Chapter 8
IIR Filters

Quick Review: Delay by n_d

$$y[n] = x[n - n_d]$$

IMPULSE RESPONSE $h[n] = \delta[n - n_d]$

SYSTEM FUNCTION $H(z) = z^{-n_d}$

FREQUENCY RESPONSE $H(e^{j\hat{\omega}}) = e^{-j\hat{\omega} n_d}$

LOGICAL THREAD

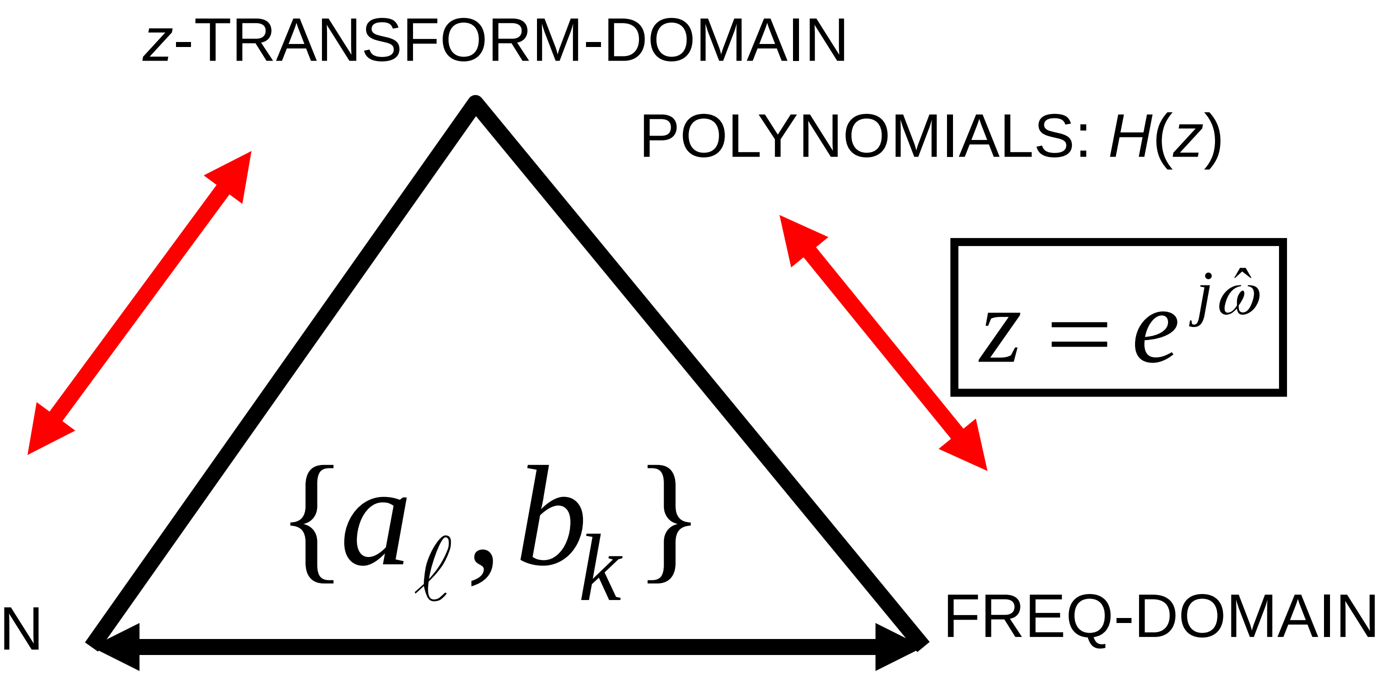
- FIND the IMPULSE RESPONSE, $h[n]$.
  - INFINITELY LONG
  - **IIR** Filters

$$\boxed{H(z) = \sum_{n=0}^{\infty} h[n] z^{-n}}$$

- EXPLOIT THREE DOMAINS:
  - Show Relationship for IIR:

$$h[n] \leftrightarrow H(z) \leftrightarrow H(e^{j\hat{\omega}})$$

THREE DOMAINS



$$y[n] = \sum_{\ell=1}^{N} a_\ell y[n-\ell] + \sum_{k=0}^{M} b_k x[n-k]$$

$$H(e^{j\hat{\omega}}) = \frac{\sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}}{1 - \sum_{\ell=1}^{N} a_\ell e^{-j\hat{\omega}\ell}}$$

ONE FEEDBACK TERM

- ADD PREVIOUS OUTPUTS.

$$\boxed{y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]}$$

PREVIOUS outputs

FEEDBACK

FIR PART of the FILTER

FEED-FORWARD

- CAUSALITY
  - NOT USING FUTURE OUTPUTS or INPUTS

FILTER COEFFICIENTS

- ADD PREVIOUS OUTPUTS.

$$\boxed{y[n] = 0.8y[n-1] + 3x[n] - 2x[n-1]}$$

FEEDBACK COEFFICIENT

SIGN CHANGE

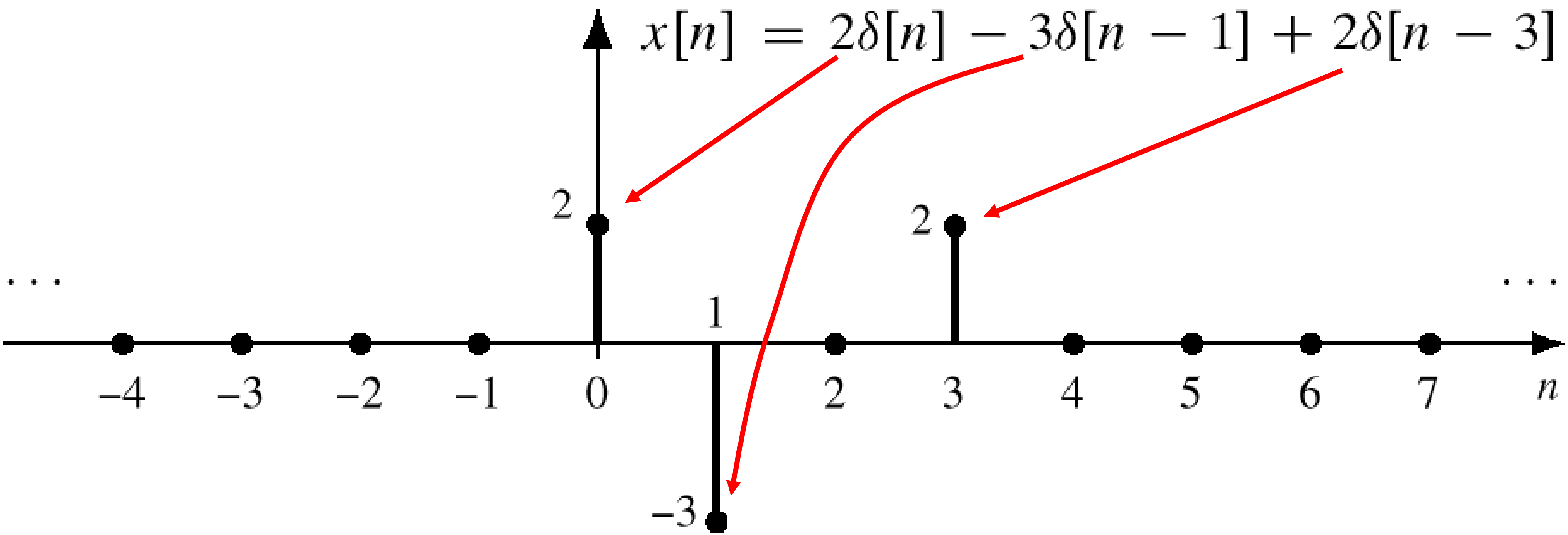
- MATLAB
  - `yy = filter([3,-2],[1,-0.8],xx)`

COMPUTE OUTPUTS.

- FEEDBACK DIFFERENCE EQUATION:

$$\boxed{y[n] = 0.8y[n-1] + 5x[n]}$$

$$x[n] = 2\delta[n] - 3\delta[n-1] + 2\delta[n-3]$$



- NEED *y*[-1] to get started.

$$y[0] = 0.8y[-1] + 5x[0]$$

AT REST CONDITION

- $y[n] = 0$, for $n<0$
- BECAUSE $x[n] = 0$, for $n<0$.

INITIAL REST CONDITIONS

1. The input must be assumed to be zero prior to some starting time n_0, i.e., $x[n] = 0$ for $n < n_0$. We say that such inputs are *suddenly applied.*
2. The output is likewise assumed to be zero prior to the starting time of the signal, i.e., $y[n] = 0$ for $n < n_0$. We say that the system is *initially at rest* if its output is zero prior to the application of a suddenly applied input.

COMPUTE $y[0]$.

- THIS STARTS THE RECURSION:

With the initial rest assumption, $y[n] = 0$ for $n < 0$,

$$y[0] = 0.8y[-1] + 5(2) = 0.8(0) + 5(2) = 10$$

- SAME with MORE FEEDBACK TERMS

$$\boxed{y[n] = a_1 y[n-1] + a_2 y[n-2] + \sum_{k=0}^{2} b_k x[n-k]}$$

COMPUTE MORE $y[n]$.

- CONTINUE THE RECURSION:

$$y[1] = 0.8y[0] + 5x[1] = 0.8(10) + 5(-3) = -7$$

$$y[2] = 0.8y[1] + 5x[2] = 0.8(-7) + 5(0) = -5.6$$

$$y[3] = 0.8y[2] + 5x[3] = 0.8(-5.6) + 5(2) = 5.52$$

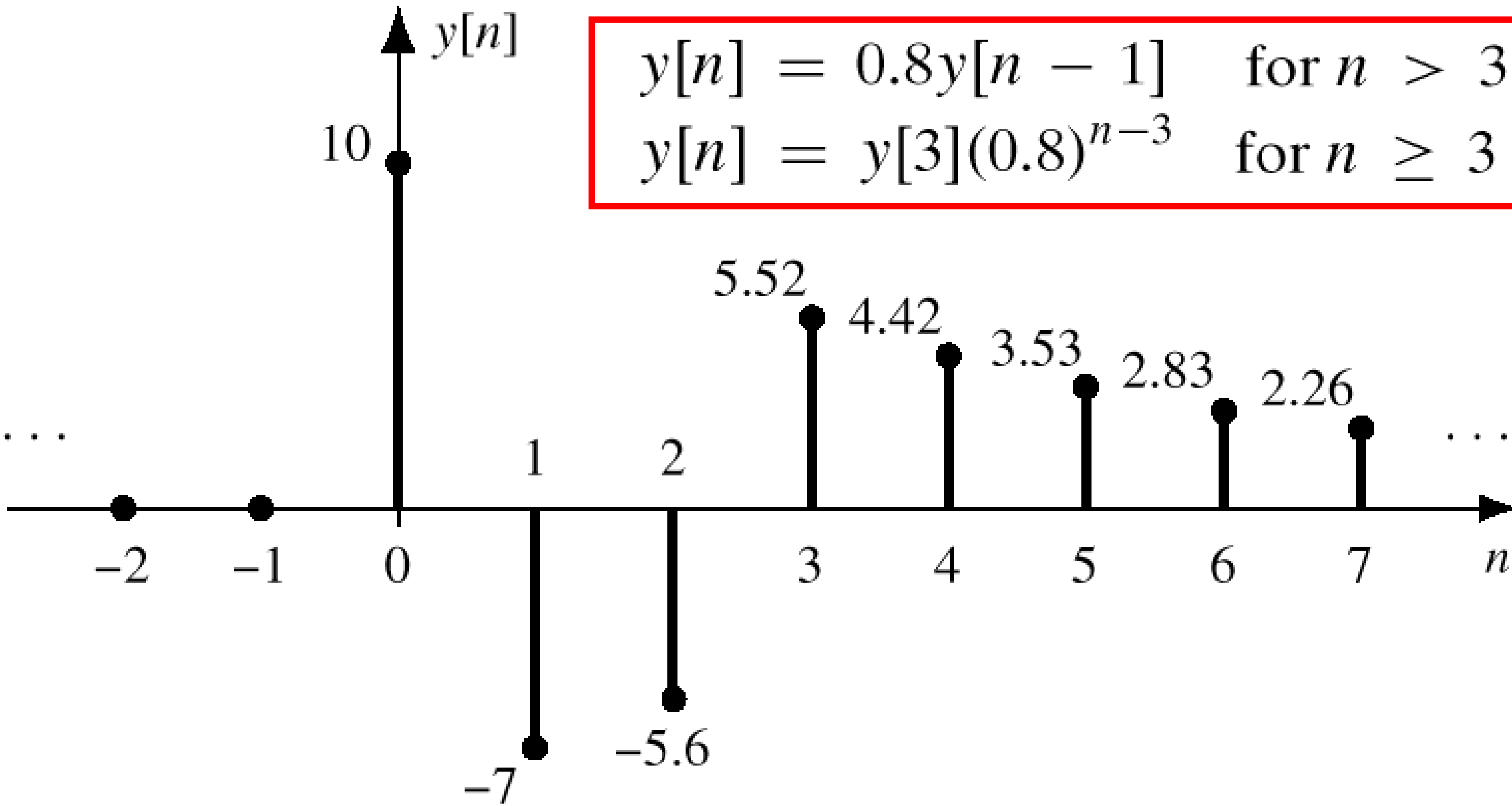
$$y[4] = 0.8y[3] + 5x[4] = 0.8(5.52) + 5(0) = 4.416$$

$$y[5] = 0.8y[4] + 5x[5] = 0.8(4.416) + 5(0) = 3.5328$$

$$y[6] = 0.8y[5] + 5x[6] = 0.8(3.5328) + 5(0) = 2.8262$$

PLOT *y*[*n*].

$$y[n] = 0.8y[n-1] \quad \text{for } n > 3$$
$$y[n] = y[3](0.8)^{n-3} \quad \text{for } n \geq 3$$



IMPULSE RESPONSE (1)

$$\boxed{y[n] = a_1 y[n-1] + b_0 x[n]} \quad \boxed{h[n] = a_1 h[n-1] + b_0 \delta[n]}$$

n	$n < 0$	0	1	2	3	4
$\delta[n]$	0	1	0	0	0	0
$h[n-1]$	0	0	b_0	$b_0(a_1)$	$b_0(a_1)^2$	$b_0(a_1)^3$
$h[n]$	0	b_0	$b_0(a_1)$	$b_0(a_1)^2$	$b_0(a_1)^3$	$b_0(a_1)^4$

From this table it is obvious that the general formula is

$$h[n] = \begin{cases} b_0(a_1)^n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$

$$\boxed{h[n] = b_0 (a_1)^n u[n]}$$

$$\boxed{u[n] = 1, \quad \text{for } n \geq 0}$$

IMPULSE RESPONSE (2)

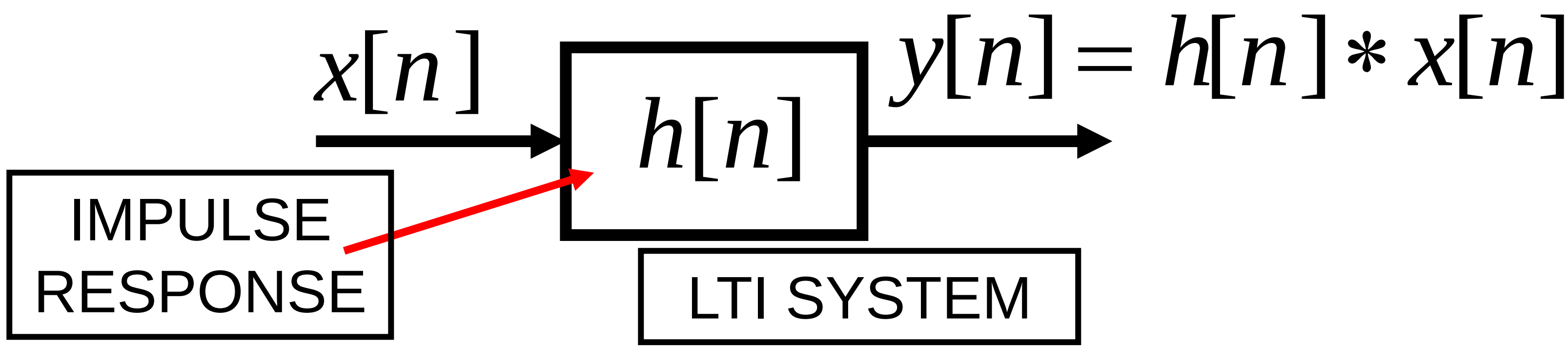
- DIFFERENCE EQUATION:

$$\boxed{y[n] = 0.8y[n-1] + 3x[n]}$$

- Find $h[n]$.

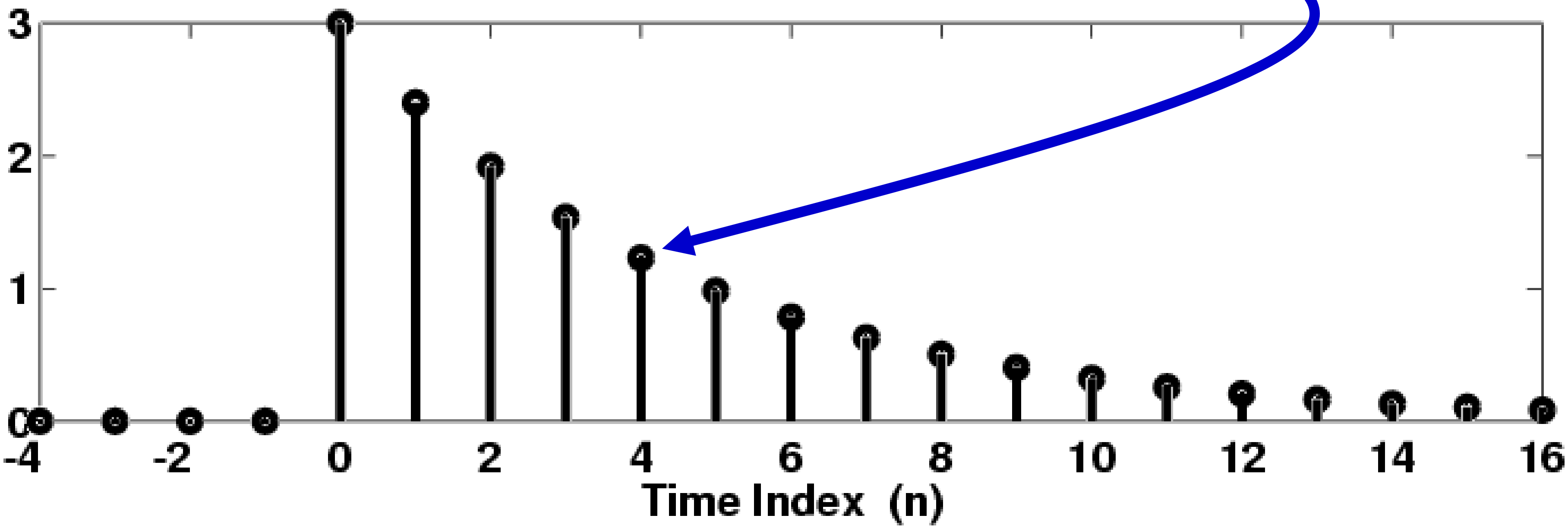
$$\boxed{h[n] = 3(0.8)^n u[n]}$$

- **CONVOLUTION** in TIME-DOMAIN



PLOT AN IMPULSE RESPONSE.

$$h[n] = b_0 (a_1)^n u[n] = 3(0.8)^n u[n]$$



Infinite-Length Signal: *h*[*n*]

- POLYNOMIAL Representation

$$\boxed{H(z)=\sum_{n=-\infty}^{\infty} h[n]z^{-n}}$$

APPLIES to Any SIGNAL.

- SIMPLIFY the SUMMATION.

$$\boxed{H(z)=\sum_{n=-\infty}^{\infty} b_0(a_1)^n u[n]z^{-n}=b_0\sum_{n=0}^{\infty} a_1^n z^{-n}}$$

Derivation of *H*(*z*)

- Recall the Sum of a Geometric Sequence:

$$\boxed{\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}}$$

- Yields a COMPACT FORM.

$$\boxed{\begin{aligned} H(z) &= b_0 \sum_{n=0}^{\infty} a_1^n z^{-n} = b_0 \sum_{n=0}^{\infty} (a_1 z^{-1})^n \\ &= \frac{b_0}{1-a_1 z^{-1}} \quad \text{if } |z| > |a_1| \end{aligned}}$$

H(*z*) = *z*-Transform{ *h*[*n*] } (1)

- FIRST-ORDER IIR FILTER:

$$\boxed{y[n] = a_1 y[n-1] + b_0 x[n]}$$

$$\boxed{h[n] = b_0 (a_1)^n u[n]}$$

$$\boxed{H(z) = \frac{b_0}{1 - a_1 z^{-1}}}$$

$H(z)$ = z-Transform{ $h[n]$ } (2)

- ANOTHER FIRST-ORDER IIR FILTER:

$$\boxed{y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]}$$

$$\boxed{h[n] = b_0 (a_1)^n u[n] + b_1 (a_1)^{n-1} u[n-1]}$$

z^{-1} is a shift.

$$\boxed{H(z) = \frac{b_0}{1 - a_1 z^{-1}} + \frac{b_1 z^{-1}}{1 - a_1 z^{-1}} = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}}$$

SUPERPOSITION

$$y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]$$

$$y[n] = y_1[n] + y_2[n]$$

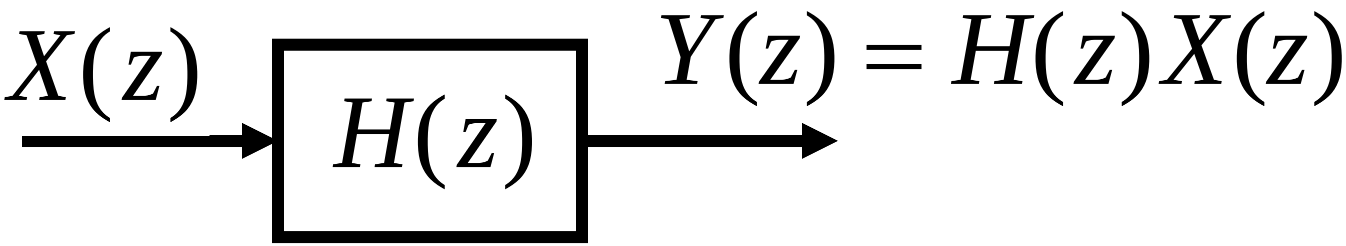
where

$$y_1[n] = a_1 y_1[n-1] + b_0 x[n]$$

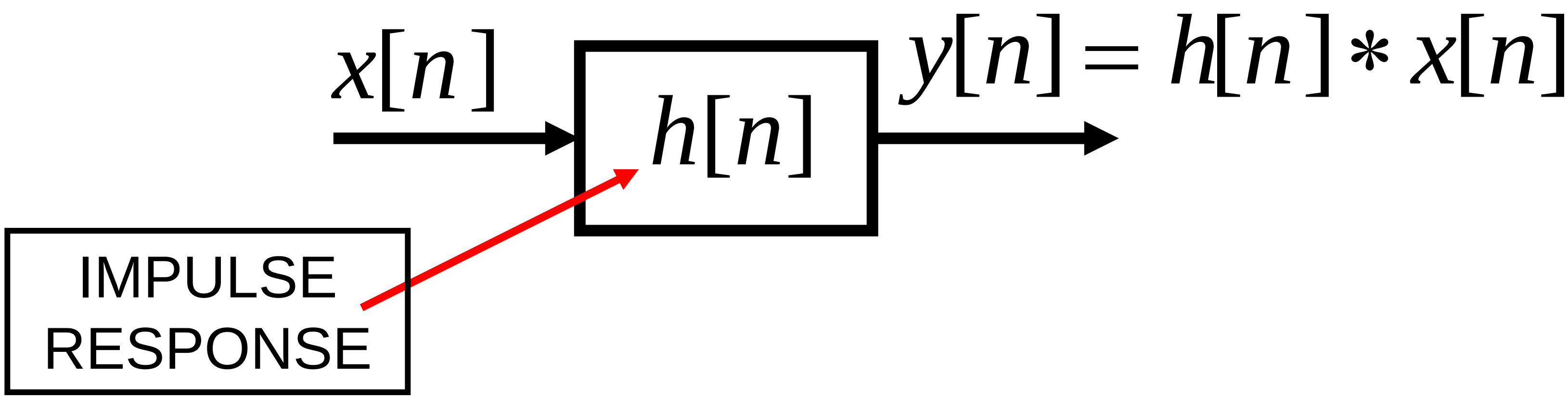
$$y_2[n] = a_1 y_2[n-1] + b_1 x[n-1]$$

CONVOLUTION PROPERTY

- MULTIPLICATION of *z*-TRANSFORMS



- CONVOLUTION in the TIME-DOMAIN



STEP RESPONSE: $x[n]=u[n]$

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$$\boxed{u[n]=1, \quad \text{for } n \geq 0}$$

n	$x[n]$	$y[n]$
$n<0$	0	0
0	1	b_0
1	1	$b_0 + b_0(a_1)$
2	1	$b_0 + b_0(a_1) + b_0(a_1)^2$
3	1	$b_0(1 + a_1 + a_1^2 + a_1^3)$
4	1	$b_0(1 + a_1 + a_1^2 + a_1^3 + a_1^4)$
$\vdots$	1	$\vdots$

DERIVE THE STEP RESPONSE.

$$y[n] = b_0(1 + a_1 + a_1^2 + \ldots + a_1^n) = b_0 \sum_{k=0}^{n} a_1^k$$

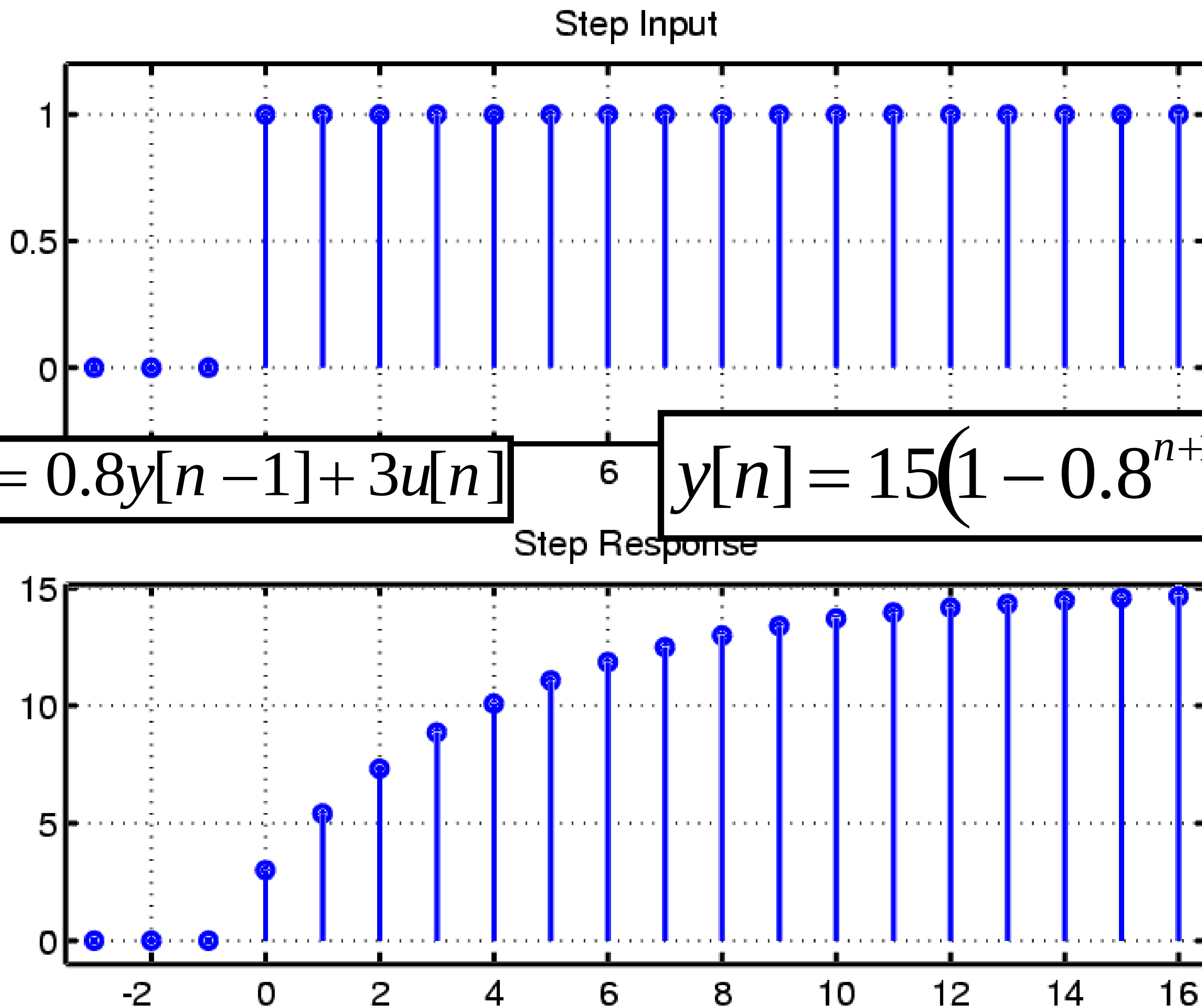
$$\sum_{k=0}^{L} r^k = \begin{cases} \dfrac{1 - r^{L+1}}{1 - r} & r \neq 1 \\ L + 1 & r = 1 \end{cases}$$

$$y[n] = b_0 \frac{1 - a_1^{n+1}}{1 - a_1} \qquad \text{for } n \geq 0, \qquad \text{if } a_1 \neq 1$$

PLOT THE STEP RESPONSE.

$$y[n] = 0.8y[n-1] + 3u[n]$$

$$y[n] = 15\left(1 - 0.8^{n+1}\right)u[n]$$



LTI: Convolution

- Output = Convolution of $x[n]$ & $h[n]$
  - NOTATION: $y[n] = x[n]*h[n]$
  - Here is the FIR case:

FINITE LIMITS

$$y[n] = \sum_{k=0}^{M} h[k]x[n-k]$$

FINITE LIMITS

Same as b_k

L-pt RUNNING AVG. $H(z)$

$$H(z) = \sum_{k=0}^{L-1} \tfrac{1}{L} z^{-k} = \frac{1 - z^{-L}}{L(1 - z^{-1})} = \frac{z^{L} - 1}{L z^{L-1}(z-1)}$$

$$z^{L} - 1 = 0 \quad \Rightarrow \quad z^{L} = 1 = e^{j2\pi k}$$

$$z = e^{j(2\pi/L)k} \qquad \text{for } k = 1, 2, \ldots L\text{-}1$$

ZEROS on UNIT CIRCLE

(z-1) in the denominator cancels k=0 term.

L-pt RUNNING AVG.: Step Response

$$y[n] = \sum_{k=0}^{L-1} \tfrac{1}{L}\, x[n-k]$$

STEP RESPONSE

$$y[n] = \sum_{k=0}^{L-1} \tfrac{1}{L}\, u[n-k] = \begin{cases} \frac{n+1}{L} & n = 0,1,2,\ldots L-1 \\ 1 & n \geq L \\ 0 & n < 0 \end{cases}$$

DELAY PROPERTY of *X*(*z*)

- DELAY in TIME<-->Multiply $X(z)$ by z^{-1}.

$$x[n] \leftrightarrow X(z)$$

$$x[n-1] \leftrightarrow z^{-1}X(z)$$

$$\text{Proof:} \quad \sum_{n=-\infty}^{\infty} x[n-1]z^{-n} = \sum_{\ell=-\infty}^{\infty} x[\ell]z^{-(\ell+1)}$$

$$= z^{-1}\sum_{\ell=-\infty}^{\infty} x[\ell]z^{-\ell} = z^{-1}X(z)$$

z-Transform of an IIR Filter

- DERIVE the SYSTEM FUNCTION $H(z)$.
  - Use **DELAY** PROPERTY.

$$\boxed{y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]}$$

$$\boxed{Y(z) = a_1 z^{-1} Y(z) + b_0 X(z) + b_1 z^{-1} X(z)}$$

EASIER with DELAY PROPERTY

Time delay of n_0 samples multiplies the z-transform by z^{-n_0}

$$x[n - n_0] \quad \Longleftrightarrow \quad z^{-n_0} X(z)$$

SYSTEM FUNCTION of an IIR FILTER (1)

- NOTE the FILTER COEFFICIENTS.

$$\boxed{Y(z) - a_1 z^{-1} Y(z) = b_0 X(z) + b_1 z^{-1} X(z)}$$

$$\boxed{(1 - a_1 z^{-1}) Y(z) = (b_0 + b_1 z^{-1}) X(z)}$$

$$H(z) = \frac{Y(z)}{X(z)} = \boxed{\frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}} = \frac{B(z)}{A(z)}$$

SYSTEM FUNCTION of an IIR FILTER (2)

- DIFFERENCE EQUATION:

$$\boxed{y[n] = 0.8y[n-1] + 3x[n] - 2x[n-1]}$$

- **READ** the FILTER COEFFICIENTS:

$$\boxed{Y(z) = \underbrace{\left(\frac{3-2z^{-1}}{1-0.8z^{-1}}\right)}_{H(z)} X(z)}$$

Implementation of first-order IIR filters (1)

- DIFFERENCE EQUATION

$$\boxed{y[n] = a_1 y[n-1] + b_0 x[n] + b_1 x[n-1]}$$

- System function

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}} = \frac{1}{1 - a_1 z^{-1}} (b_0 + b_1 z^{-1}) = \frac{1}{A(z)} B(z)$$

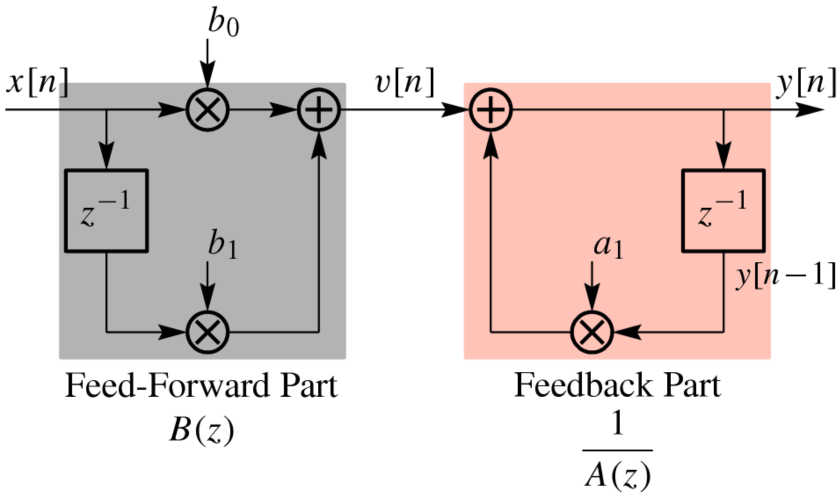
- Pair of difference equations

$$v[n] = b_0 x[n] + b_1 x[n-1]$$

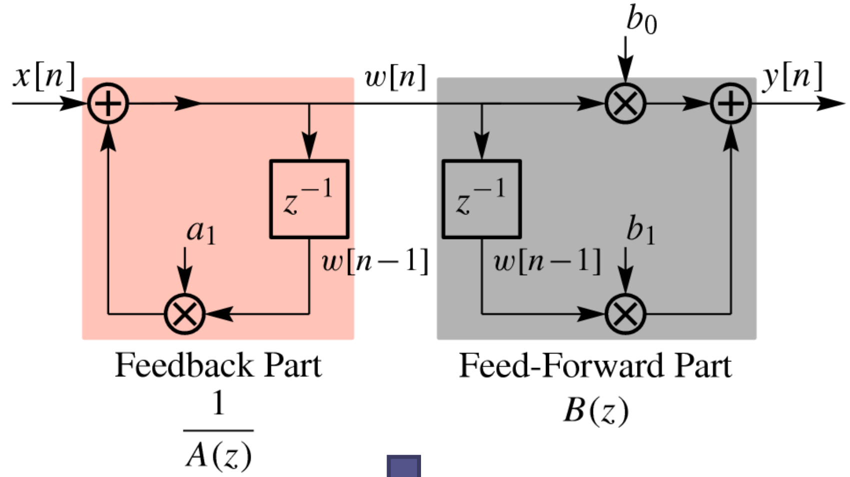
$$y[n] = a_1 y[n-1] + v[n]$$

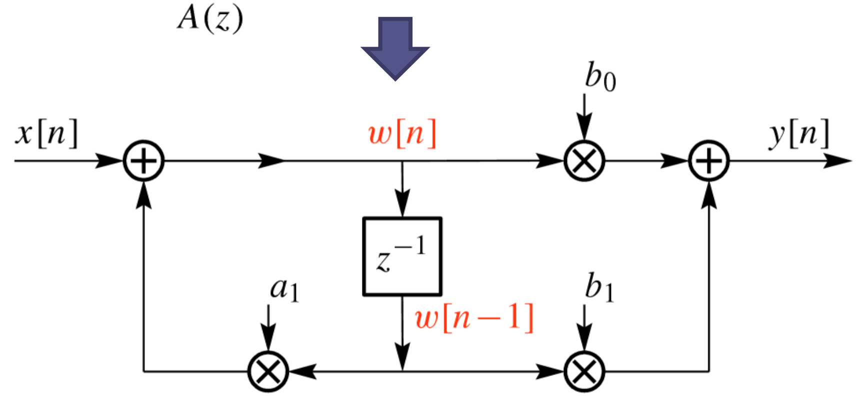
Implementation of first-order IIR filters (2)

Direct Form I Structure



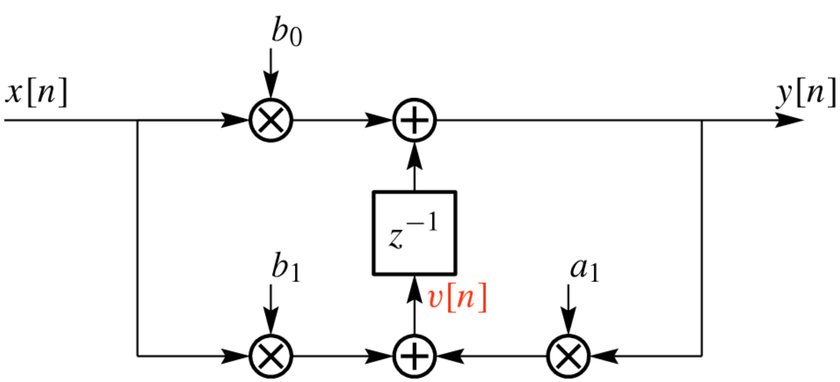
Direct Form II Structure





McClellan, Schafer and Yoder, *Signal Processing First,* ISBN 0-13-065562-7.
Pearson Prentice Hall, Inc. Upper Saddle River, NJ 07458. © 2003

Transposed Direct Form II Structure



McClellan, Schafer and Yoder, *Signal Processing First,* ISBN 0-13-065562-7.
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POLES & ZEROS

- ROOTS of the Numerator & the Denominator

$$\boxed{H(z)=\frac{b_0+b_1z^{-1}}{1-a_1z^{-1}} \rightarrow H(z)=\frac{b_0z+b_1}{z-a_1}}$$

$$\boxed{b_0z+b_1=0 \quad \Rightarrow z=-\frac{b_1}{b_0}}$$

ZERO: *H*(*z*)=0

$$\boxed{z-a_1=0 \quad \Rightarrow z=a_1}$$

POLE: *H*(*z*) → inf

EXAMPLE: Poles & Zeros

- The VALUE of $H(z)$ at POLES is **INFINITE.**

$$H(z) = \frac{2 + 2z^{-1}}{1 - 0.8z^{-1}}$$

$$H(z) = \frac{2 + 2(-1)}{1 - 0.8(-1)} = 0$$

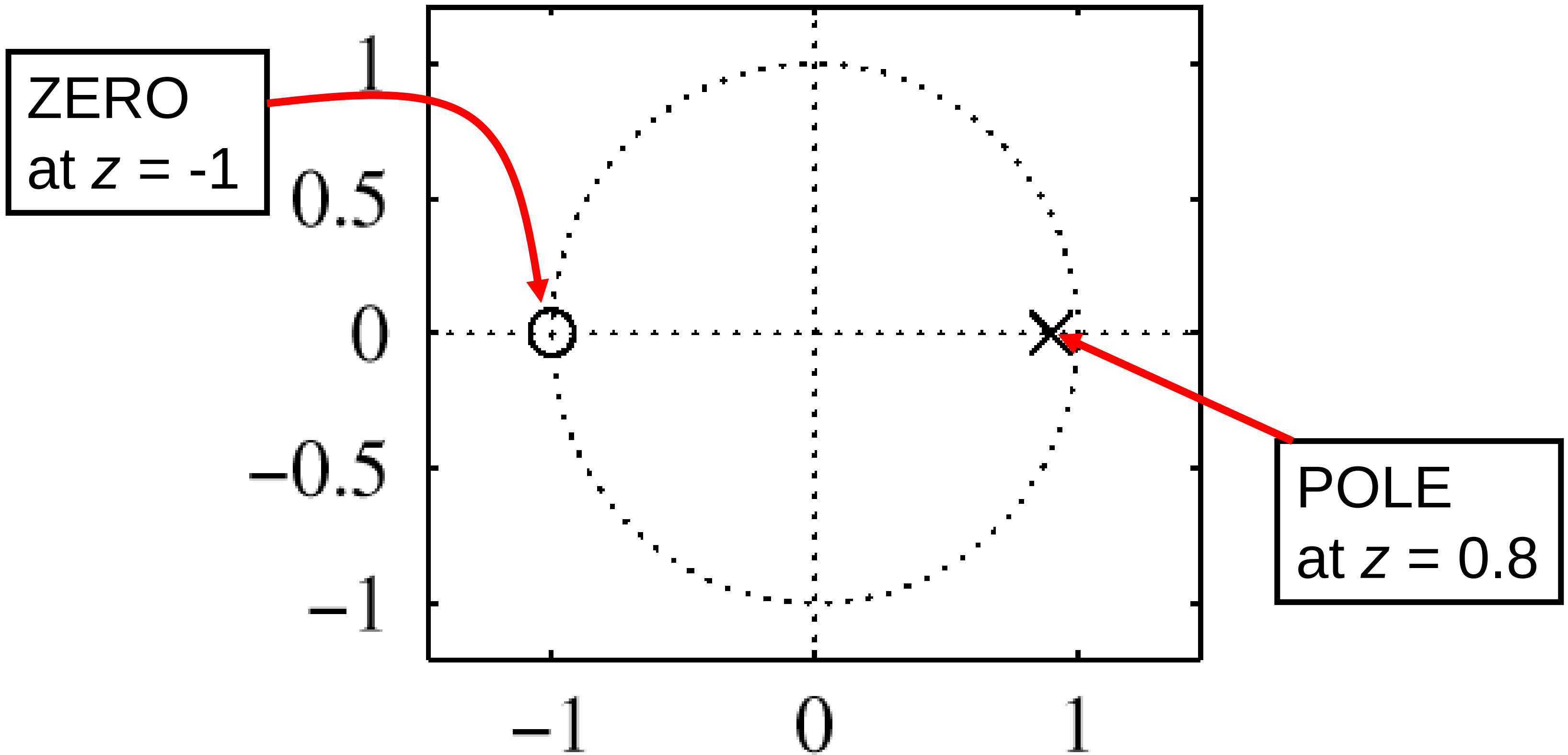
ZERO at z= -1

$$H(z) = \frac{2 + 2(\frac{4}{5})^{-1}}{1 - 0.8(\frac{4}{5})^{-1}} = \frac{\frac{9}{2}}{0} \rightarrow \infty$$

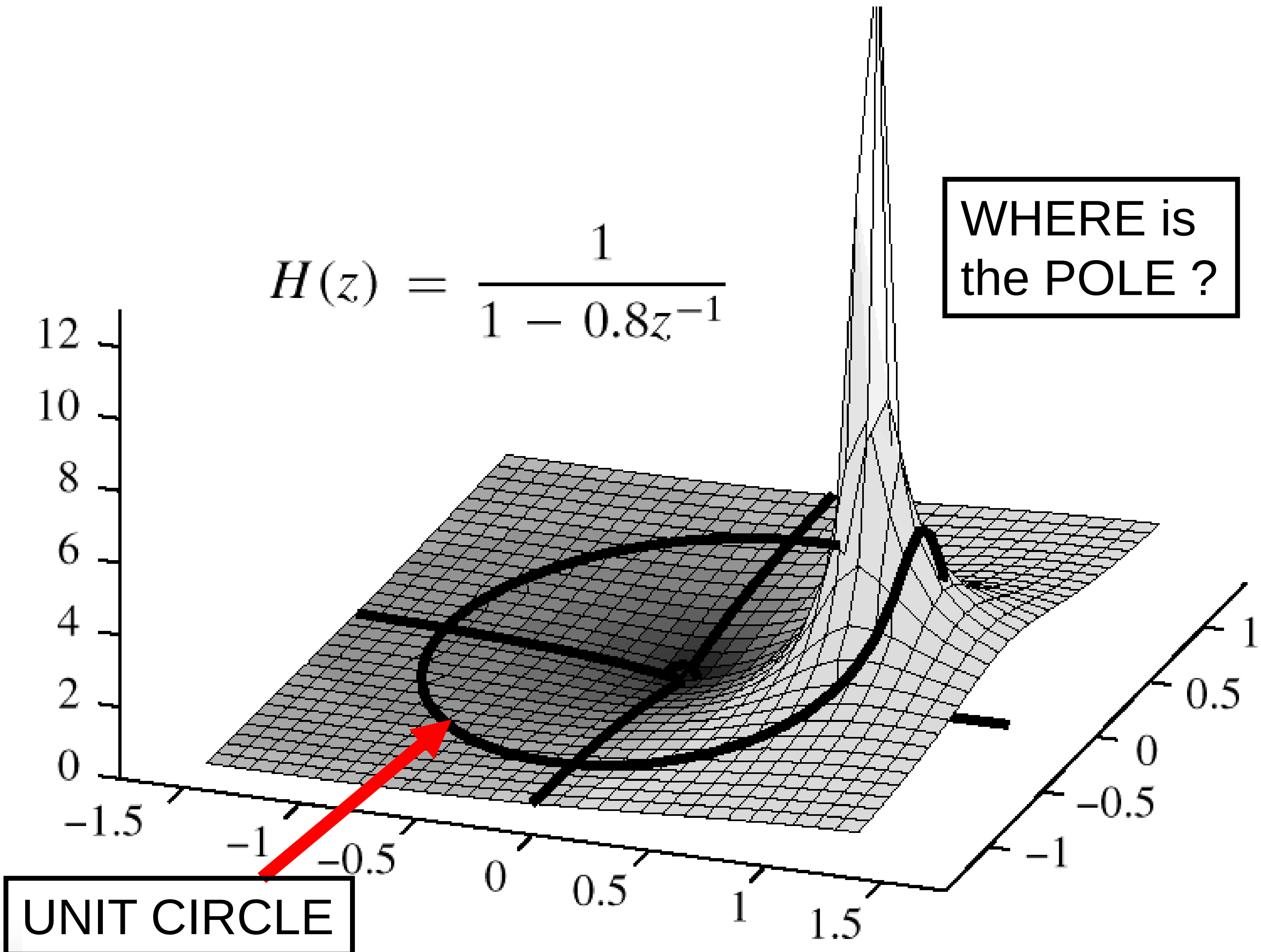
POLE at z=0.8

POLE-ZERO PLOT

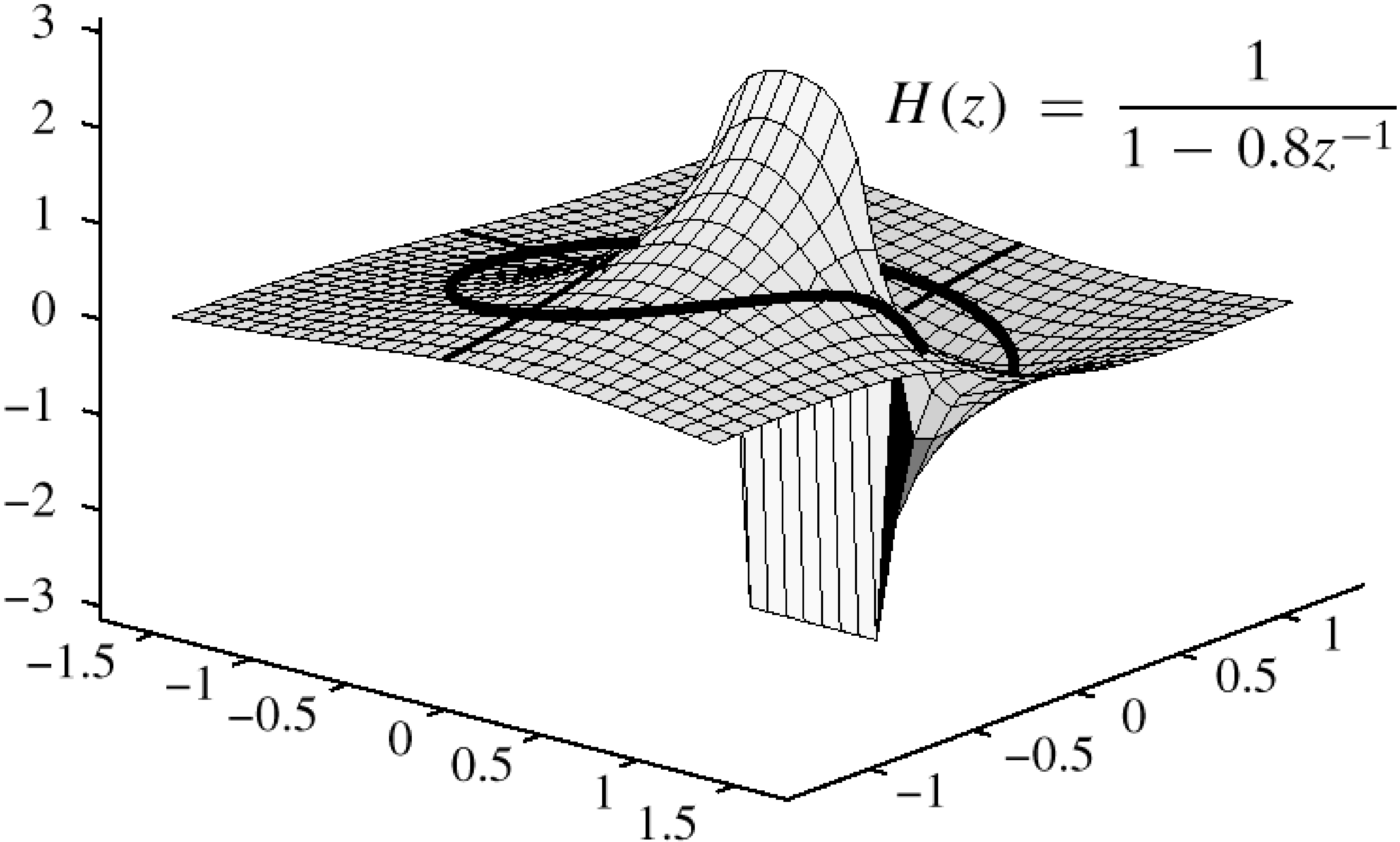
$$\boxed{\dfrac{2+2z^{-1}}{1-0.8z^{-1}}}$$



PLOT the magnitude of *H*(*z*).

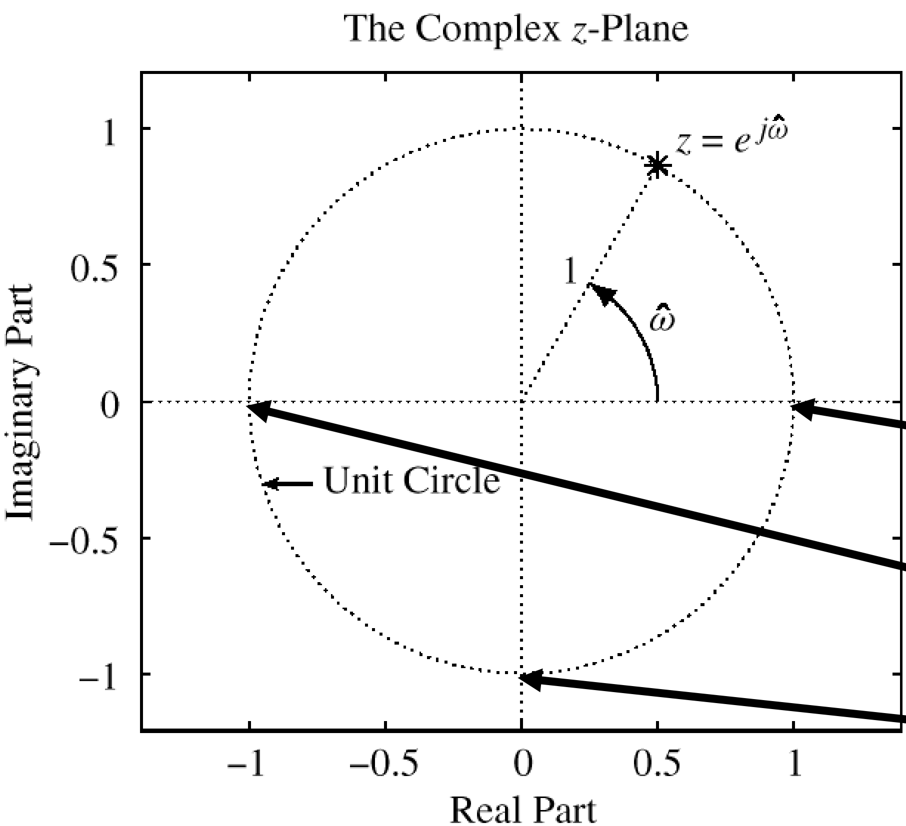


PLOT the phase of *H*(*z*).



UNIT CIRCLE

- MAPPING BETWEEN z and $\hat{\omega}$



$$\boxed{z = e^{j\hat{\omega}}}$$

$$z = 1 \quad \leftrightarrow \quad \hat{\omega} = 0$$

$$z = -1 \quad \leftrightarrow \quad \hat{\omega} = \pm\pi$$

$$z = \pm j \quad \leftrightarrow \quad \hat{\omega} = \pm\tfrac{1}{2}\pi$$

FREQUENCY RESPONSE

- EVALUATE on the UNIT CIRCLE.

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 - a_1 z^{-1}}$$

$$H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}} = \frac{b_0 + b_1 e^{-j\hat{\omega}}}{1 - a_1 e^{-j\hat{\omega}}}$$

FREQ. RESPONSE FORMULA

$$\boxed{H(z)=\frac{2+2z^{-1}}{1-0.8z^{-1}} \rightarrow H(e^{j\hat{\omega}})=\frac{2+2e^{-j\hat{\omega}}}{1-0.8e^{-j\hat{\omega}}}}$$

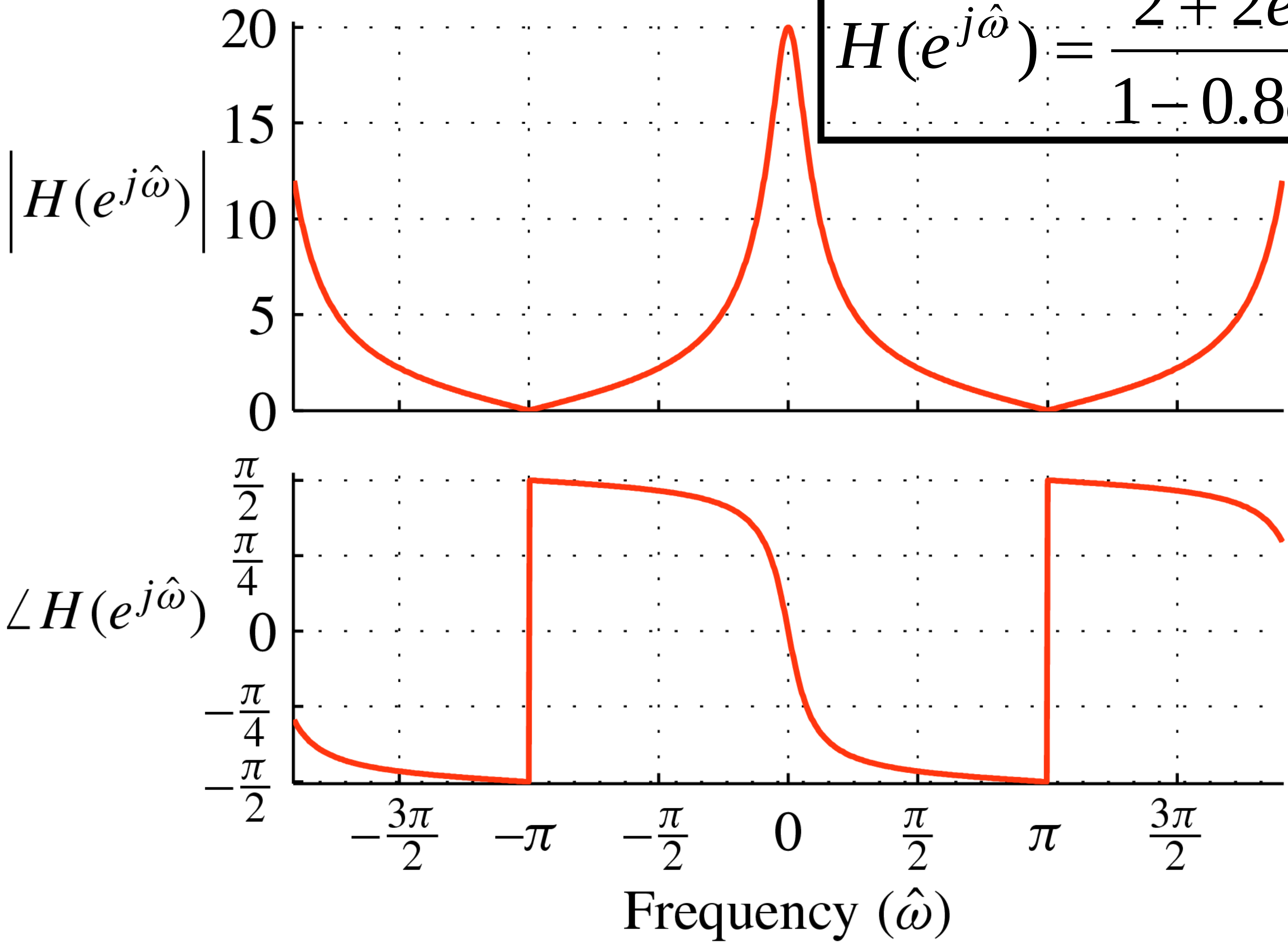
$$\boxed{\left|H(e^{j\hat{\omega}})\right|^2=}\boxed{\left|\frac{2+2e^{-j\hat{\omega}}}{1-0.8e^{-j\hat{\omega}}}\right|^2=\frac{2+2e^{-j\hat{\omega}}}{1-0.8e^{-j\hat{\omega}}}\cdot\frac{2+2e^{j\hat{\omega}}}{1-0.8e^{j\hat{\omega}}}}$$

$$\boxed{\frac{4+4+4e^{-j\hat{\omega}}+4e^{j\hat{\omega}}}{1+0.64-0.8e^{-j\hat{\omega}}-0.8e^{j\hat{\omega}}}=}\boxed{\frac{8+8\cos\hat{\omega}}{1.64-1.6\cos\hat{\omega}}}$$

$$\boxed{@\,\hat{\omega}=0,\quad \left|H(e^{j\hat{\omega}})\right|^2=\frac{8+8}{0.04}=400,\quad @\,\hat{\omega}=\pi\,?}$$

Frequency Response Plot

$$H(e^{j\hat{\omega}}) = \frac{2+2e^{-j\hat{\omega}}}{1-0.8e^{-j\hat{\omega}}}$$



SINUSOIDAL RESPONSE

- $x[n]$ is a SINUSOID. => $y[n]$ is a SINUSOID.
- Get the MAGNITUDE & the PHASE from $H(z)$.

$$\text{if} \quad x[n] = e^{j\hat{\omega}n}$$

$$\text{then} \quad y[n] = H(e^{j\hat{\omega}})e^{j\hat{\omega}n}$$

$$\text{where} \quad H(e^{j\hat{\omega}}) = H(z)\Big|_{z=e^{j\hat{\omega}}}$$

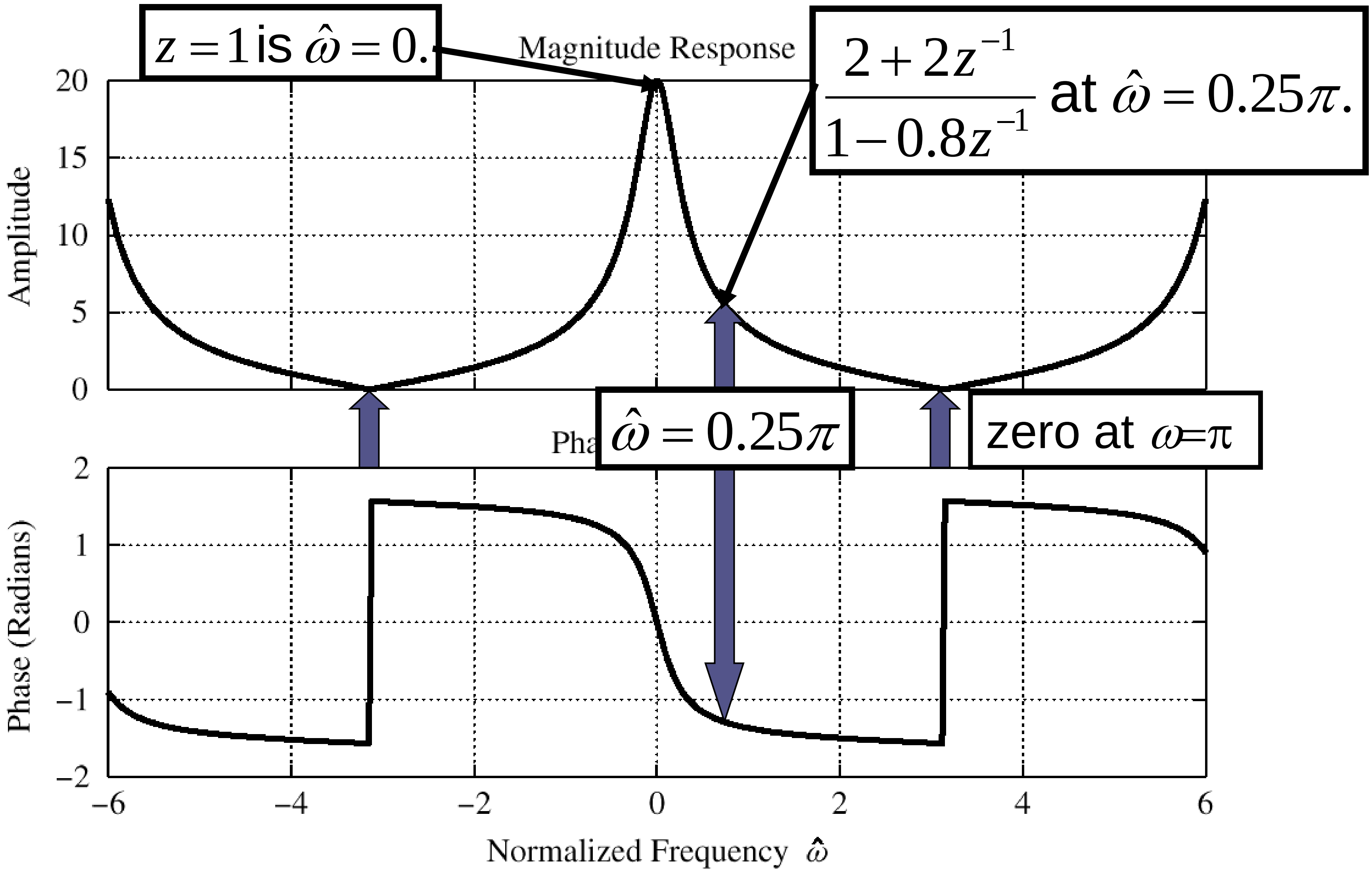
Example

- Given: $$\boxed{H(z) = \frac{2+2z^{-1}}{1-0.8z^{-1}}}$$
- Find the output, *y*[*n*], when $\boxed{x[n] = \cos(0.25\pi n)}$
  - Evaluate at $\boxed{z = e^{j0.25\pi}}$

$$\boxed{H(z) = \frac{2+2e^{-j0.25\pi}}{1-0.8e^{-j0.25\pi}} = 5.182e^{-j0.417\pi}}$$

$$\boxed{y[n] = 5.182\cos(0.25\pi n - 0.417\pi)}$$

Evaluate the FREQ. RESPONSE.



Inverse z-transform

- Given: $$\boxed{H(z)=\frac{2+2z^{-1}}{1-0.8z^{-1}}=\frac{2}{1-0.8z^{-1}}+\frac{2z^{-1}}{1-0.8z^{-1}}}$$
- Find the Impulse Response, $h[n]$.
  - Use the DELAY PROPERTY.

$$\boxed{h[n]=2(0.8)^n u[n]+2(0.8)^{n-1}u[n-1]}$$

- Use long division

$$\boxed{H(z)=\frac{2+2z^{-1}}{1-0.8z^{-1}}=\frac{4.5}{1-0.8z^{-1}}-2.5}$$

$$\boxed{h[n]=4.5(0.8)^n u[n]-2.5\delta[n]}$$

z-TRANSFORM TABLES

SHORT TABLE OF z-TRANSFORMS

	$x[n]$	$\Longleftrightarrow$	$X(z)$
1.	$ax_1[n] + bx_2[n]$	$\Longleftrightarrow$	$aX_1(z) + bX_2(z)$
2.	$x[n - n_0]$	$\Longleftrightarrow$	$z^{-n_0}X(z)$
3.	$y[n] = x[n] * h[n]$	$\Longleftrightarrow$	$Y(z) = H(z)X(z)$
4.	$\delta[n]$	$\Longleftrightarrow$	1
5.	$\delta[n - n_0]$	$\Longleftrightarrow$	z^{-n_0}
6.	$a^n u[n]$	$\Longleftrightarrow$	$\dfrac{1}{1 - az^{-1}}$

Thank you

- Homework
  - P-8.1, 3, 7, 8, 11(a,c), 13(S2,S4,S6), 14(S3,S5,S7), 15, 18
- Reading assignment
  - ~ Chapter 8