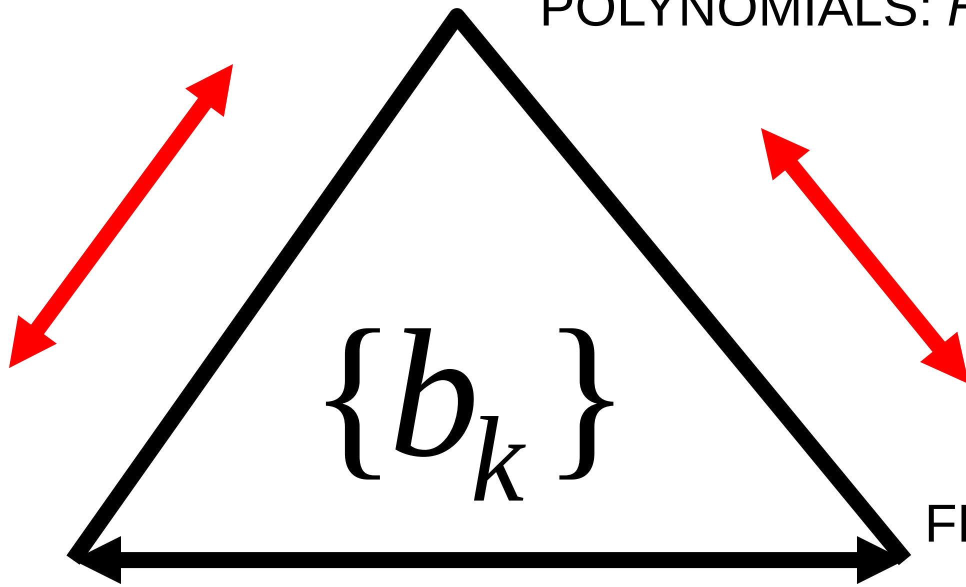
Chapter 7
z-Transforms

Domains



$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$

TRANSFORM CONCEPT

- Move to a new domain where
  - OPERATIONS are EASIER & FAMILIAR.
    - Use POLYNOMIALS.

- TRANSFORM both ways.
  - $x[n] \rightarrow X(z)$ (into the *z*-domain)
  - $X(z) \rightarrow x[n]$ (back to the time domain)

"TRANSFORM" EXAMPLE

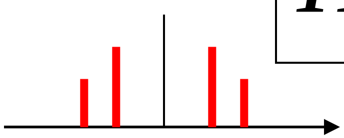
- Equivalent representations

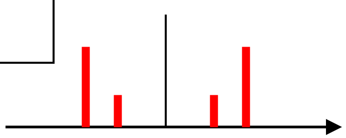
$$x[n] \longrightarrow \boxed{h[n] = \delta[n] - \delta[n-1]} \longrightarrow y[n]$$

$$\updownarrow$$

$$\boxed{H(e^{j\hat{\omega}}) = \sum_{n} h[n] e^{-j\hat{\omega}n}}$$

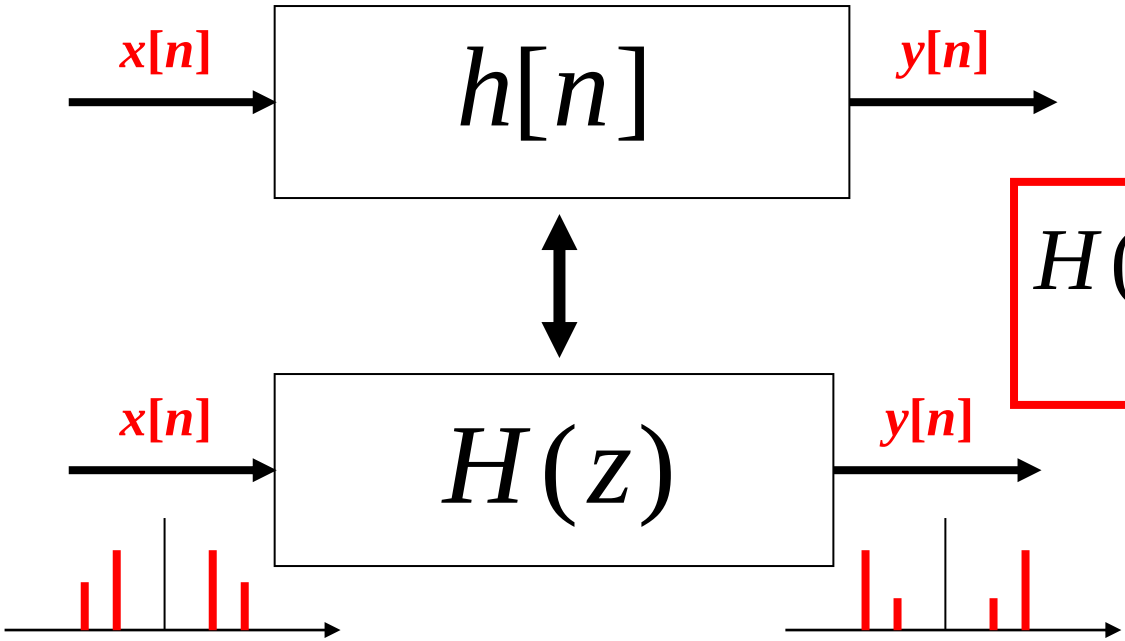
$$x[n] \longrightarrow \boxed{H(e^{j\hat{\omega}}) = 1 - e^{-j\hat{\omega}}} \longrightarrow y[n]$$





z-TRANSFORM: IDEA

- **POLYNOMIAL** REPRESENTATION



$$\boxed{H(z) = \sum_{n} h[n] z^{-n}}$$

z-Transform: DEFINITION

- POLYNOMIAL Representation of an LTI SYSTEM:

$$\boxed{H(z) = \sum_{n} h[n] z^{-n}}$$

APPLIES to Any SIGNAL.

- EXAMPLE:

$$\{h[n]\} = \{2, 0, -3, 0, 2\}$$

$$H(z) = 2z^{-0} + 0z^{-1} - 3z^{-2} + 0z^{-3} + 2z^{-4}$$

$$= 2 - 3z^{-2} + 2z^{-4}$$

$$= 2 - 3(z^{-1})^{2} + 2(z^{-1})^{4}$$

POLYNOMIAL in z^{-1}

z-Transform: EXAMPLE

- ANY SIGNAL has a *z*-Transform:

$$\boxed{X(z) = \sum_{n} x[n] z^{-n}}$$

Example 7.1

n	$n < -1$	-1	0	1	2	3	4	5	$n > 5$
$x[n]$	0	0	2	4	6	4	2	0	0

$$X(z) = ?$$

$$X(z) = 2 + 4z^{-1} + 6z^{-2} + 4z^{-3} + 2z^{-4}$$

Inverse z-Transform: EXAMPLE

Example 7.2

$$X(z) = 1 - 2z^{-1} + 3z^{-3} - z^{-5}$$

EXPONENT GIVES TIME LOCATION.

$$x[n] = \begin{cases} 0 & n < 0 \\ 1 & n = 0 \\ -2 & n = 1 \\ 0 & n = 2 \\ 3 & n = 3 \\ 0 & n = 4 \\ -1 & n = 5 \\ 0 & n > 5 \end{cases}$$

$$x[n] = ?$$

$$x[n] = \delta[n] - 2\delta[n-1] + 3\delta[n-3] - \delta[n-5]$$

z-Transform of an FIR Filter (1)

- CALLED the **SYSTEM FUNCTION**
  - $h[n]$ is the same as $\{b_k\}$.

$$\boxed{\text{SYSTEM FUNCTION}}\ H(z) = \sum_{k=0}^{M} b_k z^{-k} = \sum_{k=0}^{M} h[k] z^{-k}$$

$$y[n] = \underbrace{\sum_{k=0}^{M} b_k x[n-k]}_{\boxed{\text{FIR DIFFERENCE EQUATION}}} = \underbrace{\sum_{k=0}^{M} h[k] x[n-k]}_{\boxed{\text{CONVOLUTION}}}$$

z-Transform of an FIR Filter (1)

- Get $H(z)$ DIRECTLY from the $\{b_k\}$.
- Example 7.3 in the book:

$$y[n] = 6x[n] - 5x[n-1] + x[n-2]$$

$$\{b_k\} = \{6, -5, 1\}$$

$$H(z) = \sum_k b_k z^{-k} = 6 - 5z^{-1} + z^{-2}$$

Ex. DELAY SYSTEM

- UNIT DELAY: find $h[n]$ and $H(z)$.

$$x[n] \longrightarrow \boxed{\delta[n-1]} \longrightarrow y[n] = x[n-1]$$

$$H(z) = \sum \delta[n-1] z^{-n} = z^{-1}$$

$$x[n] \longrightarrow \boxed{z^{-1}} \longrightarrow y[n]$$

DELAY EXAMPLE

- UNIT DELAY: find *y*[*n*] via polynomials.
  - *x*[*n*] = {3,1,4,1,5,9,0,0,0,...}

n	$n<0$	0	1	2	3	4	5	6	$n>6$
$y[n]$	0	0	3	1	4	1	5	9	0

$$Y(z)=0z^{0}+3z^{-1}+z^{-2}+4z^{-3}+z^{-4}+5z^{-5}+9z^{-6}$$

$$Y(z)=z^{-1}(3+z^{-1}+4z^{-2}+z^{-3}+5z^{-4}+9z^{-5})$$

$$Y(z)=z^{-1}X(z)$$

DELAY PROPERTY

A delay of one sample multiplies the z-transform by z^{-1}.

$$x[n-1] \quad \Longleftrightarrow \quad z^{-1}X(z)$$

Time delay of n_0 *samples multiplies the z-transform by* z^{-n_0}

$$x[n-n_0] \quad \Longleftrightarrow \quad z^{-n_0}X(z)$$

GENERAL I/O PROBLEM

- Input is $x[n]$, find $y[n]$. (for an FIR filter, $h[n]$)
- How to combine $X(z)$ and $H(z)$?

Example 7.5

$$x[n] = \delta[n-1] - \delta[n-2] + \delta[n-3] - \delta[n-4]$$

$$\text{and} \quad h[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 4\delta[n-3]$$

$$X(z) = 0 + 1z^{-1} - 1z^{-2} + 1z^{-3} - 1z^{-4}$$

$$\text{and} \quad H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$$

FIR Filter = CONVOLUTION

x[n], X(z)	0	+1	-1	+1	-1			
h[n], H(z)	1	2	3	4				
	0	+1	-1	+1	-1			
		0	+2	-2	+2	-2		
			0	+3	-3	+3	-3	
				0	+4	-4	+4	-4
y[n], Y(z)	0	+1	+1	+2	+2	-3	+1	-4

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] = \sum_{k=0}^{M} h[k]x[n-k]$$

CONVOLUTION

CONVOLUTION PROPERTY

- Proof:

$$y[n] = h[n] * x[n] \quad \leftrightarrow \quad Y(z) = H(z)X(z)$$

$$y[n] = x[n] * h[n] = \sum_{k=0}^{M} h[k]x[n-k]$$

$$Y(z) = \sum_{k=0}^{M} h[k]\left(z^{-k}X(z)\right)$$

$$= \left(\sum_{k=0}^{M} h[k]z^{-k}\right) X(z) = H(z)X(z).$$

MULTIPLY
z-TRANSFORMS.

CONVOLUTION EXAMPLE (1)

- **MULTIPLY** the *z*-TRANSFORMS:

$$\boxed{H(z)X(z)}$$

- Finite-length input $x[n]$
- FIR Filter (L=4)

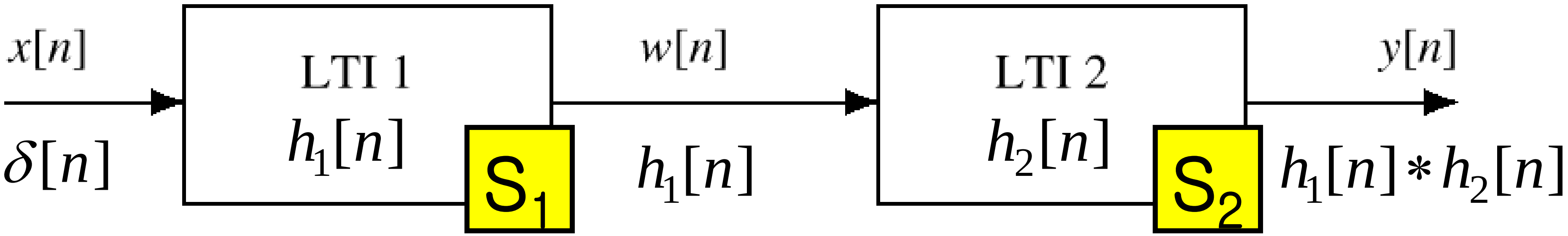
MULTIPLY z-TRANSFORMS.

$$
\begin{aligned}
Y(z) &= H(z)X(z) \\
&= (1 + 2z^{-1} + 3z^{-2} + 4z^{-3})(z^{-1} - z^{-2} + z^{-3} - z^{-4}) \\
&= z^{-1} + (-1 + 2)z^{-2} + (1 - 2 + 3)z^{-3} + (-1 + 2 - 3 + 4)z^{-4} \\
&\quad + (-2 + 3 - 4)z^{-5} + (-3 + 4)z^{-6} + (-4)z^{-7} \\
&= z^{-1} + z^{-2} + 2z^{-3} + 2z^{-4} - 3z^{-5} + z^{-6} - 4z^{-7}
\end{aligned}
$$

$\boxed{y[n] = ?}$

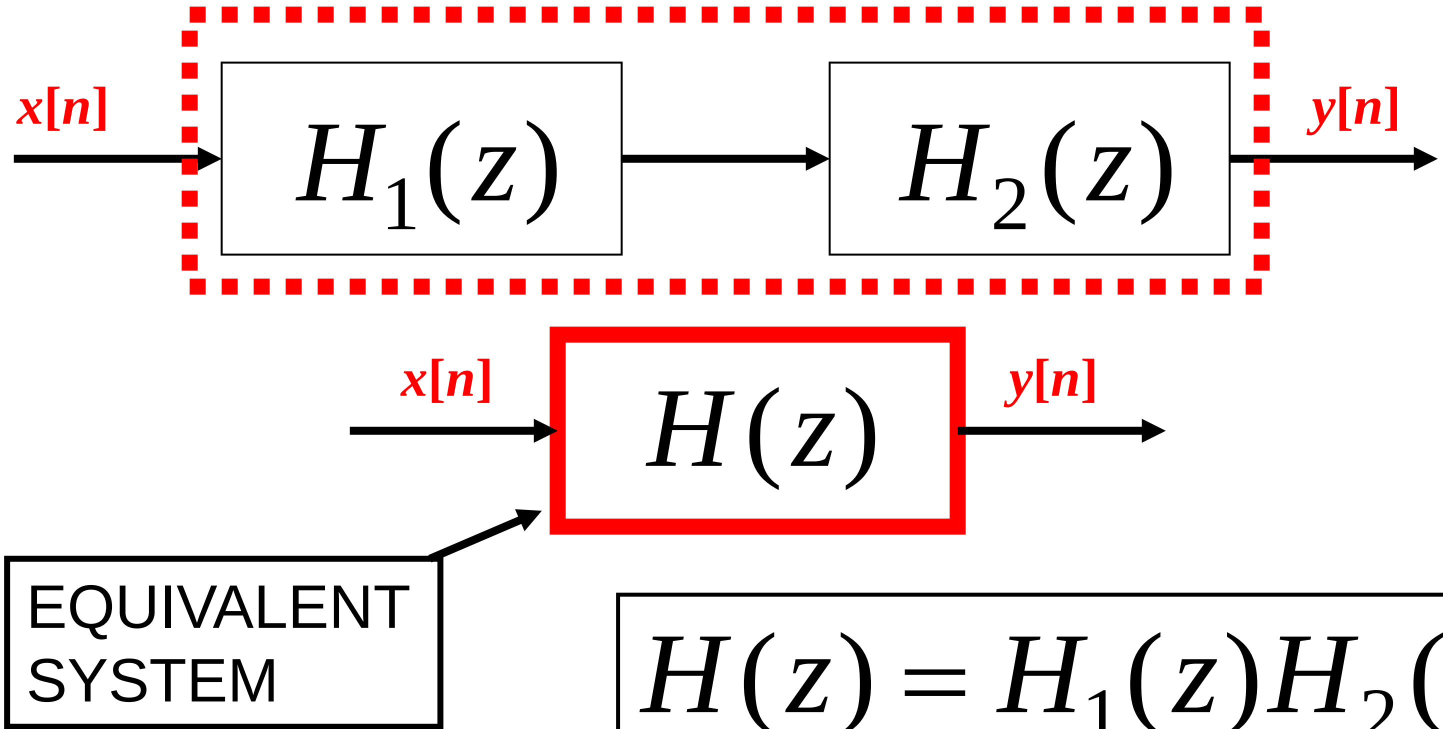
CASCADE SYSTEMS

- Does the order of S_1 & S_2 matter?
  - NO, LTI SYSTEMS can be rearranged !!!
  - Remember: $h_1[n] * h_2[n]$
  - How to combine $H_1(z)$ and $H_2(z)$?



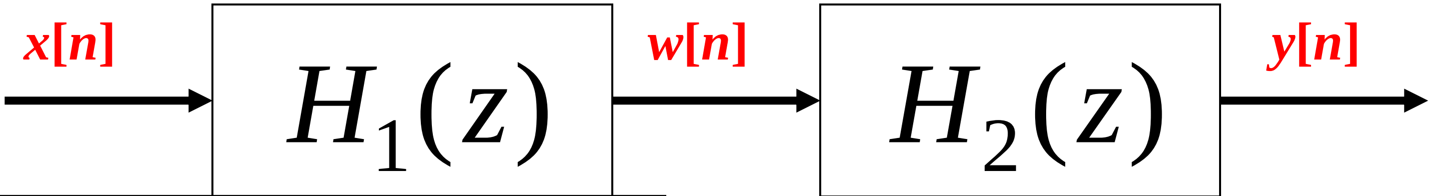
CASCADE EQUIVALENT

- Multiply the System Functions.



$$H(z) = H_1(z)H_2(z)$$

CASCADE EXAMPLE

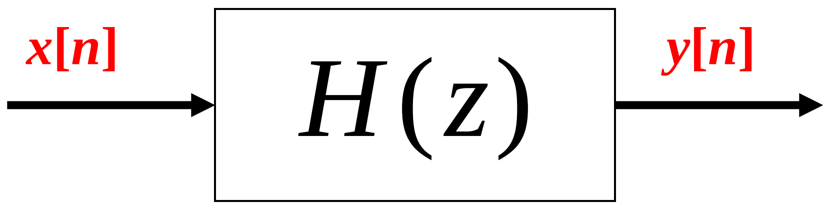


$$\boxed{w[n] = x[n] - x[n-1]}$$

$$\boxed{H_1(z) = 1 - z^{-1}}$$

$$\boxed{y[n] = w[n] + w[n-1]}$$

$$\boxed{H_2(z) = 1 + z^{-1}}$$



$$\boxed{H(z) = (1 - z^{-1})(1 + z^{-1}) = 1 - z^{-2}}$$

$$\boxed{y[n] = x[n] - x[n-2]}$$

FREQUENCY RESPONSE?

- Same Form:

$$z = e^{j\hat{\omega}}$$

$\hat{\omega}$ – Domain

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}$$

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k (e^{j\hat{\omega}})^{-k}$$

z – Domain

$$H(z) = \sum_{k=0}^{M} b_k z^{-k}$$

SAME COEFFICIENTS

CHANGE in NOTATION

- NOTATION for the FREQUENCY RESPONSE

$$H(\hat{\omega}) \leftrightarrow H(e^{j\hat{\omega}})$$

- Relate $H(z)$ to the FREQUENCY RESPONSE.

$$H(\hat{\omega}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

ANOTHER ANALYSIS TOOL

- z-Transform POLYNOMIALS are EASY !
  - ROOTS, FACTORS, etc.
- **ZEROS and POLES**
  - Zeros: $H(z) = 0$
  - Poles: $H(z) = \pm\infty$
- The z-domain is a COMPLEX plane.
  - $H(z)$ is a COMPLEX-VALUED function of a COMPLEX VARIABLE z.

ZEROS of *H*(*z*) (1)

- Find *z*, where *H*(*z*)=0.

$$H(z) = 1 - \tfrac{1}{2} z^{-1}$$

$$1 - \tfrac{1}{2} z^{-1} = 0\ ?$$

$$z - \tfrac{1}{2} = 0$$

$$\boxed{\text{Zero at}: z = \tfrac{1}{2}}$$

ZEROS of *H*(*z*) (2)

- Find z, where $H(z)=0$.
  - Interesting when z is ON the unit circle.

$$H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$$

$$H(z) = (1 - z^{-1})(1 - z^{-1} + z^{-2})$$

$$\boxed{\text{Roots}: z = 1, \tfrac{1}{2} \pm j\frac{\sqrt{3}}{2}} \qquad \boxed{e^{\pm j\pi/3}}$$

POLES of *H(z)*

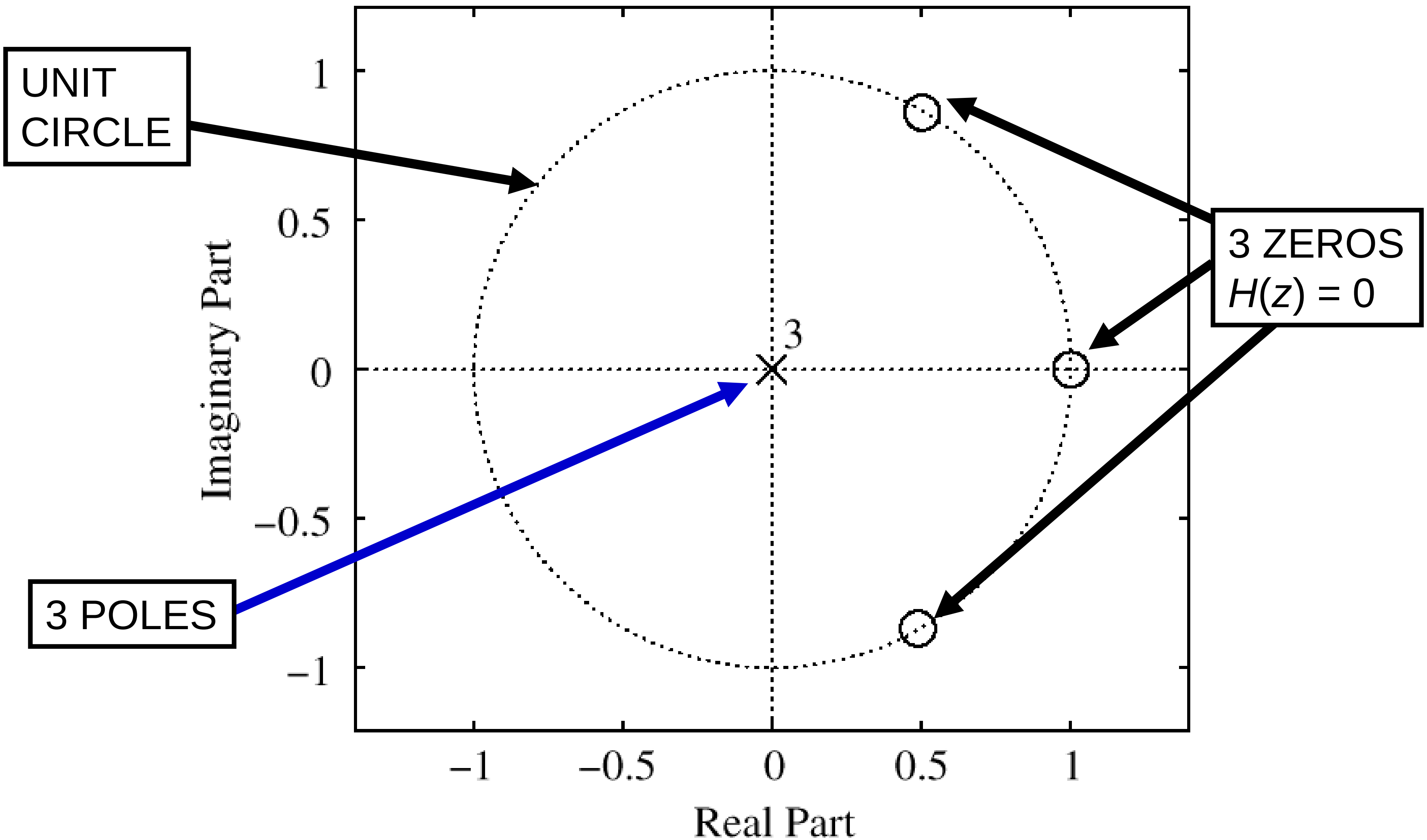
- Find *z*, where $H(z) \rightarrow \infty$.
  - Not very interesting for the FIR case

$$H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$$

$$H(z) = \frac{z^3 - 2z^2 + 2z - 1}{z^3}$$

$$\boxed{\text{Three Poles at} : z = 0}$$

PLOT ZEROS in *z*-DOMAIN



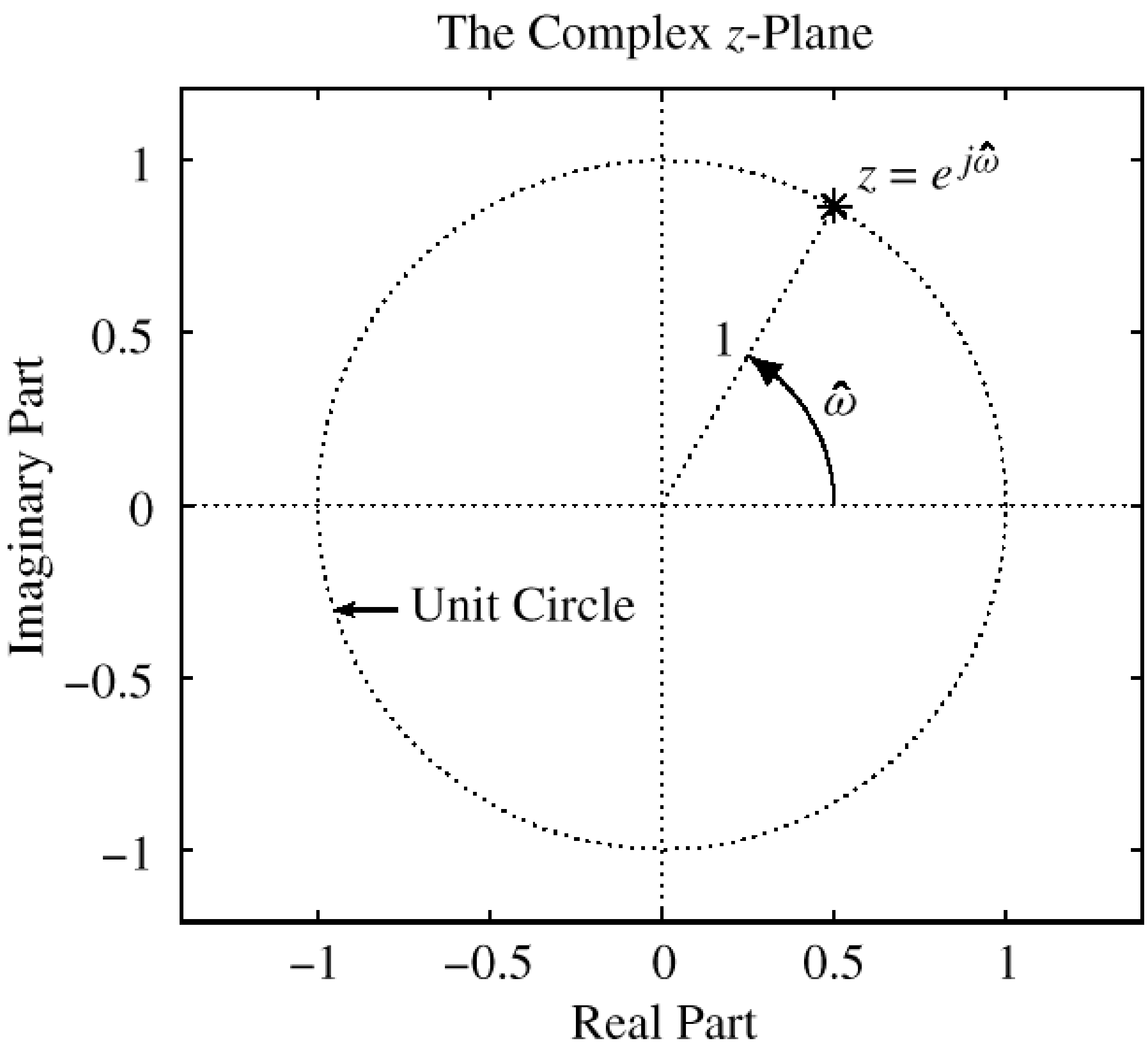
FREQ. RESPONSE from ZEROS (1)

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

- Relate $H(z)$ to the FREQUENCY RESPONSE.
- EVALUATE $H(z)$ on the **UNIT CIRCLE**.
    - The ANGLE is the same as a FREQUENCY.

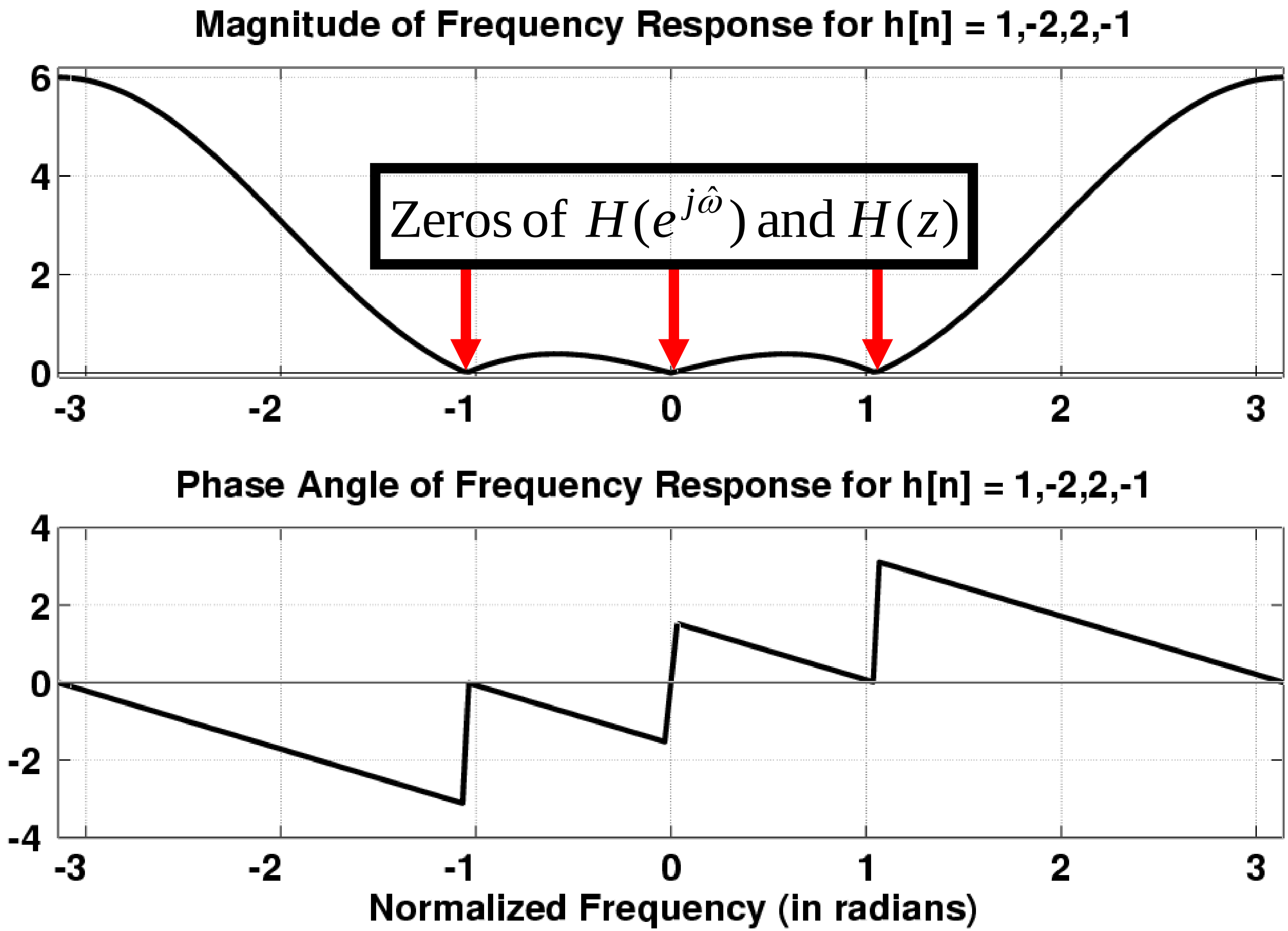
$$\boxed{\begin{array}{l} z = e^{j\hat{\omega}} \quad (\text{as } \hat{\omega} \text{ varies}) \\ \text{defines a CIRCLE, radius} = 1 \end{array}}$$

FREQ. RESPONSE from ZEROS (2)



The ANGLE is a FREQUENCY.

FIR Frequency Response

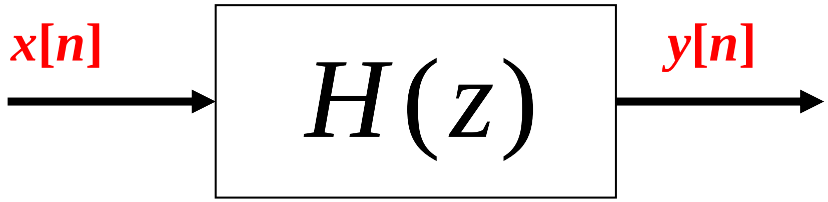


NULLING PROPERTY of *H*(*z*) (1)

- When $H(z)$=0 on the unit circle,
  - Find inputs $x[n]$ that give zero output.

$$H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$$

$$H(e^{j\hat{\omega}}) = 1 - 2e^{-j\hat{\omega}} + 2e^{-j2\hat{\omega}} - e^{-j3\hat{\omega}}$$



$$\boxed{H(e^{j\pi/3}) = ?}$$

$$x[n] = e^{j(\pi/3)n}$$

$$y[n] = H(e^{j(\pi/3)}) \cdot e^{j(\pi/3)n}$$

NULLING PROPERTY of *H*(*z*) (2)

- Evaluate *H*(*z*) at the input "frequency".

$$H(e^{j\hat{\omega}}) = 1 - 2e^{-j\hat{\omega}} + 2e^{-j2\hat{\omega}} - e^{-j3\hat{\omega}}$$

$$y[n] = H(e^{j\pi/3}) \cdot e^{j(\pi/3)n}$$

$$y[n] = \underbrace{(1 - 2e^{-j\pi/3} + 2e^{-j2\pi/3} - e^{-j3\pi/3})}_{(1-2(\frac{1}{2}-j\frac{\sqrt{3}}{2})+2(-\frac{1}{2}-j\frac{\sqrt{3}}{2})-(-1))} \cdot e^{j(\pi/3)n}$$

$$y[n] = (1 - 1 + j\sqrt{3} - 1 - j\sqrt{3} + 1) \cdot e^{j(\pi/3)n} = 0$$

NULLING FILTER

- PLACE ZEROS to make $y[n] = 0$.

$$H(z) = 1 - 2z^{-1} + 2z^{-2} - z^{-3}$$

3 ZEROS
$H(z) = 0$

the output resulting from each of the following three signals will be zero:

$H(z_1) = 0$	$x_1[n] = (z_1)^n = 1$	$y_1[n] = 0$
$H(z_2) = 0$	$x_2[n] = (z_2)^n = e^{j\pi n/3}$	$y_2[n] = 0$
$H(z_3) = 0$	$x_3[n] = (z_3)^n = e^{-j\pi n/3}$	$y_3[n] = 0$

L-pt RUNNING SUM $H(z)$

$$H(z) = \sum_{k=0}^{L-1} \tfrac{1}{L} z^{-k} = \frac{1 - z^{-L}}{L(1 - z^{-1})} = \frac{z^L - 1}{Lz^{L-1}(z-1)}$$

$$z^L - 1 = 0 \quad \Rightarrow \quad z^L = 1 = e^{j2\pi k}$$

$$z = e^{j(2\pi/L)k} \quad \text{for } k = 1, 2, \ldots L-1$$

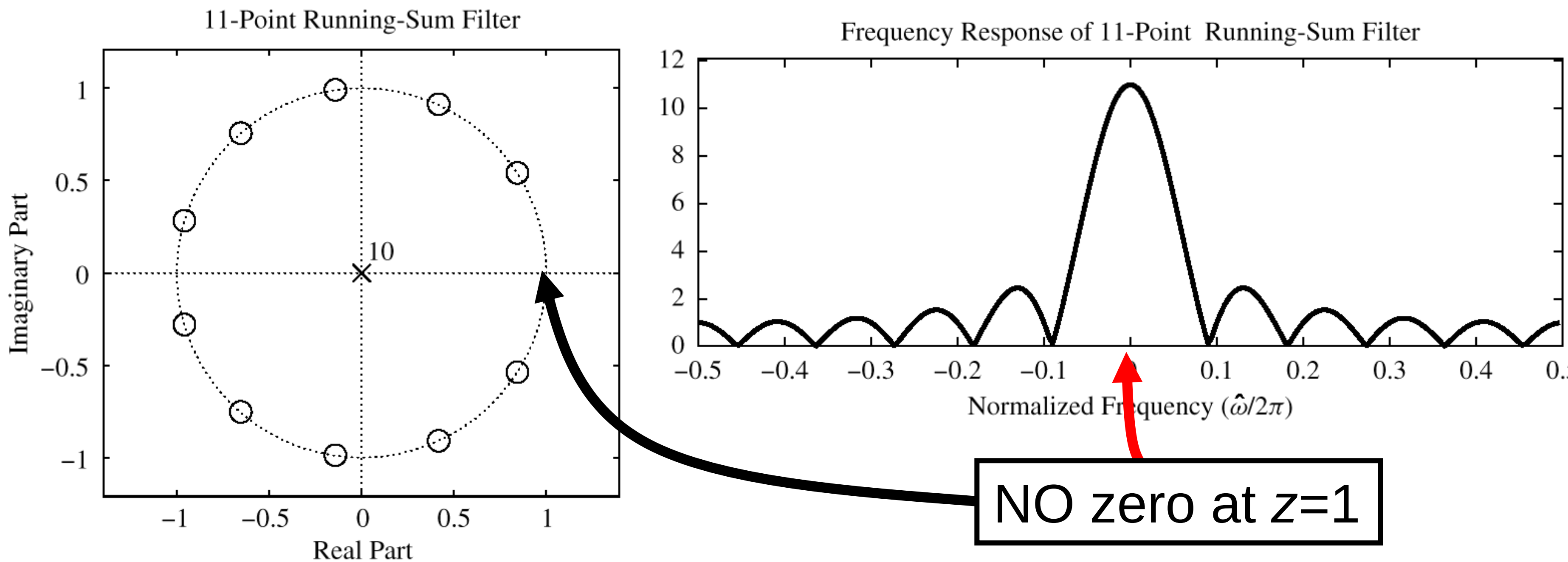
ZEROS on
the UNIT CIRCLE

(z-1) in
denominator
cancels k=0 term.

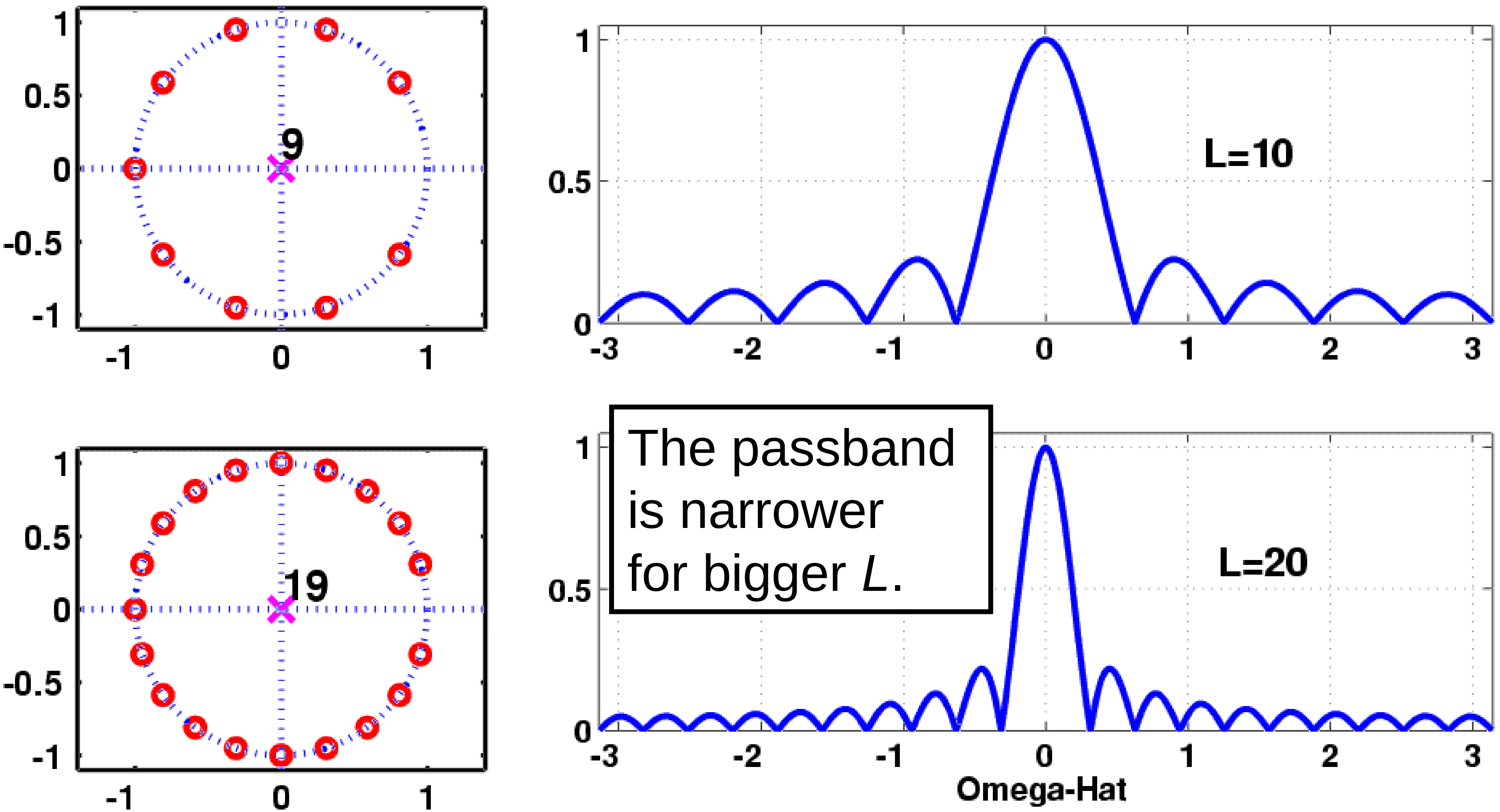
11-pt RUNNING SUM $H(z)$

$$H(z) = \sum_{k=0}^{10} z^{-k}$$

$$H(z) = (1 - e^{j2\pi/11}z^{-1})(1 - e^{j4\pi/11}z^{-1}) \cdots (1 - e^{j20\pi/11}z^{-1})$$



FILTER DESIGN: CHANGE *L*

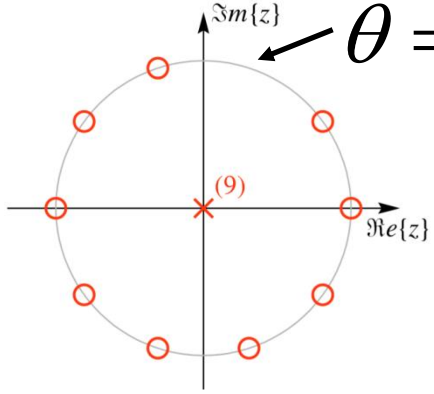


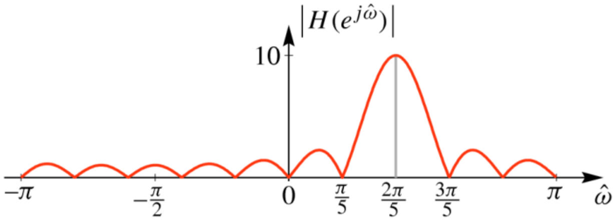
The passband is narrower for bigger *L*.

A Complex Bandpass Filter (1)

- Rotation by the angle $\theta = 2\pi k_0 / L$

$$\theta = 2\pi \times 2/10 = 0.4\pi$$





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Let $G(z) = \sum_{k=0}^{L-1} z^{-k}$.

$$H(e^{j\hat{\omega}}) = G(e^{j(\hat{\omega}-\theta)}) = \sum_{k=0}^{L-1} e^{-j(\hat{\omega}-\theta)k} = \sum_{k=0}^{L-1} e^{j\theta k} e^{-j\hat{\omega}k}$$

  - Multiply b_k by $e^{jk\theta}$.
  - Filter coefficients are complex.

A Complex Bandpass Filter (2)

- Rotation by the angle $\theta = 2\pi k_0 / L$

$$H(e^{j\hat{\omega}}) = G(e^{j(\hat{\omega}-\theta)}) = \sum_{k=0}^{L-1} e^{-j(\hat{\omega}-\theta)k} = \sum_{k=0}^{L-1} e^{j\theta k} e^{-j\hat{\omega}k}$$

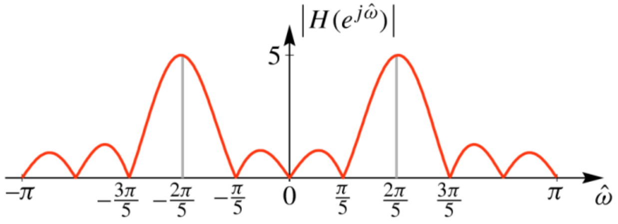
$$H(z) = \sum_{k=0}^{L-1} e^{j\theta k} z^{-k} = G(ze^{-j\theta}) = G(z/r)$$

where $r = e^{j\theta}$

  - Replace *z* by *z*/*r*.

A Bandpass Filter with Real Coefficients (1)

- Even magnitude response
  - Achieved by the sum of two complex BPFs



$$H(z) = \frac{1}{2}\left(\sum_{k=0}^{L-1} e^{j\theta k} z^{-k} + \sum_{k=0}^{L-1} e^{-j\theta k} z^{-k}\right) = \sum_{k=0}^{L-1} \cos(\theta k) z^{-k}$$

$$b_k = \cos(\theta k), k = 0,1,\ldots,L-1$$

A Bandpass Filter with Real Coefficients (2)

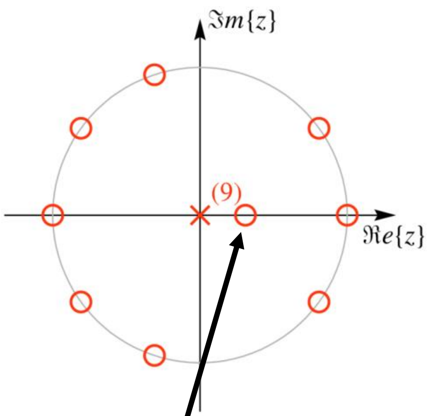
- Pole-zero distribution

$$H(z)=\frac{1}{2}\left(\sum_{k=0}^{L-1}e^{j\theta k}z^{-k}+\sum_{k=0}^{L-1}e^{-j\theta k}z^{-k}\right) \quad \text{where } p=e^{j\theta}$$

$$=\frac{1}{2}\frac{z^{L}-1}{z^{L-1}(z-p)}+\frac{1}{2}\frac{z^{L}-1}{z^{L-1}(z-p*)}$$

$$=\frac{1}{2}\frac{(z^{L}-1)(z-p*)+(z^{L}-1)(z-p)}{z^{L-1}(z-p)(z-p*)}$$

$$=\frac{(z^{L}-1)(z-\frac{1}{2}(p+p*))}{z^{L-1}(z-p)(z-p*)}$$



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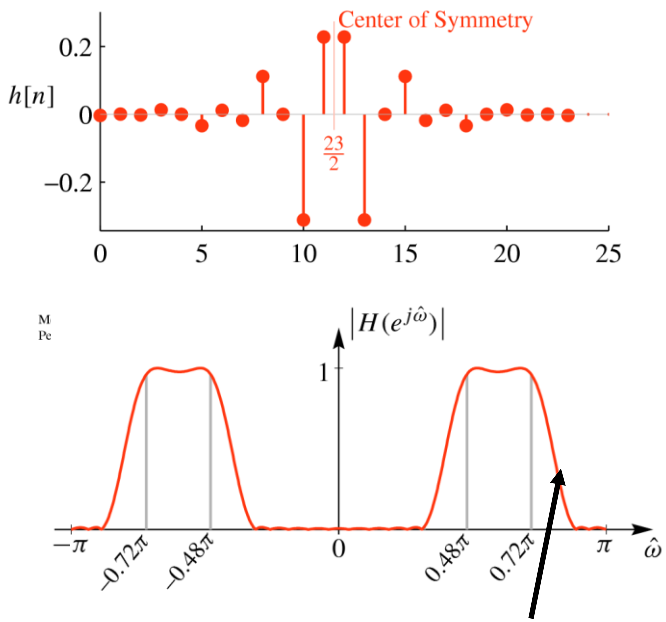
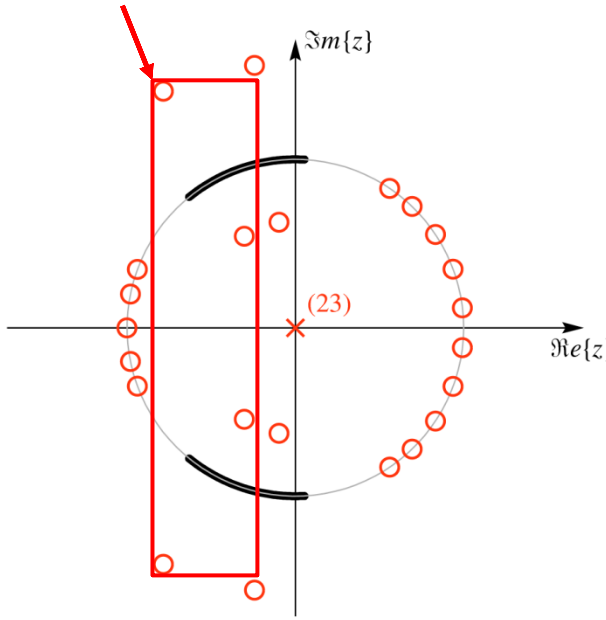
$$z=\frac{1}{2}(p+p*)=\cos(\theta)$$

Practical Bandpass Filter Design

- More sophisticated methods – a computer-aided filter-design program
  - **`firpm`** (revised **`remez`**) and **`fir1`** in the Matlab software

Groups of four zeros at the conjugate, reciprocal, and conjugate reciprocal locations





The width of the transition region is inversely proportional to *M*.

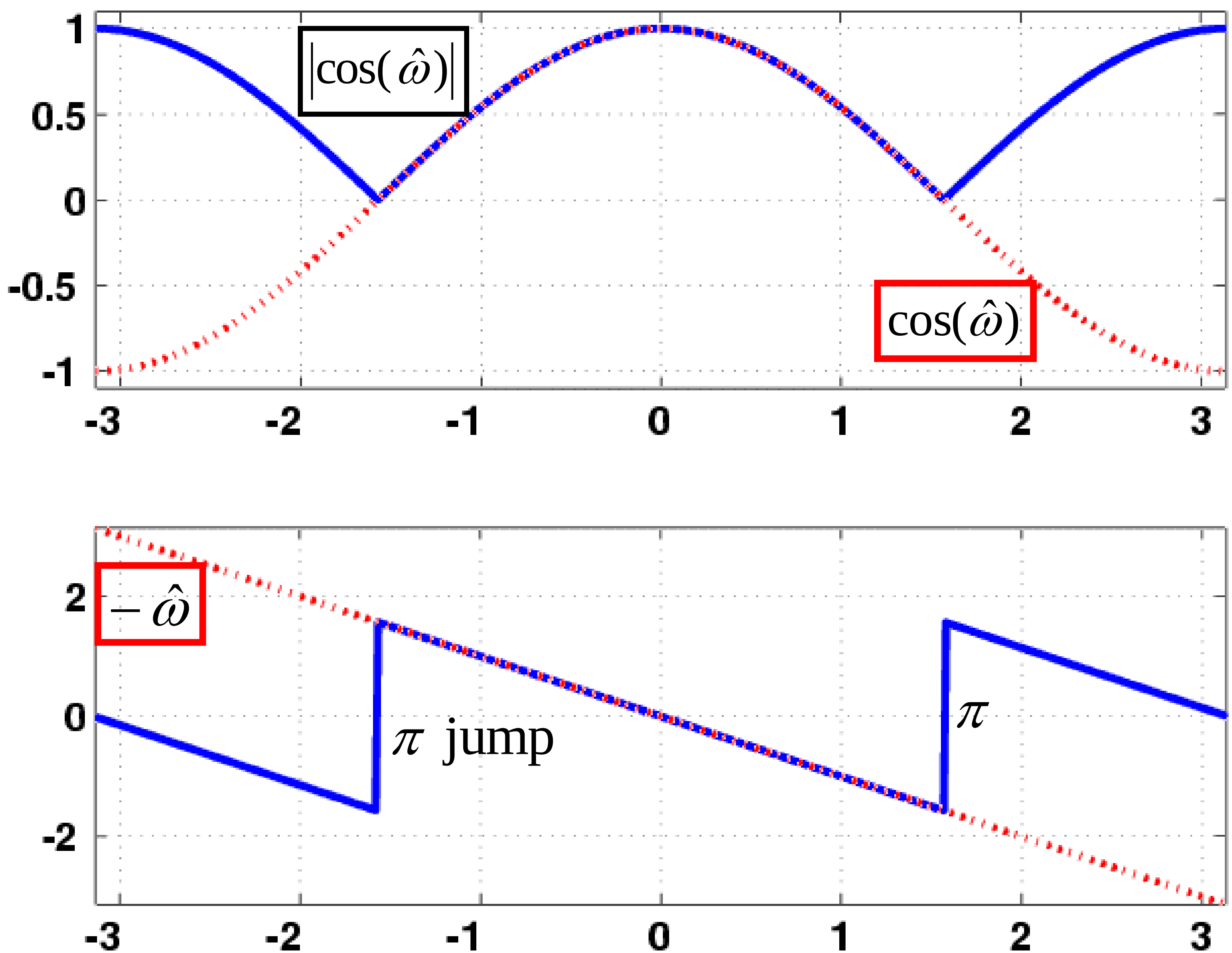
Example: FREQ. RESPONSE (1)

- Given $H(e^{j\hat{\omega}}) = e^{-j\hat{\omega}}\cos(\hat{\omega})$,
- Derive Magnitude and Phase.

$$\left|H(e^{j\hat{\omega}})\right| = \left|e^{-j\hat{\omega}}\right| \cdot \left|\cos(\hat{\omega})\right| = \left|\cos(\hat{\omega})\right|$$

$$\angle H(e^{j\hat{\omega}}) = \begin{cases} -\hat{\omega} & \cos(\hat{\omega}) \geq 0 \\ -\hat{\omega} + \pi & \cos(\hat{\omega}) < 0 \end{cases}$$

Example: FREQ. RESPONSE (2)



Example: FREQ. RESPONSE (3)

- Find *y*[*n*] when $\boxed{x[n]=\cos(0.25\pi n)}$.

$$y[n]=|H|\cos(0.25\pi n+\angle H)$$

$$\boxed{=0.707\cos(0.25\pi n-\tfrac{\pi}{4})}$$

$$H(e^{j\hat{\omega}})=e^{-j\hat{\omega}}\cos(\hat{\omega}) \qquad \boxed{\text{at } \hat{\omega}=\tfrac{\pi}{4}}$$

$$H(e^{j\pi/4})=e^{-j\pi/4}\cos(\tfrac{\pi}{4})=0.707e^{-j\pi/4}$$

Properties of Linear-Phase Filters

- Linear-phase filters

$$H(e^{j\hat{\omega}}) = R(e^{j\hat{\omega}})e^{-j\hat{\omega}N}$$

  - $R(e^{j\hat{\omega}})$ is the real function.
  - Linear phase: $\angle H(e^{j\hat{\omega}}) = -\hat{\omega}N$

- If $x[n] = e^{j(\hat{\omega}n+\phi)}$,

$$y[n] = R(e^{j\hat{\omega}})e^{j(\hat{\omega}n+\phi-\hat{\omega}N)}$$

$$= R(e^{j\hat{\omega}})e^{j(\hat{\omega}(n-N)+\phi)} = R(e^{j\hat{\omega}})x[n-N]$$

Delay of *N* samples

There is no phase distortion in *y*[*n*] from *x*[*n*].

The Linear-Phase Condition

- FIR systems
  - Symmetric filter coefficients

$$b_k = b_{M-k},\ k = 0,1,\ldots,M$$

- Example (M=4)

$$H(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} + b_1 z^{-3} + b_0 z^{-4}$$

$$H(z) = (b_0(z^2 + z^{-2}) + b_1(z^1 + z^{-1}) + b_2)z^{-2}$$

$$H(e^{j\hat{\omega}}) = (2b_0 \cos(2\hat{\omega}) + 2b_1 \cos(\hat{\omega}) + b_2)e^{-j\hat{\omega}M/2}$$

$$H(e^{j\hat{\omega}}) = R(e^{j\hat{\omega}})e^{-j\hat{\omega}M/2}$$

$$H(1/z) = z^M H(z)$$

Thank you

- Homework
  - P-7.1, 2, 3, 5, 6, 8, 10, 12, 14, 15, 16 & 18
- Reading assignment
  - ~ Section 8.3