# Bandpass signals and Analytic Signals

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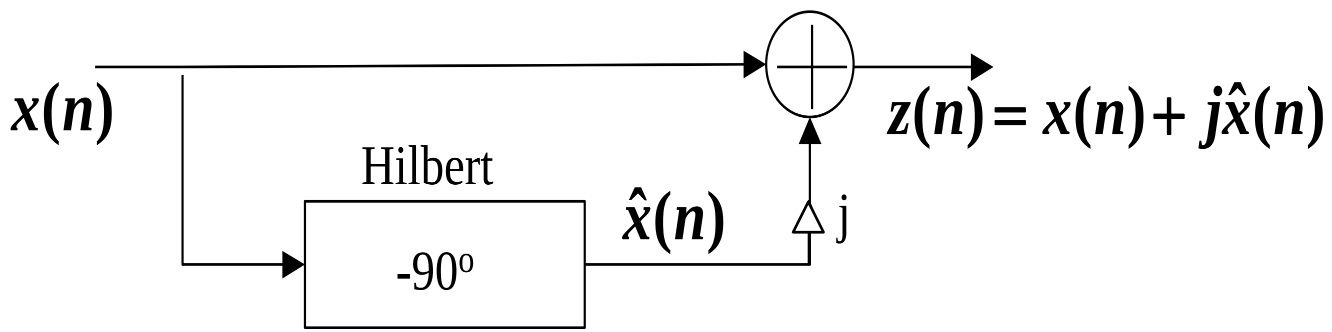
# Analytic Signal

## Definition

- The analytic signal of x(t) is defined as

$$z(t) = x(t) + j\hat{x}(t)$$

- $\boldsymbol{x(t)}$ and $\boldsymbol{\hat{x}(t)}$ are called "Hilbert pair".
  They are in quadrature relation.




Hilbert transform

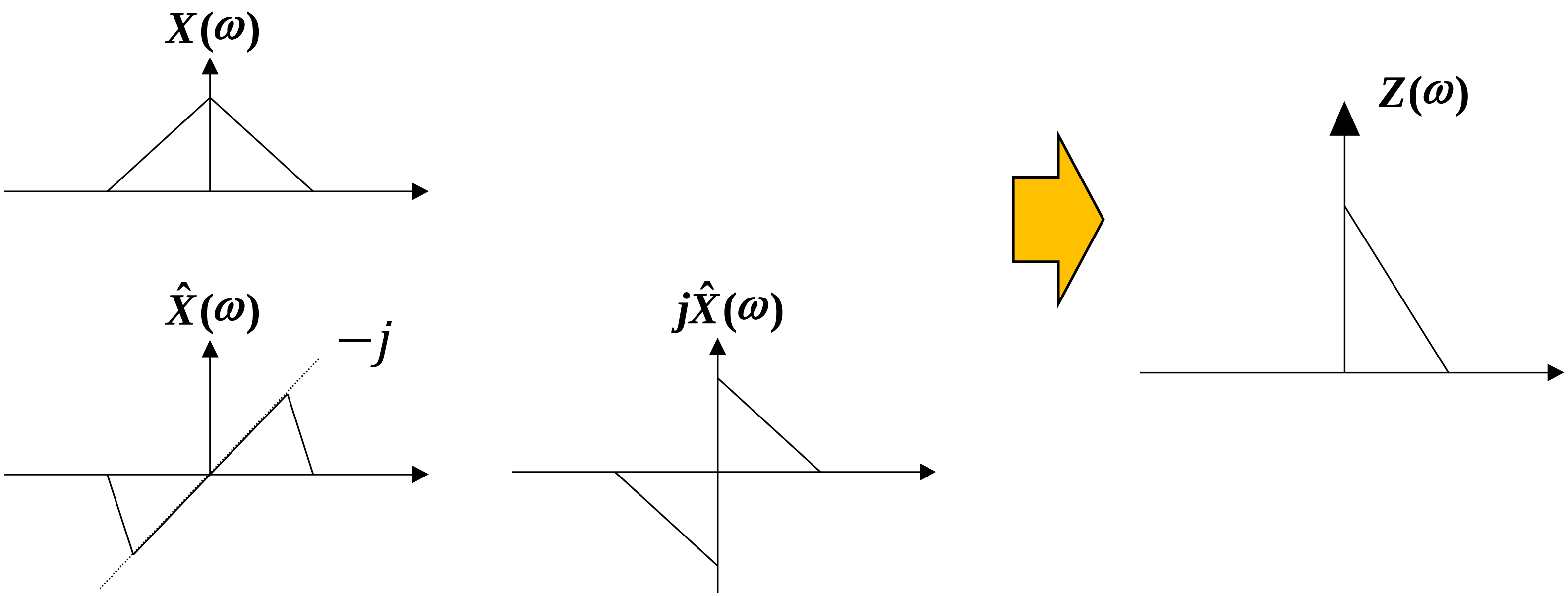
$$\mathcal{F}(H(u))(\omega) = (-i\ \mathrm{sgn}(\omega)) \cdot \mathcal{F}(u)(\omega)$$

# Analytic Signal

## Spectrum of the analytic signal

❖ **Baseband signal**

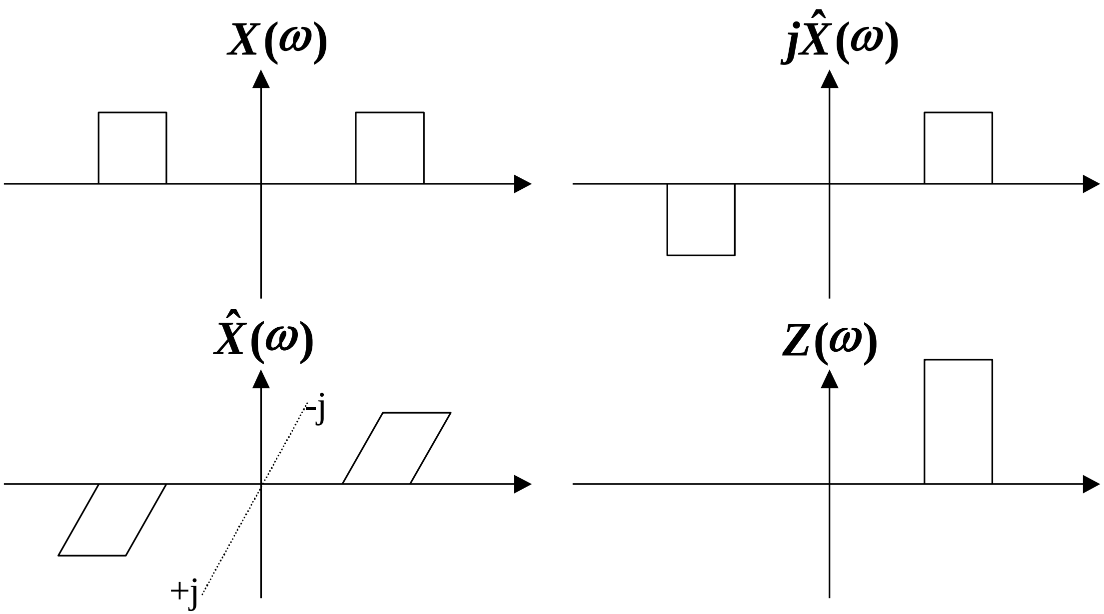
$$Z(\omega) = X(\omega) + j\hat{X}(\omega)$$



# Analytic Signal

## Spectrum of the analytic signal

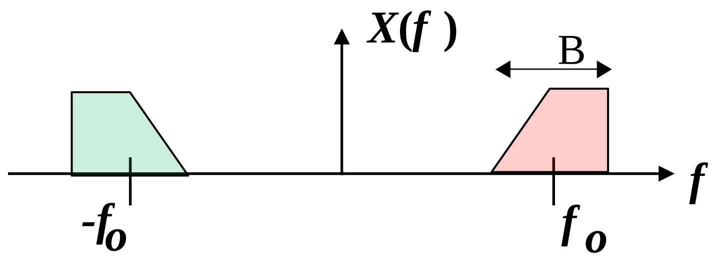
❖ **Bandpass signal**



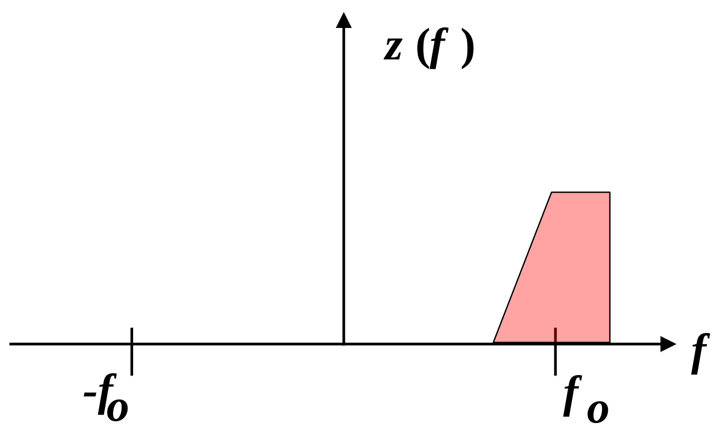
# Analytic Signal

## Quadrature demodulator

- ❖ **Bandpass signal (Analog): Spectrum**



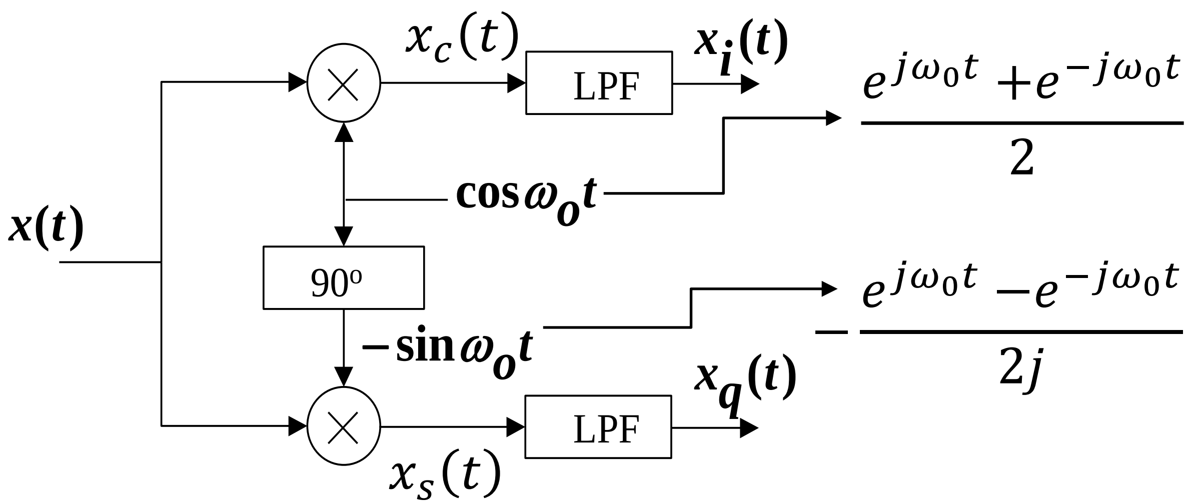

- ❖ **Spectrum of analytic signal**



# Analytic Signal

## Quadrature demodulator

### ❖ Bandpass signal (Analog): Spectrum




$$x_c(t) = x(t)cos\omega_0 t$$

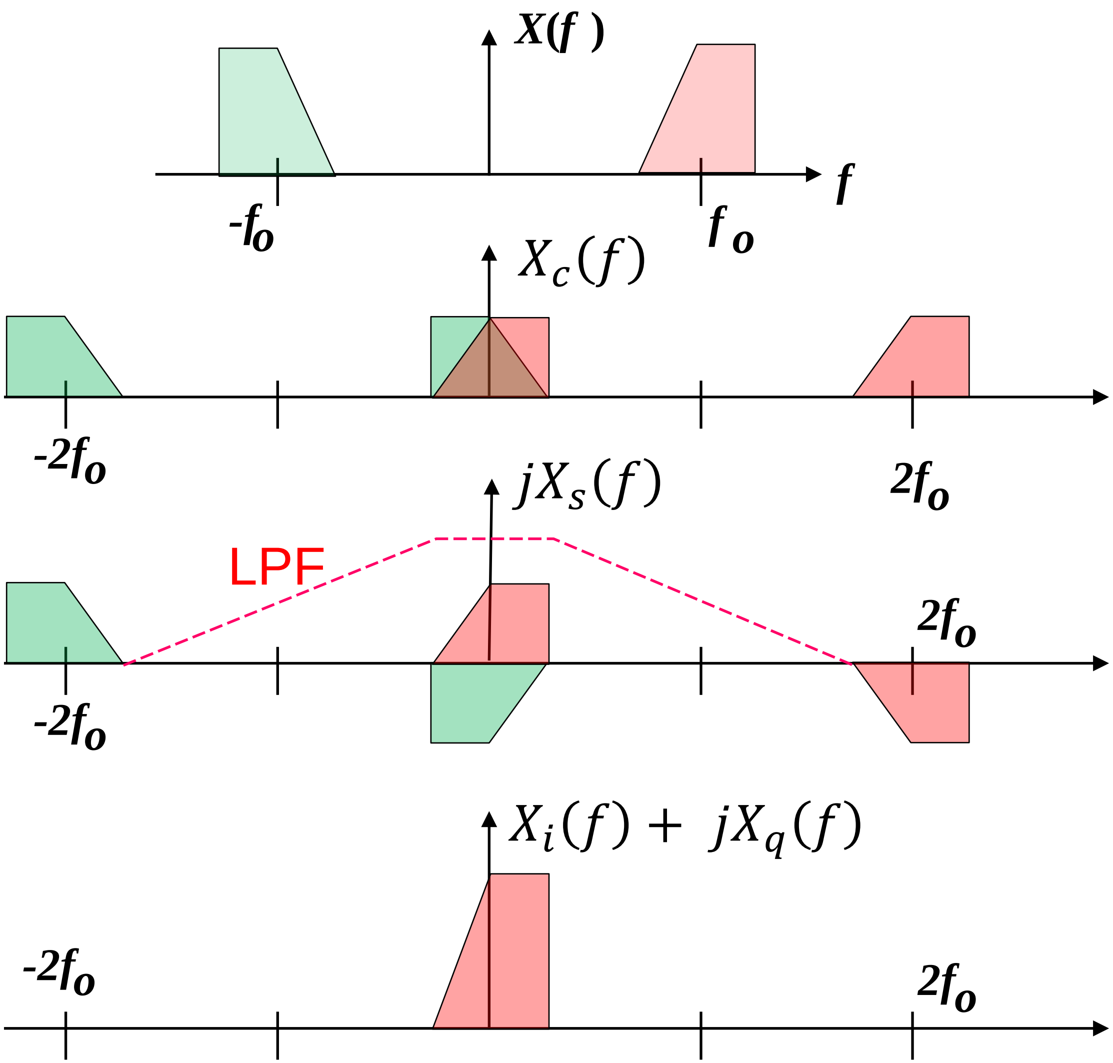
$$x_s(t) = -x(t)sin\omega_0 t$$

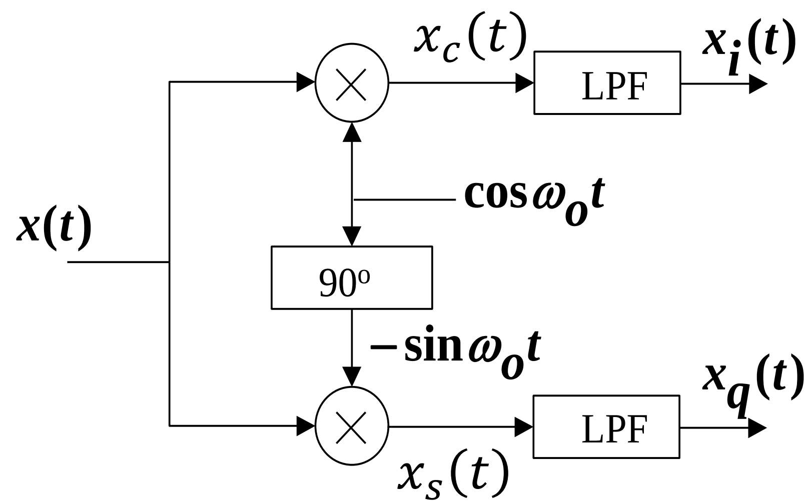
$$X_c(\omega) = \frac{1}{2\pi}X(\omega) * \frac{1}{2}(\delta(\omega-\omega_0)+\delta(\omega+\omega_0))$$

$$jX_s(\omega) = \frac{1}{2\pi}X(\omega) * \frac{1}{2}(-\delta(\omega-\omega_0)+\delta(\omega+\omega_0))$$

# Analytic Signal

## Quadrature demodulator



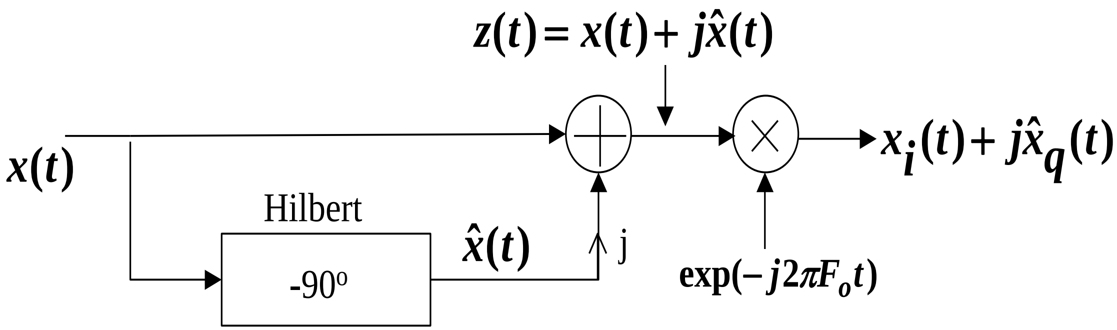

$$X_c(f) = X(f) * F\left(\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right)$$

$$jX_s(f) = X(f) * F\left(\frac{-e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right)$$

# Analytic Signal

## Quadrature demodulator

- $x_i(t) + jx_q(t)$ **can be obtained by shifting the positive freq component of $x(t)$ down to the base band.**

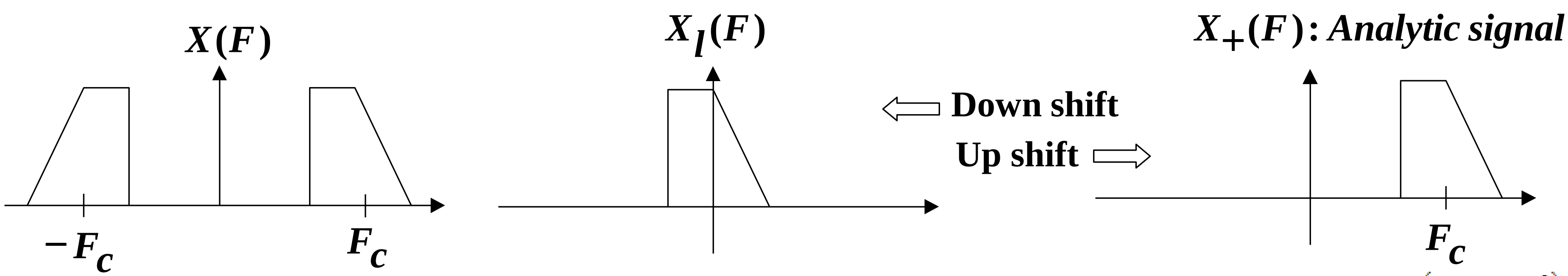



- **Envelope detection: Envelope of the bandpass signal can be obtained by**

$$Envelope\ of\ x(t) = |z(t)| = |z(t)e^{-j2\pi F_c t}| = |x_i(t) + jx_q(t)|$$

# Bandpass Signals

## Representation of bandpass signals




$$x(t) = a(t)\cos(2\pi F_c t + \theta) \qquad x_l(t) = a(t)e^{j\theta} \qquad x_+(t) = a(t)e^{j(2\pi F_c t + \theta)}$$

$$\Rightarrow \quad X_l(F) = X_+(F + F_c)$$

**Instantaneous frequency:**

$$\psi(t) = 2\pi F_c t + \theta$$

$$\hat{F}_c = \frac{1}{2\pi}\frac{d\psi(t)}{dt} = F_c$$

**Complex envelope:**

$$\begin{aligned} x_l(t) &= x_+(t)e^{-j2\pi F_c t} \\ &= [x(t) + j\hat{x}(t)]e^{-j2\pi F_c t} \end{aligned}$$

$$x(t) + j\hat{x}(t) = x_l(t)e^{j2\pi F_c t}$$

# Bandpass Signals

❖ **Complex envelope or Equivalent Lowpass signal**

$$x_l(t) = u_c(t) + ju_s(t)$$

$u_c(t)$ : *Inphase component* , $u_s(t)$ : *Quadrature component*

**a(t)cos(θ(t))** **a(t)sin(θ(t))**

❖ **Representation I: From the above results, we can get a general expression for bandpass signals as follows:**

$$x(t) = u_c(t)\cos 2\pi F_c t - u_s(t)\sin 2\pi F_c t$$

$$\hat{x}(t) = u_c(t)\sin 2\pi F_c t + u_s(t)\cos 2\pi F_c t$$

$\Leftarrow$

$$x_+(t) = x_l(t)e^{j2\pi F_c t}$$

$$x(t) = Re\left[x_l(t)e^{j2\pi F_c t}\right]$$

❖ **Representation II**

$$x_l(t) = x_+(t)e^{-j2\pi F_c t} \quad (x_+(t) = x(t) + j\hat{x}(t))$$

$$x_l(t) = a(t)e^{j\theta(t)}$$

$$u_c(t) = x(t)\cos 2\pi F_c t + \hat{x}(t)\sin 2\pi F_c t$$

$$u_s(t) = x(t)\sin 2\pi F_c t + \hat{x}(t)\cos 2\pi F_c t$$

$$a(t) = \sqrt{u_c^2(t) + u_s^2(t)} \ : \ \textit{Envelope}$$

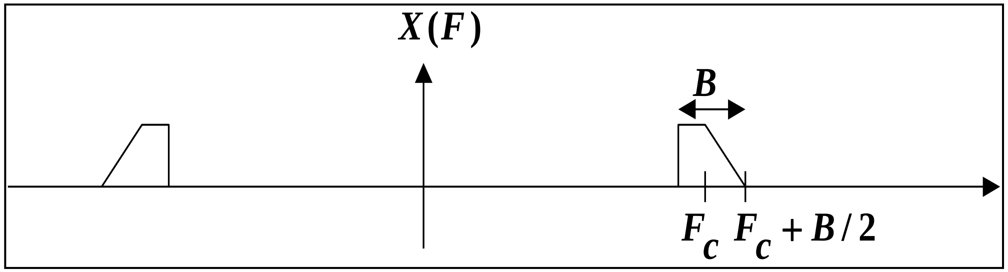
$$\theta(t) = \tan^{-1}\frac{u_s(t)}{u_c(t)} \ : \ \textit{Phase}$$

❖ **Representation III**

$$x(t) = Re\left[x_l(t)e^{j2\pi F_c t}\right] = a(t)\cos\left[2\pi F_c t + \theta(t)\right]$$

# Sampling of Bandpass Signals

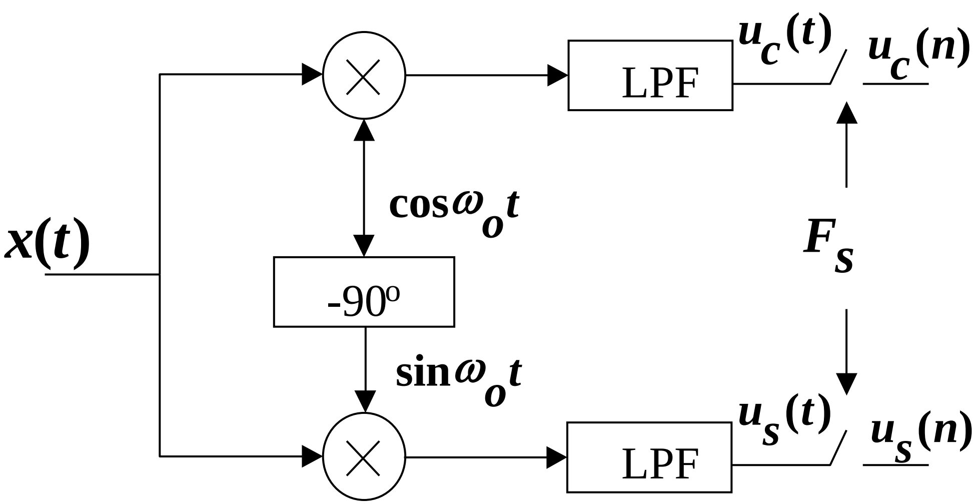
## General Uniform Sampling theorem.




$$F_s \geq F_{Nyq} = 2F_c + B$$

## Sampling techniques for Bandpass signals

❖ **Quadrature sampling**



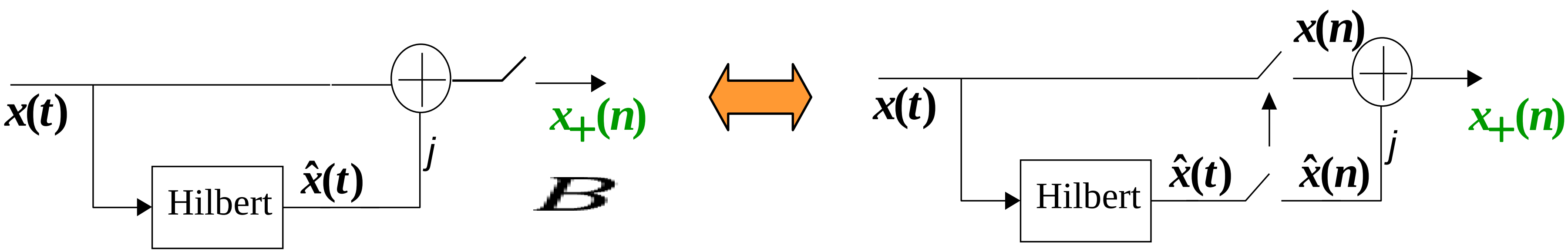

$$F_s \geq B$$

*Effective data rate = 2B*

# Sampling of Bandpass Signals

## Sampling techniques for Bandpass signals

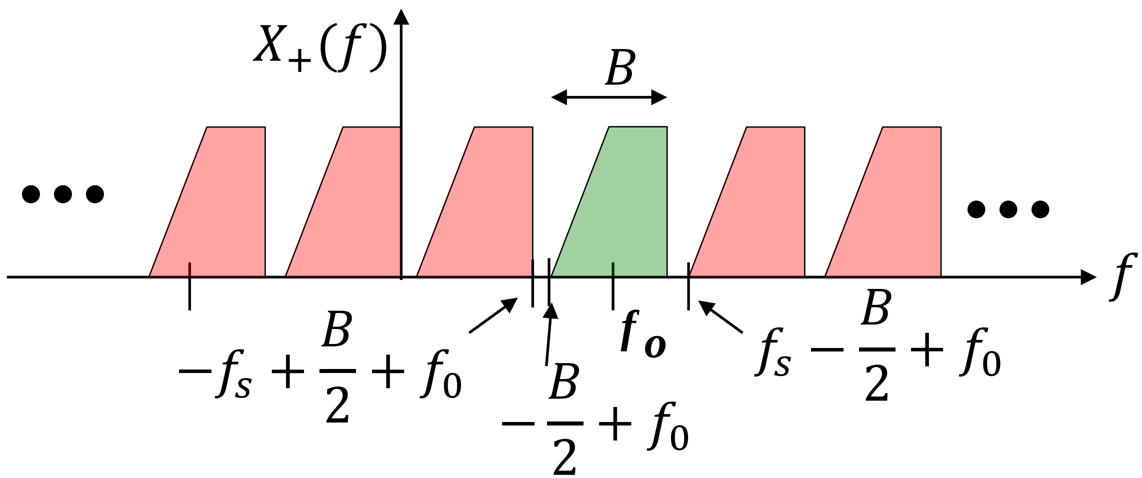
❖ **Hilbert sampling**

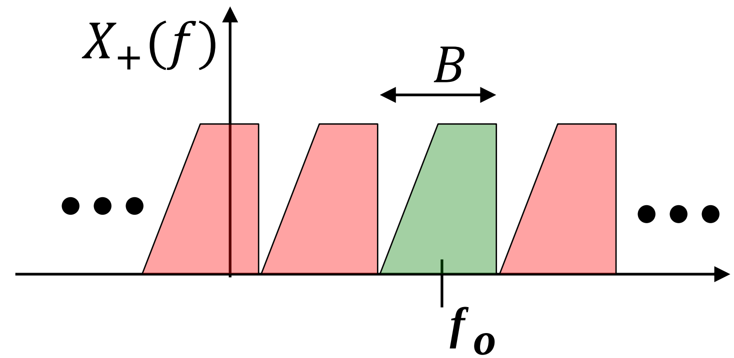



$$-f_s+\frac{B}{2}+f_0<-\frac{B}{2}+f_0 \Leftrightarrow f_s>B$$

*Effective data rate = 2B*

$$f_s=B$$

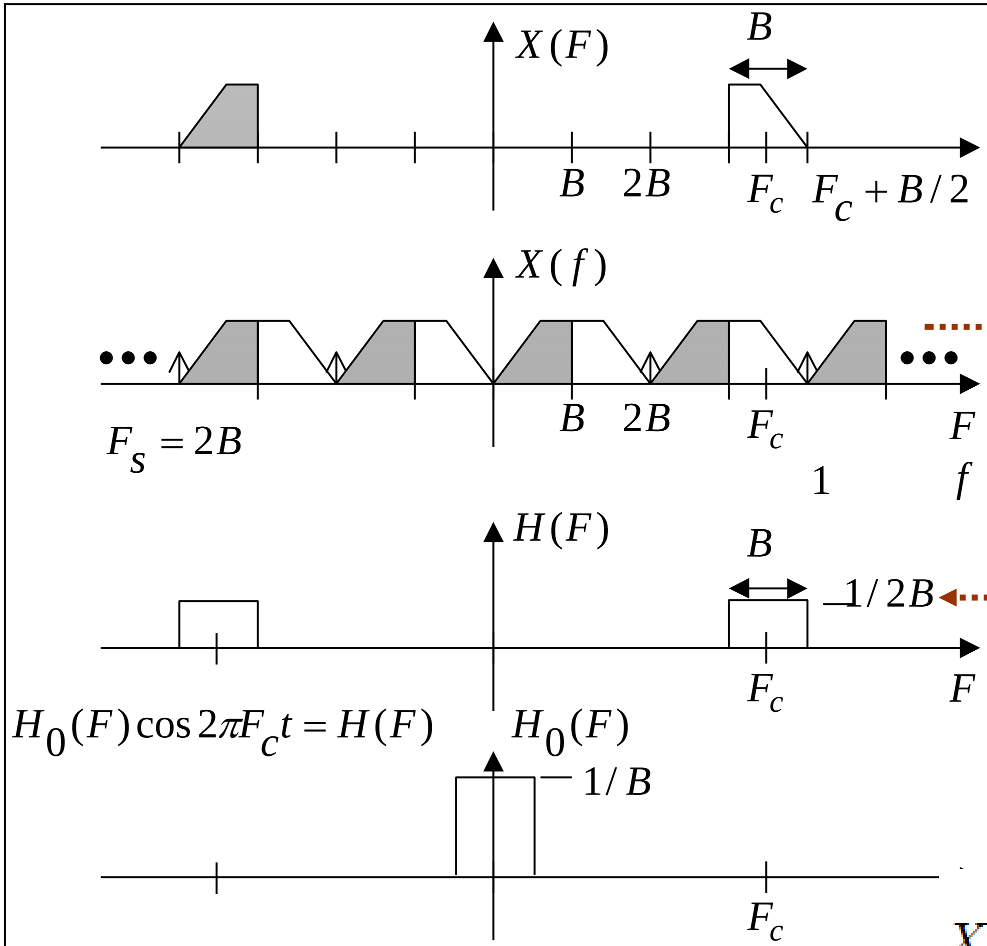


# Sampling of Bandpass Signals

## Sampling techniques for Bandpass signals

❖ Uniform sampling: Special case




$$F_c - \frac{B}{2} = kB$$

$$F_c + \frac{B}{2} = (k+1)B$$

ADC rate? 2B

How to get H(F)

$$B = 1/2T$$

$$h_0(t) = F^{-1}\left[\frac{1}{B}\, rect\left(\frac{F}{B}\right)\right] = \text{sinc}Bt$$

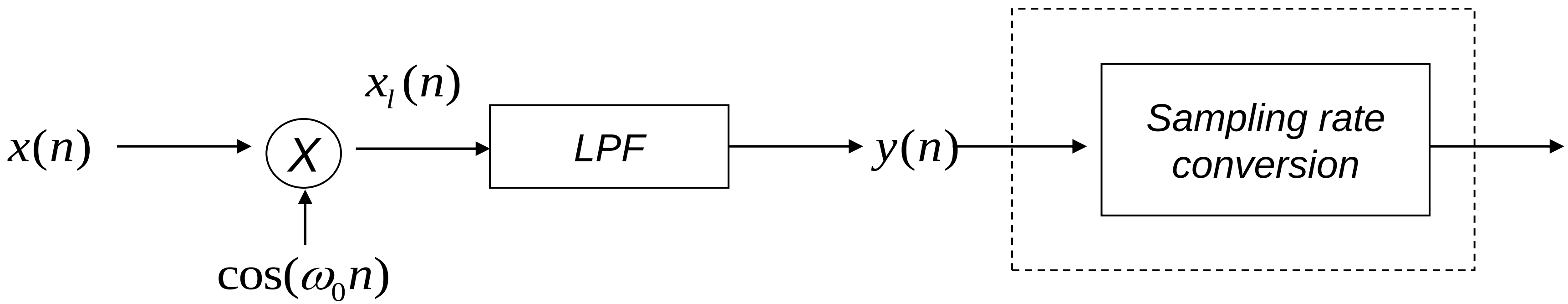
$$h(t) = h_0(t) \cdot \cos 2\pi F_c t$$

$$X(T) = \sum_{-\infty}^{\infty} \frac{\sin(\pi/2T)(t - mt)}{(\pi/2T)(t - mT)} \cos 2\pi F_c (t - mT)$$

# Envelop Detection

## Synchronous detection

- How to decide $\omega_0$ ?
- LPF Spec and Design?
- Pros and Cons of this scheme?
- Overall gain




$$x(n) = a(n)\cos(\omega_0 n + \varphi)$$

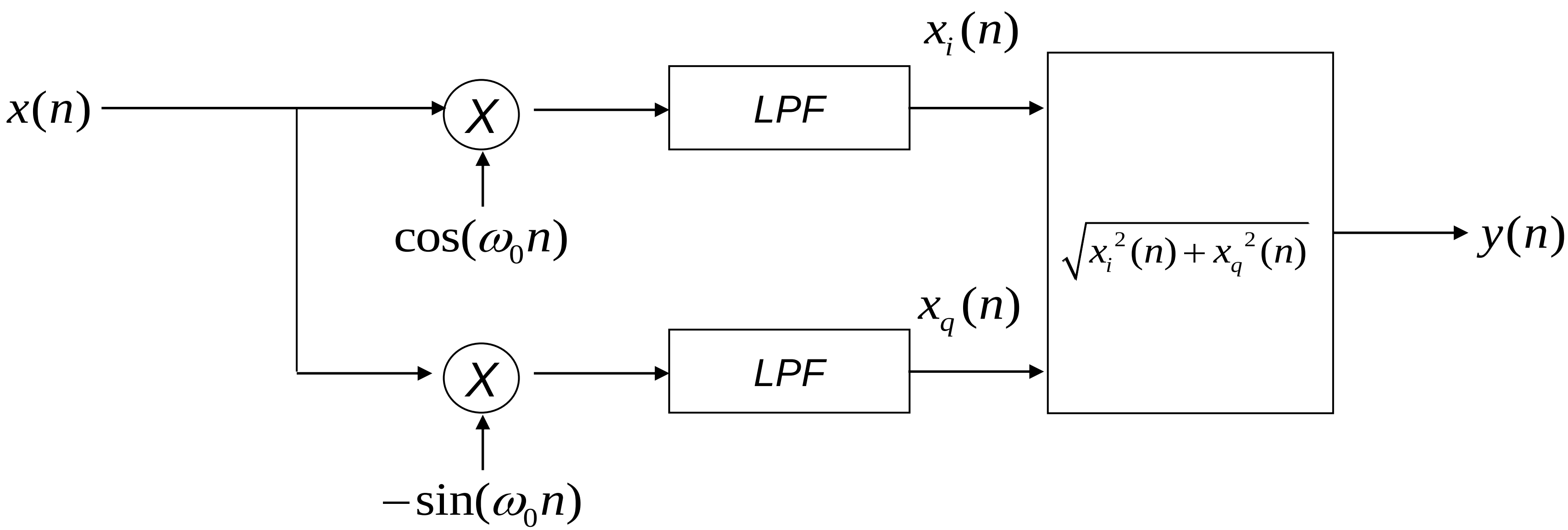
$$y(n) = \frac{1}{2}a(n)\cos(\varphi)$$

If $\varphi = \pi/2$?

# Envelop Detection

## Quadrature demodulator

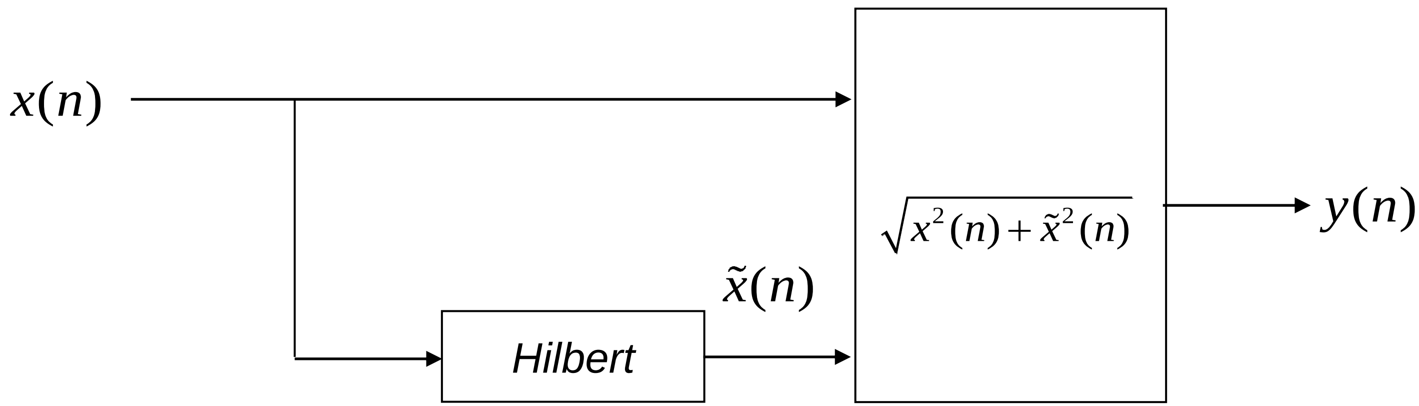
- Pros and cons of this scheme?
- Overall gain



# Envelop Detection

## Hilbert detector

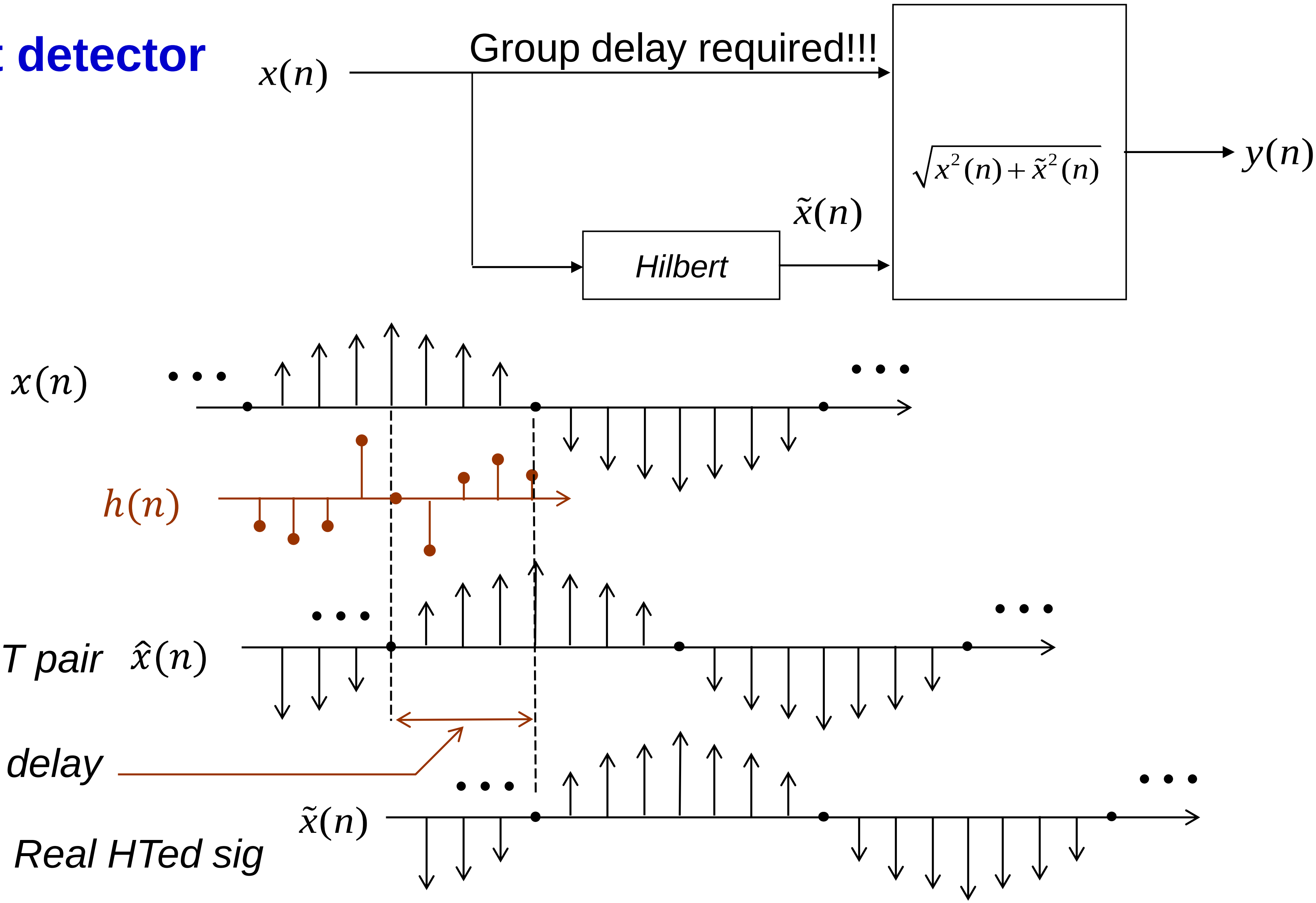
- Pros and cons of this scheme?
- Overall gain?




- When 31-tap Hilbert transformer is used, will the above scheme work?
- What happens if 30-tap filter is used?
- You should also consider the freq. response of the real Hilbert filter!

# Envelop Detection

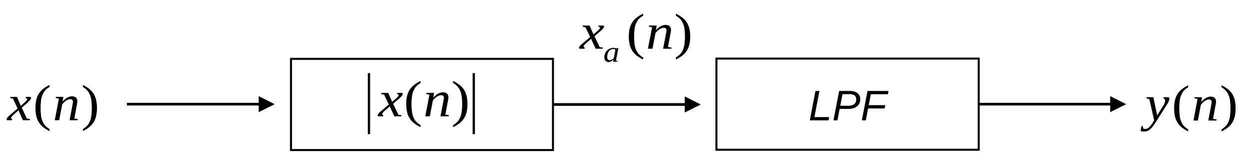
## Hilbert detector



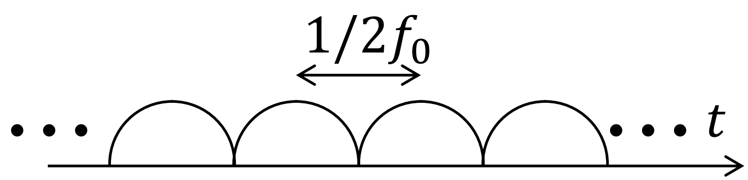
# Envelop Detection

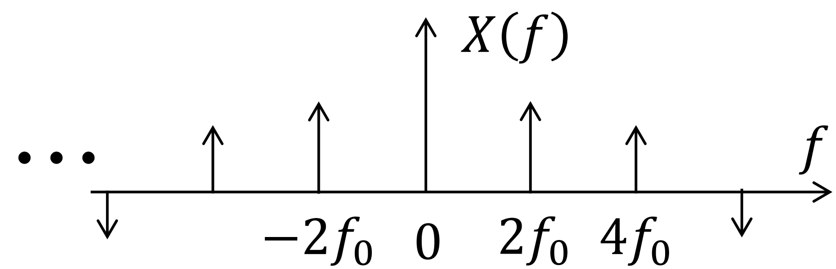
## Digital rectifier

- ❖ **Pros and cons of this scheme?**
- ❖ **Overall gain?**




$$x(t) = a(t)\cos(2\pi f_0 t) \Rightarrow |x(t)| = a(t)|\cos(2\pi f_0 t)|$$




$$X_a(\theta) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(\omega - \frac{2\pi k}{T})$$

Repeated by $F_s$

# Envelop Detection

## Square-law detector

- Pros and cons of this scheme?
- Overall gain