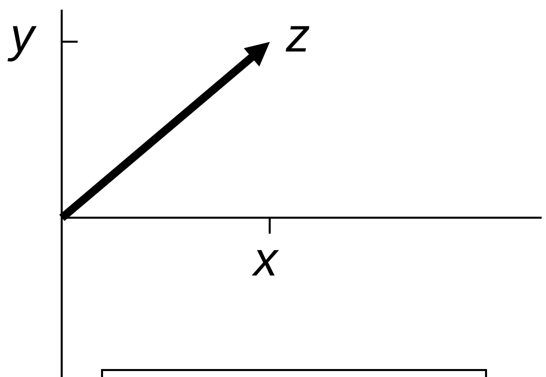
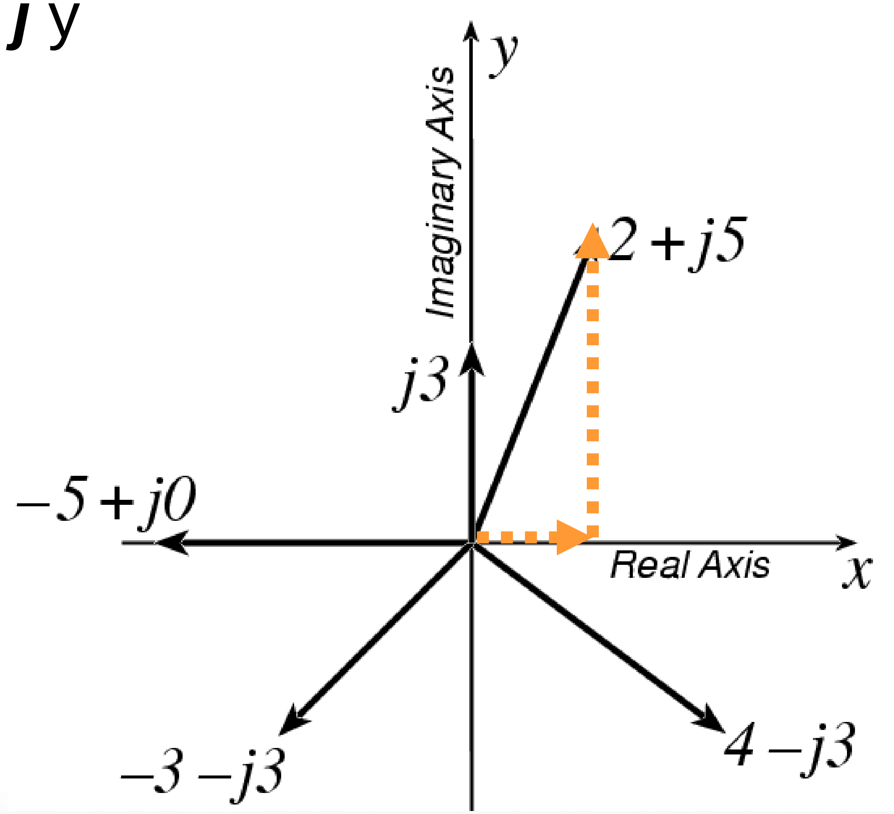
Chapter 2
Sinusoids

COMPLEX NUMBERS

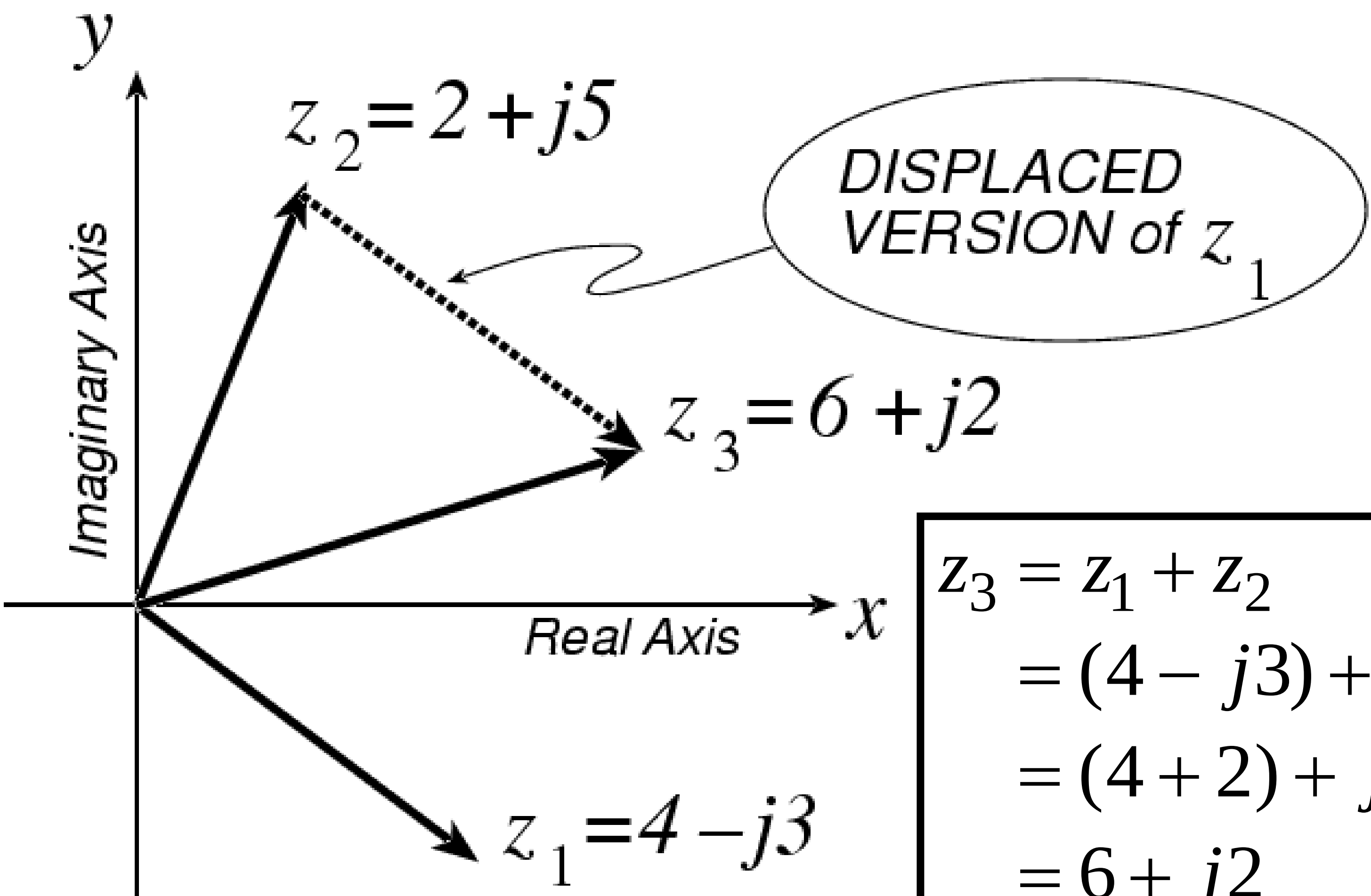
- To solve: $z^2 = -1$
  - $z = \boldsymbol{j}$
  - Math and physics use $z = \boldsymbol{i}$.
- Complex number: z = x + $\boldsymbol{j}$ y



Cartesian coordinate system



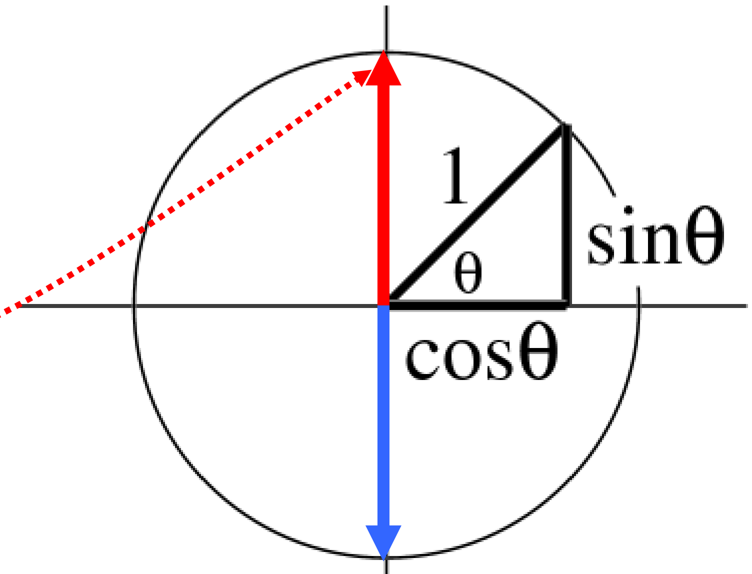
COMPLEX ADDITION = VECTOR Addition



$$
\begin{aligned}
z_3 &= z_1 + z_2 \\
&= (4 - j3) + (2 + j5) \\
&= (4 + 2) + j(-3 + 5) \\
&= 6 + j2
\end{aligned}
$$

POLAR FORM

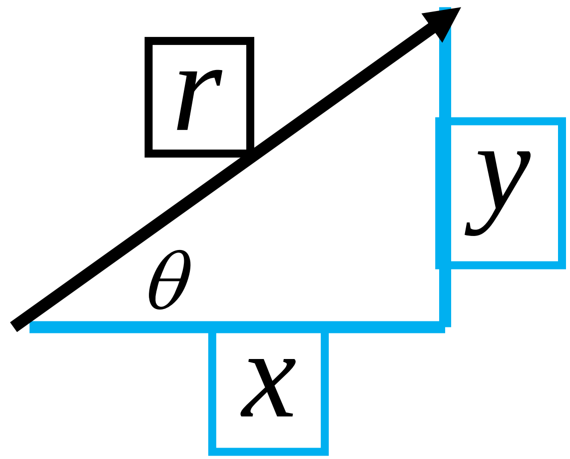
- Vector Form
  - **Length** =1
  - **Angle** = θ
- Common Values
  - ***j*** has an angle of 0.5π.
  - –1 has an angle of π.
  - – ***j*** has an angle of 1.5π.
  - Also, an angle of – ***j*** **could** be –0.5π = 1.5π –2π because the PHASE is **AMBIGUOUS**.



POLAR <–> RECTANGULAR

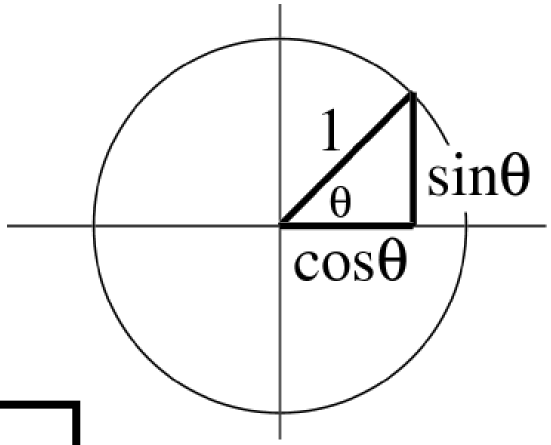
- Relate (x,y) to (r,θ).

$$\boxed{\begin{aligned} r^2 &= x^2 + y^2 \\ \theta &= \mathrm{Tan}^{-1}\left(\frac{y}{x}\right) \end{aligned}}$$



$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

Euler's FORMULA



- **Complex Exponential**
  - The real part is cosine.
  - The imaginary part is sine.
  - The magnitude is one.

$$\boxed{e^{j\theta} = \cos(\theta) + j\sin(\theta)}$$

$$\boxed{re^{j\theta} = r\cos(\theta) + jr\sin(\theta)}$$

- Using Euler's formula,

$$z = x + jy = re^{j\theta} = r\angle\theta$$

  - The magnitude of *z* (|z|): *r*
  - The argument of *z* (arg *z*) : *θ*

Euler's FORMULA

- Formula

$$e^{it} = \cos t + i \sin t$$

- Proof
  - Using Taylor Series,

$$\exp(x) = 1 + x + x^2/2! + x^3/3! + x^4/4! + x^5/5! + \ldots \langle 1 \rangle$$
$$\sin(x) = x - x^3/3! + x^5/5! - \ldots$$
$$\cos(x) = 1 - x^2/2! + x^4/4! - \ldots$$

  - Replacing x with jx in Eq. <1>,

$$\begin{aligned}\exp(jx) &= 1 + jx - x^2/2! - jx^3/3! + x^4/4! + jx^5/5! + \ldots \\ &= \left(1 - x^2/2! + x^4/4! + \ldots\right) + j\left(x - x^3/3! + x^5/5! - \ldots\right) \\ &= (\cos x) + j(\sin x)\end{aligned}$$

COMPLEX EXPONENTIAL

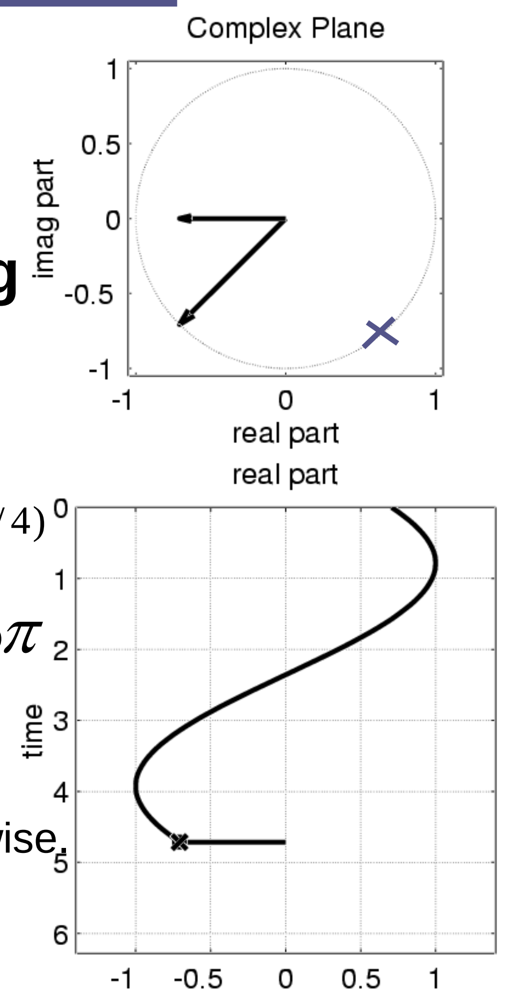
$$\boxed{e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)}$$

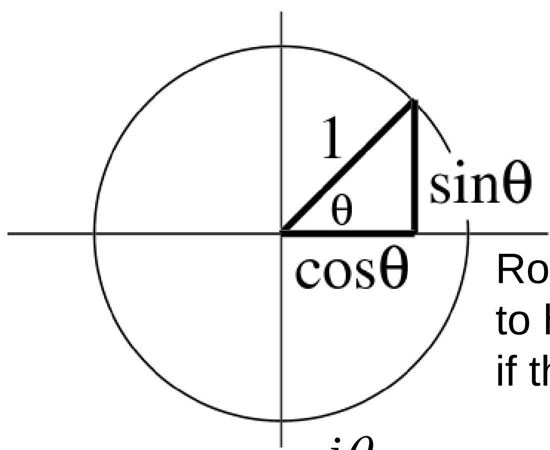
- Interpret this as a **Rotating Vector.**
  - $\boldsymbol{\theta = \omega t}$
  - Angle changes vs. time
  - ex: $\omega = 20\pi$ rad/s
  - Rotates 0.2π in 0.01 secs

Rotating Phasor

$$z(t) = e^{j(t-\pi/4)}$$

at $t = 1.5\pi$





Rotating phasors are said to have positive frequency if they rotate counterclockwise.

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

COMPLEX AMPLITUDE

- General Sinusoid

$$x(t) = A\cos(\omega t + \varphi) = \Re e\left\{Ae^{j\varphi}e^{j\omega t}\right\}$$

- **<u>Complex AMPLITUDE = *X* -> PHASOR</u>**

$$\boxed{z(t) = Xe^{j\omega t} \qquad X = Ae^{j\varphi}}$$

  - Complex exponential signal -> rotating phasor
- Then, any Sinusoid = REAL PART of $Xe^{j\omega t}$

$$x(t) = \Re e\left\{Xe^{j\omega t}\right\} = \Re e\left\{Ae^{j\varphi}e^{j\omega t}\right\}$$

Example: Complex Amplitude

- Find the COMPLEX AMPLITUDE for:

$$\boxed{x(t) = \sqrt{3}\cos(77\pi t + 0.5\pi)}$$

- Use EULER's FORMULA:

$$\begin{aligned} x(t) &= \Re e\left\{\sqrt{3}e^{j(77\pi t + 0.5\pi)}\right\} \\ &= \Re e\left\{\sqrt{3}e^{j0.5\pi}e^{j77\pi t}\right\} \end{aligned}$$

$$\boxed{X = \sqrt{3}e^{j0.5\pi}}$$

Why Complex Exponential?

- Avoid trigonometry
- Algebra, even complex, is EASIER!!!
- Can you recall $\cos(\theta_1+\theta_2)$?

- Use: real part of $e^{j(\theta_1+\theta_2)} = \cos(\theta_1+\theta_2)$

$$
\begin{aligned}
e^{j(\theta_1+\theta_2)} &= e^{j\theta_1} e^{j\theta_2} \\
&= (\cos\theta_1 + j\sin\theta_1)(\cos\theta_2 + j\sin\theta_2) \\
&= \boxed{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)} + j(\ldots)
\end{aligned}
$$

ADDITION of SINUSOIDS

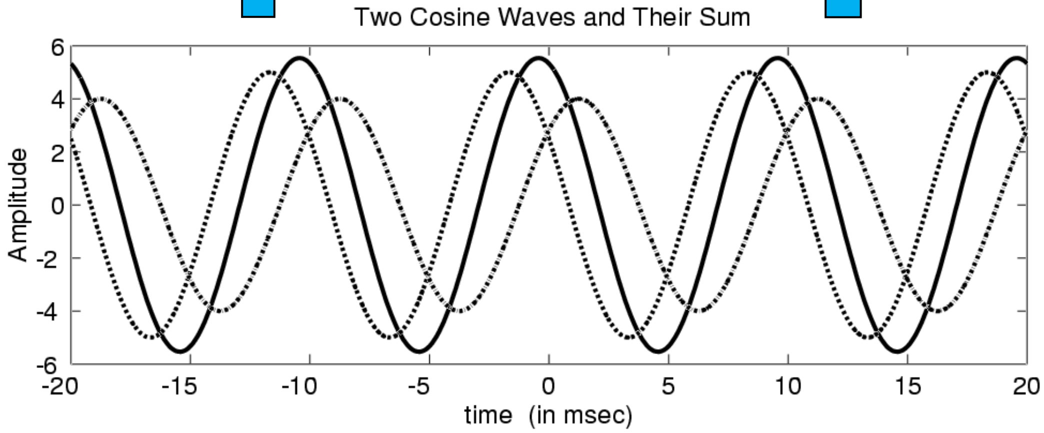
- ALL SINUSOIDS have the SAME FREQUENCY.
- HOW to GET {Amp,Phase} of the RESULT?

$$x_1(t) = 1.7\cos(2\pi(10)t + 70\pi/180)$$

$$x_2(t) = 1.9\cos(2\pi(10)t + 200\pi/180)$$

$$x_3(t) = x_1(t) + x_2(t)$$

$$= 1.532\cos(2\pi(10)t + 141.79\pi/180)$$



- The summed Sinusoid has the SAME frequency.

PHASOR ADDITION RULE

$$x(t) = \sum_{k=1}^{N} A_k \cos(\omega_0 t + \phi_k)$$

$$= A \cos(\omega_0 t + \phi)$$

Get the new complex amplitude by complex addition.

$$\sum_{k=1}^{N} A_k e^{j\phi_k} = A e^{j\phi}$$

Phasor Addition Proof

$$\sum_{k=1}^{N} A_k \cos(\omega_0 t + \phi_k) = \sum_{k=1}^{N} \Re e \left\{ A_k e^{j(\omega_0 t + \phi_k)} \right\}$$

$$= \Re e \left\{ \sum_{k=1}^{N} A_k e^{j\phi_k} e^{j\omega_0 t} \right\}$$

$$= \Re e \left\{ \left(\sum_{k=1}^{N} A_k e^{j\phi_k} \right) e^{j\omega_0 t} \right\}$$

$$= \Re e \left\{ \left(A e^{j\phi} \right) e^{j\omega_0 t} \right\} = A \cos(\omega_0 t + \phi)$$

Example 1: Addition of Sinusoids

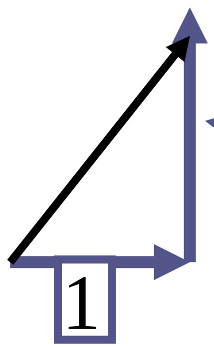
- ADD 2 SINUSOIDS:

$$x_1(t) = \cos(77\pi t)$$

$$x_2(t) = \sqrt{3}\cos(77\pi t + 0.5\pi)$$

- COMPLEX ADDITION:

$$1e^{j0} + \sqrt{3}e^{j0.5\pi}$$



$$j\sqrt{3} = \sqrt{3}e^{j0.5\pi}$$

$$1 + j\sqrt{3} = 2e^{j\pi/3}$$

- CONVERT back to cosine form:

$$x_3(t) = 2\cos(77\pi t + \tfrac{\pi}{3})$$

Example 2: Addition of Sinusoids

- Example revisited

$$x_1(t) = 1.7\cos(2\pi(10)t + 70\pi/180)$$

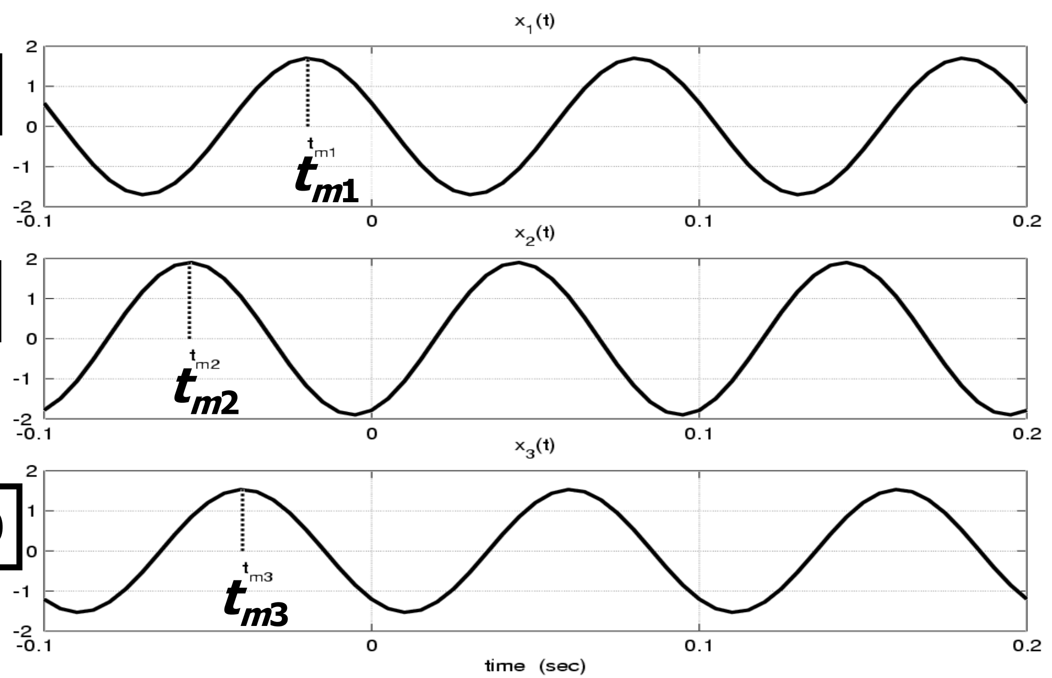
$$x_2(t) = 1.9\cos(2\pi(10)t + 200\pi/180)$$

$$x_3(t) = x_1(t) + x_2(t)$$
$$= 1.532\cos(2\pi(10)t + 141.79\pi/180)$$

$\boxed{x_1(t)}$

$\boxed{x_2(t)}$

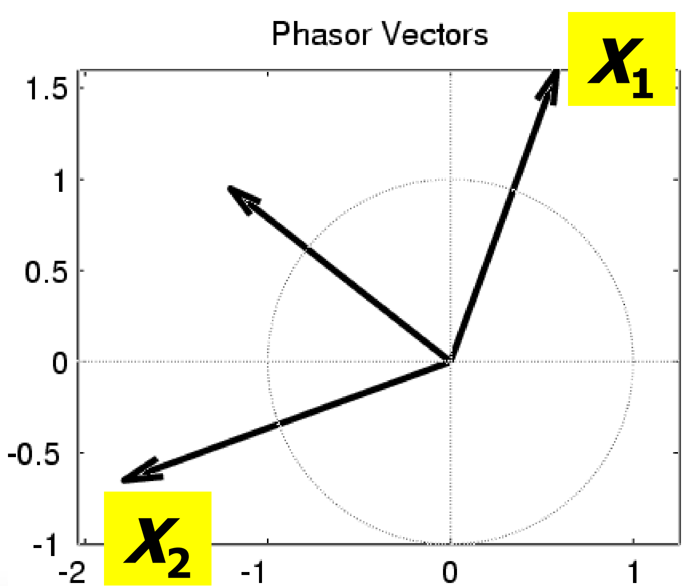
$\boxed{x_3(t) = x_1(t) + x_2(t)}$

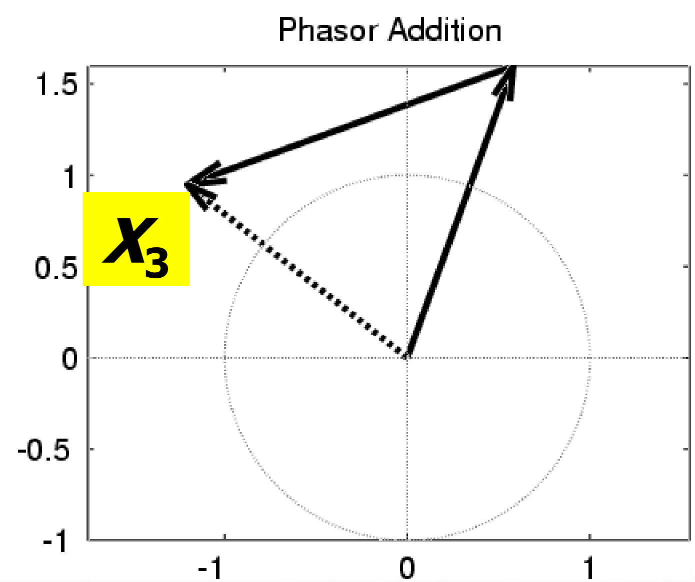


Phasor Addition

- Convert Polar to Cartesian.
  - $X_1 = 1.7e^{j\,70\pi/180} = 0.581 + j1.597$
  - $X_2 = 1.9e^{j\,200\pi/180} = -1.785 - j0.650$
  - $X_3 = X_1 + X_2 = -1.204 + j0.947$
- Convert back to Polar.
  - X_3 = 1.532 at an angle of 141.79π/180

VECTOR (PHASOR) ADD





Thank you

- Homework
  - P-2.16, 17, 18, 19, 21(a,b,c)
- Reading assignment
  - ~ Section 3-1