Chapter 3
Spectrum Representation

Inverse Euler's Formula

- Solve for a cosine (or sine)

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

$$e^{-j\omega t} = e^{j(-\omega t)} = \cos(-\omega t) + j\sin(-\omega t)$$

$$= \cos(\omega t) - j\sin(\omega t)$$

$$e^{j\omega t} + e^{-j\omega t} = 2\cos(\omega t)$$

$$\cos(\omega t) = \tfrac{1}{2}(e^{j\omega t} + e^{-j\omega t})$$

$$\sin(\omega t) = \tfrac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$$

SPECTRUM Interpretation

- Cosine = sum of 2 complex exponentials:

$$A\cos(7t) = \frac{A}{2}e^{j7t} + \frac{A}{2}e^{-j7t}$$

  - One has a positive frequency.
  - The other has a negative frequency.
  - Each amplitude is one-half the original one.
- Sine = sum of 2 complex exponentials:

$$A\sin(7t) = \frac{A}{2j}e^{j7t} - \frac{A}{2j}e^{-j7t}$$

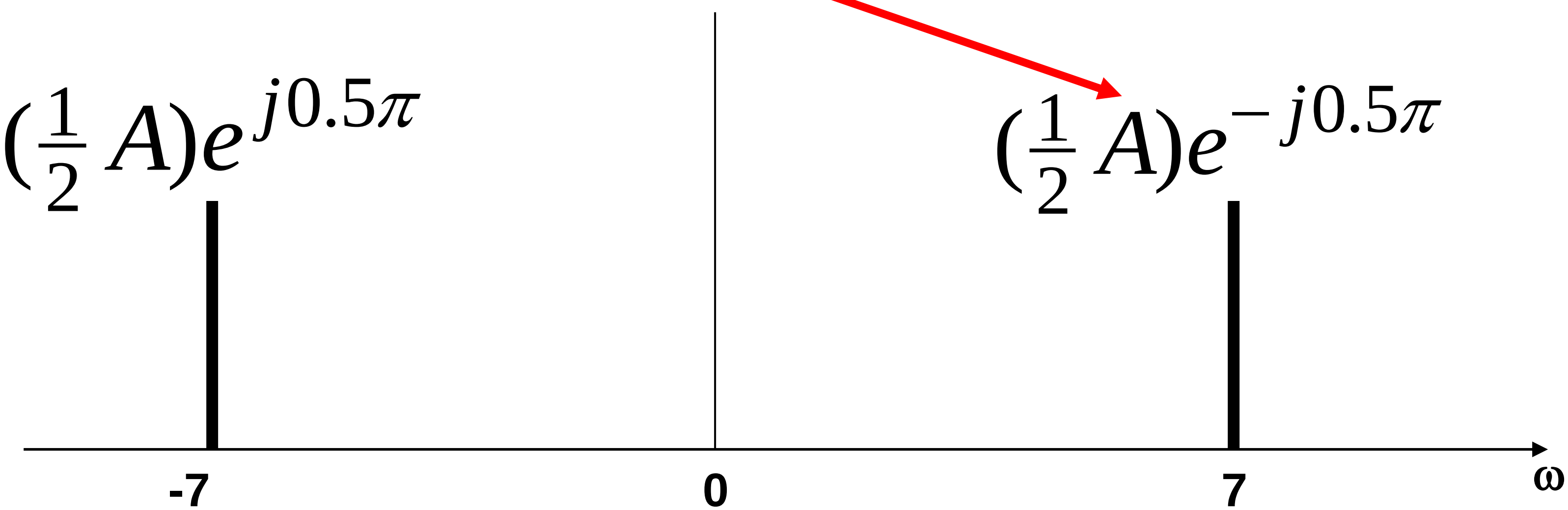
$$= \frac{1}{2}Ae^{-j0.5\pi}e^{j7t} + \frac{1}{2}Ae^{j0.5\pi}e^{-j7t}$$

  - The positive frequency term has a phase of -0.5π.
  - The negative frequency term has a phase of +0.5π.

GRAPHICAL SPECTRUM

- Spectrum
  - Compact representation of the frequency content of a signal
  - Frequency-domain representation
- Example of a sine

$$A\sin(7t) = \frac{1}{2}Ae^{-j0.5\pi}e^{j7t} + \frac{1}{2}Ae^{j0.5\pi}e^{-j7t}$$

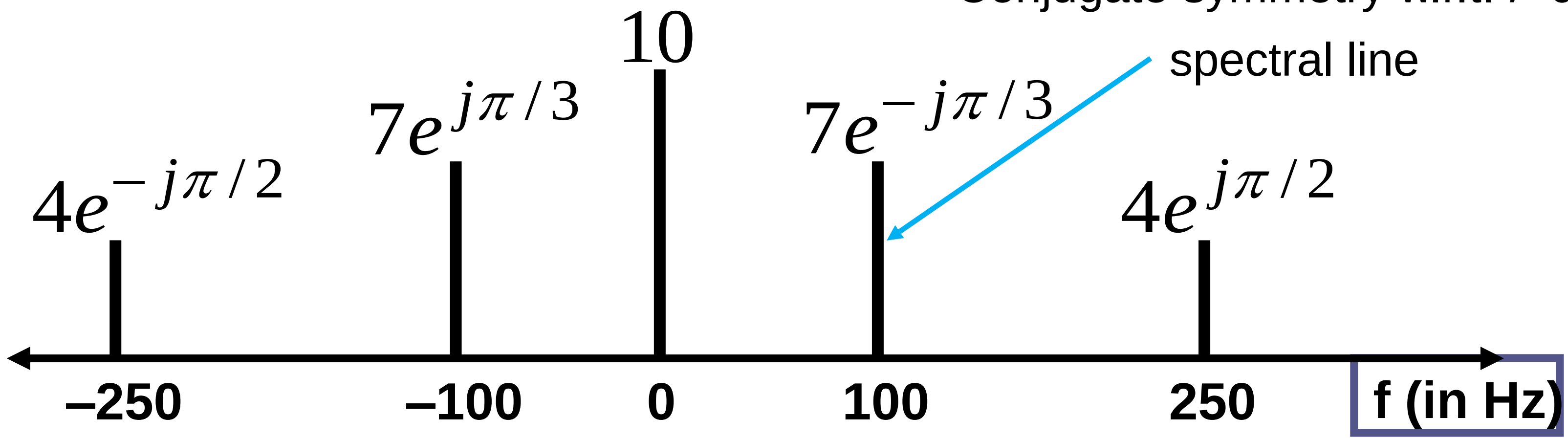


AMPLITUDE, PHASE, & FREQUENCY are shown.

Converting SPECTRUM to SINUSOID (1)

- Add the spectrum components:

Conjugate symmetry w.r.t. f=0



- Gather (A,ω,ϕ) information to get a formula for $x(t)$.

Frequency (Hz)	Amplitude	Phase
-250	4	-π/2
-100	7	π/3
0	10	0
100	7	-π/3
250	4	π/2

conjugate phase

Converting SPECTRUM to SINUSOID (2)

Frequency (Hz)	Amplitude	Phase
-250	4	-π/2
-100	7	π/3
0	10	0
100	7	-π/3
250	4	π/2

- **DC** is another name for the zero-freq. component.
- **DC** component always has ϕ=0 or π for real $x(t)$.

- Add spectral components.

$$x(t) = 10 +$$

$$7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t}$$

$$4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

Converting SPECTRUM to SINUSOID (3)

$$x(t) = 10 +$$
$$7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t}$$
$$4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

- Use Euler's Formula to get REAL sinusoids:

$$A\cos(\omega t + \varphi) = \tfrac{1}{2}Ae^{j\varphi}e^{j\omega t} + \tfrac{1}{2}Ae^{-j\varphi}e^{-j\omega t}$$

$$\boxed{\begin{aligned} x(t) = 10 + 14\cos(2\pi(100)t - \pi/3) \\ + 8\cos(2\pi(250)t + \pi/2) \end{aligned}}$$

- So, we get the general form:

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$

Summary: GENERAL FORM

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$

$$x(t) = X_0 + \sum_{k=1}^{N} \Re e\left\{X_k e^{j2\pi f_k t}\right\}$$

$$X_k = A_k e^{j\varphi_k}$$

$$\text{Frequency} = f_k$$

$$\Re e\{z\} = \frac{1}{2} z + \frac{1}{2} z^*$$

$$x(t) = X_0 + \sum_{k=1}^{N} \left\{\frac{1}{2} X_k e^{j2\pi f_k t} + \frac{1}{2} X_k^* e^{-j2\pi f_k t}\right\}$$

Period of Complex Exponential

$$x(t) = e^{j\omega t}$$

$$x(t+T) = x(t)\,?$$

The period is *T*.

$$\cancel{e^{j\omega(t+T)}} = \cancel{e^{j\omega t}}$$

$$\boxed{e^{j2\pi k} = 1}$$

$$\Rightarrow e^{j\omega T} = 1 \quad \Rightarrow \omega T = 2\pi k$$

$$\omega = \frac{2\pi k}{T} = \left(\frac{2\pi}{T}\right)k = \omega_0 k$$

k = integer

Harmonic Signal Spectrum

The periodic signal can only have : $f_k = k f_0$.

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$\boxed{X_k = A_k e^{j\varphi_k}}$$

$$f_0 = \frac{1}{T}$$

$$x(t) = X_0 + \sum_{k=1}^{N} \left\{ \tfrac{1}{2} X_k e^{j2\pi k f_0 t} + \tfrac{1}{2} X_k^* e^{-j2\pi k f_0 t} \right\}$$

FUNDAMENTAL FREQUENCY

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$f_k = k f_0 \qquad (\omega_0 = 2\pi f_0) \qquad f_0 = \frac{1}{T_0}$$

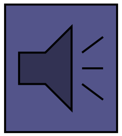
f_0 = fundamental frequency

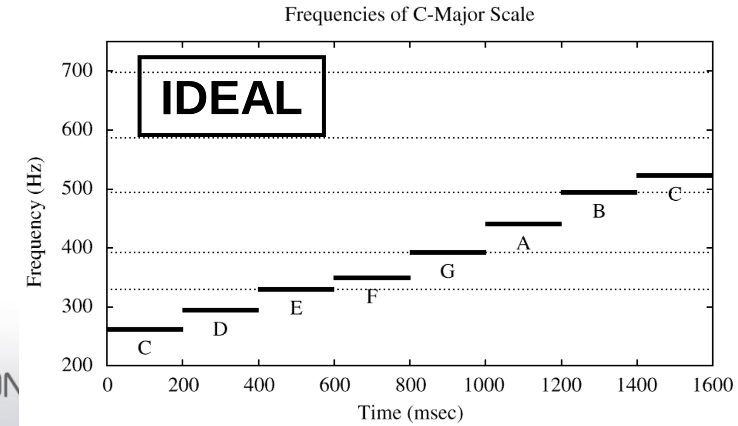
T_0 = fundamental period

TIME-VARYING FREQUENCY DIAGRAMS

- Music

- C-major SCALE: stepped frequencies
  - Frequency is constant for each note.



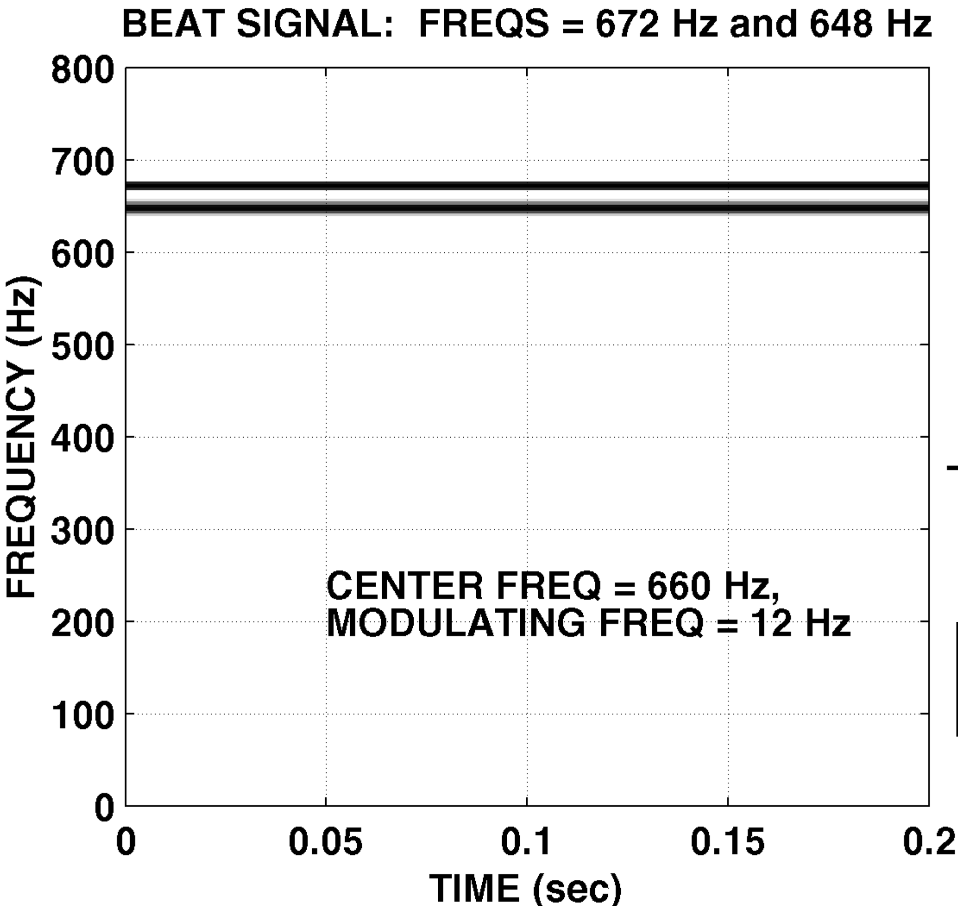


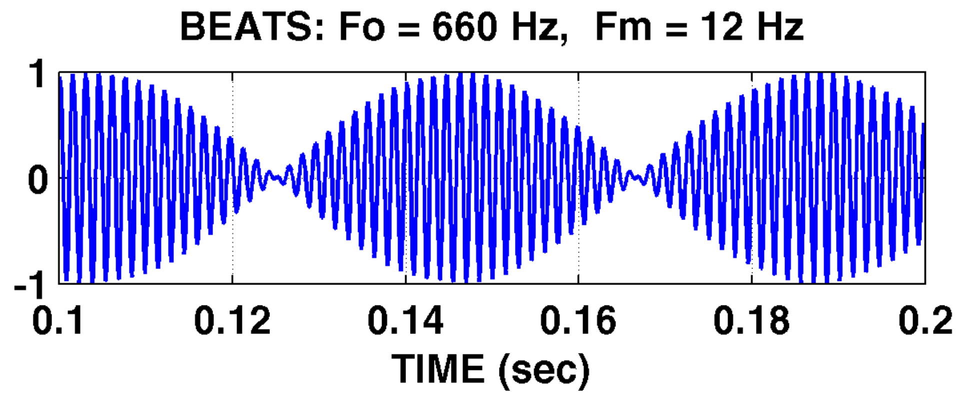
FREQUENCY ANALYSIS

- Spectrogram (a time-frequency spectrum)
  - Signals can be modelled as a sum of sinusoids whose frequencies, amplitudes, and phases vary with time.
- An **ANALYSIS** program
  - Takes $x(t)$ as input.
  - Breaks $x(t)$ into SHORT TIME SEGMENTS.
  - Produces spectrum values X_k.
    - Uses the FFT (Fast Fourier Transform).
- To generate a SPECTROGRAM
  - A MATLAB function is **`specgram.m`**.

SPECTROGRAM EXAMPLE

- Two **Constant** Frequencies: Beats





$$\boxed{\cos(2\pi(660)t)\sin(2\pi(12)t)}$$

AM Radio Signal

- Same as BEAT Notes

$$\boxed{\cos(2\pi(660)t)\sin(2\pi(12)t)}$$

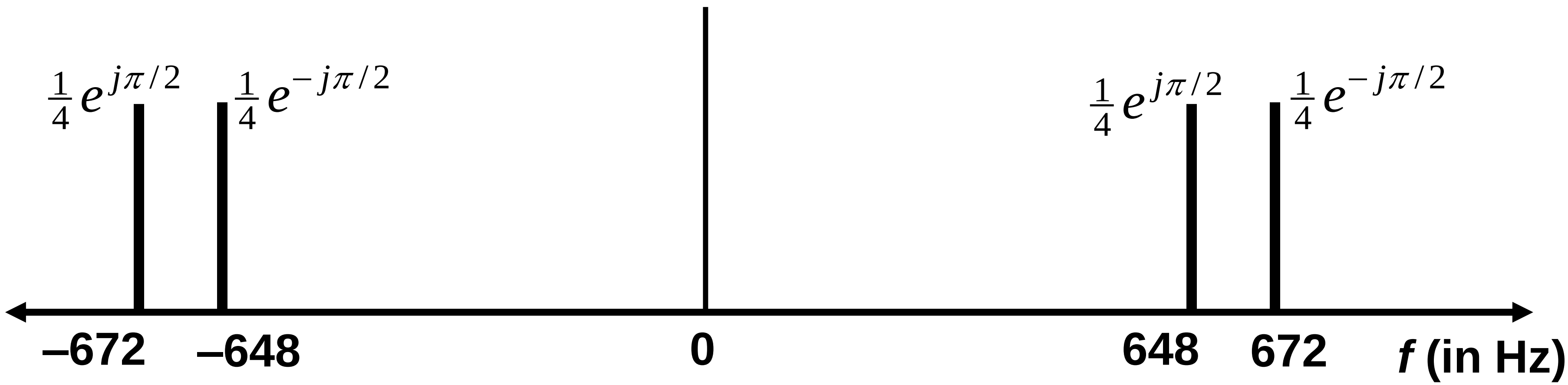
$$\frac{1}{2}\left(e^{j2\pi(660)t}+e^{-j2\pi(660)t}\right)\frac{1}{2j}\left(e^{j2\pi(12)t}-e^{-j2\pi(12)t}\right)$$

$$\frac{1}{4j}\left(e^{j2\pi(672)t}-e^{-j2\pi(672)t}-e^{j2\pi(648)t}+e^{-j2\pi(648)t}\right)$$

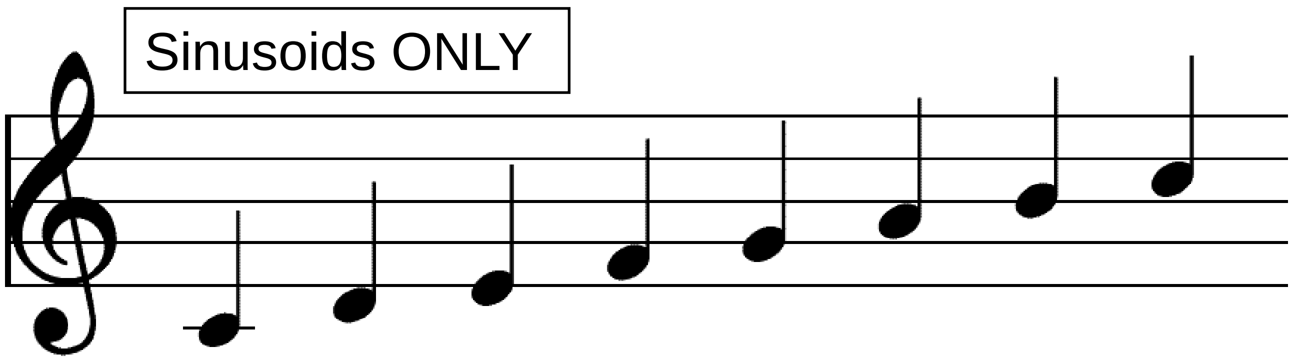
$$\boxed{\tfrac{1}{2}\cos(2\pi(672)t-\tfrac{\pi}{2})+\tfrac{1}{2}\cos(2\pi(648)t+\tfrac{\pi}{2})}$$

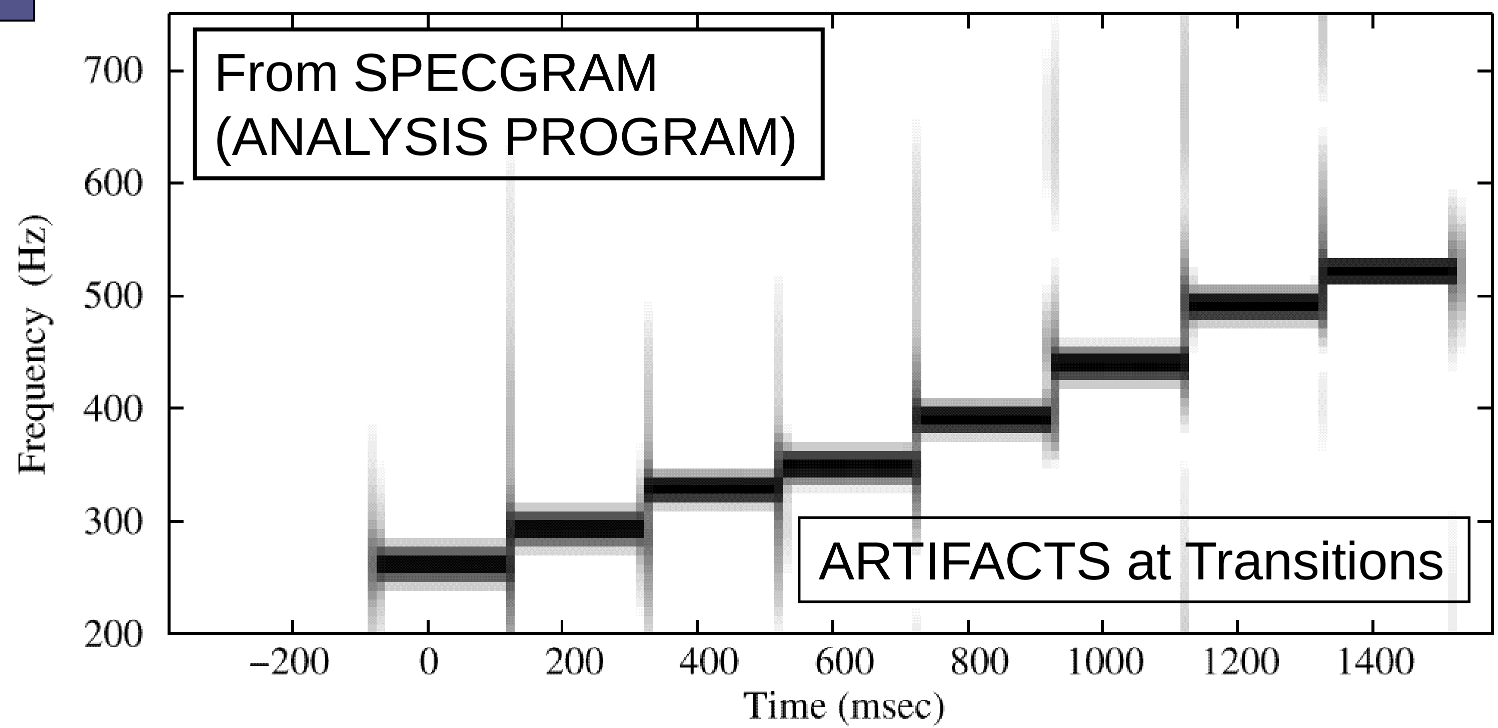
SPECTRUM of an AM SIGNAL (Beat)

- 4 complex exponentials in an AM signal:



SPECTROGRAM of C-Scale





Time-Varying Frequency

- The frequency can change with time.
  - Continuously, not stepped
- **FREQUENCY MODULATION (FM)**

Angle function

$$\boxed{x(t) = \cos(2\pi f_c t + v(t))}$$

  - Frequency variation produced by the time-varying angle function.

Linear Frequency Modulation (LFM)

- CHIRP SIGNALS
  - Linear Frequency Modulation (LFM)
  - Quadratic phase

QUADRATIC

$$x(t) = A\cos(\underbrace{\alpha t^2}_{\text{QUADRATIC}} + 2\pi f_0 t + \varphi)$$

- The frequency will change LINEARLY with time.
  - Example of Frequency Modulation (FM)

INSTANTANEOUS FREQUENCY

- Derivative of the angle function

$$\boxed{x(t) = A\cos(\psi(t)) \quad \Rightarrow \quad \omega_i(t) = \tfrac{d}{dt}\psi(t)}$$

- For Sinusoid:

$$x(t) = A\cos(2\pi f_0 t + \varphi)$$

$$\psi(t) = 2\pi f_0 t + \varphi \quad \Rightarrow \quad \omega_i(t) = \tfrac{d}{dt}\psi(t) = 2\pi f_0$$

- Chirp signals

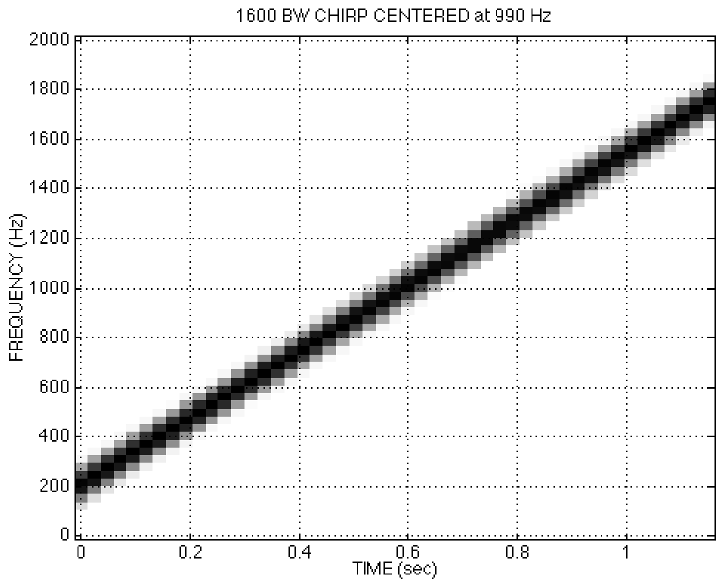
$$x(t) = A\cos(\alpha t^2 + \beta t + \varphi)$$

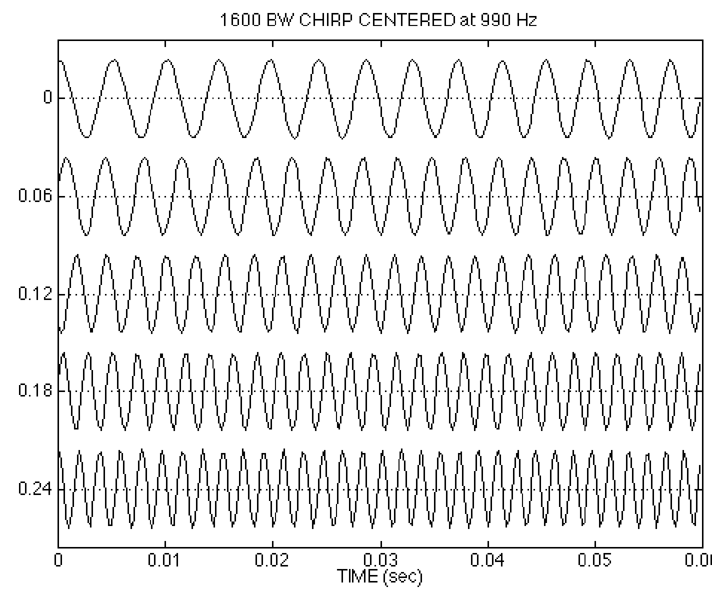
$$\Rightarrow \psi(t) = \alpha t^2 + \beta t + \varphi$$

$$\Rightarrow \omega_i(t) = \tfrac{d}{dt}\psi(t)$$

$$= 2\alpha t + \beta$$

CHIRP SPECTROGRAM & WAVEFORM





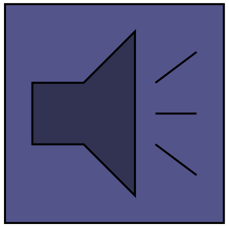
OTHER FM SIGNALS

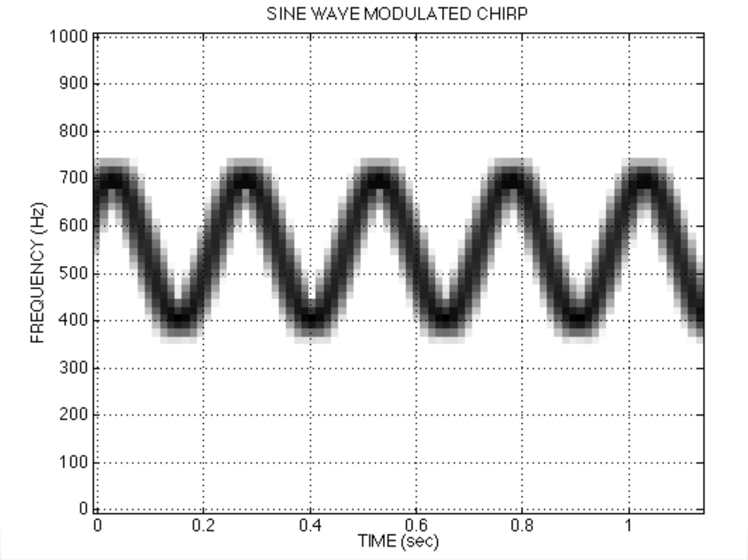
- $\psi(t)$ can be anything:

$$x(t) = A\cos(\alpha\cos(\beta t) + \varphi)$$

$$\Rightarrow \omega_i(t) = \tfrac{d}{dt}\psi(t) = -\alpha\beta\sin(\beta t)$$

- $\psi(t)$ could be speech or music:
  - FM radio broadcast
- Sine-wave FM signal





Fourier Series

- Fourier series
  - Any periodic signal can be synthesized with a sum of harmonically related sinusoids.
  - The sum may need an infinite number of terms.

- Jean Baptiste Joseph Fourier
  - Thesis (memoir) in 1807 (Napoleonic era)
    - On the Propagation of Heat in Solid Bodies

Joseph Fourier

lived from 1768 to 1830

Fourier studied the mathematical theory of heat conduction. He established the partial differential equation governing heat diffusion and solved it by using infinite series of trigonometric functions.

Find out more at:
http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Fourier.html

Fourier Series Synthesis

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}}$$

Fourier synthesis equation

$$\boxed{a_k = \tfrac{1}{2} X_k = \tfrac{1}{2} A_k e^{j\varphi_k}}$$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$X_k = A_k e^{j\varphi_k}$$

COMPLEX AMPLITUDE

STRATEGY: $x(t) \rightarrow a_k$

- **ANALYSIS**
  - Gets a representation from the signal.
  - Works for **PERIODIC** Signals.
- Fourier Series
  - The coefficients can be obtained by an INTEGRAL over one period.

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 k t}dt$$

Fourier analysis equation

INTEGRAL Property of Complex Exponential

- INTEGRAL over ONE PERIOD

$$\omega_0 = \frac{2\pi}{T_0}$$

$$\int_0^{T_0} e^{-j(2\pi/T_0)mt}dt = \frac{T_0}{-j2\pi m} e^{-j(2\pi/T_0)mt}\Big|_0^{T_0}$$

$m \neq 0$

$$= \frac{T_0}{-j2\pi m}(e^{-j2\pi m} - 1)$$

$$\int_0^{T_0} e^{-j(2\pi/T_0)mt}dt = 0$$

- PRODUCT of exp(+*j*) and exp(-*j*)

$$\boxed{\frac{1}{T_0}\int_0^{T_0} e^{j(2\pi/T_0)\ell t} e^{-j(2\pi/T_0)kt}dt = \begin{cases} 0 & k \neq \ell \\ 1 & k = \ell \end{cases}}$$

Orthogonality property

Isolate One FS Coefficient.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt}$$

$$\frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)\ell t}dt = \frac{1}{T_0}\int_0^{T_0}\left(\sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt}\right)e^{-j(2\pi/T_0)\ell t}dt$$

$$\frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)\ell t}dt = \sum_{k=-\infty}^{\infty} a_k\left(\frac{1}{T_0}\int_0^{T_0} e^{j(2\pi/T_0)kt}e^{-j(2\pi/T_0)\ell t}dt\right) = a_\ell$$

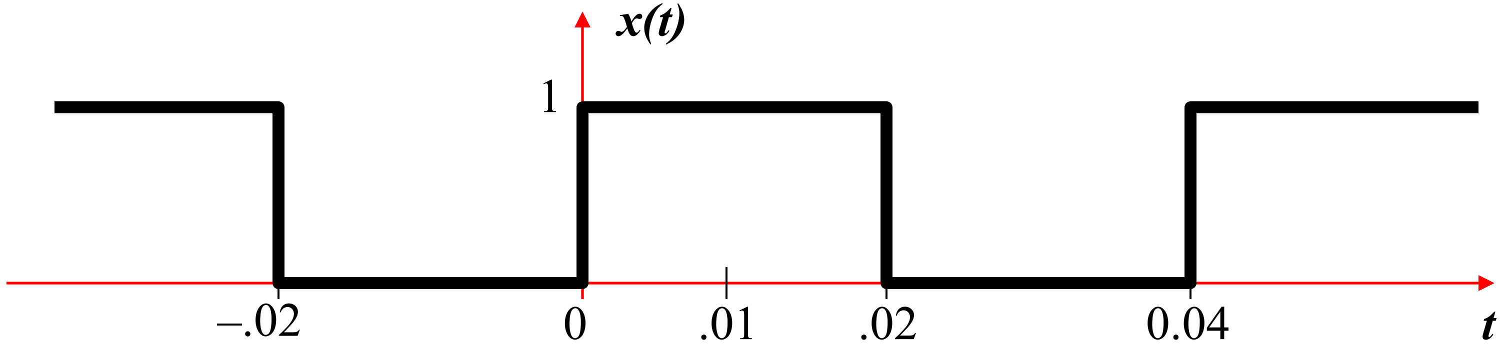
The integral is zero except for $k = \ell$.

$$\Rightarrow a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt$$

SQUARE WAVE EXAMPLE

$$x(t) = \begin{cases} 1 & 0 \le t < \frac{1}{2}T_0 \\ 0 & \frac{1}{2}T_0 \le t < T_0 \end{cases}$$

$$\text{for } T_0 = 0.04\,\text{sec.}$$



FS for a SQUARE WAVE $\{a_k\}$

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt \qquad (k \neq 0)$$

$$a_k = \frac{1}{.04}\int_0^{.02} 1e^{-j(2\pi/.04)kt}dt = \frac{1}{.04(-j2\pi k/.04)}e^{-j(2\pi/.04)kt}\Big|_0^{.02}$$

$$= \frac{1}{(-j2\pi k)}(e^{-j(\pi)k} - 1) = \frac{1-(-1)^k}{j2\pi k}$$

DC Coefficient: a_0

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt \qquad (k=0)$$

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt = \frac{1}{T_0}(\text{Area})$$

$$= \frac{1}{.04}\int_0^{.02} 1dt = \frac{1}{.04}(.02-0) = \frac{1}{2}$$

Fourier Coefficients $\{a_k\}$

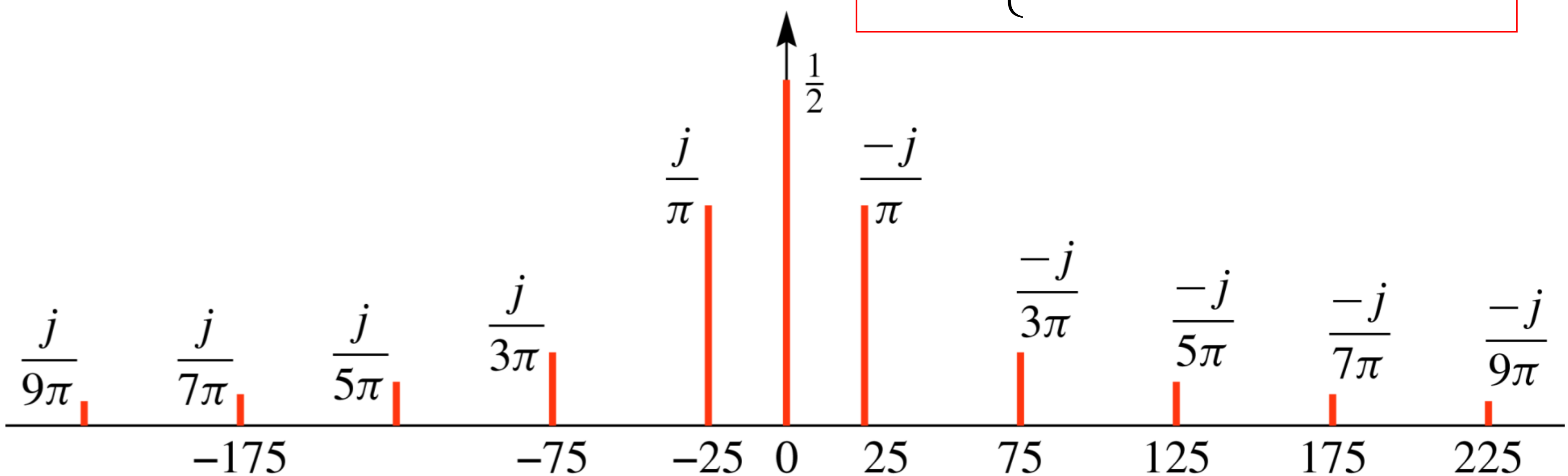
- a_k is a function of k.
  - Complex Amplitude for k-th Harmonic
  - This one doesn't depend on the period, T_0.

$$\boxed{a_k = \frac{1-(-1)^k}{j2\pi k} = \begin{cases} \dfrac{1}{j\pi k} & k = \pm 1, \pm 3, \ldots \\ 0 & k = \pm 2, \pm 4, \ldots \\ \frac{1}{2} & k = 0 \end{cases}}$$

Spectrum from Fourier Series

$$\omega_0 = 2\pi/(0.04) = 2\pi(25)$$

$$a_k = \begin{cases} \dfrac{-j}{\pi k} & k = \pm 1, \pm 3, \ldots \\ 0 & k = \pm 2, \pm 4, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$



Fourier Series Integral

- HOW do you determine a_k from $x(t)$?

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt$$

Fundamental Frequency $f_0 = 1/T_0$

$a_{-k} = a_k^*$ when $x(t)$ is real.

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt$$ (DC component)

Fourier Series Synthesis (1)

- HOW do you **APPROXIMATE** *x*(*t*) ?

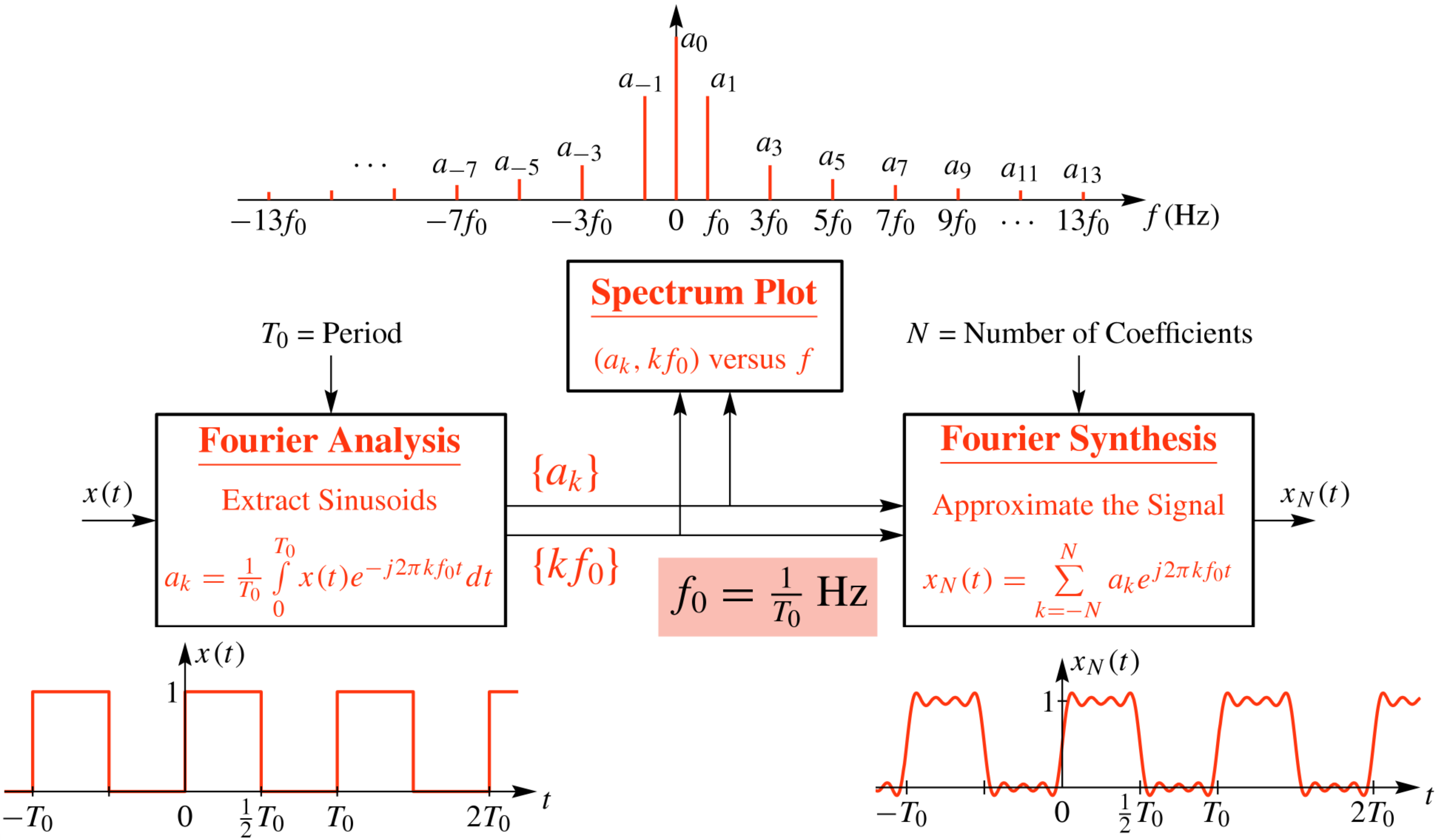
$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt$$

- Use the FINITE number of coefficients.

$$\boxed{x(t) \approx \sum_{k=-N}^{N} a_k e^{j2\pi k f_0 t}}$$

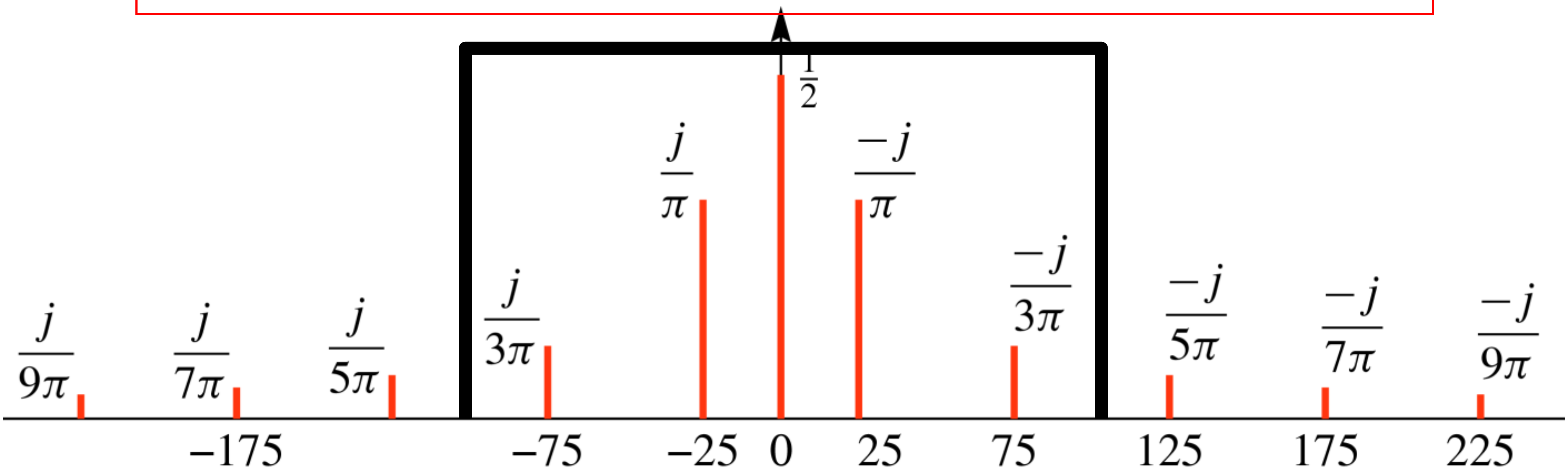
$a_{-k} = a_k^*$ when $x(t)$ is real.

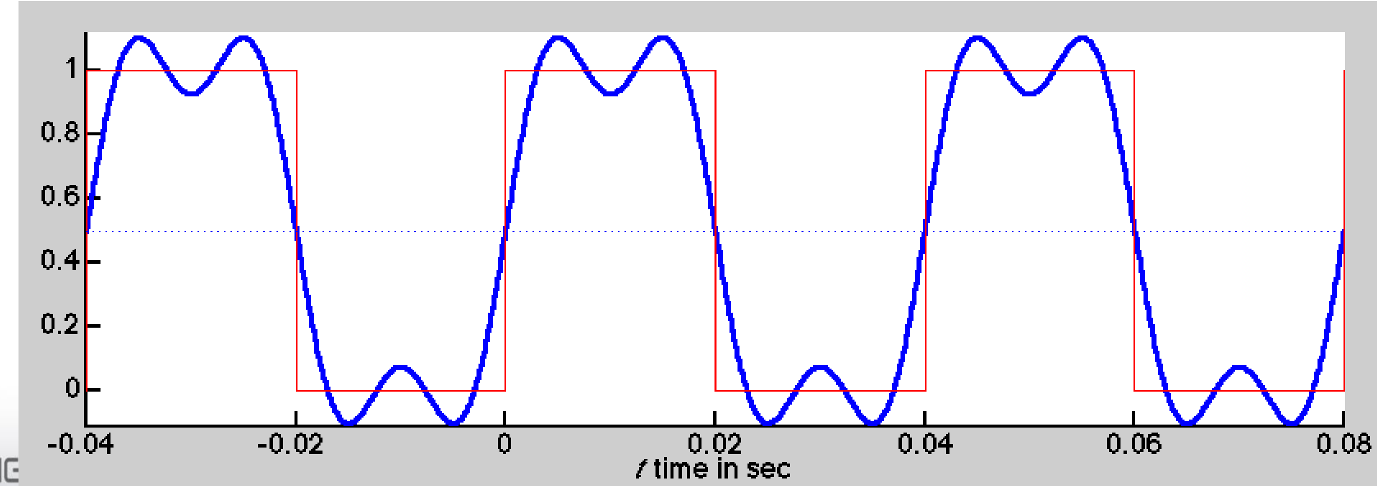
Fourier Series Synthesis (2)



Synthesis: the 1st & 3rd Harmonics

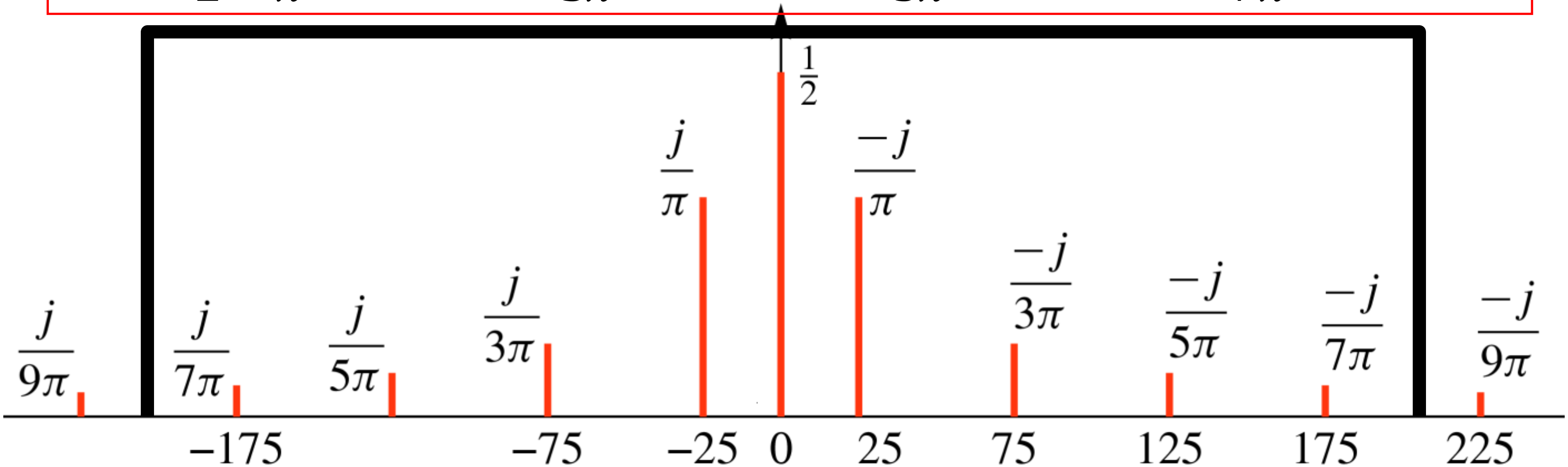
$$y(t) = \frac{1}{2} + \frac{2}{\pi}\cos(2\pi(25)t - \tfrac{\pi}{2}) + \frac{2}{3\pi}\cos(2\pi(75)t - \tfrac{\pi}{2})$$

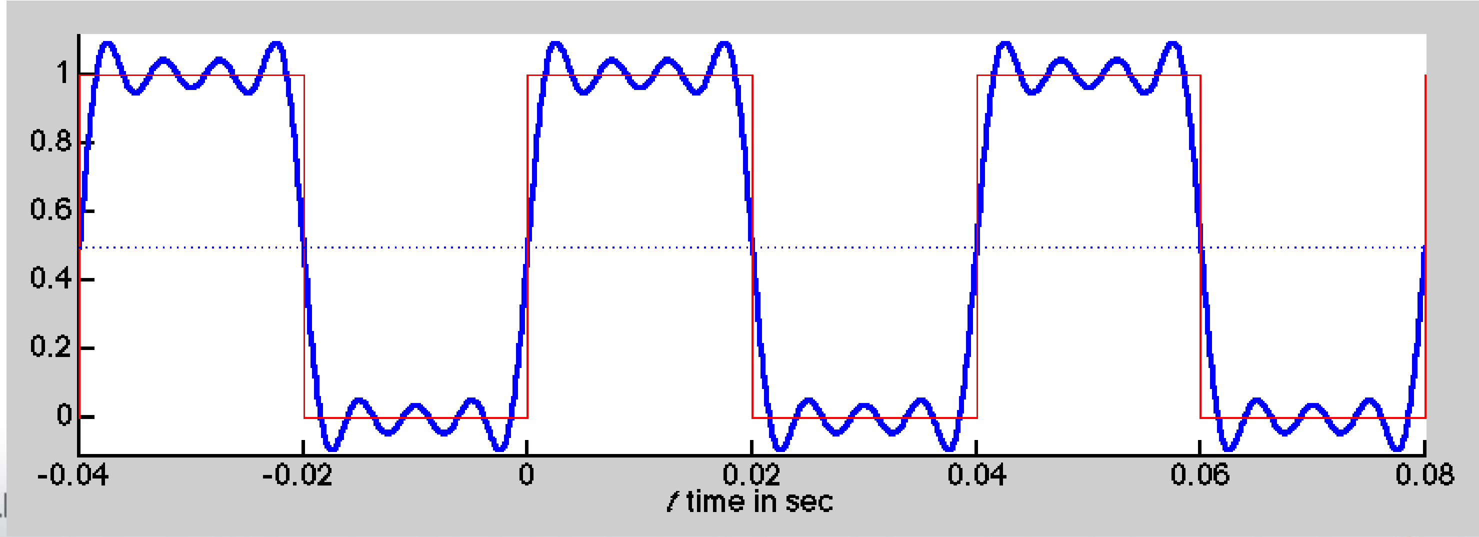




Synthesis: up to the 7th Harmonic

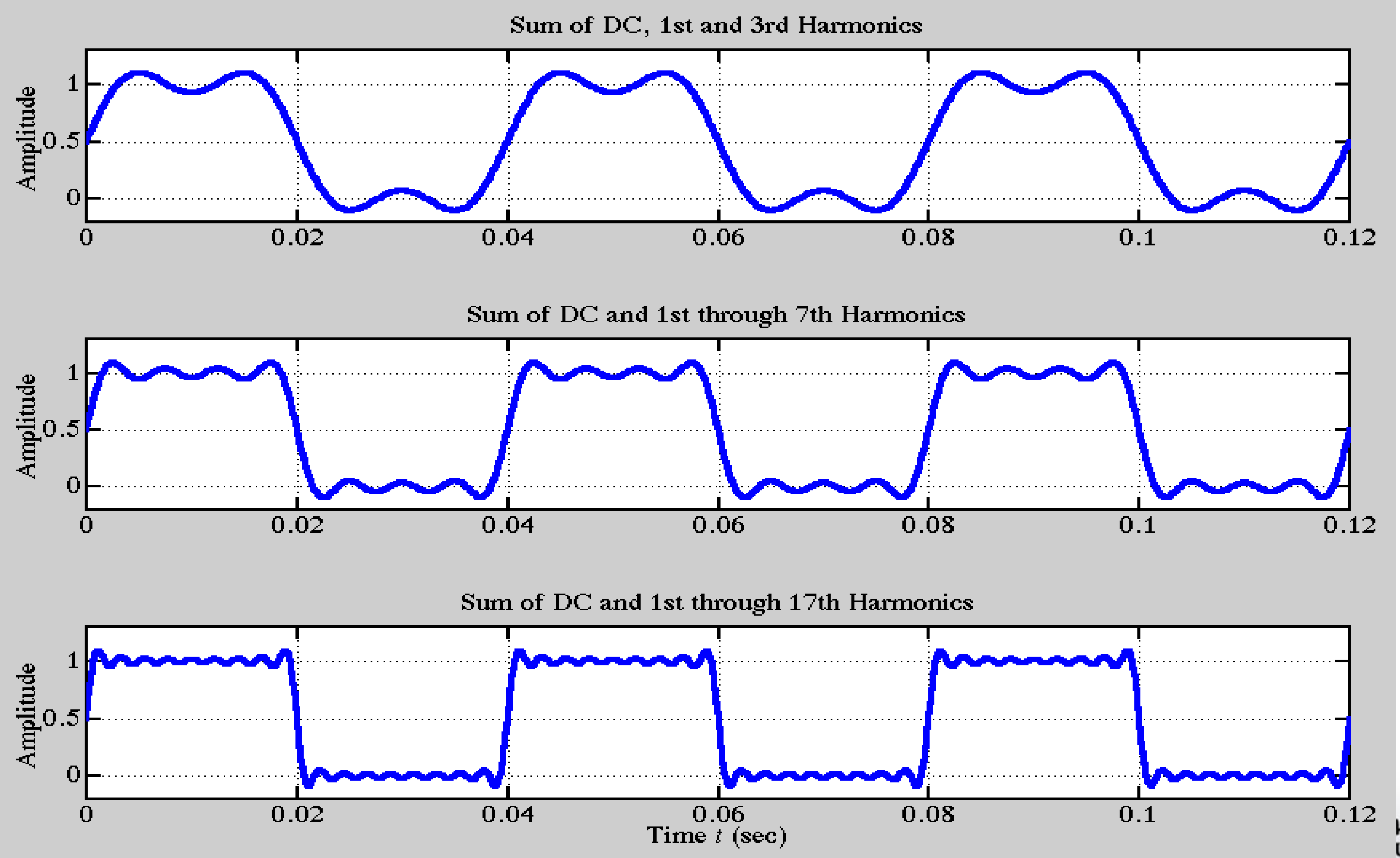
$$y(t)=\frac{1}{2}+\frac{2}{\pi}\sin(50\pi t)+\frac{2}{3\pi}\sin(150\pi t)+\frac{2}{5\pi}\sin(250\pi t)+\frac{2}{7\pi}\sin(350\pi t)$$





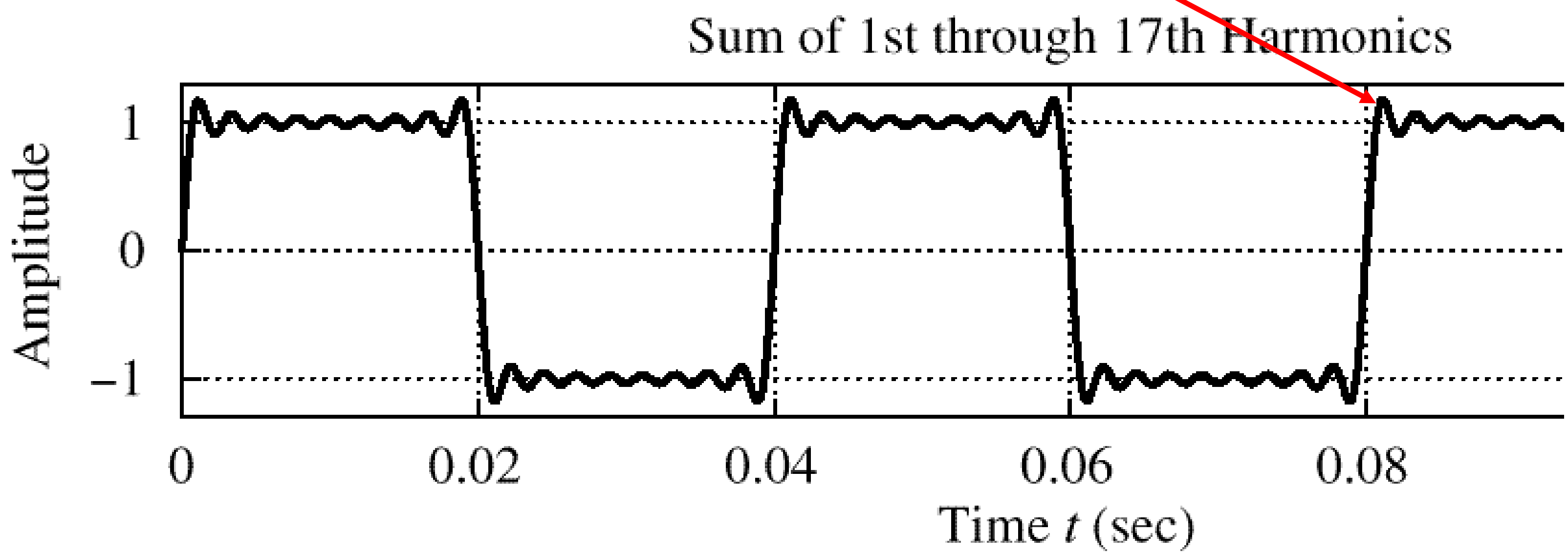
Fourier Synthesis

$$x_N(t) = \frac{1}{2} + \frac{2}{\pi}\sin(\omega_0 t) + \frac{2}{3\pi}\sin(3\omega_0 t) + \ldots$$



Gibbs' Phenomenon

- Convergence at DISCONTINUITY of $x(t)$
  - There is always an overshoot.
  - About 9% for the Square Wave case



Thank you

- Homework
  - P-3.1, 2, 4, 6, 7, 8, 9, 10, 12, 13, 14, 16, 17, 18, & 19
- Reading assignment
  - Section 4-1, 2, & 3