt{2}, \sqrt{2})$.

Hence, the four solutions are

$$(3, 1),\ (-3, -1),\ (2\sqrt{2}, \sqrt{2}),\ (-2\sqrt{2}, -\sqrt{2}),$$

where the first two are the solutions of the first system, and the second two are the solutions of the second system.

**• PROBLEM 613**

Solve the system of equations,

$$\begin{cases} x^2 + y^2 = 25, \\ xy = 12. \end{cases}$$

Solution: If the second equation of the system is multiplied by 2 and subtracted from the first we obtain $x^2 - 2xy + y^2 = 1$, which can be written as $(x - y)^2 = 1$. This last equation is equivalent to the two equations

$x - y = 1$ and $x - y = -1$ (since squaring both of these gives us back the original). Thus the solutions of the given system may be found by solving the two systems

$$(1)\begin{cases} xy = 12 \\ x - y = 1, \end{cases} \quad \text{and} \quad (2)\begin{cases} xy = 12 \\ x - y = -1. \end{cases}$$

For system (1):

$$xy = 12 \qquad (a)$$

$$x - y = 1 \qquad (b)$$

Solving equation (b) for x and then substituting this expression for x in equation (a), we obtain:

$$x = 1 + y \qquad (b')$$

$$(1 + y)y = 12 \qquad (a')$$

$$y + y^2 = 12$$

$$y^2 + y - 12 = 0$$

Factoring, $\quad (y + 4)(y - 3) = 0$

Finding all values of y which make this product zero, we set each factor equal to zero:

$$(y + 4) = 0 \qquad\qquad (y - 3) = 0$$

$$y = -4 \qquad \text{or} \qquad y = 3$$

Then from (b'), $x = 1 + y = 1 + (-4) = -3$ or

$$x = 1 + y = 1 + 3 = 4$$

The results of the original system thus far are $(-3, -4)$ and $(4, 3)$. For system (2):

$$xy = 12 \qquad (A)$$

$$x - y = -1 \qquad (B)$$

Then, as for system (1):

$$x = y - 1 \qquad (B')$$

$$(y - 1)y = 12 \qquad (A')$$

$$y^2 - y = 12$$

$$y^2 - y - 12 = 0$$

Factoring, $(y - 4)(y + 3) = 0$

Setting each factor = 0,

$$(y - 4) = 0 \qquad \text{or} \qquad (y + 3) = 0$$

$$y = 4 \qquad \text{or} \qquad y = -3$$

Then from (B'), $x = y - 1 = 4 - 1 = 3$

$$x = y - 1 = -3 - 1 = -4$$

The results for system (2) are $(3, 4)$ and $(-4, -3)$.

Thus the results for systems (1) and (2) and for the original system are

$$(4, 3); \quad (-3, -4); \quad (-4, -3); \quad (3, 4).$$

Solve the system

$$3x^2 + 4y^2 = 8 \qquad (1)$$
$$x^2 - y^2 = 5. \qquad (2)$$

Solution: Substituting $u$ for $x^2$ and $v$ for $y^2$ leads to the system of linear equations

$$3u + 4v = 8 \qquad (3)$$
$$u - v = 5 \qquad (4)$$

Multiplying both sides of equation (4) by 3 we obtain:

$$3(u - v) = 3(5)$$
$$3u - 3v = 15 \qquad (5)$$

Subtracting equation (5) from equation (3):

$$\begin{array}{r} 3u + 4v = 8 \\ \underline{-(3u - 3v = 15)} \\ 7v = -7 \end{array}$$

Dividing both sides by 7:

$$\frac{7v}{7} = \frac{-7}{7}$$
$$v = -1$$

Since $v = y^2$,

$$v = -1 = y^2$$
$$\pm\sqrt{-1} = y$$

Since $i^2 = -1$ or $i = \sqrt{-1}$,

$$\pm i = y .$$

Substituting the value $y = i$ into equation (2):

$$x^2 - (i)^2 = 5$$
$$x^2 - i^2 = 5$$
$$x^2 - (-1) = 5$$
$$x^2 + 1 = 5$$

Subtracting 1 from both sides:

$$x^2 + \cancel{1} - \cancel{1} = 5 - 1$$
$$x^2 = 4$$
$$x = \pm\sqrt{4} = \pm 2$$

Hence, two solutions of the original system of equations are: $(2,i)$, $(-2,i)$

Substituting the value $y = -i$ into equation (2):

$$x^2 - (-i)^2 = 5$$
$$x^2 - (i^2) = 5$$
$$x^2 - (-1) = 5$$
$$x^2 + 1 = 5$$

Subtracting 1 from both sides:

$$x^2 + \cancel{1} - \cancel{1} = 5 - 1$$

$$x^2 = 4$$

$$x = \pm\sqrt{4} = \pm 2$$

Hence, two other solutions of the original system of equations are:

$$(2,-1),\ (-2,-1).$$

Therefore, the solution set of the original system of equations is:

$$\{(2,1),(-2,1),(2,-1),(-2,-1)\}\ .$$

Other systems that involve quadratic equations may be solved by replacing the given system with an equivalent system that is easier to solve.

**• PROBLEM 615**

Solve $\quad x^2 + 3xy = 28,$

$\quad\quad\quad x^2 + y^2 = 20.$

Solution: Let $y = mx$ and substitute in both equations. From the first equation, we have

$$x^2 + 3mx^2 = 28.$$

Solving this equation for $x^2$: use the distributive law in relation to the $x^2$ terms,

$$x^2 + 3mx^2 = 28$$

$$(1 + 3m)x^2 = 28.$$

Divide both sides of this equation by $(1 + 3m)$:

$$\frac{\cancel{(1 + 3m)}x^2}{\cancel{(1 + 3m)}} = \frac{28}{(1 + 3m)}$$

$$x^2 = \frac{28}{1 + 3m} \qquad (1)$$

From the second equation, we have

$$x^2 + m^2x^2 = 20.$$

Solving this equation for $x^2$: use the distributive law in relation to the $x^2$ terms,

$$x^2 + m^2x^2 = 20$$

$$(1 + m^2)x^2 = 20.$$

Divide both sides of this equation by $(1 + m^2)$:

$$\frac{\cancel{(1 + m^2)}x^2}{\cancel{(1 + m^2)}} = \frac{20}{(1 + m^2)}$$

$$x^2 = \frac{20}{1 + m^2}. \qquad (2)$$

Equating the values obtained for $x^2$ in equations (1) and (2):

$$\frac{28}{(1 + 3m)} = \frac{20}{1 + m^2}$$

Multiply both sides of this equation by $(1 + 3m)$:

$$\cancel{(1 + 3m)}\left[\frac{28}{\cancel{1 + 3m}}\right] = (1 + 3m)\left[\frac{20}{1 + m^2}\right]$$

$$28 = \frac{(1 + 3m)(20)}{(1 + m^2)}$$

Multiply both sides of this equation by $\left(1 + m^2\right)$:

$$\left(1 + m^2\right)(28) = \cancel{\left(1 + m^2\right)}\left[\frac{1 + 3m}{\cancel{1 + m^2}}\right](20)$$

$$\left(1 + m^2\right)(28) = (1 + 3m)(20)$$

$$28 + 28m^2 = 20 + 60m.$$

Subtract $(20 + 60m)$ from both sides of this equation:

$$28 + 28m^2 - (20 + 60m) = 20 + 60m - (20 + 60m)$$

$$28 + 28m^2 - 20 - 60m = \cancel{20} + \cancel{60m} - \cancel{20} - \cancel{60m}$$

$$28 + 28m^2 - 20 - 60m = 0$$

or $\quad 28m^2 - 60m + 8 = 0.$

Divide both sides of this equation by 4:

$$\frac{28m^2 - 60m + 8}{4} = \frac{0}{4}$$

$$7m^2 - 15m + 2 = 0.$$

Factor the left side of this equation into a product of two polynomials:

$$(7m - 1)(m - 2) = 0. \qquad (3)$$

Whenever a product of two numbers $ab = 0$, where a and b are any two numbers, either $a = 0$ or $b = 0$. Equation (3) can be rewritten as:

$$7m - 1 = 0 \quad \text{or} \quad m - 2 = 0$$

$$7m = 1 \quad \text{or} \quad m = 2$$

$$m = \frac{1}{7}$$

Substituting these values of m in equation (1):

$$\text{for } m = \frac{1}{7}, \quad x^2 = \frac{28}{1 + 3\left(\frac{1}{7}\right)}$$

$$= \frac{28}{1 + \frac{3}{7}}$$

$$= \frac{28}{\frac{7}{7} + \frac{3}{7}}$$

$$= \frac{28}{\frac{10}{7}} \qquad (4)$$

Since division by a fraction is equivalent to multiplication of the numerator by the reciprocal of the denominator, equation (4) becomes:

$$x^2 = (\overset{14}{\cancel{28}})\,\frac{7}{\underset{5}{\cancel{10}}}$$

$$x = \frac{98}{5}$$

Taking the square root of both sides ot this equation:

$$\sqrt{x^2} = \pm\sqrt{\frac{98}{5}}$$

$$x = \frac{\pm\sqrt{98}}{\sqrt{5}}$$

$$= \pm\,\frac{\sqrt{49}\sqrt{2}}{\sqrt{5}}$$

$$x = \pm\,\frac{7\sqrt{2}}{\sqrt{5}}$$

Rationalizing the denominator by multiplying the numerator and denominator by $\sqrt{5}$:

$$x = \pm\,\frac{\sqrt{5}(7\sqrt{2})}{\sqrt{5}(\sqrt{5})}$$

$$x = \pm\,\frac{7\sqrt{10}}{5}$$

$$x = \pm\,\frac{7}{5}\sqrt{10}.$$

To calculate the y-values that correspond to these x-values, use the equation $y = mx$ (with $m = 1/7$):

When $x = \frac{7}{5}\sqrt{10}$, $y = mx = \left(\frac{1}{7}\right)\left(\frac{7}{5}\sqrt{10}\right) = \frac{1}{5}\sqrt{10}$, When $x = -\frac{7}{5}\sqrt{10}$, $y = mx = \left(\frac{1}{7}\right)\left(-\frac{7}{5}\sqrt{10}\right) = -\frac{1}{5}\sqrt{10}$.

Then, two solutions are: $\left(\frac{7}{5}\sqrt{10}, \frac{1}{5}\sqrt{10}\right)$ and $\left(-\frac{7}{5}\sqrt{10}, -\frac{1}{5}\sqrt{10}\right)$.

For m = 2 (using equation (1)):

$$x^2 = \frac{28}{1 + 3(2)} = \frac{28}{1 + 6} = \frac{28}{7} = 4$$

or $\quad x^2 = 4.$

Taking the square root of both sides of this equation:

$$\sqrt{x^2} = \pm\sqrt{4}$$
$$x = \pm 2.$$

To calculate the y-values that correspond to these x-values, use the equation $y = mx$ (with $m = 2$):

When $x = 2$, $y = mx = (2)(2) = 4$,

When $x = -2$, $y = mx = (2)(-2) = -4$.

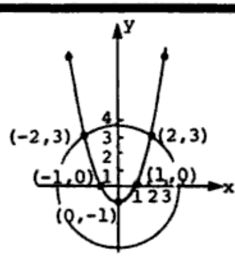
Then, two other solutions are: (2,4) and (-2,-4). Therefore, the solutions to the original pair of equations are:

$\left(\frac{7}{5}\sqrt{10}, \frac{1}{5}\sqrt{10}\right)$, $\left(-\frac{7}{5}\sqrt{10}, -\frac{1}{5}\sqrt{10}\right)$, (2,4), and (-2,-4).

**• PROBLEM 616**

Solve graphically

$$\begin{cases} x^2 + y^2 = 13, & (1) \\ \quad y = x^2 - 1 . & (2) \end{cases}$$




Solution: First, we must find the x and y intercepts. Set $y = 0$ to find the x-intercept or where the curve crosses the x-axis. Set $x = 0$ to find the y-intercept or where the curve crosses the y-axis. In Eq. (1), set $x = 0$, and find $y = \pm\sqrt{13} = \pm 3.6$. Then set $y = 0$, and find $x = \pm\sqrt{13}$. To get additional points, we solve for y.

$$x^2 + y^2 = 13$$
$$y^2 = 13 - x^2$$

$$y = \pm\sqrt{13 - x^2}$$

Then, set up a table. Choose various $x$ values and calculate the corresponding $y$ values. See Graph.

| $x$ | $\pm\sqrt{13 - x^2}$ = | $y$ |
|---|---|---|
| -3.6 | $\pm\sqrt{13 - (-3.6)^2}$ | $\approx 0$ |
| -3 | $\pm\sqrt{13 - (-3)^2}$ | $\pm 2$ |
| -2 | $\pm\sqrt{13 - (-2)^2}$ | $\pm 3$ |
| -1 | $\pm\sqrt{13 - (-1)^2}$ | $\pm 3.5$ |
| 0 | $\pm\sqrt{13 - (0)^2}$ | $\pm 3.6$ |
| 1 | $\pm\sqrt{13 - (1)^2}$ | $\pm 3.5$ |
| 2 | $\pm\sqrt{13 - (2)^2}$ | $\pm 3$ |
| 3 | $\pm\sqrt{13 - (3)^2}$ | $\pm 2$ |
| 3.6 | $\pm\sqrt{13 - (3.6)^2}$ | $\approx 0$ |

To find the domain of the relation, $\pm\sqrt{13 - x^2}$, we know that the expression, $13 - x^2$, under the square root sign must be positive in order for the expression to be real, not imaginary.

$$(13 - x^2) \geq 0$$

subtract 13 from both sides, $-x^2 \geq -13$

multiply by -1 and reverse the inequality sign,

$$x^2 \leq 13$$

Take the square root of both sides.

$$|x| \leq \sqrt{13}$$

Another way to express $|b| \leq a$ is $-a \leq b \leq +a$. Thus,

$$-\sqrt{13} \leq x \leq +\sqrt{13}$$

Thus, for the relation $y = \pm\sqrt{13-x^2}$, the domain is $\{x | -\sqrt{13} \leq x \leq \sqrt{13}\}$. The curve is a circle. The general equation of a circle is $(x-h)^2 + (y-k)^2 = r^2$, where $(h,k)$ is the center and $r$ is the radius. In this case $(0,0)$ or the origin is the center and $r^2$ is 13. Therefore, the radius $= +\sqrt{13}$.

In Eq. (2), $y$ is a quadratic function of $x$; hence the graph is a parabola. Set up a similar table for the quadratic function, $y = x^2 - 1$.

| $x$ | $x^2 - 1$ = | $y$ |
|---|---|---|
| -3 | $(-3)^2 - 1$ | 8 |
| -2 | $(-2)^2 - 1$ | 3 |
| -1 | $(-1)^2 - 1$ | 0 |
| 0 | $(0)^2 - 1$ | -1 |
| 1 | $(1)^2 - 1$ | 0 |
| 2 | $(2)^2 - 1$ | 3 |
| 3 | $(3)^2 - 1$ | 8 |

From the graphs we read the real solutions (2,3) and (-2,3). These are points of intersection for both curves.

To find the solutions algebraically substitute equation (2) into (1).

$$x^2 + y^2 = 13 \qquad (1)$$

$$y = x^2 - 1 \qquad (2)$$

$$x^2 + (x^2 - 1)^2 = 13$$
$$x^2 + x^4 - 2x^2 + 1 = 13$$
$$x^4 - x^2 = 12$$
$$x^4 - x^2 - 12 = 0$$

Substitute $z$ for $x^2$, i.e., $z = x^2$ to obtain a quadratic equation in $z$.

$$(x^2)^2 - (x^2) - 12 = 0$$
$$z^2 - z - 12 = 0$$
$$(z - 4)(z + 3) = 0$$
$$z - 4 = 0 \quad z + 3 = 0$$
$$z = 4 \qquad z = -3$$

Therefore

$$x^2 = 4 \qquad x^2 = -3$$
$$x = \pm 2 \qquad x = \pm\sqrt{-3} = \pm\sqrt{3}i$$

Find the corresponding y-values by substituting into $y = x^2 - 1$.

| | | | |
|---|---|---|---|
| $x = 2$ | $x = -2$ | $x = i\sqrt{3}$ | $x = -i\sqrt{3}$ |
| $y = (2)^2 - 1$ | $y = (-2)^2 - 1$ | $y = (i\sqrt{3})^2 - 1$ | $y = (-i\sqrt{3})^2 - 1$ |
| $= 3$ | $= 3$ | $y = i^2(3) - 1$ | $= 3(i)^2 - 1$ |
| | | $= (-1)(3) - 1$ | $= 3(-1) - 1$ |
| | | $= -4$ | $= -4$ |

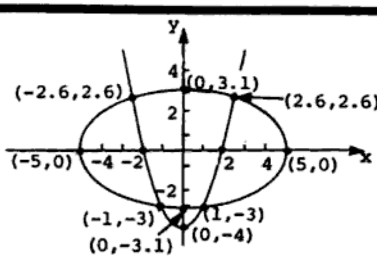
The algebraic solution gives $(2,3)$, $(-2,3)$, $(i\sqrt{3},-4)$, and $(-i\sqrt{3},-4)$. Notice that the imaginary solutions do not appear on the graph.

**• PROBLEM 617**

Obtain the simultaneous solution set of the equations

$$y = x^2 - 4 \qquad (1)$$
$$3x^2 + 8y^2 = 75 \qquad (2)$$

by the graphical method.




__Solution:__ Equation (1) is in the form of the function $ax^2 + bx + c$. Its graph is a parabola. When $a$ is positive, the curve opens upward. If it is negative, the curve opens downward. In this case $a = 1$, which is positive. Hence the graph is a parabola opening upward. We construct the parabola by means of the following table of corresponding values and show the graph in the accompanying figure.

| x | $x^2 - 4 =$ | y |
|---|---|---|
| -3 | $(-3)^2 - 4 = 9 - 4 =$ | 5 |
| -2 | $(-2)^2 - 4 = 4 - 4 =$ | 0 |
| -1 | $(-1)^2 - 4 = 1 - 4 =$ | -3 |
| 0 | $0^2 - 4 = 0 - 4 =$ | -4 |
| 1 | $(1)^2 - 4 = 1 - 4 =$ | -3 |
| 2 | $(2)^2 - 4 = 4 - 4 =$ | 0 |
| 3 | $(3)^2 - 4 = 9 - 4 =$ | 5 |

Equation (2) is of the type $ax^2 + by^2 = c$, with $a = 3$, $b = 8$, and $c = 75$. The graph is therefore an ellipse. To find the x-intercepts, set $y = 0$.

$3x^2 + 8(0)^2 = 75$

Solve for x.

$3x^2 = 75$

$x^2 = 25$

$x = \pm 5$

We have $(-5,0)$ and $(+5,0)$ as the x- intercepts

To find the y-intercepts, set $x = 0$

$$3(0)^2 + 8y^2 = 75$$

$$8y^2 = 75$$

$$y^2 = \frac{75}{8} = 9\frac{3}{8} = 9.375$$

$$y \approx \pm 3.1$$

We obtain $(0,3.1)$ and $(0,-3.1)$ for the y-intercepts.

We construct the graph and obtain the ellipse.

We now solve for the points of intersection, indicated on the graph.

$$y = x^2 - 4 \qquad (1)$$

$$3x^2 + 8y^2 = 75 \qquad (2)$$

Substitute the value of y in (1) into (2)

$$3x^2 + 8(x^2 - 4)^2 = 75$$

$$3x^2 + 8(x^2 - 8x^2 + 16) = 75$$

$$3x^2 + 8x^4 - 64x^2 + 128 = 75$$

$$8x^4 - 61x^2 + 53 = 0$$

Let $z = x^2$

$$8z^2 - 61z + 53 = 0$$

Apply the quadratic formula $z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ with

$a = 8$, $b = -61$ and $c = 53$.

$$z = \frac{-(-61) \pm \sqrt{(-61)^2 - 4(8)53}}{2(8)} = \frac{61 \pm \sqrt{3721 - 1696}}{16}$$

$$z = \frac{61 \pm 45}{16}$$

$$z = 6.625,\ 1$$

$$z = x^2 = 6.625,\ 1$$

$$x = \pm 2.57 \approx \pm 2.6,\ x = \pm 1$$

To solve for the y-values, substitute the x values.

$$y = x^2 - 4$$

For $x = \pm 1$

$$y = 1^2 - 4$$

$$y = -3$$

For $x = \pm 2.6$

$$y = (2.57)^2 - 4$$

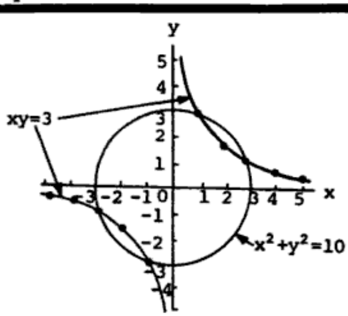
$$y = 6.6 - 4 = 2.6$$

Therefore the points of intersection are $\{(1,-3),\ (-1,-3),\ (-2.6,2.6),\ (+2.6,+2.6)\}$.

**• PROBLEM 618**

Solve $\quad xy = 3$

$\quad x^2 + y^2 = 10$




Solution: We solve the first equation for y to obtain $y = 3/x$ and substitute in the second equation to obtain

$$x^2 + \left(\frac{3}{x}\right)^2 = 10$$

Squaring: $x^2 + \dfrac{9}{x^2} = 10$

Multiplying both sides by $x^2$: $x^2\left(x^2 + \dfrac{9}{x^2}\right) = x^2\ (10)$

Distributing: $x^4 + 9 = 10x^2$

Subtracting $10x^2$ from both sides: $x^4 - 10x^2 + 9 = 0$

Factoring: $(x^2 - 1)(x^2 - 9) = 0$

Thus, $x^2 - 1 = 0$ or $x^2 - 9 = 0$

$x^2 = 1$ $\qquad$ $x^2 = 9$

$x = \pm 1$ $\qquad$ $x = \pm 3$

Therefore, $x = 1, -1, 3,$ or $-3$.

Since $y = 3/x$, substituting these values in turn in this equation, we obtain the corresponding values for y:

$x = 1 \quad x = -1 \quad x = 3 \quad x = -3$

$y = 3 \quad y = -3 \quad y = 1 \quad y = -1$

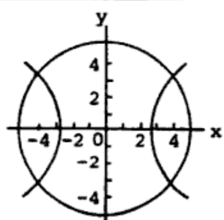
To consider this system graphically, we notice that the second equation is the equation of a circle with radius $\sqrt{10}$, whereas the graph of the first equation is a hyperbola obtained from the following table. Also, the points (1, 3) (- 1, - 3), (3, 1), (- 3, - 1) belong to both the circle and the hyperbola, and the two graphs intersect at these points.

| x | $-5$ | $-4$ | $-3$ | $-2$ | $-1$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|---|---|---|---|
| y | $-\frac{3}{5}$ | $-\frac{3}{4}$ | $-1$ | $-\frac{3}{2}$ | $-3$ | 3 | $\frac{3}{2}$ | 1 | $\frac{3}{4}$ | $\frac{3}{5}$ |

Plotting these points, and the circle with radius $\sqrt{10}$ (approximately equal to 3.16) we have the accompanying diagram.

**• PROBLEM 619**

Solve for x and y: $\begin{cases} x^2 + y^2 = 25, \\ x^2 - y^2 = 7. \end{cases}$




**Solution:** Add the following two equations to eliminate $y^2$.

$$\begin{array}{ll} (1) & x^2 + y^2 = 25 \\ (2) & \underline{x^2 - y^2 = 7} \\ & 2x^2 \qquad = 32 \end{array}$$

Divide by 2

$$x^2 = 16$$

Take the square root of both sides.

$$x = \pm 4$$

Substitute $x^2$ into $x^2 + y^2 = 25$ to solve for $y^2$.

$$16 + y^2 = 25$$

$$y^2 = 25 - 16$$

$$y^2 = 9$$

$$y = \pm 3$$

$x = 4,\ y = -3;\qquad x = -4,\ y = -3,$

$x = 4,\ y = 3;\qquad x = -4,\ y = 3;$

as can be verified by substitution in the given equations.

Graphical solution. If the graph of the first equation is constructed by finding pairs of values $(x,y)$ which satisfy the equation and plotting the corresponding points, the circle shown is obtained. In a similar manner, the hyperbola shown in the figure is obtained as the graph of the second equation. The circle and hyperbola are seen to intersect in the four points $(4,3), (-4,3), (-4,-3), (4,-3)$.

There is another way to graph these equations. The standard form of the equation of the circle whose center is at the point $c(h,k)$ and whose radius is the constant $r$ is

$$(x - h)^2 + (y - k)^2 = r^2.$$

In this case the center is $(0,0)$ and the radius is 5. Therefore, move out 5 units from the center in all directions. The circle will then intersect the axes at $(5,0), (-5,0), (0,5)$, and $(0,-5)$. Write the hyperbola in the general form,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

The hyperbola intersects one of its lines of symmetry, the x-axis, in the points $(-a,0)$ and $(a,0)$. Rewriting $x^2 - y^2 = 7$, we obtain

$$\frac{x^2}{7} - \frac{y^2}{7} = 1. \quad a^2 = 7 \text{ and } a = \pm\sqrt{7}\,.$$

Therefore the points of intersection on the x-axis are $(-\sqrt{7},0)$ and $(+\sqrt{7},0)$. This is equivalent to

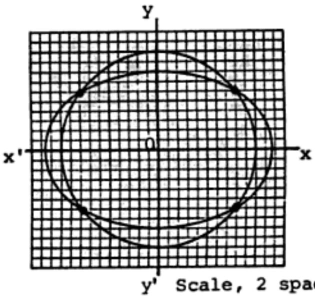
$$(-2.65,0) \text{ and } (+2.65,0).$$

**• PROBLEM 620**

Solve
$$9x^2 + 16y^2 = 288, \tag{1}$$
$$x^2 + y^2 = 25. \tag{2}$$

<u>Solution:</u> Solving for $y^2$ first (a similar method solving for $x^2$ first is equally good): Multiply Equation (2) by 9 and then subtract Equation (2) from Equation (1)

$$\begin{array}{r} 9x^2 + 16y^2 = 288 \\ \underline{-9x^2 + \ \ 9y^2 = 225} \\ 7y^2 = \ \ 63 \end{array}$$



Scale, 2 spaces=1 unit

$$y^2 = 9 \tag{3}$$

Then from Equation (2), for $y^2 = 9$, $x^2 + 9 = 25$ or

$$x^2 = 16. \tag{4}$$

From (3) and (4) $y = \pm 3$, $x = \pm 4$.

Forming all possible pairs, we have the four solutions

$$\begin{array}{llll} x = 4 & x = 4 & x = -4 & x = -4 \\ y = 3 & y = -3 & y = 3 & y = -3 \end{array}$$

Check for $x = 4$, $y = 3$:

in Eq. (1):

$$9(4)^2 + 16(3)^2 \stackrel{?}{=} 288$$
$$9(16) + 16(9) \stackrel{?}{=} 288$$
$$144 + 144 \stackrel{?}{=} 288$$
$$288 = 288$$

in Eq. (2):

$$(4^2) + (3^2) \stackrel{?}{=} 25$$
$$16 + 9 \stackrel{?}{=} 25$$
$$25 = 25$$

x and y appear squared in both Eq. (1) and Eq. (2). The other pairs of values for x and y differ from the pair checked only in sign. Therefore the other pairs also satisfy Equation (1) and Equation (2).

The Equation (1) has for its locus an oval-shaped figure called an ellipse. (Fig.) The equation (2) has a circle for its locus. The four points of intersection represent graphically the four solutions.

## MULTIVARIABLE COMBINATIONS

**• PROBLEM 621**

Solve $x + y + z = 13$ (1),
$x^2 + y^2 + z^2 = 65$ (2),

$$xy = 10 \qquad (3).$$

<u>Solution:</u> We note that the product of two binomials is

$$(x+y)^2 = (x+y)(x+y) = x^2 + xy + xy + y^2$$
$$= x^2 + 2xy + y^2 .$$

Therefore, if we were just to add (2) and (3) we would be missing an xy term. Thus, multiplying (3) by 2,

$$2xy = 20$$

Adding this to (2),

$$\begin{array}{r} x^2 + y^2 + z^2 = 65 \\ + \underline{\qquad\qquad 2xy = 20} \\ x^2 + 2xy+y^2+z^2 = 85 \end{array}$$

$$(x+y)^2 + z^2 = 85$$

Put $u$ for $x+y$; then this equation becomes

$$u^2 + z^2 = 85.$$

Also from (1),

$$u + z = 13;$$

Now to solve this system of one linear equation and one quadratic, we express one of the variables in the linear equation in terms of the other and substitute this result into the quadratic equation.

$$u^2 + z^2 = 85 \qquad (4)$$
$$u + z = 13 \qquad (5)$$

Solving (5) for $z$ by subtracting $u$ from both sides:

$$z = 13 - u \qquad (6)$$

Substituting this expressios for $z$ into (4),

$$u^2 + (13 - u)^2 = 85$$
$$u^2 + 169 - 26u + u^2 = 85$$
$$2u^2 - 26u + 169 = 85.$$

Subtracting 85 from both sides,

$$2u^2 - 26u + 169 - 85 = 0$$
$$2u^2 - 26u + 84 = 0$$

Dividing by 2, $\quad u^2 - 13u + 42 = 0$

Factor, $\quad (u - 6)(u - 7) = 0$

Whenever the product of two factors is 0, then either one or the other must equal zero. Then, $u - 6 = 0$ or $u - 7 = 0$.

Solve for $u$,

$$u = 6 \text{ or } u = 7 .$$

Substituting these values of $u$ into equation (6) to find the corresponding $z$ values

| | | | |
|---|---|---|---|
| | $u = 6$ | | $u = 7$ |
| (6) | $z = 13 - u$ | (6) | $z = 13 - u$ |
| | $z = 13 - 6$ | | $z = 13 - 7$ |
| | $z = 7$ | | $z = 6$ |

Hence, the solutions are:

$$\begin{matrix} x = 2 \\ x = 5 \\ z = 6 \end{matrix} \;;\quad \begin{matrix} x = 5 \\ y = 2 \\ z = 6 \end{matrix} \;;\quad \begin{matrix} x = 3 - \sqrt{-1} \\ y = 3 + \sqrt{-1} \\ z = 7 \end{matrix} \;;\quad \begin{matrix} x = 3 + \sqrt{-1} \\ y = 3 - \sqrt{-1} \\ z = 7 \end{matrix}$$

**• PROBLEM 622**

Eliminate $\ell$, m from the equations

$$\ell x + my = a, \; mx - \ell y = b, \; \ell^2 + m^2 = 1.$$

Solution: By squaring the first two equations and adding, we obtain:

$$(\ell x + my)^2 = a^2 \quad \text{and} \quad (mx - \ell y)^2 = b^2$$

$$(\ell x + my)(\ell x + my) = a^2; \quad (mx - \ell y)(mx - \ell y) = b^2$$

$$\ell^2x^2 + my\ell x + my\ell x + m^2y^2 = a^2 \qquad \text{and}$$

$$m^2x^2 - \ell ymx - \ell ymx + \ell^2y^2 = b^2.$$

Then,

$$\ell^2x^2 + 2\,my\ell x + m^2y^2 + \left(m^2x^2 - 2\ell ymx + \ell^2y\right) = a^2 + b^2$$

$$\ell^2x^2 + 2\,my\ell x + m^2y^2 + m^2x^2 - 2\ell ymx + \ell^2y = a^2 + b^2.$$

We observe that $my\ell x$ and $\ell ymx$ are the same; thus

$$\ell^2x^2 + m^2x^2 + m^2y^2 + \ell^2y^2 = a^2 + b^2.$$

Factoring, we obtain:

$$(\ell^2 + m^2)(x^2 + y^2) = a^2 + b^2;$$

From the third given equation we know that $\ell^2 + m^2 = 1$. Therefore, $x^2 + y^2 = a^2 + b^2$.

If $\ell = \cos\theta$, $m = \sin\theta$, the third equation is satisfied identically; that is,

$\ell^2 + m^2 = 1$ becomes $\cos^2\theta + \sin^2\theta = 1$,

which is a known identity. Again, by squaring the first two equations, which now are as follows:

$$x\cos\theta + y\sin\theta = a, \; x\sin\theta - y\cos\theta = b$$

we have:

$$(x\cos\theta + y\sin\theta)^2 = a^2$$

$$(x\sin\theta - y\cos\theta)^2 = b^2.$$

Expanding, we have $(x\cos\theta + y\sin\theta)^2 =$

$(x\cos\theta + y\sin\theta)(x\cos\theta + y\sin\theta) =$

$$\begin{array}{r} x^2y - y^3 = pxy - qy^2 \\ - \underline{\left(4x^2y \quad = qx^2 + pxy\right)} \\ x^2y - y^3 - 4x^2y = pxy - qy^2 - \left(qx^2 + pxy\right) \end{array}$$

The last equation can be rewritten as:

$$- y^3 - 3x^2y = pxy - qy^2 - qx^2 - pxy$$

$- y^3 - 3x^2y = - qy^2 - qx^2$. Multiplying each term by $- 1$ we obtain:

$$y^3 + 3x^2y = qy^2 + qx^2$$

$$y^3 + 3x^2y = q\left(x^2 + y^2\right).$$

Now, since $x^2 + y^2 = 1$, from the third equation, we have:

$$q = 3x^2y + y^3.$$

Now, adding our values for p and q we obtain:

$p + q = \left(x^3 + 3xy^2\right) + \left(3x^2y + y^3\right)$. But, this is the same as $(x + y)^3$, since

$$\begin{aligned} (x + y)^3 &= (x + y)(x + y)(x + y) \\ &= \left(x^2 + 2xy + y^2\right)(x + y) \\ &= x^3 + 2x^2y + xy^2 + x^2y + 2xy^2 + y^3 \\ &= x^3 + 3x^2y + 3xy^2 + y^3. \end{aligned}$$

Thus, $p + q = (x + y)^3$. Similarly, we subtract q from p and we obtain:

$$\begin{aligned} p - q &= \left(x^3 + 3xy^2\right) - \left(3x^2y + y^3\right) \\ &= x^3 + 3xy^2 - 3x^2y - y^3. \end{aligned}$$

But, this is the same as $(x - y)^3$, since

$$\begin{aligned} (x - y)^3 &= (x - y)(x - y)(x - y) \\ &= \left(x^2 - 2xy + y^2\right)(x - y) \\ &= x^3 - 2x^2y + xy^2 - x^2y + 2xy^2 - y^3 \\ &= x^3 - 3x^2y + 3xy^2 - y^3. \end{aligned}$$

Thus, $p - q = (x - y)^3$.

Therefore, we now have, $p + q = (x + y)^3$ and $p - q = (x - y)^3$.

Taking the cube root of both sides of each equation we have:

$$(p + q)^{\frac{1}{3}} = x + y$$

$$(p - q)^{\frac{1}{3}} = x - y$$

Now, squaring both sides of each equation gives us:

$$\left[(p + q)^{\frac{1}{3}}\right]^2 = (x + y)^2$$

$$\left[(p - q)^{\frac{1}{3}}\right]^2 = (x - y)^2$$

or

$$(p + q)^{\frac{2}{3}} = (x + y)^2; \quad (p - q)^{\frac{2}{3}} = (x - y)^2$$

Adding these equations we obtain:

$$(p + q)^{\frac{2}{3}} + (p - q)^{\frac{2}{3}} = (x + y)^2 + (x - y)^2 =$$

$$(x + y)(x + y) + (x - y)(x - y) =$$

$$x^2 + 2xy + y^2 + x^2 - 2xy + y^2 =$$

$$2x^2 + 2y^2 =$$

$$2\left(x^2 + y^2\right).$$

From the third equation we know that $x^2 + y^2 = 1$, therefore,

$$(p + q)^{\frac{2}{3}} + (p - q)^{\frac{2}{3}} = 2.$$

**• PROBLEM 624**

Eliminate x, y, z from the equations

$y^2 + z^2 = ayz, \quad z^2 + x^2 = bzx, \quad x^2 + y^2 = cxy.$

Solution: We wish to isolate a, b, and c on the right side of the three given equations. Dividing, we have:

$$\frac{y^2 + z^2}{yz} = a, \quad \frac{z^2 + x^2}{zx} = b, \quad \frac{x^2 + y^2}{xy} = c.$$

These can be rewritten as

$$\frac{y}{z} + \frac{z}{y} = a, \quad \frac{z}{x} + \frac{x}{z} = b, \quad \frac{x}{y} + \frac{y}{x} = c.$$

Multiplying together these three equations we obtain,

$$\left(\frac{y}{z} + \frac{z}{y}\right)\left(\frac{z}{x} + \frac{x}{z}\right)\left(\frac{x}{y} + \frac{y}{x}\right) = abc$$

$$\left(\frac{y}{z} \cdot \frac{z}{x} + \frac{z}{y} \cdot \frac{z}{x} + \frac{y}{z} \cdot \frac{x}{z} + \frac{z}{y} \cdot \frac{x}{z}\right)\left(\frac{x}{y} + \frac{y}{x}\right) = abc$$

$$\left(\left(\frac{y}{z} \cdot \frac{z}{x} \cdot \frac{x}{y}\right) + \left(\frac{z}{y} \cdot \frac{z}{x} \cdot \frac{x}{y}\right) + \left(\frac{y}{z} \cdot \frac{x}{z} \cdot \frac{x}{y}\right) + \left(\frac{z}{y} \cdot \frac{x}{z} \cdot \frac{x}{y}\right)\right.$$

$$+ \left(\frac{y}{z} \cdot \frac{z}{x} \cdot \frac{y}{x}\right) + \left(\frac{z}{y} \cdot \frac{z}{x} \cdot \frac{y}{x}\right) + \left(\frac{y}{z} \cdot \frac{x}{z} \cdot \frac{y}{x}\right) +$$

$$\left.\left(\frac{z}{y} \cdot \frac{x}{z} \cdot \frac{y}{x}\right)\right) = abc$$

Now, simplifying each term by multiplication, and reducing, we have:

$$1 + \frac{z^2}{y^2} + \frac{x^2}{z^2} + \frac{x^2}{y^2} + \frac{y^2}{x^2} + \frac{z^2}{x^2} + \frac{y^2}{z^2} + 1 = abc$$

or $\quad 2 + \dfrac{y^2}{z^2} + \dfrac{z^2}{y^2} + \dfrac{z^2}{x^2} + \dfrac{x^2}{z^2} + \dfrac{x^2}{y^2} + \dfrac{y^2}{x^2} = abc.$

Now, since $\frac{y}{z} + \frac{z}{y} = a$, then $\left(\frac{y}{z} + \frac{z}{y}\right)^2 = a^2$.

But we can rewrite $\left(\frac{y}{z} + \frac{z}{y}\right)^2$ as:

$$\left(\frac{y}{z} + \frac{z}{y}\right)^2 = \left(\frac{y}{z} + \frac{z}{y}\right)\left(\frac{y}{z} + \frac{z}{y}\right) =$$

$$\frac{y^2}{z^2} + \left(\frac{z}{y} \cdot \frac{y}{z}\right) + \left(\frac{y}{z} \cdot \frac{z}{y}\right) + \frac{z^2}{y^2} =$$

$$\frac{y^2}{z^2} + 1 + 1 + \frac{z^2}{y^2} = \frac{y^2}{z^2} + \frac{z^2}{y^2} + 2.$$

Similarly, since $\frac{z}{x} + \frac{x}{z} = b$, and $\frac{x}{y} + \frac{y}{x} = c$, then $\left(\frac{z}{x} + \frac{x}{z}\right)^2 = b^2$ and $\left(\frac{x}{y} + \frac{y}{x}\right)^2 = c^2$. But writing these we have:

$$\left(\frac{z}{x} + \frac{x}{z}\right)^2 = \left(\frac{z}{x} + \frac{x}{z}\right)\left(\frac{z}{x} + \frac{x}{z}\right) = \frac{z^2}{x^2} + 1 + 1 + \frac{x^2}{z^2}$$

$$= \frac{z^2}{x^2} + \frac{x^2}{z^2} + 2$$

and, $\left(\frac{x}{y} + \frac{y}{x}\right)^2 = \left(\frac{x}{y} + \frac{y}{x}\right)\left(\frac{x}{y} + \frac{y}{x}\right) = \dfrac{x^2}{y^2} + 1 + 1 + \dfrac{y^2}{x^2}$

$$= \frac{x^2}{y^2} + \frac{y^2}{x^2} + 2$$

# CHAPTER 21

# EQUATIONS AND INEQUALITIES OF DEGREE GREATER THAN TWO

**Basic Attacks and Strategies for Solving Problems in this Chapter.** **See pages 464 to 490 for step-by-step solutions to problems.**

Solving polynomial equations and inequalities of degree greater than two can be very difficult. However, rational solutions of polynomial equations of degree greater than two with integral coefficients can be solved as follows:

**DEGREE 3**

(1) If the equation is of degree 3 and has a linear factor, then the procedure is clear. Put the equation in the form

$$ax^3 + bx^2 + cx + d = 0.$$

Then,

$-b/a$ = the sum of the roots,

$c/a$ = the sum of the products of the roots taken two at a time, and

$-d/a$ = the product of the roots.

(2) If the equation is the difference of cubes, e.g.,

$$x^3 - 1 = 0,$$

then $(x-1)(x^2+x+1) = 0.$

Set $x - 1 = 0$ and $x^2 + x + 1 = 0$

and solve using standard techniques.

(3) If the equation of degree 3 has a common variable factor, then factor it out and use the zero-factor property to solve the equation using standard techniques. For example,

$$ax^3 + bx^2 + cx = 0$$

$$x\,(ax^2 + bx + c) = 0$$

$$x = 0 \quad \text{and} \quad ax^2 + bx + c = 0$$

A method for solving an inequality of degree 3 involves the procedure outlined above, except it is necessary to consider and solve each of the possible cases for which the inequality in factored form is satisfied. The intersection of the various case solutions gives the final solution of the original inequality of degree 3. Appropriately shaded areas of the various case solutions on a graph usually aid in representing the final solution.

## DEGREE 4

The solution of fourth degree equations can be solved by the same methods applied to quadratic equations. The basic approach is to first define a replacement variable $z$ as $z = x^2$, and then substitute it in the given equation of degree 4. This step converts the equation of degree 4 to a quadratic equation involving variable $z$. After the solutions have been obtained for the variable $z$, then substitute the values of $z$ into the equation $z = x^2$ and solve for $x$. The values of $x$ represent the solutions of the original equation of degree 4.

Inequalities of degree 4 are handled in a manner similar to equations of the same degree, except all the various cases that satisfy the inequality must be examined and the final solution represented as the intersection of the solutions of the cases.

# Step-by-Step Solutions to Problems in this Chapter, "Equations and Inequalities of Degree Greater than Two"

## DEGREE 3

**• PROBLEM 625**

Remove fractional coefficients from the equation

$$2x^3 - \frac{3}{2}x^2 - \frac{1}{8}x + \frac{3}{16} = 0.$$

Solution: To rewrite this equation without fractional coefficients we must find a common denominator for all the terms of the equation. Observe that a common denominator is 16. Thus,

$$2x^3 - \frac{3}{2}x^2 - \frac{1}{8}x + \frac{3}{16} = 0$$

can be rewritten as:

$$\frac{2x^3}{1} - \frac{3x^2}{2} - \frac{x}{8} + \frac{3}{16} = 0 \qquad \text{or,}$$

$$\frac{32x^3 - 24x^2 - 2x + 3}{16} = 0.$$ Multiplying both sides of the equation by 16 we obtain:

$32x^3 - 24x^2 - 2x + 3 = 0$. This is the required equation without fractional coefficients.

**• PROBLEM 626**

Solve the equation $x^3 - 16x = 0$.

Solution: Multiplying both sides by $\frac{1}{x}$, we have $x^2 - 16 = 0$. Factoring, we have $(x - 4)(x + 4) = 0$. Then all values of $x$ which make this product equal to $0$ satisfy either $x - 4 = 0$ or $x + 4 = 0$. Thus $x = 4$ or $x = -4$. Both $4$ and $-4$ satisfy the original equation since $(4)^3 - 16(4) = 0$ and $(-4)^3 - 16(-4) = 0$. However, so does the number $0$, since $(0)^3 - 16(0) = 0$. From where did this root come?

There are several logical flaws in this solution. First, we cannot multiply both sides by $1/x$ if $x = 0$. But, basically, what is wrong is that $x^3 - 16x$ is not an equivalent expression to $x^2 - 16$.

We may undo this error by writing $x^3 - 16x = 0$ in factored form as $x(x + 4)(x - 4) = 0$.

Then $\quad x = 0 \quad$ or $\quad x + 4 = 0 \quad$ or $\quad x - 4 = 0$

Hence $\quad x = 0 \quad$ or $\quad x = -4 \quad$ or $\quad x = 4$

$$x = 1;$$

$$(x^2 + x + 1) = 0$$

Solve by the quadratic formula since this equation is of the form $ax^2 + bx + c = 0$ with $a = 1$, $b = 1$, $c = 1$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-1 \pm \sqrt{1 - 4(1)}}{2}$$

$$x = -\frac{1}{2} \pm \frac{\sqrt{-3}}{2}$$

$$x = -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}.$$

The solutions are

$$x = \left\{1, \frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}\right\}.$$

**• PROBLEM 629**

Form the equation whose roots are 2, -3, and $\frac{7}{5}$.

Solution: The roots of the equation are 2, -3, and $\frac{7}{5}$. Hence, $x = 2$, $x = -3$, and $x = \frac{7}{5}$. Subtract 2 from both sides of the first equation:

$$x - 2 = 2 - 2 = 0.$$

Add 3 to both sides of the second equation:

$$x + 3 = -3 + 3 = 0.$$

Subtract $\frac{7}{5}$ from both sides of the third equation:

$$x - \frac{7}{5} = \frac{7}{5} - \frac{7}{5} = 0.$$

Hence, $(x - 2)(x + 3)\left(x - \frac{7}{5}\right) = (0)(0)(0) = 0$ or

$$(x - 2)(x + 3)\left(x - \frac{7}{5}\right) = 0.$$

Multiply both sides of this equation by 5:

$$5(x - 2)(x + 3)\left(x - \frac{7}{5}\right) = 5(0) \text{ or}$$

$$(x - 2)(x + 3)5\left(x - \frac{7}{5}\right) = 0 \text{ or}$$

$$(x - 2)(x + 3)(5x - 7) = 0$$

$$(x^2 + x - 6)(5x - 7) = 0$$

$$5x^3 - 7x^2 + 5x^2 - 7x - 30x + 42 = 0$$

$$5x^3 - 2x^2 - 37x + 42 = 0.$$

**• PROBLEM 630**

Solve the equation $24x^3 - 14x^2 - 63x + 45 = 0$, one root being double another.

Solution: A cubic equation, with the properties stated, has three roots which may be denoted by a, 2a, b.

Then we can use the following known relations between roots and coefficients of an equation to obtain equations involving the three roots. Since we can convert the given equation into one in which the coefficient of the first term is 1, we can transform the equation into one of the form, $x^3 + b_1 x^2 + b_2 x + b_3 = 0$, and then, $-b_1$ = sum of the roots

$b_2$ = sum of products of the roots taken two at a time

$(-1)^3 b_3$ = product of roots.

Dividing our given equation by 24 we obtain:

$$x^3 - \frac{14}{24}x^2 - \frac{63}{24}x + \frac{45}{24} = 0.$$

Thus, $-b_1 = -\left(-\frac{14}{24}\right) = \frac{7}{12}$ = sum of roots = a + 2a + b

$b_2 = -\frac{63}{24} = -\frac{21}{8}$ = sum of products of roots taken two at a time

$= (a)(2a) + (a)(b) + (2a)(b)$

$(-1)^3 b_3 = (-1)\frac{45}{24} = -\frac{15}{8}$ = product of roots

$= (a)(2a)(b).$

Therefore, $3a + b = \frac{7}{12}$, $2a^2 + 3ab = -\frac{21}{8}$,

$$2a^2 b = -\frac{15}{8}.$$

We can now solve the first two equations simultaneously as follows: Transpose the constant term in each equation from the right to the left side of the equal sign. Multiply each term of the first equation by 3a, and then subtract the second equation from the first.

$- b_1$ = sum of the roots

$b_2$ = sum of the products of the roots taken two at a time

$(-1)^3 b_3$ = product of the roots

Dividing each term of our given equation by 4 we obtain:

$$x^3 - \frac{24}{4}x^2 + \frac{23}{4}x + \frac{18}{4} = 0$$

Thus, $- b_1 = -\left(-\frac{24}{4}\right) = 6$ = sum of roots

$= (a - b) + a + (a + b)$

$b_2 = \frac{23}{4}$ = sum of products of roots taken two at a time $= (a-b)(a)+(a-b)(a+b)+(a)(a+b)$

$(-1)^3 b_3 = (-1)\frac{18}{4} = -\frac{9}{2}$ = product of roots

$= (a - b)(a)(a + b)$

Therefore,

$$3a = 6,\ 3a^2 - b^2 = \frac{23}{4},\ a\left(a^2 - b^2\right) = -\frac{9}{2}.$$

From the first equation we find $a = 2$, and substituting this value in the second equation we find $b = \pm\frac{5}{2}$, and since these values satisfy the third, the three equations are consistent.

Now, since the roots were denoted by $a - b$, $a$, $a + b$, our three roots are $\left(2 \pm \frac{5}{2}\right)$, 2, $\left(2 \pm \frac{5}{2}\right)$, or, $-\frac{1}{2}$, 2, $\frac{9}{2}$.

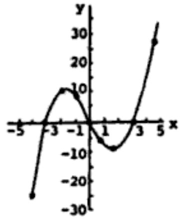
**• PROBLEM 632**

Graph the function $y = x^3 - 9x$.

Solution: Choosing values of x in the interval $-4 \le x \le 4$, we have for $y = x^3 - 9x$,

| x | −4 | −3 | −2 | −1 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|---|
| y | −28 | 0 | 10 | 8 | 0 | −8 | −10 | 0 | 28 |

Notice that for each ordered pair (x,y) listed in the table there exists a pair (−x,−y) which also satisfies the equation, indicating symmetry with respect to the origin. To




prove that this is true for all points on the curve, we substitute $(-x,-y)$ for $(x,y)$ in the given equation and show that the equation is unchanged. Thus

$$-y = (-x)^3 - 9(-x) = -x^3 + 9x$$

or, multiplying each member by -1,

$$y = x^3 - 9x$$

which is the original equation.

The curve is illustrated in the figure. The domain and range of the function have no restrictions in the set of real numbers. The x-intercepts are found from

$$y = 0 = x^3 - 9x$$

$$0 = x\left(x^2 - 9\right)$$

$$0 = x(x - 3)(x + 3)$$

$$\begin{array}{c|c|c} x = 0 & x - 3 = 0 & x + 3 = 0 \\ & x = 3 & x = -3. \end{array}$$

The curve has three x-intercepts at $x = -3$, $x = 0$, $x = 3$. This agrees with the fact that a cubic equation has three roots. The curve has a single y-intercept at $y = 0$ since for $x = 0$, $y = 0^3 - 9(0) = 0$.

**• PROBLEM 633**

Locate the roots of $x^3 - 3x^2 - 6x + 9 = 0$.

Solution: If we let $f(x)$ be a function, then a solution of the equation $f(x) = 0$ is called a root of the equation.

In this particular case let the function $f(x) = x^3 - 3x^2 - 6x + 9$ and set it equal to zero to find its roots. When $f(x) = 0$, the graph of this equation crosses the x-axis. These x-values are the roots of the function.

To locate the roots of $x^3 - 3x^2 - 6x + 9 = 0$, we consider the function $y = x^3 - 3x^2 - 6x + 9$, assign consecutive integers from - 3 to 5 to x, compute each corresponding value of y, and record the results.

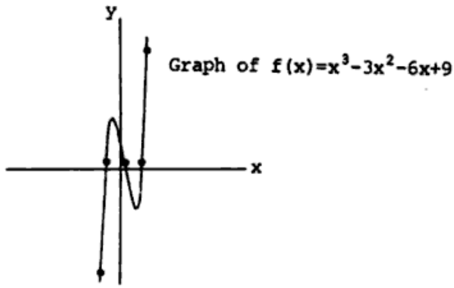



Table of Results

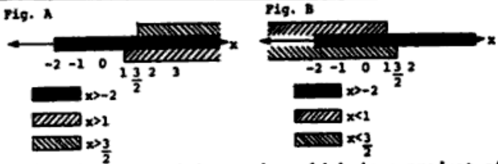
| x | $x^3 - 3x^2 - 6x + 9$ = | y |
|---|---|---|
| − 3 | $(-3)^3 - 3(-3)^2 - 6(-3) + 9 =$ | − 27 |
| − 2 | $(-2)^3 - 3(-2)^2 - 6(-2) + 9 =$ | 1 |
| − 1 | $(-1)^3 - 3(-1)^2 - 6(-1) + 9 =$ | 11 |
| 0 | $(0)^3 - 3(0)^2 - 6(0) + 9 =$ | 9 |
| 1 | $(1)^3 - 3(1)^2 - 6(1) + 9 =$ | 1 |
| 2 | $(2)^3 - 3(2)^2 - 6(2) + 9 =$ | − 7 |
| 3 | $(3)^3 - 3(3)^2 - 6(3) + 9 =$ | − 9 |
| 4 | $(4)^3 - 3(4)^2 - 6(4) + 9 =$ | 1 |
| 5 | $(5)^3 - 3(5)^2 - 6(5) + 9 =$ | 29 |

Since $f(-3) = -27$ and $f(-2) = 1$, there is an odd number of roots between $x = -3$ and $x = -2$. Since $f(-3) = -27$ which is negative and $f(-2) = 1$ is positive, the graph must cross the x-axis at least once. The function is continuous; thus the curve must connect the two points. To do this, the curve must cross from the negative to the positive side of the x-axis. By the definition of continuity, in order for the curve to traverse the axis it must intersect the axis. Each intersection point is called a zero or a root of the function.

Note that the curve must intersect the x-axis an odd number of times if it is to pass from the negative side to the positive, for if it traversed the axis an even number of times it would end up on the side on which it started.

Similarly, there is an odd number of roots between $x = 1$ and $x = 2$, and between $x = 3$ and $x = 4$. Furthermore, since the equation is of degree 3, it has exactly three roots. Observe that the curve crosses the x-axis three times, indicating the three roots of the equation. Therefore, exactly one root lies in each of the above intervals.

Solve the inequality $(x + 2)(x - 1)(2x - 3) > 0$.



**Solution:** If we have a positive number which is a product of three factors, then all three factors are positive or two are negative and one is positive (since a negative multiplied by a negative is positive). The tentative possibilities are:

(1) $x+2 > 0$, and $x-1 > 0$, and $2x-3 > 0$, $\Rightarrow x > -2$, and $x > 1$, and $x > 3/2$

(2) $x+2 > 0$, $x-1 < 0$, $2x-3 < 0$, $\Rightarrow x > -2$, $x < 1$, $x < 3/2$

(3) $x+2 < 0$, $x-1 > 0$, $2x-3 < 0$, $\Rightarrow x < -2$, $x > 1$, $x < 3/2$

(4) $x+2 < 0$, $x-1 < 0$, $2x-3 > 0$, $\Rightarrow x < -2$, $x < 1$, $x > 3/2$

Then, if "and" means intersection ($\cap$), we must find,

$$(x > -2) \cap (x > 1) \cap (x > 3/2).$$

(See the number line in Figure A). Thus, the inequalities in (1) yield, as the range satisfying all three linear inequalities, $x > 3/2$.
For (2)

$$x > -2, \quad x < 1, \text{ and } \quad x < 3/2;$$

thus we must find,

$$(x > -2) \cap (x < 1) \cap (x < 3/2)$$

(See Figure B).
Thus, the inequalities in (2) yield, as the range satisfying all three linear inequalities, $-2 < x < 1$.
In Case (3), $x < -2$, $x > 1$, and $x < 3/2$, $x > 1$ is inconsistent with $x < -2$. Thus, there are no values on the number line common to all three inequalities.
For the last alternative, (4), $x < -2$, $x < 1$, and $x > 3/2$, the last inequality, $x > 3/2$, is inconsistent with $x < -2$, and $x < 1$. Thus, again the intersection of these three inequalities is the empty set.
Hence the complete solution consists of the ranges

$$-2 < x < 1 \quad \text{and} \quad x > 3/2 .$$



Solve $x^3 + x^2 - 2x > 0$.

**Solution:** We factor $x^3 + x^2 - 2x$ and have $x(x^2 + x - 2) > 0$. Again, factor the left side of this inequality:

$$x^2 + x - 2 = (x + 2)(x - 1).$$

Therefore, $x(x^2 + x - 2) > 0$ becomes $x(x + 2)(x - 1) > 0$. Now we have four possibilities: (I) all factors positive; (II) the last two negative; (III) the first and third negative; and (IV) the first two negative. In tabular form this gives:

| I | | II | | III | | IV |
|---|---|---|---|---|---|---|
| $x > 0$ | | $x > 0$ | | $x < 0$ | | $x < 0$ |
| and | | and | | and | | and |
| $x + 2 > 0$ | or | $x + 2 < 0$ | or | $x + 2 > 0$ | or | $x + 2 < 0$ |

Solution: The given expression will be real for those real values of $x$ yielding a radicand which is greater than or equal to zero. Thus, we have to solve the inequality

$$x^3 - 3x^2 + 2x \geqq 0 .$$

There is a common factor, namely $x$, of every term in the left side of this inequality. Hence, $x$ is factored out from the left side:

$$x(x^2 - 3x + 2) \geqq 0$$

Factoring the expression in parenthesis into a product of two polynomials:

$$x(x - 2)(x - 1) \geqq 0 .$$

Only if all three factors are positive or two are negative and one positive will the product be positive. Thus, the following four cases result:

Case 1: $x \geqq 0$ , $x - 2 \geqq 0$, $x - 1 \geqq 0$ .

Case 2: $x \geqq 0$ , $x - 2 \leqq 0$, $x - 1 \leqq 0$ .

Case 3: $x \leqq 0$ , $x - 2 \leqq 0$, $x - 1 \geqq 0$ .

Case 4: $x \leqq 0$ , $x - 2 \geqq 0$, $x - 1 \leqq 0$ .

For Case 1: Solve the inequalities,

$$x \geqq 0,\ x \geqq 2,\ x \geqq 1.$$

These three inequalities are satisfied by the single inequality $x \geqq 2$, that is, the intersection of these three inequalities is the set $\{x|x \geqq 2\}$. This intersection can be noted in Diagram A. Hence, the solution set for Case 1 is $\{x|x \geqq 2\}$.

For Case 2: Solve the inequalities,

$$x \geqq 0,\ x \leqq 2,\ x \leqq 1.$$

The two inequalities $x \leqq 2$ and $x \leqq 1$ are satisfied by the single inequality $x \leqq 1$. Putting this inequality, $x \leqq 1$ and the remaining inequality $x \geqq 0$ together:

$$x \geqq 0 \text{ or } 0 \leqq x,\ x \leqq 1; \text{ that is,}$$

$$0 \leqq x \leqq 1.$$

The set $\{x|0 \leqq x \leqq 1\}$ is indicated in Diagram B. Hence, the solution set for Case 2 is $\{x|0 \leqq x \leqq 1\}$.

For Case 3: Solving the inequalities,

$$x \leqq 0,\ x \leqq 2,\ x \geqq 1 .$$

The two inequalities $x \leqq 0$ and $x \leqq 2$ are satisfied by the single inequality $x \leqq 0$. However, combining this inequality, $x \leqq 0$, with the remaining inequality $x \geqq 1$, there is no value of $x$ which is less than or equal to zero, and, at the same time, greater than or equal to 1. This is illustrated in Diagram C. Hence, there is no solution set for Case 3.

For Case 4: Solve the inequalities,

$$x \leqq 0,\ x \geqq 2,\ x \leqq 1.$$

The two inequalities $x \leqq 0$ and $x \leqq 1$ are satisfied by the single inequality $x \leqq 0$. However, combining this inequality, $x \leqq 0$, with the remaining inequality $x \geqq 2$, there is no value of $x$ which is less than or equal to zero, and, at the same time, greater than or equal to 2. This is illustrated in Diagram D. Hence, there is no solution set for Case 4.

$$= [m^2 - 4m + 3](m - 4)$$

distributing, $\quad = \left(m^3 - 4m^2 + 3m\right) - \left(4m^2 - 16m + 12\right)$

$$= m^3 - 4m^2 + 3m - 4m^2 + 16m - 12$$

$$(m - 1)(m - 3)(m - 4) = m^3 - 8m^2 + 19m - 12 \qquad (8)$$

From equations (6) and (7):

$$m^3 - 8m^2 + 19m - 12 = (m - 1)(m - 3)(m - 4) = 0$$

(i) Take $m = 1$, and substitute in either (1) or (2). From (2),

$$x^3\left(2 - 2(1) + (1)^2\right) = 1$$
$$x^3(2 - 2 + 1) = 1$$
$$x^3(0 + 1) = 1$$
$$x^3(1) = 1$$
$$x^3 = 1$$

Take the cube root of each side:

$$\sqrt[3]{x^3} = \sqrt[3]{1}$$
$$x = \sqrt[3]{1} = 1 .$$

Also, $y = mx = 1(1) = 1$.

(ii) Take $m = 3$, and substitute in (2):

thus $\quad 5x^3 = 1$. Then,

$$x^3 = 1/5 .$$

Take the cube root of each side:

$$\sqrt[3]{x^3} = \sqrt[3]{1/5}$$
$$x = \sqrt[3]{1/5}$$

and $\quad y = mx = 3x = 3\sqrt[3]{1/5}$ .

(iii) Take $m = 4$; we obtain from (2):

$$10x^3 = 1.$$

Then, $x^3 = \frac{1}{10}$ . Take the cube root of each side:

$$\sqrt[3]{x^3} = \sqrt[3]{1/10} ,$$

or

$$x = \sqrt[3]{1/10} .$$

and

$$y = mx = 4x = 4\sqrt[3]{1/10} .$$

Hence the complete solution is

$$x = 1, \sqrt[3]{1/5} , \sqrt[3]{1/10}.$$
$$y = 1, 3\sqrt[3]{1/5} , 4\sqrt[3]{1/10} .$$

**• PROBLEM 638**

Show that $x^3 > y^3$ if $x > y$.

Solution: If we subtract $y^3$ from both sides of the inequality $x^3 > y^3$ we obtain:

$$y = x^{3/2} - x$$

$$y = (-1)^{3/2} - (-1) = \left(-1^{1/2}\right)^3 + 1 = (i)^3 + 1 = i^2(i) + 1$$

$$= -1(i) + 1 = -i + 1 \neq 0 .$$

Thus the point (-1,0) is not on the graph $y = x^{3/2} - x$.

To check (4,4) we replace x by 4:

$$y = x^{3/2} - x$$

$$y = (4)^{3/2} - 4 = \left(4^{1/2}\right)^3 - 4 = 2^3 - 4 = 8 - 4 = 4$$

Thus the point (4,4) is on the graph $y = x^{3/2} - x$ .

To check (9,17) we replace x by 9:

$$y = x^{3/2} - x$$

$$y = (9)^{3/2} - 9 = \left(9^{1/2}\right)^3 - 9 = 3^3 - 9 = 27 - 9 = 18 \neq 17$$

Thus the point (9,17) is not on the graph $y = x^{3/2} - x$.

Therefore, the points (1,0) and (4,4) belong to the graph.

DEGREE 4

• PROBLEM 640

Solve the equation $x^4 - 5x^2 - 36 = 0$.

Solution: This is a fourth degree equation, but it can be solved by the same methods applied to quadratic equations. To solve $x^4 - 5x^2 - 36 = 0$, we let $z = x^2$, substitute in the given equation, and get

$$z^2 - 5z - 36 = 0$$

This is now a quadratic equation in the variable z. We solve this equation by factoring.

| | |
|---|---|
| $z^2 - 5z - 36 = 0$ | |
| $(z - 9)(z + 4) = 0$ | Factoring |
| $z - 9 = 0, \; z + 4 = 0$ | Setting both factors equal to zero |
| $z = 9, \quad z = -4$ | Solving for z. |

Hence the solution set of the equation in z is {- 4,9}. Now we replace z in $z = x^2$ by - 4 and then by 9 and get

$$x^2 = -4$$

Taking the square root of each member,

$$x = \pm \sqrt{-4} = \pm \sqrt{4(-1)} = \pm \sqrt{4}\,\sqrt{-1}$$

Solve for x.

$$x^4 - 10x^2 + 9 = 0$$

Solution: The given equation can be written as

$$(x^2)^2 - 10x^2 + 9 = 0.$$

Set $p = x^2$ and solve for p.

$$p^2 - 10p + 9 = 0.$$

Factor the left side of this equation into a product of two polynomials. Hence,

$$(p - 9)(p - 1) = 0$$

Whenever a product ab = 0, where a and b are any two numbers, either a = 0 or b = 0. Hence, either,

$$p - 9 = 0 \quad \text{or} \quad p - 1 = 0$$

$$p = 9 \quad \text{or} \quad p = 1$$

Set each value of $p = x^2$ and solve for x.

$$p = x^2 = 9$$

$$x^2 = 9$$

Take the square root of both sides.

$$\sqrt{x^2} = \sqrt{9}$$

$$x = \pm 3$$

Also,

$$p = x^2 = 1$$

$$x^2 = 1$$

Take the square root of both sides.

$$\sqrt{x^2} = \sqrt{1}$$

$$x = \pm 1$$

The solution set is {3, -3, 1, -1}.



Solve the equation $x^4 = 4x^2 - 4$.

$9x^2 + 2x^2 - 21x - 18 = 0.$

Combine like terms: $(x^2 + 3x)^2 - 7x^2 - 21x - 18 = 0.$

Factor $-7$ from the second and third terms; then we obtain:

$$(x^2 + 3x)^2 - 7(x^2 + 3x) - 18 = 0.$$

By substituting $v = x^2 + 3x$, we have a quadratic equation in $v$:

$$v^2 - 7v - 18 = 0.$$

Factor this quadratic equation in $v$, in terms of a product of two binomials:

$$(v - 9)(v + 2) = 0.$$

Whenever we have a product of two numbers such that $ab = 0$, then either $a = 0$ or $b = 0$. Thus, (1) $v - 9 = 0$ or (2) $v + 2 - 0$. Substitute the expression $(x^2 + 3x)$ for $v$ in equations (1) and (2). Then:

(3) $x^2 + 3x - 9 = 0$ or (4) $x^2 + 3x + 2 = 0$

Solve for $x$ by the quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$,

from the equation $ax^2 + bx + c = 0$. For equation (3), $a = 1$, $b = 3$, $c = -9$. Thus,

$$x = \frac{-3 \pm \sqrt{9-4(1)(-9)}}{2(1)}$$

$$x = \frac{-3 \pm 3\sqrt{5}}{2}$$

For equation (4), $a = 1$, $b = 3$, $c = 2$. Thus,

$$x = \frac{-3 \pm \sqrt{9-4(1)(2)}}{2(1)}$$

$$x = \frac{-3 \pm 1}{2} = \frac{-3 + 1}{2} \text{ or } \frac{-3 - 1}{2}$$

Therefore, the four solutions are

$$x = -1, -2, \frac{-3 \pm 3\sqrt{5}}{2}, \text{ or } x = -1, -2, \frac{-3 + 3\sqrt{5}}{2}, \frac{-3 - 3\sqrt{5}}{2}$$

**• PROBLEM 648**

Form the equation whose roots are $0$, $\pm a$, $\frac{c}{b}$.

Solution: The roots of the equation are $0$, $+a$, $-a$, and $\frac{c}{b}$. Hence, $x = 0$, $x = a$, $x = -a$, and $x = \frac{c}{b}$. Subtract 0 from both sides of the first equation:

$$x - 0 = 0 - 0 = 0.$$

Subtract $a$ from both sides of the second equation:

$$x - a = a - a = 0.$$

Add $a$ to both sides of the third equation:

$$(\sqrt{2} + \sqrt{-3})(\sqrt{2} - \sqrt{-3}) = 2 + \sqrt{2}\sqrt{-3} - \sqrt{2}\sqrt{-3} - \sqrt{-3}\,\sqrt{-3}$$

$$= 2 - (-3).$$

$$= 5.$$

So our sum $= 2\sqrt{2} = \frac{-b}{a}$, and our product $= 5 = \frac{c}{a}$.

Therefore, $a = 1$, $b = -2\sqrt{2}$, $c = 5$, and the quadratic factor corresponding to this pair of roots is $x^2 - 2\sqrt{2}x + 5$.

The second pair of roots are $-\sqrt{2} + \sqrt{-3}$ and $-\sqrt{2} - \sqrt{-3}$. Their sum is:

$-\sqrt{2} + \sqrt{-3} + (-\sqrt{2} - \sqrt{-3}) = -2\sqrt{2}$, and their product is:

$$(-\sqrt{2} + \sqrt{-3})(-\sqrt{2} - \sqrt{-3}) = 2 - \sqrt{2}\,\sqrt{-3} + \sqrt{2}\,\sqrt{-3} - \sqrt{-3}\,\sqrt{-3}$$

$$= 2 - (-3)$$

$$= 2 + 3$$

$$= 5.$$

So our sum $= -2\sqrt{2} = \frac{-b}{a}$, and our product $= 5 = \frac{c}{a}$. Therefore, $a = 1$, $b = 2\sqrt{2}$, $c = 5$, and the quadratic factor corresponding to the second pair of roots is $x^2 + 2\sqrt{2}x + 5$.

Thus the required equation is

$$(x^2 + 2\sqrt{2}x + 5)(x^2 - 2\sqrt{2}x + 5) = 0,$$

or $$x^4 + 2x^2 + 25 = 0.$$

**• PROBLEM 650**

Solve the equation $6x^4 - 13x^3 - 35x^2 - x + 3 = 0$, having given that one root is $2 - \sqrt{3}$.

<u>Solution:</u> Since $2 - \sqrt{3}$ is a root, $2 + \sqrt{3}$ is also a root. Recall that every quadratic equation may be written in the form:

$$ax^2 + bx + c = 0.$$

Also, recall the well-known formula that the sum of the roots of a quadratic equation $= -\frac{b}{a}$, and the product of the roots is $\frac{c}{a}$. Thus, finding the sum and product of our known roots will give us their corresponding equation. Our roots are $2 - \sqrt{3}$ and $2 + \sqrt{3}$. Their sum is

$2 - \sqrt{3} + (2 + \sqrt{3}) = 4$ and their product is

$(2 - \sqrt{3})(2 + \sqrt{3}) = 4 - 2\sqrt{3} + 2\sqrt{3} - 3 = 1$. So our

sum $= 4 = -\frac{b}{a}$, and our product $= 1 = \frac{c}{a}$. Therefore,

$a = 1$, $b = -4$, $c = 1$, and the equation is $x^2 - 4x + 1$.

Now, dividing this new equation into the given equation we obtain:

$$
\begin{array}{r|l}
 & 6x^2 + 11x + 3 \\
\hline
x^2 - 4x + 1 & 6x^4 - 13x^3 - 35x^2 - x + 3 \\
 & \underline{6x^4 - 24x^3 + 6x^2} \\
 & 11x^3 - 41x^2 - x + 3 \\
 & \underline{11x^3 - 44x^2 + 11x} \\
 & 3x^2 - 12x + 3 \\
 & \underline{3x^2 - 12x + 3} \\
 & 0
\end{array}
$$

Thus, $6x^4 - 13x^3 - 35x^2 - x + 3 = (x^2 - 4x + 1)(6x^2 + 11x + 3)$;

Since $6x^4 - 13x^3 - 35x^2 - x + 3 = 0$,

then $(x^2 - 4x + 1)(6x^2 + 11x + 3) = 0$,

and this means that $6x^2 + 11x + 3 = 0$.

Factoring this equation we obtain $(3x + 1)(2x + 3) = 0$. Thus, $3x + 1 = 0$ or $x = -\frac{1}{3}$; and $2x + 3 = 0$, or $x = -\frac{3}{2}$. Therefore, the roots of the given equation are

$-\frac{1}{3}, -\frac{3}{2}, 2 + \sqrt{3}, 2 - \sqrt{3}$.

**• PROBLEM 651**

Find all rational roots of the equation

$$x^4 - 4x^3 + x^2 - 5x + 4 = 0.$$

Solution: This is a fourth degree equation. We can solve it by synthetic division. Guess at a root by trying to find an x-value which will make the equation equal to zero. $x = 4$ works.

Now write the coefficients of the equation in descending powers of x. Note that if a term is missing, its coefficient is zero. In the corner box, the root 4 is placed. Bring the first coefficient down and multiply it by the root. Place the result below the next coefficient and add. Multiply the result by the root and continue as before.

$$
\begin{array}{rrrrrl}
1 & -4 & +1 & -5 & +4 & \underline{|4} \\
 & +4 & 0 & +4 & -4 & \\
\hline
1 & 0 & 1 & -1 & 0 &
\end{array}
$$

The last result is zero which indicates $(x - 4)$ is a factor and $x = 4$ is a root. The other results are the coefficients of the third degree expression when $(x - 4)$ is factored.

$$(x - 4)(x^3 + 0x^2 + x - 1) = 0$$

$$(x - 4)(x^3 + x - 1) = 0$$

To find the roots of the third degree equation, call it $g(x)$, we must set it equal to zero.

$$g(x) = x^3 + x - 1 = 0$$

Try to find where the curve of the equation crosses the x-axis which is when $y = 0$.

| x | -2 | -1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| y | -11 | -3 | -1 | 1 | 9 |

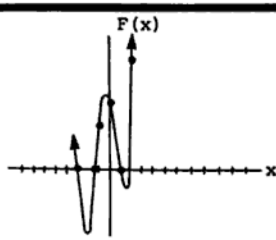
It crosses the x-axis between $x = 0$ and $x = 1$. It is an irrational root.

Since the given equation is a fourth degree equation, it has 4 roots. All of the real roots, namely the rational number 4, and an irrational number between 0 and 1, have been found. Therefore, the two remaining roots are not real; that is, they are complex numbers.

**• PROBLEM 652**

Approximate the real roots of the equation

$$x^4 + 2x^3 - 5x^2 - 4x + 6 = 0.$$




| x | -4 | -3 | -2 | -1 | 0 | 1 |
|---|---|---|---|---|---|---|
| F(x) | 70 | 0 | -6 | 4 | 6 | 0 |

**Solution:** To find the real roots of the given equation, let us sketch the graph of the related polynomial function defined by

$$F(x) = x^4 + 2x^3 - 5x^2 - 4x + 6. \quad \text{(See Fig.)}$$

When $f(x) = 0$, $x$ is a root of the equation. From the table, two real roots are $x = -3$ and $x = 1$. Reducing $F(x)$ by dividing by $(x + 3)(x - 1)$ and solving, we find that $-\sqrt{2}$ and $+\sqrt{2}$ are also roots.

Solve $x^4 + y^4 = 82$ (1),

$x - y = 2$ (2).

<u>Solution:</u> Let $x = u + v$ and $y = u - v$. From equation (2)

$$\begin{aligned} x - y &= 2 \\ u + v - (u - v) &= 2 \\ u + v - u + v &= 2 \\ 2v &= 2 \\ v &= 1 \end{aligned}$$

Substituting in equation (1),

$$\begin{aligned} x^4 + y^4 &= 82 \\ (u + v)^4 + (u - v)^4 &= 82 \end{aligned}$$

Replacing $v$ by $1$,

$$(u + 1)^4 + (u - 1)^4 = 82 \qquad (3)$$

Now, simplify the left side of equation (3),

$$(u + 1)^2 = (u+1)(u+1) = u^2 + 2u + 1$$

$$(u+1)^3 = (u+1)(u+1)^2 ,$$

or

$$(u+1)^3 = (u+1)\left(u^2+2u+1\right)$$

Distributing on the right side of this equation:

$$\begin{aligned} (u+1)^3 &= \left(u^3+2u^2+u\right) + \left(u^2+2u+1\right) \\ &= u^3 + 3u^2 + 3u + 1 . \end{aligned}$$

$$(u+1)^4 = (u+1)(u+1)^3$$

or

$$(u+1)^4 = (u+1)\left(u^3+3u^2+3u+1\right).$$

Distributing on the right side of this equation:

$$(u+1)^4 = \left(u^4+3u^3+3u^2+u\right) + \left(u^3+3u^2+3u+1\right)$$

$$(u+1)^4 = u^4 + 4u^3 + 6u^2 + 4u + 1 \qquad (4)$$

Also, $(u-1)^2 = (u-1)(u-1) = u^2 - 2u + 1$

$$(u-1)^3 = (u-1)(u-1)^2$$

or

$$(u-1)^3 = (u-1)\left(u^2-2u+1\right).$$

Distributing on the right side of this equation:

$$\begin{aligned} (u-1)^3 &= \left(u^3-2u^2+u\right) - \left(u^2-2u+1\right) \\ &= u^3 - 3u^2 + 3u - 1 \end{aligned}$$

$$(u-1)^4 = (u-1)(u-1)^3$$

or

$$(u-1)^4 = (u-1)\left(u^3-3u^2+3u-1\right)$$

Distributing on the right side of this equation:

$$(u-1)^4 = \left(u^4-3u^3+3u^2-u\right) - \left(u^3-3u^2+3u-1\right)$$

$$(u-1)^4 = u^4 - 4u^3 + 6u^2 - 4u + 1 \qquad (5)$$

Using equations (4) and (5) to simplify equation (3):

$$(u+1)^4 + (u-1)^4 = \left(u^4+4u^3+6u^2+4u+1\right) + \left(u^4-4u^3+6u^2-4u+1\right)$$

$$= u^4 + 4u^3 + 6u^2 + 4u + 1 + u^4 - 4u^3 + 6u^2 - 4u + 1$$
$$= 2u^4 + 12u^2 + 2 = 82.$$

Thus, $2u^4 + 12u^2 + 2 = 82$.

Subtract 82 from both sides of this equation:

$$2u^4 + 12u^2 + 2 - 82 = 82 - 82$$
$$2u^4 + 12u^2 - 80 = 0$$

Divide both sides of this equation by 2:

$$\frac{2u^4 + 12u^2 - 80}{2} = \frac{0}{2},$$

or

$$u^4 + 6u^2 - 40 = 0$$

Factoring the left side of this equation into a product of two polynomials:

$$(u^2 + 10)(u^2 - 4) = 0.$$

Whenever a product $ab = 0$, where $a$ and $b$ are any two numbers, either $a = 0$ or $b = 0$. Hence, either

$$u^2 + 10 = 0 \quad \text{or} \quad u^2 - 4 = 0$$
$$u^2 = -10 \quad \text{or} \quad u^2 = 4$$
$$u = \pm\sqrt{-10} \quad \text{or} \quad u = \pm\sqrt{4} = \pm 2$$

Using the equations relating $x$ and $y$ to $u$ and $v$ in order to solve for $x$ and $y$:

when $u = \sqrt{-10}$, $x = u + v = \sqrt{-10} + 1 = 1 + \sqrt{-10}$ and
$y = u - v = \sqrt{-10} - 1 = -1 + \sqrt{-10}$.

when $u = -\sqrt{-10}$, $x = u + v = -\sqrt{-10} + 1 = 1 - \sqrt{-10}$ and
$y = u - v = -\sqrt{-10} - 1 = -1 - \sqrt{-10}$.

when $u = 2$, $x = u + v = 2 + 1 = 3$ and
$y = u - v = 2 - 1 = 1$.

when $u = -2$, $x = u + v = -2 + 1 = -1$ and
$y = u - v = -2 - 1 = -3$.

Thus, the pairs of solutions are:

$$x = 1 + \sqrt{-10}, \quad y = -1 + \sqrt{-10}$$
$$x = 1 - \sqrt{-10}, \quad y = -1 - \sqrt{-10}$$
$$x = 3, \quad y = 1$$
$$x = -1, \quad y = -3.$$

**• PROBLEM 654**

If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, show that

$$\frac{a^3b + 2c^2e - 3ae^2f}{b^4 + 2d^2f - 3bf^3} = \frac{ace}{bdf}.$$

Solution: Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k$;

then $a = bk$, $c = dk$, $e = fk$.

# CHAPTER 22

# PROGRESSIONS AND SEQUENCES

**Basic Attacks and Strategies for Solving Problems in this Chapter. See pages 491 to 529 for step-by-step solutions to problems.**

Arithmetic and geometric sequences are two types of sequences or progressions that are of special interest mathematically because they fit the real world as mathematical models in many situations. The procedure for finding the terms of an arithmetic sequence, given the first term, involves adding a constant called the common difference, to the preceding term. For example,

$$3, 10, 17, 24, 31, \ldots$$

is an arithmetic sequence because after the first term 3, the other given terms are found by adding the constant 7 to each preceding term. Any additional terms in the sequence is found in a similar manner.

The procedure for finding the terms of a geometric sequence, given the first term, involves multiplying the preceding number by a constant number called the common ratio. For example,

$$3, 6, 12, 24, 48$$

is a geometric sequence because the ratio of one term to the preceding term equals the constant 2.

Another important progression is called the harmonic progression. The sequence of numbers of such a progression are reciprocals of the sequence of numbers in an arithmetic progression. Thus, the procedure for finding the next term in a harmonic progression is to first take the reciprocal of each term to get the corresponding arithmetic progression. Then, find the common differences between the terms in the arithmetic progression and use it to find the next term by adding the common difference value to the last term in the sequence. Take the reciprocal of this value to find the next term in the harmonic progression. For example, find the next term in the harmonic progression

$$1/12, 1/19, 1/26, 1/33, \ldots$$

The corresponding arithmetic progression is given by 12, 19, 26, 33, ... which has a common difference of 7. Thus, the next term in the arithmetic progression sequence is 40. The reciprocal of 40 is $1/40$. So, the next term in the harmonic progression is $1/40$.

# Step-by-Step Solutions to Problems in this Chapter, "Progressions and Sequences"

## ARITHMETIC

**• PROBLEM 655**

If the 6th term of an arithmetic progression is 8 and the 11th term is - 2, what is the 1st term? What is the common difference?

Solution: An arithmetic progression is a sequence of numbers where each term excluding the first is obtained from the preceding one by adding a fixed quantity to it. This constant amount is called the common difference.
Let a = value of first term,and d = common difference

| | | | | | | |
|---|---|---|---|---|---|---|
| Term of sequence: | $1^{st}$ | $2^{nd}$ | $3^{thrd}$ | $4^{th}$ | ... | $n^{th}$ |
| Value of term: | a | a+d | a+2d | a+3d | ... | a+(n-1)d |

Use the formula for the nth term of the sequence to write equations for the given 6th and 11th terms, to determine a and d.

11th term: $a + (11 - 1)d = - 2$

6th term: $a + (6 - 1)d = 8$. Simplifying the above equations we obtain:

$$a + 10d = - 2 \qquad (1)$$
$$a + 5d = 8 \qquad (2)$$

$5d = - 10$ Subtracting (2) from (1)

$d = - 2$ Substituting in (1)

$$a + 10(- 2) = - 2$$
$$a = 18$$

The first term is 18 and the common difference is - 2.

**• PROBLEM 656**

Find $a_n$ for the sequence 1, 4, 7, 10,...

Solution: An arithmetic progression (A.P.) is a sequence of numbers each of which, after the first, is obtained by adding to the preceding number a constant number, d, called the common difference. Thus 1, 4, 7, 10,... is an arithmetic progression because each term is obtained by adding 3 to the preceding number. The nth term, $a_n$, of an A.P. is:

$$a_n = a_1 + (n-1)d$$

where $a_1$ = first term of the progression; d = common difference; n = number of terms.

Thus, with $a_1 = 1$ and $d = 3$,

$$a_n = 1 + (n-1)3 = 1 + 3n - 3$$

$$a_n = 3n - 2$$

It is easily verified by substitution that $a_n = 3n - 2$ will suffice, i.e.

$$a_1 = 3(1) - 2 = 1$$
$$a_2 = 3(2) - 2 = 4$$
$$a_3 = 3(3) - 2 = 7$$
$$a_4 = 3(4) - 2 = 10$$
$$a_5 = 3(5) - 2 = 13 = a_4 + d = 10 + 3$$

by definition of an A.P. etc.

**• PROBLEM 657**

Find the first term of an arithmetic progression if the fifth term is 29 and d is 3.

Solution: The $n^{th}$ term, or last term, of an arithmetic progression (A.P.) is:

$$\ell = a_1 + (n-1)d \qquad (1)$$

where $a_1$ = first term of the progression
$d$ = common difference
$n$ = number of terms
$\ell$ = $n^{th}$ term, or last term.

Using this formula we can find the first term of an A.P. whose fifth term is 29 and d is 3. Since $\ell = a_5 = 29$, $d = 3$, and $n = 5$, substituting into equation (1) gives:

$$29 = a_1 + (5 - 1)3$$
$$29 = a_1 + 12$$
$$a_1 = 29 - 12 = 17.$$

Thus, the first term is 17.

**• PROBLEM 658**

If the first term of an arithmetic series is -4 and the twelfth term is 32, find the common difference.

Solution: Since the first and last terms and the number of terms are known, the formula for the $n^{th}$ term, or last term of the series

$$\ell = a_1 + (n-1)d$$

where $a_1$ = first term of the series
$n$ = number of terms
$d$ = common difference
$\ell$ = $n^{th}$ term, or last term

can be solved for d.

$\ell = a_1 + (n-1)d$ with $a_1 = -4$, $n = 12$, $\ell = 32$ gives

$$32 = -4 + (12 - 1)d$$

$$36 = 11\ d$$

$$\frac{36}{11} = d$$

**• PROBLEM 659**

Find the twelfth term of the arithmetic sequence

$$2,\ 5,\ 8,\ \ldots\ .$$

Solution: It is given that the sequence is an arithmetic sequence. The common difference d is obtained by subtracting any term from the succeeding term;

$$d = 5 - 2 = 3$$

The twelfth term, $a_{12}$, can be obtained by substituting $a_1 = 2$, $d = 3$, and $n = 12$ in the expression for the $n^{th}$ term:

$$a_1 + (n - 1)d$$

Thus,

$$a_{12} = a_1 + (n - 1)d = 2 + (12 - 1)3 = 35$$

This can be checked by completing the sequence to the twelfth term.

$$2,\ 5,\ 8,\ 2+(4-1)3,\ 2+(5-1)3,\ 2+(6-1)3,\ \ldots\ ,\ 2+(11-1)3,\ 35$$

$$2,\ 5,\ 8,\ 11,\ 14,\ 17,\ 20,\ 23,\ 26,\ 29,\ 32,\ 35$$

**• PROBLEM 660**

Given that the first term of an arithmetic sequence is 56 and the seventeenth term is 32, find the tenth term and the twenty-fifth term.

Solution: The formula for the nth term, or last term, of an arithmetic sequence is:

$$\ell = s_1 + (n-1)d$$

where $s_1$ = first term of the sequence
$d$ = common difference
$n$ = number of terms
$\ell$ = nth term, or last term.

Since we are given the first and seventeenth term of the sequence we can use this information, with $n = 17$, to find d before we proceed to find the tenth and twenty-fifth terms. Thus we find d by

$$s_{17} = s_1 + (17 - 1)d,$$

$$32 - 56 = 16d$$

$$16d = -24$$

$$d = -\frac{3}{2}$$

Thus, the 23rd term = a + 22d, and substituting for a and d we have:

$$\frac{143}{2} + 22\left(-\frac{5}{2}\right) = \frac{143}{2} - \frac{110}{2} = \frac{33}{2}.$$

**• PROBLEM 662**

If the first term of an arithmetic progression is 7, and the common difference is -2, find the fifteenth term and the sum of the first fifteen terms.

Solution: An arithmetic progression is a sequence of numbers each of which is obtained from the preceding one by adding a constant quantity to it, the common difference, d. If we designate the first term by a and the common difference by d, then the terms can be expressed as follows:

| | | | | | |
|---|---|---|---|---|---|
| terms of series | 1 | 2 | 3 | ..... | n |
| value of term | a | a + d | a + 2d | | $\ell = a + (n-1)d$ |

In this example a = 7, and d = -2. To find the fifteenth term, we have n = 15. The nth term is $a + (n-1)d$. For n = 15, $a + (n-1)d = 7 + (15 - 1)(-2) = 7 - 28 = -21$. To find the sum of the first fifteen terms apply the following formula:

$$S_n = \frac{n}{2}(a + \ell)$$

$$S_{15} = \frac{15}{2}[7 + (-21)] = \frac{15}{2}(-14) = -105.$$

**• PROBLEM 663**

Find the sum of the first sixteen terms of the arithmetic series whose first term is $\frac{1}{4}$ and common difference is $\frac{1}{2}$.

Solution: The sum of the first n terms of an arithmetic series is

$$S_n = \frac{n}{2}\left[2a_1 + (n - 1)d\right]$$

where $a_1$ = first term of the series
d = common difference
n = number of terms
$S_n$ = sum of first n terms

Hence, the sum of the first sixteen terms of the arithmetic series with $a_1 = \frac{1}{4}$, $d = \frac{1}{2}$, and n = 16, is

$$S_{16} = \frac{16}{2}\left[2\left(\frac{1}{4}\right) + (16 - 1)\frac{1}{2}\right]$$
$$= 8\left(\frac{1}{2} + \frac{15}{2}\right)$$

$$= 8\left(\frac{16}{2}\right)$$
$$= 8(8)$$
$$= 64$$

**• PROBLEM 664**

Find the sum of the first 20 terms of the arithmetic progression -9, -3, 3, ...

Solution: An arithmetic progression is a sequence in which each term after the first is formed by adding a fixed amount, called the common difference, to the preceding term. The common difference of -9, -3, 3, ... is 6 since $-9+6=-3$, $-3+6=3$, etc. If a is the first term, d is the common difference, and n is the number of terms of the arithmetic progression, then the last term (or $n^{th}$ term) $\ell$ is given by

$$\ell = a + (n-1)d \qquad (1)$$

and the sum $S_n$ of the n terms of this progression is given by

$$S_n = \frac{n}{2}[a + \ell] \qquad (2)$$

In this example,

$a = -9$, $d = 6$, $n = 20$. By equation (1):

$$\ell = -9 + (19)(6)$$
$$= -9 + 114$$
$$= 105.$$

By equation (2): $S_{20} = \frac{20}{2}\left(-9 + 105\right)$
$$= 10(96)$$
$$= 960.$$

Thus, the sum of the first 20 terms is 960.

**• PROBLEM 665**

Find the sum of the arithmetic series

$$5 + 9 + 13 + \ldots + 401$$

Solution: The common difference is $d = 9 - 5 = 4$, and the nth term, or last term, is $\ell = a + (n-1)d$, where

a = first term of the progression
d = common difference
n = number of terms
$\ell$ = nth term, or last term.

Hence, $401 = 5 + (n-1)4$. Solving for the number of terms n, we have $n = 100$. The required sum is

$$S = 5 + 9 + 13 + \ldots + 393 + 397 + 401$$

Written in reverse order, this sum is

$$S = 401 + 397 + 393 + \ldots + 13 + 9 + 5$$

Adding the two expressions for S, we have

$$2S = (5 + 401) + (9 + 397) + (13 + 393) + \ldots + (393 + 13) + (397 + 9) + (401 + 5)$$

Each term in parentheses is equal to the sum of the first and last terms; 5 + 401 = 406. There is a parenthetic term corresponding to each term of the original series; that is, there are 100 terms. Hence,

$$2S = 100(5 + 401) = 40,600 \text{ and } S = \frac{40,600}{2} = 20,300$$

In general, the sum of the first n terms of an arithmetic series is:

$$S = \frac{n}{2}(a + \ell) = \frac{n}{2}[2a + (n-1)d]$$

For this problem,

$$S = \frac{100}{2}(5 + 401) = \frac{100}{2}[2(5) + (100-1)4] = 20,300$$

**• PROBLEM 666**

Find the sum of the first 100 positive integers.

Solution: The first 100 positive integers is an arithmetic progression (A.P.), because each number after the first is obtained by adding 1, called the common difference, to the preceding number. For an A.P., the sum of the first n terms is

$$S_n = \frac{n}{2}(a + \ell)$$

where a = first number of the progression
n = number of terms
$\ell$ = $n^{th}$ term, or last term
$S_n$ = sum of first n terms.

Concerning the first 100 positive integers: there are 100 terms; hence n = 100. The first term is 1; hence, a = 1. The last term is 100; hence, $\ell$ = 100.

$$S_{100} = \frac{100(1 + 100)}{2} = 5050$$

**• PROBLEM 667**

Find the sum of the first 25 even integers.

Solution: The even integers form an arithmetic progression which is a sequence of numbers each of which is obtained from the preceding one by adding a constant quantity to it. This constant quantity is called the common difference, d. The first term of an arithmetic progression is a and the nth term is $\ell = a + (n-1)d$. In this case:

$$a = 2, n = 25, d = 2.$$

$$\ell = 2 + (25 - 1)2$$

$$= 50$$

To find the sum of the n terms of an arithmetic progression, we apply the formula

$$S_n = \frac{n}{2}(a + \ell).$$

$$S_{25} = \frac{25}{2}(2 + 50)$$

$$= 25(26)$$

$$= 650.$$

**• PROBLEM 668**

Find the sum of the first p terms of the sequence whose nth term is $3n - 1$.

Solution: By putting $n = 1$, and $n = p$, respectively, in $3n - 1$, we obtain

1st term $= 3(1) - 1 = 2$

last term = pth term $= 3p - 1$.

We can now apply the formula for the sum of the terms of an arithmetic progression, which states:

$$S_n = \frac{n}{2}(a + \ell), \text{ where}$$

$n$ = the number of terms

$a$ = the first term

$\ell$ = the last term

$S_n$ = sum of first n terms.

By subtitution,

$$S_p = \text{sum of first p terms} = \frac{p}{2}(2 + 3p - 1) =$$

$$\frac{p}{2}(3p + 1).$$

**• PROBLEM 669**

How many terms of the sequence $-9, -6, -3, \ldots$ must be taken that the sum may be 66?

Solution: To solve this problem we apply the formula for the sum of the first n terms of an arithmetic progression. The formula states:

$$S_n = \frac{n}{2}[2a + (n - 1)d], \text{ where}$$

$S_n$ = sum of the first n terms

$n$ = number of terms
$a$ = first term

d = common difference

We are given all of the above information except n. Therefore, by substituting for $S_n$, a, and d, we can solve for n. We are given that $S_n = 66$, $a = -9$, and $d = 3$, since $-9 + 3 = -6$, $-6 + 3 = -3$, ...

Hence, $\frac{n}{2}[-18 + (n - 1)3] = 66.$

Now, multiplying both sides of the equation by 2 we obtain: $n[-18 + (n - 1)3] = 132$; and simplifying the expression in brackets, we have: $n(-18 + 3n - 3) = 132$, or $n(3n - 21) = 132$. Therefore, we have: $3n^2 - 21n = 132$, and dividing each term by 3 we obtain:

$$n^2 - 7n - 44 = 0; \text{ factoring we have,}$$

$$(n - 11)(n + 4) = 0;$$

therefore, $n = 11$ or $-4$.

We can reject the negative value because there cannot be a negative number of terms in the sequence, and therefore, 11 terms must be taken so that the sum of the terms is 66.

We can check this by taking 11 terms of the series. Doing this we have:

$$-9, -6, -3, 0, 3, 6, 9, 12, 15, 18, 21;$$

the sum of which is 66.

**• PROBLEM 670**

How many terms of the sequence 26, 21, 16, ... must be taken to amount to 74?

Solution: To solve this problem we apply the formula for the sum of the first n terms of an arithmetic progression. The formula states:

$$S = \frac{n}{2}[2a + (n - 1)d] \text{ , where}$$

S = sum of the first n terms

n = number of terms

a = first term

d = common difference

We are given all of the above information except n. Therefore, by substituting for S, a, and d, we can solve for n. We are given that $S = 74$, $a = 26$, and $d = -5$, since $26 - 5 = 21$, $21 - 5 = 16$, ...

Hence, $74 = \frac{n}{2}[2(26) + (n - 1)(-5)]$ , or

Solution: In an arithmetic progression, the terms between any two other terms are called the arithmetic means between the two given terms. An arithmetic progression is a sequence of numbers where each is derived from the preceeding one by adding a constant quantity to it. The constant quantity is called the common difference. The first term of the A.P. is designed by a and the common difference by d. We express the terms of the series:

| Term of the Series | 1 | 2 | 3 | 4 | ... | n |
|---|---|---|---|---|---|---|
| Value of the Series | $a_1 = a$ | $a_2 = a+d$ | $a_3 = a_2+d$ $= (a+d)+d$ $= a+2d$ | $a_4 = a_3+d$ $= a+3d$ | ... | $a_n =$ $a+(n-1)d$ |

We are concerned with seven terms here: the first term, five arithmetic means, and the last term. In order to find the arithmetic means, we need to find the common difference, d. (We know a, which is 13.) The seventh term, $31 = a + (n-1)d = 13(7-1)d = 13 + 6d$. Thus,

$$31 = 13 + 6d$$
$$18 = 6d$$
$$d = 3$$

Consequently the five arithmetic means are

$$a_2 = 13 + 3 = 16,\ a_3 = 19,\ a_4 = 22,\ a_5 = 25,\ a_6 = 28.$$

**• PROBLEM 673**

Insert 20 arithmetic means between 4 and 67.

Solution: 'Arithmetic means' are all the terms that fall between any two given terms in an arithmetic progression. The two given terms are called the extremes. Thus, in this example, including the extremes, the number of terms will be 22; so that we have to find a sequence of 22 terms in A.P., of which 4 is the first and 67 the last.

Let d be the common difference; then, since the general nth term of an A.P. $= a + (n - 1)d$, and 67 is the 22nd term, we have:

$$67 = a + 21d, \qquad a = \text{first term.}$$

Since the first term is 4 we obtain:

$4 + 21d = 67$. Solving for d we find:

$$21d = 63$$

$$d = 3.$$

Thus, the sequence is,

$$4,\ 4 + 3,\ (4 + 3) + 3,\ \dots \quad \text{or}$$

$$4,\ 7,\ 10,\ 13,\ \dots,\ 67$$

and the 20 required means are,

7, 10, 13, 16, 19, 22, 25, 28, 31, 34, 37, 40, 43, 46, 49,

52, 55, 58, 61, 64.

• **PROBLEM 674**

Determine the first four terms and 12th term of the arithmetic progression generated by $F(x) = 2x + 3$.

**Solution:** Find the terms of the progression by letting $x = 1, 2, 3, \ldots$ etc.

$$1\text{st term} = F(1) = 2(1) + 3 = 5$$

$$2\text{nd term} = F(2) = 2(2) + 3 = 7$$

$$3\text{rd term} = F(3) = 2(3) + 3 = 9$$

$$4\text{th term} = F(4) = 2(4) + 3 = 11$$

$$12\text{th term} = F(12) = 2(12) + 3 = 27$$

The common difference, d, is found by subtracting one term from the one that immediately follows it.

The first term is denoted by a.

Note: For this progression $a = 5$ and $d = 2$. The coefficient of x in the linear function will always be the common difference for the arithmetic progression.

• **PROBLEM 675**

If an arithmetic progression is generated by the linear function $F(x) = -3x + 14$, what is the first term? What is the 15th term? What is the common difference?

**Solution:** $1\text{st term} = F(1) = -3 + 14 = 11$

$$15\text{th term} = F(15) = -3(15) + 14 = -31$$

common difference $= d = -3$, the coefficient of the linear term.

The coefficient of x in a linear function will always be the common difference, d. To verify that $d = -3$, find the second term and subtract the first term from it.

• **PROBLEM 676**

The sum of three numbers in arithmetic progression is 27, and the sum of their squares is 293; find them.

**Solution:** Let a be the middle number, d the common difference: then the three numbers are $a - d, a, a + d$.

Since the sum of the three numbers is 27 we have:

$a - d + a + a + d = 27$; or $3a = 27$. Hence, $a = 9$, and the three numbers are $9 - d, 9, 9 + d$.

Now, since the sum of the squares of the numbers is 293, we can use the following equation to solve for d:

$$(9 - d)^2 + 9^2 + (9 + d)^2 = 293.$$

Squaring, we obtain:

$$(81 - 18d + d^2) + (81) + (81 + 18d + d^2) = 293$$
$$2d^2 + 243 = 293$$
$$2d^2 = 50$$
$$d^2 = 25$$
$$d = \sqrt{25} = \pm 5.$$

Therefore, the three numbers are:

$9 \pm 5,\ 9,\ 9 \pm 5$ or

$4,\ 9,\ 14.$

**• PROBLEM 677**

The sums of n terms of two arithmetic progressions are in the ratio of $7n + 1 : 4n + 27$; find the ratio of their 11th terms.

Solution: Let the first term and common difference of the two sequences be $a_1$, $d_1$, and $a_2$, $d_2$ respectively.

The sum of the first n terms of an arithmetic progression is given by the formula:

$$S = \frac{n}{2}\,[2a + (n - 1)d]\ .$$

Since the sums of the two given progressions are in the ratio $\frac{7n + 1}{4n + 27}$ we have:

$$\frac{\frac{n}{2}\,[2a_1 + (n - 1)d_1]}{\frac{n}{2}\,[2a_2 + (n - 1)d_2]} = \frac{7n + 1}{4n + 27}\ ,\ \text{or}$$

$$\frac{2a_1 + (n - 1)d_1}{2a_2 + (n - 1)d_2} = \frac{7n + 1}{4n + 27}\ .$$

Now, using the fact that the nth term of an A.P. = $a + (n - 1)d$, we find

$$\text{11th term} = a + (11 - 1)d = a + 10d.$$

Therefore, the ratio of the 11th terms of our two progressions is:

$$\frac{a_1 + 10d_1}{a_2 + 10d_2}\ .$$

Now, we want to transform the ratio of the sums into the above ratio of the 11th terms. We can do this by dividing each term in the following proportion by 2:

$$\frac{2a_1 + (n - 1)d_1}{2a_2 + (n - 1)d_2} = \frac{7n + 1}{4n + 27} .$$

Doing this we obtain:

$$\frac{a_1 + \frac{(n - 1)}{2} d_1}{a_2 + \frac{(n - 1)}{2} d_2} = \frac{\frac{7n}{2} + \frac{1}{2}}{\frac{4n}{2} + \frac{27}{2}} .$$

Now, to obtain the ratio on the left of the equal sign in the form $\frac{a_1 + 10d_1}{a_2 + 10d_2}$, we must have: $\frac{n - 1}{2} = 10$, or $n - 1 = 20$; therefore, $n = 21$. Substituting this value in our proportion we obtain:

$$\frac{a_1 + \left(\frac{21 - 1}{2}\right) d_1}{a_2 + \left(\frac{21 - 1}{2}\right) d_2} = \frac{\frac{7(21)}{2} + \frac{1}{2}}{\frac{4(21)}{2} + \frac{27}{2}} \qquad \text{or,}$$

$$\frac{a_1 + 10d_1}{a_2 + 10d_2} = \frac{\frac{147}{2} + \frac{1}{2}}{\frac{84}{2} + \frac{27}{2}} = \frac{\frac{148}{2}}{\frac{111}{2}} = \frac{148}{2} \cdot \frac{2}{111} = \frac{148}{111} .$$

Now, since both numerator and denominator are divisible by 37, we find that the desired ratio is $4 : 3$ or $\frac{4}{3}$.

**• PROBLEM 678**

If $S_1, S_2, S_3, \dots S_p$ are the sums of n terms of an arithmetic progression whose first terms are 1, 2, 3, 4, ... and whose common differences are 1, 3, 5, 7, ... respectively, find the value of

$$S_1 + S_2 + S_3 + \dots S_p$$

Solution: We can find $S_1, S_2, S_3, \dots, S_p$ by applying the formula for the sum of the first n terms of an arithmetic progression. The formula states:

$$S = \frac{n}{2}[2a + (n - 1)d] \text{, where}$$

S = the sum

n = the number of terms

a = the first term

d = the common difference

Thus, for $S_1$, $a = 1$ and $d = 1$, we have:

$$S_1 = \frac{n}{2}[2(1) + (n - 1)1] = \frac{n}{2}(2 + n - 1) =$$

$$\frac{n}{2}(n + 1) = \frac{n(n + 1)}{2} .$$

For $S_2$, $a = 2$ and $d = 3$; thus,

$$S_2 = \frac{n}{2} [2(2) + (n - 1)3] = \frac{n}{2} (4 + 3n - 3) =$$

$$\frac{n}{2} (3n + 1) = \frac{n(3n + 1)}{2} .$$

For $S_3$, $a = 3$ and $d = 5$; thus

$$S_3 = \frac{n}{2} [2(3) + (n - 1)5] = \frac{n}{2} (6 + 5n - 5) =$$

$$\frac{n}{2}(5n + 1) = \frac{n(5n + 1)}{2} .$$

Now, to find a and d for $S_p$ we notice that a relation exists between the sum and the first term, and the sum and the common difference. For $S_1$, the first term is 1, and the difference 1, or $2(1) - 1$. For $S_2$, the first term is 2, and the difference 3, or $2(2) - 1$. For $S_3$, the first term is 3, and the difference 5, or $2(3) - 1$. Similarly, for $S_p$, the first term is p, and the common difference is $(2p - 1)$. Thus,

$$S_p = \frac{n}{2} [2p + (n - 1)(2p - 1)]$$

$$= \frac{n}{2} [2p + (2pn - 2p - n + 1)] = \frac{n}{2} (2pn - n + 1).$$

Factoring n from the first two terms in the parentheses we have:

$$\frac{n}{2}[(2p - 1) n + 1].$$

Therefore, the required sum,

$S_1 + S_2 + S_3 + \ldots + S_p$ is:

$$\frac{n(n + 1)}{2} + \frac{n(3n + 1)}{2} + \frac{n(5n + 1)}{2} + \ldots . + \frac{n[(2p - 1)n + 1]}{2}$$

Factoring $\frac{n}{2}$ from each term, we obtain:

$$\frac{n}{2}[(n + 1) + (3n + 1) + (5n + 1) + \ldots ((2p - 1)n + 1)]$$

Now, since we are adding 1, p times, we can write:

$$\frac{n}{2} [(n + 3n + 5n + \ldots + (2p - 1)n) + p].$$

Factoring n from the terms in the parentheses, we obtain:

$\frac{n}{2}$ [n(1 + 3 + 5 + ... + (2p - 1)) + p].

Let us now examine the terms of the above series: 1 + 3 + 5 + ... + (2p - 1). Notice that we can apply the formula for the sum of the first n terms of an arithmetic progression, which states the following:

$$S = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + \ell).$$

In our case it is more efficient to use the form $S = \frac{n}{2}(a + \ell)$, where S = the sum

n = the number of terms

a = first term

$\ell$ = last term

We know that n = p, a = 1, $\ell$ = (2p - 1). Thus,

$$S = \frac{p}{2}(1 + 2p - 1) \quad \text{or,} \quad S = \frac{p}{2}(2p) = p^2.$$

Therefore,

$$S_1 + S_2 + S_3 + \dots + S_p = \frac{n}{2}[n(1 + 3 + 5 + \dots$$

$$\dots + (2p - 1)) + p] =$$

$\frac{n}{2}\left[n(p^2) + p\right]$. Factoring p from both terms in the brackets we obtain: $\frac{n}{2}[p(np + 1)] =$

$$\frac{np}{2}(np + 1).$$

Note: It is of interest to observe that in the formula for the sum of n terms of an A.P.,

$$S = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + \ell),$$

we can easily derive the first formula, $\frac{n}{2}[2a + (n - 1)d]$, from the second,

$\frac{n}{2}(a + \ell)$, as follows:

Since n = the number of terms, and $\ell$ = last term, then we can use the fact that: $\ell = a + (n - 1)d$. Substituting this value for $\ell$ we obtain:

$$\frac{n}{2}(a + \ell) = \frac{n}{2}[a + (a + (n - 1)d)]$$

$$= \frac{n}{2}[2a + (n - 1)d],$$

which is precisely our first formula.

the 6th term is $\left(\frac{1}{3}\right)\left(-\frac{1}{3}\right) = -\frac{1}{9}$,

and the 7th term is

$$\left(-\frac{1}{9}\right)\left(-\frac{1}{3}\right) = \frac{1}{27}.$$

**• PROBLEM 682**

Find the 10th term of the geometric progression 3, 6, 12, 24,... .

Solution: $a_1 = 3$ and $r = \frac{6}{3} = 2$

If we let $\ell$ represent the 10th term

$$\ell = a_1 r^{n-1}$$

$$\ell = 3(2)^{10-1}$$

$$= 3(2)^9$$

$$= 3(512)$$

$$= 1536$$

**• PROBLEM 683**

The seventh term of a geometric progression is 192 and $r = 2$. Find the first four terms.

Solution: The formula for the nth term, or last term, of a geometric progression is:

$$\ell = s_1 r^{n-1}$$

where $s_1$ = first term of the progression

$r$ = common ratio

$n$ = number of terms

$\ell$ = nth term, or last term

Since we are given the seventh term and the common ratio of the progression we can use this information, with $n = 7$, to find the first term:

$$s_7 = s_1 r^{n-1}$$

$$192 = s_1(2)^{7-1} = 2^6 s_1 = 64 s_1$$

$$s_1 = \frac{192}{64} = 3$$

Then, since a geometric progression is a sequence of numbers each of which, after the first, is obtained by multiplying the preceding number by a constant number called the common ratio,

$s_1 = 3$, $s_2 = 3 \cdot 2 = 6$, $s_3 = 6 \cdot 2 = 12$, and $s_4 = 12 \cdot 2 = 24$

**• PROBLEM 684**

If the 8th term of a geometric progression is 16 and the common ratio is -3, what is the 12th term?

Solution: A geometric progression is a sequence of numbers

$$r\left(\frac{32}{9}\right) = r\left(27r^{n-1}\right)$$

$$\frac{32}{9}r = 27r\ r^{n-1}$$

$$\frac{32}{9}r = 27r^{1+n-1}$$

$$\frac{32}{9}r = 27r^{n}$$

Substituting $\frac{32}{9}r$ for $27r^n$ in equation (2),

$$\frac{665}{9} = \frac{32r/9 - 27}{r - 1}$$

$$\cancel{9}(r-1)\ \frac{665}{\cancel{9}} = 9\cancel{(r-1)}\left[\frac{32r/9 - 27}{\cancel{r-1}}\right]$$

$$(r-1)665 = 9[32r/9 - 27]$$

Distributing on the right side:

$$(r-1)665 = 32r - 243$$

Distributing on the left side:

$$665r - 665 = 32r - 243$$

Subtract $33r$ from both sides:

$$665r - 665 - 32r = \cancel{32r} - 243 - \cancel{32r}$$
$$633r - 665 = -243$$

Add 665 to both sides:

$$633r - \cancel{665} + \cancel{665} = -243 + 665$$
$$633r = 422$$

Divide both sides by 633:

$$\frac{663r}{663} = \frac{442}{633}$$

$$r = \frac{422}{633} = \frac{2(\cancel{211})}{3(\cancel{211})} = \frac{2}{3}$$

Hence, the common ratio $= r = \frac{2}{3}$.

Substituting $\frac{2}{3}$ for $r$ in equation (1):

$$\frac{32}{9} = 27\left(\frac{2}{3}\right)^{n-1}$$

Multiply both sides by $\frac{1}{27}$:

$$\frac{1}{27}\left(\frac{32}{9}\right) = \frac{1}{\cancel{27}}\left[\cancel{27}\left(\frac{2}{3}\right)^{n-1}\right]$$

$$\frac{1}{27}\left(\frac{32}{9}\right) = \left(\frac{2}{3}\right)^{n-1}$$

$$\frac{32}{243} = \left(\frac{2}{3}\right)^{n-1} \qquad (3)$$

Express the fraction on the left side as a power of 2/3. Since $32 = 2^5$ and $243 = 3^5$, equation (3) becomes:

$$\frac{32}{243} = \frac{2^5}{3^5} = \left(\frac{2}{3}\right)^5 = \left(\frac{2}{3}\right)^{n-1}.$$

Hence, $5 = n - 1$.

Add 1 to both sides:

$$5 + 1 = n - 1 + 1$$

$$6 = n .$$

Hence, the number of terms = n = 6.

• **PROBLEM 686**

Find the sum of the first ten terms of the geometric progression: 15, 30, 60, 120, ...

**Solution:** A geometric progression is a sequence in which each term after the first is formed by multiplying the preceding term by a fixed number, called the common ratio.

If a is the first term, r is the common ratio, and n is the number of terms, the geometric progression (G.P.) is

$$a,\ ar,\ ar^2,\ \ldots,\ ar^{n-1}$$

The given G.P., 15, 30, 60, 120, ..., may be written as $15, 15(2), 15\left(2^2\right), 15\left(2^3\right) \ldots$ . The sum, $S_n$, of the first n terms of the geometric progression is given by

$$S_n = \frac{a\left(1 - r^n\right)}{1 - r}, \text{ where } a = \text{first term}, \ r = \text{common ratio}, \ n = \text{number of terms.}$$

Here a = 15, r = 2, and n = 10.

$$S_{10} = \frac{15\left(1 - 2^{10}\right)}{1 - 2}$$

$$= \frac{15(1 - 1024)}{-1}$$

$$= 15(1023)$$

$$= 15,345$$

• **PROBLEM 687**

Find the sum of the first four terms of the geometric series $2 + \left(-\frac{1}{3}\right) + \frac{1}{18} + \ldots$ .

**Solution:** The ratio of any number of a geometric series to the number preceding it is constant. In this example, the common ratio $r = \frac{\left(-\frac{1}{3}\right)}{2} = -\frac{1}{6}$. Since the sum of the first n terms of a geometric series is:

$$S_n = \frac{a_1\left(r^n - 1\right)}{r - 1}, \ r \neq 1$$

where $a_1$ = first term; r = common ratio; n = number of terms; $S_n$ = sum of first n terms; then with $a_1 = 2$, $r = -\frac{1}{6}$, n = 4, the sum $S_4$ is

$$S_4 = \frac{2\left[\left(-\frac{1}{6}\right)^4 - 1\right]}{-\frac{1}{6} - 1}$$

$$S_4 = \frac{2\left(\frac{1}{1296} - 1\right)}{-\frac{7}{6}}$$

$$S_4 = \frac{185}{108}$$

Check: $2 + 2\left(-\frac{1}{6}\right) + 2\left(-\frac{1}{6}\right)^2 + 2\left(-\frac{1}{6}\right)^3$

$$= 2 + \left(-\frac{1}{3}\right) + \left(\frac{1}{18}\right) - \left(\frac{1}{108}\right)$$

$$= 2 - \frac{1}{3} + \frac{1}{18} - \frac{1}{108} = \frac{185}{108}.$$

**• PROBLEM 688**

Find the sum of the first eight terms of the geometric progression:

$$4,\ \frac{-4}{3},\ \frac{4}{9},\ \frac{-4}{27}.$$

Solution: $a_1 = 4$, $r = \frac{-4/3}{4} = \frac{-1}{3}$, and $n = 8$

$S_n = \frac{a_1(1 - r^n)}{1 - r}$. Therefore, $S_8 = \frac{4\left[1 - \left(-\frac{1}{3}\right)^8\right]}{1 - \left(-\frac{1}{3}\right)}$

$$= \frac{4\left[1 - \left(\frac{-1^8}{3^8}\right)\right]}{1 + \frac{1}{3}}$$

$$= \frac{4\left[1 - \frac{1}{3^8}\right]}{\frac{4}{3}}$$

$$= \frac{4\left[1 - \frac{1}{6561}\right]}{\frac{4}{3}}$$

$$= \frac{4\left[\frac{6561}{6561} - \frac{1}{6561}\right]}{\frac{4}{3}}$$

$$= \frac{4\left(\frac{6560}{6561}\right)}{\frac{4}{3}}$$

$$= \cancel{4}\left(\frac{3}{\cancel{4}}\right)\left(\frac{6560}{6561}\right)$$

$$= {}^{1}\cancel{3}\left(\frac{6560}{\underset{2187}{\cancel{6561}}}\right) = \frac{6560}{2187}$$

Sum the sequence $\frac{2}{3}$, $-1$, $\frac{3}{2}$, ..... to 7 terms.

Solution: To solve this problem we use the formula for finding the sum of the first n terms of a geometric progression, which states:

$$S = \frac{a\left(r^n - 1\right)}{r - 1}, \qquad r \neq 1, \text{ where}$$

S = the sum

a = the first term

r = the common ratio

n = the number of terms

Now, we are looking for S, when $a = \frac{2}{3}$, $n = 7$, and $r = -\frac{3}{2}$, since $\frac{2}{3}\left(-\frac{3}{2}\right) = -1$, $(-1)\left(-\frac{3}{2}\right) = \frac{3}{2}$, ...

Thus, $$S = \frac{\frac{2}{3}\left[\left(-\frac{3}{2}\right)^7 - 1\right]}{-\frac{3}{2} - 1}.$$ Simplifying, we obtain:

$$S = \frac{\frac{2}{3}\left(-\frac{2187}{128} - \frac{128}{128}\right)}{-\frac{3}{2} - \frac{2}{2}} = \frac{\frac{2}{3}\left(-\frac{2315}{128}\right)}{-\frac{5}{2}} = \frac{-\frac{4630}{384}}{-\frac{5}{2}}$$

$$= -\frac{4630}{384} \cdot -\frac{2}{5} =$$

$$\frac{9260}{1920} = \frac{463}{96}.$$

Thus, the sum of the first seven terms of the above sequence is $\frac{463}{96}$.



Find the sum of the first six terms of a geometric progression whose first term is $\frac{1}{3}$ and whose second term is $-1$.

Solution: A geometric progression (G.P.) is a sequence of numbers each of which, after the first, is obtained by multiplying the preceding number by a constant number called the common ratio. Thus the sequence may be represented as $a_1$, $a_1r$, $a_1r^2$, $a_1r^3$, ..., where $a_1$ = first term and r = common ratio.

For the G.P. whose first term is $\frac{1}{3}$ and whose second term is $-1$, we can find r as follows: since $a_1r$ is the second term, and $a_1 = \frac{1}{3}$,

$$-1 = a_1 r$$

$$-1 = \frac{1}{3}r$$

$$r = -3.$$

Then, since the sum of the first n terms of a G.P. is

$$S_n = \frac{a_1(r^n - 1)}{r - 1}$$

where n = number of terms, we can find the sum of the first six terms of the G.P.:

$$S_6 = \frac{1}{3}\,\frac{(-3)^6 - 1}{-3 - 1} = \frac{1}{3} \cdot \frac{729 - 1}{-4} = -\frac{182}{3}$$

**• PROBLEM 691**

Find the seventh term of a geometric series whose third term is $\frac{1}{8}$ and common ratio is 2. Find the sum of the first seven terms.

Solution: Formulas for geometric series include formulas for the $n^{th}$ term, or last term:

$$\ell = a_1 r^{n-1}$$

and the sum of the first n terms:

$$S_n = \frac{a_1(r^n - 1)}{r - 1}, \quad r \neq 1$$

where $a_1$ = first term; r = common ratio; n = number of terms; $\ell = n^{th}$ term, or last term; $S_n$ = sum of first n terms.

Since the formulas for $\ell$ and $S_n$ require the value of the first term, we will use the given information to determine $a_1$. For our series of the form $a_1, a_1r, a_1r^2, a_1r^3, \ldots$ the third term is $\frac{1}{8}$ and the ratio is 2. The first term, $a_1$, can be found using the fact that the third term is $a_1r^2$.

$$a_1 r^2 = \frac{1}{8}$$

$$a_1(2)^2 = \frac{1}{8}$$

$$4a_1 = \frac{1}{8}$$

$$a_1 = \frac{1}{32}$$

sixth term is 1/8. Find the first term and the common ratio.

Solution: A geometric progression, $a_1, a_2, a_3, \ldots, a_n$, has terms with a common ratio r, so that the sequence can be expressed by

$$a_1,\ a_1r,\ a_1r^2,\ a_1r^3,\ \ldots\ a_1r^{n-1}$$

Observe that $a_2 = a_1r$, $a_3 = a_1r^2$, $a_4 = a_1r^3$, ..., $a_n = a_1r^{n-1}$. We are given that the fourth term is ½ and the sixth term is 1/8 so that

$$a_1r^3 = \tfrac{1}{2}$$

$$a_1r^5 = 1/8$$

Dividing the second equation by the first,

$$\frac{a_1r^5}{a_1r^3} = \frac{1/8}{\tfrac{1}{2}}\ .$$

Now, $a_1/a_1 = 1$, and $r^5/r^3 = r^{5-3}$; also, since division by a fraction is equivalent to multiplication by its reciprocal,

$$\frac{1/8}{\tfrac{1}{2}} = 1/8 \cdot 2/1$$

Therefore $r^2 = \tfrac{1}{4}$, and taking the square of both sides, $r = \pm \tfrac{1}{2}$.

Thus, there are two possible common ratios, ½ and $-$ ½; and therefore there are two possible series that satisfy the given conditions:

For $r = \tfrac{1}{2}$, the first term, $a_1$, is given by:

$$a_1 = \frac{a_1r^3}{r^3} = \frac{a_4}{r^3}\ .$$

Since $a_4$ = the fourth term = ½, and $r = \tfrac{1}{2}$, the first term

$$= \frac{\tfrac{1}{2}}{(\tfrac{1}{2})^3} = 4.$$

For $r = -\tfrac{1}{2}$,

$$a_1 = \frac{a_4}{r^3} = \frac{\tfrac{1}{2}}{(-\tfrac{1}{2})^3} = -4.$$

**• PROBLEM 694**

Insert 4 geometric means between 160 and 5.

Solution: 'Geometric means' are the terms between any two given terms in a geometric progression. Thus, for this problem we have to find 6 terms in G.P. of which 160 is the first, and 5 the sixth.

We can apply the formula:

nth term = $ar^{n-1}$, where a = 1st term

r = common ratio

Thus, for the sixth term we have:

$$\text{sixth term} = 5 = 160r^{6-1}$$

$$5 = 160r^5$$

Solving for r we obtain

$$r^5 = \frac{5}{160} = \frac{1}{32}$$

$$\sqrt[5]{r} = \sqrt[5]{\frac{1}{32}}$$

$$r = \frac{1}{2}$$

Now, since $r = \frac{1}{2}$ is the common ratio, we obtain each successive term of the progression by multiplication by $\frac{1}{2}$. Thus, we have:

160, 80, 40, 20, 10, 5, and the

four required means are 80, 40, 20, 10.

**• PROBLEM 695**

Insert three geometric means between 16 and 81.

Solution: In a geometric progression, G.P., the terms between any two other terms are called the geometric means. A G.P. is a sequence of numbers in which each term after the first is obtained by multiplying the preceeding term by a fixed number called the common ratio. If we designate the first term by a and the common ratio by r, then the terms of the series can be written as follows:

$a, ar, ar^2, ar^3, \ldots, ar^{n-1}$. Note that the nth term is designated by $\ell = ar^{n-1}$.

In this example $a = 16$ and $\ell = 81$. Furthermore, $n = 5$, since we have three geometric means between 16 and 81. A geometric mean is just a term between any two other terms. Hence,

$$81 = ar^{n-1}$$

$$81 = 16r^{5-1}$$

$$81 = 16r^4$$

$$\frac{81}{16} = r^4$$

$$\frac{9}{4} = r^2$$

$$\pm\frac{3}{2} = r$$

Symbolically, the geometric means are $ar^1$, $ar^2$, and $ar^3$. If $r = +\frac{3}{2}$

then the means are

$$16\left(\frac{3}{2}\right)^1,\ 16\left(\frac{3}{2}\right)^2,\ 16\left(\frac{3}{2}\right)^3$$

or 24, 36, 54. If $r = -\frac{3}{2}$, then the means are -24, 36, -54.

**• PROBLEM 696**

> Find the first four terms of the geometric progression generated by the exponential function $f(x) = 12(3/2)^x$ if the domain of the function is the set of nonnegative integers (0,1,2,3,...).

Solution:
$$f(0) = 12\left(\frac{3}{2}\right)^0 = 12(1) = 12$$

$$f(1) = 12\left(\frac{3}{2}\right)^1 = 18$$

$$f(2) = 12\left(\frac{3}{2}\right)^2 = 12\left(\frac{9}{4}\right) = 27$$

$$f(3) = 12\left(\frac{3}{2}\right)^3 = 12\left(\frac{27}{8}\right) = \frac{81}{2}$$

The first four terms are 12, 18, 27, and $\frac{81}{2}$.

**• PROBLEM 697**

> Find three numbers in geometric progression whose sum is 19, and whose product is 216.

Solution: The three numbers of the G.P. may be denoted by $\frac{a}{r}$, a, ar; then $\frac{a}{r} \times a \times ar = 216$. Carrying out the multiplication we obtain:

$$a^3 = 216$$

$$\sqrt[3]{a^3} = \sqrt[3]{216}$$

$$a = 6$$

Thus, substituting for a, we find that the numbers are $\frac{6}{r}$, 6, 6r.

Now, since the sum of the three numbers is 19, we have:

$\frac{6}{r} + 6 + 6r = 19$, and we wish to solve for r. To do this we take r as a common denominator. Thus,

$$\frac{6 + 6r + 6r^2}{r} = 19$$

$$6 + 6r + 6r^2 = 19r$$

$$6 + 6r + 6r^2 - 19r = 0$$

$$6r^2 - 13r + 6 = 0$$

Factoring, we have

$(3r - 2)(2r - 3) = 0$; hence

$r = \frac{3}{2}$ or $\frac{2}{3}$.

Thus the numbers are 4, 6, 9.

**• PROBLEM 698**

Express $.4\overline{23}$ as a rational fraction.

Solution: We know that,

$$.4\overline{23} = .423232323...$$

This is a repeating decimal in which 23 is the repeated portion of the decimal. This is indicated by the bar above the given decimal, that is, $.4\overline{23}$.

The decimal .4232323... can be rewritten as .4 + .023 + .00023 + .0000023 + ... We can easily see this by adding each term in column form. Thus, we have:

$$\begin{array}{r} .4\phantom{000000} \\ .023\phantom{0000} \\ .00023\phantom{00} \\ +\ \ .0000023 \\ \hline .4232323 \end{array}$$

; and this sum is the desired result.

Now, .4 + .023 + .00023 + .0000023 + ... can be rewritten as:

$$\frac{4}{10} + \frac{23}{1000} + \frac{23}{100000} + \frac{23}{10000000} + \ldots$$

$$= \frac{4}{10} + \frac{23}{10^3} + \frac{23}{10^5} + \frac{23}{10^7} + \ldots$$

Factoring $\frac{23}{10^3}$ from all terms except the first, we have:

$$.4\overline{23} = \frac{4}{10} + \frac{23}{10^3}\left[1 + \frac{1}{10^2} + \frac{1}{10^4} + \ldots\ldots\right].$$

Notice that the series $1 + \frac{1}{10^2} + \frac{1}{10^4} + \ldots$ has terms which are in a geometric progression where a = first term = 1, and r = common ratio = $\frac{1}{10^2}$.

Since r is less than 1 and the series is an infinite one, we can state that:

$$S = \text{sum} = \frac{a}{1 - r}$$

$$= \frac{1}{1 - \frac{1}{10^2}}$$

Thus, $.4\overline{23} = \frac{4}{10} + \frac{23}{10^3} \cdot \frac{1}{1 - \frac{1}{10^2}}$ .

Now, we can simplify $\frac{1}{1 - \frac{1}{10^2}}$ as follows:

$$\frac{1}{1 - \frac{1}{10^2}} = \frac{1}{\left[1 - \frac{1}{100}\right]} = \frac{1}{\left[\frac{100}{100} - \frac{1}{100}\right]} = \frac{1}{\frac{99}{100}} = \frac{100}{99} .$$

Thus, substituting $\frac{100}{99}$ for $\frac{1}{1 - \frac{1}{10^2}}$ , we have:

$$4\overline{23} = \frac{4}{10} + \frac{23}{10^3} \cdot \frac{100}{99} = \frac{4}{10} + \left(\frac{23}{1000} \cdot \frac{100}{99}\right)$$

$$= \frac{4}{10} + \frac{23}{990} = \frac{396}{990} + \frac{23}{990}$$

$$= \frac{419}{990} .$$

**• PROBLEM 699**

A person has 2 parents, 4 grandparents, 8 great-grand-parents, and so on. Find the number of his ancestors during the eight generations preceding his own, provided there are no duplications.

Solution: This is a geometric progression, which is a sequence of numbers in which each term after the first is obtained by multiplying the preceding term by a fixed number. The first term is the 2 parents. It is multiplied by a fixed number 2 since each parent has two parents of his or her own. To find the number of ancestors of the eight generations preceding his own, we must add up the terms starting from the parental generation up until the eigth ancestral generation. We apply the formula for the sum of a geometric progression: the first term of it is designated by a and the fixed amount called the common ratio is r. Let $S_n$ represent the sum of n terms of the progression.

$$S_n = \frac{a - ar^n}{1 - r}$$

In this problem, a = 2, r = 2, and n = 8. By substituting these values, we get

$$S_8 = \frac{2 - 2(2^8)}{1 - 2} = \frac{2 - 2(256)}{-1} = 510$$

If $|x| < 1$, sum the series

$1 + 2x + 3x^2 + 4x^3 + \ldots\ldots$ to infinity.

Solution: The terms of the given series are nearly in the form of a geometric progression, with a common ratio of x. Since $|x| < 1$, and the required sum is to infinity, obtaining the given series in the form of a geometric progression will enable us to apply the formula for the sum of an infinite G.P. The formula states: $S = \frac{a}{1 - r}$,

where $S$ = the sum
$a$ = the first term
$r$ = common ratio

Now, to transform the given series into a G.P. we can multiply it by $(1 - x)$. Thus,

$$(1 - x)(1 + 2x + 3x^2 + 4x^3 + \ldots)$$

$$= (1 - x) + (2x - 2x^2) + (3x^2 - 3x^3) + (4x^3 - 4x^4) + \ldots$$

$$= 1 + (-x + 2x) + (-2x^2 + 3x^2) + (-3x^3 + 4x^3) - 4x^4 + \ldots$$

$$= 1 + x + x^2 + x^3 + \ldots$$

The last series, $1 + x + x^2 + x^3 + \ldots$ is our desired G.P. with $a = 1$, $r = x$. Thus,

$$S = \frac{1}{1 - x}$$

But, $\frac{1}{1 - x}$ represents the sum of: $(1 - x)(1 + 2x + 3x^2 + \ldots)$, and we want the sum of: $1 + 2x + 3x^2 + \ldots$ Therefore, dividing $\frac{1}{1 - x}$ by $(1 - x)$ gives us the required sum. Hence, the sum of the given series =

$$\frac{\frac{1}{1 - x}}{(1 - x)} = \frac{1}{1 - x} \cdot \frac{1}{1 - x} = \frac{1}{(1 - x)^2}.$$

• PROBLEM 701

Find the sum of the geometric series

$$30 + 10 + 3\tfrac{1}{3} + \ldots + 30\left(\tfrac{1}{3}\right)^{n-1} + \ldots$$

Solution: Rewriting the geometric series as

$$30 + 30\left(\tfrac{1}{3}\right) + 30\left(\tfrac{1}{3}\right)^2 + \ldots + 30\left(\tfrac{1}{3}\right)^{n-1} + \ldots,$$

it can be seen that the first term is $a_1 = 30$; the ratio is $r = \frac{1}{3}$. Hence, since the sum to infinity $(S_\infty)$ of any geometric progression in which the common ratio r is numerically less than 1 is given by

$$S_\infty = \frac{a_1}{1-r}, \text{ where } |r| < 1,$$

then $$S_\infty = \frac{30}{1-\frac{1}{3}} = 45$$

Note that the sum of the first n terms of this series differs from 45 by $\frac{1}{2}$ of the nth term. For example, when $n = 2$, $S_2 = 40$, $a_2 = 10$, and $\frac{1}{2}a_2 = 5$. Thus,

$$40 = 45 - 5$$

When $n = 3$, $S_3 = 43\frac{1}{3}$, $a_3 = 3\frac{1}{3}$, and $\frac{1}{2}a_3 = 1\frac{2}{3}$. Thus $43\frac{1}{3} =$ $= 45 - 1\frac{2}{3}$, etc.

**• PROBLEM 702**

The sum of an infinite number of terms in geometric progression is 15, and the sum of their squares is 45; find the sequence. Assume that the common ratio of the G.P. is less than 1.

Solution: The sum of any infinite geometric progression in which the common ratio is less than is:

$$S_1 = \frac{a}{1-r}$$

Now, squaring the terms of the sequence,

$a, ar, ar^2, ar^3, \ldots$

we have:

$$a^2, (ar)^2, (ar^2)^2, (ar^3)^2, \ldots =$$

$$a^2, a^2r^2, a^2r^4, a^2r^6, \ldots$$

This is a new infinite geometric progression with $a^2$ as the first term, and $r^2$ as the common ratio, and $r^2 < 1$. Therefore,

$$S_2 = \frac{a^2}{1-r^2} .$$

We are given that the sum of the terms, or $S_1$, is 15, and the sum of the squares of the terms, or $S_2$, is 45. Thus,

$$\frac{a}{1-r} = 15, \quad \text{and} \quad \frac{a^2}{1-r^2} = 45$$

We must now solve for a and for r. This can be done in the following manner: Multiply both sides of the equation $\frac{a}{1 - r} = 15$ by $(1 - r)$. Thus, we obtain: $a = 15(1 - r)$.

Now, multiply both sides of the equation $\frac{a^2}{1 - r^2} = 45$ by $(1 - r^2)$. Thus, we obtain:

$$a^2 = 45(1 - r^2) \qquad (1)$$

Squaring the equation $a = 15(1 - r)$ will give us a value for $a^2$ in terms of r. Substituting this value in equation (1) will give us an equation in r alone. Then, we can solve for r. Thus,

$a^2 = (15 - 15r)^2$, and expanding we have:

$$a^2 = (15 - 15r)(15 - 15r)$$

$$= 225 - 225r - 225r + 225r^2$$

$$= 225r^2 - 450r + 225$$

Now, by substitution:

$$225r^2 - 450r + 225 = 45(1 - r^2)$$

$$225r^2 - 450r + 225 = 45 - 45r^2$$

Subtracing 45 from both sides of the equation, and adding $45r^2$ to both sides we obtain:

$$225r^2 + 45r^2 - 450r + 225 - 45 = 45 - 45 - 45r^2 + 45r^2$$

$$270r^2 - 450r + 180 = 0$$

Dividing each term by 90, we have:

$$\frac{270r^2}{90} - \frac{450r}{90} + \frac{180}{90} = \frac{0}{90}$$

$$3r^2 - 5r + 2 = 0$$

Factoring gives us:

$(3r - 2)(r - 1) = 0$, and this means that either $3r - 2 = 0$, or $r - 1 = 0$.

To solve the first equation for r we first add 2 to both sides of $3r - 2 = 0$, and then divide by 3. Thus,

$$3r - 2 + 2 = 0 + 2$$

$$3r = 2$$

$$\frac{3r}{3} = \frac{2}{3}$$

$$r = \frac{2}{3}$$

To solve the second equation we add 1 to both sides of $r - 1 = 0$. Thus,

$$r - 1 + 1 = 0 + 1$$

$$r = 1$$

Therefore, we have two values for r, $r = \frac{2}{3}$ , $r = 1$.

But notice that the second value, $r = 1$, can be rejected. This is so because if we substitute this into either of our original equations, $\frac{a}{1 - r} = 15$ or $\frac{a^2}{1 - r^2} = 45$, we obtain $\frac{a}{0}$ , which is an undefined expression, and also because the formula for the sum, $S = \frac{a}{1 - r}$ , holds only for progessions where the common ratio is less than 1.

Thus, we have $r = \frac{2}{3}$ , and to solve for a we substitute this value into

$$a = 15(1 - r)$$

Thus, $a = 15\left(1 - \frac{2}{3}\right)$

$$= 15 \left(\frac{3}{3} - \frac{2}{3}\right)$$

$$= 15 \left(\frac{1}{3}\right) = \frac{15}{3} = 5$$

Therefore, we have a = first term of the sequence = 5, and r = common ratio = $\frac{2}{3}$ . Thus the terms of the sequence are: $5, 5\left(\frac{2}{3}\right), \left(5 \cdot \frac{2}{3}\right)\frac{2}{3}, \ldots$ and the sequence is:
$5, \frac{10}{3}, \frac{20}{9}, \ldots\ldots$

**• PROBLEM 703**

> **Convert the repeating decimal .477477 ... to a fraction.**

Solution: This is a geometric progression. We can compute the rational equivalent by first determining the common ratio and then using the formula for the $n^{th}$ partial sum of a geometric series. The common ratio is computed by dividing any term by the term immediately preceding it. Therefore r, the common ratio, is:

$$\frac{.000477}{.477} = .001$$

Allow $a_1$ to be .477. Then the sum of the geometric progression, $S_n$, is: $a_1 + a_1 r + a_1 r^2 + \ldots + a_1 r^{n-1}$ . Now we can compute $S_n$ using the formula.

$$S_n = \frac{a_1 - a_1 r^n}{1 - r} = \frac{.477 - .477(.001)^n}{1 - r}$$

by taking the limit of $S_n$ as $n \to \infty$. We then compute the rational expression to which this geometric progression converges.

**• PROBLEM 705**

Find the 9th term of the harmonic progression $3, 2, \frac{3}{2}, \ldots$ .

Solution: The terms of a harmonic progression that lie between two given terms are called the harmonic means between these terms. If a single harmonic mean is inserted between two numbers, it is called the harmonic mean of the numbers.
A harmonic progression (H.P.) is a sequence of numbers whose reciprocals are in arithmetic progression, (A.P.). The terms of the A.P. are $\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \ldots$. We find the ninth term of the A.P. and take its reciprocal to find the corresponding term in the H.P. The formula for the nth term, $a_n$, of an A.P. is $a_1 + (n-1)d$ where $a_1$ is the first term and $d$ is the common difference, the constant quantity added to each term to form the progression. Hence, if $a_n = a_1 + (n-1)d$, $a_1 = 1/3$ and $n = 9$, to find $d$ subtract the first term, 1/3, from the second term, 1/2.

$$d = \frac{1}{2} - \frac{1}{3} = \frac{3}{6} - \frac{2}{6} = \frac{1}{6} .$$

Thus,

$$a_9 = \frac{1}{3} + (9-1)\frac{1}{6} = \frac{1}{3} + 8\left(\frac{1}{6}\right) = \frac{1}{3} + \frac{4}{3} = \frac{5}{3} .$$

Therefore, the ninth term of the harmonic progression is $\frac{3}{5}$ .

**• PROBLEM 706**

Insert three harmonic means between $\frac{1}{10}$ and $\frac{1}{42}$ .

Solution: The harmonic means are those terms of a harmonic progression that lie between two given terms. These two terms are 1/10 and 1/42. A harmonic progression (H.P.) is a sequence of numbers whose reciprocals are in arithmetric progression. Therefore, we consider the corresponding arithmetic progression, A.P. We shall first insert three arithmetic means between 10 and 42. The formula for the nth term is $a_n = a_1 + (n-1)d$ where $a_1$ is the first term of the A.P. and $d$ is the common difference, the fixed amount added to each term to form the progression. The nth term is 42 where $n = 5$ since there are 3 terms to be inserted between 10 and 42. $a_1 = 10$. Thus, $a_5 = 10 + (5-1)d = 42$. Solve for d.

$$\begin{aligned} 10 + 4d &= 42 \\ 4d &= 42 - 10 \\ 4d &= 32 \\ d &= 8 \end{aligned}$$

Now we can find the terms of the A.P. by adding 8 to the first term and continuing in the same manner with each succeeding term. Thus, we obtain:

10, 10 + 8 = 18, 18 + 8 = 26, 26 + 8 = 34, 34 + 8 = 42 or

10, 18, 26, 34, 42.

The arithmetic progression is 10, 18, 26, 34, 42 .

If $a^2$, $b^2$, $c^2$ are in arithmetic progression, show that $b + c$, $c + a$, $a + b$ are in harmonic progression.

Solution: We are given that $a^2$, $b^2$, $c^2$ are in arithmetic progression. By this we mean that each new term is obtained by adding a constant to the preceding term.

By adding $(ab + ac + bc)$ to each term, we see that,

$$a^2 + (ab + ac + bc),\ b^2 + (ab + ac + bc),$$

$$c^2 + (ab + ac + bc)$$

are also in arithmetic progression. These three terms can be rewritten as

$$a^2 + ab + ac + bc,\ b^2 + bc + ab + ac,$$

$$c^2 + ac + bc + ab$$

Notice that:

$$a^2 + ab + ac + bc = (a + b)(a + c)$$

$$b^2 + bc + ab + ac = (b + c)(b + a)$$

$$c^2 + ac + bc + ab = (c + a)(c + b)$$

Therefore, the three terms can be rewritten as:

$$(a + b)(a + c),\ (b + c)(b + a),\ (c + a)(c + b),$$

which are also in arithmetic progression.

Now, dividing each term by $(a + b)(b + c)(c + a)$, we obtain:

$$\frac{1}{b + c},\ \frac{1}{c + a},\ \frac{1}{a + b},$$ which are in

arithmetic progression.

Recall that a sequence of numbers whose reciprocals form an arithmetic progression, is called a harmonic progression. Thus, since $\frac{1}{b + c}, \frac{1}{c + a}, \frac{1}{a + b}$ is an arithmetic progression, $b + c$, $c + a$, $a + b$ are in harmonic progression.



Find the first six terms of the sequence determined by the function $g(x)$, where $x = 1, 2, 3, 4, 5, 6$.

$$g(x) = \frac{x^2}{x!},\ x \text{ a positive integer}$$

Solution:

# CHAPTER 23

# MATHEMATICAL INDUCTION

**Basic Attacks and Strategies for Solving Problems in this Chapter.** **See pages 530 to 538 for step-by-step solutions to problems.**

The procedure for doing mathematical induction, a method of proving that a statement, formula, or theorem is true for every integer

$$N \geq N_1,$$

where $N$ is a given integer, is as follows:

(1) Show that the statement is true for the smallest integer, $N$, for which it is claimed to be true.

(2) Next, assume that the statement is true for some integer, call it $k$, which is greater than or equal to $N_1$.

(3) Finally, show (using the assumption made in Step 2) that the statement is true for the next integer, $k + 1$.

For example, prove that $2^N > N$ for all positive integer values of $N$.

**PROOF**

(1) If $N = 1$, then

$$2^N > N$$

becomes $2^1 > 1$, which is a true statement.

(2) Assume that for

$$N = k, 2^k > k$$

is true.

(3) We must prove the statement, if $2^k > k$, then

$$2^{k+1} > k + 1$$

for all positive integer values of $k$. This means we should be able to start with $2^k > k$ and from that deduce that

$$2^{k+1} > k + 1.$$

To show this we first note that

$$2^k > k \Rightarrow 2(2^k) > 2k \Rightarrow 2^{k+1} > 2k.$$

We know that $k \geq 1$ because we are working with positive integers. So,

$$k + k \geq k \Rightarrow 2k \geq k + 1.$$

Since

$$2^{k+1} > 2k \quad \text{and} \quad 2k \geq k + 1,$$

by the transitivity property, we can conclude it is true that

$$2^{k+1} > k + 1.$$

# Step-by-Step Solutions to Problems in this Chapter, "Mathematical Induction"

• **PROBLEM 710**

Prove by mathematical induction

$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$.

Solution: Mathematical induction is a method of proof. The steps are:
(1) The verification of the proposed formula or theorem for the smallest value of n. It is desirable, but not necessary, to verify it for several values of n.
(2) The proof that if the proposed formula or theorem is true for $n = k$, some positive integer, it is true also for $n = k+1$. That is, if the proposition is true for any particular value of n, it must be true for the next larger value of n.
(3) A conclusion that the proposed formula holds true for al' values of n.

Proof: Step 1. Verify:

For $n = 1$: $1^2 = \frac{1}{6}(1)(1+1)[2(1)+1] = \frac{1}{6}(1)(2)(3) = \frac{1}{6}(6) = 1$

$1 = 1 \checkmark$

For $n = 2$: $1^2 + 2^2 = \frac{1}{6}(2)(2+1)[2(2)+1] = \frac{1}{6}(2)(3)(5) = \frac{1}{6}(6)(5)$

$1 + 4 = (1)(5)$

$5 = 5 \checkmark$

For $n = 3$: $1^2 + 2^2 + 3^2 = \frac{1}{6}(3)(3+1)[2(3)+1]$

$1 + 4 + 9 = \frac{1}{6}(3)(4)(7) = \frac{1}{6}(12)(7) = 14$

$14 = 14 \checkmark$

Step 2. Let k represent any particular value of n. For $n = k$, the formula becomes

$$1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{1}{6}k(k+1)(2k+1). \quad (A)$$

For $n = k+1$, the formula is

$$1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = \frac{1}{6}(k+1)[(k+1) + 1][2(k+1) + 1]$$

$$= \frac{1}{6}(k+1)(k+2)(2k+3). \quad (B)$$

We must show that if the formula is true for $n = k$, then it must be true for $n = k+1$. In other words, we must show that (B) follows from (A). The left side of (A) can be converted into the left side of (B) by merely adding $(k+1)^2$. All that remains to be demonstrated is that when $(k+1)^2$ is added to the right side of (A), the result is the right side of (B).

$$1^2 + 2^2 + \ldots + k^2 + (k+1)^2 = \frac{1}{6}k(k+1)(2k+1) + (k+1)^2$$

Factor out $(k+1)$:

$$1^2 + 2^2 + 3^2 + \ldots + k^2 + (k+1)^2 = (k+1)\left[\frac{1}{6}k(2k+1) + (k+1)\right]$$

$$= (k+1)\left[\frac{k(2k+1)}{6} + \frac{(k+1)6}{6}\right]$$

$$= (k+1)\ \frac{2k^2 + k + 6k + 6}{6}$$

$$= \frac{(k+1)(2k^2 + 7k + 6)}{6}$$

$$= \frac{1}{6}(k+1)(k+2)(2k+3),$$

since $\qquad 2k^2 + 7k + 6 = (k+2)(2k+3).$

Thus, we have shown that if we add $(k+1)^2$ to both sides of the equation for $n = k$, then we obtain the equation or formula for $n = k+1$. We have thus established that if (A) is true, then (B) must be true; that is, if the formula is true for $n = k$, then it must be true for $n = k+1$. In other words, we have proved that if the proposition is true for a certain positive integer $k$, then it is also true for the next greater integer $k+1$.

Step 3. The proposition is true for $n = 1,2,3$ (Step 1). Since it is true for $n = 3$, it is true for $n = 4$ (Step 2, where $k = 3$ and $k+1 = 4$). Since it is true for $n = 4$, it is true for $n = 5$, and so on, for all positive integers $n$.

**• PROBLEM 711**

Prove:

$$1\cdot2 + 2\cdot3 + 3\cdot4 + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3}\ .$$

Solution by mathematical induction: The steps for a proof by mathematical induction are:

I) check validity of formula for $n = 1$

II) assume the formulation is true for $n = p$

III) prove it is true for $n = p + 1$.

I. For $n = 1$ the formula gives

$$1(1+1) = 1(2) = 2 = \frac{1(1+1)(1+2)}{3} = \frac{1\cdot2\cdot3}{3} = 2$$

which is correct and completes Step I.

II) Assume the formula is true for $n = p$:

$$1\cdot2 + 2\cdot3 + 3\cdot4 + \ldots + p(p+1) = \frac{p(p+1)(p+2)}{3}.$$

Prove the formula is true for $n = p + 1$, that is, prove

$$1\cdot2 + 2\cdot3 + \ldots + p(p+1) + (p+1)(p+2) = \frac{(p+1)(p+2)(p+3)}{3},$$

$(p+1)(p+2)$ is added to both members of the first equation in this step; this gives

$$[1\cdot2 + 2\cdot3 + \ldots + p(p+1)] + (p+1)(p+2) =$$

$$= \left[\frac{p(p+1)(p+2)}{3}\right] + (p+1)(p+2)$$

Factoring out $(p+1)(p+2)$, $\qquad = (p+1)(p+2)\left(\frac{p}{3} + 1\right)$

$$= (p+1)(p+2)\left(\frac{p}{3} + 1\right)\left(\frac{3}{3}\right)$$

$$= \frac{(p+1)(p+2)(p+3)}{3},$$

$$= \frac{(p+1)[(p+1) + 1][(p+2) + 1]}{3}$$

which is of the same form as the result we assumed to be true for p terms, p+1 taking the place of p. Since the statement is true for $n = 1$ and $n = p + 1$ assuming it was true for $n = p$, then the statement is true for all n.

**• PROBLEM 712**

Prove by mathematical induction that, for all positive integral values of n,

$$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} \; .$$

Solution: Step 1. The formula is true for $n = 1$, since $1 = \frac{1(1+1)}{2} = 1$.

Step 2. Assume that the formula is true for $n = k$. Then, adding (k+1) to both sides,

$$1 + 2 + 3 + \ldots + k + (k+1) = \frac{k(k+1)}{2} + (k+1) = \frac{(k+1)(k+2)}{2}$$

which is the value of $\frac{n(n+1)}{2}$ when (k+1) is substituted for n.

Hence if the formula is true for $n = k$, we have proved it to be true for $n = k + 1$. But the formula holds for $n = 1$; hence it holds for $n = 1 + 1 = 2$. Then, since it holds for $n = 2$, it holds for $n = 2 + 1 = 3$, and so on. Thus the formula is true for all positive integral values of n.

**• PROBLEM 713**

Prove by mathematical induction that, for all positive integral values of n,

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1} \; .$$

Solution: Step 1. The formula is true for $n = 1$, since

$$\frac{1}{(2-1)(2+1)} = \frac{1}{2+1} = \frac{1}{3} \; .$$

Step 2. Assume that the formula is true for $n = k$. Then

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1} \; .$$

Add the (k+1)th term, which is $\frac{1}{(2k+1)(2k+3)}$, to both sides of the above equation. Then

$$\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2k+1)(2k+3)} = \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} \; .$$

The right hand side of this equation $= \frac{k(2k+3)+1}{(2k+1)(2k+3)} = \frac{k+1}{2k+3}$, which is the value of $\frac{n}{2n+1}$ when n is replaced by (k+1).

Hence if the formula is true for $n = k$, it is true for $n = k + 1$.

But the formula holds for $n = 1$; hence it holds for $n = 1 + 1 = 2$. Then, since it holds for $n = 2$, it holds for $n = 2 + 1 = 3$, and so on. Thus the formula is true for all positive integral values of $n$.

**• PROBLEM 714**

Using mathematical induction, prove that

$$x^{2n} - y^{2n} \text{ is divisible by } x + y.$$

Solution:

(1) The theorem is true for $n = 1$, since $x^2 - y^2 = (x-y)(x+y)$ is divisible by $x + y$.

(2) Let us assume the theorem true for $n = k$, a positive integer; that is, let us assume

(A) $\quad x^{2k} - y^{2k}$ is divisible by $x + y$.

We wish to show that, when (A) is true,

(B) $\quad x^{2k+2} - y^{2k+2}$ is divisible by $x + y$.

Now $x^{2k+2} - y^{2k+2} = \left(x^{2k+2}-x^2y^{2k}\right) + \left(x^2y^{2k}-y^{2k+2}\right)$

$$= x^2\left(x^{2k} - y^{2k}\right) + y^{2k}\left(x^2 - y^2\right).$$

In the first term $\left(x^{2k} - y^{2k}\right)$ is divisible by $(x+y)$ by assumption, and in the second term $\left(x^2 - y^2\right)$ is divisible by $(x+y)$ by Step (1); hence, if the theorem is true for $n = k$, a positive integer, it is true for the next one $n = k + 1$.

(3) Since the theorem is true for $n = k = 1$, it is true for $n = k + 1 = 2$; being true for $n = k = 2$, it is true for $n = k + 1 = 3$; and so on, for every positive integral value of $n$.

**• PROBLEM 715**

Prove by mathematical induction that

$$1 + 7 + 13 + \ldots + (6n - 5) = n(3n - 2).$$

Solution: (1) The proposed formula is true for $n = 1$, since $1 = 1(3 - 2)$.

(2) Assume the formula to be true for $n = k$, a positive integer; that is, assume

(A) $\quad 1 + 7 + 13 + \ldots + (6k - 5) = k(3k - 2)$.

Under this assumption we wish to show that

(B) $\quad 1 + 7 + 13 + \ldots + (6k - 5) + (6k + 1) = (k+1)(3k+1)$.

When $(6k+1)$ is added to both members of (A), we have on the right

$$k(3k-2) + (6k+1) = 3k^2 + 4k + 1 = (k+1)(3k+1);$$

hence, if the formula is true for $n = k$ it is true for $n = k + 1$.

(3) Since the formula is true for $n = k = 1$ (Step 1), it is true for $n = k + 1 = 2$; being true for $n = k = 2$ it is true for $n = k + 1 = 3$; and so on, for every positive integral value of $n$.

**• PROBLEM 716**

> Prove by mathematical induction that
> $$1 + 5 + 5^2 + \ldots + 5^{n-1} = \tfrac{1}{4}\left(5^n - 1\right).$$

Solution: (1) The proposed formula is true for $n = 1$, since $1 = \frac{1}{4}(5-1)$.

(2) Assume the formula to be true for $n = k$, a positive integer; that is assume

(A) $$1 + 5 + 5^2 + \ldots + 5^{k-1} = \tfrac{1}{4}\left(5^k - 1\right).$$

Under this assumption we wish to show that

(B) $$1 + 5 + 5^2 + \ldots + 5^{k-1} + 5^k = \tfrac{1}{4}\left(5^{k+1} - 1\right).$$

When $5^k$ is added to both members of (A), we have on the right
$$\tfrac{1}{4}\left(5^k - 1\right) + 5^k = \frac{5}{4}\left(5^k\right) - \frac{1}{4} = \frac{1}{4}\left(5 \cdot 5^k - 1\right) = \frac{1}{4}\left(5^{k+1} - 1\right);$$
hence, if the formula is true for $n = k$ it is true for $n = k + 1$.

(3) Since the formula is true for $n = k = 1$ (Step 1), it is true for $n = k + 1 = 2$; being true for $n = k = 2$ it is true for $n = k + 1 = 3$; and so on, for every positive integral value of $n$.

**• PROBLEM 717**

> Prove by mathematical induction that
> $$\frac{5}{1\cdot2\cdot3} + \frac{6}{2\cdot3\cdot4} + \frac{7}{3\cdot4\cdot5} + \ldots + \frac{n+4}{n(n+1)(n+2)} = \frac{n(3n+7)}{2(n+1)(n+2)} \quad .$$

Solution:

(1) The formula is true for $n = 1$, since $\frac{5}{1\cdot2\cdot3} = \frac{1(3+7)}{2\cdot2\cdot3} = \frac{5}{6}$.

(2) Assume the formula to be true for $n = k$, a positive integer; that is, assume

(A) $$\frac{5}{1\cdot2\cdot3} + \frac{6}{2\cdot3\cdot4} + \ldots + \frac{k+4}{k(k+1)(k+2)} = \frac{k(3k+7)}{2(k+1)(k+2)} \quad .$$

Under this assumption we wish to show that

(B) $$\frac{5}{1\cdot2\cdot3} + \frac{6}{2\cdot3\cdot4} + \ldots + \frac{k+4}{k(k+1)(k+2)} + \frac{k+5}{(k+1)(k+2)(k+3)}$$
$$= \frac{(k+1)(3k+10)}{2(k+2)(k+3)} \; .$$

When $\frac{k+5}{(k+1)(k+2)(k+3)}$ is added to both members of (A), we have on the right
$$\frac{k(3k+7)}{2(k+1)(k+2)} + \frac{k+5}{(k+1)(k+2)(k+3)} = \frac{1}{(k+1)(k+2)}\left[\frac{k(3k+7)}{2} + \frac{k+5}{k+3}\right]$$

$$= \frac{1}{(k+1)(k+2)} \frac{k(3k+7)(k+3)+2(k+5)}{2(k+3)} = \frac{1}{(k+1)(k+2)} \frac{3k^3+16k^2+23k+10}{2(k+3)}$$

$$= \frac{1}{(k+1)(k+2)} \frac{(k+1)^2(3k+10)}{2(k+3)} = \frac{(k+1)(3k+10)}{2(k+2)(k+3)} \quad ;$$

hence, if the formula is true for $n = k$ it is true for $n = k + 1$.

(3) Since the formula is true for $n = k = 1$ (Step 1), it is true for $n = k + 1 = 2$; being true for $n = k = 2$, it is true for $n = k + 1 = 3$; and so on, for all positive integral values of $n$.

**• PROBLEM 718**

> Let $x$ be any real number. Show that $|\sin nx| \le n|\sin x|$ for every positive integer $n$.

__Solution:__ Let $\{P_n\}$ be the sequence in which $P_n$ is the statement "$|\sin nx| \le n|\sin x|$." Recall the following theorem: If $\{P_n\}$ is a sequence of statements that possesses the properties

(i) $P_1$ is true and

(ii) for each index $k$ such that $P_k$ is true, the statement $P_{k+1}$ is also true, then $P_n$ is a true statement for every positive integer $n$. $P_1$ is the statement "$|\sin x| \le |\sin x|$," which is surely true. Now we must show that the truth of the statement $P_k$ implies that statement $P_{k+1}$ also is true. For any $k$, we have

$$\begin{aligned} |\sin(k+1)x| &= |\sin(kx+x)| \\ &= |\sin kx \cos x + \cos kx \sin x| \\ &\le |\sin kx \cos x| + |\cos kx \sin x| \\ &\le |\sin kx| + |\sin x|. \end{aligned}$$

Hence, if $P_k$ is true $\Big($that is, if $|\sin kx| \le k|\sin x|\Big)$, we see that

$$\begin{aligned} |\sin(k+1)x| &\le |\sin kx| + |\sin x| \le k|\sin x| + |\sin x| \\ &= (k+1)|\sin x| \ . \end{aligned}$$

Thus, the statement $P_{k+1}$,

$$|\sin(k+1)x| \le (k+1)|\sin x|,$$

is true whenever $P_k$ is true. The two conditions of the above theorem are satisfied, so we conclude that statement $P_n$ is true for any positive integer $n$.

**• PROBLEM 719**

> Prove by mathematical induction that the number of straight lines determined by $n > 1$ points, no 3 on the same straight line, is $\frac{1}{2}n(n-1)$.

__Solution:__

(1) The theorem is true when $n = 2$, since $\frac{1}{2}\cdot 2(2-1) = 1$ and two

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+2)}{(r-1)!}a^{n-r+1}x^{r-1}$$

$$+ \ldots + x^n$$

for positive integral values of $n$.

Solution: Step 1. The formula is true for $n = 1$.

Step 2. Assume the formula is true for $n = k$. Then

$$(a+x)^k = a^k + ka^{k-1}x + \frac{k(k-1)}{2!}a^{k-2}x^2 + \ldots + \frac{k(k-1)\ldots(k-r+2)}{(r-1)!}a^{k-r+1}x^{r-1}$$

$$+ \ldots + x^k .$$

Multiply both sides by $a+x$. The multiplication on the right may be written

$$a^{k+1} + ka^kx + \frac{k(k-1)}{2!}a^{k-1}x^2 + \ldots + \frac{k(k-1)\ldots(k-r+2)}{(r-1)!}a^{k-r+2}x^{r-1}$$

$$+ \ldots + ax^k$$

$$+ a^kx + ka^{k-1}x^2 + \ldots + \frac{k(k-1)\ldots(k-r+3)}{(r-2)!}a^{k-r+2}x^{r-1} + \ldots + x^{k+1} .$$

Since

$$\frac{k(k-1)\ldots(k-r+2)}{(r-1)!}a^{k-r+2}x^{r-1} + \frac{k(k-1)\ldots(k-r+3)}{(r-2)!}a^{k-r+2}x^{r-1}$$

$$= \frac{k(k-1)\ldots(k-r+3)}{(r-2)!}a^{k-r+2}x^{r-1}\left\{\frac{k-r+2}{r-1} + 1\right\} = \frac{(k+1)k(k-1)\ldots(k-r+3)}{(r-1)!}a^{k-r+2}x^{r-1},$$

the produc may be written

$$(a+x)^{k+1} = a^{k+1} + (k+1)a^kx + \ldots + \frac{(k+1)k(k-1)\ldots(k-r+3)}{(r-1)!}a^{k-r+2}x^{r-1}$$

$$+ \ldots + x^{k+1}$$

which is the binomial formula with $n$ replaced by $k+1$.

Hence if the formula is true for $n = k$, it is true for $n = k + 1$. But the formula holds for $n = 1$; hence it holds for $n = 1 + 1 = 2$, and so on. Thus the formula is true for all positive integral values of $n$.

# CHAPTER 24

# FACTORIAL NOTATION

**Basic Attacks and Strategies for Solving Problems in this Chapter.** **See pages 539 to 541 for step-by-step solutions to problems.**

If $n$ is any positive integer, then the symbol $n!$, read "$n$ factorial," represents a product of the integers from 1 up to and including $n$ itself. Thus, in general,

$$n! = n(n-1)(n-2)(n-3)\ldots(2)\,1.$$

For example,

$$5! = 5(5-1)(5-2)(5-3)(5-4)$$

$$5! = 5 * 4 * 3 * 2 * 1$$

$$5! = 120.$$

Note that the $n!$ expansion may stop at any point. For example,

$$5! = 5 * 4 * 3!$$

$$5! = 20 * 3!$$

Thus, in general, if $r < n$, we can write

$$n! = n(n-1)\ldots r!,$$

where, in the above example $n = 5$ and $r = 3$.

**Solution:**

(a) Note $n! = n\cdot(n-1)\cdot(n-2)\cdot(n-3)\cdot \ldots \cdot 1$ ;

also $n! = n\cdot(n-1)!$ or $n\cdot(n-1)\cdot(n-2)!$ , etc.

Thus, $\frac{8!}{11!} = \frac{\cancel{8!}}{11\cdot10\cdot9\cdot\cancel{8!}} = \frac{1}{11\cdot10\cdot9} = \frac{1}{990}$ .

(b) Similarly,

$$\frac{5!-8!}{4!-7!} = \frac{5\cdot4! - 8\cdot7\cdot6\cdot5\cdot4!}{4! - 7\cdot6\cdot5\cdot4!}$$

Factoring 4!, $= \frac{\cancel{4!}[5 - (8\cdot7\cdot6\cdot5)]}{\cancel{4!}[1 - (7\cdot6\cdot5)]}$

$$= \frac{5 - 1680}{1 - 210}$$

$$= \frac{-1675}{-209}$$

$$= \frac{1675}{209} .$$

**• PROBLEM 726**

Find the value of $\frac{5!6!}{4!7!}$ .

**Solution:** Apply the definition of factorial: If n is any positive integer, the symbol n! is the product of the integers from 1 up to and including n.

Also if r and n are both positive integers and r is less than n, then $n! = n\cdot(n-1)\ldots(r+2)(r+1)r!$ . Use these two ideas to expand each factorial.

$$n! = 1\cdot2\cdot3\ldots n$$

$5! = (4!)(5)$ and $7! = (6!)(7)$ Substituting the values of 5! and 7! we have

$\frac{5!6!}{4!7!} = \frac{(4!)(5)(6!)}{(4!)(6!)(7)}$ . Dividing the common factors 4! and 6!

$$= \frac{5}{7} .$$

**• PROBLEM 727**

If n and r are positive integers, and $r < n$, show that $n! = r!(r+1)(r+2)\cdot \ldots \cdot n$.

**Solution:** By definition of factorial,

$$r! = 1\cdot2\cdot \ldots \cdot r .$$

Then,

$$r!(r+1)(r+2)\cdot \ldots \cdot n = (1\cdot2\cdot \ldots \cdot r)(r+1)(r+2)\cdot \ldots \cdot n$$

$$r!(r+1)(r+2)\cdot \ldots \cdot n = 1\cdot2\cdot \ldots \cdot n \qquad (1)$$

Again, by definition of factorial,

$$n! = 1 \cdot 2 \cdot \ \ldots \ \cdot n \ .$$

Hence, equation (1) becomes:

$$r!(r+1)(r+2) \cdot \ \ldots \ \cdot n = n!$$

or

$$n! = r!(r+1)(r+2) \cdot \ \ldots \ \cdot n \ .$$

# CHAPTER 25

# BINOMIAL THEOREM/EXPANSION

**Basic Attacks and Strategies for Solving Problems in this Chapter. See pages 542 to 554 for step-by-step solutions to problems.**

It is possible to expand the binomial expression of

$(x + y)^n$,

where $n$ is any positive integer, without showing all of the intermediate steps of multiplying and combining similar terms. To do this, first observe some patterns in the following examples:

$(x + y)^1 = x + y$

$(x + y)^2 = x^2 + 2xy + y^2$

$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

The patterns which are evident are:

(1) In each case the expansion of $(x + y)^n$ has $n + 1$ terms.

(2) In the expansions of $(x + y)^n$, the first term is $x^n$ and in each term thereafter the power of $x$ decreases by 1 until finally the last term has no factors of $x$ at all. The powers of $y$ behave just the opposite; that is, in the first term there is no power of $y$ and in each term thereafter the power of $y$ increases by 1 until finally the last term is $y^n$.

(3) The coefficients form a pattern known as Pascal's Triangle as follows:

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

Note that in each row the first and last entry is 1, and each of the interior entries is the sum of the two closest numbers in the row above it. Beginning with the second row, 1 1,

one sees the coefficients of the expansion $(x + y)^n$.

Another method of obtaining the coefficients in the expansion $(x + y)^n$ is to use the formula

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

For instance, expand $(x + y)^4$. Using the above formula for the coefficients and the pattern for the power the expansion is given by:

$$(x+y)^4 = \binom{4}{0} x^4 + \binom{4}{1} x^3y + \binom{4}{2} x^2y^2 + \binom{4}{3} xy^3 + \binom{4}{4} y^4,$$

where the coefficients are calculated using the aforementioned formula.

This method of generating an expansion of $(x + y)^n$ is often called the binomial theorem which is stated as follows: For any binomial $(x + y)$ and any positive integer $n$,

$$(x+y)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1}y + \binom{n}{2} x^{n-2}y^2 + \ldots + \binom{n}{n} y^n.$$

Once the binomial theorem is known, expansion of all types of binomial expressions, even if $n$ is a negative integer or a fraction, can be achieved.

# Step-by-Step Solutions to Problems in this Chapter, "Binomial Theorem/Expansion"

• **PROBLEM 728**

Find the binomial expansion of $(2x - 5)^4$.

Solution: The generalized form of the binomial expansion is

$$(a+b)^n = a^n + {}_nC_1 a^{n-1}b + {}_nC_2 a^{n-2}b^2 + {}_nC_3 a^{n-3}b^3 + \ldots + {}_nC_{n-1} a^1 b^{n-1} + {}_nC_n a^0 b^n$$

Here we take $a = 2x$, $b = -5$, and $n = 4$.

$$(2x-5)^4 = (2x)^4 + {}_4C_1(2x)^3(-5)^1 + {}_4C_2(2x)^2(-5)^2 + {}_4C_3(2x)^1(-5)^3 + {}_4C_4(2x)^0(-5)^4$$

$$= 16x^4 + {}_4C_1\ 8x^3\cdot(-5) + {}_4C_2\ 4x^2(25) + {}_4C_3\ 2x(-125) + {}_4C_4 1(-5)^4$$

Note that ${}_nC_r = \frac{n!}{r!(n-r)!}$. Therefore

$$(2x-5)^4 = 16x^4 + \frac{4!}{1!3!}8x^3(-5) + \frac{4!}{2!2!}(4x^2)(25) + \frac{4!}{3!1!}(2x)(-125) + \frac{4!}{4!0!}(1)(-5)^4$$

$$= 16x^4 + \frac{4\cdot\cancel{3!}}{\cancel{3!}}8x^3(-5) + \frac{\overset{2}{\cancel{4}}\cdot 3\cdot\cancel{2!}}{\underset{1}{\cancel{2}}\cdot 1\ \cancel{2!}}(100x^2) + \frac{4\cdot\cancel{3!}}{\cancel{3!}\cancel{1!}}(-250x) + 625$$

$$= 16x^4 - 160x^3 + 600x^2 - 1{,}000x + 625.$$

• **PROBLEM 729**

Expand $(2z - 3y)^4$.

Solution: Use the binomial theorem.

$$(u+v)^n = u^n + nu^{n-1}v + \frac{n(n-1)}{2}v^{n-2}v^2 + \frac{n(n-1)(n-2)}{2\cdot 3}u^{n-3}v^3 + \ldots + v^n$$

where $u = 2z$
$v = -3y$
$n = 4$

Therefore,

$$(2z - 3y)^4 = (2z)^4 + 4(2z)^3(-3y) + 6(2z)^2(-3y)^2 + 4(2z)(-3y)^3 + (-3y)^4$$

$$= 16z^4 + 4(8z^3)(-3y) + 6(4z^2)(9y^2) + 4(2z)(-27y^3) + 81y^4$$

$$= 16z^4 - 96z^3y + 216z^2y^2 - 216zy^3 + 81y^4$$

• **PROBLEM 730**

Find the expansion of $(a - 2x)^7$.

Solution: Use the binomial formula:

$$(u + v)^n = u^n + nu^{n-1}v + \frac{n(n-1)}{2} u^{n-2}v^2 + \frac{n(n-1)(n-2)}{2\cdot 3} u^{n-3}v^3 + \ldots + v^n$$

and substitute a for u and (-2x) for v and 7 for n to obtain:

$$\begin{aligned}(a-2x)^7 &= [a+(-2x)]^7\\ &= a^7 + 7a^6(-2x) + \frac{7\cdot\cancel{6}^{\,3}}{\cancel{2}_{\,1}} a^5(-2x)^2 + \frac{7\cdot\cancel{6}\cdot5}{\cancel{2}\cdot\cancel{3}} a^4(-2x)^3\\ &\quad + \frac{7\cdot\cancel{6}\cdot5\cdot\cancel{4}}{\cancel{2}\cdot\cancel{3}\cdot\cancel{4}} a^3(-2x)^4 + \frac{7\cdot\cancel{6}^{\,3}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}}{\cancel{2}\cdot\cancel{3}\cdot\cancel{4}\cdot\cancel{5}} a^2(-2x)^5\\ &\quad + \frac{7\cdot\cancel{6}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}}{\cancel{2}\cdot\cancel{3}\cdot\cancel{4}\cdot\cancel{5}\cdot\cancel{6}} a^1(-2x)^6 + \frac{\cancel{7}\cdot\cancel{6}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot1}{\cancel{2}\cdot\cancel{3}\cdot\cancel{4}\cdot\cancel{5}\cdot\cancel{6}\cdot\cancel{7}} a^0(-2x)^7\end{aligned}$$

$$(a - 2x)^7 = a^7 - 14a^6x + 84a^5x^2 - 280a^4x^3 + 560a^3x^4 - 672a^2x^5 + 448a\,x^6 - 128x^7.$$

**• PROBLEM 731**

Find the expansion of $(x + y)^6$.

Solution: Use the Binomial Theorem which states that

$$(a+b)^n = \frac{1}{0!} a^n + \frac{n}{1!} a^{n-1}b + \frac{n(n-1)}{2!} a^{n-2}b^2 + \ldots + nab^{n-1} + b^n .$$

Replacing a by x and b by y:

$$(x+y)^6 = \frac{1}{0!} x^6 + \frac{6}{1!} x^5y + \frac{6\cdot5}{2!} x^4y^2 + \frac{6\cdot5\cdot4}{3!} x^3y^3 + \frac{6\cdot5\cdot4\cdot3}{4!} x^2y^4 + \frac{6\cdot5\cdot4\cdot3\cdot2}{5!} x^1y^5 + \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{6!} x^0y^6$$

$$\begin{aligned}&= \frac{1}{1} x^6 + \frac{6}{1} x^5y + \frac{\cancel{6}^{\,3}\cdot5}{\cancel{2}_{\,1}\cdot1} x^4y^2 + \frac{\cancel{6}^{\,\cancel{2}^{\,1}}\cdot5\cdot4}{\cancel{3}_{\,1}\cdot\cancel{2}_{\,1}\cdot1} x^3y^3 + \frac{\cancel{6}^{\,3}\cdot5\cdot\cancel{4}\cdot\cancel{3}}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}_{\,1}\cdot1} x^2y^4\\ &\quad + \frac{6\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}}{\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot1} xy^5 + \frac{\cancel{6}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}}{\cancel{6}\cdot\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot\cancel{1}} y^6\end{aligned}$$

$$(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6 .$$

**• PROBLEM 732**

Expand $(x + 2y)^5$.

Solution: Apply the binomial theorem. If n is a positive integer, then

$$(a + b)^n = \binom{n}{0}a^nb^0 + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r$$

$$+ \dots + \begin{bmatrix} n \\ n \end{bmatrix} b^n.$$

Note that $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ and that $0! = 1$. Then, we obtain:

$$\begin{aligned}
(x + 2y)^5 &= \binom{5}{0}x^5(2y)^0 + \binom{5}{1}x^4(2y)^1 + \binom{5}{2}x^3(2y)^2 \\
&\quad + \binom{5}{3}x^2(2y)^3 + \binom{5}{4}x^1(2y)^4 + \binom{5}{5}x^0(2y)^5 \\
&= \frac{\cancel{5!}}{0!\cancel{5!}} x^5 + \frac{5!}{1!4!} x^4 2y + \frac{5!}{2!3!}x^3\left(4y^2\right) \\
&\quad + \frac{5!}{3!2!}x^2\left(8y^3\right) + \frac{5!}{4!1!}x\left(16y^4\right) + \frac{5!}{5!0!}1\left(32y^5\right) \\
&= x^5 + \frac{5 \cdot \cancel{4!}}{\cancel{4!}}x^4 2y + \frac{5 \cdot \overset{2}{\cancel{4}} \cdot \cancel{3!}}{\underset{1}{\cancel{2}} \cdot 1 \cdot \cancel{3!}}x^3\left(4y^2\right) \\
&\quad + \frac{5 \cdot \overset{2}{\cancel{4}} \cdot \cancel{3!}}{\cancel{3!}\,\underset{1}{\cancel{2}} \cdot 1}x^2\left(8y^3\right) + \frac{5 \cdot \cancel{4!}}{\cancel{4!}1!}x\left(16y^4\right) + \frac{\cancel{5!}}{\cancel{5!}0!}\left(32y^5\right) \\
&= x^5 + 10x^4y + 40x^3y^2 + 80x^2y^3 + 80xy^4 + 32y^5.
\end{aligned}$$

**• PROBLEM 733**

Expand $\left(3p^2 - 2q^{\frac{1}{2}}\right)^4$ by means of the binomial theorem.

**Solution:** We apply the binomial theorem: If $n$ is a positive integer, then

$$(a + b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \dots + \binom{n}{r} a^{n-r}b^r + \dots + \binom{n}{n} a^0 b^n$$

where $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

We identify $3p^2$ with $a$, $-2q^{\frac{1}{2}}$ with $b$, and $n$ with 4. The binomial theorem then gives us

$$\begin{aligned}
\left(3p^2 - 2q^{\frac{1}{2}}\right)^4 &= \binom{4}{0}\left(3p^2\right)^4\left(-2q^{\frac{1}{2}}\right)^0 + \binom{4}{1}\left(3p^2\right)^3\left(-2q^{\frac{1}{2}}\right) + \binom{4}{2}\left(3p^2\right)^2\left(-2q^{\frac{1}{2}}\right)^2 \\
&\quad + \binom{4}{3}\left(3p^2\right)^1\left(-2q^{\frac{1}{2}}\right)^3 + \binom{4}{4}\left(3p^2\right)^0\left(-2q^{\frac{1}{2}}\right)^4 \\
&= \frac{4!}{0!(4)!}3^4p^8 + \frac{4!}{1!3!}3^3p^6\left(-2q^{\frac{1}{2}}\right) + \frac{4!}{2!2!}3^2p^4(-2)^2q \\
&\quad + \frac{4!}{3!1!}3p^2(-2)^3q^{3/2} + \frac{4!}{4!0!}(-2)^4q^2 \\
&= 81p^8 + 4 \cdot \frac{3!}{3!} 27p^6\left(-2q^{\frac{1}{2}}\right) + \frac{\overset{2}{\cancel{4}} \cdot 3 \cdot \cancel{2!}}{\cancel{2!}\cancel{2}} 9p^4 4q \\
&\quad + \frac{4 \cdot \cancel{3!}}{\cancel{3!}1!} 3p^2(-8)q^{3/2} + 16q^2
\end{aligned}$$

$$= 81p^8 - 216p^6q^{\frac{1}{2}} + 216p^4q - 96p^2q^{3/2} + 16q^2 .$$

**• PROBLEM 734**

Give the expansion of $\left(r^2 - \frac{1}{s}\right)^5$.

<u>Solution:</u> Write the given expression as the sum of two terms raised to the 5th power:

$$\left(r^2 - \frac{1}{s}\right)^5 = \left[r^2 + \left(-\frac{1}{s}\right)\right]^5 \qquad (1)$$

The Binomial Theorem can be used to expand the expression on the right side of equation (1). The Binomial Theorem is stated as:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1 \cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}a^{n-3}b^3$$

$$+ \ldots + nab^{n-1} + b^n,$$ where a and b are any two numbers.

Let $a = r^2$, $b = -\frac{1}{s}$, and $n = 5$. Then, using the Binomial Theorem:

$$\left(r^2 - \frac{1}{s}\right)^5 = \left[r^2 + \left(-\frac{1}{s}\right)\right]^5$$

$$= \left(r^2\right)^5 + 5\left(r^2\right)^{5-1}\left(-\frac{1}{s}\right) + \frac{5(5-1)}{1 \cdot 2}\left(r^2\right)^{5-2}\left(-\frac{1}{s}\right)^2$$

$$+ \frac{5(5-1)(5-2)}{1 \cdot 2 \cdot 3}\left(r^2\right)^{5-3}\left(-\frac{1}{s}\right)^3$$

$$+ \frac{5(5-1)(5-2)(5-3)}{1 \cdot 2 \cdot 3 \cdot 4}\left(r^2\right)^{5-4}\left(-\frac{1}{s}\right)^4$$

$$+ \frac{5(5-1)(5-2)(5-3)(5-4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}\left(r^2\right)^{5-5}\left(-\frac{1}{s}\right)^5$$

$$= r^{10} - \frac{5\left(r^2\right)^4}{s} + \frac{5(\overset{2}{\cancel{4}})}{1 \cdot \underset{1}{\cancel{2}}}\left(r^2\right)^3\left(\frac{1}{s^2}\right) - \frac{5(\overset{2}{\cancel{4}})(\cancel{3})}{1 \cdot \underset{1}{\cancel{2}} \cdot \cancel{3}}\left(r^2\right)^2\left(\frac{1}{s^3}\right)$$

$$+ \frac{5(\cancel{4})(\cancel{3})(\cancel{2})}{1 \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4}}\left(r^2\right)^1\left(\frac{1}{s^4}\right) - \frac{\cancel{5}(\cancel{4})(\cancel{3})(\cancel{2})(\cancel{1})}{\cancel{1} \cdot \cancel{2} \cdot \cancel{3} \cdot \cancel{4} \cdot \cancel{5}}\left(r^2\right)^0\left(\frac{1}{s^5}\right)$$

$$= r^{10} - \frac{5r^8}{s} + \frac{10r^6}{s^2} - \frac{10r^4}{s^3} + \frac{5r^2}{s^4} - (1)(1)\left(\frac{1}{s^5}\right)$$

$$\left(r^2 - \frac{1}{s}\right)^5 = r^{10} - \frac{5r^8}{s} + \frac{10r^6}{s^2} - \frac{10r^4}{s^3} + \frac{5r^2}{s^4} - \frac{1}{s^5}$$

Find the first five terms of the expansion of $(1+x)^{-2}$.

**Solution:** The Binomial Theorem states that:

$$(a+b)^n = \frac{1}{0!}a^n + \frac{n}{1!}a^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \ldots + nab^{n-1} + b^n .$$

This theorem can be used to find the first five terms of the expansion of $(1+x)^{-2}$. Replacing a by 1 and b by x, the expression $(1+x)^{-2}$ becomes:

$$(1+x)^{-2} = \frac{1}{0!}1^{-2} + \frac{-2}{1!}1^{-3}x + \frac{(-2)(-3)}{2!}1^{-4}x^2 + \frac{(-2)(-3)(-4)}{3!}1^{-5}x^3$$

$$+ \frac{(-2)(-3)(-4)(-5)}{4!}1^{-6}x^4 + \ldots + (-2)1x^{-3} + x^{-2} .$$

Writing only the first five terms of this expansion:

$$(1+x)^{-2} = \frac{1}{0!}1^{-2} + \frac{-2}{1!}1^{-3}x + \frac{(-2)(-3)}{2!}1^{-4}x^2 + \frac{(-2)(-3)(-4)}{3!}1^{-5}x^3$$

$$+ \frac{(-2)(-3)(-4)(-5)}{4!}1^{-6}x^4 + \ldots$$

$$= \frac{1}{1}\left(\frac{1}{1^2}\right) - 2x\left(\frac{1}{1^3}\right) + \frac{6x^2}{2\cdot1}\left(\frac{1}{1^4}\right) + \frac{(-24)x^3}{3\cdot2\cdot1}\left(\frac{1}{1^5}\right) + \frac{\overset{30}{\cancel{120}}x^4}{\underset{1}{\cancel{4}}\cdot3\cdot2\cdot1}\left(\frac{1}{1^6}\right) + \ldots$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 + \ldots \qquad (1)$$

Hence, the right side of equation (1) represents the first five terms of the expansion of $(1+x)^{-2}$.

• PROBLEM 736

Find the first four terms of the expansion of $(2-1)^{\frac{1}{2}}$.

**Solution:** The Binomial Theorem states that:

$$(a+b)^n = \frac{1}{0!}a^n + \frac{n}{1!}a^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \ldots + nab^{n-1} + b^n .$$

This theorem can be used to expand the expression $(2-1)^{1/2}$. Replacing a by 2 and b by $-1$, the expression $(2-1)^{1/2}$ becomes:

$$(2-1)^{1/2} = [2+(-1)]^{1/2} = \frac{1}{0!}2^{1/2} + \frac{1/2}{1!}2^{\frac{1}{2}-1}(-1) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(2)^{\frac{1}{2}-2}(-1)^2$$

$$+ \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(2)^{\frac{1}{2}-3}(-1)^3 + \ldots + \tfrac{1}{2}(2)(-1)^{\frac{1}{2}-1} + (-1)^{\frac{1}{2}}$$

$$= \frac{1}{1}\sqrt{2} + \tfrac{1}{2}(2)^{-\frac{1}{2}}(-1) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2\cdot1}(2)^{-\frac{3}{2}}(1)$$

$$+ \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3\cdot2\cdot1}(2)^{-\frac{5}{2}}(-1) + \ldots + 1(-1)^{-\frac{1}{2}} + \sqrt{-1}$$

$$= \sqrt{2} - \tfrac{1}{2}\left(\frac{1}{2^{\frac{1}{2}}}\right) - \frac{1}{8}\left(\frac{1}{2^{\frac{3}{2}}}\right) - \frac{3}{48}\left(\frac{1}{2^{\frac{5}{2}}}\right) + \ldots + 1\left(\frac{1}{\sqrt{-1}}\right) + \sqrt{-1} .$$

The last result is true because of the law of exponents which states that

$$(N)^{-a/b} = \frac{1}{(N)^{a/b}}$$

where a and b are any positive integers. Writing only the first theorem, express $(1.01)^5$ as the sum of two numbers raised to the fifth power. Then,

$$(1.01)^5 = (1 + .01)^5$$

Now, let a = 1, b = .01, and n = 5 in the Binomial Theorem. Calculating the first four terms of this theorem with these substitutions:

$$\begin{aligned}(1.01)^5 &= (1 + .01)^5 \\ &= (1)^5 + 5(1)^{5-1}(.01) + \frac{5(5-1)}{1\cdot 2} \times \\ &\quad (1)^{5-2}(.01)^2 + \frac{5(5-1)(5-2)}{1\cdot 2\cdot 3} \times \\ &\quad (1)^{5-3}(.01)^3 + \cdots \\ &= 1 + 5(1)^4(.01) + \frac{5(\cancel{4}^{2})}{\cancel{2}_{1}}(1)^3(.01)^2 \\ &\quad + \frac{5(4)(\cancel{3}^{1})}{\cancel{6}_{2}}(1)^2(.01)^3 + \cdots \\ &= 1 + 5(1)(.01) + 10(1)(.0001) + 10(1) \\ &\quad (.000001) + \cdots \\ &= 1 + 0.05 + 0.001 + 0.00001 \\ &= 1.05101\end{aligned}$$

**• PROBLEM 745**

Use the binomial formula with $n = 1/3$ to find an approximation to $\sqrt[3]{28}$.

Solution: To apply the binomial formula, try to express $\sqrt[3]{28}$ as the sum of two numbers raised to a power. Note that the formula is simplified if one of the numbers is one.

$$\sqrt[3]{28} = (28)^{(1/3)} = (27+1)^{1/3} .$$

We can write the expansion of $(x+y)^{1/3}$ to four terms and later substitute for x and y. We write out the binomial expansion to four terms when n = 1/3.

$$\begin{aligned}(x+y)^{1/3} &= x^{1/3} + \frac{1}{3}x^{(1/3)-1}y + \frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)}{1\cdot 2}x^{(1/3)-2}y^2 \\ &\quad + \frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{1\cdot 2\cdot 3}x^{(1/3)-3}y^3 + \cdots \\ &= x^{1/3} + \frac{1}{3}x^{-2/3}y + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2}x^{-5/3}y^2\end{aligned}$$

$$+ \frac{\left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)\left(\frac{-5}{3}\right)}{1\cdot 2\cdot 3} x^{-8/3} y^3 + \ldots$$

$$= x^{1/3} + \frac{1}{3} x^{-2/3} y + \left(\frac{-2}{9}\right)\left(\frac{1}{2}\right) x^{-5/3} y^2$$

$$+ \frac{\overset{5}{\cancel{10}} \cdot 1}{3\cdot 3\cdot 3\cdot 1\cdot \underset{1}{\cancel{2}}\cdot 3} x^{-8/3} y^3 + \ldots$$

$$(x+y)^{1/3} = x^{1/3} + \frac{1}{3} x^{-2/3} y - \frac{1}{9} x^{-5/3} y^2 + \frac{5}{81} x^{-8/3} y^3 + \ldots \qquad (1)$$

In this case, $n$ is fractional. We obtain an infinite series and we can expand it for the first few terms if $|x| < |y|$. $x = 27$ and $y = 1$, and $|y| < |x|$, i.e., $|1| < |27|$. Therefore, using equation (1) with $x = 27$, $y = 1$ and $n = 1/3$ (writing only the first four terms):

$$\sqrt[3]{28} = (28)^{1/3}$$
$$= (27 + 1)^{1/3}$$
$$= 27^{1/3} + 1/3\left(27^{-2/3}\right)(1) - \frac{1}{9}\left(27^{-5/3}\right)\left(1^2\right) + \frac{5}{81}\left(27^{-8/3}\right)\left(1^3\right)$$
$$= 3 + \frac{1}{3}\left(\frac{1}{9}\right) - \frac{1}{9}\left(\frac{1}{243}\right) + \frac{5}{81}\left(\frac{1}{6,561}\right)$$
$$= 3 + 0.037037 - 0.000457 + 0.000009$$
$$= 3.036589$$

# CHAPTER 26

# LOGARITHMS AND EXPONENTIALS

**Basic Attacks and Strategies for Solving Problems in this Chapter.** **See pages 555 to 614 for step-by-step solutions to problems.**

Logarithms are exponents and the properties of logarithms are actually the properties of exponents. For any positive number $b$ other than 1

$$y = f(x) = b^x$$

is called the base $b$ exponential function. On the other hand, for any positive number $b$ other than 1, the logarithm to the base $b$ of $x$, denoted by $\log_b x$, is defined to be the power of $b$ that equals $x$. In other words,

$$\log_b x = y \quad \text{if and only if} \quad b^y = x.$$

The base $b$ exponential function and the base $b$ logarithmic function are inverses of each other. The base $b$ exponential expression (in exponential form) can be written in base $b$ logarithmic form. For instance, in general,

$$b^y = x$$

is equivalent to

$$\log_b x = y,$$

whereas $b$ is the base for both the exponent and log. Also, the exponent $y$ in $b^y$ is the value of the log, and $x$, the value of $b^y$, is the number of which the log is taken. In a similar manner, a logarithm statement can be written in exponential form. For example,

$$\log_2 8 = 3$$

written in exponential form is $2^3 = 8$.

To find the value of the expression $\log_b x$, first set

$$\log_b x = y,$$

which is equivalent to the exponential form $b^y = x$. By trial and error or by using the calculator, find the value of $y$ that satisfies $b^y = x$.

To find the value of $\log_b x^y$, recall that

$$\log_b x^y = y(\log_b x).$$

When finding logs with particular values and bases (including base 10 and $e$), we can use tables, calculators, and the various properties of logarithms.

To solve a logarithmic equation, the following general steps are suggested:

(1) Write the equivalent exponential equation for the given logarithmic equation.

(2) Evaluate the power that results.

(3) Solve the resulting equation for the variable. For instance,

$$\log_3(2x - 3) = 4 \Rightarrow 2x - 3 = 3^4 \text{ or}$$

$$2x - 3 = 81 \Rightarrow x = 42.$$

In order to solve an exponential equation in which the bases on both sides of the equation can be written as powers of the same base, $b$, follow these steps:

(1) Write the base on both sides as powers of $b$.

(2) Simplify the exponents on each side, if possible.

(3) Set the exponents on each side equal.

(4) Solve the resulting equation for the variable.

For example,

$$9^2 = 27^x \Rightarrow (3^2)^2 = (3^3)^x \Rightarrow 3^4 = 3^{3x}.$$

$$\Rightarrow 4 = 3x \Rightarrow x = {}^4/_3.$$

We could have obtained the same result by noting that $3^2 = 9$. Using a log table to find $\log_{10} 9$, we observe:

$\log_{10} 9 = .9542$, as above.

• **PROBLEM 749**

Write $\frac{1}{2} = \log_9 3$ in exponential form.

Solution: The statement $\log_b x = y$ is equivalent to the statement $b^y = x$, where $b$ is the base and $y$ the exponent. The latter form is the exponential form. Thus, $\frac{1}{2} = \log_9 3$ in exponential form is $9^{\frac{1}{2}} = 3$, where the base is 9 and the exponent is $\frac{1}{2}$.

• **PROBLEM 750**

If $\log_3 N = 2$, find $N$.

Solution: The equation $\log_x a = y$ is equivalent to the equation $x^y = a$. Thus $\log_3 N = 2$ is equivalent to the equation $3^2 = N$. $3^2 = 9$, hence $N = 9$.

• **PROBLEM 751**

Find the value of x if $\log_4 64 = x$.

Solution:

$$\log_b u = v$$

is equivalent to,

$$b^v = u,$$

thus the exponential equivalent of

$$\log_4 64 = x \quad \text{is } 4^x = 64.$$

Since,

$$4^3 = 4 \cdot 4 \cdot 4 = 64$$

$$\log_4 64 = 3.$$

That is,

$$x = 3.$$

• **PROBLEM 752**

Find $\log_3 729$.

Solution: Since we are working with log base 3, we check whether 729 has factors of 3.

$$\begin{aligned} 729 &= 3\cdot 243 = 3\cdot(3\cdot 81) = 3\cdot[3\cdot(3\cdot 27)] = 3[3\cdot 3(3\cdot 9)] \\ &= [3\cdot 3\cdot 3\cdot 3\cdot(3\cdot 3)] \\ &= 3^6 \end{aligned}$$

$$b^y = x, \ \log_b \frac{1}{25} = -2$$

is equivalent to

$$b^{-2} = \frac{1}{25}. \quad x^{-y} = \frac{1}{x^y}$$

thus,

$$b^{-2} = \frac{1}{b^2}. \text{ Therefore } b^{-2} = \frac{1}{25} \text{ is equivalent to } \frac{1}{b^2} = \frac{1}{25}.$$

Cross multiply to obtain the equivalent equation ,

$$b^2 = 25.$$

Take the square root of both sides. Thus,

$$b = \pm 5$$

• **PROBLEM 756**

> Express the logarithm of $\dfrac{\sqrt{a^3}}{c^5 b^2}$ in terms of log a, log b and log c.

Solution: We apply the following properties of logarithms:

$$\log_b(P \cdot Q) = \log_b P + \log_b Q$$

$$\log_b(P/Q) = \log_b P - \log_b Q$$

$$\log_b\left(P^n\right) = n \log_b P$$

$$\log_b\left(\sqrt[n]{P}\right) = \frac{1}{n} \log_b P$$

Therefore,

$$\log \frac{\sqrt{a^3}}{c^5 b^2} = \log \frac{a^{3/2}}{c^5 b^2}$$

$$= \log a^{3/2} - \log\left(c^5 b^2\right)$$

$$= 3/2 \log a - \left(\log c^5 + \log b^2\right)$$

$$= 3/2 \log a - \log c^5 - \log b^2$$

$$= 3/2 \log a - 5 \log c - 2 \log b \ .$$

• **PROBLEM 757**

> If $\log_{10} 3 = .4771$ and $\log_{10} 4 = .6021$, find $\log_{10} 12$.

Solution: Since $12 = 3 \times 4$,

$$\log_{10} 12 = \log_{10}(3)(4).$$

Since $\log_b(xy) = \log_b x + \log_b y$

$$\log_{10}(3 \times 4) = \log_{10} 3 + \log_{10} 4$$

$$= .4771 + .6021$$

statement $b^y = a$. Thus, $x = \log_6 36$ is equivalent to $6^x = 36$. $6^2 = 36$, thus $x = 2$ and $\log_6 36 = 2$. Replacing $\log_6 36$ by 2 we obtain $\log_b 16 = 2$, or equivalently $b^2 = 16$. Thus $b = \sqrt{16} = 4$.

**• PROBLEM 766**

Evaluate $\log_{10} \sqrt[3]{7}$.

**Solution:** Since $\sqrt[a]{x} = x^{1/a}$, $\sqrt[3]{7} = 7^{\frac{1}{3}}$, and

$$\log_{10} \sqrt[3]{7} = \log_{10} 7^{\frac{1}{3}}$$

Recall the property of logarithms: $\log_b x^a = a \log_b x$.

Thus, $\log_{10} 7^{\frac{1}{3}} = \frac{1}{3} \log_{10} 7$.

From the table of common logarithms we find that $\log_{10} 7 = .8451$, thus

$$\frac{1}{3} \log_{10} 7 = \frac{1}{3} (.8451) = .2817$$

Therefore, $\log_{10} \sqrt[3]{7} = .2817$.

**• PROBLEM 767**

Find $\log \left(4 \frac{2}{7}\right)$.

**Solution:** $4 \frac{2}{7} = \frac{30}{7}$, thus $\log \left(4 \frac{2}{7}\right) = \log \frac{30}{7}$.

Since $\log_b \frac{x}{y} = \log_b x - \log_b y$,

$$\log \frac{30}{7} = \log 30 - \log 7.$$

Reducing 30 to prime factors,

$$\log \frac{30}{7} = \log (2 \times 3 \times 5) - \log 7$$

Recalling $\log_b xyz = \log_b x + \log_b y + \log_b z$,

$$\log \frac{30}{7} = \log 2 + \log 3 + \log 5 - \log 7.$$

From a log table we find

$$\log 2 = .3010$$
$$\log 3 = .4771$$
$$\log 5 = .6990$$
$$\log 7 = .8451$$

Hence, $\log \left(4 \frac{2}{7}\right) = .3010 + .4771 + .6990 - .8451 = .6320$.

Find the logarithm of 258, using a trig table.

**Solution:** Since our log tables only give values of logarithms between 1.00 and 9.99 we must express 258 in terms of some number between 1 and 9.99 multiplied by a power of ten. Hence

$$258 = 2.58 \times 100 = 2.58 \times 10^2$$

and $\quad \log 258 = \log\left(2.58 \cdot 10^2\right)$

since $\quad \log_a BC = \log_a B + \log_a C$

$$\log_{10}\left(2.58 \cdot 10^2\right) = \log_{10} 2.58 + \log_{10} 10^2 \ .$$

Since $\log_{10} x = a$ means by definition $10^a = x$ ,
we note $\log_{10} 10^2 = 2$ because $10^2 = 10^2$ ;
hence $\log_{10}\left(2.58 \cdot 10^2\right) = \log_{10} 2.58 + 2$ . From our trig. table we read $\log 2.58 = .4116$. Hence $\log 258 = .4116 + 2 = 2.4116$.



Evaluate $\dfrac{\log_{10} 12}{\log_{10} 5}$ .

**Solution:** First calculate $\log_{10} 12$.

$$\log_{10} 12 = \log_{10}(1.2 \times 10)$$

By the law of logarithms which states that $\log_b(x \cdot y) = \log_b x + \log_b y$,

$$\log_{10} 12 = \log_{10}(1.2 \times 10) = \log_{10} 1.2 + \log_{10} 10$$
$$= 0.0792 + 1$$
$$= 1.0792$$

The $\log_{10} 1.2$ was obtained from a table of common logarithms, base 10. Also, $\log_{10} 5 = 0.6990$. This value was also obtained from a table of common logarithms, base 10.

$$\frac{\log_{10} 12}{\log_{10} 5} = \frac{1.0792}{0.6990} = 1.544.$$



Find the logarithm of 30,700.

**Solution:** First express 30,700 in scientific notation. $30{,}700 = 3.07 \times 10^4$. 4 is the characteristic. To find the mantissa, see a table of common logarithms of numbers The mantissa is 4871. Thus $\log 30{,}700 = 4 + .4871 = 4.4871$.

• **PROBLEM 771**

Find $\log 0.0364$ .

Solution: $0.0364 = 3.64 \times 10^{-2}$ . Therefore, the characteristic, the power of 10, is -2. From a table of logarithms, the mantissa for 3.64 is 0.5611. Therefore, $\log 0.0364 = -2 + 0.5611 = -1.4389$.

• **PROBLEM 772**

Find N if $\log N = 0.7917 - 3$.

Solution: Using a table of logarithms, the mantissa .7917 is found to correspond to the number 6.19. Therefore the antilogarithm is 6.19. Then, since the characteristic is -3,

$$N = 6.19 \times 10^{-3} = 0.00619.$$

• **PROBLEM 773**

What is the value of $\log 0.0148$?

Solution: $0.0148 = 1.48 \times 10^{-2}$. The characteristic is the exponent of 10. Hence, the characteristic is -2. The mantissa for 148 can be found in a table of logarithms. The mantissa is 0.1703. Therefore, $\log 0.0148 = -2 + 0.1703 = -2.0000 + 0.1703 = -1.8297$. Notice that the number 0.0148 is less than 1. Therefore, the value of $\log 0.0148$ must be negative, as it was found to be.

• **PROBLEM 774**

Determine x when $\log x = 3.1818$.

Solution: $\text{Log}_{10}\ x = 3.1818$ is equivalent to $10^{3.1818} = x$. Since $a^{x+y} = a^x \cdot a^y$, $x = 10^{3.1818} = 10^3 \cdot 10^{.1818}$

$= 1{,}000 \cdot 10^{.1818}$.

We look in the body of the log table for the mantissa 0.1818 and find it in row 15 and column 2, so that the digits of x are 1.52. Thus

$$x = 1{,}000 \times 1.52$$
$$= 1520.$$

• **PROBLEM 775**

Find $\text{Antilog}_{10}(0.8762 - 2)$.

Solution: Let $N = \text{Antilog}_{10}(0.8762 - 2)$. The following relationship between log and antilog exists: $\log_{10}x = a$ is the equivalent of $x = \text{antilog}_{10}a$ . Therefore, corresponds to the mantissa 0.5019 is approximately 3.18. Then $x^3 + 1 = 4.18$ and $x^3 - 1 = 2.18$. Let

$$N = \sqrt{\frac{x^3 + 1}{x^3 - 1}} = \sqrt{\frac{4.18}{2.18}}\,.$$

Take the logarithm of both sides of the above equation.

$$\log N = \log\sqrt{\frac{4.18}{2.18}} = \log\left(\frac{4.18}{2.18}\right)^{\frac{1}{2}}.$$

By the law of the logarithm of a power of a positive number which states that

$$\log a^n = n \log a,\ \log\left(\frac{4.18}{2.18}\right)^{\frac{1}{2}} = \frac{1}{2}\log\left(\frac{4.18}{2.18}\right).$$

By the law of the logarithm of a quotient which states that $\log \frac{a}{b} = \log a - \log b$, $\log\left(\frac{4.18}{2.18}\right) = \log 4.18 - \log 2.18$. Hence,

$$\log N = \log\left(\frac{4.18}{2.18}\right)^{\frac{1}{2}} = \frac{1}{2}\log\left(\frac{4.18}{2.18}\right) = \frac{1}{2}\left(\log 4.18 - \log 2.18\right)$$

$$= \frac{1}{2}(0.6212 - 0.3385).$$

Note that the values for the two logs were found in a table of logarithms. Therefore,

$$\log N = \frac{1}{2}(0.6212 - 0.3385) = \frac{1}{2}(0.2827) = 0.1414.$$

$$N = \text{antilog } 0.1414 = 1.38.$$

Note that in a table of logarithms, the number that corresponds to the mantissa 0.1414 is approximately 1.38.

## INTERPOLATIONS

**• PROBLEM 781**

Use linear interpolation to find log 5.723.

Solution: Since 5.723 is .3 of the way from 5.72 to 5.73, we argue that log 5.723 is approximately .3 of the way from log 5.72 to log 5.73.

This is the basic idea involved in linear interpolation. We obtain log 5.72 and log 5.73 from a table of common logarithms, and find the mantissas to be 7574 and 7582, respectively. We now use interpolation to find the mantissa for 5.723.

$$
\begin{array}{cc}
N & \log N \\
.01\left[\begin{array}{l} .004\left[\begin{array}{l}3.120\\3.124\end{array}\right.\\ \qquad 3.130\end{array}\right. & \left.\begin{array}{r}\left.\begin{array}{r}.4942\\x\end{array}\right]d\\ .4955\end{array}\right] .0013
\end{array}
$$

Setting up the following proportion:

$$\frac{.004}{.01} = \frac{d}{.0013} .\quad \text{Cross multiply to obtain } (.01)d = (.004)(.0013)$$

$$d = (.0013)\left(\frac{.004}{.01}\right) = \frac{(1.3 \times 10^{-3})(4 \times 10^{-3})}{1 \times 10^{-2}}$$

$$= \frac{5.2 \times 10^{-6}}{1 \times 10^{-2}}$$

$$= 5.2 \times 10^{[-6-(-2)]} = 5.2 \times 10^{-4}$$

$$= (5.2)(0.0001)$$

$$d = 0.00052$$

Therefore, $x = \log 3.124 = 0.4942 + 0.00052 = 0.49472$. Hence, the mantissa for 3.124 is 0.49472. Therefore, $\log 0.003124 = -3 + 0.49472$

$$= -2.50528$$

or

$$7.49472 - 10 .$$

**• PROBLEM 788**

Find the logarithm of 3614.0.

Solution: $3,614 = 3.614 \times 10^3$ , hence our characteristic (exponent of 10) is 3. To determine the mantissa of 3.614, since our log tables only give us values for 3.61 and 3.62, we make the following interpolation:

$$
\begin{array}{cc}
N & \log N \\
0.01\left[\begin{array}{l} 0.004\left[\begin{array}{l}3.61\\3.614\end{array}\right.\\ \qquad 3.62\end{array}\right. & \left.\begin{array}{r}\left.\begin{array}{r}0.5575\\x\end{array}\right]d\\ 0.5587\end{array}\right] .0012
\end{array}
$$

The following proportion is set up:

$$\frac{0.004}{0.01} = \frac{d}{0.0012} ,\quad \text{cross multiplying we obtain } (.01)d = (.004)(.0012)$$

or $$d = \frac{0.004}{0.01}(0.0012) = \frac{(4 \times 10^{-3})(1.2 \times 10^{-3})}{1 \times 10^{-2}} = \frac{4.8 \times 10^{-6}}{1 \times 10^{-2}}$$

$$= 4.8 \times 10^{-6-(-2)} = (4.8) \times 10^{-4} = (4.8)(0.0001)$$

$$d = 0.00048$$

Hence, $x = \log 3.614 = 0.5575 + d$

$$= 0.5575 + 0.00048$$

$$= 0.5580$$

Therefore, the mantissa for 3614 is 0.5580. The characteristic is 3. Hence, the logarithm of 3614.0 is $3 + .5580 = 3.5580$.

denominator by 1000 (moving the decimal 3 places to the right). Thus,

$$\frac{.2}{1} = \frac{y}{.0012}$$

$y = (.2)(.0012)$, by cross multiplying. Therefore, $y = .00024$.

$$\log .3612 = (9.5575 - 10) + .00024$$

$$= 9.55774 - 10$$

We must now find log 456.53. Since 456.53 > 1, and for a number greater than 1, the characteristic is positive and is one less than the number of digits before the decimal point, the characteristic is 2. By use of the log table we find the mantissa for 456 is .6590, and for 457 is .6599. Hence, by interpolation,

$$\begin{array}{cc} & \underline{x} \qquad\qquad \underline{\log x} & & \\ 1 & \left[ \begin{array}{cc} .53 \left[ \begin{array}{cc} 456 & 2.6590 \\ 456.53 & ? \end{array} \right] y \\ \quad 457 \qquad 2.6599 \end{array} \right] & .0009 \end{array}$$

$$\frac{.53}{1} = \frac{y}{.0009}$$

$$y = (.53)(.0009)$$

$$y = .00047$$

$$\log 456.53 = 2.6590 + .00047 = 2.65947$$

Thus, $\log x = (9.55774 - 10) - (2.65947)$

$$\log x = 6.89827 - 10$$

We again use interpolation to obtain x. Observe that the characteristic is - 4. Thus, the desired number will be less than 1, and have 3 zeros following the decimal (by the rule that for a number less than 1 the characteristic is negative and is one more than the number of zeros following the decimal point). Thus, by interpolation:

$$\begin{array}{cc} & \underline{x} \qquad\qquad \underline{\log x} & & \\ .000001 & \left[ \begin{array}{cc} y \left[ \begin{array}{cc} .000791 & 6.8982 - 10 \\ ? & 6.89827 - 10 \end{array} \right] .00007 \\ \quad .000792 \qquad 6.8987 - 10 \end{array} \right] & .0005 \end{array}$$

$$\frac{y}{.000001} = \frac{.00007}{.0005}$$

$$\frac{y}{.000001} = \frac{.7}{5}$$

$$5.06 \times 71.32 = \text{antilog}[\log(5.06 \times 7.132 \times 10)] \qquad (1)$$

Evaluating the expression in the brackets:

$$\log(5.06 \times 7.132 \times 10) = \log 5.06 + \log 7.132 + \log 10.$$

This is true because of the following law of exponents:

$$\log abc = \log a + \log b + \log c.$$

Using a table of common logarithms to find the value of log 5.06 and noting that $\log 10 = 1$,

$$\begin{aligned}\log(5.06 \times 7.132 \times 10) &= 0.7042 + (\log 7.132) + 1 \\ &= 1.7042 + \log 7.132 \qquad (2)\end{aligned}$$

We now evaluate log 7.132. The numbers that appear in a table of common logarithms which are closest to the number 7.132 are 7.13 and 7.14. The mantissa that corresponds to the number 7.132 will be found by interpolation.

| | | Number | Logarithm | | |
|---|---|---|---|---|---|
| .01 | .002 | 7.13 | 0.8531 | d | .0006 |
| | | 7.132 | x | | |
| | | 7.14 | 0.8537 | | |

Now, setting up the following proportion:

$$\frac{d}{.0006} = \frac{.002}{.01}$$

Cross-multiplying,

$$\begin{aligned} d &= .0006\left(\frac{.002}{.01}\right) \\ &= \left(6 \times 10^{-4}\right)\left(\frac{2 \times 10^{-3}}{1 \times 10^{-2}}\right) = \frac{12 \times 10^{-4+(-3)}}{1 \times 10^{-2}} \\ &= \frac{12 \times 10^{-7}}{1 \times 10^{-2}} = 12 \times 10^{-7-(-2)} \\ &= 12 \times 10^{-5} \\ &= \left(1.2 \times 10^{1}\right) \times 10^{-5} = 1.2 \times 10^{1+(-5)} \\ &= 1.2 \times 10^{-4} \\ &= 1.2 \times 0.0001 \\ &= 0.00012 \\ &\approx 0.0001 \end{aligned}$$

Hence,

$$\begin{aligned}\log 7.132 = x &= 0.8531 + 0.0001 \\ &= 0.8532.\end{aligned}$$

Therefore, equation (2) becomes:

$$\begin{aligned}\log(5.06 \times 7.132 \times 10) &= 1.7042 + 0.8532 \\ &= 2.5574\end{aligned}$$

Equation (1) becomes:

$$5.06 \times 71.32 = \text{antilog}[2.5574] = M \qquad (3)$$

By definition, antilog[2.5574] is equivalent to $\log M = 2.5574$. The characteristic is 2. The mantissa is 0.5574. The number that corresponds to this mantissa will be multiplied to $10^2$ or 100. The mantissas which appear in a table of logarithms and are closest to the mantissa 0.5574 are 0.5563 and 0.5575. The number that corresponds to the mantissa 0.5574 can be found by interpolation.

Since $x^{\frac{1}{2}} = \sqrt{x}$

$$\log_{10} 10{,}000^{\frac{1}{2}} = \log_{10}\sqrt{10{,}000}$$
$$= \log_{10} 100$$
$$= 2$$

$$\tfrac{1}{2}\log_{10} 10{,}000 = \tfrac{1}{2}\log_{10} 10^4$$
$$= \tfrac{1}{2}\cdot 4$$
$$= 2$$

Thus $\log_{10} 10{,}000^{\frac{1}{2}} = 2 = \frac{1}{2}\log_{10} 10{,}000$ , and

$$\log_{10} 10{,}000^{1/2} = \tfrac{1}{2}\log_{10} 10{,}000 .$$

**• PROBLEM 804**

The graph of an exponential function $f$ contains the point (2,9). What is the base of $f$?

Solution: Since $f$ is an exponential function, we know that $f(x) = b^x$ , where b, the base, is a positive number that we are to determine. An exponential function $f$ may also be written as $y = f(x) = b^x$. Since the exponential function $f$ contains the point (2,9),

$$9 = f(2) = b^2 \quad \text{or} \quad b^2 = 9$$
$$\sqrt{b^2} = \sqrt{9}$$
$$b = 3 .$$

Note that only the positive square root was taken, since for the base b, a positive number, is desired.

**• PROBLEM 805**

If f is the logarithmic function with base 4, find f(4), $f\left(\frac{1}{4}\right)$, and f(8).

Solution: Since f is the logarithmic function with base 4, then $y = f(x) = \log_4 x$. The values f(4), $f\left(\frac{1}{4}\right)$, and f(8) can be found by replacing x by 4, $\frac{1}{4}$, and 8 in the logarithmic function $y = f(x) = \log_4 x$. Hence, $f(4) = \log_4$. Let $N_1 = f(4) = \log_4 4$. By definition, $\log_x a = N$ is equivalent to $x^N = a$. Therefore, $N_1 = \log_4 4$ is equivalent to $4^{N_1} = 4$. Since $4^1 = 4$, $N_1 = 1$. Then , $N_1 = 1 = f(4)$.

For the second value $f\left(\frac{1}{4}\right)$, $f\left(\frac{1}{4}\right) = \log_4 \frac{1}{4}$. Let $N_2 = f\left(\frac{1}{4}\right) = \log_4 \frac{1}{4}$. Hence, $N_2 = \log_4 \frac{1}{4}$ is equivalent to $4^{N_2} = \frac{1}{4}$. Since $4^{-1} = \frac{1}{4^1} = \frac{1}{4}$, $N_2 = -1$. Then, $N_2 = -1 = f\left(\frac{1}{4}\right)$.

For the third value f(8), $f(8) = \log_4 8$. Let $N_3 = f(8) = \log_4 8$. Hence, $N_3 = \log_4 8$ is equivalent to $4^{N_3} = 8$. Since $4 = 2^2$, $N_3 = \log_4 8$ is equivalent to $\left(2^2\right)^{N_3} = 8$ or

$2^{2N_3} = 8$. Since $2^3 = 8$, $2N_3 = 3$. Dividing both sides of the equation $2N_3 = 3$ by 2:

$$\frac{2N_3}{2} = \frac{3}{2} \text{ or } N_3 = \frac{3}{2}.$$

Then, $N_3 = \frac{3}{2} = f(8)$.

**• PROBLEM 806**

Solve the equation $\log_3(x^2 - 8x) = 2$.

Solution: The expression $\log_b a = y$ is equivalent to $b^y = a$. Hence, $\log_3(x^2 - 8x) = 2$ is equivalent to $3^2 = x^2 - 8x$. Therefore,

$$3^2 = x^2 - 8x$$

or

$$9 = x^2 - 8x .$$

Subtract 9 both sides of this equation:

$$9 - 9 = x^2 - 8x - 9$$

$$0 = x^2 - 8x - 9.$$

Factoring this equation:

$$0 = (x - 9)(x + 1).$$

Whenever the product $ab = 0$ where $a$ and $b$ are any two numbers, either $a = 0$ or $b = 0$. Hence, either

$$x - 9 = 0 \quad \text{or} \quad x + 1 = 0$$

$$x = 9 \quad \text{or} \quad x = -1.$$

**• PROBLEM 807**

Solve the equation $2^x = 7$ for x.

Solution: By taking logarithms of both sides of the equation, we obtain the equation

$$\log 2^x = \log 7.$$

Using the rule $\log M^x = x \log M$, we obtain:

$$x \log 2 = \log 7$$

From a table on common logarithms the $\log 2 = .3010$ and the $\log 7 = .8451$. So our equation becomes $.3010x = .8451$. Hence,

$$x = \frac{.8451}{.3010} = 2.808.$$

Remark 1. Since $2^2 = 4$ and $2^3 = 8$, it should be obvious at the start that the solution of the equation $2^x = 7$ is a number between 2 and 3.

Remark 2. Since $x \log 2 = \log 7$, it follows that $x = \log 7/\log 2$. It should be emphasized that the expression $\log 7/\log 2$ is a quotient. We do not evaluate this quotient by looking up log 2 and log 7 in the table and subtracting; we look up the two numbers and divide. We can divide with the aid of logarithms, but it still will be division.

**• PROBLEM 808**

Solve the equation $2^x = 3^{x+1}$ for x.

Solution: We take logarithms of both sides of the equation to obtain the equation:

$$\log\left(2^x\right) = \log\left(3^{x+1}\right)$$

Using the rule $\log_b M^x = x \log_b M$, we obtain:

$$x \log 2 = (x + 1) \log 3.$$

Hence, $x \log 2 = x \log 3 + \log 3$, by distributing log 3; and

$$x \log 2 - x \log 3 = \log 3,$$

or in other words $x(\log 2 - \log 3) = \log 3$

$$x = \frac{\log 3}{\log 2 - \log 3}.$$

Using a table of common logarithms we obtain:

$$x = \frac{.4771}{.3010 - .4771} = \frac{.4771}{-.1761} = -2.709.$$

**• PROBLEM 809**

Solve for x in the equation $7^{2x-1} - 5^{3x} = 0$.

Solution: Writing the equation as $7^{2x-1} = 5^{3x}$, and equating logarithms of both members, we have

$$\log 7^{2x-1} = \log 5^{3x}$$

Recall $\log x^y = y \log x$, thus,

$$(2x - 1)\log 7 = 3x \log 5$$

Looking up log 7 and log 5 in our log table and substituting,

$$(2x - 1)(0.8451) = 3x(0.6990).$$

Hence

$$1.6902x - 0.8451 = 2.097x, \quad 0.4068x = -0.8451$$

and

$$x = -2.077.$$

**• PROBLEM 810**

Solve the equation

$$2^{3x} = 4^{x+1}.$$

Solution: This is an exponential equation, an equation involving one or more unknowns in an exponent. We solve it by logarithms.

$$2^{3x} = 4^{x+1}$$

Taking logarithms of both sides:

$$\log\left(2^{3x}\right) = \log\left(4^{x+1}\right)$$

Apply $\log p^q = q \log p$: $\quad 3x \log 2 = (x+1)\log 4$

Now $\log 4 = \log 2^2 = 2 \log 2$, and consequently

$$3x \log 2 = (x+1)\log 2^2$$

$$3x \log 2 = 2(x+1)\log 2,$$

Divide by log 2: $\quad 3x = 2x + 2,$

Subtract 2x from both sides:

$$x = 2.$$

The solution can be verified by substituting the value $x = 2$ into the original equation, $2^{3x} = 4^{x+1}$. Thus,

$$2^{3\cdot 2} \stackrel{?}{=} 4^{2+1}$$

$$2^6 = 4^3$$

$$2^6 = \left(2^2\right)^3$$

$$2^6 = 2^6$$

Thus, the solution is $x = 2$.

**• PROBLEM 811**

Solve $5^{2x} = 7^{x+1}$.

Solution: If we take the common logarithm of each member of the given equation, we have

$$\log 5^{2x} = \log 7^{x+1}$$

Recall $\log x^y = y \log x$. Thus $\log 5^{2x} = 2x \log 5$ and $\log 7^{x+1} = (x+1) \log 7$. Making these substitutions we obtain,

$$2x \log 5 = (x + 1)\log 7$$

Distribute, $\quad 2x \log 5 = x \log 7 + \log 7$

Add $(-x \log 7)$ to both sides,

$$2x \log 5 - x \log 7 = \log 7$$

Factor $x$ from the left member,

$$x(2 \log 5 - \log 7) = \log 7$$

Since $y \log x = \log x^y$

$$2 \log 5 = \log 5^2 = \log 25$$

Thus replacing $2 \log 5$ by $\log 25$ we obtain,

$$x(\log 25 - \log 7) = \log 7$$

Divide both members by $\log 25 - \log 7$,

$$x = \frac{\log 7}{\log 25 - \log 7}$$

Looking up $\log 7$ in a log table we observe $\log 7 = .8451$. Since

$$25 = 2.5 \times 10^1$$
$$\begin{aligned}\log 25 &= \log(2.5 \times 10^1)\\ &= \log 2.5 + \log 10^1\\ &= \log 2.5 + 1\\ &= .3979 + 1\\ &= 1.3979\end{aligned}$$

Substituting these values for our logarithms,

$$= \frac{.8451}{1.3979 - .8451}$$
$$= \frac{.8451}{.5528} = 1.529$$

**• PROBLEM 812**

Solve the "exponential" equation: $2^{0.4x} = 7.$

Solution: Taking the log of both sides of the given equation,

$$\log 2^{0.4x} = \log 7.$$

Since $\log_b y^a = a \log_b y$, $\log 2^{0.4x} = 0.4x \log 2.$

Thus $$0.4x \log 2 = \log 7$$

or, $$x = \frac{\log 7}{0.4 \log 2}$$

In a log table we find $\log 7 = .8451$ and $\log 2 = .3010.$
Thus

$$x = \frac{.8451}{(0.4)(.3010)} = \frac{.8451}{.1204} = 7.02\ .$$

Or, if we wish, we may complete the solution by using logarithms again:

$$\log 0.8451 = \log 8.451 \times 10^{-1} = .9269 - 1$$
$$\log 0.1204 = \log 1.204 \times 10^{-1} = .0806 - 1$$

Since $\log_b \frac{y}{z} = \log_b y - \log_b z$ ,

$$\log x = \log \frac{.8451}{.1204} = \log .8451 - \log .1204$$

$$= (.9269 - 1) - (.0806 - 1)\ .$$

Thus $$\log x = .8463$$

We look up the mantissa, 0.8463, in a table of Common Logarithms and find its corresponding number to be 7020. We adjust the decimal point by noting the characteristic 0 of 0.8463 is one less than the number of digits to the left of the decimal point of the number we seek. In this case, therefore, there should be one digit to the left of the decimal point. Hence,

$$x = 7.020\ .$$

Solution: We can use a fundamental property of logarithms to simplify the left-hand side of this equation.

The logarithm of the product of two or more positive numbers is equal to the sum of the logarithms of the several numbers. If P, Q, and R are positive numbers, then $\log(P \cdot Q \cdot R) = \log P + \log Q + \log R$.

$$\begin{aligned} 2\log x - \log 10x &= 2\log x - (\log 10 + \log x) \\ &= 2\log x - \log 10 - \log x \\ &= \log x - \log 10. \end{aligned}$$

But log 10 means that base 10 raised to what power = 10, or $10^? = 10$; and $10^1 = 10$. Thus, $\log 10 = 1$, and the equation becomes: $\log x - 1 = 0$.

Rewriting this equation:

$$\log x - 1 = 0$$

$$\log x = 1$$

Since the problem is in base 10, $\log x = 1$ can be rewritten as,

$$10^1 = x. \text{ Thus } x = 10.$$

• PROBLEM 817

Solve $\log(40x - 1) - \log(x - 1) = 3$.

Solution: By the law of the logarithm of a quotient of two numbers which states that $\log \frac{a}{b} = \log a - \log b$, $\log(40x - 1) - \log(x - 1) = \log \frac{40x - 1}{x - 1} = 3$. Hence, $\log_{10} \frac{40x - 1}{x - 1} = 3$. By the definition of a logarithm, if $\log_b N = x$, then $b^x = N$. Therefore, $\log_{10} \frac{40x - 1}{x - 1} = 3$ means $10^3 = \frac{40x - 1}{x - 1}$. Thus $1000 = \frac{40x - 1}{x - 1}$. Multiply both sides by $(x - 1)$:

$$(x - 1)1000 = \cancel{(x - 1)}\left(\frac{40x - 1}{\cancel{x - 1}}\right)$$

$$(x - 1)1000 = 40x - 1.$$

Distributing, $1000x - 1000 = 40x - 1$. Subtract $40x$ from both sides to obtain:

$$1000x - 1000 - 40x = \cancel{40x} - 1 - \cancel{40x}.$$

Combining terms, $960x - 1000 = -1$. Add 1000 to both sides to obtain:

$$960x - \cancel{1000} + \cancel{1000} = -1 + 1000.$$

Combining terms, $960x = 999$. Divide both sides by 960:

$$\frac{\cancel{960}x}{\cancel{960}} = \frac{999}{960}$$

$$x = \frac{999}{960} = \frac{333}{320}.$$

• **PROBLEM 818**

Solve $\log_2(x - 1) + \log_2(x + 1) = 3$.

Solution: Applying a property of logarithms, $\log_b x + \log_b y = \log_b xy$, to

$$\log_2(x - 1) + \log_2(x + 1) = 3$$

we get $\log_2[(x - 1)(x + 1)] = 3$. $\log_b x = y$ is equivalent to $b^y = x$ by definition, thus $\log_2[(x - 1)(x + 1)] = 3$ is equivalent to

$$(x - 1)(x + 1) = 2^3 = 8$$

$$x^2 = -1 = 8$$

$$x^2 - 9 = 0$$

$$0 = x^2 - 9 = x^2 - 3^2.$$

Thus we apply the formula for the difference of two squares, $a^2 - b^2 = (a + b)(a - b)$, replacing a by x and b by 3 and obtain $0 = x^2 - 3^2 = (x + 3)(x - 3)$. Whenever the product of two numbers $ab = 0$, either $a = 0$ or $b = 0$. Thus

$(x + 3)(x - 3) = 0$ means either

$$x + 3 = 0 \text{ or } x - 3 = 0$$

and $\qquad x = -3$ or $\quad x = 3$.

Therefore, $\{3,-3\}$ is the possible solution set, but we must check each in the given equation. This is necessary because we have not defined the logarithm of a negative number and, consequently, must rule out any value of x which would require the use of the logarithm of a negative number.

Check: Replacing x by 3 in our original equation

$$\log_2(x - 1) + \log_2(x + 1) = 3$$

$$\log_2(3 - 1) + \log_2(3 + 1) = 3$$

$$\log_2 2 + \log_2 4 = 3$$

$$1 + 2 = 3 \text{ since } 2^1 = 2 \text{ and } 2^2 = 4$$

$$3 = 3.$$

Replacing x by (-3) in our original equation

$$\log_2(x - 1) + \log_2(x + 1) = 3$$

$$\log_2(-3 - 1) + \log_2(-3 + 1) = 3$$

$$\log_2(-4) + \log_2(-2) = 3.$$

$x = -3$ cannot be accepted as a root because we have not defined the **logarithm** of a negative number. Thus our solution set is $\{3\}$.

**• PROBLEM 819**

> Solve the equation $\log 2 + 2 \log x = \log(5x + 3)$.

Solution: By the law of the logarithm of a power of a positive number which states that

$$\log a^n = n \log a, \; 2 \log x = \log x^2.$$

Hence, $\log 2 + 2 \log x = \log 2 + \log x^2 = \log(5x + 3)$. Therefore, $\log 2 + \log x^2 = \log(5x + 3)$. By the law of the logarithm of a product of two numbers which states that $\log(a \cdot b) = \log a + \log b$, $\log 2 + \log x^2 = \log 2x^2$. Therefore, $\log 2x^2 = \log(5x + 3)$. Hence, $2x^2 = 5x + 3$. Subtract $5x$ from both sides to obtain:

$$2x^2 - 5x = \cancel{5x} + 3 - \cancel{5x}.$$

Combining terms, $2x^2 - 5x = 3$. Subtract 3 from both sides to obtain:

$$2x^2 - 5x - 3 = 3 - 3.$$

Combining terms, $2x^2 - 5x - 3 = 0$. Factoring the left side of this equation into two polynomial factors, $(2x + 1)(x - 3) = 0$. Whenever $a \cdot b = 0$ where a and b are any two real numbers, either $a = 0$ or $b = 0$. Therefore, either

$$2x + 1 = 0 \quad \text{or} \quad x - 3 = 0.$$

$$2x = -1$$

$$\text{and} \quad x = -\tfrac{1}{2} \quad \text{or} \quad x = 3.$$

Since the domain of the logarithmic function is the set of positive real numbers, it is important to check all proposed solutions of a logarithmic equation. In this example, the given equation is satisfied for $x = 3$, but $x = -\frac{1}{2}$ is not a solution since $\log\left(-\frac{1}{2}\right)$ is not defined in the relation $\log 2 + 2 \log\left(-\frac{1}{2}\right) = \log \frac{1}{2}$.

**• PROBLEM 820**

> Solve for x: $2 \log (3 - x) = \log 2 + \log (22 - 2x)$.

Solution: We shall rewrite the equation in the form $\log M = \log N$ and then state that $M = N$. (If two numbers

Recall that $\log 10 = 1$ (since $10^1 = 10$). Therefore,

$$\log \frac{x^2}{30 - 2x} = \log 10.$$

Now since the logarithms of the two quantities are equal, the quantities are equal. Thus,

$$\frac{x^2}{30 - 2x} = 10.$$

Cross multiplying, we obtain:

$$x^2 = 10(30 - 2x) = 300 - 20x$$

Subtracting $x^2$ from both sides:

$$x^2 + 20x - 300 = 0$$

Factoring:

$$(x + 30)(x - 10) = 0$$

Setting each factor to zero:

$$x + 30 = 0, \qquad x - 10 = 0 \ ,$$

or

$$x = -30, \qquad x = 10$$

It is important that all solutions of a logarithmic equation be checked. Here we see that the negative number $x = -30$ is not permissible, for the first term of equation (1) has no meaning if $x$ is negative, because $2 \log x$ becomes $2 \log(-30)$, and the log of a negative quantity is meaningless. The value $x = 10$, however, when substituted in the left member of (1) gives us $2 \log 10 - \log 10 = 2 - 1 = 1$. Therefore, $x = 10$ is a valid solution.

**• PROBLEM 822**

Solve the equation $\log_{10}(x^2 + 3x) + \log_{10}5x = 1 + \log_{10}2x$.

Solution: We first subtract $\log_{10}2x$ from both sides of our equation so as to have the right-hand side free of logarithmic expressions and obtain

$$\log_{10}(x^2 + 3x) + \log_{10}5x - \log_{10}2x = 1$$

By the law of logarithms which states that $\log_b \frac{x}{y} = \log_b x - \log_b y$, $\log_{10}5x - \log_{10}2x = \log_{10} \frac{5x}{2x}$. Also, by the law of exponents which states that $\log_b(x \cdot y) = \log_b x + \log_b y$,

$$\begin{aligned}\log_{10}(x^2 + 3x) + \log_{10}5x - \log_{10}2x &= \log_{10}(x^2 + 3x) + \log_{10} \frac{5x}{2x} \\ &= \log_{10}(x^2 + 3x)\left(\frac{5x}{2x}\right) \\ &= \log_{10} \frac{5x(x^2 + 3x)}{2x} = 1\end{aligned}$$

Hence, $\log_{10}\frac{5x(x^2 + 3x)}{2x} = 1$ or $\log_{10} \frac{5(x^2 + 3x)}{2} = 1$. The expression $\log_b a = y$ is equivalent to $b^y = a$. Therefore, $\log_{10} \frac{5(x^2 + 3x)}{2} = 1$ is equivalent to $10^1 = \frac{5(x^2 + 3x)}{2}$. Hence, distributing:

$$10 = \frac{5x^2 + 15x}{2}$$

Multiply both sides of this equation by 2.

$$\log(x \cdot y) = \log x + \log y,$$

$$\log 100x = \log 100 + \log x.$$

Now, since log 100 can be equivalently written as $\log_{10} 100$, then $\log_{10} 100 = x$ or $10^x = 100$; and we can replace log 100 by 2 $(10^2 = 100)$.

Thus, we have: $2 + \log x$.

We can therefore write our equation as

$$(\log x)^2 = 2 + \log x,$$

and so it is equivalent to the equation

$$(\log x)^2 - \log x - 2, \quad \text{and factoring:}$$

$$= (\log x - 2)(\log x + 1) = 0.$$

Now, $\{(\log x - 2)(\log x + 1) = 0\}$

$$= \{x | \log x = 2 \text{ or } \log x = -1\}$$

$$= \{\log x = 2\} \cup \{\log x = -1\}.$$

Recall that when no base is expressed it is assumed to be 10. Thus, the equation $\log x = 2$ means $10^2 = x$, or $x = 100$; and $\log x = -1$ means $10^{-1} = x$, or $x = \frac{1}{10}$. Thus,

$$\{100\} \cup \left\{\frac{1}{10}\right\} = \left\{100, \frac{1}{10}\right\},$$

and this is the set of numbers that solves the given equation.

**• PROBLEM 824**

Solve the equation $27^{x^2+1} = 243$.

Solution: We seek all numbers which satisfy the equation. If $x$ is such a number, then

$$27^{x^2+1} = 243$$

Then, taking logarithms to the base 3 of both sides we have

$$\log_3 27^{x^2+1} = \log_3 243$$

Since $\log_b x^r = r \log_b x$, it follows that

$$(x^2 + 1)\log_3 27 = \log_3 243.$$

Note that the expression $\log_b x = y$ is equivalent to $b^y = x$. Hence, $\log_3 27 = N$ is equivalent to $3^N = 27$. Therefore, $N = 3$ and $\log_3 27 = 3$. Also, $\log_3 243 = M$ is equivalent to $3^M = 243$. Therefore, $M = 5$ and $\log_3 243 = 5$. Hence,

$$(x^2 + 1)3 = 5$$

or, by the commutative property of multiplication,

$$3(x^2 + 1) = 5.$$

Divide both sides of the equation by 3.

$$\frac{3(x^2 + 1)}{3} = \frac{5}{3}$$

$$x^2 + 1 = \frac{5}{3}$$

Subtract 1 from both sides of the equation.

$$x^2 + 1 - 1 = \frac{5}{3} - 1$$

$$x^2 = \frac{5}{3} - 1 = \frac{5}{3} - \frac{3}{3} = \frac{2}{3}$$

Therefore, $x = \pm\sqrt{\frac{2}{3}}$, i.e., $x = \sqrt{\frac{2}{3}}$ or $x = -\sqrt{\frac{2}{3}}$.

To check that each of these numbers satisfies the given equation, substitute each number for $x$ in the given equation. Substituting $\sqrt{\frac{2}{3}}$ for $x$:

$$\begin{aligned}(27)^{(\sqrt{2/3})^2 + 1} &= (27)^{(2/3)+1} = 27^{5/3} = \sqrt[3]{27^5} = \left(\sqrt[3]{27}\right)^5\\ &= (3)^5\\ &= 243 \checkmark\end{aligned}$$

Substituting $-\sqrt{\frac{2}{3}}$ for $x$:

$$\begin{aligned}(27)^{(-\sqrt{2/3})^2+1} &= (27)^{(2/3)+1} = 27^{5/3} = \sqrt[3]{27^5} = \left(\sqrt[3]{27}\right)^5\\ &= (3)^5\\ &= 243 \checkmark\end{aligned}$$

**• PROBLEM 825**

Solve $2^{x+1} = 7^{x+2}$.

__Solution:__ Take the logarithm of each side of the equation. $\log 2^{x+1} = \log 7^{x+2}$. By the law of the logarithm of a power of a positive number which states that

$\log a^n = n \log a$, $\log 2^{x+1} = (x+1)\log 2$ and $\log 7^{x+2} = (x+2)\log 7$. Therefore: $(x+1)\log 2 = (x+2)\log 7$. Distributing, $x \log 2 + \log 2 = x \log 7 + 2 \log 7$. Subtract $x \log 7$ from both sides to obtain:

$$x \log 2 + \log 2 - x \log 7 = \cancel{x \log 7} + 2 \log 7 - \cancel{x \log 7}$$

or $x \log 2 - x \log 7 + \log 2 = 2 \log 7$. Now, subtract $\log 2$ from both sides to obtain:

$$x \log 2 - x \log 7 + \cancel{\log 2} - \cancel{\log 2} = 2 \log 7 - \log 2$$

or $x \log 2 - x \log 7 = 2 \log 7 - \log 2$. Factoring out the common factor $x$ from the left side:

$$x(\log 2 - \log 7) = 2 \log 7 - \log 2.$$

Dividing both sides by $\log 2 - \log 7$:

$$\frac{x\,\cancel{(\log 2 - \log 7)}}{\cancel{\log 2 - \log 7}} = \frac{2 \log 7 - \log 2}{\log 2 - \log 7}$$

This is the required inverse function.

The form of relation (2) would seem to indicate that, if $x = x_1$ is the value corresponding to $y = y_1$, then $y = -y_1$ will also yield $x = x_1$. However, equation (1) shows that $y$ cannot be replaced by $-y$ without creating thereby a different functional relation. In fact, $y$ as given by (1) will be non-negative for every real value of $x$ in its permissible range, $0 < x \leq e^{-\frac{1}{2}}$. $x > 0$ because $\ln 0$ does not exist. If $x \neq e^{-\frac{1}{2}}$, for example $x = e$, the

$$\ln\left(1 + \sqrt{1 - e^2 e^4}\right)$$

does not exist. The functional relation $y = \ln\left(1 - \sqrt{1 - e^2 x^4}\right) - 2\ln x - 1$ yields only non-positive values of $y$ and likewise leads to the same inverse relation (2). If you select an $x$ from the domain $0 < x \leq e^{-\frac{1}{2}}$, for example $x = e^{-\frac{1}{2}}$, and substitute it in $y = \ln\left(1 - \sqrt{1 - e^2 x^4}\right) - 2\ln x - 1$, then $y = \ln\left(1 - \sqrt{1 - e^2\, e^{-4/2}}\right) - 2\ln e^{-\frac{1}{2}} - 1$

$$= \ln\left(1 - \sqrt{1 - e^0}\right) - 2(-\tfrac{1}{2}) - 1$$

$$y = \ln(1) + 1 - 1 = 0\,.$$

Therefore, $y$ is a non-positive value.

Our example thus illustrates the fact that all conclusions drawn from a deduced inverse function must be checked against the original relation.

**• PROBLEM 829**

> Solve for $x$ in the equation
> $$3251 = 2184(1.02)^x\,.$$

**Solution:** Taking the logarithm of each member of the given equation we obtain

$$\log 3251 = \log\left[2184(1.02)^x\right]$$

Since $\log(ab) = \log a + \log b$,

$$\log 3251 = \log 2184 + \log(1.02)^x$$

since $\log a^b = b \log a$,

$$\log 3251 = \log 2184 + x \log 1.02.$$

Adding ($-\log 2184$) to both sides,

$$x \log 1.02 = \log 3251 - \log 2184,$$

dividing both sides by $\log 1.02$,

$$x = \frac{\log 3251 - \log 2184}{\log 1.02}$$

Solving for our logarithms:

$$3{,}251 = 3.251 \times 10^3$$

Thus the characteristic is 3 and we interpolate to find log 3.251:

| | | Number | log | | |
|---|---|---|---|---|---|
| | | 3.250 | 5119 | | |
| .01 | .001 | 3.251 | ? | x | 13 |
| | | 3.260 | 5132 | | |

$$\log a^b = b \log a \text{ and,}$$

$$\log \frac{a}{b} = \log a - \log b. \text{ Thus,}$$

from $\log (.3)^x < \log \frac{4}{3}$ we have:

$$x \log (.3) < \log 4 - \log 3,$$

Express log (.3). The characteristic of the common logarithm of any positive number smaller than 1 is negative and is obtained by adding one more than the number of zeros between the decimal point and the first digit. The mantissa is obtained by looking it up in a table of common logarithms. For .3, the characteristic is - 1 since it is less than one and there are no zeros between the decimal point and the first digit. Its mantissa is .4771. Thus log .3 = (.4771 - 1). The log 4 and log 3 can be obtained from a table of mantissas of common logarithms.

Thus, solving the inequality we obtain:

$$x\,(.4771 - 1) < .6021 - .4771,$$

$$-\,.5229x < .1250,$$

$$x > -\,.239.$$

Therefore, $\{x > -\,.239\}$ is the solution set of our given inequality.

**• PROBLEM** 831

| | | |
|---|---|---|
| Solve | $5^{x+y} = 100$ | (1) |
| and | $2^{2x-y} = 10$ | (2) |
| for x and y. | | |

Solution: If we equate the common logarithms of the members of each of (1) and of (2), we get

$$\log 5^{x+y} = \log 100$$

$$\log 2^{2x-y} = \log 10.$$

Recalling that $\log_b a^y = y \text{ lob}_b\, a$,

$$\log 5^{x+y} = (x + y)\log 5 \text{ and } \log 2^{2x-y} = (2x - y)\log 2.$$

We also know $\log_{10} 100 = 2$ $\left(\text{since } 10^2 = 100\right)$ and

$$\log_{10} 10 = 1 \left(\text{since } 10^1 = 10\right).$$

Substituting these values into equations (1) and (2) we obtain

$$(x + y)\log 5 = 2 \qquad (3)$$

$$(2x - y)\log 2 = 1. \qquad (4)$$

If we solve these equations for x + y and 2x - y we obtain

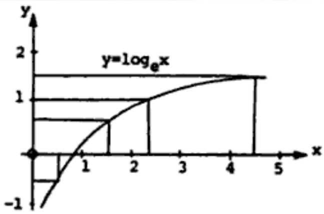
See a table of common mantissas of logarithms for the number corresponding to 0.0300. Note that the characteristic of 0.0300 which is zero, will be one less than the number of digits to the left of the decimal point of the corresponding number . Therefore, there is one digit to the left of the decimal point. Thus, the number is 1.072. Then we see that

$$N = 1.072 \times 10^9 = 1,072,000,000,$$

so a single cell is potentially capable of producing about a billion organisms in a 10-hour period.

**• PROBLEM 833**

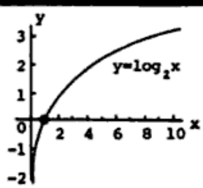
From the given graph find as well as you can (a) $\log_e 1.5$, (b) $\log_e .5$, (c) the number x for which $\log_e x = 1.5$, and (d) the value of e.




Solution: The smooth curve drawn is $y = \log_e x$. For (a) $\log_e 1.5$ and (b) $\log_e .5$, find the corresponding x-values, $x = 1.5$ and $x = .5$, and move along these vertical lines until you reach the curve, $y = \log_e x$. Then find the y-values from the corresponding projections onto the y - axis. We find that (a) $\log_e 1.5 = .4$, and (b) $\log_e .5 = -.7$. For (c), we are given the ordinate. Move along the horizontal line, $y = 1.5$, up to the curve of $\log_e x$, and then find its abscissa, which is 4.5. Thus, (c) $x = 4.5$. The number e satisfies the equation $\log_e x = 1$, and from the figure, it appears that $e = 2.7$ (actually, to 5 decimal places, $e = 2.71828$).

**• PROBLEM 834**

Construct the graph of $y = \log_2 x$ .



Solution: The equations $u = \log_b v$ and $v = b^u$ are equivalent. Therefore, the relation $y = \log_2 x$ is equivalent to $x = 2^y$. Hence we assume values of $y$ and compute the corresponding values of $x$, getting the table:

| x: | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | 1 | 2 | 4 | 8 |
|---|---|---|---|---|---|---|---|
| y: | -3 | -2 | -1 | 0 | 1 | 2 | 3 |

For example, if $y = -3$, then $x = 2^y = 2^{-3} = \frac{1}{2^3} = \frac{1}{8}$.

The points corresponding to these values are plotted on the coordinate system in the figure. The smooth curve joining these points is the desired graph of $y = \log_2 x$. It should be noted that the graph lies entirely to the right of the y-axis. The graph of $y = \log_b x$ for any $b > 1$ will be similar to that in the figure. Some of the properties of this function which can be noted from the graph are:

I. $\log_b x$ is not defined for negative values of $x$ or zero.

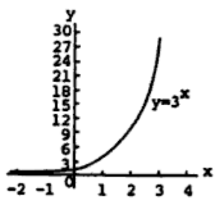
II. $\log_b 1 = 0$.

III. If $x > 1$, then $\log_b x > 0$.

IV. If $0 < x < 1$, then $\log_b x < 0$.

**• PROBLEM 835**

Construct the graph of $y = 3^x$.




Solution: Assume values of $x$ and compute the corresponding values of $y$ by substituting into $y = 3^x$, obtaining the following table of values:

| x: | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| y: | $\frac{1}{27}$ | $\frac{1}{9}$ | $\frac{1}{3}$ | 1 | 3 | 9 | 27 |

The points corresponding to these pairs of values are plotted on the coordinate system of the figure and these points are joined by a smooth curve, which is the desired graph of the function. Note that the values of $y$ are all positive. Furthermore, if $x < 0$, then $y$ increases to a small extent as $x$ does. If $x > 0$, $y$ increases at a more rapid rate.

**• PROBLEM 836**

Graph the following functions: (A) $y = 2^x$, (B) $y = 4^x$,

(C) $4^{-x}$, (D) $y = 3 \cdot 2^x$.

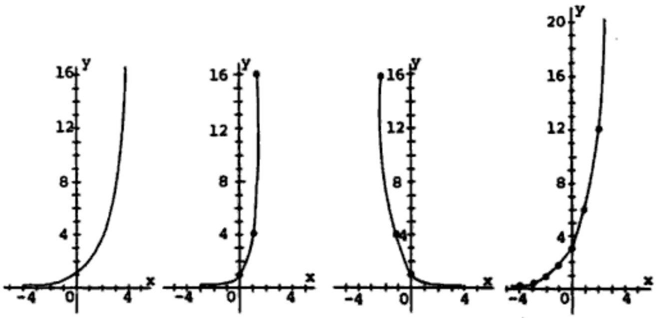


Fig. A Fig. B Fig. C Fig. D

(A) Solution: When graphing a function $y = f(x)$, set up a table consisting of two columns: one for x and one for y. Choose values for x and find the corresponding value for y. In this problem if:

$x = -4$, then $y = 2^x = 2^{-4} = \frac{1}{2^4} = \frac{1}{16}$.

Similarly, find other y values for different values of x. It is best to choose negative and positive values of x centering around and including zero to determine the nature of the graph.

| x | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|---|
| y | $\frac{1}{16}$ | $\frac{1}{8}$ | $\frac{1}{4}$ | $\frac{1}{2}$ | 1 | 2 | 4 | 8 | 16 |

Plot these points and draw a smooth curve through them. This is the graph of the exponential function $y = 2^x$.

Note that from the table and graph constructed, as you increase x by 1 each time moving from $x = -4$ to $x = 0$, the y values increase slightly. However, when you move through the positive values of x, the change in y is much greater for each unit change in x.

See Figure A.

(B) Solution: Construct a table in the same manner as problem A. The table of values for the integers - 3 to 3 can be determined to be:

| x | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| y | $\frac{1}{64}$ | $\frac{1}{16}$ | $\frac{1}{4}$ | 1 | 4 | 16 | 64 |

Then plot these points and draw a smooth curve.

Figure B is the graph of $y = 4^x$ although it is not practical to plot the points corresponding to $x = -3$ or $x = 3$ on this coordinate system.

When this curve is compared to the graph of $y = 2^x$ (Figure A), we see that the general shape is the same.

Both curves pass through the point (0,1), that is, both have a y-intercept of 1. If we consider the curves to the left of the y-axes, we see that the curve of Figure B approaches the x-axis faster than the curve of Figure A. If we consider the same negative value on both curves, the y-value in Figure B is smaller than in Figure A. If $x = -3$, then $y = 2^{-3} = \frac{1}{8}$ for Figure A and $y = 4^{-3} = \frac{1}{64}$ for Figure B. Thus (the) point $\left(-3, \frac{1}{64}\right)$ is closer to the x-axis than $\left(-3, \frac{1}{8}\right)$.

(C) Solution: Obtain a table of ordered pairs as in Examples A and B. Plot the points and draw the smooth curve by connecting them.

$$f(-3) = 4^{-(-3)} = 4^3 = 64$$

$$f(-2) = 4^{-(-2)} = 4^2 = 16$$

$$f(-1) = 4^{-(-1)} = 4^1 = 4$$

$$f(0) = 4^0 = 1$$

$$f(1) = 4^{-1} = \frac{1}{4}$$

$$f(2) = 4^{-2} = \frac{1}{4^2} = \frac{1}{16}$$

$$f(3) = 4^{-3} = \frac{1}{4^3} = \frac{1}{64}$$

| x | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| y | 64 | 16 | 4 | 1 | $\frac{1}{4}$ | $\frac{1}{16}$ | $\frac{1}{64}$ |

Figure C is the graph of $y = 4^{-x}$.

The graph of $y = 4^x$ of the Figure B and the graph of $y = 4^{-x}$ of Figure C are mirror images of each other.

(D) Solution: In Example A we determined the values of $2^x$ for x an integer and $-4 \le x \le 4$. The values of y for this function then must be three times the corresponding values of y of Example A.

| x | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|---|---|
| y | $\frac{3}{16}$ | $\frac{3}{8}$ | $\frac{3}{4}$ | $\frac{3}{2}$ | 3 | 6 | 12 | 24 | 48 |

The graph of this function is shown in Figure D.

From these four examples we can see some of the features of the graph of $y = ab^x$, $a > 0$, and $b > 0$. The y-intercept of the function is a: If $a > 1$, the curve will approach the x-axis to the left of the y-axis and the y value increases as the x value increases. The graph will be in quadrants I and II.

The y-intercept of Examples A, B, and C is 1 since $a = 1$. It is true for Examples A and B that the curve approaches the x-axis as x becomes more negative and the y-value increases as x increases. However when a = 1 in Example C, the reverse occurs. As x decreases, y increases and the curve approaches the x-axis as x becomes more positive.

# CHAPTER 27

# TRIGONOMETRY

**Basic Attacks and Strategies for Solving Problems in this Chapter.** **See pages 615 to 656 for step-by-step solutions to problems.**

In trigonometry it is convenient to think of an angle in terms of rotating a ray about its endpoint from some original position (vertical side) to a new position (terminal side). The size of an angle is usually described using degree or radian measure.

If an angle $A$ is in standard position and the point $(x, y)$ is any point on the terminal side $r$ of $A$, other than the origin, then six trigonometric functions of angle $A$ are defined as follows:

$\sin A = {}^{y}/_{r}$

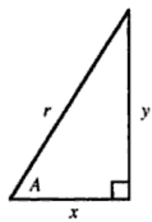
$\cos A = {}^{x}/_{r}$

$\tan A = {}^{y}/_{x}$

$\cot A = {}^{x}/_{y}$

$\sec A = {}^{r}/_{x}$

and $\csc A = {}^{r}/_{y}$.




An easy way to find the trigonometric function of an angle between 0 and 90 degrees, inclusive, is to use trigonometric function tables and/or a calculator. However, it may be necessary to first change degrees-minutes-seconds measure of the angle to the decimal form or radian form before using tables or the calculator. For example, $75° \ 12' \ 36''$ in decimal form is given by

$$75° + (12/60)° + (36/3600)° = 75° + (.2)° + (.01)° = 75.21°$$

and in radian form is about 1.31 radians.

If the angle is larger than 90 degrees, then the values of the trigonometric functions of the angle are found by first determining the corresponding or related reference angle for the given angle. Then, the values of the trigonometric functions of the reference angle are equal to the trigonometric functions of the original angle. The trigonometric functions of any angle $A > 90$ degrees are equal to those of the reference angle associated with angle $A$, except possibly for the sign. The sign can be determined by considering the quadrant in what the terminal side of angle $A$ lies. If the angle is larger than 360° or a negative angle (e.g., – 480°) smaller than – 360°, then a coterminal angle should be found before finding the reference angle. For example, to find tan (– 480°) note that the coterminal angles for – 480° is 240° since

$$-480° + 2(360°) = 240°.$$

The reference angle for 240° is 60° in the third quadrant and the tangent is positive in this quadrant. Thus,

$$\tan(-480°) = +\tan 60° = \text{square root of } 3.$$

There are two general types of trigonometric equations — identities and conditional. Trigonometric identity equations, like algebraic identities, are statements which are true for all permissible replacements of the variable, this is, for any angle for which the function is defined. Examples of identity equations are

$$\sec^2 x = \tan^2 x + 1 \quad \text{and} \quad \tan(x - y) = (\tan x - \tan y) / (1 + \tan x \cdot \tan y).$$

The second type of trigonometric equations, conditional equations, are true only for specific replacements of the variable. Examples of conditional equations are

$$4 \cos^2 x = 1 \quad \text{and} \quad \sin 2x + \sin x = 0.$$

Any trigonometric identity can be proved by three different procedures:

(1) The left-hand member may be reduced to the right-hand member,

(2) The right-hand member may be reduced to the left-hand member, or

(3) Each side may be separately reduced to the same form as long as the steps are reversible.

The series of steps used in the simplification process is by no means unique. So, no matter which side you start on, there may be a choice of next steps of which some may result in a shorter proof than others. Some suggestions for handling identities are as follows:

(1) Memorize the basic trigonometric relations and be able to recognize when they should be used.

(2) It is usually better to start with the side of the identity that appears to be more complicated. If both sides of the identity appear to be equally complicated, express each function in terms of sines or cosines and simplify.

(3) Look for places to use algebraic manipulations that do not involve division by zero.

(4) Keep in mind where you are heading and what steps might get you there.

Solving a triangle using trigonometric functions is straight forward if the triangle is a right triangle. The basic trigonometric relationships — sine, cosine, tangent and their respective reciprocals, cosecant, secant, and cotangent, are used. However, if there is an oblique triangle (that is, it either has three acute angles or one obtuse angle and two acute angles), then the solution technique involves the use of the law of sines or law of cosines. The law of sines is given by:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C},$$

where $a$, $b$, and $c$ represent the sides of the triangle and $A$, $B$, and $C$ represent the corresponding angle. In any event, we must be given three parts of the triangle, at least one of which is a side, in order to use the aforementioned laws. The law of cosines is given by:

$$a^2 = b^2 + c^2 - 2bc(\cos A);$$

$$b^2 = a^2 + c^2 - 2ac(\cos B); \text{ and}$$

$$c^2 = a^2 + b^2 - 2ab(\cos C),$$

where $a$, $b$, and $c$ represent the sides of the triangle and $A$, $B$, and $C$ represent the corresponding angles.

# Step-by-Step Solutions to Problems in this Chapter, "Trigonometry"

## ANGLES AND TRIGONOMETRIC FUNCTIONS

**• PROBLEM 837**

> If tan θ = 3.8436, find θ.

<u>Solution:</u> Looking through a table of trigonometric functions under the vertical column marked tan, it is found that the angle θ = 75°35' corresponds to the number 3.8436.

**• PROBLEM 838**

> Complete the following table:

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| Width of θ in radians | 0 | $\frac{1}{6}\pi$ | $\frac{1}{4}\pi$ | | $\frac{1}{2}\pi$ | $\frac{2}{3}\pi$ | | | $\pi$ |
| Width of θ in degrees | $0^\circ$ | | | $60^\circ$ | | | $135^\circ$ | $150^\circ$ | |

<u>Solution:</u> If an angle θ is A degrees wide and also t radians wide, then the numbers A and t are related by the equation:

$$\frac{A}{180^\circ} = \frac{t}{\pi} \qquad (1)$$

Thus, equation (1) can be used to complete the table. For column (2):

$$\frac{A}{180^\circ} = \frac{1/6\,\cancel{\pi}}{\cancel{\pi}}$$

$$\frac{A}{180^\circ} = 1/6$$

Multiplying both sides by $180^\circ$ ,

$$\cancel{180^\circ}\left(\frac{A}{\cancel{180^\circ}}\right) = 180^\circ(1/6)$$

$$A = 30^\circ$$

For column (3):

$$\frac{A}{180^\circ} = \frac{\frac{1}{4}\cancel{\pi}}{\cancel{\pi}}$$

$$\frac{A}{180^\circ} = \frac{1}{4}$$

Multiplying both sides by $180^\circ$,

$$\cancel{180^\circ}\left(\frac{A}{\cancel{180^\circ}}\right) = 180^\circ\left(\tfrac{1}{4}\right)$$

$$A = 45^\circ$$

For column (4):

$$\frac{60^\circ}{180^\circ} = \frac{t}{\pi}$$

$$\frac{1}{3} = \frac{t}{\pi}$$

Multiplying both sides by $\pi$,

$$\pi(1/3) = \cancel{\pi}(t/\cancel{\pi})$$

$$1/3\ \pi = t$$

For column (5):

$$\frac{A}{180^\circ} = \frac{\frac{1}{2}\cancel{\pi}}{\cancel{\pi}}$$

$$\frac{A}{180^\circ} = \frac{1}{2}$$

Multiplying both sides by $180^\circ$,

$$\cancel{180}^\circ\left(\frac{A}{180^\circ}\right) = 180^\circ(\tfrac{1}{2})$$

$$A = 90^\circ$$

For column (6):

$$\frac{A}{180^\circ} = \frac{2/3\ \cancel{\pi}}{\cancel{\pi}}$$

$$\frac{A}{180^\circ} = 2/3$$

Multiplying both sides by $180^\circ$,

$$\cancel{180}^\circ\left(\frac{A}{\cancel{180}^\circ}\right) = 180^\circ(2/3)$$

$$A = 120^\circ$$

For column (7):

$$\frac{135^\circ}{180^\circ} = \frac{t}{\pi}$$

$$\frac{27}{36} = \frac{t}{\pi}$$

$$\frac{3}{4} = \frac{t}{\pi}$$

Multiplying both sides by $\pi$,

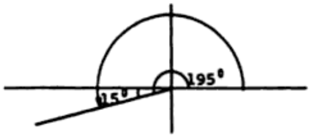
$$\pi(3/4) = \cancel{\pi}(t/\cancel{\pi})$$

$$3/4\ \pi = t$$

For column (8):

$$\frac{150^\circ}{180^\circ} = \frac{t}{\pi}$$

$$\frac{50}{60} = \frac{t}{\pi}$$




**Solution:** The reference angle for 195° is 15°. Also, 195° is a third quadrant angle (see Figure)

In the third quadrant, the sine and cosine functions are negative, but the tangent and the cotangent functions are positive. Therefore, sin 195° = sin 15° = -0.2588, cos 195° = cos 15° = -0.9659, tan 195° = tan 15°= 0.2679, and cot 195° = cot 15° = 3.7321. (Note that the values obtained for the trigonometric functions were found in a table of trigonometric functions.)

**• PROBLEM 842**

Find (a) tan 30° (b) tan 90°

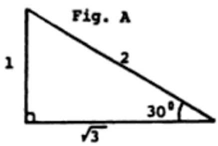




Fig. B

**Solution:** a) Recall $\tan\theta = \frac{\sin\theta}{\cos\theta}$, by definition; hence

$$\tan 30° = \frac{\sin 30°}{\cos 30°}$$

Looking at a 30-60 right triangle we find the values of sin 30° and cos 30°:

$$\sin = \frac{\text{opposite side}}{\text{hypotenuse}}\ ;\ \sin 30° = \tfrac{1}{2}$$

$$\cos = \frac{\text{adjacent side}}{\text{hypotenuse}}\ ;\ \cos 30° = \frac{\sqrt{3}}{2}$$

hence, $$\frac{\sin 30°}{\cos 30°} = \frac{\tfrac{1}{2}}{\frac{\sqrt{3}}{2}}$$

Multiplying numerator and denominator by 2:

$$= \frac{2(\tfrac{1}{2})}{2\left(\frac{\sqrt{3}}{2}\right)} = \frac{1}{\sqrt{3}} \quad \text{therefore } \tan 30° = \frac{1}{\sqrt{3}}$$

b) To determine tan 90° we can use the same method as in part a:

$$\tan 90° = \frac{\sin 90°}{\cos 90°}\text{, by definition of tangent.}$$

$= \frac{1}{0}$ .

However, since the quotient $\frac{1}{0}$ is undefined, we must therefore conclude that the tangent of 90° does not exist.

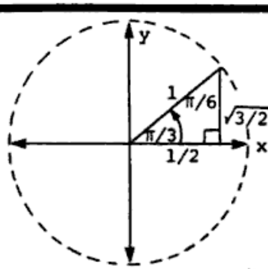
We may also observe the graph of the tangent function to determine tan 90°.

$$\tan 90° = \tan \frac{90 \cdot \pi}{180} \text{ radians} = \tan \frac{\pi}{2}$$

We observe that the tangent function does not exist on the line $\frac{\pi}{2}$ (the graph of the tangent function is asymptodic to the line $x = \frac{\pi}{2}$, but never touches it).

**• PROBLEM 843**

> **Calculate the values of the six trigonometric functions at the point $\frac{1}{3}\pi$ .**




**Solution:** To find the trigonometric point $P\left(\frac{1}{3}\pi\right)$, proceed around the unit circle in a counterclockwise direction, since $\frac{\pi}{3}$ is a positive angle. Recall that $\sin 60^{\circ}$ i.e., $\sin(\pi/3) = \sqrt{3}/2$. Now, using the Pythagorean theorem and the fact that the hypotenuse is unity because it is a unit circle we can compute the third side, which we find to be 1/2 (see figure). Therefore, the coordinates of the trigonometric point $P\left(\frac{1}{3}\pi\right)$ are $(1/2, 1/2\sqrt{3})$. Hence, we apply the following equations:

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{\text{hypotenuse}}{\text{adjacent side}}$$

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} \qquad \csc\theta = \frac{1}{\sin\theta} = \frac{\text{hypotenuse}}{\text{opposite side}}$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\text{adjacent side}}{\text{opposite side}}$$

Thus,

$$\cos\frac{1}{3}\pi = \frac{1}{2}, \qquad \sec\frac{1}{3}\pi = 2,$$

$$\sin\frac{1}{3}\pi = \frac{1}{2}\sqrt{3}, \qquad \csc\frac{1}{3}\pi = 2/\sqrt{3} = 2/\sqrt{3} \cdot \sqrt{3}/\sqrt{3} = 2/3\sqrt{3}$$

$$\tan\frac{1}{3}\pi = \sqrt{3}, \qquad \cot\frac{1}{3}\pi = 1/\sqrt{3} = 1/\sqrt{3} \cdot \sqrt{3}/\sqrt{3} = 1/3\sqrt{3}$$

**• PROBLEM 844**

**What primary angle is coterminal with the angle of $-743°$?**

Solution: $$-743° = \alpha - n \cdot 360° \qquad (1)$$

$$-743° = \alpha - 3 \cdot 360° = \alpha - 1080° \qquad (2)$$

Multiply both sides of equation (2) by $-1$.

$$-1(-743°) = -1(\alpha - 1080°)$$

$$743° = -\alpha + 1080°$$

$$743° = 1080° - \alpha \qquad (3)$$

Note that the positive integer value chosen for n results in an angle (in equation (3)) which is larger than but closest to the angle of 743°. Also,

$$0° \le \alpha \le 360°.$$

From equation (3),

$$\alpha = 1080° - 743° = 337°.$$

**• PROBLEM 845**

**What primary angle is coterminal with the angle of 1243°?**

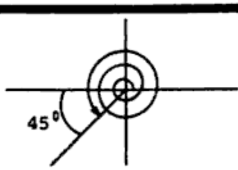
Solution: Using the formula, $\beta = \alpha + n \cdot 360°$, obtain $1243° = \alpha + n \cdot 360°$. In this formula choose the largest positive integer for n which, when multiplied by 360, is closest to but smaller than the given angle. Also, $\alpha$ is an angle between 0° and 360°, that is

$$0° \le \alpha \le 360°.$$

Since $3 \cdot 360° = 1080°$ and $4 \cdot 360° = 1440°$, $n = 3$, and $1243° = \alpha + 1080°$ or $\alpha = 1243° - 1080° = 163°$. Thus, the angles 1243° and 163° are coterminal.

**• PROBLEM 846**

**What primary angle is coterminal with the angle of $5\frac{1}{4}\pi$ radians?**

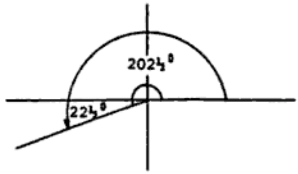



Solution: The figure illustrates the angle of $5\frac{1}{4}\pi$ radians.

Note that the angle of $5\frac{1}{4}\pi$ radians has a reference angle of 45°. However, we seek a primary angle (an angle between 0° and 360°) which is coterminal with $5\frac{1}{4}\pi$; that is, which has the same terminal side as an angle of $5\frac{1}{4}\pi$ radians. Also, since a primary angle is a positive angle, its initial side is the positive x-axis and the angle revolves in the counter-clockwise direction. Therefore, a primary angle with the same terminal side as an angle of $5\frac{1}{4}\pi$ radians is $(180° + 45°) = 225° = (\pi + \pi/4)$ radians $= 5/4\ \pi$ radians.

**• PROBLEM 847**

> Find $\sin 202\frac{1}{2}°$ and $\tan 202\frac{1}{2}°$.

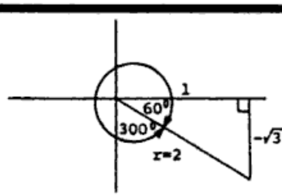



Solution: $202\frac{1}{2}°$ is a third quadrant angle. Thus its reference angle is $22\frac{1}{2}°$ $(202\frac{1}{2}° - 180° = 22\frac{1}{2}°)$. See the figure.

In the third quadrant, the sine function is negative while the tangent function is positive. Therefore, using a table of trigonometric functions, $\sin 202\frac{1}{2}° = -\sin 22\frac{1}{2}° = -\sin 22°30' = -0.3827$ and $\tan 202\frac{1}{2}° = \tan 22\frac{1}{2}° = \tan 22°30' = 0.4142$.

**• PROBLEM 848**

> Find the values of the trigonometric functions of an angle of 300°.




Solution: An angle of 300° is a fourth quadrant angle and its reference angle is an angle of 60°. In the fourth quadrant the sine, tangent, cotangent and cosecant functions are negative. This yields

$$\sin 300° = \sin 60° = -\frac{\sqrt{3}}{2},$$

$$\cos 300° = \cos 60° = \frac{1}{2},$$

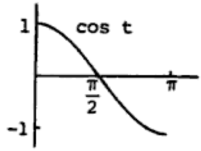
$$\tan 300° = \tan 60° = -\sqrt{3},$$

$$\cot 300° = \cot 60° = -\frac{\sqrt{3}}{3},$$

$$\sec 300° = \sec 60° = \frac{1}{\cos 60°} = \frac{1}{\frac{1}{2}} = 2, \text{ and}$$

$$\csc 300° = \csc 60° = \frac{1}{\sin 60°} = \frac{1}{-\sqrt{3}/2} = -\frac{2}{\sqrt{3}}$$

If $0 < t < \frac{1}{2}\pi$, and $\tan t = \frac{1}{2}\sqrt{5}$, find $\cos t$.




Solution: One way to find cos t is to express $\frac{1}{2}\sqrt{5}$ as a decimal, use a table of trigonometric functions to solve the equation $\tan t = \frac{1}{2}\sqrt{5}$ for t, and then again use the table to find cos t. But it is much easier to use some trigonometric identities to solve this problem.

Since we are given that $\tan t = \frac{1}{2}\sqrt{5} = \frac{\sqrt{5}}{2}$, we can use the identity $\sec^2 t = 1 + \tan^2 t$ to find $\sec^2 t$. Substituting, we obtain:

$$\sec^2 t = 1 + (\tan t)^2$$

$$\sec^2 t = 1 + \left(\frac{\sqrt{5}}{2}\right)^2 \quad \text{, or}$$

$$\sec^2 t = 1 + \frac{5}{4} = \frac{4}{4} + \frac{5}{4} = \frac{9}{4}$$

Thus, since $\cos t = \frac{1}{\sec t}$, then

$$\cos^2 t = 1/\sec^2 t = \frac{1}{\frac{9}{4}} = \frac{4}{9} \text{ ; therefore}$$

$$\cos t = \sqrt{\frac{4}{9}} = \pm \frac{2}{3}$$

Since $0 < t < \frac{1}{2}\pi$, we know that $\cos t > 0$ (see graph) and hence we reject the negative value, $-\frac{2}{3}$, and it follows from the last equation that $\cos t = \frac{2}{3}$.

## TRIGONOMETRIC INTERPOLATIONS



Find $\cos 37°12'$.

**Solution:** See a table of natural trigonometric functions, which is constructed in terms of multiples of ten seconds. The cosine of 37°12' lies between 37°10' and 37°20'. Therefore, we must interpolate. The cosine decreases as the angle increases, so we form our proportion as follows, where

x = the cosine of the angle 37°12'

d = the difference between the cos 37°10' and cos 37°12':

$$
10' \left[ \begin{array}{l} \cos 37^\circ 10' = 0.7969 \\ 2' \left[ \cos 37^\circ 12' = \quad x \right] d \\ \cos 37^\circ 20' = 0.7951 \end{array} \right] -0.0018
$$

$$\frac{2}{10} = \frac{d}{-0.0018}$$

Cross multiply to obtain:

$$10d = 2(-0.0018)$$

$$d = .2(-0.0018)$$
$$= -0.00036$$
$$d \approx -0.0004$$

Thus,
$$x = 0.7969 - 0.0004$$
$$= 0.7965$$

Since the cosine is positive in the first quadrant,

$$\cos 37^\circ 12' = 0.7965$$

Remember that results obtained by interpolation are approximations. You should not use an answer that is more accurate than the original data, in this case, four significant digits.

**• PROBLEM 856**

Find the value of tan 38°46' by use of interpolation.

**Solution:** Since 38°46' is between 38°40' and 38°50', we assume that tan 38°36' is between tan 38°40' and tan 38°50'. In fact, since 38°46' is six-tenths of the way from 38°40' toward 38°50', we assume that tan 38°46' is six-tenths of the way from tan 38°40' = .8002 toward tan 38°50' = .8050. Using these assumptions we perform the following interpolation:

$$
10' \left[ \begin{array}{l} 6' \left[ \begin{array}{l} \tan 38^\circ 40' = .8002 \\ \tan 38^\circ 46' = \quad ? \end{array} \right] c \\ \tan 38^\circ 50 = .8050 \end{array} \right] .0048
$$

Set up the proportion $\frac{c}{.0048} = \frac{6}{10}$

$$10c = 6(.0048)$$

$$c = \frac{6}{10}(.0048) = .0029$$

$$\tan 38°46' = .8002 + .0029 = .8031$$

Therefore, c was added because tan θ increases from

$$\theta = 38°40' \text{ to } \theta = 38°50'.$$

**• PROBLEM 857**

Find tan 63°19.27'.

Solution: The value of tan 63°19.27' can be found by interpolating the values of tan 63°19' and tan 63°20'.

$$\begin{array}{ll|ll}
 & \text{Degree} & \text{Value of function} & \\
1 \left[\begin{array}{l} .27\left[\begin{array}{l} 63°19' \\ 63°\ 19.27' \end{array}\right. \\ \quad 63°\ 20' \end{array}\right. & & \left.\begin{array}{r} \left.\begin{array}{r} 1.9897 \\ x \end{array}\right] d \\ 1.9912 \end{array}\right] 0.0015 &
\end{array}$$

Now, set up the following proportion:

$$\frac{d}{0.0015} = \frac{.27}{1} = .27$$

$$d = (.27)(0.0015)$$

$$d = 0.0004.$$

Therefore, $x = \tan 63°19.27' = \tan 63°19' + d$

$$= 1.9897 + 0.0004$$

$$\tan 63°19.27' = 1.9901.$$

**• PROBLEM 858**

Find θ if sin θ = .6212, and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

Solution: Since 0.6212 is not found in the sine table, we proceed by finding the two numbers closest to .6212, one greater and the other less than it, and interpolating.

$$10'\left[\begin{array}{l} c\left[\begin{array}{l} \sin 38°20' \\ \sin \theta \end{array}\right. \\ \quad \sin 38°30' \end{array}\right. \begin{array}{l} \left.\begin{array}{l} = .6302 \\ = .6212 \end{array}\right] .0010 \\ = .6225 \end{array} \Bigg] .0023$$

We set up the proportion $\frac{c}{10'} = \frac{.0010}{.0023}$.

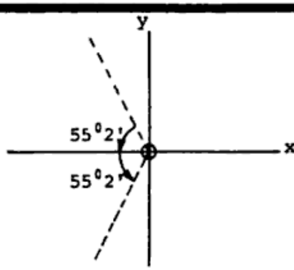
$$c = \frac{.0010}{.0023}(10') = \frac{10}{23}(10')$$

$$\theta = 4' \text{ to the nearest minute.}$$

Thus, $\theta = 38°20' + 4'$

$$= 38°24'.$$

Find $\theta$ if $\cos\theta = -0.5731$ and $0 \le \theta \le 360°$.




Solution: Since 0.5731 is not found in the cosine table, we proceed by finding the two numbers closest to .5731, one greater and the other less than it, and interpolating. Notice that the cosine function decreases as the angle increases.

$$10'\left[\;d\left[\begin{array}{ll}\cos 55°0' & = 0.5736 \\ \cos x & = 0.5731\end{array}\right]0.0005 \\ \cos 55°10' = 0.5712\right]0.0024$$

Setting up the proportion, $\frac{d}{10} = \frac{0.0005}{0.0024} = \frac{5}{24}$. Cross multiplying, $24d = 50$

$$d = 2' \text{ to the nearest minute.}$$

Thus, $x = 55°0' + 2' = 55°2'$. Since we are given a negative cosine, and cosine is negative in the second and third quadrants, we know that our reference angle $55°2'$ appears in the second or third quadrant (see diagram). Hence

$$= 180° - 55°2' = 124°58', \text{ or}$$

$$= 180° + 55°2' = 235°2'.$$



Find a solution of the equation $\cos t = .6241$.

Solution: We see that .6241 is not found in a trigonometric table; therefore, interpolation is necessary. We choose the two entries in the trigonometric table which .6241 lies between, and arrange the numbers as follows:

$$\cos .89 = .6294,$$

$$\cos t = .6241,$$

$$\cos .90 = .6216.$$

We obtain,

$$.01\left[\begin{array}{l} x\left[\begin{array}{r}\cos .89 = .6294\\ \cos t = .6241\end{array}\right]53\\ \quad\cos .90 = .6216\end{array}\right]78$$

by subtracting .89 from .90 (.90 - .89 = .01) and calling the difference between .89 and t, x. Also, .6294 - 6241 = .0053, and .6294 - .6216 = .0078. Since both .0053 and .0078 are four decimal places we can rewrite them as 53 and 78, without changing the value of the following proportion. Now,

$$\frac{x}{.01} = \frac{53}{78}$$

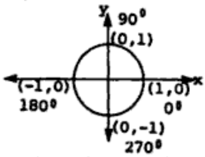
$$78x = 53(.01) = .53$$

$$x = \frac{.53}{78} \simeq .007$$

The cost lies between .89 and .90; thus, t = .89 +.007 = .897.

**• PROBLEM 861**

Find log sin 36°41′.




Solution: From a table of logarithms of trigonometric functions we find log sin 36°40′ and log sin 36°50′. Then, by the process of interpolation we find: log sin 36°41′ = 9.77626 - 10. Since all values of the sine function for acute angles are in the range of $0 \leq \sin x \leq 1$, the characteristic is negative. (Recall that for a number less than one, the characteristic is negative.) The range of sine can be seen by inspecting the accompanying figure. Sine is given by the y coordinate; cos is given by the x coordinate. Observe that y value varies from 0 to 1, as the angle varies from 0° to 90°.

**• PROBLEM 862**

Find log cos 49°13.6′.

Solution: First consult a table of logarithms of trigonometric functions.

Notice that 49°13.6′ lies between 49°10′ and 49°20′, so that the log of 49°13.6′ will occur between the logs of 49°10′ and 49°20′ and can be determined by interpolation.

$$\begin{array}{lll} & \text{angle} & \text{log cosine}\\ 10'\left[\begin{array}{l} \\ 3.6'\left[\begin{array}{l}\\ \end{array}\right.\end{array}\right. & \begin{array}{l}49°20'\\ 49°13.6'\\ 49°10'\end{array} & \left.\begin{array}{l}9.8140^{-10}\\ \left.\begin{array}{l}\\ 9.8155^{-10}\end{array}\right]x\end{array}\right] .0015 \end{array}$$

To combine terms, we convert cos θ into a fraction whose denominator is cos θ, thus

$$= \frac{\sin^2 \theta}{\cos \theta} + \left(\frac{\cos \theta}{\cos \theta}\right) \cdot \cos \theta. \quad \text{(Note that}$$

cos θ/cos θ equals one, so the equation is unaltered)

$$= \frac{\sin^2 \theta}{\cos \theta} + \frac{\cos^2 \theta}{\cos \theta}$$

$$= \frac{\sin^2 \theta + \cos^2 \theta}{\cos \theta}.$$

Recall the identity $\sin^2 \theta + \cos^2 \theta = 1$; hence,

$$= \frac{1}{\cos \theta}$$

$$= \sec \theta.$$

**• PROBLEM 865**

Reduce the expression $\dfrac{\tan x - \cot x}{\tan x + \cot x}$ to one involving only sin x.

Solution: Since, by definition, $\tan x = \dfrac{\sin x}{\cos x}$ and

$$\cot x = \frac{1}{\tan x} = \frac{1}{\sin x/\cos x} = \frac{\cos x}{\sin x},$$

$$\frac{\tan x - \cot x}{\tan x + \cot x} = \frac{\dfrac{\sin x}{\cos x} - \dfrac{\cos x}{\sin x}}{\dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x}}$$

$$= \frac{\dfrac{\sin x(\sin x)}{\sin x(\cos x)} - \dfrac{\cos x(\cos x)}{\cos x(\sin x)}}{\dfrac{\sin x(\sin x)}{\sin x(\cos x)} + \dfrac{\cos x(\cos x)}{\cos x(\sin x)}}$$

$$= \frac{\dfrac{\sin^2 x - \cos^2 x}{\sin x \cos x}}{\dfrac{\sin^2 x + \cos^2 x}{\sin x \cos x}}$$

$$= \frac{\sin^2 x - \cos^2 x}{\sin x \cos x} \times \frac{\sin x \cos x}{\sin^2 x + \cos^2 x}$$

$$= \frac{\sin^2 x - \cos^2 x}{\sin^2 x + \cos^2 x}$$

Since $\sin^2 x + \cos^2 x = 1$ or $\cos^2 x = 1 - \sin^2 x$,

$$\frac{\tan x - \cot x}{\tan x + \cot x} = \frac{\sin^2 x - \cos^2 x}{\sin^2 x + \cos^2 x} = \frac{\sin^2 x - \cos^2 x}{1}$$

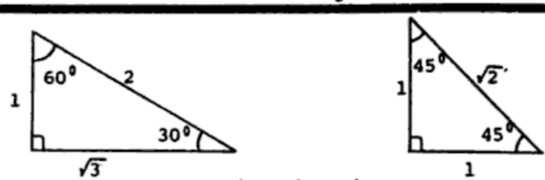
$$= \sin^2 x - \cos^2 x$$

$$= \sin^2 x - (1 - \sin^2 x)$$

$$= \sin^2 x - 1 + \sin^2 x$$

$$= 2 \sin^2 x - 1.$$

**Find $\sin 105^\circ$ without the use of a trig. table.**




**Solution:** We note that $105^\circ = 60^\circ + 45^\circ$ and find the sine of the sum of two angles.

$$\sin 105^\circ = \sin(60^\circ + 45^\circ)$$

Using the formula for the sine of the sum of two numbers,

$$\sin(x+y) = \sin x \cos y + \cos x \sin y, \ \sin(60^\circ + 45^\circ)$$
$$= \sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ .$$

Now we must find the values of $\sin 60^\circ$, $\cos 45^\circ$, $\cos 60^\circ$, and $\sin 45^\circ$. Observing a 30-60 and 45-45 right triangle we note:

$$\sin = \frac{\text{opposite}}{\text{hypotenuse}} ; \quad \text{thus,} \quad \sin 60^\circ = \frac{\sqrt{3}}{2}$$
$$\sin 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos = \frac{\text{adjacent}}{\text{hypotenuse}} ; \quad \text{thus,} \quad \cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$
$$\cos 60^\circ = \frac{1}{2}$$

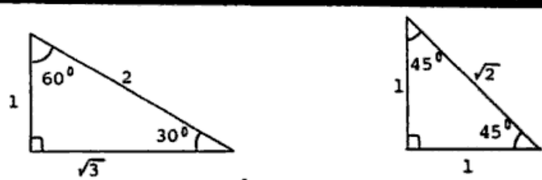
Substituting, we obtain:

$$\frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{1}{2} \cdot \frac{\sqrt{2}}{2}$$

Multiply the fractions (recall $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$) to obtain $\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$ $= \frac{\sqrt{6} + \sqrt{2}}{4}$. Therefore, $\sin 105^\circ = \frac{\sqrt{6} + \sqrt{2}}{4}$ .



**Find $\cos 15^\circ$, without the use of a Trig. table.**




**Solution:** We express $\cos 15^\circ$ as the cosine of the difference of two angles whose cosine and sine we know. Since we know $15^\circ = 45^\circ - 30^\circ$, then $\cos 15^\circ = \cos(45^\circ - 30^\circ)$. Now we apply the formula for the cosine of the difference of two angles, which states:

$$\cos(u - v) = \cos u \cos v + \sin u \sin v\ .$$

Thus, $\cos 15^\circ = \cos(45^\circ - 30^\circ) = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$.
Now, we must find the values for $\cos 45^\circ$, $\cos 30^\circ$, $\sin 30^\circ$. This can be accomplished by observing the 45-45 and 30-60 right triangles.

Since $\cos = \dfrac{\text{adjacent}}{\text{hypothenuse}}$ we find: $\cos 45^\circ = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

$$\cos 30^\circ = \frac{\sqrt{3}}{2}\ ,$$

and since $\sin = \dfrac{\text{opposite}}{\text{hypotenuse}}$ we find: $\sin 45^\circ = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$

$$\sin 30^\circ = \frac{1}{2}$$

Substituting, we obtain:

$$\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2}\ .$$

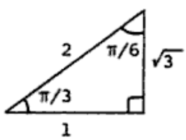
Multiplying the fractions (recall $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$) we obtain:

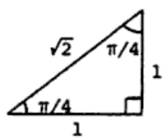
$$\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{\sqrt{6} + \sqrt{2}}{4}\ .$$

Therefore, $\cos 15^\circ = \dfrac{\sqrt{6} + \sqrt{2}}{4}$ .

**• PROBLEM 868**

Find $\cos \dfrac{1}{12}\pi$.




<u>Solution:</u> Express $\cos \dfrac{\pi}{12}$ in terms of angles whose values of the trigonometric functions are known.

$$\begin{aligned}\cos \frac{1}{12}\pi &= \cos\left(\frac{4}{12}\pi - \frac{3}{12}\pi\right)\\ &= \cos\left(\frac{1}{3}\pi - \frac{1}{4}\pi\right)\end{aligned}$$

Now apply the difference formula for the cosine of two angles, $\alpha$ and $\beta$. $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.
In this example, $\alpha = \dfrac{1}{3}\pi$ and $\beta = \dfrac{1}{4}\pi$.

$$= \frac{\sqrt{2}(-\sqrt{3} + 1)}{4}$$

Hence, $\sin 195^\circ = \frac{\sqrt{2}(1 - \sqrt{3})}{4}$.

**• PROBLEM 873**

> Derive a formula for $\sin 3\alpha$ in terms of $\sin \alpha$.

Solution: We may regard $3\alpha$ as $\alpha + 2\alpha$, and use the addition formula for the sine of two angles.

$$\sin(a + b) = \sin a \cos a + \cos a \sin b$$

where $a = \alpha$ and $b = 2\alpha$.

$$\sin 3\alpha = \sin(\alpha + 2\alpha) = \sin \alpha \cos 2\alpha + \cos \alpha \sin 2\alpha .$$

Now replace $\sin 2\alpha$ and $\cos 2\alpha$ by the expressions

$$\cos 2a = \cos^2 a - \sin^2 a$$

and $$\sin 2a = 2 \sin a \cos a .$$

We find that

$$\begin{aligned}\sin 3\alpha &= (\sin \alpha)\left(\cos^2\alpha - \sin^2\alpha\right) + (\cos \alpha)(2 \sin \alpha \cos \alpha)\\ &= \sin \alpha \cos^2\alpha - \sin^3\alpha + 2 \sin \alpha \cos^2\alpha\\ &= 3 \sin \alpha \cos^2\alpha - \sin^3\alpha .\end{aligned}$$

Finally, since we wish a result involving only $\sin \alpha$, replace $\cos^2\alpha$ by $1 - \sin^2\alpha$; then

$$\begin{aligned}\sin 3\alpha &= 3(\sin \alpha)\left(1 - \sin^2\alpha\right) - \sin^3\alpha\\ &= 3 \sin \alpha - 3 \sin^3\alpha - \sin^3\alpha\\ &= 3 \sin \alpha - 4 \sin^3\alpha .\end{aligned}$$

This is the desired identity.

**• PROBLEM 874**

> Simplify the expression
>
> $$2 \sin\left(\frac{3\pi}{2} - \theta\right) - 3 \cos(\pi + \theta) - \tan(-\theta) + \cot\left(\frac{\pi}{2} + \theta\right) .$$

Solution: Apply the following laws of subtraction and addition of trigonometric functions:

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$
$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

Derive $\cot(\alpha + \beta)$ from the addition formula for the tangent:

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1-\tan \alpha \tan \beta}$$

Since $\cot(\alpha + \beta) = \frac{1}{\tan(\alpha + \beta)}$, we have: $\cot(\alpha + \beta)$

$$= \frac{1 - \tan \alpha \tan \beta}{\tan \alpha + \tan \beta}$$

$$= \frac{1 - \frac{1}{\cot \alpha \cot \beta}}{\frac{1}{\cot \alpha} + \frac{1}{\cot \beta}}$$ Multiplying all terms by $\cot \alpha \cot \beta$,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \qquad \cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos A = \frac{23^2 + 32^2 - 17^2}{(2)(23)(32)} \qquad \cos B = \frac{17^2 + 32^2 - 23^2}{(2)(17)(32)}$$

$$\cos A = 0.8587 \qquad \cos B = 0.7206.$$

$\angle A = 31°$ (nearest degree) $\quad \angle B = 44°$ (nearest degree)

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos C = \frac{17^2 + 23^2 - 32^2}{(2)(17)(23)}$$

$$\cos C = -\,0.2634$$

$\angle C = 105°$ (nearest degree)

It is not necessary to calculate angle C using the law of cosines when angles A and B are known. Angle C can be found instead by the following:

$\angle A + \angle B + \angle C = 180°$ since the sum of the angles of a triangle is 180°,

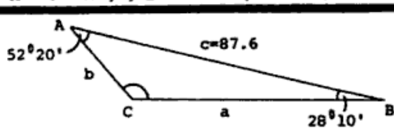
$$\text{or } 31° + 44° + \angle C = 180°$$

$$75° + \angle C = 180°$$

$$\angle C = 180° - 75° = 105°.$$

**• PROBLEM 882**

Solve triangle ABC for $\angle$ C, a, and b, given
$\angle A = 52°20'$, $\angle B = 28°10'$, $c = 87.6$




Solution: Given two angles, the third is determined, since the three angles of a triangle equal 180°. Thus,

$$\angle A + \angle B + \angle C = 180°.$$

$$\angle C = 180° - (\angle A + \angle B)$$

$$\angle C = 180° - 80°30' = 179°60' - 80°30' = 99°30'.$$

Sides a and b can be determined from the law of sines. In any triangle, the sides are proportional to the sines of the opposite angles (see figure). Then:

$$\frac{a}{\sin A} = \frac{c}{\sin C} \qquad \frac{b}{\sin B} = \frac{c}{\sin C}$$

Now, since $\angle$ A, $\angle$ B, and side C were given, and we previously determined $\angle$ C, substituting into these proportions, and using a table of trigonometric functions, will give us a and b. Thus,

$$\frac{\sin B}{\text{side } b} = \frac{\sin C}{\text{side } c}$$

to find side b:

$$\frac{\sin 113^\circ 40'}{b} = \frac{\sin 23^\circ}{78}$$

Cross multiplying we obtain

$$b \sin 23^\circ = 78 \sin 113^\circ 40'$$

$$b = \frac{78 \sin 113^\circ 40'}{\sin 23^\circ}$$

Substituting in the values $\sin 113^\circ 40' = 0.9159$ and $\sin 23^\circ = 0.3907$

we obtain

$$b = 78\left(\frac{0.9159}{0.3907}\right) = 183$$

Hence if we choose $< A = 43^\circ 20'$ then

$$< B = 113^\circ 40' \text{ and}$$

$$\text{side } b = 183$$

If, however, we choose $< A = 136^\circ 40'$, then since

$$< A + < B = 157^\circ$$

$$136^\circ 40' + < B = 157^\circ$$

and

$$< B = 157^\circ - 136^\circ 40' = 20^\circ 20'$$

Applying the law of sines to find side b gives us:

$$\frac{\sin B}{\text{side } b} = \frac{\sin C}{\text{side } c}$$

$$\frac{\sin 20^\circ 20'}{b} = \frac{\sin 23^\circ}{78}$$

Cross multiplying gives us

$$b \sin 23^\circ = 78 \sin 20^\circ 20'$$

$$b = \frac{78 \sin 20^\circ 20'}{\sin 23^\circ}$$

Substituting in $\sin 20^\circ 20' = 0.3475$ and $\sin 23^\circ = 0.3907$ we obtain

$$b = \frac{78(0.3475)}{0.3907} = 69$$

Hence if we choose $< A = 136^\circ 40'$, then $< B = 20^\circ 20'$, and

side $b = 69$ .

• **PROBLEM 887**

Find all the sides and angles of triangle ABC, given $a = 43$, $b = 32$, $< B = 67^\circ$ .

**Solution:** Draw triangle ABC, filling in the given information. Thus

**Solution:** The Law of Tangents states:

$$\frac{b - c}{b + c} = \frac{\tan \frac{1}{2}(\beta - \gamma)}{\tan \frac{1}{2}(\beta + \gamma)}$$

We use this relation here to find the remaining angles, $\beta$ and $\gamma$. We know two sides, b and c, and we can derive the two unknown angles from the law of tangents and the fact that the sum of the angles in a triangle is 180°. From this last fact, we know the sum of the unknown angles:

$$\beta + \gamma = 180° - \alpha, \text{ or } \beta + \gamma = 180° - 42°19.8'$$

$$= 179°60.0' - 42°19.8' = 137°40.2'$$

$\frac{1}{2}(\beta + \gamma) = \frac{1}{2}(137°40.2')$. Now dividing by 2 we have 68°, with 1° remaining. Since 1° = 60′, we have 60′ + 40.2′ = 100.2′, and divided by 2 = 50.1′. Thus,

$$\tfrac{1}{2}(\beta + \gamma) = 68°50.1'.$$

Now, since $b - c = 16.351 - 11.189 = 5.162, b+c=27.540,$

$$\tfrac{1}{2}(\beta + \gamma) = 68°50.1',$$

substituting into the law of tangents and solving for $\tan \frac{1}{2}(\beta - \gamma)$:

$$\tan \tfrac{1}{2}(\beta - \gamma) = \frac{5.162}{27.54} \tan 68°50.1'$$

Now, using a table of values of trigonometric functions we find that tan 68°50′ = 2.583. We accept this value also as tan 68°50.1′, because the two angles differ only by .1, and even interpolation will not give us a different value. Thus, we have:

$$\tan \tfrac{1}{2}(\beta - \gamma) = \frac{5.162}{27.54}(2.583),$$

and performing the multiplication:

$$\tan \tfrac{1}{2}(\beta - \gamma) = .4841$$

Now, we again refer to a table of trigonometric functions. We are looking for the angle whose tangent is .4841 (or the arctangent of .4841). We find that, tan 25°50′= .4841. Thus,

$$\tfrac{1}{2}(\beta - \gamma) = \text{Arctan } .4841, \text{ or}$$

$$\tfrac{1}{2}(\beta - \gamma) = 25°50'$$

We now solve the equations

$$\tfrac{1}{2}(\beta + \gamma) = 68°50.1'$$

$$\tfrac{1}{2}(\beta - \gamma) = 25°50'$$

simultaneously for $\beta$ and $\gamma$ by distributing $\frac{1}{2}$, and then adding:

$$\frac{c}{\sin \gamma} = \frac{b}{\sin \beta},$$ and substituting:

$$\frac{c}{\sin 153°44'} = \frac{47}{\sin 5°33'}$$

$$c = \sin 153°44' \left(\frac{47}{\sin 5°33'}\right)$$ . Taking the

logarithm of both sides, we obtain:

$$\log c = \log \sin 153°44' + (\log 47 - \log \sin 5°33')$$

To solve for log c we must find log sin 153°44', and substitute this value, and those found previously for log 47 and log sin 5°33'. To obtain sin 153°44' we first observe the coordinate axis, (see fig. 2), and discover that sin 153°44' = sin 26°16'. We now use interpolation:

$$10\left[\begin{array}{l} 6\left[\begin{array}{lll} 26°10' & \quad & 9.6444 - 10 \\ 26°16' & & ? \end{array}\right] y \\ \quad\; 26°20' \qquad\quad 9.6470 - 10 \end{array}\right] .0026$$

$$\begin{array}{cc} \underline{x} & \underline{\log \sin x} \end{array}$$

$$\frac{6}{10} = \frac{y}{.0026}$$

$$10y = .0156$$

$$y = .00156$$

$$\log \sin 26°16' = \log \sin 153°44' = 9.6444 - 10 + .00156$$

$$= 9.64596 - 10$$

Thus, since log 47 was found to equal 1.67210 and log sin 5°33' = 8.98547 - 10, we have:

$$\log c = 9.64596 - 10 + [1.67210 - (8.98547 - 10)]$$

$$= 9.64596 - 10 + [11.67210 - 10 - (8.98547 - 10)]$$

$$= 9.64596 - 10 + (2.68663)$$

$$= 9.64596 - 10 + (12.68663 - 10)$$

$$= 22.33259 - 20$$

$$\log c = 2.33259$$

Interpolating, to find c:

$$\begin{array}{cc} \underline{x} & \underline{\log x} \end{array}$$

$$1\left[\begin{array}{l} y\left[\begin{array}{lll} 215. & \quad & 2.3324 \\ ? & & 2.33259 \end{array}\right] .00019 \\ \quad 216. \qquad\quad 2.3345 \end{array}\right] .0021$$

# CHAPTER 28

# INVERSE TRIGONOMETRIC FUNCTIONS

**Basic Attacks and Strategies for Solving Problems in this Chapter. See pages 657 to 670 for step-by-step solutions to problems.**

The inverses of functions defined by formulas involving trigonometric functions can be determined in the same way we determine the inverses of other functions. That is, interchange the independent and dependent variables and solve for the new dependent variable. Note that the elements of the ranges of the inverse trigonometric functions are angles in standard position.

When evaluating the inverse function of the sine, cosine, and tangent, it helps to first remember that the arc (sine/cosine/tangent), respectively, of $x$ is the angle (or number) whose sine/cosine/tangent, respectively, is $x$. Secondly, to find the value of an inverse trigonometric function, apply its definition and determine which quadrant(s) the angle terminates. Then, calculate or find the $y$ value (angle) which is the solution. The use of a sketched graph is helpful to visualize the positions of the angle.

For example, to find the value of

$$\sin^{-1}(^1/_2) \quad \text{or} \quad \arcsin(^1/_2).$$

Set $y = \sin^{-1}(^1/_2)$. Then, by definition of the inverse sine function, $\sin y = {}^1/_2$. Since

$$\sin {}^{\pi}/_6 = {}^1/_2 \quad \text{and} \quad {}^{\pi}/_6$$

is in the range of $y = \sin^{-1}x$, then

$$\sin(^1/_2) = {}^{\pi}/_6.$$

A calculator can be used to easily find the values of inverse trigonometric functions.

The domains of the other trigonometric functions: $\csc x$, $\sec x$, and $\cot x$, can be restricted so that each has an inverse function. A calculator can be used to find the value of the functions. If appropriate keys are not on the calculator, then we can use the sine, cosine, and tangent keys to find the values.

To find the value of composite function, first find the value of the argument function. After this value is determined, then take the trigonometric function of this value. For example, evaluate sin ($\cos^{-1} \frac{1}{2}$). This expression means the sine of the angle between 0 and $\pi$, inclusive, whose cosine is $\frac{1}{2}$. The angle $\frac{\pi}{3}$ is the value whose cosine is $\frac{1}{2}$. Then,

$$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}.$$

Therefore,

$$\sin(\cos^{-1} \tfrac{1}{2}) = \frac{\sqrt{3}}{2}.$$

**Solution:** $y = \arctan(-\sqrt{3})$; $\tan y = -\sqrt{3}$; $y = 120^\circ, 300^\circ$. Since the tangent has period $\pi$, we have

$$\arctan(-\sqrt{3}) = n\pi + \frac{2}{3}\pi .$$

**• PROBLEM 893**

Does the sine function have an inverse?

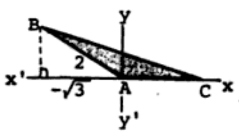
**Solution:** The domain of the sine function is the set of real numbers, and its range is the interval $\{-1 \le y \le 1\}$. For each number $y$ in this interval, the equation $y = \sin x$ has infinitely many solutions. For example, for the number $0$ in $\{-1 \le y \le 1\}$,

$$0 = \sin 0^\circ = \sin 360^\circ = \sin 720^\circ \;\ldots\; .$$

Recall that a function has an inverse if for each number $y$ in the range of the function there is only one number $x$ in the domain of the function such that $y = f(x)$. Thus, the sine function does not have an inverse.

**• PROBLEM 894**

In $\triangle ABC$, $A = \text{arc}\cos\left(-\frac{\sqrt{3}}{2}\right)$. What is the value of $A$ expressed in radians?



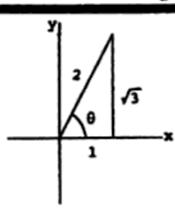

**Solution:** The expression "arc $\cos\left(-\frac{\sqrt{3}}{2}\right)$" means "the angle whose cosine equals $-\frac{\sqrt{3}}{2}$." Angles whose cosine equals $-\frac{\sqrt{3}}{2}$ are

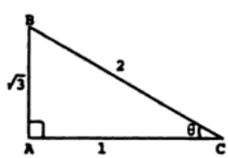
$150^\circ$, $210^\circ$, $-150^\circ$, and $-210^\circ$.

Since the principal value of an arc cosine of an angle is the positive angle having the smallest numerical value of the angle, $150^\circ$, or $\frac{5\pi}{6}$, is the principal value of angle $A$.

**• PROBLEM 895**

Evaluate: (a) $\sin^{-1}\frac{\sqrt{3}}{2}$, (b) $\tan^{-1}\left(-\sqrt{3}\right)$.



Since the principal value of an arc sine or an arc tangent of an angle is the smallest numerical value of the angle,

$$\text{arc sin } \tfrac{1}{2} = 30^\circ \text{ and arc tan } 1 = 45^\circ.$$

Hence, the ratio

$$\frac{\text{arc sin } \frac{1}{2}}{\text{arc tan } 1} = \frac{30^\circ}{45^\circ} = \frac{2}{3} .$$

**• PROBLEM 898**

Find $\sin^{-1} 0.4075$.

Solution: The inverse sine of 0.4075 is the angle, x, whose sine is 0.4075. Since 0.4075 is not found in the sine table, we proceed by finding the two numbers closest to .4075 (one greater and the other less than) which do appear in the table, and interpolating.

$$10'\left[\begin{array}{l} d\left[\begin{array}{l}\sin 24^\circ 0' = 0.4067 \\ \sin x \quad\;\;\, = 0.4075\end{array}\right] 0.0008 \\ \sin 24^\circ 10' = 0.4094 \end{array}\right] 0.0027$$

Setting up the proportion, $\frac{d}{10} = \frac{0.0008}{0.0027} = \frac{8}{27}$. Cross multiplying, $27d = 80$

$$d = 3' \text{ to the nearest minute.}$$

Therefore,

$$\sin^{-1} 0.4075 = 24^\circ 0' + 3' = 24^\circ 3'$$

Each function value is the function value of many angles. For example, a negative cosine is the cosine of an angle in the second quadrant and also of an angle in the third quadrant. But we defined the inverse relation so that it is a function. Hence, $\sin^{-1} 0.4075$ defines a unique angle, namely, 24°3'.

**• PROBLEM 899**

Find Arccot 10.365.

Solution: The problem is to find the angle θ whose cotangent function has the value 10.365 . In a table of trigonometric functions, the number 10.365 does not appear in the vertical column headed by cot. We find two numbers, 10.385 and 10.354, one of which is larger and the other smaller than 10.365.

In such a case, we must interpolate to obtain the proper angle. It must lie between 5°30' and 5°31'. The form for interpolation can be used.

| Angles | Values of Function |
|---|---|
| $1'=60''\left[\begin{array}{l} x\left[\begin{array}{l}5^\circ 30' \\ \theta\end{array}\right. \\ 5^\circ 31'\end{array}\right.$ | $\left.\begin{array}{l}\left.\begin{array}{l}10.385 \\ 10.365\end{array}\right] -0.020 \\ 10.354\end{array}\right] -0.031$ |

Note that the negative values found in the values of the cotangent function were obtained by subtracting a number from the one below it. The same was done for the angles. Illustrating this point: $5°31' - 5°30' = 1' = 60"$, $\theta - 5°30' = x$, $10.354 - 10.385 = -0.031$, $10.365 - 10.385 = -0.020$. Now, the following proportion is set up:

$$\frac{x}{60"} = \frac{-0.020}{-0.031}$$

or $x = \dfrac{-0.020}{-0.031}\,(60") = (.645)(60") = 38.7"$

Therefore, $\theta = 5°30'0" + 0°0'38.7" = 5°30'38.7"$.

Hence, Arccot $10.365 = \theta = 5°\ 30'38.7"$.

**• PROBLEM 900**

| Find Arccos 0.74652. |
|---|

__Solution:__ The problem is to find the angle $\theta$ whose cosine function has value 0.74652. In a table of trigonometric functions, the number 0.74652 does not appear in the column headed by cos. We find two numbers, 0.74664 and 0.74644, one of which is larger and the other smaller than 0.74652. In such a case, we must interpolate to obtain the proper angle. It must lie between 41°42' and 41°43'. The form for interpolation may be used.

$$\begin{array}{lcl} \text{Angles} & \qquad\qquad & \text{Values of Function} \\ 1'\left[\begin{array}{l} x\left[\begin{array}{l}41°42'\\ \theta\end{array}\right.\\ 41°43'\end{array}\right. & & \left.\begin{array}{l}\left.\begin{array}{l}0.74664\\ 0.74652\end{array}\right]-0.00012\\ 0.74644\end{array}\right]-0.00020 \end{array}$$

Note that the negative values found in the values of the cosine function were obtained by subtracting a number from the one below it. The same was done for the angles. Illustrating this point: $41°43' - 41°42' = 1'$, $\theta - 41°42' = x$, $0.74652 - 0.74664 = -0.00012$, and $0.74644 - 0.74664 = -0.00020$.

Now the following proportion is set up:

$$\frac{x}{1'} = \frac{-0.00012}{-0.00020}$$

$$\text{or } x = \frac{-0.00012}{-0.00020}(1') = \frac{-0.00012}{-0.00020}(60")$$

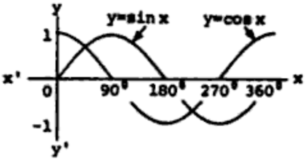
$$= (.6)(60") = 36.0" = 36"$$

Therefore, $x = 41°42'\ 0" + 0°\ 0'36" = 41°42'36"$.

Hence, Arccos 0.74652 $= \theta = 41°42'36"$.

**• PROBLEM 901**

| Evaluate $\cos[\text{arc sin } (-1)]$. |
|---|




Solution: The expression "arc sin (-1)" means "the angle whose sine equals -1." Between $0^\circ$ and $360^\circ$, the only angle whose sine equals -1 is $270^\circ$.

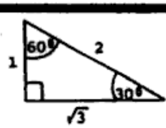
Hence, $\cos[\text{arc}\sin(-1)] = \cos 270^\circ$.

The value of $\cos 270^\circ = 0$.

Note: In problems of this type, a sketch of $y = \sin x$ and $y = \cos x$ is very useful.

**• PROBLEM 902**

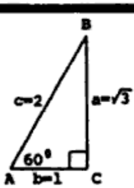
Evaluate $\cos\left(\sin^{-1} \frac{1}{2}\right)$.




Solution: Inverse sine of $\frac{1}{2}$ is the angle whose sin is $\frac{1}{2}$. From our diagram of a $30° - 60° - 90°$ right triangle we observe $\sin 30° = \frac{1}{2}$ so $\sin^{-1} \frac{1}{2} = 30°$. Thus $\cos\left(\sin^{-1} \frac{1}{2}\right) = \cos(30°)$. Consulting our diagram we see $\cos 30° = \frac{\sqrt{3}}{2}$. Therefore $\cos\left(\sin^{-1} \frac{1}{2}\right) = \frac{\sqrt{3}}{2}$.

**• PROBLEM 903**

Express in radical form the positive value of $\sin(\text{arc}\cos \frac{1}{2})$.




Solution: The expression "arc cos ½" means "the angle whose cosine equals ½." In the diagram, note that A is the angle whose cosine is ½.

Since $60^\circ$ is the angle whose cosine equals ½, $A = 60^\circ$. The positive value of $\sin(\text{arc}\cos \frac{1}{2}) = \sin 60^\circ = \frac{\sqrt{3}}{2}$.

**• PROBLEM 909**

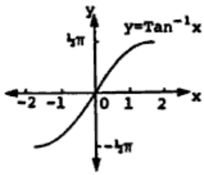
Find $\sin\left(\frac{1}{2} \text{Arccos } \frac{5}{15}\right)$.

**Solution:** Let $\text{Arccos } \frac{5}{13} = \alpha$; then we have

$$\sin\left(\frac{1}{2}\alpha\right) = \sqrt{\frac{1 - \cos\alpha}{2}} = \sqrt{\frac{1 - \frac{5}{13}}{2}} = \sqrt{\frac{8}{26}} = \frac{2}{13}\sqrt{13}.$$

**• PROBLEM 910**

Find sin(Arctan x), where $x$ may be any real number.




**Solution:** Let $t = \text{Arctan } x$. Then $\tan t = x$, and $-\frac{1}{2}\pi < t < \frac{1}{2}\pi$. Also $t$ and $x$ have the same sign (see Figure). Now, we express $\sin t$ in terms of $x$ as follows:

$$\text{since } \tan t = x = \frac{\sin t}{\cos t}, \text{ and}$$

$$\sin t = \frac{\sin t}{\cos t} \cdot \cos t,$$

we write

$$\sin t = x \cos t$$

$$\sin^2 t = x^2 \cos^2 t,$$

and by the identity

$$\sin^2 t + \cos^2 t = 1,$$

$$\sin^2 t = x^2(1 - \sin^2 t).$$

Solving for $\sin^2 t$,

$$1 = \frac{x^2(1 - \sin^2 t)}{\sin^2 t}$$

$$1 = \frac{x^2 - x^2\sin^2 t}{\sin^2 t}$$

$$1 = \frac{x^2}{\sin^2 t} - \frac{x^2\sin^2 t}{\sin^2 t}$$

$$1 = \frac{x^2}{\sin^2 t} - x^2$$

$$1 + x^2 = \frac{x^2}{\sin^2 t}$$

$$\sin^2 t(1 + x^2) = x^2$$

$$\sin^2 t = \frac{x^2}{1 + x^2}.$$

Thus, $x = 1$ does not satisfy the given equation. (Notice that outside the restricted values for arccos we have $\arccos 0 = -\pi/2$ also, which, together with $\arcsin 1 = \pi/2$, makes $x = 1$ a solution.)

Hence $x = 0$ is the only solution of the given equation when only the restricted values are permitted (recall that for the inverse sine function,

$$-\frac{\pi}{2} \le (\arcsin x = \alpha) \le \frac{\pi}{2}$$

and for the inverse cos function

$$0 \le (\arccos(1 - x) = \beta) \le \pi).$$

# CHAPTER 29

# TRIGONOMETRIC EQUATIONS

**Basic Attacks and Strategies for Solving Problems in this Chapter. See pages 671 to 705 for step-by-step solutions to problems.**

The process of solving trigonometric equations is very similar to the process of solving algebraic equations. Because of the periodic nature of trigonometric functions, trigonometric equations will not typically have a finite number of solutions unless one restricts the search for solutions to some specified interval.

The simplest kind of trigonometric equation has the linear form

$$\operatorname{trg} x = c,$$

where trg is one of the trigonometric functions, $c$ is a given number, and $x$ is the variable whose value is to be determined. Such equations can be solved using memorized values or the inverse trigonometric function operations on a calculator.

If the trigonometric equation is more complex, then the first step in the solution process is to set the equation equal to zero and factor (if possible) the other side of the equation. Then set each factor equal to zero and solve each resulting equation.

If the equation is not factorable and is in quadratic form, then other techniques for solving algebraic equations should be tried, such as, the quadratic formula and completing the square. If more than one trigonometric function is involved in a trigonometric equation and/or different multiples of the angle and the factoring method is not applicable, then it is necessary to use identities to obtain an equation involving only one trigonometric function. After identities have been applied, then solve the equation using the techniques mentioned above.

General guidelines for proving trigonometric identities involve the following steps:

(1) Begin work on the side of the statement which is more complicated.

(2) Look for trigonometric substitutions involving the basic identities that may help simplify things.

(3) Look for algebraic operations, such as adding fractions, the distributive property, or factoring, that may simplify the side you are working with or that will at least lead to an expression that will be easier to simplify.

(4) If you cannot think of anything else to do, change everything to sines and cosines and see if that helps.

(5) Always keep an eye on the side that you are not working with to be sure you are working toward it. There is a certain sense of direction that accompanies a successful proof.

**Solution:** Since the unknown quantity is involved in the radicand, squaring of both sides to eliminate the radical is suggested. Thus, we obtain $1 + \sin^2 x =$

$2 \sin^2 x$. Hence, $\sin^2 x - 1 = 0$,

or $\sin^2 x = 1$

$$\sqrt{\sin^2 x} = \pm\sqrt{1}$$

$$\sin x = \pm 1$$

When $\sin x = 1$ on $[0,2\pi]$, $x = \pi/2$. When $\sin x = -1$ on $[0,2\pi]$, $x = 3\pi/2$.

The complete solution set seems to be $\{\pi/2, 3\pi/2\}$. Since we squared both sides of the equation, we should try each element in the original equation. When $x = \pi/2$, we obtain $\sqrt{1+1} = \sqrt{2}\cdot 1$. When $x = 3\pi/2$, we obtain $\sqrt{1+1} = \sqrt{2}(-1)$. The second element does not satisfy the original equation, hence does not belong to the solution set. An extraneous root was introduced by squaring the equation, it would seem. Thus, the solution set is $\{\pi/2\}$.

**• PROBLEM 922**

Find one number in the solution set of the equation $\tan t = \sqrt{3}$.

Fig. A

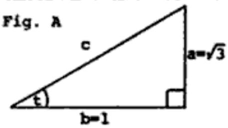
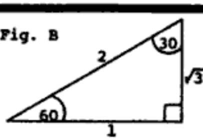



Fig. B




**Solution:** We know that the

$$\tan t = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{\sqrt{3}}{1} \quad \text{(see figure A).}$$

The third side, c, can be found by the Pythagorean theorem $c^2 = a^2 + b^2$

$$c^2 = (\sqrt{3})^2 + 1^2 = 3 + 1 = 4$$

$$c = \sqrt{4} = \pm 2$$

Reject $-2$, since a side of a triangle cannot be negative.

Thus, $c = +2$. The student should recall the trigonometric functions of the $30^\circ$, $60^\circ$, $90^\circ$ triangle, that is,

$$\sin 30^\circ = \frac{\text{opposite}}{\text{hypotenuse}} = \tfrac{1}{2} = \cos 60^\circ = \frac{\text{adjacent}}{\text{hypotenuse}} \quad \text{(see figure B).}$$

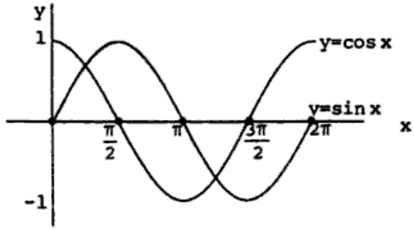
Thus, $t = 60^\circ$ because $\cos t = \frac{1}{2}$, and $\sin t = \frac{\sqrt{3}}{2}$; that is,

$$\tan 60^\circ = \frac{\sin 60^\circ}{\cos 60^\circ} = \frac{\sqrt{3}/2}{1/2} = \left(\frac{\sqrt{3}}{2}\right)\left(\frac{2}{1}\right) = \sqrt{3}$$

**• PROBLEM 923**

Find the solution set on $[0,\pi]$ for the equation
$\tan x \sin x - \sin x - \tan x + 1 = 0$.

**Solution:** This equation can be factored to obtain
$(\sin x - 1)(\tan x - 1) = 0$.
The values of x satisfying this equation may be found by setting each factor equal to zero.




Solution: $\sin\theta + 2\tan\theta = 0$

Since $\tan\theta = \frac{\sin\theta}{\cos\theta}$, by substitution: $\sin\theta + 2\frac{\sin\theta}{\cos\theta} = 0$

Multiplying both sides by $\cos\theta$ gives us $\cos\theta\left(\sin\theta + 2\frac{\sin\theta}{\cos\theta}\right) = 0(\cos\theta)$

Distributing we obtain,

$$\sin\theta\cos\theta + 2\sin\theta = 0$$

Factoring out $\sin\theta$ gives us

$$\sin\theta(\cos\theta + 2) = 0$$

If $xy = 0$, either $x = 0$ or $y = 0$, hence either $\sin\theta = 0$ or $\cos\theta + 2 = 0$. Subtracting 2 from each side of the latter gives us $\cos\theta = -2$ Thus $\sin\theta = 0$ or $\cos\theta = -2$

On the given interval, $0 \le \theta \le 2\pi$, $\sin\theta = 0$ when $\theta = 0, \theta = \pi$ or $\theta = 2\pi$; and $\cos\theta = -2$ for no angles of $\theta$ (cos is only defined on the interval $|-1,1|$).

Thus the solution set is $\{0,\pi,2\pi\}$. Check these values. If we do not restrict $\theta$ as we have done, then the solutions may be expressed as $\theta = 0 + k\pi$, where $k$ is any integer. That is, by adding any integral multiple of $\pi$ to the angle $\theta$, we obtain an angle coterminal with either zero or $\pi$. (By coterminal we mean the initial sides of the angles lie on the positive branch of the x-axis and the terminal sides coincide.)

**• PROBLEM 931**

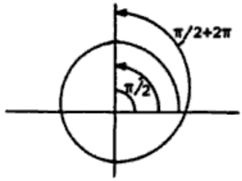
Solve the equation

$$\sin^2\theta + 2\cos\theta - 1 = 0$$

for non-negative values of $\theta$ less than $2\pi$.

Solution: Two trigonometric functions of the unknown $\theta$ itself appear in this equation. Accordingly, we make use of the identity connecting these functions, namely, $\sin^2\theta + \cos^2\theta = 1$, to transform it into an equation in-

Solve the equation $\sin^2 x - 4\sin x + 3 = 0$.




Solution: Factoring the left side of the given equation into a product of two trigonometric functions

$$(\sin x - 3)(\sin x - 1) = 0 \qquad (1)$$

Whenever a product $ab = 0$ (where $a$ and $b$ are any two numbers) either $a = 0$ or $b = 0$. Hence, either

$$\sin x - 3 = 0 \quad \text{or} \quad \sin x - 1 = 0$$

$$\sin x = 3 \quad \text{or} \quad \sin x = 1 \quad ,$$

and so our desired solution set is the union

$$\{\sin x = 3\} \cup \{\sin x = 1\}. \qquad (2)$$

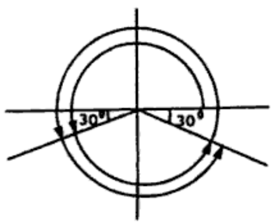
The first of these sets is the empty set because $\sin x$ takes on values only between $-1$ and $1$; that is, $-1 \le \sin x \le 1$. For the second set, $\sin \pi/2 = 1$. Note that any integral multiple of $2\pi$ added to $\pi/2$ will result in the same reference angle. Hence,

$$\sin\frac{\pi}{2} = \sin\left(\frac{\pi}{2} + 2\pi\right) = \sin\left(\frac{\pi}{2} - 2\pi\right) = \sin\left(\frac{\pi}{2} + 4\pi\right) = 1 \ .$$

Therefore, the solution set for the second set is $\left\{\frac{\pi}{2} + 2\pi k\text{, where } k \text{ is } 0, \pm 1, \pm 2, \ldots\right\}$. Since the first set has no solution set, the solution set for the second set is the solution set of the given equation. Hence, $\left\{\frac{\pi}{2} + 2\ k\text{, where } k = 0, \pm 1, \pm 2\right\}$ is the solution set of the given equation.



Determine all angles $x$, $0^\circ \le x < 360^\circ$, such that $\sin 2x = -\frac{1}{2}$.




Solution: To determine all values of $x$ such that $0^\circ \le x < 360^\circ$ and $\sin 2x = -\frac{1}{2}$, we must determine all values of $2x$ such that

$$2 \cdot 0^\circ \le 2 \cdot x < 2 \cdot 360^\circ \quad \text{and } \sin 2x = -\tfrac{1}{2} \ ,$$

They are 135° in the second quadrant, and 315° in the fourth quadrant. This is through one revolution or 360°. Through a second revolution (720°) the chosen angles are 495° in the second quadrant and 675° in the fourth (see Figure).

The angles are 135°, 315°, 495°, 675°. According to the equation each angle is twice the required angle. Therefore, the values of x which we are interested in are

$$\frac{135°}{2} = 67.5°$$

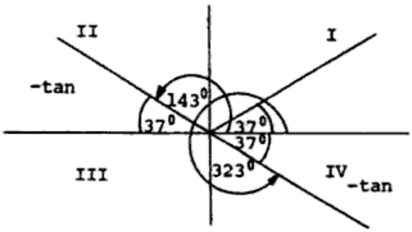
$$\frac{315°}{2} = 157.5°$$

$$\frac{495°}{2} = 247.5°$$

$$\frac{675°}{2} = 337.5°$$

**• PROBLEM 936**

> Solve the equation
>
> $$3 \tan \theta + \sec \theta + 1 = 0$$
>
> for non-negative values of $\theta$ less than $2\pi$, that is, $0 \le \theta < 2\pi$.




Solution: Here the two functions involved are connected by the identity $\sec^2\theta = 1 + \tan^2\theta$. This can be derived from $\sin^2\theta + \cos^2\theta = 1$, by dividing by $\cos^2\theta$.

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$
$$\tan^2\theta + 1 = \sec^2\theta$$

Since both functions appear to the first degree in the equation, the introduction of an irrationality is unavoidable. Therefore a redundancy may arise, and all solutions obtained must be checked.

We choose to eliminate $\sec\theta$; thus, solving for $\sec\theta$ in the given equation,

$$3 \tan \theta + \sec \theta + 1 = 0$$
$$3 \tan \theta + 1 = -\sec \theta$$

Solving for $\sec\theta$ in the identity $\sec^2\theta = \tan^2\theta + 1$, and substituting:

$$\sec \theta = \pm \sqrt{1 + \tan^2\theta}$$

$$3 \tan\theta + 1 = -\sec\theta = \pm\sqrt{1+\tan^2\theta}$$

Squaring both sides:

$$(3 \tan\theta + 1)^2 = \left(\pm\sqrt{1+\tan^2\theta}\right)^2$$

$$9 \tan^2\theta + 6 \tan\theta + 1 = 1 + \tan^2\theta$$

Subtract $\tan^2\theta + 1$ from both sides to obtain:

$$9 \tan^2\theta + 6 \tan\theta + 1 - \left(\tan^2\theta + 1\right) = 1 + \tan^2\theta - \left(\tan^2\theta + 1\right)$$

$$8 \tan^2\theta + 6 \tan\theta = 0$$

Factor $2 \tan\theta$ from the left side:

$$(2 \tan\theta)(4 \tan\theta + 3) = 0. \text{ Thus,}$$

$$2 \tan\theta = 0, \quad \text{or} \quad 4 \tan\theta + 3 = 0$$

$$\tan\theta = 0, \quad \text{or} \quad \tan\theta = -3/4$$

Therefore, we must find the angles whose tangent is 0, and the angle whose tangent is $(-3/4) = -.75$, where the angle is greater than or equal to 0 and less than $2\pi$ (or $360^\circ$).

The angle whose tangent is 0 is $0^\circ$ and $180^\circ$. Thus, for $\tan\theta = 0$ we find $\theta = 0^\circ$, and $\theta = 180^\circ$.

Now, we must find $\theta$ such that $\tan\theta = -.75$. Referring to a table of trigonometric functions, we find $\tan 37^\circ \approx .75$. Thus, $37^\circ$ is our reference angle. Since the tangent function is negative in the second and fourth quadrants, $\theta = 143^\circ, 323^\circ$ (see figure).

Therefore, the roots of the equation are $\theta = 0^\circ, 180^\circ, 143^\circ, 323^\circ$. But, before accepting these roots as solutions to the original equation, we substitute each root into the equation as a check for validity. Thus, when $\theta = 0^\circ$, $3 \tan\theta + \sec\theta + 1 = 0$ becomes,

$$3 \tan 0^\circ + \sec 0^\circ + 1 = 0, \text{ and}$$

since $\tan 0^\circ = 0$ and $\sec 0^\circ = \frac{1}{\cos 0^\circ} = 1$, we have:

$$3(0) + 1 + 1 \stackrel{?}{=} 0 ; \quad 2 \neq 0$$

Thus, $\theta = 0^\circ$ is an extraneous root.

when $\theta = 180^\circ$, we have

$3 \tan 180^\circ + \sec 180^\circ + 1 = 0$, since

$\tan 180^\circ = 0$, and $\sec 180^\circ = \cos 180^\circ = -1$

we have $\quad 0 + (-1) + 1 = 0$

Therefore, $\theta = 180^\circ$ is a solution of the given equation.

When $\theta = 143^\circ$, $3 \tan\theta + \sec\theta + 1 = 0$ becomes,

$$3 \tan 143^\circ + \sec 143^\circ + 1 = 0; \text{ and}$$

since $\sec 143^\circ = -\sec 37^\circ$ ( the sign is negative because sec is negative in the second quadrant), we have:

$$3 \tan 143^\circ - \sec 37^\circ + 1 = 0$$

From a table of trig functions we find that $\sec 37^\circ \approx 1.25$, and we found previously that $\tan 143^\circ \approx -.75$ ; thus:

$$3(-.75) - 1.25 + 1 \stackrel{?}{=} 0$$

$$-2.25 - .25 \stackrel{?}{=} 0$$

$$-2.5 \neq 0$$

Therefore, $\theta = 143^\circ$ is not a solution of the given equation,

When $\theta = 323^\circ$, we have: $\quad 3 \tan 323^\circ + \sec 323^\circ + 1 = 0$;

and since sec is positive in Quadrant IV, $\sec 323^\circ = \sec 37^\circ$. We have previously found that $\tan 323^\circ \approx -.75$. Thus,

$$3\tan(323^\circ) + \sec 37^\circ + 1 \stackrel{?}{=} 0$$

$$3(-.75) + 1.25 + 1 = 0$$

$$-2.25 + 2.25 = 0$$

$$0 = 0$$

Therefore, $\theta = 323^\circ$ is a solution of the given equation. Thus, the solutions of the given equation are

$$\theta = 180^\circ,\ 323^\circ \text{ for } 0 \le \theta < 2\pi.$$

**• PROBLEM 937**

Solve the equation

$$2\sin 2\theta + \cos 2\theta + 2\sin\theta = 1$$

for non-negative values of $\theta$ less than $2\pi$.

Solution: Here we have functions of both $\theta$ itself and $2\theta$. Hence we first transform so as to get an equivalent equation involving functions of $\theta$ only. We set $\sin 2\theta = 2\sin\theta\cos\theta$ and $\cos 2\theta = 1 - 2\sin^2\theta$, the latter form being chosen so that the constant terms in the equation will cancel.

This leads to

$$2\sin 2\theta + \cos 2\theta + 2\sin\theta = 1$$

$$2(2\sin\theta\cos\theta) + \left(1 - 2\sin^2\theta\right) + 2\sin\theta = 1$$

$$4\sin\theta\cos\theta + 1 - 2\sin^2\theta + 2\sin\theta = 1$$

$$4\sin\theta\cos\theta - 2\sin^2\theta + 2\sin\theta = 0 \quad \text{(subtracting 1 from both sides)}$$

$$(2\sin\theta)(2\cos\theta - \sin\theta + 1) = 0 \quad \text{(Factoring out } 2\sin\theta\text{)}$$

$$2(\sin\theta)(2\cos\theta - \sin\theta + 1) = 0.$$

Whenever a product of two numbers $ab = 0$ either $a = 0$ or $b = 0$. Hence $2\sin\theta = 0$ or $2\cos\theta - \sin\theta + 1 = 0$. When the factor $\sin\theta$ is set equal to zero, the last equation is satisfied, whence we get

$$\theta = 0, \quad \theta = \pi.$$

It is easy to verify these solutions in the original equation.

Check:

$$\theta = 0$$

$$2\sin 2(0) + \cos 2(0) + 2\sin(0) = 1$$

$$2\sin(0) + \cos(0) + 2\sin(0) = 1$$

$$2\cdot 0 \qquad 1 \quad + \quad 2\cdot 0 = 1$$

$$1 = 1$$

$\theta = \pi$

$$2 \sin 2(\pi) + \cos 2(\pi) + 2 \sin \pi = 1$$

$$2 \sin 2\pi + \cos 2\pi + 2 \sin \pi = 1$$

$$2(0) + 1 + 2(0) = 1$$

$$1 = 1$$

To find possible solutions of the equation $2 \cos \theta - \sin \theta + 1 = 0$, we transpose to get $2 \cos \theta + 1 = \sin \theta$.

From the Pythagorean relation $\cos^2 \theta + \sin^2 \theta = 1$, we obtain an expression for $\sin \theta$.

$$\sin^2 \theta = 1 - \cos^2 \theta$$

$$\sin \theta = \pm \sqrt{1 - \cos^2 \theta}.$$

Substitute this into the factor $2 \cos \theta - \sin \theta + 1 = 0$.

$$2 \cos \theta - \sin \theta + 1 = 2 \cos \theta \pm \sqrt{1 - \cos^2 \theta} + 1 = 0$$

Transpose $\pm\sqrt{1 - \cos^2 \theta}$,

$$2 \cos \theta + 1 = \pm\sqrt{1 - \cos^2 \theta}.$$

To solve this radical equation, we square both sides:

$$4 \cos^2 \theta + 4 \cos \theta + 1 = 1 - \cos^2 \theta.$$

Subtract 1 and add $\cos^2 \theta$ to both sides to obtain:

$$5 \cos^2 \theta + 4 \cos \theta = 0$$

$$\cos \theta(5 \cos \theta + 4) = 0.$$

Set both factors equal to zero.

$$\cos \theta = 0 \qquad 5 \cos \theta + 4 = 0$$

$$5 \cos \theta = -4$$

$$\cos \theta = 0 \qquad \cos \theta = -\frac{4}{5}.$$

Note that all values of $\theta$ obtained must be checked because the process of rationalizing a radical equation may lead to extraneous roots.

Corresponding to $\cos \theta = 0$, $\theta = \frac{\pi}{2}$ and $\theta = \frac{3}{2}\pi$. Substitute these values into the given equation, $2 \cos \theta - \sin \theta + 1 = 0$.

Verify both values of $\theta$:

For $\theta = \frac{\pi}{2}$.

$$2 \cos \theta - \sin \theta + 1 = 0$$

a relation involving only one trigonometric function. Multiplying by -1 (or transposing), we have

$$2 \sin^2 x - \sin x < 0,$$

or $$(\sin x)(2 \sin x - 1) < 0.$$

Now for a product to be negative, then one factor must be negative and the other must be positive. There are two cases to be considered.

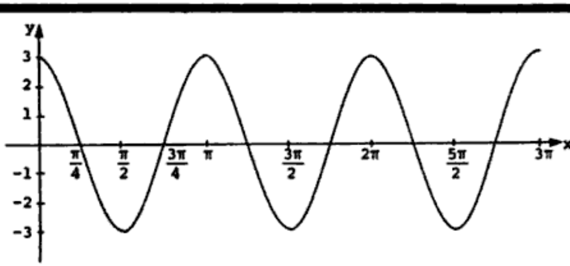
| Case I | | Case II |
|---|---|---|
| $\sin x > 0$ | or | $\sin x < 0$ |
| $2 \sin x - 1 < 0$ | | $2 \sin x - 1 > 0.$ |

For Case I, when $\sin x > 0$, the angle is in quadrant I or II and $x > 0$. Consider the second restriction, $2 \sin x - 1 < 0$. Then $\sin x < \frac{1}{2}$. Thus, $x < \frac{\pi}{6}$. Combine these two restrictions: $0 < x < \frac{\pi}{6}$. Since $\frac{\pi}{6}$ is a reference angle in quadrant II, then $\frac{5\pi}{6} < x < \pi$ since $0 < \sin x < \frac{1}{2}$.

For Case II, when $\sin x < 0$, the angle is in quadrant III or IV. Also, if $2 \sin x - 1 > 0$, then $\sin x > \frac{1}{2}$. Thus $x > \frac{7}{6}\pi$ and $x < \frac{11}{6}\pi$, $\frac{7}{6}\pi < x < \frac{11}{6}\pi$.

**• PROBLEM 941**

Sketch three periods of the graph $y = 3 \cos 2x$.

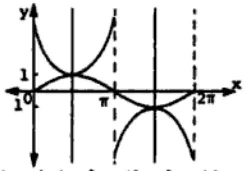



<u>Solution:</u> The coefficient of the function is 3, which means that the maximum and minimum values are 3 and - 3, respectively. The period of the cosine function is the coefficient of x multiplied by $\frac{\pi}{2}$ radians. Therefore, the period of the cosine function given in this problem is

$$2 \left(\frac{\pi}{2} \text{ radians}\right) = \pi \text{ radians}$$

and with this knowledge, we sketch the curve as in the Figure.

Graph $y = \csc x$, $0 \le x \le 2\pi$.




Solution: To plot points for the function cosecant of $x$, first find the y-values of the reciprocal function, the sine of the angle $x$.

| x | 0 | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |
|---|---|---|---|---|---|
| sin x | 0 | 1 | 0 | -1 | 0 |
| csc x | not defined | 1 | not defined | -1 | not defined |

Since the sine and cosecant are reciprocals, we state the following conclusions based on properties of real numbers.

(1) For $0 \le x \le \frac{\pi}{2}$, $0 \le \sin x \le 1$ and $\csc x \geq 1$.

In fact, as sin x increases, csc x decreases. For example:

$\sin 0^\circ = \sin 0 = 0$ $\qquad$ $\csc 0 = \text{undefined}$

$\sin 30^\circ = \sin \frac{\pi}{6} = \frac{1}{2} = .5000$ $\qquad$ $\csc \frac{\pi}{6} = 2$

$\sin 45^\circ = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \approx \frac{1.414}{2} = .7070$ $\qquad$ $\csc \frac{\pi}{4} = \sqrt{2} \approx 1.414$

$\sin 90^\circ = \sin \frac{\pi}{2} = 1.000$ $\qquad$ $\csc \frac{\pi}{2} = 1$

(2) For $\frac{\pi}{2} \le x \le \pi$, sin x decreases from 1 to 0. Hence, csc x will increase from 1 to very large values. We can observe this from specific examples:

$\sin \frac{\pi}{2} = 1$ $\qquad$ $\csc \frac{\pi}{2} = 1$

$\sin \frac{2}{3}\pi = \frac{\sqrt{3}}{2} \approx .87$ $\qquad$ $\csc \frac{2}{3}\pi \approx 1.15$

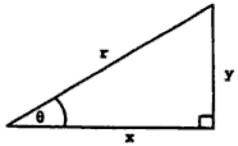
$\sin \frac{3}{4}\pi \approx .707$ $\qquad$ $\csc \frac{3}{4}\pi \approx 1.4$

$\sin \frac{5}{6}\pi = .500$ $\qquad$ $\csc \frac{5}{6}\pi = 2$

$\sin \pi = 0$ $\qquad$ $\csc \pi = \text{undefined}$

(3) For $\pi \le x \le \frac{3\pi}{2}$, sin x decreases from 0 to -1. Hence, csc x will be increasing and will increase from very large negative values to -1.

(4) For $\frac{3\pi}{2} \le x \le 2\pi$, $-1 \le \sin x \le 0$, and the graph will be increasing. Hence, csc x will decrease from -1 to very large negative values. The student can verify conclusions (3) and (4) in a similar manner to that used for (1) and (2), that is, choosing specific angles between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, and then between $\frac{3\pi}{2}$ and $2\pi$.




$$\sec\theta = \frac{1}{\cos\theta} = \frac{\text{hypotenuse}}{\text{adjacent side}} = \frac{r}{x}$$

Substitute these expressions into the identity:

$$\sec^2\theta - \tan^2\theta = \left(\frac{r}{x}\right)^2 - \left(\frac{y}{x}\right)^2$$
$$= \frac{r^2}{x^2} - \frac{y^2}{x^2}$$
$$= \frac{r^2 - y^2}{x^2}$$

By the Pythagorean Theorem, $r^2 = x^2 + y^2$; substitute $x^2$ for $r^2 - y^2$; since $x^2 + y^2 = r^2$,

$$x^2 = r^2 - y^2.$$

Thus, $\sec^2\theta - \tan^2\theta$

$$= \frac{x^2}{x^2}$$
$$= 1$$

**• PROBLEM 946**

Prove the following two identities:

(1) $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$ (2) $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$ .

Solution: To prove identity (1), we use the cosine difference formula, which states: $\cos(u - v) = \cos u \cos v + \sin u \sin v$ . Thus,

$$\cos\left(\frac{\pi}{2} - \theta\right) = \cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta$$

Now, we must find the values for $\cos\frac{\pi}{2}$ and $\sin\frac{\pi}{2}$ . Since $\frac{\pi}{2} = 90^\circ$, and we know that $\cos 90^\circ = 0$ and $\sin 90^\circ = 1$, by substitution we have:

$$\cos\left(\frac{\pi}{2} - \theta\right) = 0\cdot\cos\theta + 1\cdot\sin\theta$$
$$= \sin\theta, \text{ the desired result.}$$

To prove identity (2) note that

$$\cos\left[\frac{\pi}{2} - \left(\frac{\pi}{2} - \theta\right)\right] = \cos\left(\frac{\pi}{2} - \frac{\pi}{2} + \theta\right) = \cos\theta\ .$$

Since $\cos\theta = \cos\left[\frac{\pi}{2} - \left(\frac{\pi}{2} - \theta\right)\right]$ we can use the difference formula, which states: $\cos(u-v) = \cos u \cos v + \sin u \sin v$ . Thus, we have:

$$\cos\theta = \cos\left[\frac{\pi}{2} - \left(\frac{\pi}{2} - \theta\right)\right] = \cos\frac{\pi}{2}\cos\left(\frac{\pi}{2} - \theta\right) + \sin\frac{\pi}{2}\sin\left(\frac{\pi}{2} - \theta\right)$$

Now since $\cos\frac{\pi}{2} = 0$ and $\sin\frac{\pi}{2} = 1$, by substitution we obtain:

$$\cos^4\beta - \sin^4\beta = (\cos^2\beta + \sin^2\beta)(\cos^2\beta - \sin^2\beta)$$
$$= (1)(\cos^2\beta - \sin^2\beta)$$
$$= \cos^2\beta - \sin^2\beta$$
$$= (1 - \sin^2\beta) - \sin^2\beta$$
$$= 1 - 2\sin^2\beta$$

Therefore, $\cos^4\beta - \sin^4\beta = 1 - 2\sin^2\beta$.

**• PROBLEM 951**

Prove the identity $1 + \sin 2x = (\sin x + \cos x)^2$.

Solution: To prove this identity, start with the right side of the equation.

$$(\sin x + \cos x)^2 = (\sin x + \cos x)(\sin x + \cos x)$$
$$= \sin^2 x + 2\sin x\cos x + \cos^2 x$$
$$= \sin^2 x + \cos^2 x + 2\sin x\cos x$$
$$= 1 + 2\sin x\cos x, \text{since}$$
$$\left(\sin^2 x + \cos^2 x\right) = 1 .$$

But $\sin 2x = 2\sin x\cos x$, and $(\sin x + \cos x)^2 =$

$1 + 2\sin x\cos x$. Therefore $(\sin x + \cos x)^2 = 1 + \sin 2x$.

**• PROBLEM 952**

Prove the identity $\csc 2x = \dfrac{\csc x}{2\cos x}$.

Solution: Starting with the right side of the identity,

$$\frac{\csc x}{2\cos x} = \frac{1/\sin x}{2\cos x}, \text{ since } \csc x = \frac{1}{\sin x}. \quad \text{Hence,}$$

$$\frac{\csc x}{2\cos x} = \frac{1}{\sin x}\cdot\frac{1}{2\cos x} = \frac{1}{2\sin x\cos x}.$$

Using the double-angle formula, $\sin 2\alpha = 2\sin\alpha\cos\alpha$,

$$\frac{\csc x}{2\cos x} = \frac{1}{2\sin x\cos x} = \frac{1}{\sin 2x}.$$

Again, since $\csc x = \dfrac{1}{\sin x}$, $\dfrac{\csc x}{2\cos x} = \dfrac{1}{\sin 2x} = \csc 2x$. Since we have proved the left side equal to the right,

$$\frac{\csc x}{2\cos x} = \csc 2x .$$

**• PROBLEM 953**

Prove the identity $\dfrac{\sin^2\theta + \cos^2\theta}{\cos^2\theta} = \sec^2\theta$.

Solution: The fraction $\dfrac{a+c}{b} = \dfrac{a}{b} + \dfrac{c}{b}$, by the definition of addition

$$\frac{\cos^3 x - \cos x + \sin x}{\cos x} = \frac{\cos^3 x}{\cos x} - \frac{\cos x}{\cos x} + \frac{\sin x}{\cos x} .$$

But $\frac{\cos^3 x}{\cos^1 x} = \cos^{3-1} x = \cos^2 x$

$\frac{\cos x}{\cos x} = 1$

and $\frac{\sin x}{\cos x} = \tan x.$

Thus, replacing these values we obtain:

$$= \cos^2 x - 1 + \tan x.$$

Recall the identity $\sin^2 \theta + \cos^2 \theta = 1$.

Subtracting $\sin^2 \theta$ from both sides gives us

$$\cos^2 \theta = 1 - \sin^2 \theta,$$

and subtracting 1 from both sides we obtain

$$\cos^2 \theta - 1 = - \sin^2 \theta;$$

thus replacing $\cos^2 x - 1$ by $- \sin^2 x$ we have:

$$\frac{\cos^3 x - \cos x + \sin x}{\cos x} = - \sin^2 x + \tan x$$

$$= \tan x - \sin^2 x.$$

• **PROBLEM 956**

Show that $\tan t + \cot t = \csc t \sec t$.

Solution: Since $\tan t = \frac{\sin t}{\cos t}$ and $\cot t = \frac{\cos t}{\sin t}$ ,

by substitution we have:

$$\tan t + \cot t = \frac{\sin t}{\cos t} + \frac{\cos t}{\sin t}$$

Since multiplying by $1 \equiv \frac{\sin t}{\sin t}$ and $1 \equiv \frac{\cos t}{\cos t}$

does not alter the value of either fraction, we perform this multiplication, and obtain:

$$\frac{\sin t}{\cos t}\left(\frac{\sin t}{\sin t}\right) + \frac{\cos t}{\sin t}\left(\frac{\cos t}{\cos t}\right)$$

$$= \frac{\sin^2 t}{\sin t \cos t} + \frac{\cos^2 t}{\sin t \cos t}$$

$$= \frac{\sin^2 t + \cos^2 t}{(\sin t)(\cos t)}$$

$$= \frac{1 - \cos^2 \theta}{\sin \theta(1 + \cos \theta)}$$

Since $\cos^2 \theta + \sin^2 \theta = 1$ or $\sin^2 \theta = 1 - \cos^2 \theta$, then

$$\frac{1 - \cos \theta}{\sin \theta} = \frac{1 - \cos^2 \theta}{\sin \theta(1 + \cos \theta)} = \frac{\sin^2 \theta}{\sin \theta(1 + \cos \theta)}$$

$$= \frac{\sin \theta}{1 + \cos \theta}$$

Note that this method starts with the left side of the identity to be proved. The following is another method which can be used to prove the identity.

(b) $$\frac{\sin \theta}{1 + \cos \theta} = \frac{\sin \theta(1 - \cos \theta)}{(1 + \cos \theta)(1 - \cos \theta)}$$

$$= \frac{\sin \theta(1 - \cos \theta)}{1 + \cos \theta - \cos \theta - \cos^2 \theta}$$

$$= \frac{\sin \theta(1 - \cos \theta)}{1 - \cos^2 \theta}$$

Again, since $\sin^2 \theta = 1 - \cos^2 \theta$,

$$\frac{\sin \theta}{1 + \cos \theta} = \frac{\sin \theta(1 - \cos \theta)}{1 - \cos^2 \theta}$$

$$= \frac{\sin \theta(1 - \cos \theta)}{\sin^2 \theta}$$

$$= \frac{1 - \cos \theta}{\sin \theta}$$

Note that this second method starts with the right side of the identity to be proved.

**• PROBLEM 959**

Prove that

$$\frac{\cos A}{\csc A - 1} + \frac{\cos A}{\csc A + 1} = 2 \tan A$$

is an identity.

Solution: The left member is the more complicated; hence, we shall work with it and begin by performing the indicated addition. The lowest common denominator is $(\csc A - 1)(\csc A + 1) = \csc^2 A - 1$. Thus

$$\frac{\cos A}{\csc A - 1} + \frac{\cos A}{\csc A + 1} = \left(\frac{\csc A + 1}{\csc A + 1}\right) \frac{\cos A}{\csc A - 1} + \left(\frac{\csc A - 1}{\csc A - 1}\right) \frac{\cos A}{\csc A + 1}$$

$$= \frac{(\csc A + 1)\cos A}{\csc^2 A - 1} + \frac{(\csc A - 1)\cos A}{\csc^2 A - 1}$$

$$= \frac{\csc A \cos A + \cos A}{\csc^2 A - 1} + \frac{\csc A \cos A - \cos A}{\csc^2 A - 1}$$

$$= \frac{\csc A \cos A + \cos A + \csc A \cos A - \cos A}{\csc^2 A - 1}$$

$$= \frac{\csc A \cos A + \csc A \cos A}{\csc^2 A - 1}$$

Recall the trignometric identity $\csc^2 A - 1 = \cot^2 A$,

$$= \frac{2 \cos A \csc A}{\cot^2 A}$$

replace $\csc A$ by $\frac{1}{\sin A}$,

$$= \frac{(2 \cos A)/(\sin A)}{\cot^2 A}$$

replace $\frac{\cos A}{\sin A}$ by $\cot A$,

$$= \frac{2 \cot A}{\cot^2 A}$$

cancelling out $\cot A$,

$$= \frac{2}{\cot A}$$

Replace $\cot A$ by $\frac{1}{\tan A}$, $= \frac{2}{\frac{1}{\tan A}}$

Multiply numerator and denominator by $\tan A$,

$$= 2 \tan A$$

We have thus proved

$$\frac{\cos A}{\csc A - 1} + \frac{\cos A}{\csc A + 1} = 2 \tan A \text{ is an identity.}$$

**• PROBLEM 960**

Prove that the following equation is an identity:

$$\frac{1 - \sin x}{\cos x} = \frac{\cos x}{1 + \sin x}$$

Solution: This problem may be approached in a variety of ways. One method is based on the fact that two fractions are equal if their cross products are equal. That is,

$$\frac{1 - \sin x}{\cos x} = \frac{\cos x}{1 + \sin x} \text{ if } (1 - \sin x)(1 + \sin x) = (\cos x)(\cos x).$$

$$(1 - \sin x)(1 + \sin x) = 1 - \sin^2 x$$

Now, recall the trigonometric identity $\cos^2 \theta = 1 - \sin^2\theta$; thus $(1 - \sin x)(1 + \sin x) = \cos^2 x = (\cos x)\cdot(\cos x)$, and since the cross products are equal, we have proven the original fractions equivalent.

**• PROBLEM 961**

Prove the identity: $\sec A \csc A = \tan A + \cot A$.

Solution: One approach to the proof of identities, when many functions are involved, is to express the given functions in terms of fewer functions. In this case, suppose we express each of the given trigonometric functions in terms of sine and cosine functions. We will work in parallel columns, with each side of the given equation:

Since sec A = 1/cos A and csc A = 1/sin A,

sec A csc A

$$= \frac{1}{\cos A} \cdot \frac{1}{\sin A}$$

$$= \frac{1}{\cos A \sin A}$$

Since tan A = sin A/cos A and cot A = 1/tan A = 1/sin A/cos A = cos A/sin A, tan A + cot A

$$= \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}.$$

To combine these fractions, we convert them into fractions with the least common denominator (LCD) cos A sin A; thus:

$$= \left(\frac{\sin A}{\sin A}\ \frac{\sin A}{\cos A}\right) + \left(\frac{\cos A}{\cos A}\ \frac{\cos A}{\sin A}\right)$$

$$= \frac{\sin^2 A + \cos^2 A}{\cos A \sin A}.$$

Recall the trigonometric identity $\sin^2 A + \cos^2 A = 1$; replacing $\sin^2 A+\cos^2 A$ by 1 we obtain:

$$= \frac{1}{\cos A \sin A}$$

Now, since we have proved that both sides of the given equation are equal to the same expression, we are tempted to say that they are therefore equal to each other, and that we have therefore proved what we set out to prove.

We have indeed, except for one detail. We have not considered the values of A for which the given expressions and those which we substituted are meaningful.

This aspect of the proof of a trigonometric identity rarely leads to trouble, and may therefore usually be omitted. The careful student, however, will want to be prepared to investigate this question.

Thus we note that sec A and tan A are defined if and only if A is a real number of degrees not an odd multiple of 90; csc A and cot A are defined if and only if A is a real number of degrees not an even multiple of 90. Both sides of the equation are therefore defined if and only if A is a real number of degrees not an integer multiple of 90.

Each of the substitutions made in the parallel columns above is valid if A is such a number.

Therefore sec A csc A and tan A cot A are equal whenever both are defined, and the equation sec A csc A = tan A + cot A is an identity.

**• PROBLEM 962**

Show that tan (− v) = − tan v for every number v in the domain of the tangent function.

**Solution:** Working on the left side of the equation, we note that $\cos 2\theta$ can be rewritten as $\cos(\theta + \theta)$, to which we apply the formula for the cosine of the sum of two angles, $\cos(u + v) = \cos u \cos v - \sin u \sin v$. Thus, $\cos 2\theta = \cos(\theta + \theta) = \cos\theta\cos\theta - \sin\theta\sin\theta$ $= \cos^2\theta - \sin^2\theta$.

Replacing $\cos 2\theta$ by $\cos^2\theta - \sin^2\theta$ we obtain,

$$\frac{\cos 2\theta}{\cos\theta} = \frac{\cos^2\theta - \sin^2\theta}{\cos\theta}$$

Divide numerator and denominator by $\cos^2\theta$,

$$= \frac{\dfrac{\cos^2\theta - \sin^2\theta}{\cos^2\theta}}{\dfrac{\cos\theta}{\cos^2\theta}}$$

$$= \frac{\dfrac{\cos^2\theta}{\cos^2\theta} - \dfrac{\sin^2\theta}{\cos^2\theta}}{\dfrac{\cos\theta}{\cos^2\theta}}$$

$$\frac{1 - \dfrac{\sin^2\theta}{\cos^2\theta}}{\dfrac{\cos\theta}{\cos^2\theta}}$$

Recall that $\tan\theta = \sin\theta/\cos\theta$; thus

$$\tan^2\theta = \left(\frac{\sin\theta}{\cos\theta}\right)^2 = \frac{\sin^2\theta}{\cos^2\theta} \quad .$$

Substituting $\tan^2\theta$ for $\sin^2\theta/\cos^2\theta$ we obtain,

$$= \frac{1 - \tan^2\theta}{\dfrac{\cos\theta}{\cos^2\theta}}$$

$$= \frac{1 - \tan^2\theta}{\dfrac{1}{\cos\theta}} \quad .$$

Since $1/\cos\theta = \sec\theta$, replace $1/\cos\theta$ by $\sec\theta$; thus

$$\frac{1 - \tan^2\theta}{\sec\theta} \quad .$$

Therefore, we have shown that

$$\frac{\cos 2\theta}{\cos\theta} = \frac{1 - \tan^2\theta}{\sec\theta} \quad .$$

**• PROBLEM 965**

Prove that $\dfrac{\cos\theta}{1 - \sin\theta} = \dfrac{1 + \sin\theta}{\cos\theta}$ .

**Solution:** $\dfrac{\cos\theta}{1 - \sin\theta} = \dfrac{\cos\theta}{1 - \sin\theta} \cdot \dfrac{1 + \sin\theta}{1 + \sin\theta}$, since multiplication by 1 does not change the value of the fraction. Performing the multiplication we obtain: $\dfrac{\cos\theta(1 + \sin\theta)}{(1 - \sin\theta)(1 + \sin\theta)}$

Multiplying the terms in the numerator we obtain:

$$\frac{\sin^2\theta - \sin^4\theta + \sin^2\theta}{1-\sin^2\theta} = \frac{1-\left(1-2\sin^2\theta + \sin^4\theta\right)}{1-\sin^2\theta}$$

$$\frac{2\sin^2\theta - \sin^4\theta}{1-\sin^2\theta} = \frac{1-\left(1-2\sin^2\theta + \sin^4\theta\right)}{1-\sin^2\theta}$$

$$\frac{2\sin^2\theta - \sin^4\theta}{1-\sin^2\theta} = \frac{1-1+2\sin^2\theta - \sin^4\theta}{1-\sin^2\theta}$$

$$\frac{2\sin^2\theta - \sin^4\theta}{1-\sin^2\theta} = \frac{2\sin^2\theta - \sin^4\theta}{1-\sin^2\theta}$$

Since $\sin^2\theta + \tan^2\theta = \dfrac{2\sin^2\theta - \sin^4\theta}{1-\sin^2\theta}$ and

$$\sec^2\theta - \cos^2\theta = \frac{2\sin^2\theta - \sin^4\theta}{1-\sin^2\theta}$$

$$\sin^2\theta + \tan^2\theta = \sec^2\theta - \cos^2\theta$$

because two expressions equal to the same expression are equal to each other.

**• PROBLEM 967**

Prove the identity $\dfrac{\sec x + 1}{\sec x - 1} = \cot^2 \dfrac{x}{2}$.

Solution: Starting with the left side of the identity,

$$\frac{\sec x + 1}{\sec x - 1} = \frac{\frac{1}{\cos x} + 1}{\frac{1}{\cos x} - 1} \quad \text{since } \sec x = \frac{1}{\cos x}$$

Hence,

$$\frac{\sec x + 1}{\sec x - 1} = \frac{\frac{1}{\cos x} + \frac{\cos x}{\cos x}}{\frac{1}{\cos x} - \frac{\cos x}{\cos x}} \quad \text{for } 1 = \frac{\cos x}{\cos x}$$

Combining fractions,

$$= \frac{\frac{1 + \cos x}{\cos x}}{\frac{1 - \cos x}{\cos x}}$$

Dividing by a fraction is equivalent to multiplying by its reciprocal, hence,

$$= \frac{1 + \cos x}{\cos x} \cdot \frac{\cos x}{1 - \cos x}$$

Cancelling $\cos x$ from numerator and denominator we obtain:

$$\frac{\sec x + 1}{\sec x - 1} = \frac{1 + \cos x}{1 - \cos x} = \left(\pm\sqrt{\frac{1 + \cos x}{1 - \cos x}}\right)^2$$

Looking at the right side of the identity:

from the formula for the tangent of a half angle,

$$\tan \tfrac{1}{2}\theta = \tan \frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}}, \quad \text{and}$$

$$\cot \tfrac{1}{2}x = \cot \frac{x}{2} = \frac{1}{\tan \frac{x}{2}} = \frac{1}{\pm\sqrt{\frac{1 - \cos x}{1 + \cos x}}}$$

$$= \frac{1}{\pm\frac{\sqrt{1 - \cos x}}{\sqrt{1 + \cos x}}}$$

$$= (1)\left(\pm\frac{\sqrt{1 + \cos x}}{\sqrt{1 - \cos x}}\right) = \pm\frac{\sqrt{1 + \cos x}}{\sqrt{1 - \cos x}}$$

Hence, $\cot \frac{x}{2} = \pm\sqrt{\frac{1 + \cos x}{1 - \cos x}}$

Therefore, $\frac{\sec x + 1}{\sec x - 1} = \left(\cot \frac{x}{2}\right)^2 = \cot^2 \frac{x}{2}$.

**• PROBLEM 968**

Prove the identity

$$\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \sec \theta + \tan \theta.$$

<u>Solution:</u> Factor out $\frac{1}{2}$.

$$\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \tan \frac{1}{2}\left(\frac{\pi}{2} + \theta\right).$$

Then, apply the half-angle formula for the tangent.

$$\tan \frac{1}{2}\theta_1 = \pm\sqrt{\frac{1 - \cos \theta_1}{1 + \cos \theta_1}}.$$

Rationalize the denominator to obtain

$$\tan \frac{1}{2}\theta_1 = \pm\sqrt{\frac{1 - \cos \theta_1}{1 + \cos \theta_1}}\frac{\sqrt{1 - \cos \theta_1}}{\sqrt{1 - \cos_1}} = \frac{1 - \cos \theta_1}{\pm\sqrt{1 - \cos^2 \theta_1}}$$

$$= \frac{1 - \cos \theta_1}{\sqrt{\sin^2 \theta_1}}$$

$$\tan \frac{1}{2}\theta_1 = \frac{1 - \cos \theta_1}{\sin \theta_1}.$$

Replace $\theta_1$ by $\frac{\pi}{2} + \theta$.

$$\tan \frac{1}{2}\left(\frac{\pi}{2} + \theta\right) = \frac{1 - \cos\left(\frac{\pi}{2} + \theta\right)}{\sin\left(\frac{\pi}{2} + \theta\right)}$$

Apply the formula for the sum of the sine of two angles and the cosine of the sum of two angles.

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\cos\left(\frac{\pi}{2} + \theta\right) = \cos\frac{\pi}{2}\cos\theta - \sin\frac{\pi}{2}\sin\theta$$

$$= 0(\cos\theta) - \sin\theta = -\sin\theta.$$

$$\sin\left(\frac{\pi}{2} + \theta\right) = \sin\frac{\pi}{2}\cos\theta + \cos\frac{\pi}{2}\sin\theta$$

$$= 1\cos\theta + 0(\sin\theta)$$

$$= \cos\theta$$

Substitute these two results.

$$\tan\frac{1}{2}\theta_1 = \frac{1 - \cos\theta_1}{\sin\ _1} = \tan\frac{1}{2}\left(\frac{\pi}{2} + \theta\right) = \frac{1 - \cos\left(\frac{\pi}{2} + \theta\right)}{\sin\left(\frac{\pi}{2} + \theta\right)}$$

$$= \frac{1 - (-\sin\theta)}{\cos\theta} = \frac{1 + \sin\theta}{\cos\theta}$$

$$= \frac{1}{\cos\theta} + \frac{\sin\theta}{\cos\theta}$$

$$= \sec\theta + \tan\theta.$$

# CHAPTER 30

# POLAR COORDINATES

**Basic Attacks and Strategies for Solving Problems in this Chapter. See pages 706 to 711 for step-by-step solutions to problems.**

Some problems in analytic geometry, especially involving motion about a point, are difficult to solve using the $x$-$y$ coordinate system. In fact, such equations as

$$x^2 + y^2 - 2x = 2\sqrt{x^2 + y^2}$$

are unwieldly in the rectangular coordinate system, but are manageable in the polar coordinate system.

To set up the polar coordinate system in a plane, begin with a fixed point $O$ (called the origin or pole) and a directed half-line (called the polar axis) with its endpoint at $O$ drawn horizontally to the right. To each point $P$ in the plane, the polar coordinates $(r, \theta)$ can be assigned. The angle $\theta$ has the polar axis as its initial side and the half-line $OP$ as its terminal side. The number $r$ indicates the distance from the pole to the point $P$. There is an infinite number of ordered pairs associated with $(r, \theta)$

For instance, a point whose coordinates are $(3, 45°)$ may be described by

$$(3, 405°), (3, 765°), (-3, 225°), (-3, -135°), (3, -315°)$$

to name the function. Polar coordinate paper is useful in plotting points.

We can superimpose an $xy$-plane on the $r\theta$-plane with the positive $x$-axis coinciding with the polar axis. Then, the following polar-rectangular relationships can be deduced.

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$\tan\theta = {}^{y}/_{x} \qquad r^2 = x^2 + y^2$$

These relationships provide the basis for changing polar equations to equations in rectangular form and vice versa. For example, using the above relationships

$$x^2 + y^2 - 2x = 2\sqrt{x^2 + y^2}$$

# Step-by-Step Solutions to Problems in this Chapter, "Polar Coordinates"

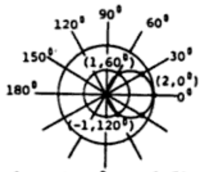
**• PROBLEM 969**

What is the graph of $\rho = 3$?

**Solution:** This equation says that for all values of $\theta$, $\rho = 3$. Thus for $\theta = 0^\circ$, $\rho = 3$; $\theta = 30^\circ$, $\rho = 3$; etc. The graph is a circle of radius 3 with the center at the origin.

**• PROBLEM 970**

Draw the graph of $\rho = 2\cos\theta$.



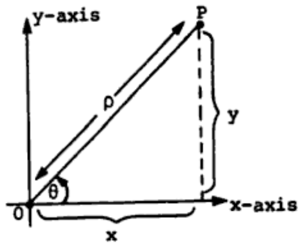

**Solution:** We assign values to $\theta$ and find the corresponding values of $\rho$, giving the following table:

| $\theta$ | $\cos\theta$ | $\rho=2\cos\theta$ |
|---|---|---|
| 0° | 1 | 2 |
| 30° | .87 | 1.74 |
| 60° | .5 | 1 |
| 90° | 0 | 0 |
| 120° | -.5 | -1 |
| 150° | -.87 | -1.74 |
| 180° | -1 | -2 |
| Values from 180° to 360° give the same points. (Check this.) | | |

We then plot the points $(\rho,\theta)$ and draw a smooth curve through them. We get the graph of the figure. The equation which defines the path of P may involve only one of the variables $(\rho,\theta)$. In that case the variable which is not mentioned may have any and all values.

**• PROBLEM 971**

Transform the equation $x^2 + y^2 - x + 3y = 3$ to a polar equation.




**Solution:** Ordinarily, when we wish to locate a point in a plane, we draw a pair of perpendicular axes and measure specified signed distances from the axes. The points are designated by pairs in terms of (x,y). These are called rectangular coordinates.

Another way is to designate a point in terms of polar coordinates. $(\rho,\theta)$ are the polar coordinates of a point P where $\rho$ is the radius vector of P and $\theta$ is the angle that is made with the positive x-axis and the radius vector, OP. (See diagram.)

If P is designated by the coordinates (x,y) in rectangular coordinates and by $(\rho,\theta)$ in polar coordinates, then the following relationships hold:

$$\text{Cos } \theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{x}{\rho} \quad \text{or} \quad x = \rho \text{ Cos } \theta$$

$$\text{Sin } \theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{y}{\rho} \quad \text{or} \quad y = \rho \text{ Sin } \theta$$

Now in this example, we replace x by $\rho$ Cos $\theta$ and y by $\rho$ Sin $\theta$ to obtain:

$$(\rho \text{ Cos } \theta)^2 + (\rho \text{ Sin } \theta)^2 - (\rho \text{ Cos } \theta) + 3(\rho \text{ Sin } \theta) = 3$$

$$\rho^2 \text{ Cos}^2 \theta + \rho^2 \text{ Sin}^2 \theta - \rho \text{ Cos } \theta + 3\rho \text{ Sin } \theta = 3$$

Factor out $\rho^2$ and $-\rho$.

$$\rho^2(\text{Cos}^2 \theta + \text{Sin}^2 \theta) - \rho(\text{Cos } \theta - 3\text{Sin } \theta) = 3$$

Apply the identity $\text{Cos}^2 \theta + \text{Sin}^2 \theta = 1$. Then,

$$\rho^2 - \rho(\text{Cos } \theta - 3 \text{ Sin } \theta) = 3$$

**• PROBLEM 972**

Transform the equation $\rho = 2 \cos \theta$ to rectangular coordinates.

**Solution:**

$$\rho = \sqrt{x^2 + y^2}, \qquad \cos \theta = \frac{x}{\rho} = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\rho = 2 \cos \theta$$

$$\sqrt{x^2 + y^2} = \frac{2x}{\sqrt{x^2 + y^2}}$$

$$x^2 + y^2 = 2x$$

Transform the equation $xy = 4$ to polar coordinates.

Solution:

$$x = \rho \cos \theta$$

$$y = \rho \sin \theta$$

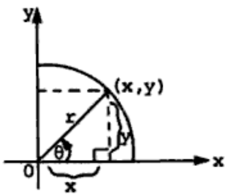
$$xy = 4$$

$$\rho \cos \theta \cdot \rho \sin \theta = 4$$

$$\rho^2 \cos \theta \sin \theta = 4$$

• PROBLEM 974

Transform the equation $r = 4/(2 - 3 \sin \theta)$ to an equation in cartesian coordinates.




Solution: The given equation is in polar coordinates. This is another system of coordinates where a point (x, y) lies on a circle of radius r whose center is the origin. (see Figure.)

We want to replace r and $\sin \theta$ by rectangular coordinates. Observe the following needed substitutions which can be derived from the diagram.

$$\sin \theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{y}{r}$$

Pythagorean Identity $\quad x^2 + y^2 = r^2$

Solving for r: $\quad \sqrt{x^2 + y^2} = r$

Then we proceed as follows:

$r = \dfrac{4}{2 - 3y/r}$ replacing $\sin \theta$ by $y/r$

$= \dfrac{4r}{2r - 3y}$ simplifying the complex fraction

$2r - 3y = 4$ multiplying each member by $(2r - 3y)/r$

$2\sqrt{x^2 + y^2} = 4 + 3y$ replacing r by $\sqrt{x^2 + y^2}$ and adding 3y to each member

| | |
|---|---|
| $4x^2 + 4y^2 = 9y^2 + 24y + 16$ | equating the squares of each member |
| $4x^2 - 5y^2 - 24y = 16$ | adding $-9y^2 - 24y$ to each member |

**• PROBLEM 975**

Convert the equation $r = \tan\theta + \cot\theta$ to an equation in cartesian coordinates.

Fig. A

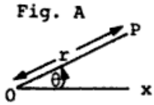
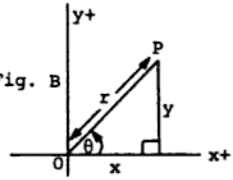



Fig. B




Solution: The given equation is expressed in polar coordinates $(r, \theta)$ where r is the radius vector, OP, and $\theta$ is the angle that r makes with the polar axis, OX. O is the fixed point called the pole. See figure A.

Since $\tan\theta \neq -\cot\theta$, then $r \neq 0$, and the graph of

$$r = \tan\theta + \cot\theta$$

does not pass through the pole. If r were equal to zero, then the curve would pass through (0,0). Therefore in the transformation of this equation to cartesian coordinates, we must remember that $(x, y) \neq (0,0)$. Now we must convert all expressions of r and $\theta$ into rectangular coordinates (x, y). If P is designated by the coordinates (x, y) in rectangular coordinates and by $(r, \theta)$ in polar coordinates, then the following relationships hold true: (see Figure B).

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{y}{x}$$

$$\cot\theta = \frac{\text{adjacent side}}{\text{opposite side}} = \frac{x}{y}$$

By the Pythagorean Identity $x^2 + y^2 = r^2$

Solve for r: $r = \sqrt{x^2 + y^2}$

Substitute these values for r, $\tan\theta$, and $\cot\theta$.

$$\sqrt{x^2 + y^2} = \frac{y}{x} + \frac{x}{y}$$

$$xy\sqrt{x^2 + y^2} = x^2 + y^2$$

Divide by $\sqrt{x^2 + y^2}$

$$xy = \frac{x^2 + y^2}{\sqrt{x^2 + y^2}}$$

Rationalize the denominator by multiplying the numerator and denominator by $\sqrt{x^2 + y^2}$

$$xy = \frac{x^2 + y^2}{\sqrt{x^2 + y^2}} \; \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2}}$$

$$xy = \frac{x^2 + y^2}{x^2 + y^2} \sqrt{x^2 + y^2}$$

$$xy = \sqrt{x^2 + y^2}$$

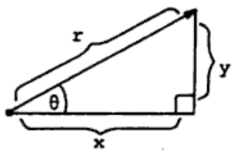
Squaring both sides, we obtain:

$$x^2y^2 = x^2 + y^2$$

where $x \neq 0$ and $y \neq 0$.

**• PROBLEM 976**

> Discuss the graph of the equation
>
> $$r = 4 \cos \theta - 2 \sin \theta .$$




Solution: Instead of plotting points of the form $(r,\theta)$, we will write this equation in terms of cartesian coordinates $x$ and $y$. Cartesian coordinates $(x,y)$ and polar coordinates $(r,\theta)$ are related by the following equations: (noting the figure,)

$$\sin \theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{y}{r} \qquad (1)$$

$$\cos \theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{x}{r} \qquad (2)$$

and by the Pythagorean Theorem,

$$r^2 = x^2 + y^2 . \qquad (3)$$

First, we multiply both sides of the given equation by $r$ to obtain the equation

$$r^2 = 4r \cos \theta - 2r \sin \theta \qquad (4)$$

From equations (1), (2), and (3), the new equation (equation (4)) becomes:

$$x^2 + y^2 = 4\cancel{r}\left(\frac{x}{\cancel{r}}\right) - 2\cancel{r}\left(\frac{y}{\cancel{r}}\right)$$

$$x^2 + y^2 = 4x - 2y \qquad (5)$$

Subtract $(4x - 2y)$ from both sides of equation (5):

$$x^2 + y^2 - (4x - 2y) = 4x - 2y - (4x - 2y)$$
$$x^2 + y^2 - 4x + 2y = 0$$
$$x^2 - 4x + y^2 + 2y = 0 \qquad (6)$$

Now complete the square in both $x$ and $y$. This is done by taking half the coefficient of the $x$ term (or $y$ term) and then squaring this value. The result is then added to both sides of equation (6). Completing the square in $x$:

$$[\tfrac{1}{2}(-4)]^2 = (-2)^2 = 4 .$$

Hence, equation (6) becomes:

$$(x^2 - 4x + 4) + y^2 + 2y = 0 + 4$$

or

$$(x^2 - 4x + 4) + y^2 + 2y = 4$$

or

$$(x - 2)^2 + y^2 + 2y = 4 \qquad (7)$$

Completing the square in $y$:

$$[\tfrac{1}{2}(2)]^2 = (1)^2 = 1.$$

Hence, equation (7) becomes:

$$(x - 2)^2 + (y^2 + 2y + 1) = 4 + 1$$

or

$$(x - 2)^2 + (y^2 + 2y + 1) = 5$$

or

$$(x - 2)^2 + (y + 1)^2 = 5 \qquad (8)$$

Also, note that the equation of a circle is:

$$(x - h)^2 + (y - k)^2 = r^2$$

where $(h,k)$ are the coordinates of the center of the circle and $r$ is the radius of the circle. Equation (8) is in the form for the equation of circle where:

$x - 2$ corresponds to $x - h$; i.e., $h = 2$,

$y + 1$ corresponds to $y - k$; that is, $y + 1 = y - k$

$$y + 1 - y = y - k - y$$
$$1 = -k$$
$$-1(1) = (-1)(-k)$$
$$-1 = k;$$

and $r^2$ corresponds to 5; that is; $r = \sqrt{5}$ .

Therefore, the original equation given in polar coordinates $(r,\theta)$ represents a circle of center $(h,k) = (2,-1)$ and radius $= r = \sqrt{5}$.

# CHAPTER 31

# VECTORS AND COMPLEX NUMBERS

**Basic Attacks and Strategies for Solving Problems in this Chapter.** **See pages 712 to 741 for step-by-step solutions to problems.**

To find the sum of two vectors $A$ and $B$, written as $A + B$, we place the initial point of vector $B$ at the terminal point of vector $A$. The vector with the same initial point as $A$ and the same terminal point as $B$ is the vector sum $A + B$.

Another way to find the sum of two vectors $A$ and $B$ is to use the parallelogram rule. To apply the rule place vectors $A$ and $B$ so that their initial points coincide. Then, complete a parallelogram which has $A$ and $B$ as sides. The diagonal of the parallelogram with the same initial point as $A$ and $B$ is the vector sum $A + B$.

The length of a vector $V = (x, y)$, expressed in rectangular components, is called the magnitude, denoted by $|V|$ and defined by

$$|V| = \sqrt{x^2 + y^2}$$

The magnitude of the resultant sum of two vectors $A$ and $B$ can be determined by using the formula for the Law of Cosines

$$|V|^2 = x^2 + y^2 - 2xy\cos\theta$$

$$|V| = \sqrt{x^2 + y^2 - 2xy\cos\theta},$$

where $\theta$ is each of the angles of the parallelogram adjacent to the angle at the initial point of the diagonal vector $V$.

The standard or rectangular form of a complex number is

$$z = a + bi.$$

The trigonometric polar form of a complex number is

$$z = r(\cos\theta + i\sin\theta),$$

when

$$r = \sqrt{a^2 + b^2}, \ \cos\theta = \frac{a}{r}, \ \sin\theta = \frac{b}{r},$$

# Step-by-Step Solutions to Problems in this Chapter, "Vectors and Complex Numbers"

## VECTORS

**• PROBLEM 977**

Which of the following vectors are equal to $\overrightarrow{MN}$ if $M = (2, 1)$ and $N = (3, -4)$?
(a) $\overrightarrow{AB}$, where $A = (1, -1)$ and $B = (2, 3)$
(b) $\overrightarrow{CD}$, where $C = (-4, 5)$ and $D = (-3, 10)$
(c) $\overrightarrow{EF}$, where $E = (3, -2)$ and $F = (4, -7)$.

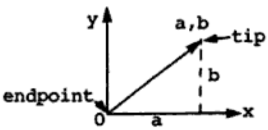


Fig. A: (a-0,b-0) represents the vector.

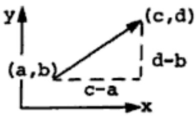


Fig. B: (c-a,d-b) represents the vector.

Solution: With each ordered pair in the plane there can be associated a vector from the origin to that point.

The vector is determined by subtracting the coordinates of the endpoint from the corresponding coordinates of the tip. As for $\overrightarrow{MN}$, the tip is the point corresponding to the second letter of the alphabetical notation, N, while the endpoint is the point corresponding to the first, M. In this problem the vectors are of a general nature wherein their endpoints do not lie at the origin.

We first find the ordered pair which represents $\overrightarrow{MN}$.

$$\overrightarrow{MN} = (3 - 2, -4 - 1) = (1, -5)$$

Now, we find the ordered pair representing each vector.

(a) $\overrightarrow{AB} = (2 - 1, 3 - (-1)) = (1, 4)$

(b) $\overrightarrow{CD} = ((-3) - (-4), 10 - 5) = (1, 5)$

(c) $\overrightarrow{EF} = (4 - 3, -7 - (-2)) = (1, -5)$

Only $\overrightarrow{EF}$ and $\overrightarrow{MN}$ are equal.

• **PROBLEM 978**

> A force of 315 lbs. is acting at an angle of 67° with the horizontal. What are its horizontal and vertical components?

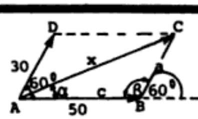

**Solution:** Construct the figure shown.

OR = vector force = c.
b = OA = horizontal component.
a = OB = vertical component.

| In △OAR: c = 315; α = 67°. | | |
|---|---|---|
| $\frac{a}{c} = \sin\alpha$, or $a = c\sin\alpha$. | log c = 2.49831<br>log sin α = 9.96403 − 10<br>log a = 2.46234 | a = 289.96 lbs. |
| $\frac{b}{c} = \cos\alpha$, or $b = c\cos\alpha$. | log c = 2.49831<br>log cos α = 9.59188 − 10<br>log b = 2.09019 | b = 123.08 lbs. |

• **PROBLEM 979**

> Two forces of 50 lbs. and 30 lbs. have an included angle of 60°. Find the magnitude and direction of their resultant.




**Solution:** Construct the parallelogram and label it as in the figure. Since $\overline{AD}$ is parallel to $\overline{BC}$ we have $\angle ABC = \beta = 180° - 60° = 120°$.
By the law of cosines:

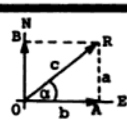
$$x^2 = c^2 + a^2 - 2\,ac\cos\beta$$
$$= 2500 + 900 - 2(50)(30)(-\tfrac{1}{2})$$
$$= 2500 + 900 + 1500 = 4900.$$
$$x = 70 \text{ lbs.}$$

$$\cos\alpha = \frac{x^2 + c^2 - a^2}{2xc} = \frac{4900 + 2500 - 900}{2(70)(50)} = \frac{13}{14} = .9286.$$

$$\alpha = 21°47'.$$

• **PROBLEM 980**

> Two forces act simultaneously on a body free to move. One force of 112 lbs. is acting due east, while the other of 88 lbs. is acting due north. Find the magnitude and direction of their resultant.



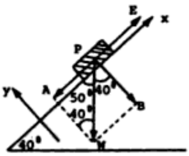
**Solution:** Construct the figure shown.

OA = b = 112 lbs.
OB = 88 lbs. = RA = a.

| In △OAR: a = 88; b = 112. | | |
|---|---|---|
| $\frac{a}{b} = \tan\alpha$. | log a = 1.94448<br>log b = 2.04922<br>log tan α = 9.89526 − 10 | $\boxed{\alpha = 38°9'25''}$ |
| $\frac{a}{c} = \sin\alpha$, or<br>$c = \frac{a}{\sin\alpha}$ | log a = 11.94448 − 10<br>log sin α = 9.79086 − 10<br>log c = 2.15362 | $\boxed{c = 142.44}$ |
| Therefore the resultant is 142.44 lbs. and its direction is 38°9'25" north of east. | | |

**• PROBLEM 981**

Find the force required to prevent a 500-pound object from sliding down a 40° incline, disregarding friction.




Solution: The weight of the object acts as a 500-pound force vertically downward. The component of the force parallel to the plane tends to force the object down the incline. The component perpendicular to the plane tends to force the object against the plane. The required force is parallel to the plane, equal in magnitude and opposite in direction to the parallel component. Observing the figure, the triangle WPB is similar to the triangle made by the inclined plane and the ground. Angle WPB therefore equals 40°. It follows then that angle APW equals 50°, as it is complementary to angle WPB. By super posing the coordinate axes with the x axis parallel to the inclined plane, observe that the component of the force to be determined lies parallel to the inclined plane and can be found by multiplying the magnitude of weight by the cos of 50°. Here cos 50° can be calculated using the rule cos θ = adjacent/hypotenuse for right triangles. The adjacent side is $\overrightarrow{AP}$ and the hypotenuse is $\overrightarrow{PW}$. Therefore cos 50° = AP/PW or (PW) cos 50° = AP, where Ap and PW represent the magnitudes of the force vectors.

$$\overrightarrow{PW} = 500, \quad \measuredangle APW = 90° - 40° = 50°$$

$$\cos \measuredangle APW = \frac{AP}{PW}, \quad \text{where } AP = |\overrightarrow{AP}|$$

$$AP = PW \cos \measuredangle APW$$

$$AP = 500 \cos 50°$$

$$AP = 500(0.6428) = 321$$

The required force $\overrightarrow{PE}$ is 321 pounds.

Find the magnitude and direction of the force necessary to counteract the effect of a force of 60 pounds and a force of 40 pounds that act on a point at an angle of 60° with each other.

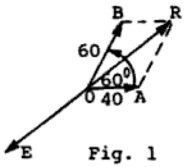



Fig. 1

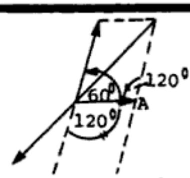



Fig. 2

Solution: The problem requires finding the equilibrant of the two forces. The equilibrant is equal in magnitude, but opposite in direction, to the resultant force. We find the resultant force of two vectors by using the parallelogram law of addition of vectors. Two vectors are drawn and the parallelogram is completed. The resultant is drawn connecting the two opposite vertices (see figure 1). We indicate the magnitude of a vector as the corresponding segment. In the parallelogram, resultant $\overrightarrow{OR}$ may be found by solving triangle OAR.

$$OA = BR = 40$$
$$AR = OB = 60$$
$$\angle A = 180° - 60° = 120° \text{(see figure 2)}$$

By the Law of Cosines:

$$(OR)^2 = (OA)^2 + (AR)^2 - [2(OA)(AR)\cos 120°]$$
$$(OR)^2 = (40)^2 + (60)^2 - [(2)(40)(60)(-\tfrac{1}{2})]$$
$$(OR)^2 = 7{,}600$$
$$OR = 87$$

From the Law of Sines,

$$\sin\angle AOR = \frac{60 \sin 120°}{87} = \frac{60(0.8660)}{87} = 0.5972$$

$\angle AOR = 37°$ (to the nearest degree)

$\angle AOE = 180° - 37° = 143°$, and the required

force is a force of 87 pounds, 143° from the 40-pound force in the opposite direction from the 87-pound force.

## RECTANGULAR AND POLAR/TRIGONOMETRIC FORMS OF COMPLEX NUMBERS

• PROBLEM 983

Find the amplitude and the modulus of 5 - 3i.

Solution: The complex number 5 - 3i is expressed in the form a + ib. Here the modulus is the length (distance) r from the origin 0 to the point (a,b) and the amplitude is the angle θ, measured clockwise, that

Fig. A

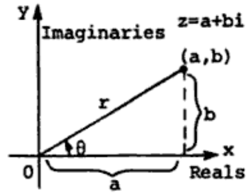
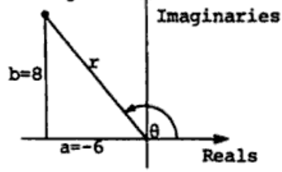



Fig. B




**Solution:** We are given a complex number z in the form of a + bi, where a and b are real numbers and (a,b) is the corresponding point in the cartesian plane. The value of a is found on the real axis and b is located on the imaginary axis. (See Figure A). Now let r denote the distance between the origin and the point which represents z and let θ be an angle in standard position whose terminal side contains the point z. We want to express a and b in terms of θ.

$$\cos\theta = \frac{a}{r} \quad\Rightarrow\quad a = r\cos\theta$$

$$\sin\theta = \frac{b}{r} \quad\Rightarrow\quad b = r\sin\theta$$

Thus, $a + bi = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$.

In this example, $a = -6$ and $b = 8$ (see Figure B). To find r, apply the Pythagorean theorem.

$$r^2 = a^2 + b^2 = (-6)^2 + (8)^2 = 100\ ;$$

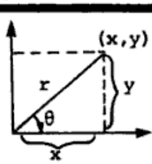
thus, $r = 10$.

For θ: $\tan\theta = \frac{b}{a} = \frac{8}{-6} = -\frac{4}{3} \approx -1.333 \ldots$. First look up in a table of trigonometric functions the reference angle whose tangent is 4/3. It is $53.1^\circ$. But θ is in Quadrant II as we note from Figure B. Hence $\theta = 180^\circ - 53.1^\circ = 126.9^\circ$. Therefore, since $a + bi = r(\cos\theta + i\sin\theta)$,

$$-6 + 8i = 10(\cos 126.9^\circ + i\sin 126.9^\circ).$$

**• PROBLEM 986**

Express each of the following in trigonometric form.
(a) $-\sqrt{2} + \sqrt{2}i$ (b) $3 - 4i$ (c) $2 + i$




**Solution:** In the plane, a complex number is represented as x + iy.

Therefore the angle θ can be defined as $\arctan\frac{y}{x} = \theta$ or $\tan\theta = \frac{y}{x}$.

In part a the x coordinate is negative and the y coordinate is positive, therefore θ must lie in the second quadrant.

$$r_1^2 = 4^2 + (-4)^2 = 32$$

$$r_1 = \sqrt{32} = 4\sqrt{2}$$

$$r_2^2 = (\sqrt{3})^2 + (-1)^2 = 3 + 1 = 4$$

$$r_2 = 2$$

Since all three sides of both triangles are known, any of the trigonometric functions can be used to determine $\theta$. i.e. $\sin\theta = x/r$; $\cos\theta = y/r$; $\tan\theta = x/y$

Once $\theta_1$ and $\theta_2$ have been determined the multiplication can be performed according to the formula

$$r_1(\cos\theta_1 + i\sin\theta_1) \cdot r_2(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1r_2\big(\cos[\theta_1 + \theta_2] + i\sin[\theta_1 + \theta_2]\big).$$

$$\tan\theta_1 = \frac{4}{-4} \rightarrow \theta_1 = \tan^{-1}(-1) = 315°, \text{ fourth quadrant.}$$

$$\tan\theta_2 = \frac{\sqrt{3}}{-1} \rightarrow \theta_2 = \tan^{-1}(-\sqrt{3}) = 330°, \text{ fourth quadrant.}$$

Changing $4 - 4i$ and $\sqrt{3} - i$ to polar form, we obtain $4 - 4i = 4\sqrt{2}\,(\cos 315° + i\sin 315°)$ and $\sqrt{3} - i = 2(\cos 330° + i\sin 330°)$. Thus, $(4 - 4i)(\sqrt{3} - i) = 4\sqrt{2}\,(\cos 315° + i\sin 315°) \cdot 2(\cos 330° + i\sin 330°) = 8\sqrt{2}\,(\cos 645° + i\sin 645°) = 8\sqrt{2}(\cos 285° + i\sin 285°)$.

**• PROBLEM 988**

Express each of the following in rectangular form, $a + bi$.   (a) $3(\cos 30° + i\sin 30°)$

(b) $10(\cos 180° + i\sin 180°)$

Solution: The complex numbers as given are in the trigonometric form

$$r(\cos\theta + i\sin\theta)$$

in part a, $\theta = 30°$,

(a) $3(\cos 30° + i\sin 30°) = \frac{3}{2}\sqrt{3} + \frac{3}{2}i$

in part b, $\theta = 180°$,

(b) $10(\cos 180° + i\sin 180°) = 10(-1 + i \cdot 0) = -10$

Check: $r^2 = x^2 + y^2$

part a: $(3)^2 = \left(\frac{3\sqrt{3}}{2}\right)^2 + \left(\frac{3}{2}\right)^2 = \frac{27}{4} + \frac{9}{4} = 9$

part b: $(10)^2 = (-10)^2$

**• PROBLEM 989**

Find $[2(\cos 30° + i\sin 30°)][8(\cos 60° + i\sin 60°)]$.

**Solution:** The complex numbers are written in the form $r(\cos\theta + i\sin\theta)$.

Therefore, the division of these two numbers is performed by dividing the first modulus by the second, and subtracting the second angle from the first according to the formula:

$$\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$= \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$$

$$r_1 = 8$$

$$r_2 = 2$$

$$\frac{r_1}{r_2} = \frac{8}{2} = 4$$

$$\theta_1 = \pi/2$$

$$\theta_2 = \pi/6$$

$$\theta_1 - \theta_2 = \pi/2 - \pi/6 = \pi/3 = \text{ampltitude}$$

$$8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) \div 2\cos\left(\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

$$= 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

Check $8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 8(0 + i) = 8i.$

$$2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \sqrt{3} + i$$

$$\frac{8i}{\sqrt{3} + i} = \frac{8i}{\sqrt{3} + i} \cdot \frac{\sqrt{3} - i}{\sqrt{3} - i} = \frac{8 + 8\sqrt{3}i}{3 - i^2}$$

$$= \frac{8 + 8\sqrt{3}i}{4} = 2 + 2\sqrt{3}i$$

$$4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 4\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 2 + 2\sqrt{3}i$$

**• PROBLEM 992**

Show $$\frac{r_1(\cos\theta + i\sin\theta)}{r_2(\cos\phi + i\sin\phi)}$$

$$= \frac{r_1}{r_2}\left[\cos(\theta - \phi) + i\sin(\theta - \phi)\right]$$

For finding n roots just raise the complex number to a rational power which corresponds, i.e. 1/n:

$$(r[\cos(\theta + 2k\pi) + i \sin(\theta + 2k\pi)])^{1/n}$$

$$= r^{1/n}\left(\cos\left(\frac{\theta + 2k\pi}{n}\right) + i \sin\left(\frac{\theta + 2k\pi}{n}\right)\right)$$

Therefore, for the fifth roots of 2:

$$2^{\frac{1}{5}} = (2\ [\cos(0 + 2k\pi) + i \sin(0 + 2k\pi)])^{\frac{1}{5}}$$

$$= 2^{\frac{1}{5}}\left(\cos\frac{2k\pi}{5} + i \sin\frac{2k\pi}{5}\right) \quad k = 0, 1, 2, \ldots$$

Now substitute values of k from 0 to 4, inclusive.

For $k = 0$, $\quad r_0 = 2^{\frac{1}{5}}(\cos 0 + i \sin 0)$

For $k = 1$, $\quad r_1 = 2^{\frac{1}{5}}\left(\cos\frac{2\pi}{5} + i \sin\frac{2\pi}{5}\right)$

For $k = 2$, $\quad r_2 = 2^{\frac{1}{5}}\left(\cos\frac{4\pi}{5} + i \sin\frac{4\pi}{5}\right)$

For $k = 3$, $\quad r_3 = 2^{\frac{1}{5}}\left(\cos\frac{6\pi}{5} + i \sin\frac{6\pi}{5}\right)$

For $k = 4$, $\quad r_4 = 2^{\frac{1}{5}}\left(\cos\frac{8\pi}{5} + i \sin\frac{8\pi}{5}\right)$

For $k > 4$, we obtain the same cycle of values. Hence, the fifth roots of 2 are those values designated as $r_0$, $r_1$, $r_2$, $r_3$, and $r_4$.

**• PROBLEM 996**

Find the four fourth roots of 9. The polar form of 9 is $9(\cos 0^\circ + i \sin 0^\circ)$.

**Solution:** 9 is a real number. This implies that its imaginary component equals zero, (i.e., $9 + 0i$). In its polar representation, the angle $\theta$ must be such that the $\sin\theta = 0$ and the $\cos\theta = 1$, $0^\circ$ satisfies this relationship. Therefore

$$9 = 9(\cos 0^\circ + i \sin 0^\circ).$$

To find the $4^{th}$ roots of 9, we use the formula

$$w^b = r^b(\cos\theta + i \sin\theta)^b = r^b\left(\cos(b\theta) + i\sin(b\theta)\right)$$

where b is any rational number. We must also keep in mind that since cos and sin are periodic, whenever $\theta$ satisfies a given relationship, $\theta + 2k\pi$, where k is an integer, satisfies that relationship also, when we seek the roots of a number, i.e., $w^{1/n}$, we allow k to assume values from $0,\ldots,n-1$. In our problem $\theta = 0^\circ$, $n = 4$ and we

**Find the equations for $\sin 2\theta$ and $\cos 2\theta$ from the de Moivre equation with $n = 2$.**

Solution: Let a complex number be expressed in polar form, $r(\cos\theta + i\sin\theta)$ where $r$ is the radius vector and $\theta$ is the angle made between the x-axis and $r$. Then de Moivre's Theorem states that if $n$ is any rational number, then

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta).$$

Furthermore we can see that:

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta).$$

Also the complex exponential function is:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

If we substitute $n = 2$,

$$(\cos\theta + i\sin\theta)^2 = (\cos 2\theta + i\sin 2\theta) = \left(e^{i\theta}\right)^2$$
$$= e^{i\theta}\cdot e^{i\theta} = e^{i\theta+i\theta} = e^{2i\theta}$$

Expand the expression $(\cos\theta + i\sin\theta)^2$.

$$(\cos\theta + i\sin\theta)^2 = (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)$$
$$= \cos^2\theta + 2i\sin\theta\cos\theta + i^2\sin^2\theta$$

Noting that $i^2 = -1$, we obtain:

$$(\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta \qquad (1)$$

Furthermore by de Moivre's Theorem for the case $n = 2$, we have

$$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta \qquad (2)$$

Equate the right side of equations (1) and (2) to obtain:

$$\cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta = \cos 2\theta + i\sin 2\theta$$
$$\cos^2\theta - \sin^2\theta + i2\sin\theta\cos\theta = \cos 2\theta + i\sin 2\theta$$

Equate the real and imaginary parts.

$$\cos^2\theta - \sin^2\theta = \cos 2\theta \qquad (3)$$
$$i2\sin\theta\cos\theta = i\sin 2\theta \qquad (4)$$

Note that, after dividing both sides by $i$, equation (4) becomes:

$$2\sin\theta\cos\theta = \sin 2\theta$$

Therefore, the expressions for $\sin 2\theta$ and $\cos 2\theta$ are:

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\sin 2\theta = 2\sin\theta\cos\theta.$$

## OPERATIONS WITH COMPLEX NUMBERS



**Express each of the following as the product of $i$ and a real number.**

**(a) $2i^5$ (b) $\frac{-5}{-7}$ (c) $\sqrt{-81}$**

Solution: Recalling that $\sqrt{-1} = i$ $\left(\text{or } -1 = i^2\right)$:

(a) $2i^5 = 2 \cdot i^4 \cdot i = 2 \cdot 1 \cdot i = 2i$

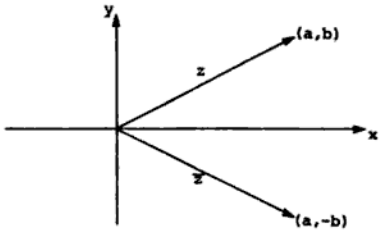
(b) $\dfrac{-5}{i^7} = \dfrac{-5}{i^4 \cdot i^2} \cdot \dfrac{1}{i} = \dfrac{-5}{1 \cdot -1} \cdot \dfrac{1}{i} = \dfrac{-5}{1 \cdot -1} \cdot \dfrac{-i^2}{i} = -5i$

Note that $1 = -(-1) = -\left(i^2\right) = -i^2$ . Hence $\dfrac{1}{i} = \dfrac{-i^2}{i}$

(c) $\sqrt{-81} = \sqrt{(-1)(81)} = \sqrt{-1} \cdot \sqrt{81} = 9i$

**• PROBLEM 1001**

What is the conjugate of $3 - 2i$ and the conjugate of $5 + 7i$?




Solution: Any complex number may be interpreted as an ordered pair in the plane with the real component designated by the $x$ value and the imaginary part designated by the $y$ value. The conjugate of a complex number is that number which when multiplied by the original complex number yields a product which is purely real. Geometrically, the complex conjugate is a reflection of the complex number through the x-axis. The complex conjugate of $3 - 2i$ is $3 + 2i$

i.e., $$(3 - 2i)(3 + 2i) = 13 .$$

The conjugate of $5 + 7i$ is $5 - 7i$

$$(5 + 7i)(5 - 7i) = 74 .$$

The conjugate of a pure real number $a$, which can be written $a + 0i$, is merely itself or $a - 0i$. Geometrically we see that the reflection of a real number is actually itself. The conjugate of a pure imaginary number $bi$ is $-bi$. The conjugate of a complex number $a + bi$ is $a - bi$.

**• PROBLEM 1002**

Add $(3 + 4i)$ and $(2 - 5i)$.

Solution: Numbers of the form $a + bi$, where $a$ and $b$ are real numbers, are called complex numbers. In the complex number $a + bi$, $a$ is called the real part and $bi$ is called the imaginary part. To add two complex numbers, add the real parts and add the pure imaginary parts. Therefore: we have

$$\begin{aligned}(3 + 4i) + (2 - 5i) &= (3 + 2) + (4 - 5)i \\ &= 5 + (-i) \\ &= 5 - i\end{aligned}$$

Or we may treat the problem as the sum of two binomials:

$$\begin{aligned}(3 + 4i) + (2 - 5i) &= 3 + 4i + 2 - 5i \\ &= 5 - i\end{aligned}$$

• PROBLEM 1003

Perform the indicated operations: $(-2 + 3i) + [5 + (-6)i]$.

Solution: Addition of complex numbers is defined in the following way:

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$

where $a,b,c,d$ are real numbers. Thus,

$$\begin{aligned}(-2 + 3i) + [5 + (-6)i] &= (-2 + 5) + [3 + (-6)]i \\ &= 3 + (-3)i \\ &= 3 - 3i.\end{aligned}$$

• PROBLEM 1004

Write each of the following in the form $a + bi$.

a) $(2 + 4i) + (3 + i)$

b) $(2 + i) - (4 - 2i)$

c) $(4 - i) - (6 - 2i)$

d) $3 - (4 + 2i)$

Solution:

a) $$\begin{aligned}(2 + 4i) + (3 + i) &= 2 + 4i + 3 + i \\ &= (2 + 3) + (4i + i) \\ &= 5 + 5i\end{aligned}$$

b) $$\begin{aligned}(2 + i) - (4 - 2i) &= 2 + i - 4 + 2i \\ &= (2 - 4) + (i + 2i) \\ &= -2 + 3i\end{aligned}$$

c) $$\begin{aligned}(4 - i) - (6 - 2i) &= 4 - i - 6 + 2i \\ &= (4 - 6) + (-i + 2i) \\ &= -2 + i\end{aligned}$$

d) $$\begin{aligned}3 - (4 + 2i) &= 3 - 4 - 2i \\ &= (3 - 4) - 2i \\ &= -1 - 2i\end{aligned}$$

• PROBLEM 1005

Find the product $(2 + 3i)(-2 - 5i)$.

**Solution:** Using the following method: product of first elements, + product of outer elements + product of inner elements + product of last elements:

$$(2 + 3i)(-2 - 5i) = 2(-2) + 2(-5i) + 3i(-2) + 3i(-5i)$$
$$= -4 - 10i - 6i - 15i^2$$
$$= -4 - 16i - 15i^2$$

Recall $i^2 = -1$, hence, $= -4 - 16i - 15(-1)$

$$= -4 - 16i + 15$$
$$= 11 - 16i$$

The same result is obtained by using the distributive law.

$$(2 + 3i)(-2 - 5i) = (2 + 3i)(-2) - (2 + 3i)5i$$
$$= -4 - 6i - 10i - 15i^2 = 11 - 16i.$$

In other words, if one multiples $2 + 3i$ and $-2 - 5i$ as if they were polynomials and replaces $i^2$ by $-1$, then the correct product is obtained.

**• PROBLEM 1006**

Multiply $(3 + 4i)$ by $(5 + 2i)$.

**Solution:** To multiply two complex numbers, form the product treating $i$ as an ordinary number and then replace $i^2$ by $-1$. Hence,

$$(3 + 4i)(5 + 2i) = 15 + 20i + 6i + 8i^2 = 15 + 26i + 8i^2$$
$$= 15 + 26i - 8 = 7 + 26i$$

Or we may treat the problem as the product of two binomials:

$$(3 + 4i)(5 + 2i) = (3 + 4i)5 + (3 + 4i)2i$$
$$= 15 + 20i + 6i + 8i^2$$
$$= 15 + 26i + 8(-1)$$
$$= 15 + 26i - 8$$
$$= 7 + 26i$$

**• PROBLEM 1007**

Compute the sum and product of the complex numbers $3 + 2i$ and $1 - 3i$.

**Solution:**

$$(3 + 2i) + (1 - 3i) = 3 + 2i + 1 - 3i = 3+1 +2i - 3i= 4-i$$
$$(3 + 2i)(1 - 3i) = 3(1) +2i(1) + 3(-3i) + 2i(-3i)$$
$$= 3 + 2i - 9i - 6i^2$$
$$(3 + 2i)(1 - 3i) = 3 - 7i - 6i^2 \qquad (1)$$

Since $i^2 = -1$, equation (1) becomes:

$$(3 + 2i)(1 - 3i) = 3 - 7i - 6(-1)$$
$$= 3 - 7i + 6$$
$$= 9 - 7i.$$

**• PROBLEM 1008**

Find the values of the following expressions:

a. $(2 + 3i) + (6 - 2i)$

b. $(2 - i)(1 + 3i)$

c. $i - (2 + 3i)$

**Solution:** a) $(2 + 3i) + (6 - 2i) = 2 + 3i + 6 - 2i$

$$= 2 + 6 + 3i - 2i$$
$$= 8 + i$$

b) $(2 - i)(1 + 3i) = 2(1) - i(1) + 2(3i) - i(3i)$

$$= 2 - i + 6i - 3i^2 \qquad (1)$$

Since $i^2 = -1$, equation (1) becomes:

$$(2 - 1)(1 + 3i) = 2 - i + 6i - 3(-1)$$
$$= 2 - i + 6i + 3$$
$$= 2 + 3 - i + 6i$$
$$= 5 + 5i$$

c) $i - (2 + 3i) = i - 2 - 3i$

$$= -2 + i - 3i$$
$$= -2 - 2i$$

**• PROBLEM 1009**

Express each of the following complex numbers in the form $a + bi$, where $a$ and $b$ are real:

(a) $7$ (b) $2i$ (c) $\sqrt{-3}$

(d) $\dfrac{1 + \sqrt{-3}}{2}$ (e) $(3 + 2i)(3 - 2i)$ (f) $\dfrac{1 + i}{1 - i}$

**Solution:**

(a) $7 = 7 + 0i \quad (a = 7, \quad b = 0)$

(b) $2i = 0 + 2i \quad (a = 0, \quad b = 2)$

(c) $\sqrt{-3} = \sqrt{(-1)(3)} = \sqrt{-1} \cdot \sqrt{3} = i\sqrt{3} = 0 + i\sqrt{3} \quad (a = 0, b = \sqrt{3})$

(d) $\dfrac{1 + \sqrt{-3}}{2} = \dfrac{1 + i\sqrt{3}}{2} = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2} i \quad \left(a = \dfrac{1}{2}, \; b = \dfrac{\sqrt{3}}{2}\right)$

(e) $(3 + 2i)(3 - 2i) = 9 - 6i + 6i - 4i^2 = 9 - 4i^2 = 9 - 4(-1)$

either number is the complex conjugate of the other.

$$\frac{6 - 5i}{3 + 4i} \cdot \frac{3 - 4i}{3 - 4i} = \frac{18 - 15i - 24i + 20i^2}{9 + 16} = \frac{-2 - 39i}{9 + 16}$$

$$= - \frac{2 + 39i}{25}.$$

Check: We check by multiplying the quotient by the divisor. The product must be the dividend.

$$\left(- \frac{2 + 39i}{25}\right)(3 + 4i) = -\frac{1}{25}(6 + 125i - 156)$$

$$= - \frac{1}{25}(125i - 150)$$

$$= 6 - 5i.$$

• **PROBLEM 1012**

Find the real and imaginary parts of

$$(2 + 3i) \div (3 + 4i)$$

Solution: In order to divide one complex number by another, the denominator must be converted to a real number. This can be done by multiplying the numerator and the denominator by the conjugate of the denominator. The complex numbers a + bi and a - bi are conjugates of each other. Therefore, the conjugate of the denominator 3 + 4i is 3 - 4i. Then,

$$\frac{2 + 3i}{3 + 4i} = \frac{(2 + 3i)(3 - 4i)}{(3 + 4i)(3 - 4i)}$$

$$= \frac{6 + 9i - 8i - 12i^2}{9 + \cancel{12i} - \cancel{12i} - 16i^2}$$

$$\frac{6 + i - 12(-1)}{9 - 16(-1)}, \qquad \text{since } i^2 = -1$$

$$= \frac{6 + i + 12}{9 + 16}$$

$$= \frac{18 + i}{25}$$

$$= \frac{18}{25} + \frac{1}{25} i$$

Hence, the real part of the quotient is $\frac{18}{25}$ and the imaginary part is $\frac{1}{25}$ i.

• **PROBLEM 1013**

Divide $2 - \sqrt{2}i$ by $2 - i$:

__Solution:__ This problem involves dividing one complex number by another. To perform this division, the numerator and the denominator are multiplied by the conjugate of the denominator. The conjugate of the denominator $2 - i$ is $2 + i$. Then,

$$\frac{2 - \sqrt{2}\, i}{2 - i} = \frac{(2 - \sqrt{2}\, i)(2 + i)}{(2 - i)(2 + i)}$$

$$= \frac{4 - 2\sqrt{2}\, i + 2i - \sqrt{2}\, i^2}{4 - \cancel{2i} + \cancel{2i} - i^2}$$

$$= \frac{4 + (2 - 2\sqrt{2})i - \sqrt{2}\,(-1)}{4 - (-1)}, \text{since } i^2 = -1$$

$$= \frac{4 + (2 - 2\sqrt{2})i + \sqrt{2}}{4 + 1}$$

$$= \frac{4 + \sqrt{2} + (2 - 2\sqrt{2})i}{5}$$

$$= \frac{4 + \sqrt{2}}{5} + \frac{(2 - 2\sqrt{2})}{5}\, i$$

**• PROBLEM 1014**

Simplify $\frac{3 - 5i}{2 + 3i}$.

__Solution:__ To simplify $\frac{3 - 5i}{2 + 3i}$ means to write the fraction without an imaginary number in the denominator. To achieve this, we multiply the fraction by another fraction which is equivalent to unity, (so that the value of the original fraction is unchanged) which will transform the expression in the denominator to a real number. A fraction with this property must have the complex conjugate of the expression in the denominator of the original fraction as its numerator and denominator. The complex conjugate must be chosen because of its special property that when multiplied by the original complex number the result is real.

Note: $a + bi$; its complex conjugate is $a - bi$ or they can be said to be conjugates of each other. To multiply notice that $(a + bi)(a - bi)$ is the factored form of the difference of two squares. Thus we obtain

$$(a)^2 - (bi)^2;\; i^2 = -1;\; (a)^2 - (-1)(b)^2 \text{ or } a^2 + b^2.$$

$$\frac{3 - 5i}{2 + 3i} \cdot \frac{2 - 3i}{2 - 3i} = \frac{6 - 9i - 10i + 15i^2}{4 - 9i^2}$$

$$= \frac{6 - 19i - 15}{4 + 9}$$

$$= \frac{-9 - 19i}{13} \text{ or } \frac{-9}{13} - \frac{19}{13}i$$

Since the resulting fraction has a rational number in the denominator, we have rationalized the denominator.

**• PROBLEM 1015**

Simplify: (a) $4i - 7i^3$ (b) $\frac{2-3i}{5i}$

**Solution:** (a) Factor i in the expression to obtain,

$$i\left(4 - 7i^2\right)$$

$$i^2 = -1$$

and we obtain,

$$i(4 - 7[-1]) = i(4 + 7) = 11i$$

(b) Rationalize the denominator by multiplying the original fraction by a fraction equivalent to unity which will cause the imaginary expression in the denominator of the original fraction to be eliminated.

$\frac{i}{i}$ is suitable because $\frac{i}{i} = 1$ and $i^2 = -1$.

$$\frac{2-3i}{5i} = \frac{2-3i}{5i} \cdot \frac{i}{i} = \frac{2i - 3i^2}{5i^2} = \frac{2i + 3}{-5}$$

**• PROBLEM 1016**

Expand $(2 + 3i)^3$.

**Solution:** The identity $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ is still valid in the case of complex numbers. Thus, replacing a by 2 and b by 3i we obtain

$$\begin{aligned}(2 + 3i)^3 &= 2^3 + 3 \cdot 2^2(3i) + 3 \cdot 2(3i)^2 + (3i)^3 \\ &= 8 + 3 \cdot 4 \cdot 3 \cdot i + 3 \cdot 2(3^2i^2) + 3^3i^3 \\ &= 8 + 36i + 6(9)i^2 + 27i^3 \\ &= 8 + 36i + 54i^2 + 27i^3.\end{aligned}$$

Recalling that $i^2 = -1$, since $i = \sqrt{-1}$ and $i^2 = \sqrt{-1}\sqrt{-1} = -1$; and $i^3 = i^2(i) = (-1)i = -i$, we obtain:

$$\begin{aligned}&= 8 + 36i + 54(-1) + 27(-i) \\ &= 8 + 36i - 54 - 27i \\ &= 9i - 46.\end{aligned}$$

**• PROBLEM 1017**

Evaluate $x^2 - 2x + 6$ for $x = 3 + 2i$.

**Solution:** Substituting the given value, we get

$$x^2 - 2x + 6 = (3 + 2i)^2 - 2(3 + 2i) + 6$$
$$= (3 + 2i)(3 + 2i) - 6 - 4i + 6$$

Since $(a + b)(c + d) = ac + ad + bc + bd$

$$x^2 - 2x + 6 = (3)(3) + 6i + 6i + (2i)(2i) - 6 - 4i + 6$$
$$= 9 + 12i + (2i)^2 - 6 - 4i + 6$$

Since $(ab)^2 = a^2b^2$

$$(2i)^2 = 2^2(i)^2$$

By definition $i^2 = -1$, hence $(2i)^2 = 4(-1) = -4$ and $x^2 - 2x + 6$

$$= 9 + 12i - 4 - 6 - 4i + 6$$

Combine terms, $= 5 + 8i.$

**• PROBLEM 1018**

Show that $\left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)^3 = 1.$

**Solution:** Factor out $\frac{1}{2}$:

$$\left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)^3 = \left[\frac{1}{2}(-1 + \sqrt{3}i)\right]^3$$
$$= \left(\frac{1}{2}\right)^3(-1 + \sqrt{3}i)^3 = \frac{1}{8}(-1 + \sqrt{3}i)^3.$$

Then we apply the identity $(a + b)^3 = (a + b)(a + b)(a + b)$ $= \left(a^2 + 2ab + b^2\right)(a + b) = a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3$ $= a^3 + 3a^2b + 3ab^2 + b^2$. Let $a = -1$, $b = \sqrt{3}i$, then

$$(-1 + \sqrt{3}i)^3 = (-1)^3 + 3(-1)^2\sqrt{3}i + 3(-1)(\sqrt{3}i)^2 + (\sqrt{3}i)^3$$
$$= -1 + 3\sqrt{3}i - 9i^2 + 3\sqrt{3}i^3$$
$$= -1 + 3\sqrt{3}i + 9 - 3\sqrt{3}i = 8.$$

since $i^2 = -1$ and $i^3 = i^2\ i = (-1)i = -i$. Hence

$$\left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)^3 = \frac{1}{8} \cdot 8 = 1.$$

**• PROBLEM 1019**

Show that $\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^4 = -1.$

**Solution:** Factor out $\frac{1}{\sqrt{2}}$:

$$\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^4 = \left[\frac{1}{\sqrt{2}}(1 + i)\right]^4 = \left(\frac{1}{\sqrt{2}}\right)^4(1 + i)^4 = \frac{1}{4}(1 + i)^4$$

$$3 + 4i = x^2 - y^2 + 2xyi$$

Equate the real and imaginary parts of both members:

$$3 = x^2 - y^2 \qquad (3)$$

$$4i = 2xyi \qquad (4)$$

Dividing both sides of equation (4) by i:

$$\frac{4i}{i} = \frac{2xyi}{i}$$

$$4 = 2xy \qquad (5)$$

Therefore, our equations are:

$$3 = x^2 - y^2 \qquad (3)$$

$$4 = 2xy \qquad (5)$$

Solving equation (5) for x:
dividing both sides by 2y,

$$\frac{4}{2y} = \frac{2xy}{2y}$$

$$\frac{2}{y} = x$$

Note that the above operation assumes that $y \neq 0$ since division by 0 is undefined. (Also, in our original expression $\sqrt{3 + 4i} = x + yi$, there is assumed to be an an imaginary part; namely, yi. If y were equal to 0, then there would be no imaginary part since $yi = 0(i) = 0$. Hence, y cannot equal 0.)

Substituting the expression for x in equation (3):

$$3 = \left(\frac{2}{y}\right)^2 - y^2$$

$$3 = \frac{4}{y^2} - y^2$$

Obtaining a common denominator of $y^2$ for the two terms on the right side of this equation:

$$3 = \frac{4}{y^2} - \frac{y^2(y^2)}{y^2}$$

$$3 = \frac{4}{y^2} - \frac{y^4}{y^2}$$

$$3 = \frac{4 - y^4}{y^2}$$

Multiplying both sides by $y^2$:

$$y^2(3) = y^2\left(\frac{4 - y^4}{y^2}\right)$$

$$3y^2 = 4 - y^4$$

Subtracting $\left(4 - y^4\right)$ from both sides:

$$3y^2 - \left(4 - y^4\right) = 4 - y^4 - \left(4 - y^4\right)$$

$$3y^2 - 4 + y^4 = 0$$

$$\text{or } y^4 + 3y^2 - 4 = 0$$

Factoring the left side of this equation as a product of two polynomials:

$$\left(y^2 + 4\right)\left(y^2 - 1\right) = 0$$

Whenever a product ab = 0 where a and b are any two numbers, either a = 0 or b = 0. Therefore,

$$\text{either } y^2 + 4 = 0 \quad \text{or } y^2 - 1 = 0$$
$$y^2 = -4 \quad \text{or} \quad y^2 = 1$$
$$y = \pm\sqrt{1}$$
$$y = \pm 1$$

Note that there is no real solution to $y^2 = -4$ since there is no real number y whose square is -4.

Substituting y = -1 in equation (3):

$$3 = x^2 - (-1)^2$$
$$3 = x^2 - (1)$$
$$3 = x^2 - 1$$

Add 1 to both sides:

$$3 + 1 = x^2 - 1 + 1$$
$$4 = x^2$$

Take the square root of both sides:

$$\pm\sqrt{4} = x$$
$$\pm 2 = x.$$

Hence, the two solutions appear to be (-2,-1) and (2,-1).
Substituting y = 1 in equation (3):

$$3 = x^2 - (1)^2$$
$$3 = x^2 - 1$$

Add 1 to both sides:

$$3 + 1 = x^2 - 1 + 1$$
$$4 = x^2$$

Take the square root of both sides:

$$\pm\sqrt{4} = x \quad \text{or } \pm 2 = x$$

Hence, the two additional solutions appear to be (-2,1) and (2,1). For the four solutions obtained:

when $(x,y) = (-2,-1)$, $\sqrt{3 + 4i} = -2 + (-1)i = -2-i$,

when $(x,y) = (2,-1)$, $\sqrt{3 + 4i} = 2 + (-1)i = 2 - i$,

when $(x,y) = (-2,1)$, $\sqrt{3 + 4i} = -2 + 1i = -2 + i$,

when $(x,y) = (2,1)$, $\sqrt{3 + 4i} = 2 + 1i = 2 + i$.

Checking the four solutions of $\sqrt{3 + 4i}$; namely, $-2-i$, $2-i$, $-2+i$, $2+i$, using equation 1:

for $-2-i$, $3 + 4i = (-2-i)(-2-i)$

$$3 + 4i = 4 + 2i + 2i + i^2$$

$3 + 4i = 4 + 4i - 1$. Note that $i^2 = -1$.

$3 + 4i = 3 + 4i$ ✓

Therefore, $-2-i$ is a solution to $\sqrt{3 + 4i}$.

For $2-i$, $3 + 4i = (2 - i)(2 - i)$

$3 + 4i = 4 - 2i - 2i + i^2$

$3 + 4i = 4 - 4i - 1$ since $i^2 = -1$

$3 + 4i \neq 3 - 4i$

Therefore, $2 - i$ is not a solution to $\sqrt{3 + 4i}$.

For $-2+i$, $3 + 4i = (-2 + i)(-2 + i)$

$3 + 4i = 4 - 2i - 2i + i^2$

$3 + 4i = 4 - 4i - 1$ since $i^2 = -1$

$3 + 4i \neq 3 - 4i$

Therefore, $-2 + i$ is not a solution to $\sqrt{3 + 4i}$.

For $2 + i$, $3 + 4i = (2 + i)(2 + i)$

$3 + 4i = 4 + 2i + 2i + i^2$

$3 + 4i = 4 + 4i - 1$ since $i^2 = -1$

$3 + 4i = 3 + 4i$ ✓

Therefore, $2 + i$ is a solution to $\sqrt{3 + 4i}$.

Hence, the only two solutions to $\sqrt{3 + 4i}$ are: $-2-i$ and $2+i$.

**• PROBLEM 1021**

Show that $(a + bi) + (c + di) = (c + di) + (a + bi)$.

Solution: Use the assoiative, distributive, and commutative laws. Associate the corresponding components of the complex numbers, i.e., associate the real and imaginary parts respectively.

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$= a + c + bi + di$$

$$= a + bi + c + di$$

$$= (c + di) + (a + bi)$$

We would suspect that zero is still the additive identity, but zero is a real number. Recall that the real number 5 and the complex number $5 + 0i$ represent the same number. Then the additive identity should be $0 + 0i$. Let us see whether it is. Recall that adding the additive identity to a number does not change the number. Applying the definition of addition.

$$(a + bi) + (0 + 0i) = (a + 0) + (b + 0)i$$

$$= a + bi$$

This verifies that $0 + 0i$ is the additive identity.

Given $f(x) = x^3 + x + 1$, evaluate $f(1 + i)$.

Solution: $f(1 + i)$ indicates that $1 + i$ should be subsituted for x.

$$f(1 + i) = (1 + i)^3 + (1 + i) + 1$$
$$= (1 + i)(1 + i)(1 + i) + (1 + i) + 1$$
$$= \left(1 + 2i + i^2\right)(1 + i) + (1 + i) + 1$$
$$= 1 + 2i + i^2 + i + 2i^2 + i^3 + (1 + i) + 1$$
$$= 1 + 3i + 3i^2 + i^3 + (1 + i) + 1$$
$$= 1 + 3i + 3i^2 + i^3 + 1 + i + 1$$

Note that $i^2 = -1, i^3 = i^2(i) = (-1)(i) = -i$. Then

$$f(1 + i) = 1 + 3i + 3(-1) + (-1)i + 1 + i + 1$$
$$= 3i.$$



Find the real numbers a and b such that

$$(a + bi) + (2 - 3i) = 2(-2 + i).$$

Solution: Subtract $(2 - 3i)$ from both sides of the given equation.

$$(a+bi) + \cancel{(2-3i)} - \cancel{(2-3i)} = 2(-2+i) - (2-3i)$$
$$a+bi = -4 + 2i - 2 + 3i$$
$$= -4 - 2 + 2i + 3i$$
$$a+bi = -6 + 5i$$

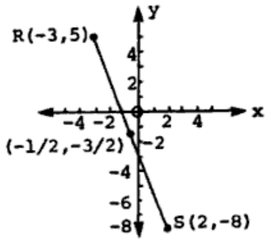
Therefore, $a = -6$ and $b = 5$.

# Step-by-Step Solutions to Problems in this Chapter, "Analytic Geometry"

POINTS ON LINE SEGMENTS

• PROBLEM 1024

Find the midpoint of the segment from R(-3,5) to S(2,-8).




Solution: The midpoint of a line segment from $(x_1,y_1)$ to $(x_2,y_2)$ is given by

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right),$$

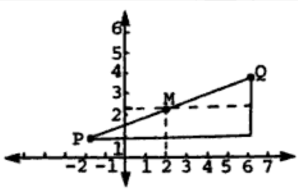
the abscissa being one half the sum of the abscissas of the endpoints and the ordinate one half the sum of the ordinates of the endpoints. Let the coordinates of the midpoint be $P(x_0,y_0)$. Then,

$$x_0 = \frac{1}{2}(-3 + 2) = -\frac{1}{2} \quad y_0 = \frac{1}{2}[5 + (-8)] = \frac{1}{2}(-3) = -\frac{3}{2}.$$

Thus, the midpoint is $P\left(-\frac{1}{2}, -\frac{3}{2}\right)$.

• PROBLEM 1025

What are the coordinates of the midpoint of a line segment joining P(-2,1) and Q(6,4)?



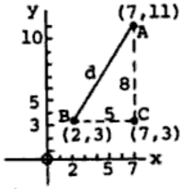


Fig. A

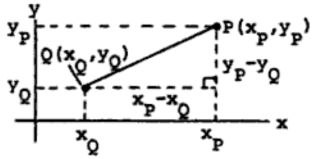


Fig. B

$$(\overline{AB})^2 = (5)^2 + (8)^2 = 25 + 64 = 89$$

Taking the square root of both sides, $\overline{AB} = \sqrt{89}$. Thus, the distance between (2,3) and (7,11), $\overline{AB}$, is $\sqrt{89}$.

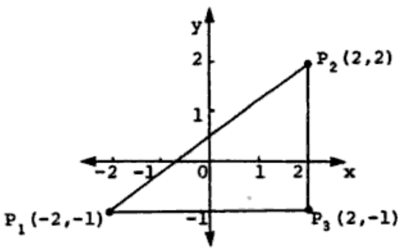
Generalizing, suppose we replace these points by P and Q with coordinates $(x_P, y_P)$ and $(x_Q, y_Q)$, respectively (see Figure B). Then what we have done in this problem would amount to using the following general formula for the distance between P and Q:

$$d(P,Q) = \sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2}$$

This formula continues to hold true in all possible positions of P and Q.

**• PROBLEM 1031**

The point $P_1$ has coordinates (-2,-1) and the point $P_2$ has coordinates (2,2). Find the distance $\overline{P_1P_2}$ between these points.



**Solution:** The points $P_1$ and $P_2$ are plotted in the accompanying figure. Let $P_3$ be the point (2,-1). It is apparent that the points $P_1, P_2$ and $P_3$ are the vertices of a right triangle, $P_3$ being the vertex of the right angle. Since the points $P_2$ and $P_3$ lie in the same vertical line, you can easily see that the distance between them is 3 units. Similarly, the distance $\overline{P_1P_3} = 4$. Now according to the Pythagorean Theorem which states that the sum of the squares of the legs of a right triangle is equal to the square of the hypotenuse,

$$\overline{P_1P_2}^2 = \overline{P_1P_3}^2 + \overline{P_3P_2}^2 = (4)^2 + (3)^2 = 16 + 9 = 25.$$

Taking the square root of both sides, $\overline{P_1P_2} = 5$.

The concept of the distance between two points is so important that a formula has been developed for it. The distance between 2 points $(x_1,y_1)$ and $(x_2,y_2)$ is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\ .$$

If we apply this formula to our case, replacing $(x_1,y_1)$ by (-2,-1) and $(x_2,y_2)$ by (2,2) we obtain

$$d = \sqrt{[2-(-2)]^2 + [2-(-1)]^2} = \sqrt{(4)^2 + (3)^2} = \sqrt{16 + 9} = \sqrt{25} = 5$$

as before.

**• PROBLEM 1032**

Find the distance between P(5,3) and Q(8,7).

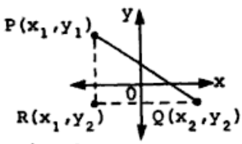


Fig. 1

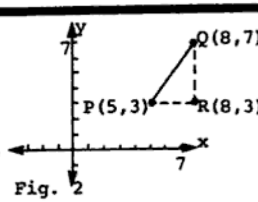


Fig. 2

Solution: Any two points $P(x_1,y_1)$ and $Q(x_2,y_2)$ not on a line parallel to either axis can be used to locate a third point. We do this by selecting for x-ordinate the x-ordinate of one of the points and y-ordinate the y-ordinate of the other point. The third point shown in Figure 1 is $R(x_1,y_2)$. The choice of R is such that triangle PQR is a right triangle with right angle at R. The distance

$$PR = |y_1 - y_2|;\ RQ = |x_1 - x_2|.$$

Since $\overline{PQ}$ is the hypotenuse of the right triangle, the distance is given as

$$PQ = \sqrt{(PR)^2 + (RQ)^2} = \sqrt{(y_1 - y_2)^2 + (x_1 - x_2)^2}$$

The point R(8,3) is opposite the hypotenuse determined by the points P(5,3) and Q(8,7) (see Figure 2). $PR = |5 - 8|$, $RQ = |3 - 7|$. Hence

$$PQ = \sqrt{|5 - 8|^2 + |3 - 7|^2} = \sqrt{|-3|^2 + |-4|^2}$$

$$= \sqrt{(3)^2 + (4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

We have proved the following theorem:

THEOREM: The distance d between any two points $(x_1,y_1)$ and $(x_2,y_2)$ is given by the formula

$$d = \sqrt{(y_1 - y_2)^2 + (x_1 - x_2)^2}$$

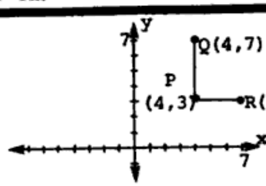
**• PROBLEM 1033**

Find the distance from the origin to the point (x,y).

$= -9$

## • PROBLEM 1035

Given the three points P(4,3), Q(4,7), and R(7,3). Find the lengths of $\overline{PQ}$ and $\overline{PR}$.




**Solution:** Points P and Q have the same x-coordinate and lie along a line parallel to the y-axis. Therefore the length of $\overline{PQ} = |y_P - y_Q|$. P and R have the same y-coordinate and lie along a line parallel to the x-axis. Hence the length of $\overline{PR} = |x_P - x_R|$.

$$\overline{PQ} = |3 - 7| = 4 \quad \text{and} \quad \overline{PR} = |4 - 7| = 3.$$

We could have used the distance formula

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

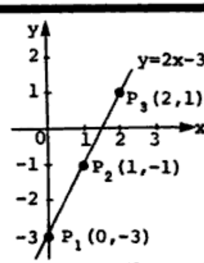
Then

$$\overline{PQ} = \sqrt{(4 - 4)^2 + (3 - 7)^2} = 4$$

$$\overline{PR} = \sqrt{(4 - 7)^2 + (3 - 3)^2} = 3$$

## • PROBLEM 1036

Suppose $f = \{(x, 2x - 3)\}$. Choose any three points of the graph of $f$ and show that they lie in a line.




**Solution:** We are asked to choose any three points, so let us arbitrarily take $x = 0$, $x = 1$, and $x = 2$, and find their corresponding $f(x)$ values:

| x | f(x) = 2x-3 | f(x) |
|---|---|---|
| 0 | $f(0) = 2(0)-3 = -3$ | -3 |
| 1 | $f(1) = 2(1)-3 = 2 - 3 = -1$ | -1 |
| 2 | $f(2) = 2(2)-3 = 4 - 3 = 1$ | 1 |

Thus, we have three points $P_1(0,-3)$, $P_2(1,-1)$, and $P_3(2,1)$ shown on the accompanying graph of $f$ (see figure). The points are collinear; that is, lie in a line, if the distance $\overline{P_1P_3}$ is equal to the sum of the distances $\overline{P_1P_2}$ and $\overline{P_2P_3}$. (The shortest path between 2 points is a line). Applying the formula for the distance between 2 points $(x_1,y_1)$ and $(x_2,y_2)$, $d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$, we find

$$\overline{P_1P_3} = \sqrt{(2-0)^2 + [1-(-3)]^2} = \sqrt{2^2+4^2} = \sqrt{4 + 16}$$

$$= \sqrt{20} = \sqrt{4 \cdot 5} = \sqrt{4} \cdot \sqrt{5} = 2\sqrt{5}$$

$$\overline{P_1P_2} = \sqrt{(1-0)^2 + [-1-(-3)]^2} = \sqrt{1^2 + (2)^2} = \sqrt{1 + 4} = \sqrt{5}$$

$$\overline{P_2P_3} = \sqrt{(2-1)^2 + [1-(-1)]^2} = \sqrt{(1)^2+(2)^2} = \sqrt{1 + 4} = \sqrt{5}$$

and therefore $\overline{P_1P_3} = \overline{P_1P_2} + \overline{P_2P_3} = 2\sqrt{5}$. Therefore we have shown that 3 randomly selected points on the graph $f = \{(x,2x-3)\}$ lie in a line.

**• PROBLEM 1037**

Use the distance formula to determine whether the points A(0,-3), B(8,3), and C(11,7) are collinear.

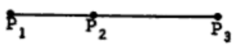



Fig. A

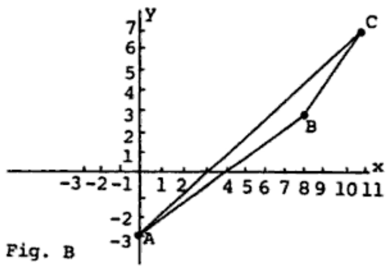



Fig. B

**Solution:** If three points $P_1$, $P_2$, $P_3$ are in such a position that $\overline{P_1P_2} + \overline{P_2P_3} = \overline{P_1P_3}$ then the three points lie on a straight line and we say that the points are collinear (Figure A).

Thus we find the distances between points A and B, A and C, and B and C to determine whether the sum of any two of these is equivalent to the third, making A, B, and C collinear. Using the formula for the distance between two points $(x_1,y_1)$ and $(x_2,y_2)$,

$$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

The distance between (0,-3) and (8,3) is

$$d_1 = \sqrt{(8-0)^2 + [3-(-3)]^2} = \sqrt{8^2 + 6^2} = \sqrt{64 + 36} = \sqrt{100} = 10$$

The distance between (0,-3) and (11,7) is

$$d_2 = \sqrt{(11-0)^2 + [7-(-3)]^2} = \sqrt{(11)^2 + (10)^2} = \sqrt{121 + 100}$$
$$= \sqrt{221} \simeq 14.74$$

The distance between (8,3) and (11,7) is

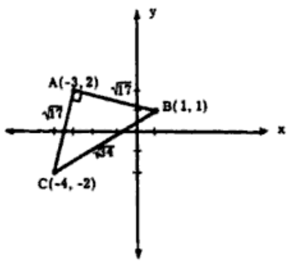
$$d_3 = \sqrt{(11-8)^2 + (7-3)^2} = \sqrt{(3)^2 + (4)^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

Since 5 + 10 = 15 and 15 > 14.74, the three points form a triangle as opposed to a straight line. Thus the points are not collinear.

Plotting the points on a graph, and attaching them we also observe that the points form a triangle, not a line. (Figure B).

**• PROBLEM 1038**

> Show that the triangle with (-3, 2), (1, 1), and (-4, -2) as vertices is an isosceles triangle.




Solution: If we can show that two sides of the triangle are equal in length, then the triangle is isosceles. This can be done by applying the formula for the distance between two points, $(x_1, y_1)$ and $(x_2, y_2)$:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Let the given points be designated as A, B, and C respectively. Then

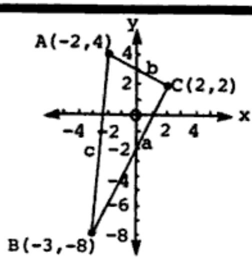
$$|AB| = \sqrt{(1 + 3)^2 + (1 - 2)^2} = \sqrt{17},$$
$$|AC| = \sqrt{(-4 + 3)^2 + (-2 -2)^2} = \sqrt{17}.$$

Hence $|AB| = |AC|$, and the triangle is isosceles. Furthermore,

$$|BC| = \sqrt{(-4 - 1)^2 + (-2 - 1)^2} = \sqrt{34}.$$

Since $|BC|^2 = |AB|^2 + |AC|^2$ $\left(\sqrt{34}^2 = \sqrt{17}^2 + \sqrt{17}^2 \text{ or } 34 = = 17 + 17\right)$, the Theorem of Pythagoras holds, and ABC is a right triangle, with the right angle at A. (See figure.)

Show that the points A(-2, 4), B(-3, -8), and C(2,2) are vertices of a right triangle.




**Solution:** If triangle ABC is a right triangle, then $a^2 + b^2 = c^2$; that is, the sum of the square of the legs equals the square of the hypotenuse by the Pythagorean Theorem.

Thus we compute the distance from B to C which is side a,

the distance from C to A which is side b,

and the distance from A to B which is side c.

The formula for the distance between two points $(x_1, y_1)$ and $(x_2, y_2)$ is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Thus the distance from B to C, from (-3, -8) to (2,2), is

$$\sqrt{[2 - (-3)]^2 + [2 - (-8)]^2} = \sqrt{(2+3)^2 + (2+8)^2}$$
$$= \sqrt{5^2 + 10^2}$$
$$= \sqrt{25 + 100}$$
$$= \sqrt{125}$$

Hence side a = $\sqrt{125}$

The distance from C to A, from (2,2) to (-2,4), is

$$\sqrt{(-2 - 2)^2 + (4 - 2)^2} = \sqrt{(-4)^2 + 2^2}$$
$$= \sqrt{16 + 4}$$
$$= \sqrt{20}$$

Hence side b = $\sqrt{20}$

The distance from A to B, from (-2,4) to (-3, -8), is

$$\sqrt{[-3 - (-2)]^2 + (-8 - 4)^2} = \sqrt{(-3 + 2)^2 + (-12)^2}$$

$$= \sqrt{(-1)^2 + (-12)^2}$$

$$= \sqrt{1 + 144}$$

$$= \sqrt{145}$$

Hence side $c = \sqrt{145}$

Now, if triangle ABC is a right triangle, $a^2+b^2 = c^2$.
Replacing,

a by $\sqrt{125}$, b by $\sqrt{20}$, and c by $\sqrt{145}$

we obtain,

$$\left(\sqrt{125}\right)^2 + \left(\sqrt{20}\right)^2 = \left(\sqrt{145}\right)^2$$

Since, $\left(\sqrt{a}\right)^2 = \sqrt{a}\ \sqrt{a} = \sqrt{a} \cdot \sqrt{a} = \sqrt{a^2} = a$

$$\left(\sqrt{125}\right)^2 = 125$$

$$\left(\sqrt{20}\right)^2 = 20$$

and $\left(\sqrt{145}\right)^2 = 145$

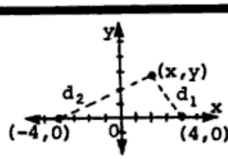
Thus $a^2 + b^2 = c^2$ is equivalent to,

$$125 + 20 = 145$$
$$145 = 145$$

Therefore, triangle ABC is indeed a right triangle.

**• PROBLEM 1040**

Find the equation for the set of points the sum of whose distances from (4,0) and from (-4,0) is 10.




Solution: We find the desired equation by choosing an arbitrary point (x,y) and computing the sum of its distances from (4,0) and (-4,0) (see accompanying figure). Applying the distance formula for the distance between two points $(a_1,b_1)$ and $(a_2,b_2)$, $d = \sqrt{(a_1-a_2)^2+(b_1-b_2)^2}$, we find that the distance from (x,y) to (4,0) is

$$d_1 = \sqrt{(x-4)^2 + y^2}$$

and the distance from (x,y) to (-4,0) is

$$d_2 = \sqrt{(x + 4)^2 + y^2}.$$

We are given that the sum of the distances $d_1 + d_2 = 10$. Hence, the required equation for the set of points is

$$\sqrt{(x - 4)^2 + y^2} + \sqrt{(x + 4)^2 + y^2} = 10$$

$$\sqrt{(x - 4)^2 + y^2} = 10 - \sqrt{(x + 4)^2 + y^2} .$$

Squaring both sides,

$$\left(\sqrt{(x - 4)^2 + y^2}\right)^2 = \left(10 - \sqrt{(x + 4)^2 + y^2}\right)^2.$$

Since $(\sqrt{a}\,)^2 = \sqrt{a}\,\sqrt{a} = \sqrt{a \cdot a} = \sqrt{a^2} = a$ ,

$$\left(\sqrt{(x - 4)^2 + y^2}\right)^2 = (x - 4)^2 + y^2 .$$

Thus $\quad (x-4)^2 + y^2 = 100 - 20\sqrt{(x+4)^2 + y^2} + (x+4)^2 + y^2$

$$x^2 - 8x + 16 + y^2 = 100 - 20\sqrt{(x+4)^2 + y^2} + x^2 + 8x + 16 + y^2$$

Adding $-\left(100 + x^2 + 8x + 16 + y^2\right)$ to both members,

$$- 16x - 100 = - 20\sqrt{(x + 4)^2 + y^2}$$

Dividing both sides by -4, $4x + 25 = 5\sqrt{(x + 4)^2 + y^2}$

Squaring again,

$$(4x + 25)(4x + 25) = \left(5\sqrt{(x + 4)^2 + y^2}\right)^2$$

$$16x^2 + 200x + 625 = 25\left[\sqrt{(x + 4)^2 + y^2}\right]^2$$

$$16x^2 + 200x + 625 = 25\left[(x + 4)^2 + y^2\right]$$

$$16x^2 + 200x + 625 = 25\left(x^2 + 8x + 16 + y^2\right)$$

$$16x^2 + 200x + 625 = 25x^2 + 200x + 400 + 25y^2$$

Adding $-(16x^2 + 200x + 400)$ to both members,

$$225 = 9x^2 + 25y^2.$$

Dividing both members by 225, we can write the last equation in the form

$$\frac{9x^2}{225} + \frac{25y^2}{225} = \frac{225}{225}$$

or

$$\frac{x^2}{25} + \frac{y^2}{9} = 1 ,$$

which is the standard form of the equation of an ellipse. This is the desired equation.

**• PROBLEM 1041**

Find the equation for the set of points the difference of whose distances from (5,0) and (-5,0) is 6 units.

$$(5x-9)^2 = \left(3\sqrt{(x-5)^2 + y^2}\right)^2$$

$$(5x-9)(5x-9) = 3^2\left(\sqrt{(x-5)^2 + y^2}\right)^2$$

$$25x^2 - 90x + 81 = 9[(x-5)^2 + y^2]$$

$$25x^2 - 90x + 81 = 9(x^2 - 10x + 25 + y^2)$$

$$25x^2 - 90x + 81 = 9x^2 - 90x + 225 + 9y^2$$

Adding $-(9x^2 - 90x + 9y^2)$ to both sides,

$$25x^2 - 90x + 81 - (9x^2 - 90x + 9y^2) = 9x^2 - 90x + 225 + 9y^2 - (9x^2 - 90x + 9y^2)$$

$$25x^2 - 9x^2 - 9y^2 + 81 = 225$$

$$16x^2 - 9y^2 + 81 = 225$$

Adding -81 to both sides,

$$16x^2 - 9y^2 = 144.$$

Dividing both sides by 144,

$$\frac{16x^2}{144} - \frac{9y^2}{144} = \frac{144}{144}$$

or

$$\frac{x^2}{9} - \frac{y^2}{16} = 1 ,$$

which is the standard form for the equation of a hyperbola.

From the form of the equation we can determine its graph. When the center is at the origin and its vertices are at (a,0) and (-a,0), the equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 .$$

In this case, the vertices are (3,0) and (-3,0) since $a^2 = 9$ and $a = \pm 3$ (see figure).

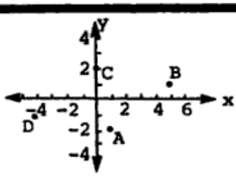
However, when the vertices lie on the y-axis, the equation of the hyperbola is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$

The vertices would then be (0,a) and (0,-a).

**• PROBLEM 1042**

Plot the points (1, - 2) and (5, 1) in the xy- plane. What ordered pair corresponds to point C in the Figure? If points A, B, and C of the Figure are three vertices of a parallelogram, what are the coordinates of the fourth vertex in the third quadrant?



**Solution:** In the Figure, the point (1, -2) is 1 unit to the right of the origin and 2 units below the x-axis. Therefore it is located at point A. The point (5, 1) is located at point B, 5 units to the right of the origin and 1 unit above the x-axis.

The abscissa of point C is 0 and its ordinate is 2. Therefore, the ordered pair is (0, 2).

There are three possible locations for a fourth vertex. In quadrant III the vertex is the intersection of lines parallel to AB and BC, respectively. Since point C is 5 units to the left of point B and 1 unit above it, the required vertex D will be 5 units to the left of point A and 1 unit above it. Its coordinates are (- 4, - 1).

## CIRCLES, ARCS, AND SECTORS

**• PROBLEM 1043**

Write equations of the following circles:

(a) With center at ( -1, 3) and radius 9.

(b) With center at (2, -3) and radius 5.

**Solution:** The equation of the circle with center at (a,b) and radius r is

$$(x - a)^2 + (y - b)^2 = r^2.$$

(a) Thus, the equation of the circle with center at (-1, 3) and radius 9 is

$$[x - (-1)]^2 + (y - 3)^2 = 9^2$$

$$(x + 1)^2 + (y - 3)^2 = 81$$

(b) Similarly the equation of the circle with center at (2, -3) and radius 5 is

$$(x - 2)^2 + [y - (-3)]^2 = 5^2$$

$$(x - 2)^2 + (y + 3)^2 = 25$$

**• PROBLEM 1044**

Find the center and radius of the circle

$$x^2 - 4x + y^2 + 8y - 5 = 0 \qquad (1)$$

**Solution:** We can find the radius and the coordinates of the center by completing the square in both x and y. To complete the square in either variable, take half the coefficient of the variable term (i.e., generally used for the equation of a circle with center at (3,4) and radius 5. See Figure A.

We can generalize the result of this example and find the equation for any circle with center (a,b) and radius r. The required circle is the set of points (x,y) at a distance r from (a,b),

$$\sqrt{(x-a)^2 + (y-b)^2} = r$$

$$(x-a)^2 + (y-b)^2 = r^2$$

as shown in Figure B.

**• PROBLEM 1046**

> If $\theta$ is an angle of $30^\circ$, what is its width in radians?

**Solution:** If an angle $\theta$ is $A$ degrees wide and also $t$ radians wide, then the numbers $A$ and $t$ are related by the equation:

$$\frac{A}{180^\circ} = \frac{t}{\pi} \qquad (1)$$

If the given angle of $30^\circ$ is $t$ radians wide, then by using equation (1):

$$\frac{30^\circ}{180^\circ} = \frac{t}{\pi}$$

$$\frac{1}{6} = \frac{t}{\pi}$$

Multiplying both sides by $\pi$,

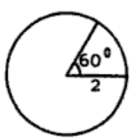
$$\pi\left(\frac{1}{6}\right) = \cancel{\pi}\left(\frac{t}{\cancel{\pi}}\right)$$

$$\frac{1}{6}\pi = t$$

Hence, the angle's width in radians is $\frac{1}{6}\pi$.

**• PROBLEM 1047**

> Find the area of a sector in which the measure of the central angle is $60^\circ$ and the radius of the circle is 2.



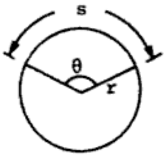

**Solution:** Note the diagram.

The central angle, $60^\circ$ is $\frac{1}{6} \times (360^\circ)$; that is, $60^\circ = \frac{1}{6} \times$ (circumference of circle). Therefore, one-sixth of the area of the circle is covered. Using the fact that the area of the total circumference of the circle is $\pi r^2$, or $A = \pi r^2$ : the area covered by the sector given in this problem is

$$A = \frac{1}{6}\pi r^2 = \frac{1}{6}\pi(2)^2 = \frac{1}{6}\pi(4) = \frac{4\pi}{6}$$
$$= \frac{2\pi}{3}.$$

• **PROBLEM 1048**

Find the length of the minor arc of a circle of radius 1 whose central angle has a measure of 90°.




Solution: For problems dealing with arc length of a circle, the following information is helpful. On a circle of radius r, a central angle of θ radians intercepts an arc of length

$$s = r\theta;$$

that is, arc length = radius x central angle in radians. (See Figure). Therefore, in this problem:

$$\text{length of minor arc} = s = (1)(90^\circ)$$
$$= (1)\left(\frac{\pi}{2}\text{ radians}\right)$$
$$= \frac{\pi}{2}\text{ radians}$$

We could have obtained this answer by noting that a central angle of 90° yields an arc equal to $\frac{1}{4}$ the circumference, since

$$90^\circ = \tfrac{1}{4}(360^\circ) = \tfrac{1}{4} \times (\text{circumference of circle})$$
$$= \tfrac{1}{4}(2\pi\text{ radians})$$
$$= \frac{2\pi}{4}\text{ radians}$$
$$= \frac{\pi}{2}\text{ radians}$$

• **PROBLEM 1049**

What length of arc is subtended by a central angle of 75° on a circle 13.7 inches in radius?

Solution: Let θ denote a central angle in a circle of radius r, and let s be the length of the intercepted arc, measured in the same units as the radius. Then if θ is an angle measured in radians, the length of the arc, s, is $s = r\theta$.

We must express the given angle in radians. Since $2\pi$ radians = 360°, then 1° = $2\pi/360$ radians = $\pi/180$ radians. Thus, 75° = 75 · $\pi/180$ radians = $5/12\ \pi$ radians.

Substituting this value into the relation $s = r\theta$, we have x-axis and y-axis satisfy the given set $\{(x,y) | xy = 0\}$.

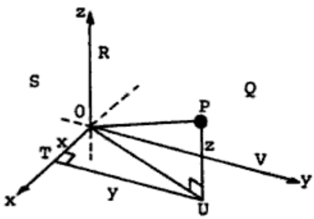
c) For the set $\{(x,y) | x^2 + y^2 < 4\}$, note that the equation $x^2 + y^2 = 4$ is a circle of radius = 2 and center (0,0). Hence, the set $\{(x,y) | x^2 + y^2 < 4\}$ consists of all points inside the circle of radius = 2 (including center (0,0), since $0^2 + 0^2 < 4$). The points on the circumference of the circle are not included.

d) For the set $\{(x,y) | 0 \le x \le 1, 0 \le y \le 1\}$, all points bounded by the vertical lines $x = 0$ and $x = 1$ and by the horizontal lines $y = 0$ and $y = 1$ will be included in the set. Note the figure. All points of the square are in the set $\{(x,y) | 0 \le x \le 1, 0 \le y \le 1\}$, including the boundary points.

## SPACE RELATED PROBLEMS

**• PROBLEM 1052**

Find the distance of the point (x, y, z) from the origin 0.




**Solution:** From the given diagram we see that point (x, y, z) is labeled P. Then OP is the distance of (x, y, z) from the origin 0. Thus, we wish to find OP. Referring to the figure, consider triangle OUP, in which the angle OUP is a right angle. From Pythagoras' theorem,

$$OP^2 = OU^2 + UP^2$$

Consider triangle OTU in which the angle OTU is a right angle. Using Pythagoras' theorem again,

$$OU^2 = OT^2 + TU^2$$

Substituting this value of $OU^2$ in the first equation

$$OP^2 = (OT^2 + TU^2) + UP^2$$

But OT = x, TU = y, UP = z, and so

$$OP^2 = x^2 + y^2 + z^2$$

The distance of the points (x, y, z) from the origin 0 is therefore

$$OP = \sqrt{x^2 + y^2 + z^2}.$$

$$18 - \cancel{z}^2 + \cancel{z}^2 = 4 + z^2$$

$$18 = 4 + z^2$$

Subtract 4 from both sides of this equation.

$$18 - 4 = \cancel{4} + z^2 - \cancel{4}$$

$$14 = z^2$$

Take the square root of both sides of this equation.

$$\pm\sqrt{14} = \sqrt{z^2}$$

$$\pm\sqrt{14} = z$$

Therefore,

$$z = \sqrt{14} \quad \text{or} - \sqrt{14}.$$

Hence, the solutions are $(0, 3, \sqrt{14})$ and $(0, 3, -\sqrt{14})$.

**• PROBLEM 1055**

Find the solutions of the equation in xyz-space where the

a. first and second components are zero,

b. first and third components are zero, and

c. second and third components are zero.

$$x^2 + 3y - z = 4$$

Solution: a) In this case, the first and second components are zero. In xyz-space, the point described is $(0, 0, z)$. Once the third component, z, is found, a solution will be found. Letting $x = 0$ and $y = 0$ in the given equation, since the first and second components are zero:

$$(0)^2 + 3(0) - z = 4$$

$$0 + 0 - z = 4$$

$$-z = 4$$

Multiplying both sides of this equation by -1:

$$-1(-z) = -1(4)$$

$$z = -4$$

Therefore, the solution in xyz-space is: $(0, 0, -4)$

# CHAPTER 33

# PERMUTATIONS

**Basic Attacks and Strategies for Solving Problems in this Chapter. See pages 766 to 775 for step-by-step solutions to problems.**

The basic counting procedure for objects is the multiplication principle. It is given by the following statement:

> If one event can occur in $m$ ways and a second event can occur in $n$ ways, then both events can occur in $mn$ ways, provided the outcome of the first event does not influence the outcome of the second.

For instance, to apply this counting principle, consider this example.

How many different meals can be selected if a restaurant offers 3 salads, 5 main dishes, and 2 desserts?

Since the first event can occur in 3 ways, the second event can occur in 5 ways, and the third event can occur in 2 ways, then there are

$$3 \cdot 5 \cdot 2 = 30$$

different meals to select from.

Often we are concerned with the arrangement of a set of distinct objects in some specific order. Such an arrangement of a group of objects is called a permutation of the objects. The number of permutations of $n$ distinct things taken all at a time, denoted by

$$P(n, n),$$

is equal to $n!$. The number of permutations of $n$ distinct things taken $r$ at a time, where

$$0 \leq r \leq n,$$

is given by

$$P(n, r) = n!/(n - r)!.$$

For instance, the number of ways 12 distinct objects can be arranged taken 3 at a

# Step-by-Step Solutions to Problems in this Chapter, "Permutations"

• **PROBLEM 1056**

Find ${}_9P_4$ .

Solution: Using the general formula for permutations of b different things taken a at a time, ${}_bP_a = \frac{b!}{(b-a)!}$ , we substitute 9 for b and 4 for a. Hence ${}_9P_4 = \frac{9!}{(9-4)!} = \frac{9!}{5!}$ . Evaluating our factorials, we obtain:

$$ {}_9P_4 = \frac{9\cdot8\cdot7\cdot6\cdot(5\cdot4\cdot3\cdot2\cdot1)}{(5\cdot4\cdot3\cdot2\cdot1)} $$

cancelling 5! in the numerator and denominator:

$$ \begin{aligned} {}_9P_4 &= 9\cdot8\cdot7\cdot6 \\ &= 3{,}024 \ . \end{aligned} $$

• **PROBLEM 1057**

Evaluate each of the following symbols:
(a) $5!$ (b) $\frac{7!}{4!}$ (c) $P(6,2)$ (d) $P(9,2)$

Solution:

(a) Recalling $n! = n\cdot(n-1)\cdot(n-2)\cdot(n-3)\cdots 1$,

$$ 5! = 5\cdot4\cdot3\cdot2\cdot1 = 120 $$

(b) Recalling $n! = n\cdot(n-1)! = n\cdot(n-1)\cdot(n-2)! = \cdots$

$$ \frac{7!}{4!} = \frac{7\cdot6\cdot5\cdot\cancel{4!}}{\cancel{4!}} = 210 $$

(c) Recalling $P(n,r) = \frac{n!}{(n-r)!}$

$$ P(6,2) = \frac{6!}{(6-2)!} = \frac{6!}{4!} = \frac{6\cdot5\cdot\cancel{4!}}{\cancel{4!}} = 30 $$

(d) Similarly $P(9,2) = \frac{9!}{(9-2)!} = \frac{9!}{7!} = \frac{9\cdot8\cdot\cancel{7!}}{\cancel{7!}} = 72$

• **PROBLEM 1058**

Calculate the number of permutations of the letters a,b,c,d taken two at a time.

Solution: The first of the two letters may be taken in 4 ways (a,b,c, or d). The second letter may therefore be selected from the remaining three letters in 3 ways. By the fundamental principle the total number of ways of selecting two letters is equal to the product of the number of ways of selecting each letter, hence

$$p(4,2) = 4 \cdot 3 = 12.$$

The list of these permutations is:

| | | | |
|---|---|---|---|
| ab | ba | ca | da |
| ac | bc | cb | db |
| ad | bd | cd | dc. |

**• PROBLEM 1059**

Calculate the number of permutations of the letters a,b,c,d taken four at a time.

Solution: The number of permutations of the four letters taken four at a time equals the number of ways the four letters can be arranged or ordered. Consider four places to be filled by the four letters. The first place can be filled in four ways choosing from the four letters. The second place may be filled in three ways selecting one of the three remaining letters. The third place may be filled in two ways with one of the two still remaining. The fourth place is filled one way with the last letter. By the fundamental principle, the total number of ways of ordering the letters equals the product of the number of ways of filling each ordered place, or $4 \cdot 3 \cdot 2 \cdot 1 = 24 = P(4,4) = 4!$ (read 'four factorial').

In general, for n objects taken r at a time,

$$P(n,r) = n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!} \quad (r < n).$$

For the special case where $r = n$,

$$P(n,n) = n(n-1)(n-2)\ldots(3)(2)(1) = n!,$$

since $(n-r)! = 0!$ which $= 1$ by definition.

**• PROBLEM 1060**

How many permutations of two letters each can be formed from the letters a,b,c,d,e? Actually write these permutations.

Solution: We recall the general formula for the number of permutations of n different things taken r at a time ${}_nP_r = n!/(n-r)!$. The number of permutations of 2 letters that can be formed from the 5 given letters is ${}_5P_2$.

$${}_5P_2 = \frac{5!}{(5-2)!} = \frac{5!}{3!} = \frac{5 \cdot 4 \cdot \cancel{3!}}{\cancel{3!}} = 20$$

Thus, the 20 permutations are:

| | | | |
|---|---|---|---|
| ab | ac | ad | ae |
| ba | bc | bd | be |
| ca | cb | cd | ce |

da db dc de

ea eb ec ed

**• PROBLEM 1061**

> Determine the number of permutations of three elements taken from a set of four elements {a, b, c, d}.

**Solution:**

**Method A**

In this example we can use the formula for permutations $P_a^b = \frac{b!}{(b-a)!}$ :

hence, $P_3^4 = \frac{4!}{(4-3)!} = \frac{4!}{1!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{1} = 24$

**Method B**

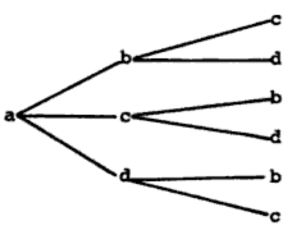
If you do not recall the formula for permutations you may determine the number of possible permutations of 3 elements taken from a set of four elements by recalling **the fundamental principle: If an act can be performed in m** ways and if, after this first act has been performed, a second act can be performed in n ways then the number of ways in which both acts can be performed, in the order given is m × n ways.

Thus, there are 4 ways of filling our first box of 3 elements × 3 ways of filling our second box × 2 ways of filling our third box

| | | | | | |
|---|---|---|---|---|---|
| 4 (a or b or c or d) | × | 3 (the 3 remaining letters) | × | 2 (the 2 remaining letters) | = 24. |

**Method C**

We can also determine the number of permutations using a tree diagram:




Hence our permutations are:

= abc (1)

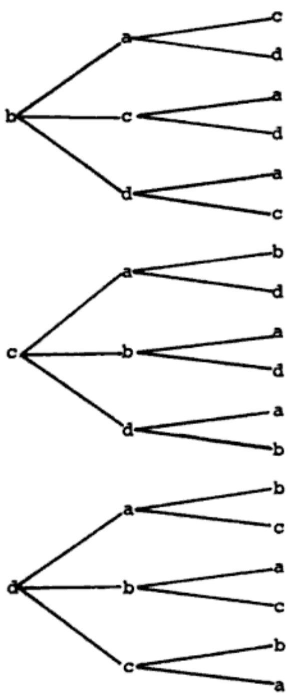
= abd (2)

= acb (3)

= acd (4)

= adb (5)

= adc (6)



= bac (7)
= bad (8)
= bca (9)
= bcd (10)
= bda (11)
= bdc (12)
= cab (13)
= cad (14)
= cba (15)
= cbd (16)
= cda (17)
= cdb (18)
= dab (19)
= dac (20)
= dba (21)
= dbc (22)
= dcb (23)
= dca (24)

**• PROBLEM 1062**

In how many ways may 3 books be placed next to each other on a shelf?

Solution: We construct a pattern of 3 boxes to represent the places where the 3 books are to be placed next to each other on the shelf:

| | | |
|---|---|---|

Since there are 3 books, the first place may be filled in 3 ways. There are then 2 books left, so that the second place may be filled in 2 ways. There is only 1 book left to fill the last place. Hence our boxes take the following form:

| | | |
|---|---|---|
| 3 | 2 | 1 |

The Fundamental Principle of Counting states that if one thing can be done in a different ways and, when it is done in any one of these ways, a second thing can be done in b different ways, and a third thing can be done in c ways, ... then all the things in succession can be done in a×b×c ... different ways. Thus the books can be arranged in $3 \cdot 2 \cdot 1 = 6$ ways.

If a group of 26 members is to elect a president and a secretary, in how many ways could the 2 officers be elected?

Solution: The group consists of 26 members, anyone of the 26 can serve as president. After the president has been elected, there are still 25 other members that could be elected as the secretary. The Fundamental Principle of Counting states that, if one thing can be done in a different ways, and a second thing can be done in b different ways, then the two things in succession can be done in a · b different ways. Therefore the number of ways the two officers can be chosen is (26)(25) or 650 ways.

The fundamental principle can be extended to more than two events. The total number of ways the successive events could be performed is the product of the numbers of ways each of the events could be performed.

This can also be seen by using the following approach. Since the arrangement of officers is important (x serving as president and y serving as secretary is different than y serving as president and x serving as secretary), this is a permutations problem. Recalling the general formula for the number of permutations of n things taken r at a time, ${}_nP_r = n!/(n-r)!$ , we replace n by 26 and r by 2 to obtain

$$ {}_{26}P_2 = \frac{26!}{(26-2)!} = \frac{26!}{24!} = \frac{26 \cdot 25 \cdot \cancel{24!}}{\cancel{24!}} = 650 $$

Thus, once again we find there are 650 ways to elect a president and secretary from the 26 members.

• PROBLEM 1066

How many telephone numbers of four different digits each can be made from the digits 0,1,2,3,4,5,6,7,8,9?

Solution: A different arrangement of the same four digits produces a different telephone number. Since we are concerned with the order in which the digits appear, we are dealing with permutations.

There are ten digits to choose from and four different ones are to be chosen at a time. The general formula for the number of permutations of n things taken r at a time is

$$ P(n,r) = \frac{n!}{(n-r)!} . $$

Here $n = 10$, $r = 4$, and the desired number is

$$ P(10,4) = \frac{10!}{(10-4)!} = \frac{10!}{6!} = \frac{10\cdot9\cdot8\cdot7\cdot6!}{6!} $$

$$ = 5040 $$

Thus 5040 telephone numbers of four digits each can be made from the 10 digits.

# CHAPTER 34

# COMBINATIONS

**Basic Attacks and Strategies for Solving Problems in this Chapter. See pages 776 to 789 for step-by-step solutions to problems.**

In many instances we are interested in a collection of items, but the order in which they are arranged is not important. In such cases we are dealing with combinations rather than permutations. The number of combinations of $n$ distinct things taken $r$ at a time is given by

$$\binom{n}{r} \quad \text{or} \quad c(n,r) = \frac{n!}{(n-r)!r!}$$

For example, the number of four-member committees that can be formed from a group of nine people is found by using the above formula as follows:

$$\begin{aligned} c(9,4) &= 9!/(9-4)!\,4! \\ &= 9!/5!\,4! \\ &= 9 * 8 * 7 * 6 * 5!/5!(4 * 3 * 2 * 1) \\ &= 126 \text{ four-member committees} \end{aligned}$$

In some applied situations, both knowledge of the fundamental counting principle and the use of combinations are necessary to solve the problem.

5,1 5,2 5,3 5,4 5,5 5,6

6,1 6,2 6,3 6,4 6,5 6,6

**• PROBLEM 1079**

How many different bridge hands are possible?

Solution: Since the order in which the cards are dealt is immaterial, we are dealing with combinations, thus we are interested in determining the number of combinations of the 52 cards taken 13 at a time. Recall the general formula for the number of combinations of n items taken r at a time,

$$C(n,r) = \frac{n!}{r!(n-r)!} \qquad \text{and}$$

$$C(52,13) = \frac{52!}{13!(52-13)!}$$

$$= \frac{52!}{13!39!}$$

$$= \frac{(52)(51)(50)(49)(48)(47)(46)(45)(44)(43)(42)(41)(40)\cancel{39!}}{(13)(12)(11)(10)(9)(8)(7)(6)(5)(4)(3)(2)(1)\ \cancel{39!}}$$

$$= 635{,}013{,}559{,}600.$$

**• PROBLEM 1080**

In how many ways can we select a committee of 3 from a group of 10 people?

Solution: The arrangement or order of people chosen is unimportant. Thus this is a combinations problem. Recalling the general formula for the number of combinations of n different things taken r at a time

$$C(n,r) = \frac{n!}{r!(n-r)!},$$

the number of committees of 3 from a group of 10 people is

$$C(10,3) = \frac{10!}{3!(10-3)!} = \frac{10!}{3!7!} = \frac{10\cdot\overset{3}{\cancel{9}}\cdot\overset{4}{\cancel{8}}\cdot\cancel{7!}}{\cancel{3}\cdot\cancel{2}\cdot1\cdot\cancel{7!}} = 120.$$

**• PROBLEM 1081**

How many committees of four members each can be formed from a group of seven persons?

Solution: This is a problem in combinations, rather than permutations, since the order is of no consequence. Thus, a committee consisting of Smith, Jones, Young, and Robinson is the same as a committee consisting of Smith, Robinson, Jones, and Young. The number of combinations of n different objects taken r at a time is equal to:

$$\frac{n(n-1)\ldots(n-r+1)}{1 \cdot 2 \cdots r}.$$

In this example, $n = 7$, $r = 4$, therefore

$$c(7,4) = \frac{7 \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4} = 35.$$

Alternately: The first member may be selected from the seven persons in 7 ways. The second member may be selected from the remaining six people in 6 ways. The third member may be selected in 5 ways from the remaining five people. The fourth member may be selected from the remaining four people in 4 ways. By the fundamental principle the total number of ways of picking the four members is equal to the product of the number of ways of picking each member, or $7 \cdot 6 \cdot 5 \cdot 4$ ways. This is a permutation of 7 people selected 4 at a time. To account for the number of ways in which the same four-person committee is selected, but in a different order, divide $7 \cdot 6 \cdot 5 \cdot 4$ by the number of ways in which the same committee of four can be selected. This equals a permutation of 4 people selected 4 at a time. This, by the fundamental principle, applied as above, equals $4 \cdot 3 \cdot 2 \cdot 1$. Then

$$\frac{7 \cdot 6 \cdot 5 \cdot 4}{4 \cdot 3 \cdot 2 \cdot 1} = 35.$$

**• PROBLEM 1082**

How many baseball teams of nine members can be chosen from among twelve boys, without regard to the position played by each member?

Solution: Since there is no regard to position, this is a combinations problem (if order or arrangement had been important it would have been a permutations problem). The general formula for the number of combinations of $n$ things taken $r$ at a time is

$$C(n,r) = \frac{n!}{r!(n-r)!} \quad .$$

We have to find the number of combinations of 12 things taken 9 at a time. Hence we have

$$C(12,9) = \frac{12!}{9!(12-9)!} = \frac{12!}{9!3!} = \frac{12 \cdot 11 \cdot 10 \cdot \cancel{9!}}{3 \cdot 2 \cdot 1 \cdot \cancel{9!}} = 220$$

Therefore, there are 220 possible teams.

**• PROBLEM 1083**

A manufacturer produces 7 different items. He packages assortments of equal parts of 3 different items. How many different assortments can be packaged?

Solution: Since we are not concerned with the order of the items, we are dealing with combinations. Thus the number of assortments is the number of combinations of 7 items taken 3 at a time. Recall the general formula for the number of combinations of $n$ items taken $r$ at a time,

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

$$C(7,3) = \frac{7!}{3!(7-3)!}$$

$$= \frac{7!}{3!4!}$$

$$= \frac{7 \cdot \cancel{6} \cdot 5 \cdot \cancel{4}!}{\cancel{3} \cdot \cancel{2} \cdot \cancel{4}!}$$

$$= 35$$

Thus, 35 different assortments can be packaged.

**• PROBLEM 1084**

> A man and his wife decide to entertain 24 friends by giving 4 dinners with 6 guests each. In how many ways can the first group be chosen?

Solution: In the first group we are considering one dinner and there are 6 people out of 24 friends to be invited. We must find the number of ways to choose 6 out of 24. We are dealing with combinations. To select r things out of n objects, we use the definition of combinations:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} = c(n,r)$$

$$c(24,6) = \binom{24}{6} = \frac{24!}{6!18!} = \frac{24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19 \cdot 18!}{6!\ 18!}$$

$$= \frac{24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$$= 134{,}596$$

**• PROBLEM 1085**

> A lady has 12 friends. She wishes to invite 3 of them to a bridge party. How many times can she entertain without having the same 3 people again?

Solution: Since no reference has been made to order or arrangement (for example, the order in which the guests arrive or their seating arrangement at the bridge table), the problem is one of combinations rather than permutations. Recall the general formula for the number of combinations of n different things taken r at a time,

$$C(n,r) = \frac{n!}{r!(n-r)!} \ .$$

Thus the number of ways of selecting 3 friends from 12 is

$$C(12,3) = \frac{12!}{3!(12-3)!} = \frac{12!}{3!9!} = \frac{\overset{4}{\cancel{12}} \cdot 11 \cdot \overset{5}{\cancel{10}} \cdot \cancel{9!}}{\cancel{3} \cdot \cancel{2} \cdot 1 \cdot \cancel{9!}} = 220$$

In evaluating C(12,3), observe that the numerator and denominator of the fraction are first divided by the larger factorial in the denominator. (9! cancels in our fraction.) Thus, the lady can entertain 220 times without having the same 3 people.

**• PROBLEM 1086**

> A Sunday school class of 12 members is to be seated on seven chairs and a bench that accommodates five persons. In how many ways can the bench be occupied?

Solution: If we are concerned with the order of people on the bench (so that we consider the same five people sitting in different arrangements as distinct ways), then this is a permutations problem. Recalling the general formula for the number of permutations of n elements taken r at a time

$$p(n,r) = \frac{n!}{(n-r)!}$$

we find the number of permutations of 12 elements taken 5 at a time, or p(12,5). Thus

$$p(12,5) = \frac{12!}{(12-5)!} = \frac{12!}{7!} = 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 = 95{,}040$$

If we are not concerned with the order of the people on the bench this becomes a combinations problem. Recalling the general formula for the number of combinations of n elements taken r at a time

$$c(n,r) = \frac{n!}{r!(n-r)!}$$

we find the number of combinations of 12 elements taken 5 at a time, or c(12,5).

$$c(12,5) = \frac{12!}{5!(12-5)!} = \frac{12!}{5!7!} = \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7!}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 7!} = 792.$$

**• PROBLEM 1087**

How many different sums of money can be obtained by choosing two coins from a box containing a penny, a nickel, a dime, a quarter, and a half dollar?

Solution: The order makes no difference here, since a selection of a penny and a dime is the same as a selection of a dime and a penny, insofar as a sum of money is concerned. This is a case of combinations, then, rather than permutations. Then the number of combinations of n different objects taken r at a time is equal to:

$$\frac{n(n-1)\ldots(n-r+1)}{1 \cdot 2 \cdots r}.$$

In this example, n = 5, r = 2, therefore

$$C(5,2) = \frac{5 \cdot 4}{1 \cdot 2} = 10.$$

As in the problem of selecting four committee members from a group of seven people, a distinct two coins can be selected from five coins in

$\frac{5 \cdot 4}{1 \cdot 2}$ = 10 ways (applying the fundamental principle).

**• PROBLEM 1088**

In how many ways can 5 prizes be given away to 4 boys, when each boy is eligible for all the prizes?

Solution: Any one of the prizes can be given in 4 ways; and then any one of the remaining prizes can also be given in 4 ways, since it may be obtained by the boy who has already received a prize. Thus two prizes can be given away in $4^2$ ways, three prizes in $4^3$ ways, and so on. Hence the 5 prizes can be given away in $4^5$, or 1024 ways.

How many "words" each consisting of two vowels and three consonants, can be formed from the letters of the word "integral"?

Solution: To find the number of ways to choose vowels or consonants from letters, we use combinations. The number of combinations of n different objects taken r at a time is defined to be

$$C(n,r) = \frac{n!}{r!(n-r)!} .$$

Then, we first select the two vowels to be used, from among the three vowels in integral; this can be done in $C(3,2) = 3$ ways. Next, we select the three consonants from the five in integral; this yields $C(5,3) = 10$ possible choices. To find the number of ordered arrangements of 5 letters selected five at a time, we need to find the number of permutations of choosing r from n objects. Symbolically, it is $P(n,r)$ which is defined to be

$$P(n,r) = \frac{n!}{(n-r)!}$$

We permute the five chosen letters in all possible ways, of which there are $P(5,5) = 5! = 120$ arrangements. Finally, to find the total number of words which can be formed, we apply the Fundamental Counting Principle which states that if one event can be performed in m ways, another one in n ways, and another in k ways, then the total number of ways in which all events can occur is $m \times n \times k$ ways. Hence the total number of possible words is, by the fundamental principle

$$C(3,2)C(5,3)P(5,5) = 3 \cdot 10 \cdot 120 = 3600.$$



From 12 books in how many ways can a selection of 5 be made, (1) when one specified book is always included, (2) when one specified book is always excluded?

Solution: Here the formula for combinations is appropriate: the number of combinations of n things taken r at a time:

$$C(n,r) = {}_nC_r = \frac{n!}{r!(n-r)!}$$

where $n = 11$, and $r = 4$.

(1) Since the specified book is to be included in every selection, we have only to choose 4 out of the remaining 11.

Hence the number of ways $= {}^{11}C_4$

$$\begin{aligned} {}_{11}C_4 &= \frac{11!}{4!(11-4)!} \\ &= \frac{11!}{4!7!} \\ &= \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot \cancel{7}!}{4 \cdot 3 \cdot 2 \cdot 1 \cdot \cancel{7}!} \\ &= \frac{11 \times 10 \times 9 \times 8}{1 \times 2 \times 3 \times 4} \\ &= 330. \end{aligned}$$

(2) Since the specified book is always to be excluded, we have to select the 5 books out of the remaining 11.

can be chosen:

(a) If each committee is to have exactly 3 men?
(b) If each committee is to have at least 3 men?

**Solution:**
(a) The order in which the people on the committee are chosen is unimportant, thus this is a problem involving combinations. The general formula for the number of combinations of n different things taken r at a time is $C(n,r) = n!/r!(n-r)!$. Thus, the number of ways to choose 3 men from 10 men is $C(10,3)$

$$= \frac{10!}{3!(10-3)!} = \frac{10!}{3!7!} = \frac{10\cdot\cancel{9}^{3}\cdot\cancel{8}^{4}\cdot\cancel{7!}}{\cancel{3}\cdot\cancel{2}\cdot1\cdot\cancel{7!}} = 120.$$

The number of ways to choose 2 women from 6 women is $C(6,2)$

$$= \frac{6!}{2!\,(6-2)!} = \frac{6!}{2!4!} = \frac{\cancel{6}^{3}\cdot5\cdot\cancel{4!}}{\cancel{2}\cdot1\cdot\cancel{4!}} = 15.$$

The Fundamental Principle of Counting states that if the first of two independent acts can be performed in a ways, and if the second act can be performed in b ways, then the number of ways of performing the two acts in the order stated is ab. Thus by the fundamental principle, the number of ways to choose the committee is $C(10,3)\cdot C(6,2) = 120\cdot5 = 1{,}800$.

(b) If the committee is to contain at least 3 men, the possibilities are 3 men and 2 women, 4 men and 1 woman, 5 men and no women.

We have just shown that the number of committees consisting of 3 men and 2 women is 1,800. The number of committees containing 4 men and 1 woman is

$$C(10,4)\cdot C(6,1) = \frac{10!}{4!(10-4)!}\cdot\frac{6!}{1!(6-1)!} = \frac{10!}{4!6!}\cdot\frac{6!}{1!5!}$$

$$= \frac{10\cdot\cancel{9}^{3}\cdot\cancel{8}\cdot7\cdot\cancel{6!}}{\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot1\cdot\cancel{6!}}\bullet\frac{6\cdot\cancel{5!}}{1\cdot\cancel{5!}} = 210.6 = 1{,}260\ .$$

The number of committees consisting of 5 men is

$$C(10,5) = \frac{10!}{5!(10-5)!} = \frac{10!}{5!5!} = \frac{\cancel{10}^{2}\cdot\cancel{9}^{3}\cdot\cancel{8}^{2}\cdot7\cdot\cancel{6}^{3}\cdot\cancel{5!}}{\cancel{5}\cdot\cancel{4}\cdot\cancel{3}\cdot\cancel{2}\cdot1\cdot\cancel{5!}} = 252$$

The probability that any of several mutually exclusive events will occur is the sum of the probabilities of the separate events.

Hence the number of committees containing at least 3 men is

$$1{,}800 + 1{,}260 + 252 = 3{,}312.$$

**• PROBLEM 1095**

From 7 Englishmen and 4 Americans a committee of 6 is to be formed; in how many ways can this be done, (1) when the committee contains exactly 2 Americans, (2) at least 2 Americans?

**Solution:** (1) : Case (1) is when we choose exactly 2 Americans and thus we need 4 Englishmen in order to have

# CHAPTER 35

# PROBABILITY

**Basic Attacks and Strategies for Solving Problems in this Chapter. See pages 790 to 810 for step-by-step solutions to problems.**

The set $S$ of all possible outcomes of a given experiment is called the sample space for the experiment. Any subset of the sample space $S$ is an event. The probability of an event $E$, denoted by $P(E)$, is the ratio of the number of outcomes in the sample space $S$ which satisfies event $E$ compared to the total number of outcomes in sample space $S$. Thus,

$$P(E) = \frac{n(e)}{n(s)} = \frac{\text{number of outcomes satisfying event } E}{\text{number of outcomes in sample space } S}$$

This definition is the procedure for calculating the probability of an event $E$. For instance, consider the experiment of rolling a single die and finding the probability of the event $E$ = "the number of spots showing on the upper face is greater than three." This is done by first noting that

$n(E) = 3$ since $E = \{4, 5, 6\}$

and $n(S) = 6$ since $S = \{1, 2, 3, 4, 5, 6\}$.

Thus,

$P(E) = n(E)/n(S) = 3/6 = 1/2.$

Calculating probabilities by listing and counting the elements of a sample space, as well as the event, is not always practical. Instead, one should use the counting principles developed in Chapters 33 and 34 to determine the number of elements in the sample space and event, respectively.

Some guidelines for understanding a probability that involves two or more independent events $A$ and $B$ are given below. In a probability statement:

(1) The word "or" usually means to add the probabilities of each event, say $A$ and $B$, if the events are mutually exclusive, that is,

$P(A \text{ or } B) = P(A) + P(B).$

(b) The probability of drawing a black ball,

$$P(B) = \frac{\text{number of ways of drawing a black ball}}{\text{number of ways of selecting a ball}}$$

$$P(B) = \frac{4}{12} = \frac{1}{3}.$$

(c) The probability that either one of two mutually exclusive events will occur is the sum of the probabilities of the separate events. Thus the probability of drawing either a white [P(W)] or a black ball [P(B)] is P(W) + P(B).

$$P(W) = \frac{\text{number of ways of drawing a white ball}}{\text{number of ways of selecting a ball}}$$

$$= \frac{6}{12} = \frac{1}{2}.$$

$P(B) = \frac{1}{3}$ [shown in part (b)].

$$\text{Thus, } P(W \text{ or } B) = P(W) + P(B) = \frac{6}{12} + \frac{4}{12}$$

$$= \frac{10}{12}$$

$$= \frac{5}{6}.$$

**• PROBLEM 1110**

Determine the probability of getting 6 or 7 in a toss of two dice.

**Solution:** Let A = the event that a 6 is obtained in a toss of two dice

B = the event that a 7 is obtained in a toss of two dice.

Then, the probability of getting 6 or 7 in a toss of two dice is

$$P(A \text{ or } B) = P(A \cup B).$$

The union symbol "$\cup$" means that A and/or B can occur. Now $P(A \cup B) = P(A) + P(B)$ if A and B are mutually exclusive. Two or more events are said to be mutually exclusive if the occurrence of any one of them excludes the occurrence of the others. In this case, we cannot obtain a six and a seven in a single toss of two dice. Thus, A and B are mutually exclusive.

To calculate P(A) and P(B), use the following table.

Note: There are 36 different tosses of two dice.

A = a 6 is obtained in a toss of two dice

$$= \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$$

B = a 7 is obtained in a toss of two dice

$= \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}.$

$$P(A) = \frac{\text{number of ways to obtain a 6 in a toss of two dice}}{\text{number of ways to toss two dice}}$$

$$= \frac{5}{36}$$

$$P(B) = \frac{\text{number of ways to obtain a 7 in a toss of two dice}}{\text{number of ways to toss two dice}}$$

$$= \frac{6}{36} = \frac{1}{6}.$$

Therefore, $P(A \cup B) = P(A) + P(B) = \frac{5}{36} + \frac{6}{36} = \frac{11}{36}.$

**• PROBLEM 1111**

A penny is to be tossed 3 times. What is the probability there will be 2 heads and 1 tail?

Solution: We start this problem by constructing a set of all possible outcomes:

| | | |
|---|---|---|
| We can have heads on all 3 tosses: | (HHH) | |
| head on first 2 tosses, tail on the third: | (HHT) | (1) |
| head on first toss, tail on next two: | (HTT) | |
| • | (HTH) | (2) |
| • | (THH) | (3) |
| | (THT) | |
| • | (TTH) | |
| | (TTT) | |

Hence there are eight possible outcomes (2 possibilities on first toss x 2 on second x 2 on third = 2 x 2 x 2 = 8).

We assume that these outcomes are all equally likely and assign the probability 1/8 to each. Now we look for the set of outcomes that produce 2 heads and 1 tail. We see there are 3 such outcomes out of the 8 possibilities (numbered (1), (2), (3) in our listing). Hence the probability of 2 heads and 1 tail is 3/8.

**• PROBLEM 1112**

Find the probability of throwing two sixes in one toss of a pair of dice.

Solution: To find the probability of throwing two sixes in one toss of a pair of dice, first we express it symbolically.

P(throwing two sixes in one toss of a pair of dice) =

P(throwing a six in one toss of a die) X P(throwing a six in one toss of a die).
This is true because the event of tossing a die is independent of tossing another die. That is, the occurrence of one event has no effect upon the occurrence or non-occurrence of the other. Now,

P(throwing a six in one toss) =

$$\frac{\text{number of ways to obtain a six}}{\text{number of ways to obtain any face value of a die}} = \frac{1}{6} .$$

Hence, the probability of obtaining two sixes is $\left(\frac{1}{6}\right)\left(\frac{1}{6}\right) = \frac{1}{36}$ .

**• PROBLEM 1113**

What is the probability of obtaining two aces on two successive throws of a die?

Solution: The throw of a die results in 6 different but equally likely face values. An ace can be obtained only when a certain 1 of the 6 faces shows. Therefore the probability of obtaining an ace in one throw is $\frac{1}{6}$. In a toss of two dice, the fall of either does not affect the fall of the other. Thus the two events, consisting of two sucessive throws, are independent. The probability that all of a set of independent events will happen in a single trial is the product of their separate probabilities. Therefore, the probability is $\frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$.

**• PROBLEM 1114**

If a pair of dice is tossed twice, find the probability of obtaining 5 on both tosses.

Solution: We obtain 5 in one toss of the two dice if they fall with either 3 and 2 or 4 and 1 uppermost, and each of these combinations can appear in two ways. The ways to obtain 5 in one toss of the two dice are:

$$(1,4),(4,1),(3,2), \text{ and } (2,3).$$

Hence we can throw 5 in one toss in four ways. Each die has six faces and there are six ways for a die to fall. Then the pair of dice can fall in $6 \cdot 6 = 36$ ways. The probability of throwing 5 in one toss is:

$$\frac{\text{the number of ways to throw a 5 in one toss}}{\text{the number of ways that a pair of dice can fall}} = \frac{4}{36} = \frac{1}{9} .$$

Now the probability of throwing a 5 on both tosses is:

P(throwing five on first toss and throwing five on second toss).

"And" implies multiplication if events are independent, thus
p(throwing 5 on first toss and throwing 5 on second toss)

= p(throwing 5 on first toss) X p(throwing 5 on second toss)

Since the results of the two tosses are independent. Consequently, the probability of obtaining 5 on both tosses is

$$\left(\frac{1}{9}\right)\left(\frac{1}{9}\right) = \frac{1}{81} .$$

**• PROBLEM 1115**

A bag contains 4 black and 5 blue marbles. A marble is drawn and then replaced, after which a second marble is drawn. What is the probability that the first is black and second blue?

Solution: Let C = event that the first marble drawn is black.

D = event that the second marble drawn is blue.

The probability that the first is black and the second is blue can be expressed symbolically:

$$P(C \text{ and } D) = P(CD).$$

We can apply the following theorem. If two events A and B, are independent, then the probability that A and B will occur is,

$$P(A \text{ and } B) = P(AB) = P(A) \cdot P(B).$$

Note that two or more events are said to be independent if the occurrence of one event has no effect upon the occurrence or non-occurrence of the other. In this case the occurrence of choosing a black marble has no effect on the selection of a blue marble and vice versa; since, when a marble is drawn it is then replaced before the next marble is drawn. Therefore, C and D are two independent events.

$$P(CD) = P(C) \cdot P(D)$$

$$P(C) = \frac{\text{number of ways to choose a black marble}}{\text{number of ways to choose a marble}}$$

$$= \frac{4}{9}.$$

$$P(D) = \frac{\text{number of ways to choose a blue marble}}{\text{number of ways to choose a marble}}$$

$$= \frac{5}{9}.$$

$$P(CD) = P(C) \cdot P(D) = \frac{4}{9} \cdot \frac{5}{9} = \frac{20}{81}.$$

**• PROBLEM 1116**

A box contains 4 black marbles, 3 red marbles, and 2 white marbles. What is the probability that a black marble, then a red marble, then a white marble is drawn without replacement?

Solution: Here we have three dependent events. There is a total of 9 marbles from which to draw. We assume on the first draw we will get a black marble. Since the probability of drawing a black marble is the

$$\frac{\text{number of ways of drawing a black marble}}{\text{number of ways of drawing 1 out of (4+3+2) marbles}},$$

$$P(A) = \frac{4}{4+3+2} = \frac{4}{9}$$

There are now 8 marbles left in the box.

On the second draw we get a red marble. Since the probability of drawing a red marble is

$$\frac{\text{number of ways of drawing a red marble}}{\text{number of ways of drawing 1 out of the 8 remaining marbles}}$$

(b) p (first ball will be white and the second red)

= p (first ball will be white) p(the second ball will be red)

$$= \left(\frac{\text{number of ways to choose a white ball}}{\text{number of ways to choose a ball}}\right)\left(\frac{\text{number of ways to choose a red ball}}{\text{number of ways to choose a ball after the removal of the first}}\right)$$

$$= \frac{\cancel{5}^{1}}{16} \cdot \frac{7}{\cancel{15}_{3}} = \frac{7}{48}$$

(c) p (three balls drawn in the order white, black, red)

= p (first ball is white) p(second ball is black) p(third ball is red)

$$= \left(\frac{\text{number of ways to choose that the first ball is white}}{\text{number of ways to choose the first ball}}\right)\left(\frac{\text{number of ways to choose that second one is black}}{\text{number of ways to choose the second one}}\right)$$
$$\left(\frac{\text{number of ways to choose that the third one is red}}{\text{number of ways to choose the third one}}\right)$$

$$= \frac{\cancel{5}^{1}}{\cancel{16}_{4}} \quad \frac{\cancel{4}^{1}}{\cancel{15}_{3}} \quad \frac{\cancel{7}^{1}}{\cancel{14}_{2}} = \frac{1}{24}$$

**• PROBLEM 1118**

What is the chance of throwing a number greater than 4 with an ordinary die whose faces are numbered from 1 to 6?

Solution: If an event can happen in $s$ ways and fail to happen in $f$ ways, and if all these ways $(s + f)$ are assumed to be equally likely, then the probability (p) that the event will happen is

$$p = \frac{s}{s + f} = \frac{\text{(successful ways)}}{\text{(total ways)}}$$

In our case there are 6 possible ways in which the die can fall (1,2,3,4,5, or 6). Of these, two are favorable to the event required, 5 or 6, therefore the required chance $= \frac{2}{6} = \frac{1}{3}$.

**• PROBLEM 1119**

A bag contains 10 red, 15 green, and 5 yellow beads. If a single bead is drawn from the bag what is the probability (a) that the bead is red, and (b) that the bead is not red?

Solution: (a) If an event can happen in $s$ ways and fail to happen in $f$ ways, and if all these ways $(s + f)$ are assumed to be equally likely, then the probability (p) that the event will happen is

$$p = \frac{s}{s + f} = \frac{\text{(successful ways)}}{\text{(total ways)}}$$

There are $10 + 15 + 5 = 30$ beads in the bag, any one of which could be drawn from the bag. Ten beads are red, therefore $s = 10$. The total of nonred beads is 20, $f = 20$.

$$p = \frac{s}{s + f}$$
$$= \frac{10}{10 + 20}$$

The probability that a 7 will appear,

$$p(7) = \frac{\text{number of possible ways of obtaining a 7}}{\text{number of ways that 2 dice can be thrown}}$$

$$p(7) = \frac{6}{36} = \frac{1}{6}\,.$$

**• PROBLEM 1124**

If two dice are cast, what is the probability the sum will be less than 5?

Solution: If A, B, and C are mutually exclusive events, that is, their intersection is the null set, then $P(A \cup B \cup C) = P(A) + P(B) + P(C)$. Since the obtaining of sums of 2, 3, and 4 are mutually exclusive events, the probability of obtaining a sum less than 5 is the sum of the probabilities of obtaining a sum of 2,3, and 4. To obtain the sum of 2 with 2 die, we have the following possibilities: (1,1).

Similarly for the sum of 3, we have: (1,2) and (2,1).

For the sum of 4, we obtain: (1,3), (3,1), and (2,2).

Thus $P_1$ = probability of obtaining a sum of 2

$$= \frac{\text{number of ways to obtain a sum of 2}}{\text{number of ways to throw 2 dice}}$$

$$= \frac{1}{36}$$

$P_2$ = probability of obtaining a sum of 3

$$= \frac{\text{number of ways to obtain a sum of 3}}{\text{number of ways to throw 2 dice}}$$

$$= \frac{2}{36} = \frac{1}{18}.$$

$P_3$ = probability of obtaining a sum of 4

$$= \frac{\text{number of ways to obtain a sum of 4}}{\text{number of ways to throw 2 dice}}$$

$$= \frac{3}{36} = \frac{1}{12}.$$

The probability of obtaining a sum less than 5 is

$$P_1 + P_2 + P_3 = \frac{1}{36} + \frac{1}{18} + \frac{1}{12}.$$

$$= \frac{1}{36} + \frac{2}{36} + \frac{3}{36} = \frac{6}{36} = \frac{1}{6}$$

**• PROBLEM 1125**

Find the probability that when a pair of dice are thrown, the sum of the two up faces is greater than 7 or the same number appears on each face.

Solution: The sample space consists of 36 equally likely outcomes as shown in the accompanying figure. Those outcomes that give a sum greater than 7 are

$$G = \{(6,2),(6,3),(6,4),(6,5),(6,6),(5,3),(5,4),(5,5),(5,6),(4,4),(4,5),(4,6),(3,5),(3,6),(2,6)\}$$

Let us call the possible outcomes which are circled above set A. Then the elements of set A, A = {(1,1), (1,2), (1,3), (2,1), (2,2), (3,1)} are all the possible ways of obtaining four or less.

The probability of obtaining 4 or less,

$$P[(x,y) \le 4] = \frac{\overbrace{\text{number of ways of obtaining 4 or less}}^{\text{number of elements in set A}}}{\text{number of ways the dice may land}}$$

$$= \frac{6}{36} = \frac{1}{6}.$$

**• PROBLEM 1127**

> The probability that A wins a certain game is $\frac{2}{3}$. If A plays 5 games, what is the probability that A will win (a) exactly 3 games? (b) at least 3 games?

Solution: We shall apply the following theorem. If P is the probability that an event will happen in a single trial and q is the probability that this event will fail in this trial, then ${}_nC_r\, p^r q^{n-r}$ is the probability that this event will happen exactly r times in n trials. ${}_nC_r$, the number of combinations of n different objects taken r at a time, is

$${}_nC_r = \frac{n!}{r!(n-r)!}.$$

Note that p + q = 1.

(a) We are given the probability of a success, p, which is winning a game: $p = \frac{2}{3}$. Therefore from p + q = 1, $q = 1 - p = 1 - \frac{2}{3} = \frac{1}{3}$. The number of ways of winning 3 games out of 5 is

$${}_5C_3 = \binom{5}{3} = \frac{5!}{3!2!} = \frac{5 \cdot \cancel{4}^{2} \cdot \cancel{3!}}{\cancel{3!}\,\underset{1}{\cancel{2}} \cdot 1} = 10.$$

Thus, the probability of A winning 3 games is

$${}_nC_r\, p^r q^{n-r} = 5C_3\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^2 = 10\ \frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{1}{3}\cdot\frac{1}{3} = \frac{80}{243}$$

(b) To win at least 3 games A must win either exactly 3 or exactly 4 or all 5 games. In order that A will win at least 3 games, we must calculate the probability that A will win three games, four games, and five games.

$$P = {}_5C_3\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^2 + {}_5C_4\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^1 + {}_5C_5\left(\frac{2}{3}\right)^5\left(\frac{1}{3}\right)^0.$$

$$= \frac{5!}{3!2!}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^2 + \frac{5!}{4!1!}\left(\frac{2}{3}\right)^4\left(\frac{1}{3}\right)^1 + \frac{5!}{5!0!}\left(\frac{2}{3}\right)^5$$

$$= \frac{5 \cdot \cancel{4}^{2} \cdot \cancel{3}}{\cancel{3!}\,\underset{1}{\cancel{2}} \cdot 1}\ \frac{8}{243} + \frac{5 \cdot \cancel{4!}}{\cancel{4!}\ 1!}\ \frac{16}{243} + \frac{32}{243}$$

$$= 10 \cdot \frac{8}{243} + 5 \cdot \frac{16}{243} + \frac{32}{243} = \frac{192}{243} = \frac{64}{81}.$$

**A die is tossed five times. What is the probability that an ace will appear: (a) at least twice; (b) at least once?**

Solution: This is a problem involving repeated trials of an experiment. The experiment is "tossing a die five times". Apply the following theorem: If $p$ is the probability that an event will happen in a single trial and $q$ is the probability that this event will fail in this trial, then

$${}_nC_r p^r q^{n-r}$$

is the probability that this event will happen exactly $r$ times in $n$ trials.

(a) To find the probability that an ace will occur at least twice, find the probability that it will occur twice, or three times, or four times, or five times. The sum (the word "or" implies addition in set notation) of these probabilities will be the probability that an ace will happen at least twice. p = probability that an ace will occur in a given trial

$$= \frac{\text{number of ways to obtain an ace}}{\text{number of ways to obtain any face of a die}}$$

$$= \frac{1}{6}$$

An experiment can only succeed or fail, hence the probability of success, p, plus the probability of failure, q, is one; $p+q = 1$. Then $q = 1-p = 1 - 1/6 = 5/6$. Therefore, using ${}_nC_r\ p^r q^{n-r}$, p (at least two aces) =

$${}_5C_2\ (1/6)^2(5/6)^3 + {}_5C_3(1/6)^3(5/6)^2$$

$$+\ {}_5C_4\ (1/6)^4(5/6)^1 + {}_5C_5\ (1/6)^5(5/6)^0$$

${}_nC_r$ is a symbol for a combination of $n$ things, $r$ at a time, where $r$ objects are chosen from $n$ objects.

$${}_nC_r = \frac{n!}{r!\,(n-r)!}$$

Apply this formula. Then,

$${}_5C_2(1/6)^2(5/6)^3 + {}_5C_3(1/6)^3(5/6)^2$$

$$+\ {}_5C_4(1/6)^4(5/6)^1 + {}_5C_5(1/6)^5(5/6)^0$$

$$= \frac{5!}{2!3!}\left(\frac{125}{6^5}\right) + \frac{5!}{2!3!}\left(\frac{25}{6^5}\right) + \frac{5!}{4!1!}\left(\frac{5}{6^5}\right) + \frac{5!}{5!0!}\left(\frac{1}{6^5}\right)$$

$$= \frac{5\cdot \overset{2}{\cancel{4}}\cdot\cancel{3!}}{\underset{1}{\cancel{2}}\cdot 1\cdot\cancel{3!}}\left(\frac{125}{6^5}\right) + \frac{5\cdot \overset{2}{\cancel{4}}\cdot\cancel{3!}}{\underset{1}{\cancel{2}}\cdot 1\cdot\cancel{3!}}\left(\frac{25}{6^5}\right) + \frac{5\cdot\cancel{4!}}{\cancel{4!}1!}\left(\frac{5}{6^5}\right) + \frac{1}{6^5}$$

$$= 10\left(\frac{125}{6^5}\right) + 10\left(\frac{25}{6^5}\right) + 5\left(\frac{5}{6^5}\right) + \frac{1}{6^5}$$

$$= \frac{1250 + 250 + 25 + 1}{6^5} = \frac{1526}{7776} = \frac{763}{3888}$$

Therefore, the probability that an ace will appear at least twice is

$$\frac{763}{3888}.$$

(b) An ace can be obtained at least once by tossing one ace, 2 aces, 3 aces,..., or 5 aces. Hence, the probability of obtaining at least one ace is the sum of the individual probabilities of obtaining one, two, three,..., up to five aces. Apply the same method as in part (a).

$$\begin{aligned}
p(\text{at least one ace}) &= {}_5C_1(1/6)^1(5/6)^4 + {}_5C_2(1/6)^2(5/6)^3 \\
&\quad + {}_5C^3(1/6)^3(5/6)^2 + {}_5C_4(1/6)^4(5/6)^1 \\
&\quad + {}_5C_5(1/6)^5(5/6)^0 \\
&= \frac{5!}{1!4!}\left(\frac{625}{6^5}\right) + \frac{5!}{2!3!}\left(\frac{125}{6^5}\right) + \frac{5!}{3!2!}\left(\frac{25}{6^5}\right) + \frac{5!}{4!1!}\left(\frac{5}{6^5}\right) \\
&\quad + \frac{\cancel{5!}}{\cancel{5!}0!}\left(\frac{1}{6^5}\right) = \frac{5\cdot\cancel{4!}}{1\cdot\cancel{4!}}\left(\frac{625}{6^5}\right) + \frac{5\cdot\overset{2}{\cancel{4}}\cdot\cancel{3!}}{\underset{1}{\cancel{2}}\cdot1\cdot\cancel{3!}}\left(\frac{125}{6^5}\right) + \frac{5\cdot\overset{2}{\cancel{4}}\cdot\cancel{3!}}{\cancel{3!}\underset{1}{\cancel{2!}}}\left(\frac{25}{6^5}\right) \\
&\quad + \frac{5\cdot\cancel{4!}}{\cancel{4!}\cdot1}\left(\frac{5}{6^5}\right) + \left(\frac{1}{6^5}\right) = 5\left(\frac{625}{6^5}\right) + 10\left(\frac{125}{6^5}\right) + 10\left(\frac{25}{6^5}\right) \\
&\quad + 5\left(\frac{5}{6^5}\right) + \frac{1}{6^5} \\
&= \frac{3125 + 1250 + 250 + 25 + 1}{6^5} = \frac{4651}{7776}
\end{aligned}$$

An alternate, shorter method, is to calculate the probability of failure, (obtaining no aces) and subtract this from one. This is true because $q+p = 1$, hence $q = 1-p$.

$$p\ (\text{at least one ace}) = 1 - p\ (\text{no aces})$$

$$p\ (\text{no aces}) = {}_5C_0(1/6)^0(5/6)^5 = \frac{\cancel{5!}}{0!\cancel{5!}}\left(\frac{5^5}{6^5}\right) = \frac{3125}{7776}$$

Thus,

$$p\ (\text{at least one ace}) = 1 - p\ (\text{no aces}) = 1 - \frac{3125}{7776} = \frac{4651}{7776}$$

Therefore, the probability that an ace appears at least once is

$$\frac{4651}{7776}.$$

**• PROBLEM 1130**

> A coin is tossed 3 times, and 2 heads and 1 tail fall. What is the probability that the first toss was heads?

__Solution:__ This problem is one of conditional probability. Given two events, $p_1$ and $p_2$, the probability that event $p_2$ will occur on the condition that we have event $p_1$ is

$$P(p_2/p_1) = \frac{P(p_1 \text{ and } p_2)}{P(p_1)} = \frac{P(p_1 p_2)}{P(p_1)}$$

Define

$p_1$: 2 heads and 1 tail fall,

$p_2$: the first toss is heads.

$$P(p_1) = \frac{\text{number of ways to obtain 2 heads and 1 tail}}{\text{number of possibilities resulting from 3 tosses}}$$

$$= (\{H,H,T\},\{H,T,H\},\{T,H,H\}) / (\{H,H,H\},\{H,H,T\},\{H,T,T\},\{H,T,H\},$$
$$\{T,T,H\},\{T,H,T\},\{T,H,H\},\{T,T,T\})$$

# CHAPTER 36

# SERIES

**Basic Attacks and Strategies for Solving Problems in this Chapter. See pages 811 to 817 for step-by-step solutions to problems.**

A series is associated with any sequence. It is the indicated sum of the terms of a finite or infinite sequence. If the sequence is finite and is an arithmetic progression, then the finite series is given by

$$S_n = (n/2)\,[2a + (n-1)d],$$

where $a$ is the first term, $n$ is the last term, and $d$ is the common difference. If the sequence is finite and is a geometric progression, then the finite series is given by

$$S_n = (a - ar^n) / (1 - r),$$

where $r \neq 1$, $a$ is the first term, $n$ is the last term, and $r$ is the common ratio.

A series for which the general term is known can be represented by the sigma or summation symbol. For example,

$$S_n = 4 + 7 + 10 + \ldots + (3n + 1)$$

can be written as

$$S_n = \sum_{j=1}^{n} (3j + 1).$$

For an infinite series, the question of whether it converges or diverges is pertinent. For an infinite series,

$$S_\infty = \sum_{j=1}^{\infty} s_j$$

converges if and only if

$$S_1, S_2, \ldots, S_n, \ldots,$$

the corresponding sequence of partial sums, converges. If the sequences of partial sums converge to the number $L$

$$(\lim_{n\to\infty} S_n = L)$$

then $L$ is said to be the sum of the infinite series and thus

$$S_\infty = \sum_{j=1}^{\infty} s_j = L.$$

For example,

$$\sum_{j=1}^{\infty} [1/j - 1/(j+1)]$$

converges to

$$L = 1 - 1/(n+1)$$

because the $n^{th}$ partial sum of this series is given by

$$\begin{aligned} S_n &= (1/1 - 1/2) + (1/2 - 1/3) + \ldots + [1/(n-1) - 1/n] + [1/n - 1/(n+1)] \\ &= 1 - 1/(n+1). \end{aligned}$$

An infinite series that does not converge is said to diverge (or, if the sequence of partial sums diverges, then the series is said to diverge). For example, the series

$$\sum_{j=1}^{\infty} a^j$$

diverges because the $n^{th}$ partial sum of this series,

$$S_n = 1 + 2^1 + 2^2 + \ldots + 2^n = (1 - 2^{n+1}) / (1 - 2) = 2^{n+1} - 1$$

does not yield a real number value $L$ as $n$ approaches $\infty$.

# Step-by-Step Solutions to Problems in this Chapter, "Series"

• **PROBLEM 1132**

Find the numerical value of the following:

a) $\sum_{j=1}^{7} (2j+1)$ b) $\sum_{j=1}^{21} (3j-2)$ .

Solution: If $A(r)$ is some mathematical expression and $n$ is a positive integer, then the symbol $\sum_{r=0}^{n} A(r)$ means "Successively replace the letter $r$ in the expression $A(r)$ with the numbers $0,1,2,\ldots,n$ and add up the terms. The symbol $\Sigma$ is the Greek letter sigma and is a shorthand way to denote "the sum". It avoids having to write the sum $A(0) + A(1) + A(2) + \ldots + A(n)$.

a) For a) successively replace $j$ by $1,\ldots,7$ and add up the terms.

$$\begin{aligned}\sum_{j=1}^{7} (2j+1) &= \bigl(2(1)+1\bigr) + \bigl(2(2)+1\bigr) + \bigl(2(3)+1\bigr) + \bigl(2(4)+1\bigr) + \bigl(2(5)+1\bigr)\\ &\quad + \bigl(2(6)+1\bigr) + \bigl(2(7)+1\bigr)\\ &= (2+1) + (4+1) + (6+1) + (8+1) + (10+1) + (12+1) + (14+1)\\ &= 3 + 5 + 7 + 9 + 11 + 13 + 15\\ &= 63 .\end{aligned}$$

b) For b) successively replace $j$ by $1,2,3,\ldots,21$ and add up the terms.

$$\begin{aligned}\sum_{j=1}^{21} (3j-2) &= \bigl(3(1)-2\bigr) + \bigl(3(2)-2\bigr) + \bigl(3(3)-2\bigr) + \bigl(3(4)-2\bigr) + \bigl(3(5)-2\bigr)\\ &\quad + \bigl(3(6)-2\bigr) + \bigl(3(7)-2\bigr) + \bigl(3(8)-2\bigr) + \bigl(3(9)-2\bigr)\\ &\quad + \bigl(3(10)-2\bigr) + \bigl(3(11)-2\bigr) + \bigl(3(12)-2\bigr) + \bigl(3(13)-2\bigr)\\ &\quad + \bigl(3(14)-2\bigr) + \bigl(3(15)-2\bigr) + \bigl(3(16)-2\bigr) + \bigl(3(17)-2\bigr)\\ &\quad + \bigl(3(18)-2\bigr) + \bigl(3(19)-2\bigr) + \bigl(3(20)-2\bigr) + \bigl(3(21)-2\bigr)\\ &= (3-2) + (6-2) + (9-2) + (12-2) + (15-2) + (18-2) + (21-2)\\ &\quad + (24-2) + (27-2) + (30-2) + (33-2) + (36-2)\\ &\quad + (39-2) + (42-2) + (45-2) + (48-2) + (51-2)\\ &\quad + (54-2) + (57-2) + (60-2) + (63-2)\\ &= 1 + 4 + 7 + 10 + 13 + 16 + 19 + 22 + 25 + 28 + 31 + 34\\ &\quad + 37 + 40 + 43 + 46 + 49 + 52 + 55 + 58 + 61\\ &= 651 .\end{aligned}$$

• **PROBLEM 1133**

Determine the general term of the sequence:

$$\frac{1}{2}, \frac{1}{12}, \frac{1}{30}, \frac{1}{56}, \frac{1}{90}, \ldots$$

**Solution:** To make use of the ratio test, we find the nth term of the given series, and the (n+1)th term. If we let the first term, $1 = u_1$, then $\frac{2!}{2^2} = u_2$, $\frac{3!}{3^3} = u_3$, etc., up to $u_n + u_{n+1}$. We examine the terms of the series to find the law of formation, from which we conclude:

$$u_n = \frac{n!}{n^n} \quad \text{and,} \quad u_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}.$$

Forming the ratio $\frac{u_{n+1}}{u_n}$ we obtain:

$$\frac{(n+1)!}{(n+1)^{n+1}} \times \frac{n^n}{n!}$$

$$\frac{(n+1)(n!)}{(n+1)^n(n+1)} \times \frac{n^n}{n!} = \frac{n^n}{(n+1)^n}.$$

Now, we find $\lim_{n\to\infty}\left|\frac{n^n}{(n+1)^n}\right|$. This can be rewritten as:

$$\lim_{n\to\infty} \frac{n^n}{\left[n\left(1+\frac{1}{n}\right)\right]^n} = \lim_{n\to\infty} \frac{n^n}{n^n \cdot \left(1+\frac{1}{n}\right)^n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n}.$$

We now use the definition:

$$e = \lim_{x\to 0}(1 + x)^{\frac{1}{x}}.$$

If we let $x = \frac{1}{n}$ in this definition, we have:

$$\lim_{\frac{1}{n}\to 0}\left(1 + \frac{1}{n}\right)^{\frac{1}{\frac{1}{n}}} = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n,$$ which is what we have above. Therefore, $\lim_{n\to\infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n} = \frac{1}{e}$. Since $e \simeq 2.7$, $\frac{1}{e} \simeq \frac{1}{2.7}$ which is less than 1.

Hence, by the ratio test, the given series in convergent.

• **PROBLEM 1138**

Test the series:

$$1 - \frac{3^2}{2^2} + \frac{3^4}{2^2 \cdot 4^2} - \frac{3^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$$

by means of the ratio test. If this test fails, use another test.

**Solution:** To make use of the ratio test, we find the nth term of the given series, and the (n+1)th term. If we let the first term, $1 = u_1$, then $\frac{3^2}{2^2} = u_2$, $\frac{3^4}{2^2 \cdot 4^2} = u_3$, etc. up to $u_n \pm u_{n+1}$. We examine the terms of the series to find the law of formation, from which we conclude:

$$u_n = \frac{3^{2n-2}}{2^2 \cdot 4^2 \cdot \ldots (2n-2)^2},$$

and

$$u_{n+1} = \frac{3^{2(n+1)-2}}{2^2 \cdot 4^2 \cdot \ldots (2n-2)^2 [2(n+1)-2]^2}$$

$$= \frac{3^{2n}}{2^2 \cdot 4^2 \cdot \ldots (2n-2)^2 (2n)^2}.$$

Forming the ratio $\frac{u_{n+1}}{u_n}$, we obtain:

$$\frac{3^{2n}}{2^2 \cdot 4^2 \cdot \ldots (2n-2)^2 (2n)^2} \times \frac{2^2 \cdot 4^2 \cdot \ldots (2n-2)^2}{3^{2n-2}}$$

$$= \frac{3^{2n}}{(2n)^2 \times 3^{2n-2}} = \frac{3^{2n-(2n-2)}}{(2n)^2} = \frac{3^2}{4n^2}.$$

Now, we find:

$$\lim_{n\to\infty} \left| \frac{3^2}{4n^2} \right| = 0.$$

Since $\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = 0$ and $0 < 1$, the given series converges.

# Step-by-Step Solutions to Problems in this Chapter, "Decimal/Fractional Conversions Scientific Notation"

**• PROBLEM 1139**

Use scientific notation to express each number.
(a) 4,375 (b) 186,000 (c) 0.00012 (d) 4,005

Solution: A number expressed in scientific notation is written as a product of a number between 1 and 10 and a power of 10. The number between 1 and 10 is obtained by moving the decimal point of the number (actual or implied) the required number of digits. The power of 10, for a number greater than 1, is positive and is one less than the number of digits before the decimal point in the original number. The power of 10, for a number less than 1, is negative and is one more than the number of zeros immediately following the decimal point in the original number. Hence,

(a) $4,375 = 4.375 \times 10^3$ (b) $186,000 = 1.86 \times 10^5$

(c) $0.00012 = 1.2 \times 10^{-4}$ (d) $4,005 = 4.005 \times 10^3$

**• PROBLEM 1140**

Express $\frac{6,400,000}{400}$ in scientific notation.

Solution: In order to solve this problem, we express the numerator and denominator as the product of a number between 1 and 10 and a power of 10. This is known as scientific notation. Thus

$$6,400,000 = 6.4 \times 1,000,000 = 6.4 \times 10^6$$
$$400 = 4 \times 100 = 4 \times 10^2$$

Thus,

$$\frac{6,400,000}{400} = \frac{6.4 \times 10^6}{4.0 \times 10^2}$$

Since $\frac{ab}{cd} = \frac{a}{c} \cdot \frac{b}{c}$ : $= \frac{6.4}{4.0} \times \frac{10^6}{10^2}$

Since $\frac{a^x}{a^y} = a^{x-y}$ : $= 1.6 \times 10^4$

**• PROBLEM 1141**

Write $\frac{2}{7}$ as a repeating decimal.

Solution: To write a fraction as a repeating decimal divide the numerator by the denominator, until a pattern of repeated digits appears.

$$2 \div 7 = .285714285714...$$