Discrete-Time Fourier Transform, Sampling Theorem, and Reconstruction

Fourier Series for CT periodic signal

Definition

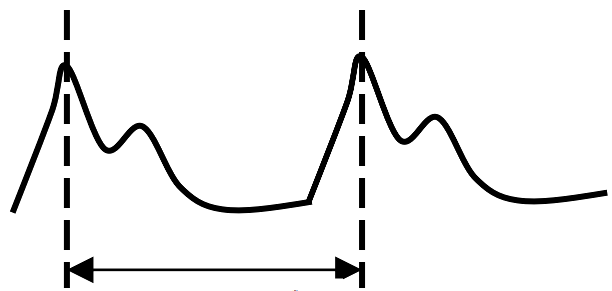
$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t} \qquad c_k = \frac{1}{T_p}\int_{T_p} x(t) e^{-j2\pi k F_0 t} dt$$

- **Period:**

$$T_p = \frac{1}{F_0}$$

- **For real signal,**

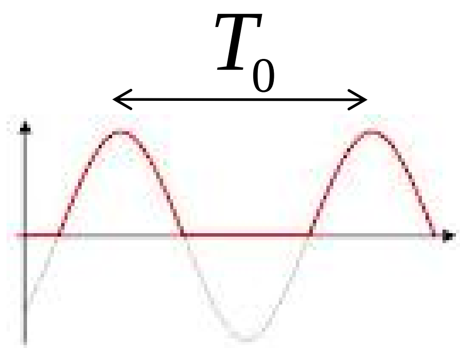
$$c_{-k} = c_k^*$$

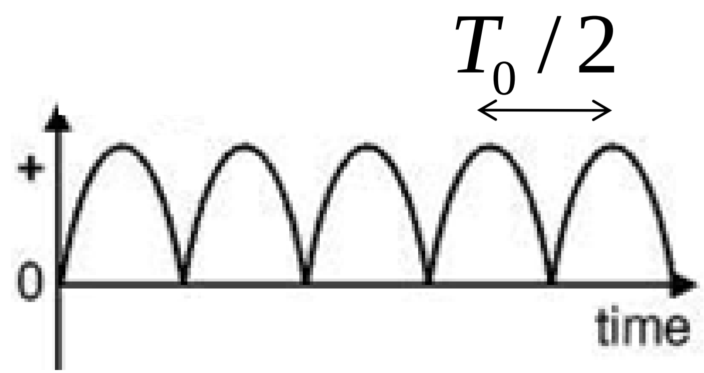


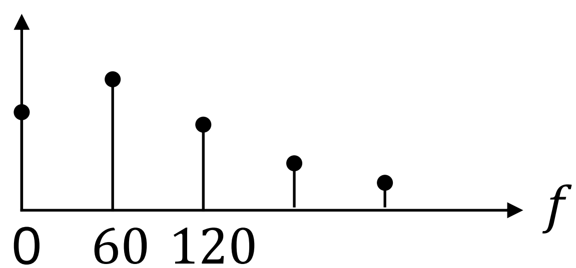
$$T_p = \frac{1}{F_0}$$

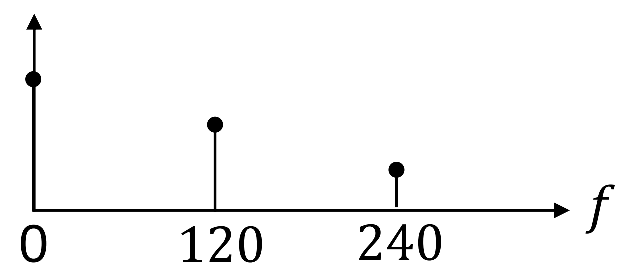
Fourier Series for CT periodic signal

Example: Full, half – wave recifiers









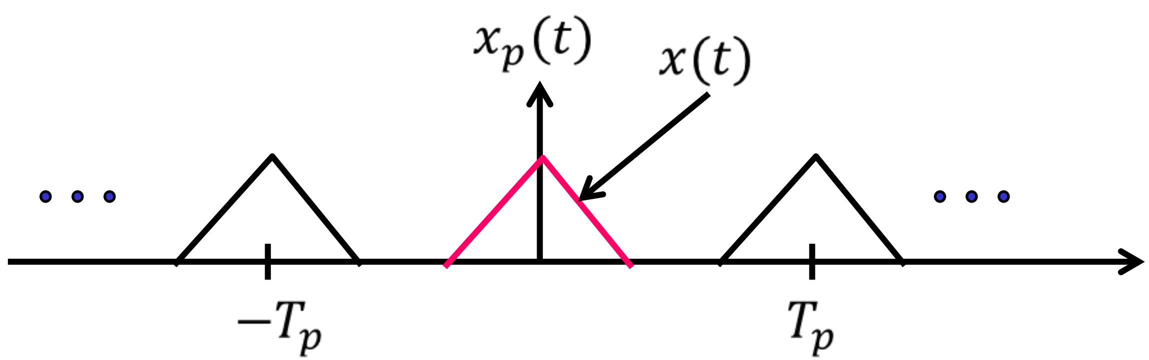
CTFT vs. CTFS

To get aperiodic signal from periodic signal:

$$x(t) = \lim_{T_p \to \infty} x_p(t) \qquad x_p(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t} \qquad F_0 = \frac{1}{T_p}$$

Conversely, we can model a periodic as follows :

$$x_p(t) = \tilde{x}(t) = x(t + kT_p)$$



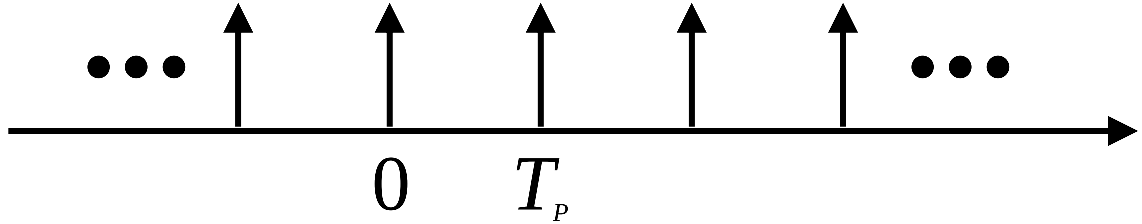
Questions:

1. How do they different in their spectral components?
2. Are their spectra similar ? How are they different?

CTFT vs. CTFS

Impulse train

$$\delta_p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_p)$$



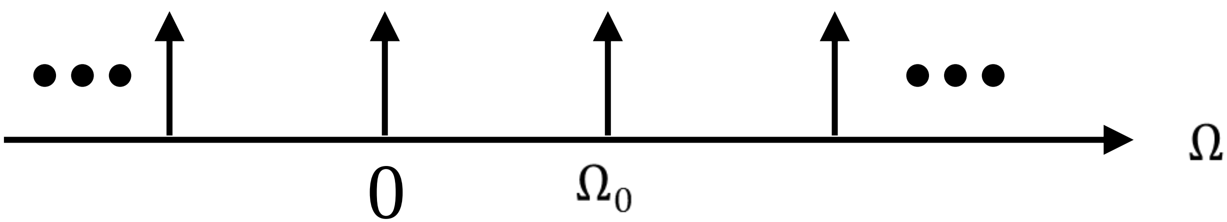
❖ **FS expansion**

$$c_k = \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} \delta(t) e^{-jk\Omega_0 t} dt = \frac{1}{T_p}$$

$$\Rightarrow \quad \delta_p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_p) = \frac{1}{T_p} \sum_{k=-\infty}^{\infty} e^{jk\Omega_0 t}$$

(Refer to p.8 in ch2_1_1.)

❖ **FT :** $\delta_p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_p) \longleftrightarrow \Omega_0 \sum_{n=-\infty}^{\infty} \delta(\Omega - n\Omega_0)$

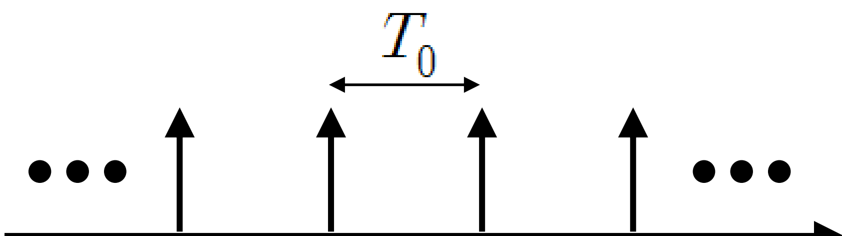


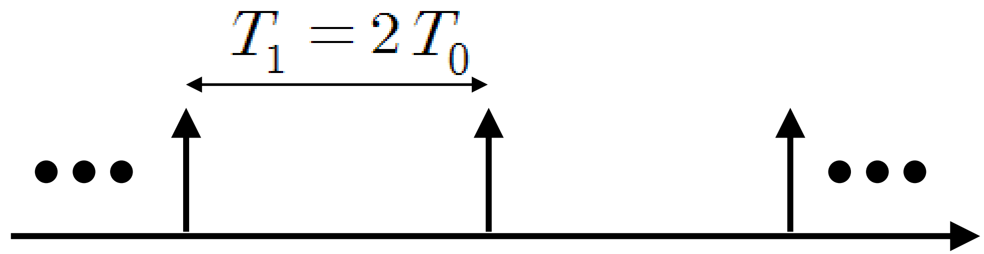
$$\boxed{\Omega_0 = 2\pi F_0 \ , \quad F_0 = \frac{1}{T_p}}$$

CTFT vs. CTFS

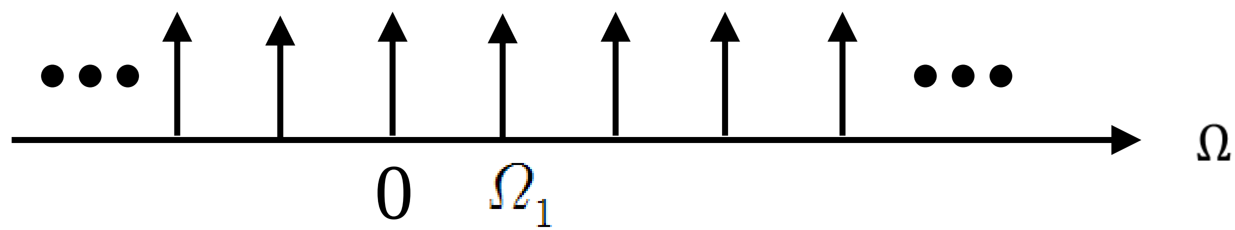
Impulse train

$$\boxed{\delta_p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_p) \longleftrightarrow \Omega_0 \sum_{n=-\infty}^{\infty} \delta(\Omega - n\Omega_0) \qquad \Omega_0 = 2\pi F_0 \;, \quad F_0 = \frac{1}{T_p}}$$

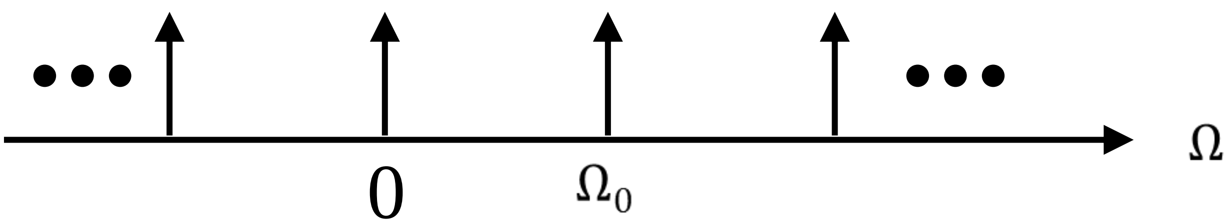




$$\Omega_1 = 2\pi F_1 (= 2\pi / T_1) = 2\pi F_0 / 2$$



$$\Omega_0 = 2\pi F_0 = 2\pi / T_0$$



CTFT vs. CTFS

CTFT vs. CTFS

- **Consider the periodic signal,**

$$f_p(t) = \sum_{n=-\infty}^{\infty} f(t - nT_p) = f(t) * \left\{ \sum_{n=-\infty}^{\infty} \delta(t - nT_p) \right\} = f(t) * \delta_p(t)$$

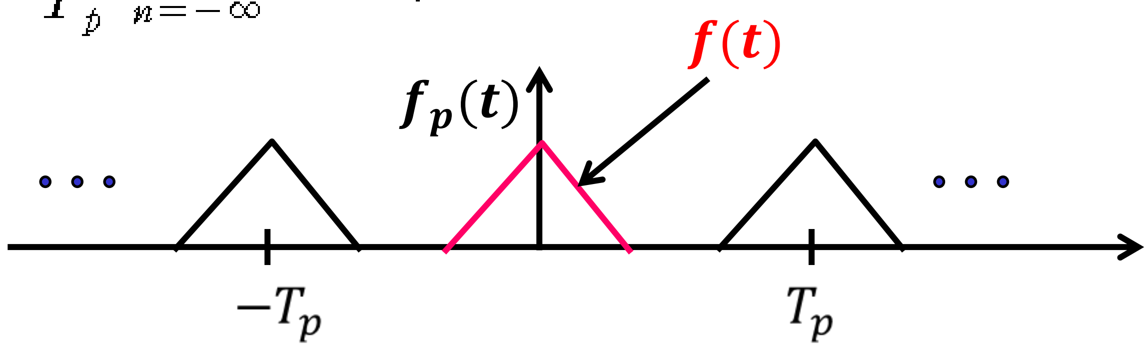
- **Then,**

$$F_p(\Omega) = F(\Omega) \cdot \Omega_0 \sum_{n=-\infty}^{\infty} \delta(\Omega - n\Omega_0) = \Omega_0 \sum_{n=-\infty}^{\infty} F(n\Omega_0)\delta(\Omega - n\Omega_0)$$

$$f_p(t) = \frac{\Omega_0}{2\pi} \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} F(n\Omega_0)\delta(\Omega - n\Omega_0) e^{j\Omega t} d\Omega$$

$$\Rightarrow \quad f_p(t) = \sum_{n=-\infty}^{\infty} f(t - nT_p) = \frac{1}{T_p} \sum_{n=-\infty}^{\infty} F(n\Omega_0) e^{jn\Omega_0 t}$$

$$\boxed{\begin{aligned} f_p(t) &= \sum_{n=-\infty}^{\infty} c_n e^{jn\Omega_0 t} \\ c_n &= \frac{1}{T_p} F(n\Omega_0) \end{aligned}}$$



CTFT vs. CTFS

A signal and its repeated version have spectra with the same shape.

Spectral outline is preserved!!

$$\boxed{c_k = \frac{1}{T_p} X(k\Omega_0)}$$

More specifically, the spectrum of $x_p(t) = x(t + kT_p)$ **is sampled version of** $X(F)$**.**

The sampling rate is $F_0 = 1/T_p$**.**

Repetition in Time Domain results in Sampling in freq. domain.

Fourier Series for CT periodic signal

Fourier coefficients of a rectangular pulse train

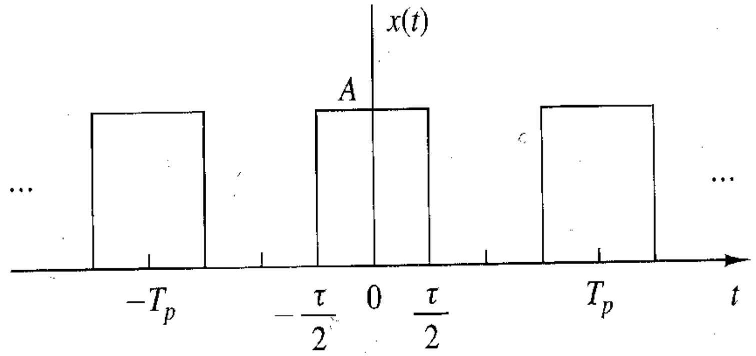


Figure 4.3 Continuous-time periodic train of rectangular pulses.

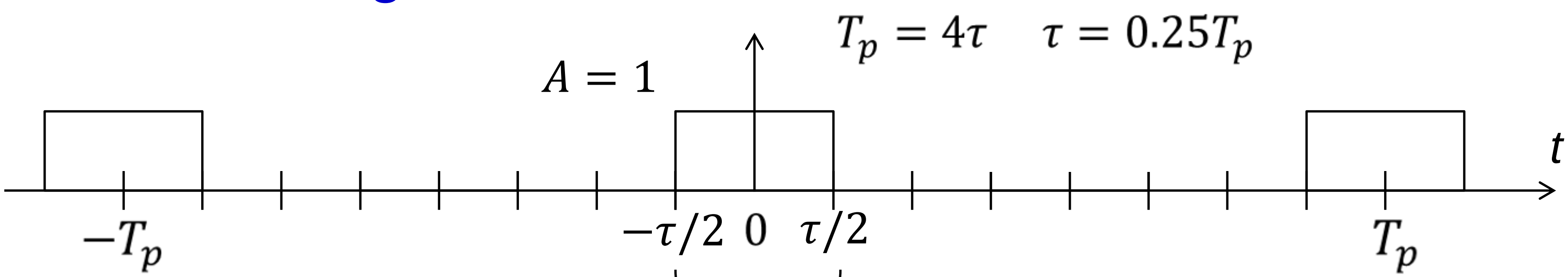
$$c_0 = \frac{1}{T_p}\int_{-T_p/2}^{T_p/2} x(t)dt = \frac{1}{T_p}\int_{-\tau/2}^{\tau/2} A dt = \frac{A\tau}{T_p}$$

$$c_k = \frac{1}{T_p}\int_{-\tau/2}^{\tau/2} A e^{-j2\pi k F_0 t} dt = \frac{A}{T_p}\frac{1}{-j2\pi k F_0} e^{-j2\pi k F_0 t}\Big|_{-\tau/2}^{\tau/2}$$

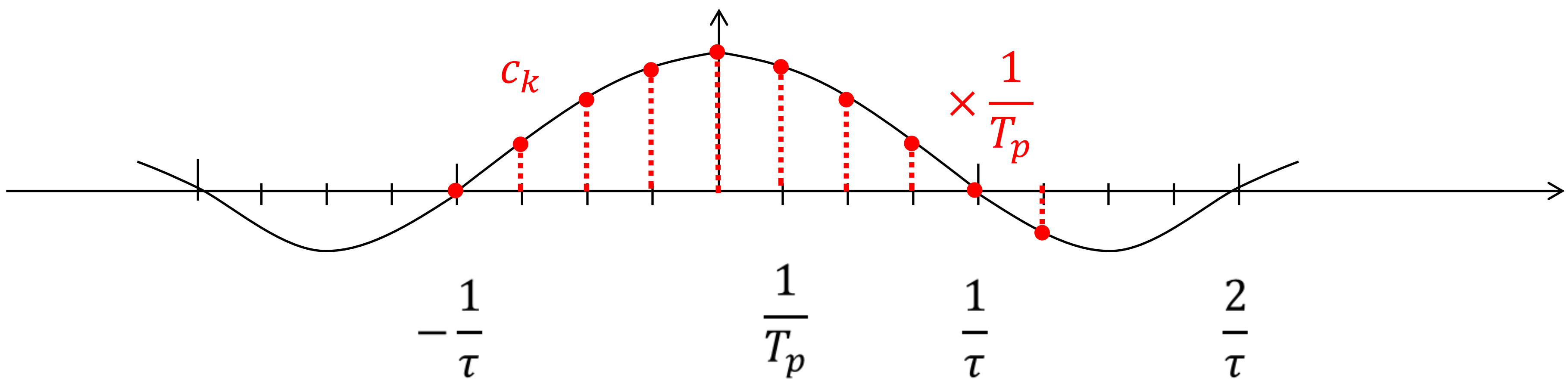
$$= \frac{A}{T_p}\frac{e^{j\pi k F_0 \tau} - e^{-j\pi k F_0 \tau}}{+j2\pi k F_0} = A\frac{\tau}{T_p}\frac{\sin(\pi k F_0 \tau)}{\pi k F_0 \tau} = A\frac{\tau}{T_p} sinc(k F_0 \tau)$$

Fourier Series for CT periodic signal

FT of a rectangular function

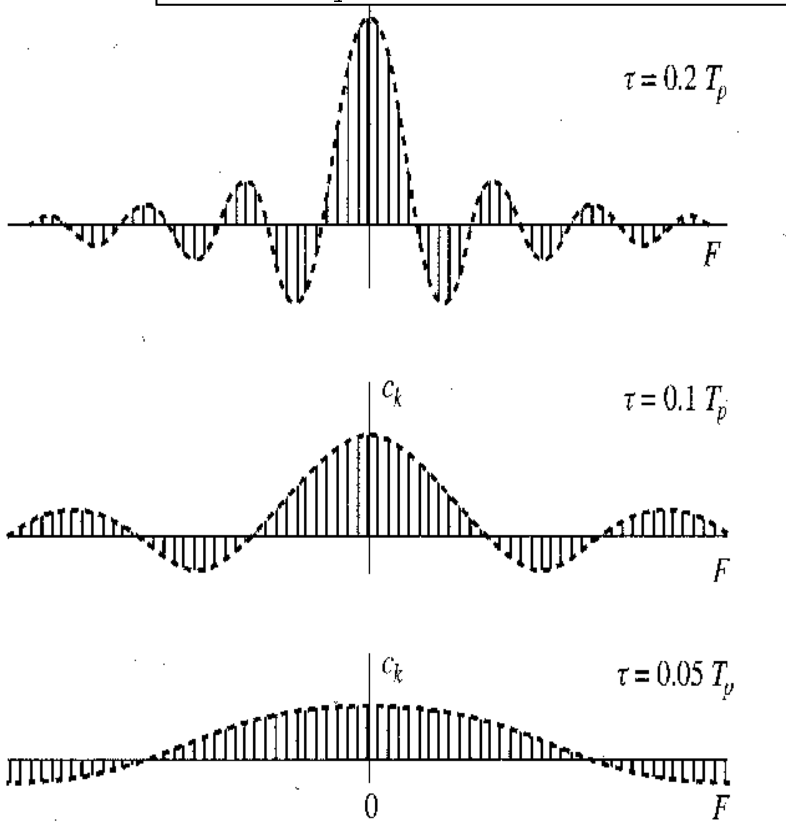


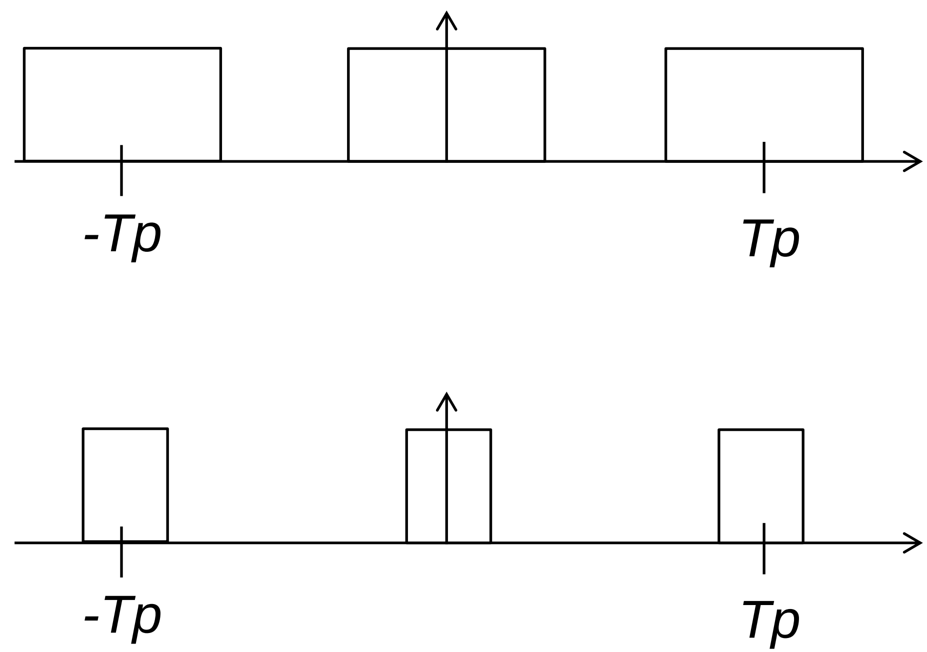
$rect\left(\frac{t}{\tau}\right) \leftrightarrow \boxed{\tau \cdot sinc(F\tau)}$ (Refer to p.9 in ch2_1_1.)



Fourier Series for CT periodic signal

$$c_k = \frac{1}{T_p}\int_{-\tau/2}^{\tau/2} A e^{-j2\pi k F_0 t} dt = A\frac{\tau}{T_p}\frac{\sin \pi k F_0 \tau}{\pi k F_0 \tau}, \quad k = \pm 1, \pm 2, \ldots$$



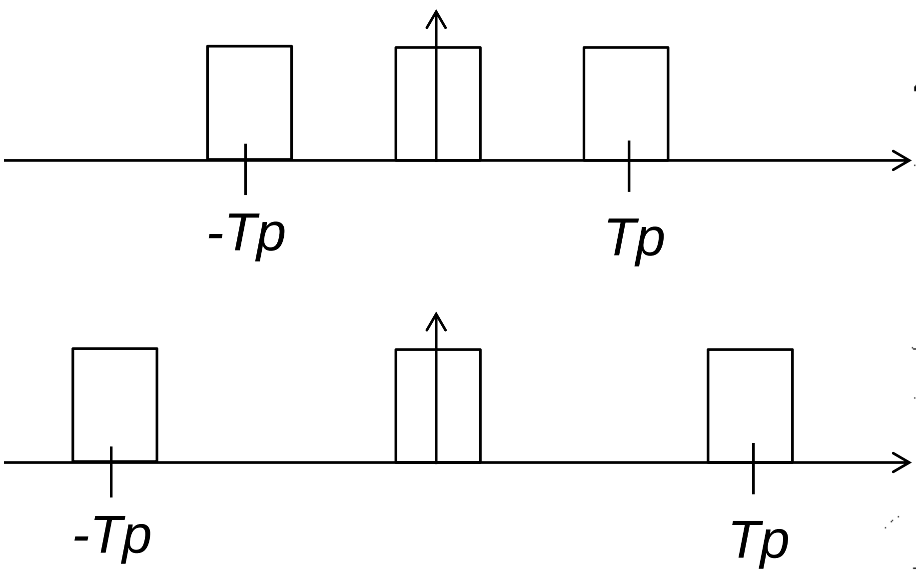


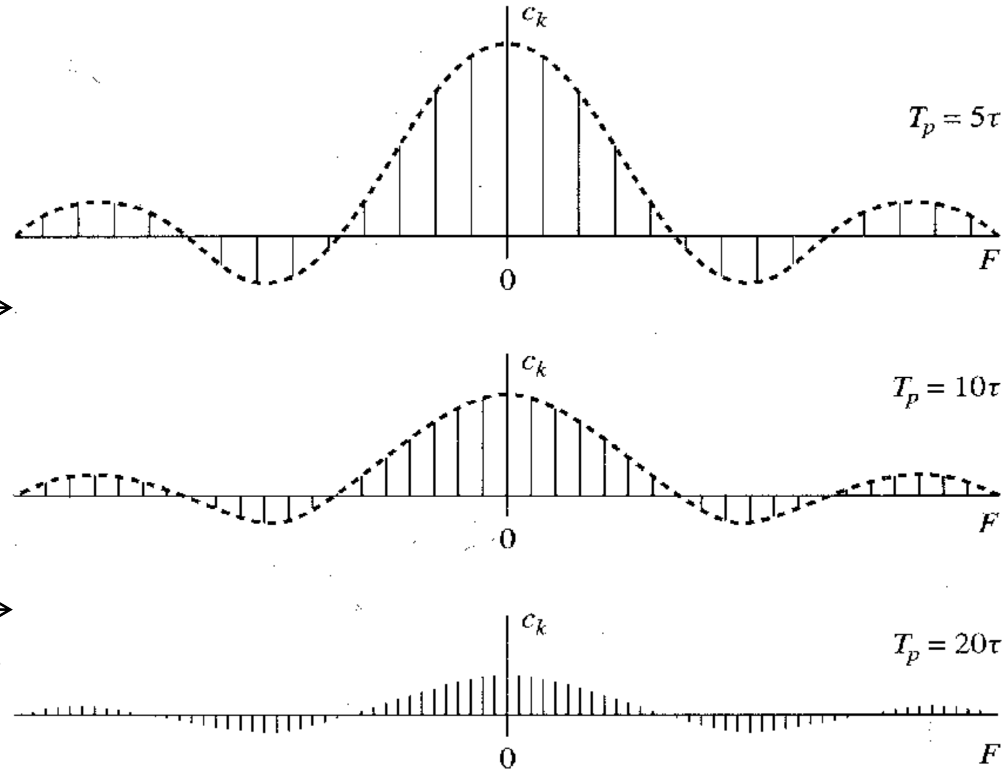
Relation with CTFT?

$$A rect(\frac{t}{\tau}) \leftrightarrow A\tau \operatorname{sinc}(F\tau) \quad F = kF_0 = k\frac{1}{T_p} \quad A\tau \rightarrow \frac{A\tau}{T_p} \quad => c_k$$

Fourier Series for CT periodic signal

$$c_k = \frac{1}{T_p}\int_{-\tau/2}^{\tau/2} A e^{-j2\pi k F_0 t} dt = A\frac{\tau}{T_p}\frac{\sin \pi k F_0 \tau}{\pi k F_0 \tau}, \quad k = \pm 1, \pm 2, \ldots$$





Relation with CTFT?

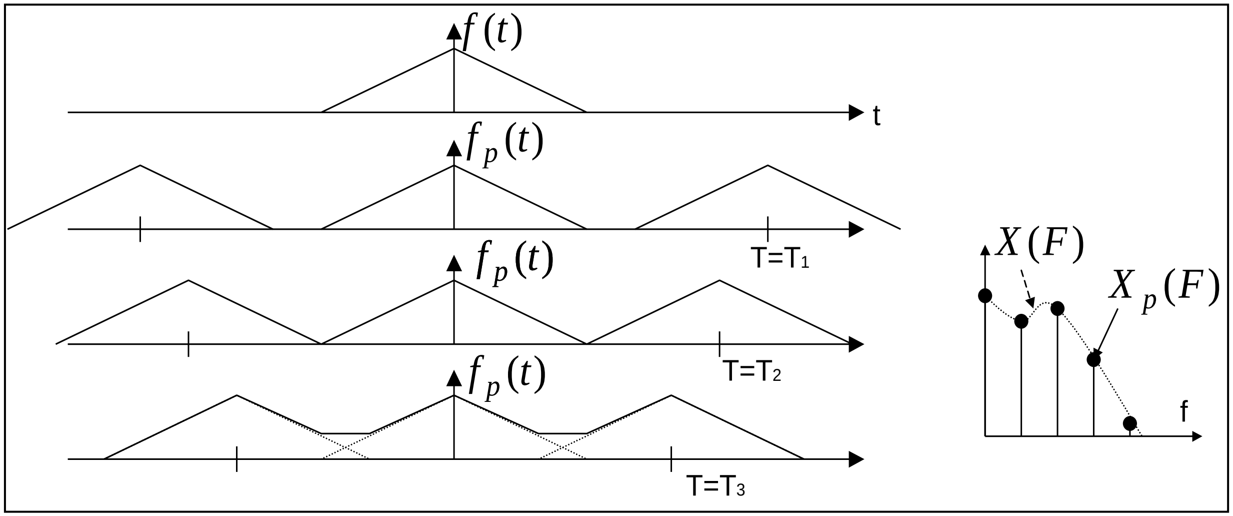
$$A\,rect\left(\frac{t}{\tau}\right) \leftrightarrow A\tau\,\mathrm{sinc}(F\tau) \quad F = kF_0 = k\frac{1}{T_p} \quad A\tau \rightarrow \frac{A\tau}{T_p} \quad => c_k$$

CTFT vs. CTFS

Time-domain Aliasing

❖ **When a periodic signal is obtained from an aperiodic signal,**

$$f_p(t) = \sum_{n=-\infty}^{\infty} f(t - nT_p) = f(t) * \left\{ \sum_{n=-\infty}^{\infty} \delta(t - nT_p) \right\} = f(t) * \delta_p(t)$$



Waveform distortion due to time-domain aliasing

Lower frequency resolution as T gets smaller

DTFT

Definition

$$X(\Theta) = F\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Theta n}$$ *(Θ: Digital angular frequency)*

❖ **Uniform convergence condition: A sufficient condition**

$$\lim_{N\to\infty} |X(\omega) - X_N(\omega)| = 0 \text{ for all } \omega$$ *: Error tends toward zero.*

where $X_N(\omega) = \sum_{n=-N}^{N} x(n)e^{-j\omega n}$

- Uniform convergence is guaranteed if $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$

❖ **Weaker condition: Mean square convergence condition**

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty \quad \Rightarrow \quad \lim_{N\to\infty} \int_{-\pi}^{\pi} \left| X(\Theta) - \sum_{n=-N}^{N} x[n]e^{-j\Theta n} \right|^2 d\Theta = 0$$

❖ $X(\Theta) = X(\Theta + 2\pi)$ **: Periodic function with a period of** 2π

DTFT

Inverse DT (Discrete-Time) Fourier transform

$$x(n)=\frac{1}{2\pi}\int_{2\pi} X(\Theta)e^{j\Theta n}d\Theta$$

$$\int_{\Theta_0}^{\Theta_0+2\pi} X(\Theta)e^{j\Theta m}d\Theta=\int_{\Theta_0}^{\Theta_0+2\pi}\sum_{n=-\infty}^{\infty}x(n)e^{-j\Theta(n-m)}d\Theta$$

$$\Rightarrow \sum_{n=-\infty}^{\infty}x(n)\int_{\Theta_0}^{\Theta_0+2\pi} e^{-j\Theta(n-m)}d\Theta = 2\pi\, x(m)$$

$$\boxed{\int_{\Theta_0}^{\Theta_0+2\pi} e^{-j\Theta(n-m)}d\Theta=\begin{cases}2\pi, & n=m\\ 0, & n\neq m\end{cases}}$$

$$x(m)=\frac{1}{2\pi}\int_{2\pi} X(\Theta)e^{j\Theta m}d\Theta$$

Note:

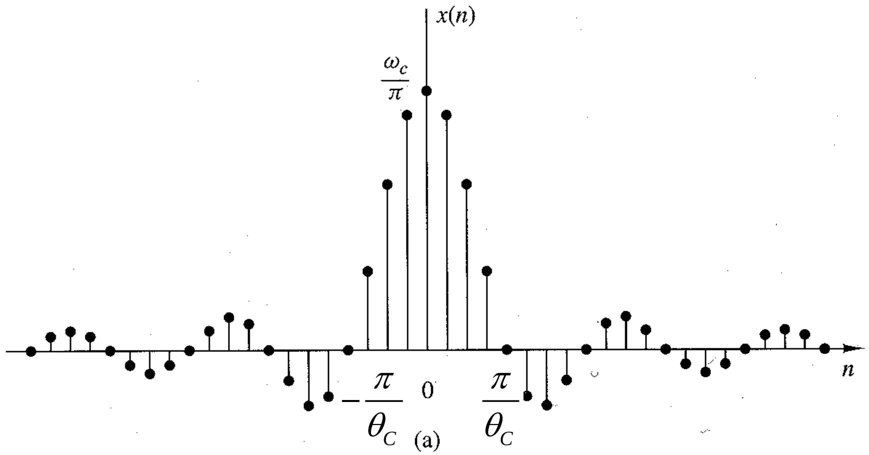
$$\delta(n)\leftrightarrow 1 \quad \Leftrightarrow \quad x(n)=\frac{1}{2\pi}\int_{2\pi} e^{j\Theta n}d\Theta=\delta(n)$$

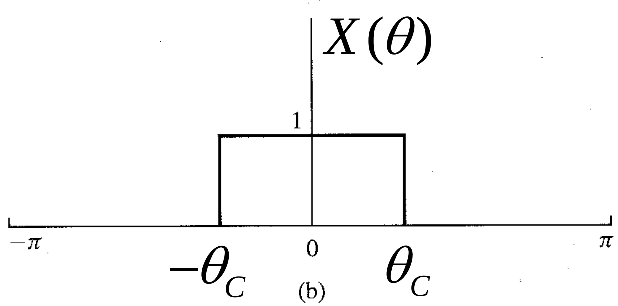
DTFT Example: Sinc function

$$X(\Theta) = \begin{cases} 1, & |\Theta| \le \Theta_c \\ 0, & \Theta_c < |\Theta| \le \pi \end{cases}$$

$$\begin{aligned} x(n) &= \frac{1}{2\pi}\int_{\Theta_c}^{\Theta_c} e^{j\Theta n}\, d\Theta \\ &= \frac{(e^{j\Theta_c n} - e^{-j\Theta_c n})}{2\pi j n} \\ &= \frac{\sin \Theta_c n}{n\pi} \end{aligned}$$

: not absolute summable





DTFT

DTFT Example: Truncation

- Gibbs phenomenon due to truncation.

$$X_N(\Theta) = \sum_{n=-N}^{N} \frac{\sin \Theta_c n}{\pi n} e^{-j\Theta n}$$

- FT of a sinc function exists,
  but the infinite series does not converge
  uniformly for all θ

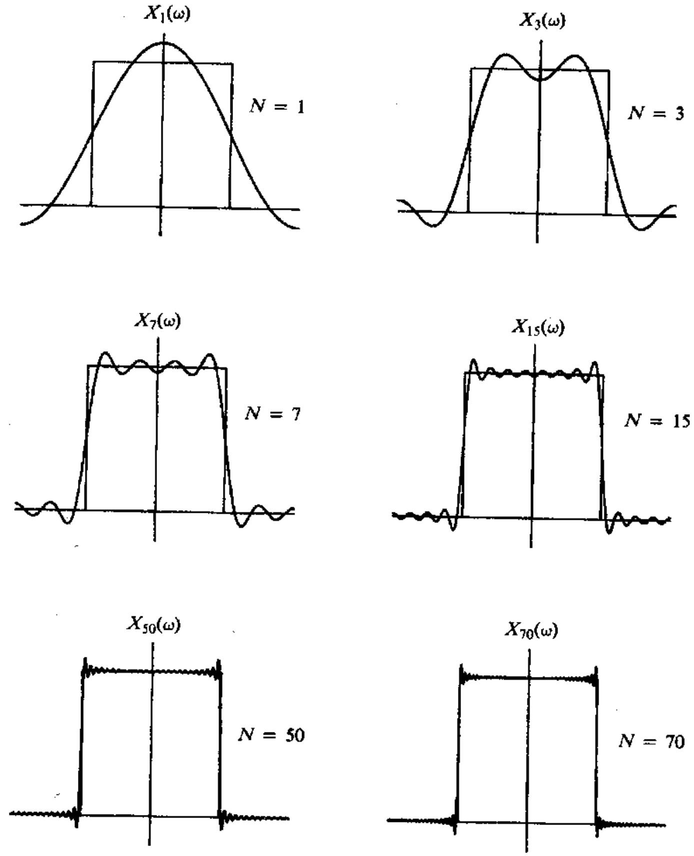


Figure 4.14 Illustration of convergence of the Fourier transform and the Gibbs phenomenon at the point of discontinuity.

DTFT

Properties

❖ Symmetry properties

$\omega \rightarrow \theta$

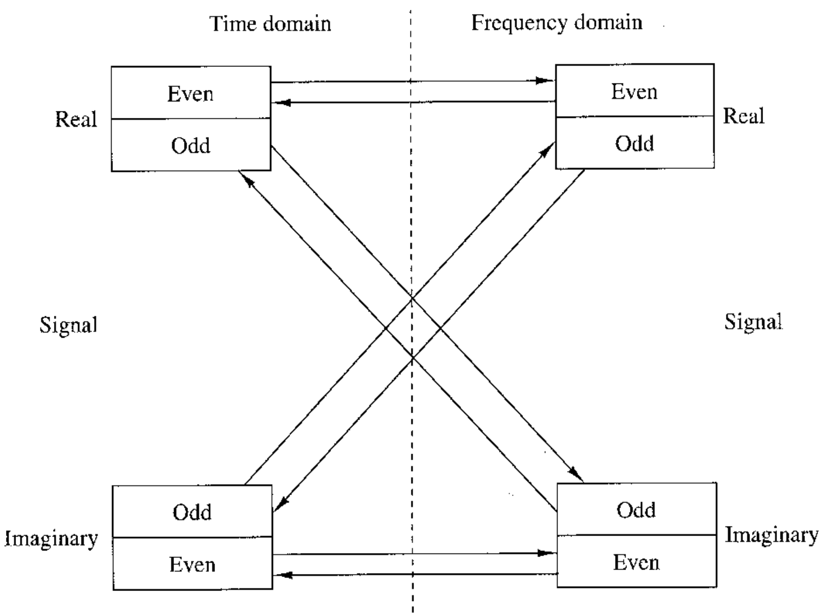


Figure 4.29 Summary of symmetry properties for the Fourier transform.

TABLE 4.4 SYMMETRY PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Sequence	DTFT
$x(n)$	$X(\omega)$
$x^*(n)$	$X^*(-\omega)$
$x^*(-n)$	$X^*(\omega)$
$x_R(n)$	$X_e(\omega) = \frac{1}{2}[X(\omega) + X^*(-\omega)]$
$jx_I(n)$	$X_o(\omega) = \frac{1}{2}[X(\omega) - X^*(-\omega)]$
$x_e(n) = \frac{1}{2}[x(n) + x^*(-n)]$	$X_R(\omega)$
$x_o(n) = \frac{1}{2}[x(n) - x^*(-n)]$	$jX_I(\omega)$
Real Signals	
Any real signal $x(n)$	$X(\omega) = X^*(-\omega)$ $X_R(\omega) = X_R(-\omega)$ $X_I(\omega) = -X_I(-\omega)$ $\lvert X(\omega)\rvert = \lvert X(-\omega)\rvert$ $\measuredangle X(\omega) = -\measuredangle X(-\omega)$
$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$ (real and even)	$X_R(\omega)$ (real and even)
$x_o(n) = \frac{1}{2}[x(n) - x(-n)]$ (real and odd)	$jX_I(\omega)$ (imaginary and odd)

DTFT

Properties

$\omega \longrightarrow \Theta$

TABLE 4.5 PROPERTIES OF THE FOURIER TRANSFORM FOR DISCRETE-TIME SIGNALS

Property	Time Domain	Frequency Domain
Notation	$x(n)$	$X(\omega)$
	$x_1(n)$	$X_1(\omega)$
	$x_2(n)$	$X_2(\omega)$
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(\omega) + a_2X_2(\omega)$
Time shifting	$x(n-k)$	$e^{-j\omega k}X(\omega)$
Time reversal	$x(-n)$	$X(-\omega)$
Convolution	$x_1(n) * x_2(n)$	$X_1(\omega)X_2(\omega)$
Correlation	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$S_{x_1x_2}(\omega) = X_1(\omega)X_2(-\omega)$ $= X_1(\omega)X_2^*(\omega)$ [if $x_2(n)$ is real]
Wiener–Khintchine theorem	$r_{xx}(l)$	$S_{xx}(\omega)$
Frequency shifting	$e^{j\omega_0 n}x(n)$	$X(\omega - \omega_0)$
Modulation	$x(n)\cos\omega_0 n$	$\frac{1}{2}X(\omega + \omega_0) + \frac{1}{2}X(\omega - \omega_0)$
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\lambda)X_2(\omega - \lambda)d\lambda$
Differentiation in the frequency domain	$nx(n)$	$j\frac{dX(\omega)}{d\omega}$
Conjugation	$x^*(n)$	$X^*(-\omega)$
Parseval's theorem	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(\omega)X_2^*(\omega)d\omega$	

DTFT

Energy density spectrum of aperiodic signals

$$\boxed{\omega \to \Theta}$$

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$E_x = \sum_{n=-\infty}^{\infty} x(n)x^*(n) = \sum_{n=-\infty}^{\infty} x(n)\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(\omega)\, e^{-j\omega n} d\omega\right]$$

$$E_x = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(\omega)\left[\sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}\right] d\omega$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

DTFT

Bandwidth of DT signals

$\omega \rightarrow \theta$

- **A DT signal is said to be band-limited if**

$$X(\omega) = 0 \quad for \;\; all \;\; |\omega| \geq \omega_c < \pi$$

- **Periodic, repeated spectrum**
- **Low-Pass signal.**
  - Symmetric signal→Real spectrum
- **Multiplication by a complex exponential**
  - Shifted spectrum
    Unsymmetric spectrum
  - Complex signal
- **Modulation**

$$x(n)\cos\omega_0 n \overset{F}{\Longleftrightarrow} \frac{1}{2}[X(\omega+\omega_0)+X(\omega-\omega_0)]$$

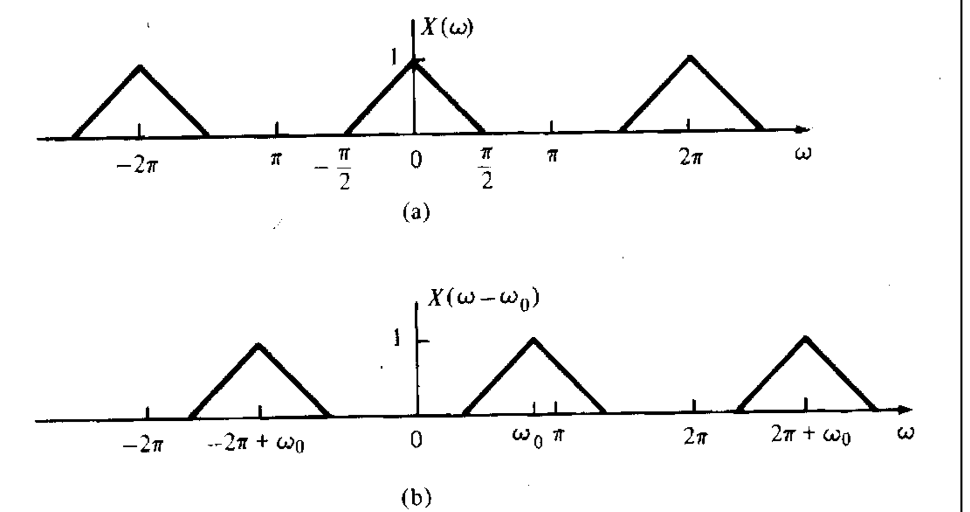


Figure 4.35 Illustration of the frequency-shifting property of the Fourier transform.

DTFT

Frequency response of DT systems

❖ (relaxed) LTI system and convolution sum

$$L[e^{j\Theta n}] = H(\Theta)e^{j\Theta n}$$

$$L[\delta(n)] = L\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\Theta n} d\Theta\right\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} L[e^{j\Theta n}]d\Theta = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\Theta)e^{j\Theta n}d\Theta = h(n)$$

$$L[x(n)] = L\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right] = \sum_{k=-\infty}^{\infty} x(k)L[\delta(n-k)]$$

$$= \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad : convolution\ \ sum$$

$$\boxed{Y(\Theta) = H(\Theta)X(\Theta) \quad S_{yy}(\Theta) = |Y(\Theta)|^2 = |H(\Theta)|^2 S_{xx}(\Theta) = |H(\Theta)|^2 |X(\Theta)|^2}$$

Sampling

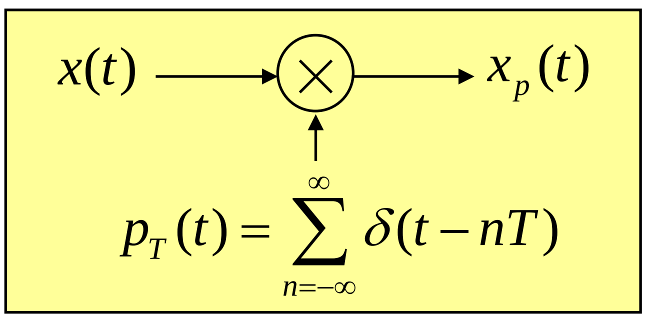
Uniform sampling

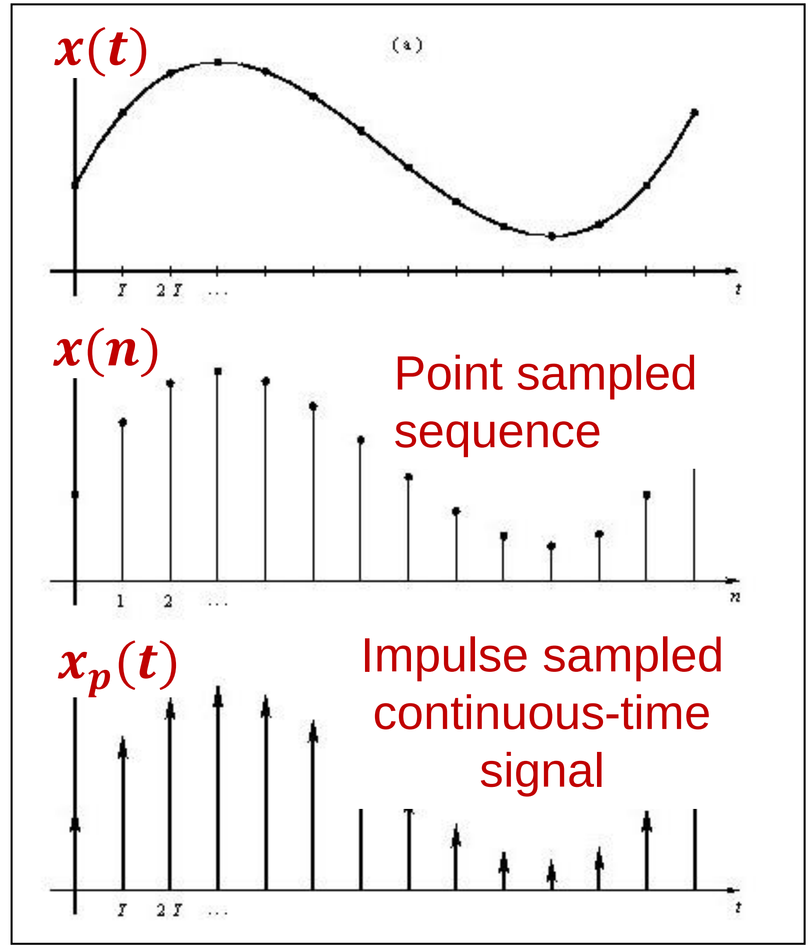
- (uniform) Sampling interval: T
- Sampling frequency (or rate): $f_s = 1/T$
- Angular sampling frequency: $\omega_s = 2\pi/T$

Sampled signal

$$x(n) = x(t)\big|_{t=nT} = x(nT)$$

Sampled signal: a CT signal model





$$x_p(t) = x(t)p_T(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

Sampling

Relationship between CTFT and DTFT

❖ **Sampled signal: a CT signal model**

$$
\begin{aligned}
F\{x_p(t)\} &= \int_{-\infty}^{\infty} x_p(t) e^{-j\omega t} dt \\
&= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(t)\delta(t-nT) e^{-j\omega t} dt \\
&= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \delta(t-nT) dt \\
&= \sum_{n=-\infty}^{\infty} x(nT) e^{-j\omega n T}
\end{aligned}
$$

$$\boxed{\Theta = \omega T}$$

$$X(\Theta) = F\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Theta n}$$

Sampling

Relationship between CTFT and DTFT

❖ **Sampled signal: a CT signal model**

$$x_p(t) = x(t)p_T(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t-nT) = \boxed{\sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)}$$

$$X_p(\omega) = \int_{-\infty}^{\infty} x_p(t)e^{-j\omega t}dt = \sum_{n=-\infty}^{\infty} x(nT)e^{-jn\omega T}$$

$$\boxed{\delta_p(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT_p) \longleftrightarrow \Omega_0 \sum_{n=-\infty}^{\infty} \delta(\Omega - n\Omega_0)} \quad \boxed{\Omega \to \omega}$$

$$X_p(\omega) = \frac{1}{2\pi} P_T(\omega) * X^F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^F(\lambda)\left[\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\omega-\lambda-\frac{2\pi k}{T}\right)\right]d\lambda$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} X^F\left(\omega - \frac{2\pi k}{T}\right)$$

$$\underbrace{\sum_{n=-\infty}^{\infty} x(nT)e^{-jn\omega T}}_{X_p(\omega) \leftrightarrow x_p(t)} = \frac{1}{T}\sum_{n=-\infty}^{\infty} \underbrace{X^F}_{X(\omega) \leftrightarrow x(t)}\left(\omega - \frac{2\pi k}{T}\right)\Big|_{\omega=\theta/T} = X(\theta)$$

$\uparrow$
$x(n)$

Sampling

Relationship between CTFT and DTFT

❖ **Point-sampled signal when regarded as a DT signal**

$$X_p(\omega) =$$

$$\boxed{\sum_{n=-\infty}^{\infty} x(nT)e^{-jn\omega T} = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\omega - \frac{2\pi k}{T}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(\omega - k\omega_s)}$$

$$X(\Theta) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X^F\left(\frac{\Theta - 2\pi k}{T}\right)$$

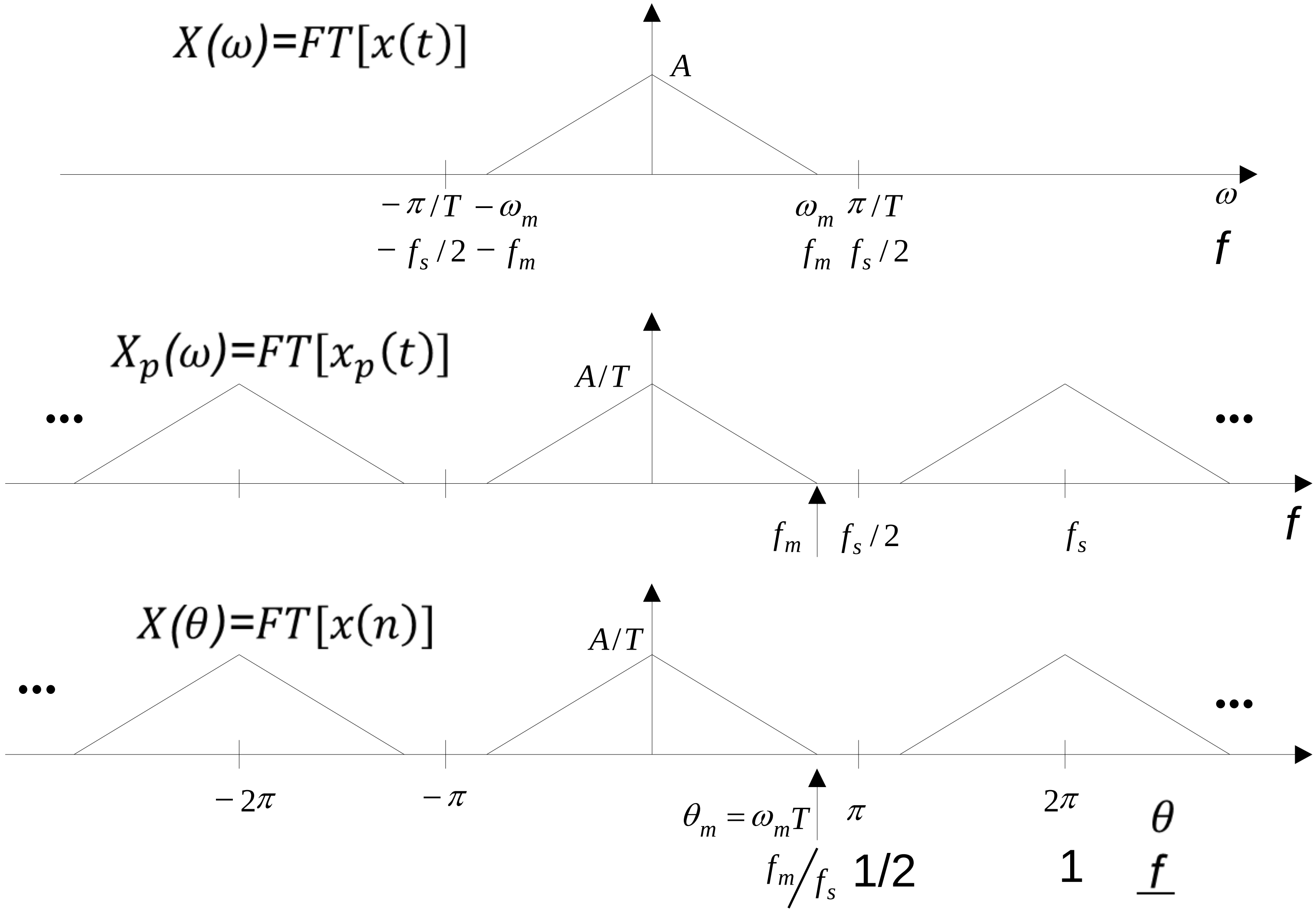
$$\Theta = \omega T \Rightarrow \underline{f} = fT = f/f_s$$

$$X(\Theta) = X(\Theta + 2\pi)$$

Sampling in time domain ➡ Repetition in freq. domain

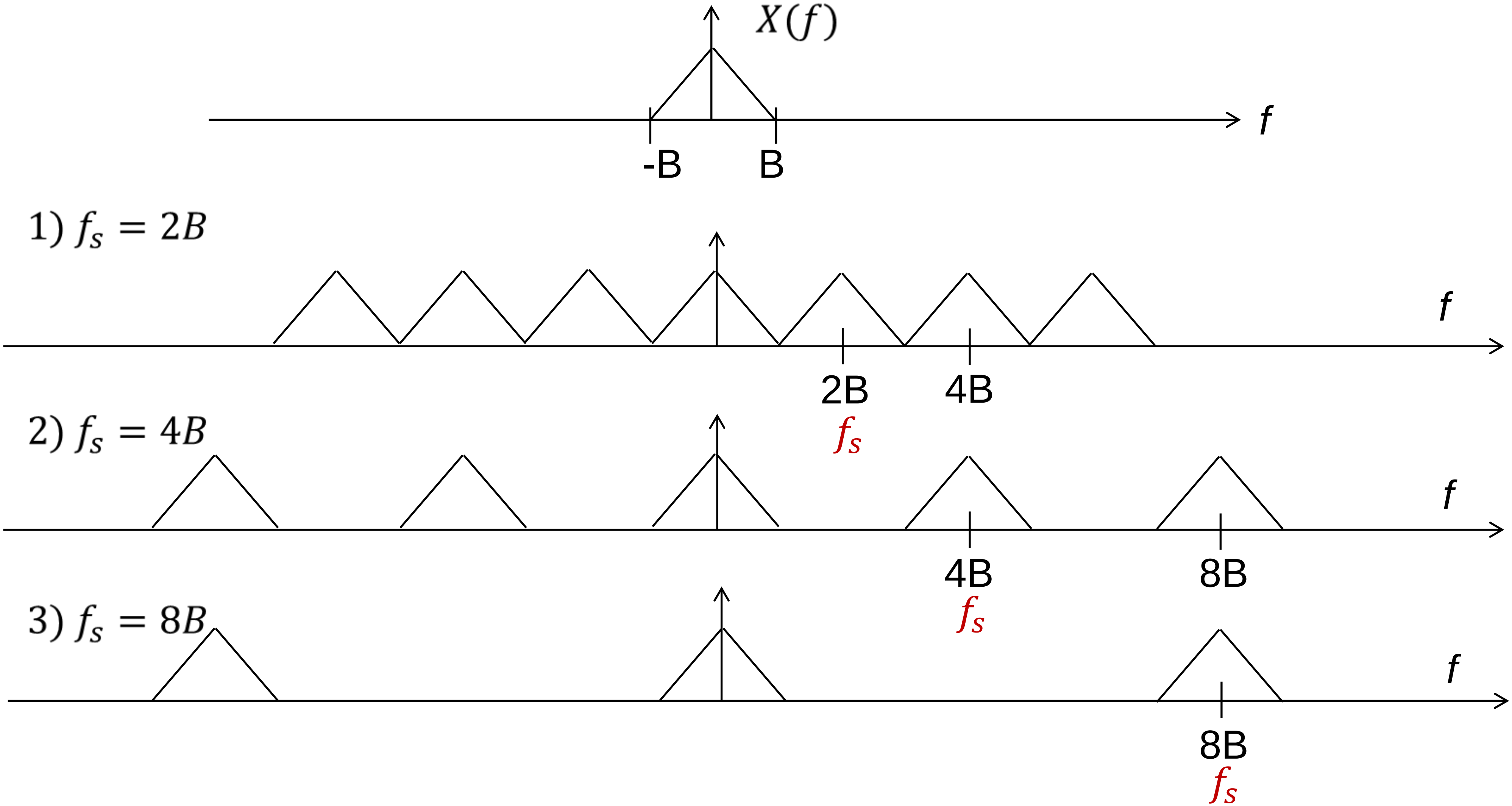
Sampling

Spectrum of a sampled signal $X^{f}(\Theta) = X^{f}(\omega T) = \frac{1}{T} X^{F}(\omega)$



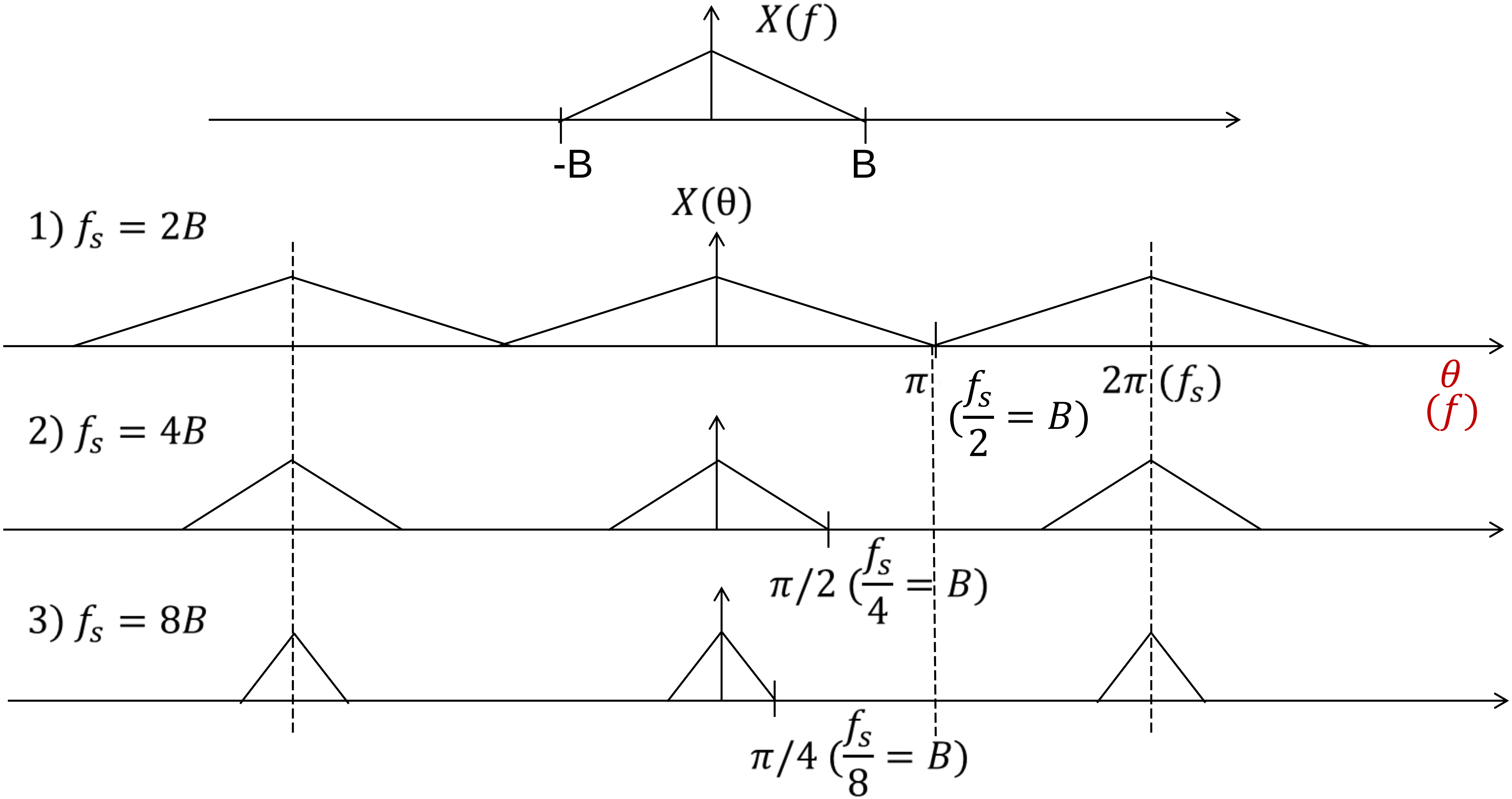
Sampling

CTFT spectra for different sampling rates: impulse sampled signal



Sampling

DTFT spectrum for different sampling rates: Sampled sequence



Sampling

Condition under which the replicas (repeated spectra) do not overlap

- $x(t)$ **must be band limited, that is,**

$$X(\omega) = 0 \quad \text{for} \quad |\,\omega\,| \geq \omega_m$$

$$f_s \geq 2 \cdot f_m \quad or \quad \omega_m / \pi$$

- **Bandwidth of x(t) :** $\omega_m \; or \; f_m$
- **Nyquist rate (= twice the highest frequency) :** $2 \cdot f_m$
- **Aliasing: The phenomenon that happens when**

  $X(\Theta) = X(\omega T) = \dfrac{1}{T} X(\omega)$ **not hold.**

  The replicas overlap and $f_s \geq 2 \cdot f_m$ **not hold either.**

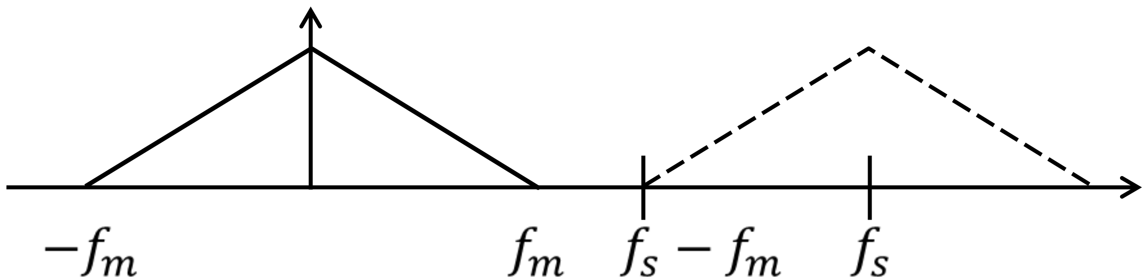
  Can never reconstruct $X(\omega) \; or \; x(t)$ **from** $X(\Theta) \; or \; x(nT)$

Sampling

Sampling Theorem

A band limited CT signal, with highest frequency f_m, can be uniquely recovered from its samples if the sampling rate satisfies

$$f_s \geq f_{Nyquist} = 2f_m, \qquad i.e., 2 \cdot (highest\ frequency)$$



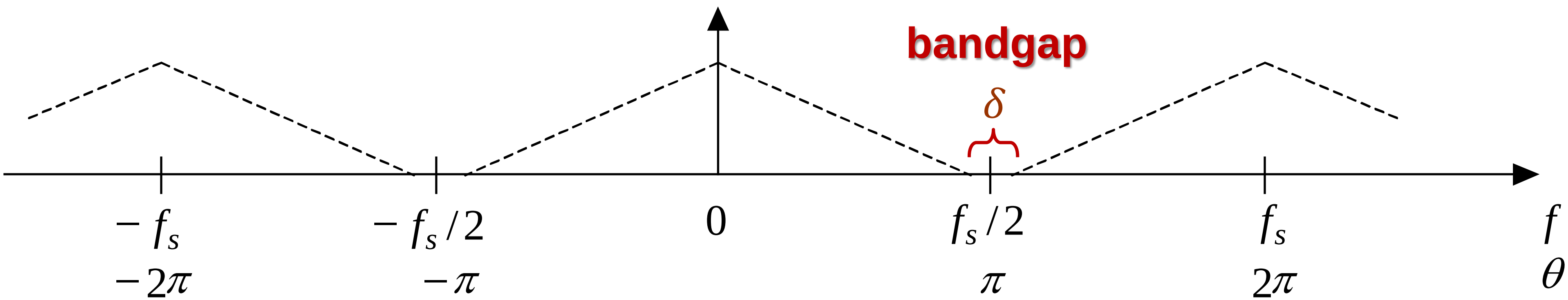
$$f_s - f_m \geq f_m$$

Sampling

Common sampling rule

❖ **Practical sampling rate to achieve a necessary bandgap.**

$$f_s \geq 2 \cdot f_m + \delta \quad where\ \delta \geq 0 \qquad \delta \geq 0.1 \cdot (2 \cdot f_m)$$

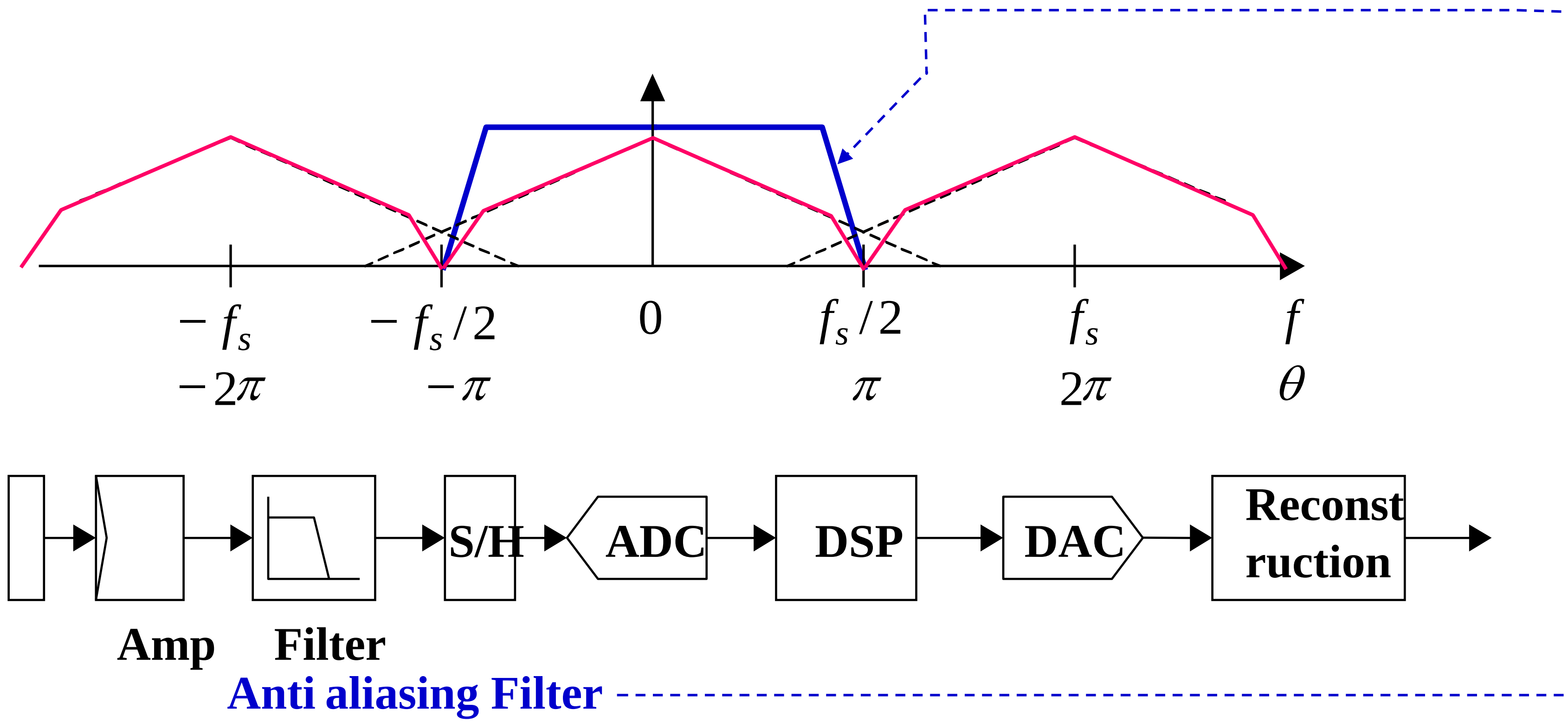


Sampling

Common sampling rule

- ❖ **When Sampling Theorem can't be met**

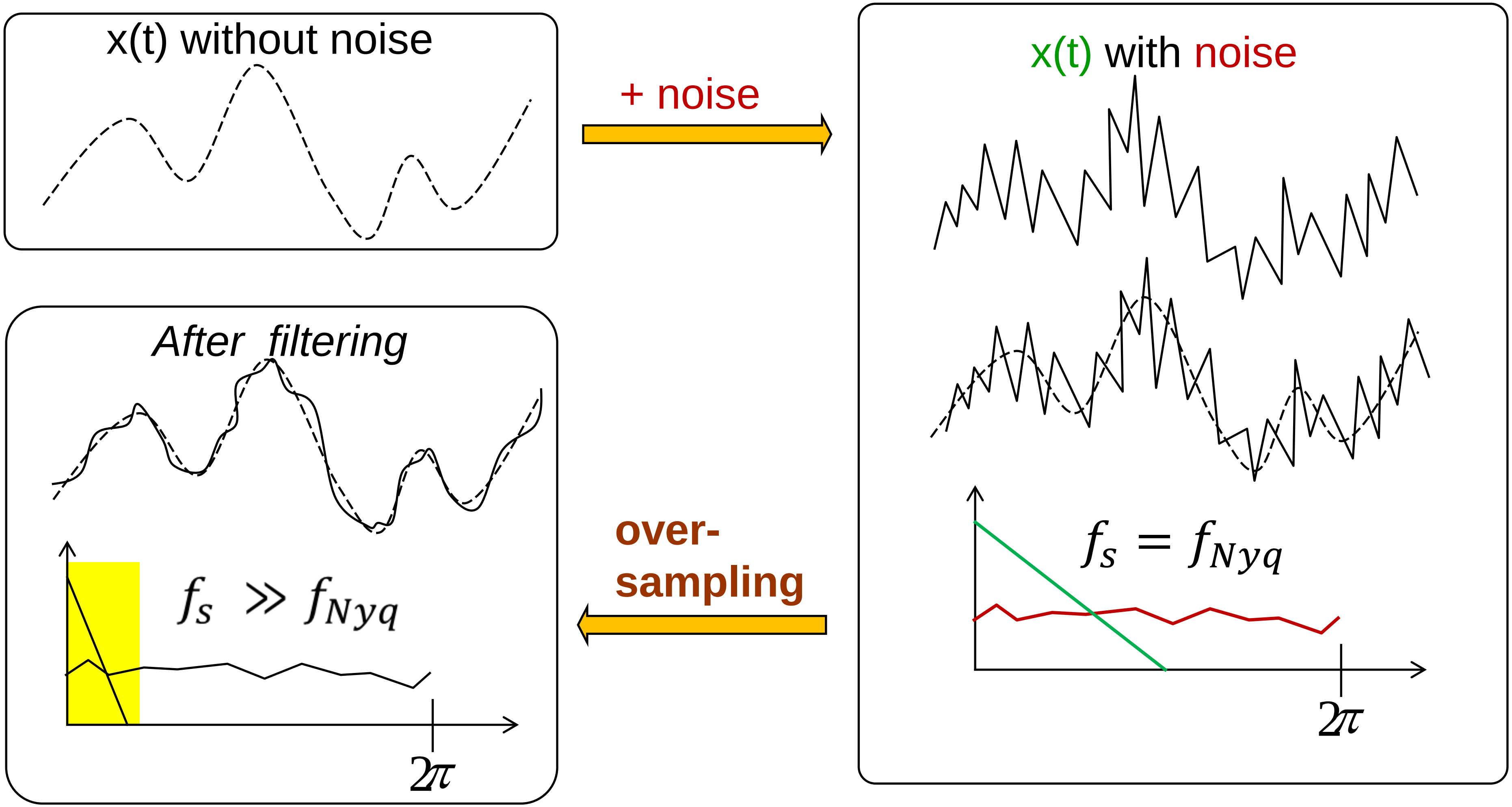
  ➔ **Anti-alising filter is commonly used in front of ADC**



Sampling

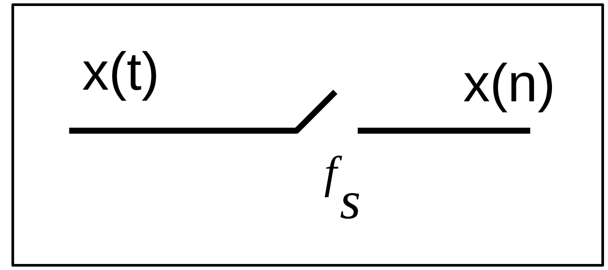
Common sampling rule

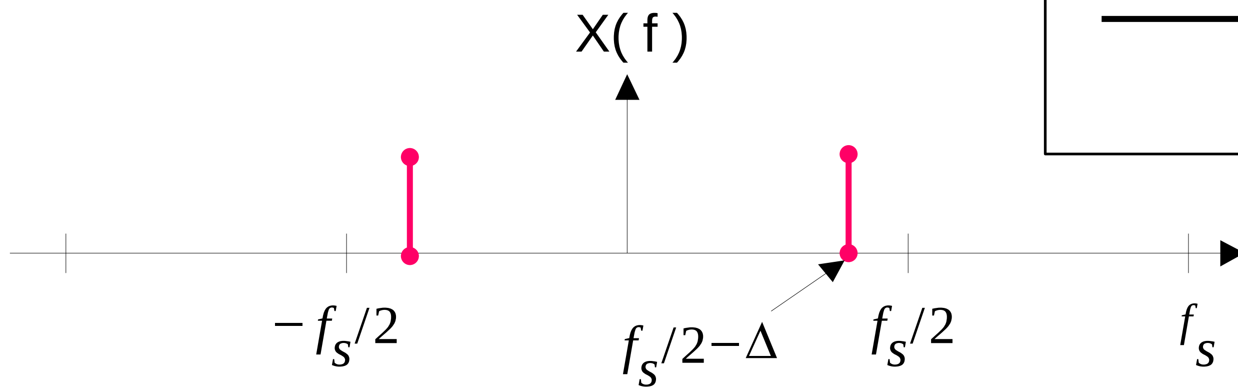
❖ **Oversampling for noise reduction.**

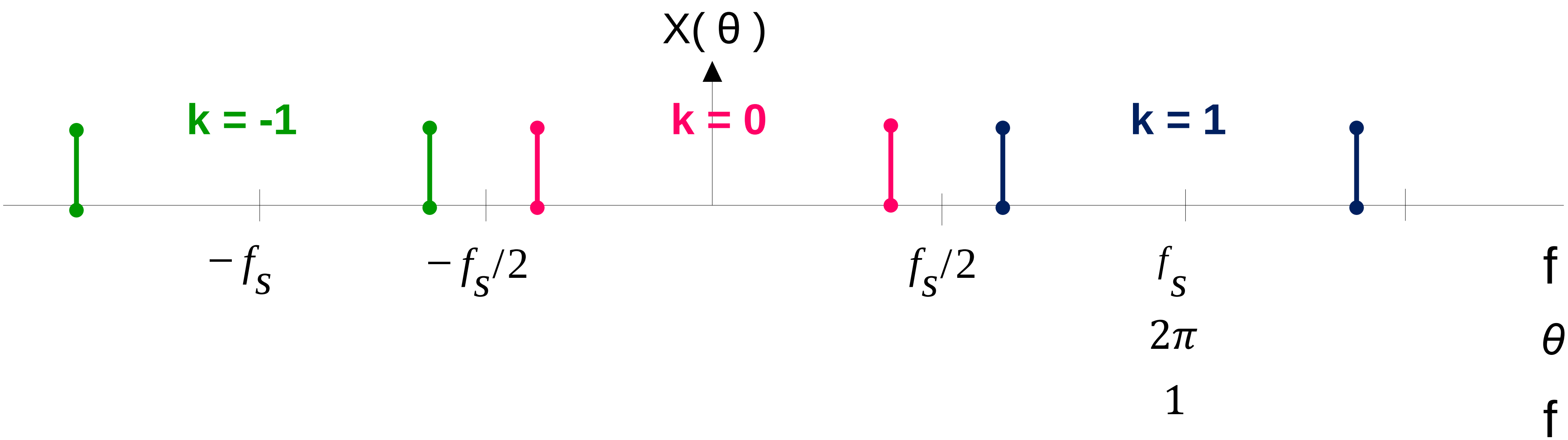


Sampling

Sampling without aliasing

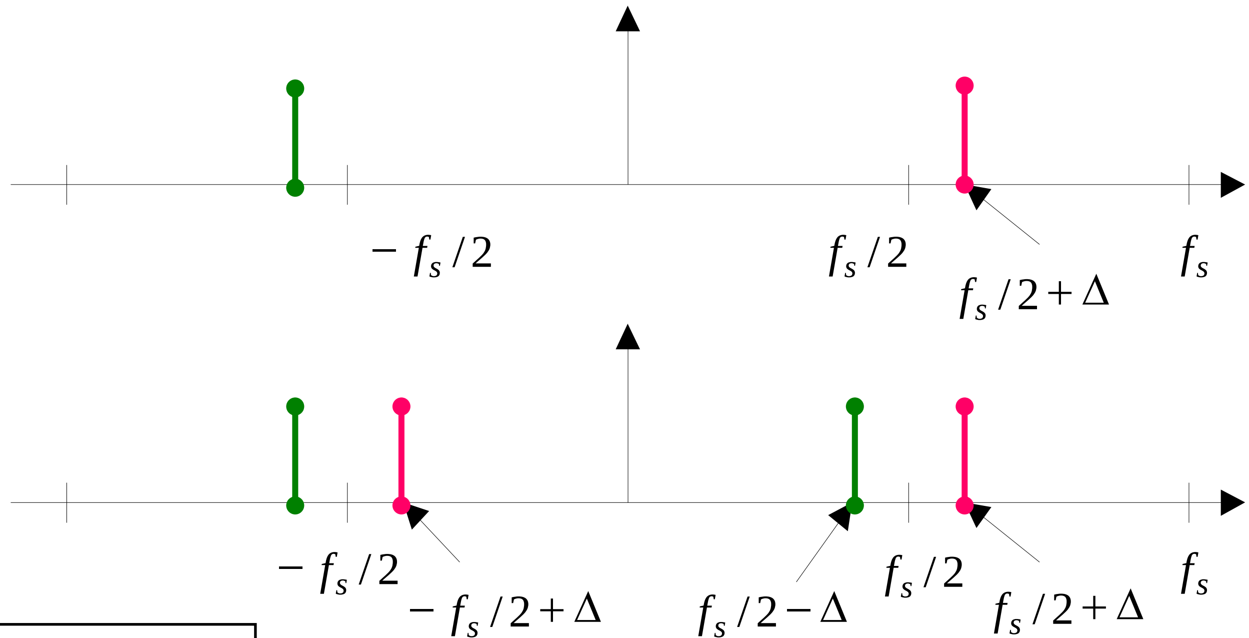






Sampling

Frequency Aliasing



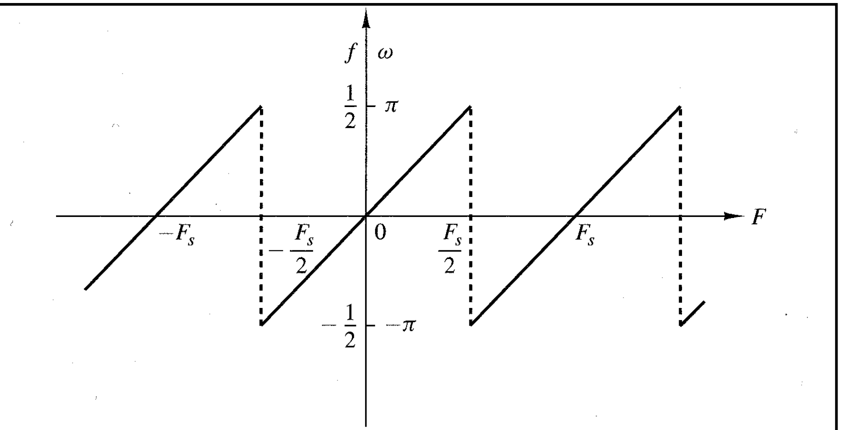
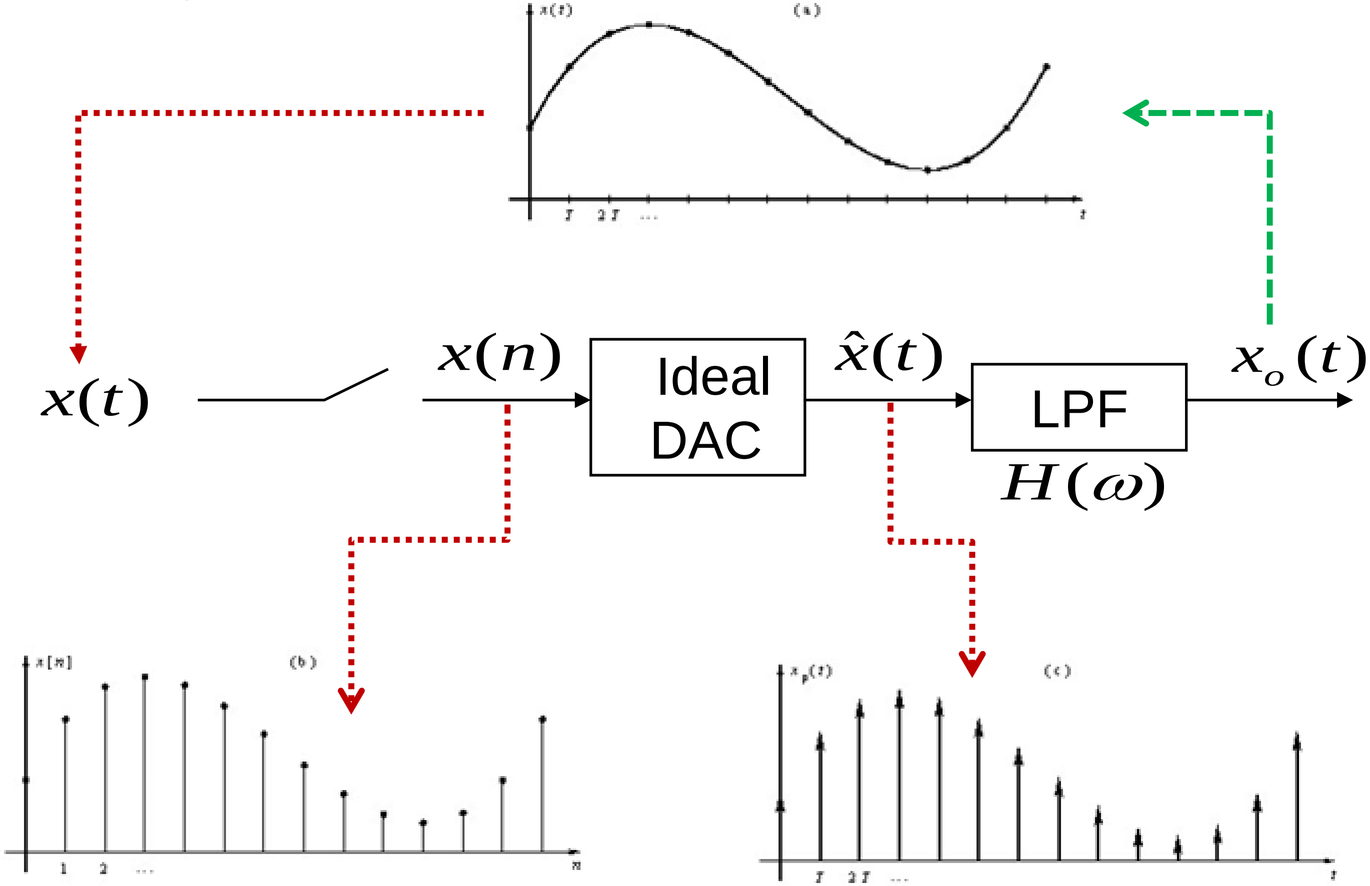


Figure 1.17 Relationship between the continuous-time and discrete-time frequency variables in the case of periodic sampling.

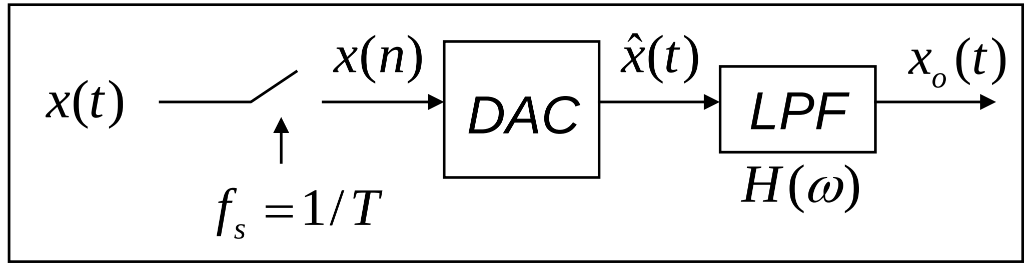
Reconstruction

With an ideal DAC

- **Block diagram**



Reconstruction



With an ideal DAC

- **Input bandwidth:** f_m
- **Output bandwidth:**

$$B = f_s / 2$$

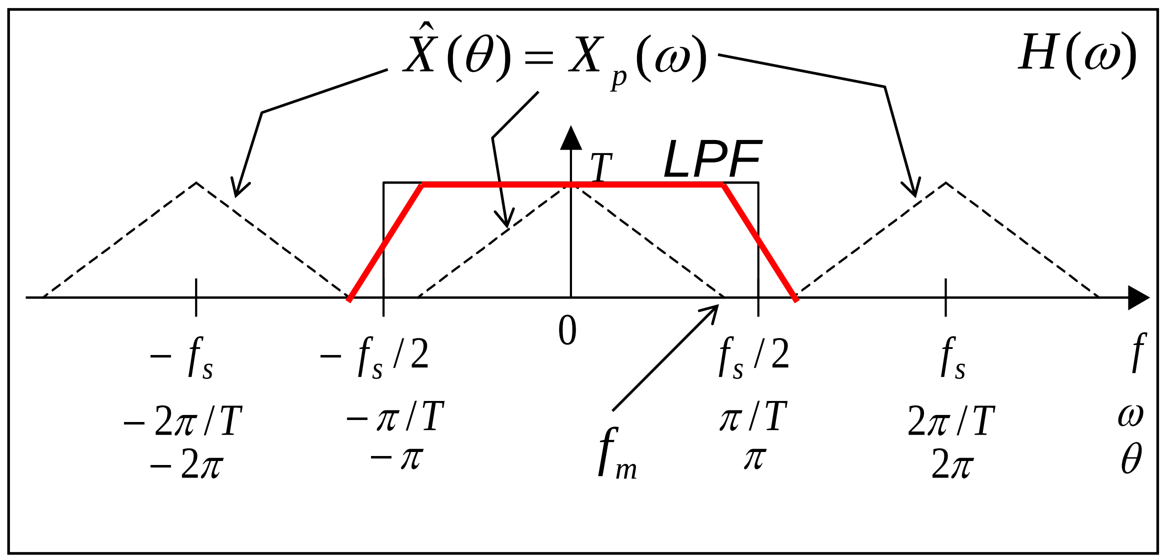
- **Let's assume we can use an Ideal LPF**

$$x(t) = x_o(t) \text{ if } f_m \le B$$

$$H(\omega) = T \cdot rect\left(\frac{\omega T}{2\pi}\right) \quad or \quad H(f) = T \cdot rect(f / f_s)$$

$$\begin{aligned} X(\omega) &= X_p(\omega) H(\omega) \\ &= X_p(\omega) T \cdot rect\left(\frac{\omega T}{2\pi}\right) \end{aligned}$$

$$\Rightarrow h(t) = \operatorname{sin}c\left(\frac{t}{T}\right)$$

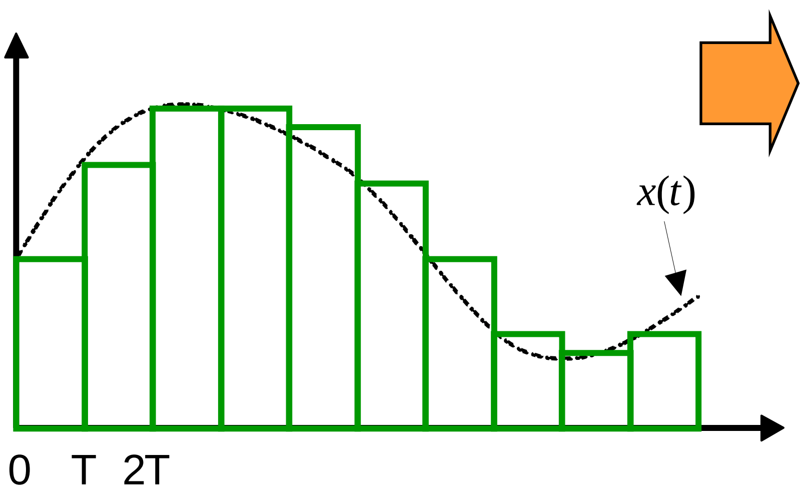


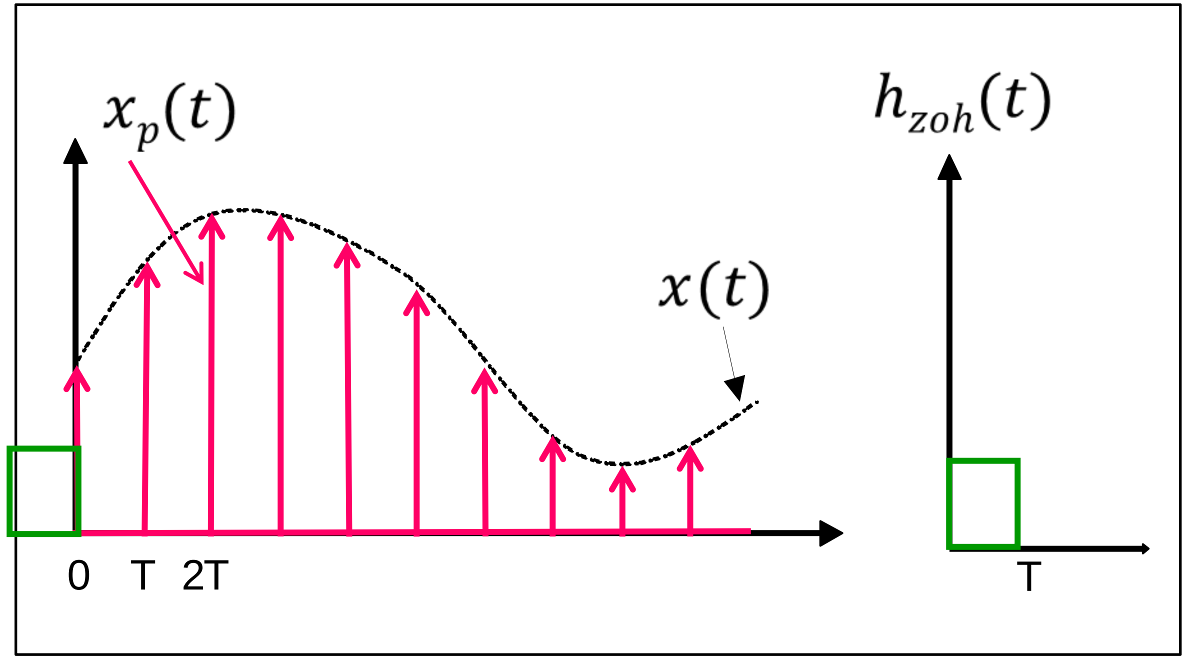
Reconstruction

Practical approach: With a real DAC

- **Note that** $x_p(t) = \sum_{n=-\infty}^{\infty} x(t)\delta(t-nT)$ **can't exist in reality.**

- **Zero-order Hold (ZOH) using DAC (Digital-to-Analog Converter)**

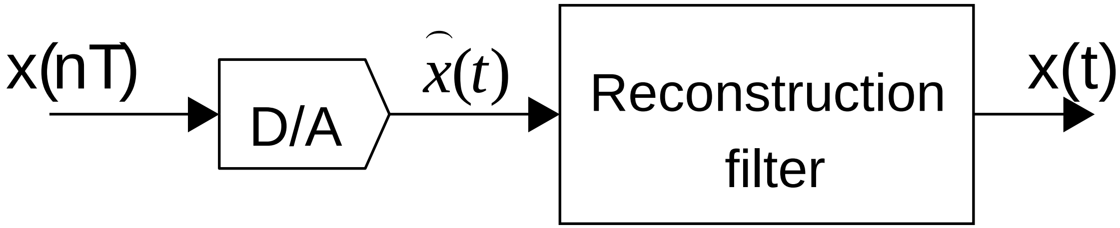




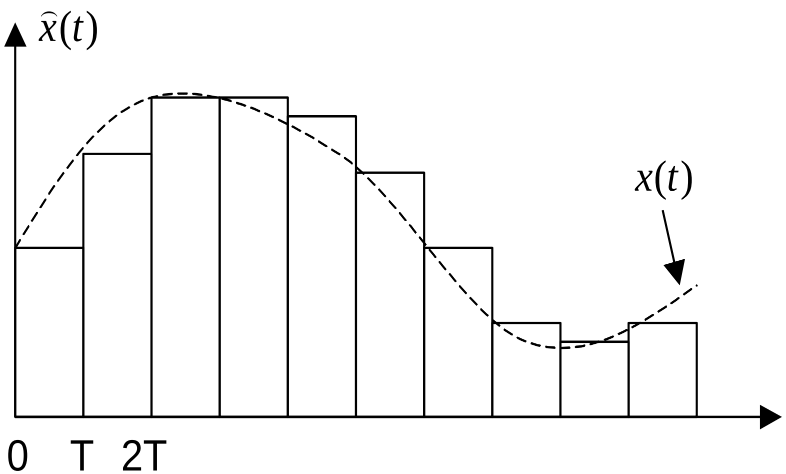
Reconstruction

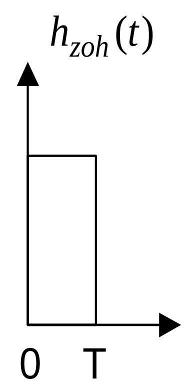
Practical approach: With a real DAC

- **Zero-order Hold (ZOH) using DAC (Digital-to-Analog Converter)**



$$h_{opt}(t) = rect\left(\frac{t - T/2}{T}\right) \longleftrightarrow H_{zoh}(\omega) = T \cdot \mathrm{sinc}\left(\frac{\omega T}{2\pi}\right) e^{-j0,5\omega T}$$



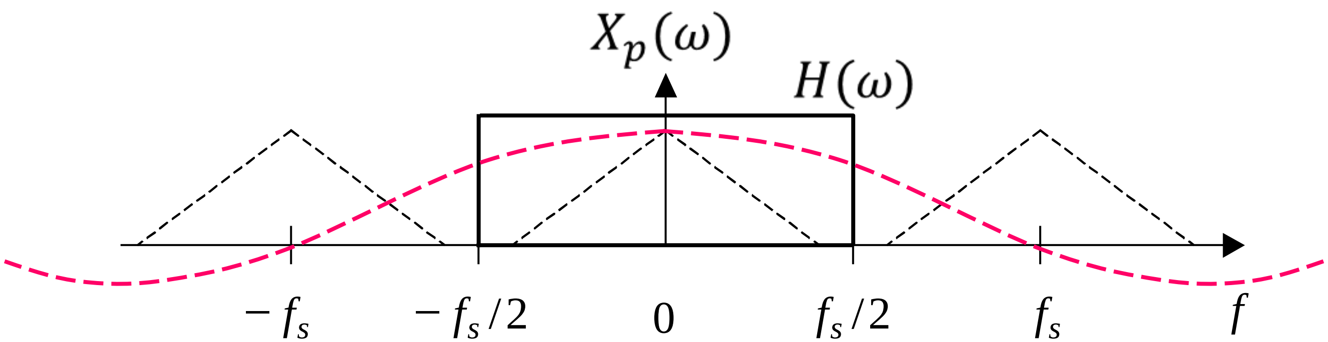


Note: Can you exactly describe the frequency spectrum of the DAC output?

Reconstruction

Practical approach: With a real DAC

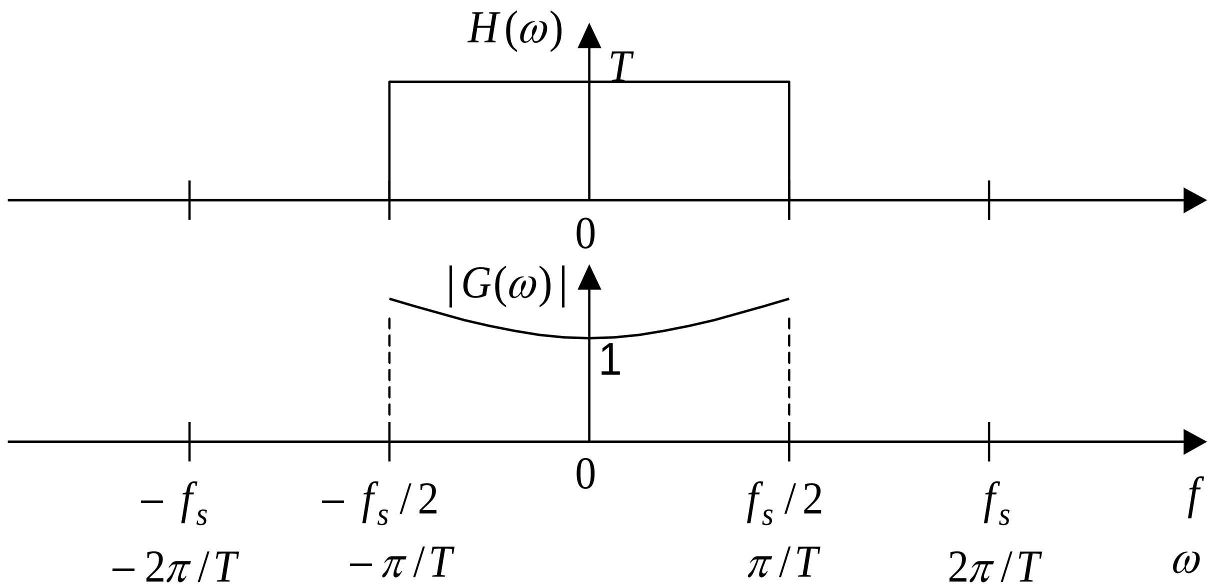
- **Zero-order Hold (ZOH) using DAC (Digital-to-Analog Converter)**



Reconstruction

Practical approach: With a real DAC

❖ **Amplitude equalizer**



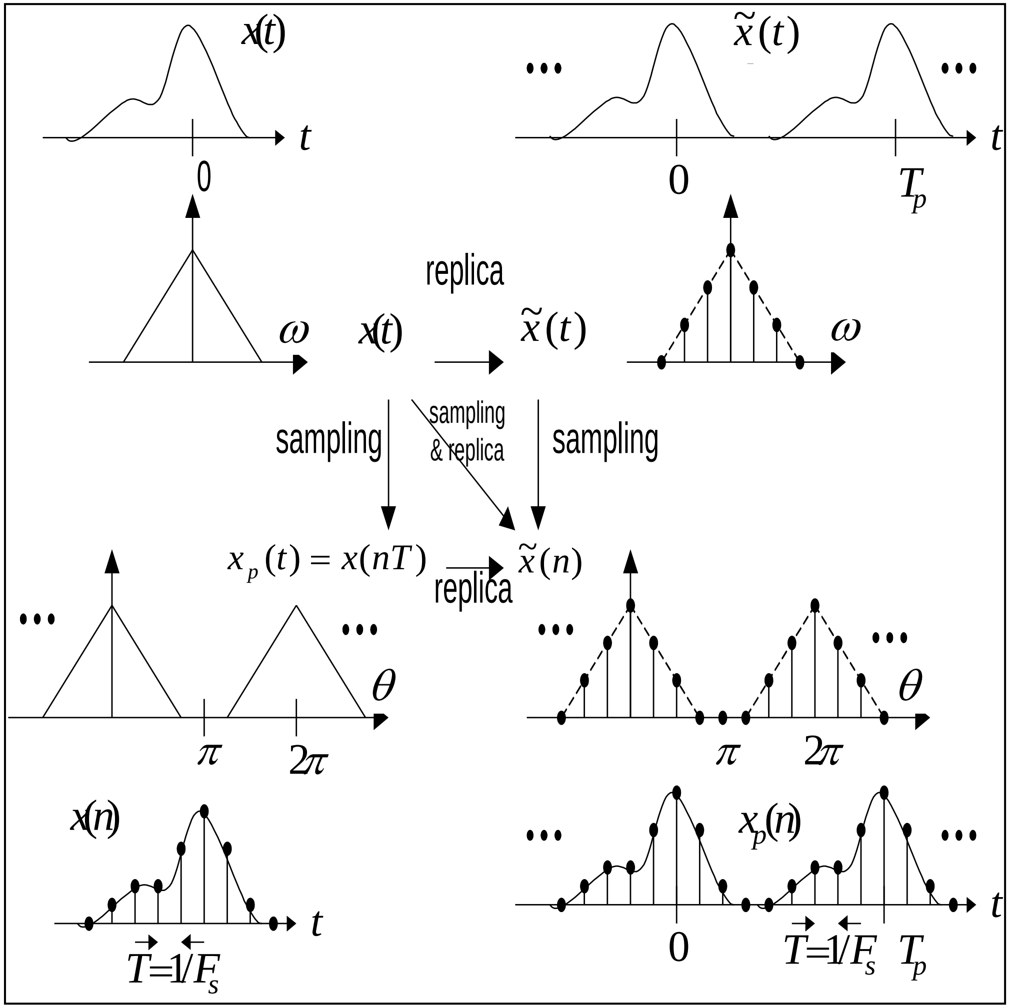
Physical Aspects of Sampling and Reconstruction

❖ **Read the text, pp. 62 - pp. 71.**

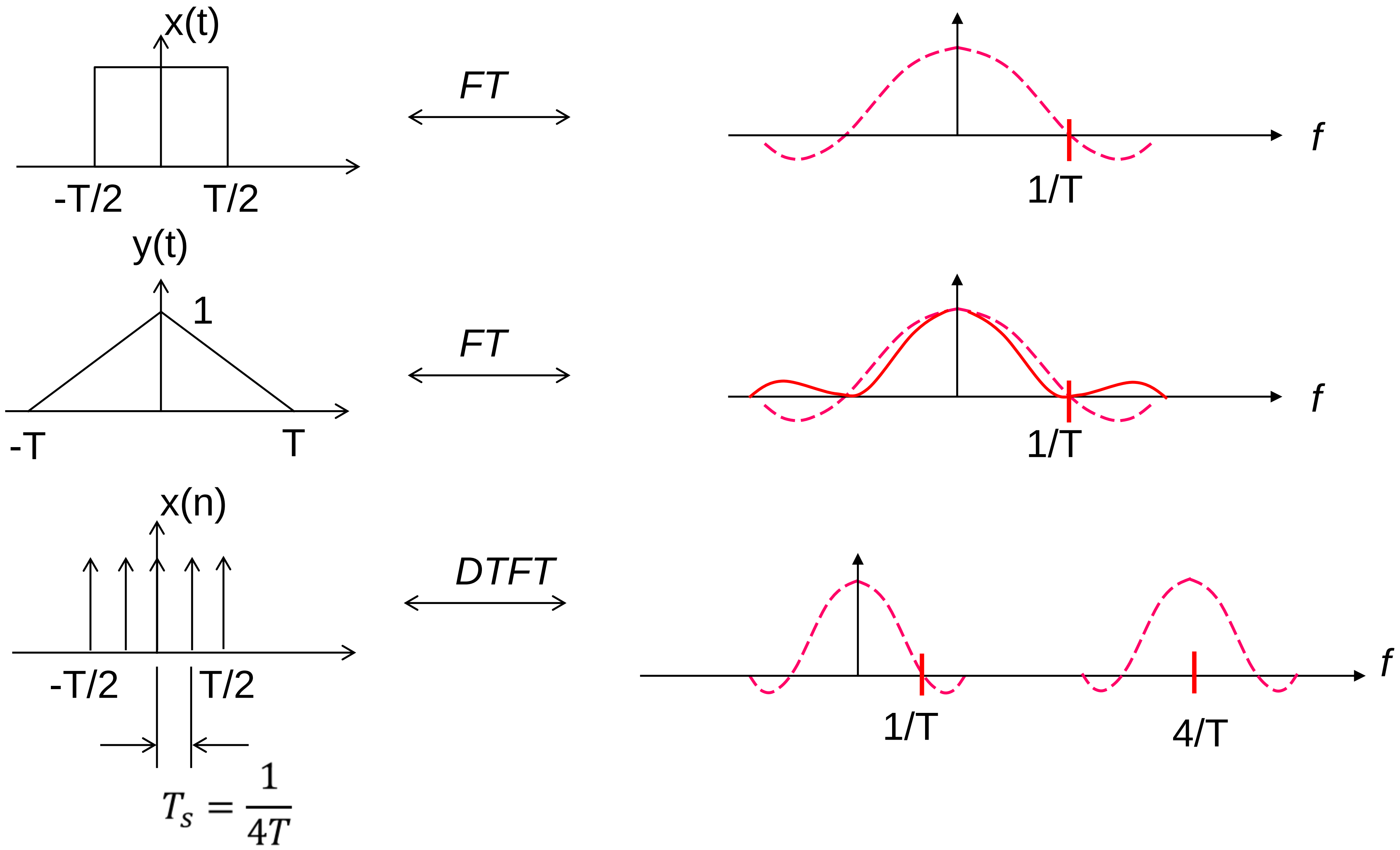
Summary: Transforms for spectral analysis

Relationship between CTFT and DTFT

- **CTFT**
  - For CT nonperiodic signals
- **CTFS**
  - For CT periodic signals
- **DTFT**
  - For DT nonperiodic signals
- **DFT(Discrete Fourier Transform)**
  - For DT periodic signals

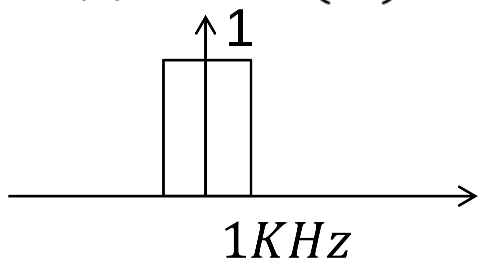


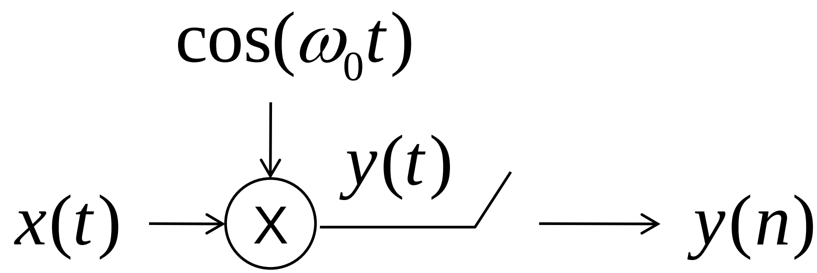
Exercise : DTFT



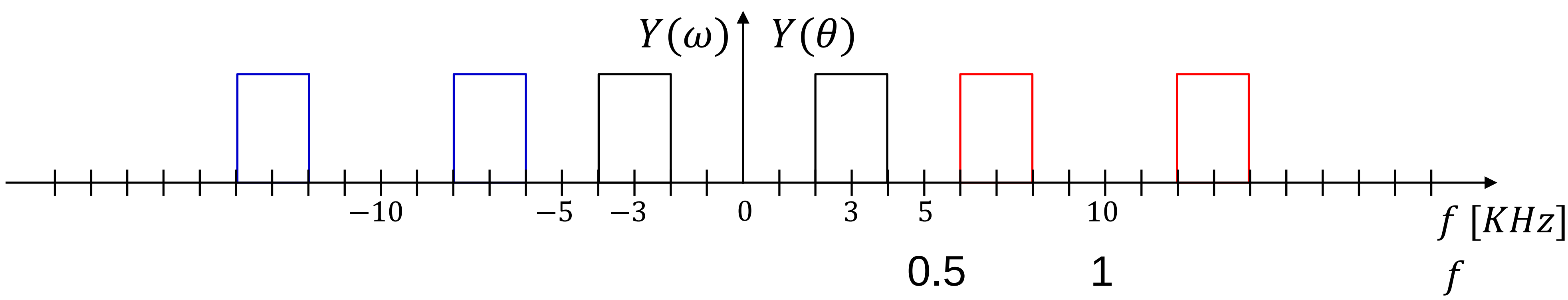
Exercise : FT and DTFT spectra

$x(t) \leftrightarrow X(\omega)$



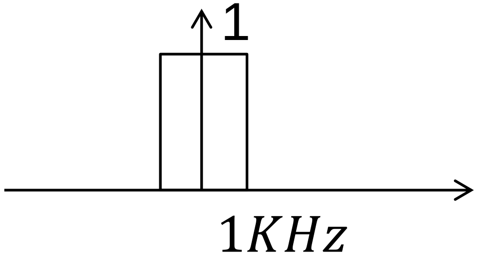


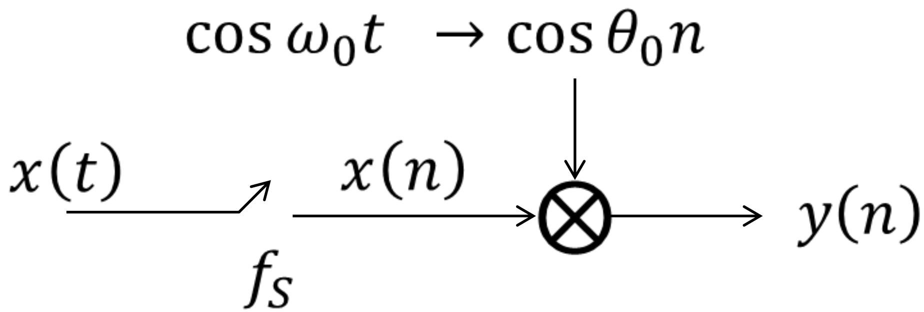
1. Determine the minimum sampling frequency.
2. Draw spectra of $y(t)$ and $y(n)$ when $f_0 = 3KHz$, $f_S = 10KHz$



Exercise : FT and DTFT spectra

$x(t) \leftrightarrow X(\omega)$





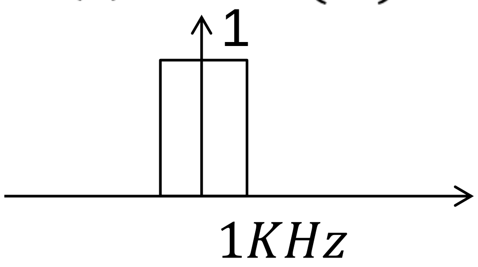
1. Determine the minimum sampling frequency. $f_s \geq 2KHz$?
2. Determine $\theta_0, \underline{f_0}$ and draw spectra of $x(n)$ and $y(n)$

   when 1) $f_S = 10KHz, f_0 = 3KHz$ $\quad \theta_0 = 2\pi f_0/f_s = 2\pi \cdot (\frac{3}{10.}) = 0.6\pi$

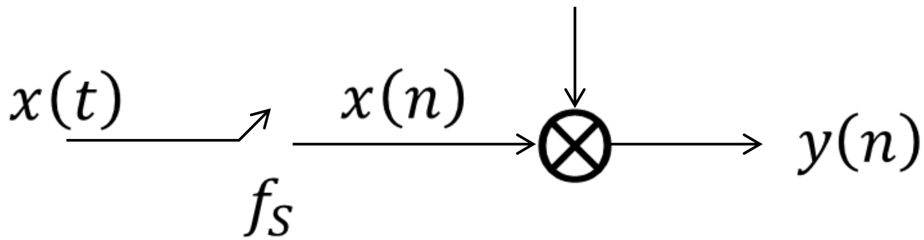
   2) $f_S = 5KHz, f_0 = 3KHz$ $\quad \theta_0 = 2\pi f_0/f_s = 2\pi \cdot (\frac{3}{5}) = 1.2\pi$

Exercise : FT and DTFT spectra

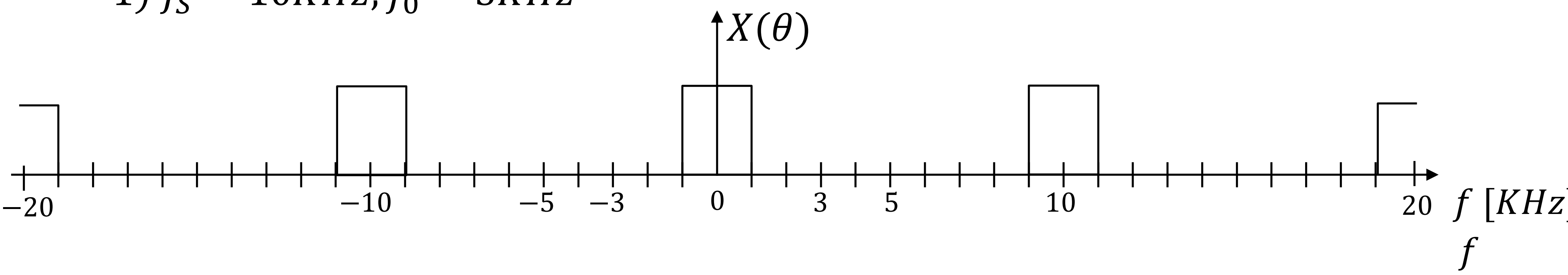
$x(t) \leftrightarrow X(\omega)$

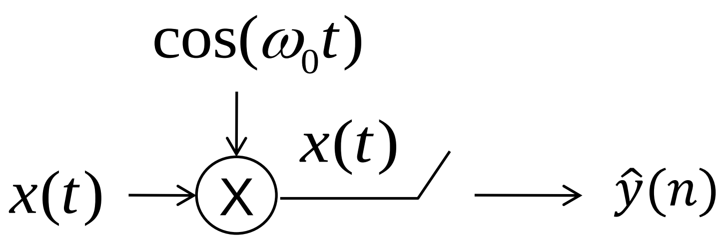


$\cos \omega_0 t \rightarrow \cos \theta_0 n$



1) $f_S = 10KHz, f_0 = 3KHz$



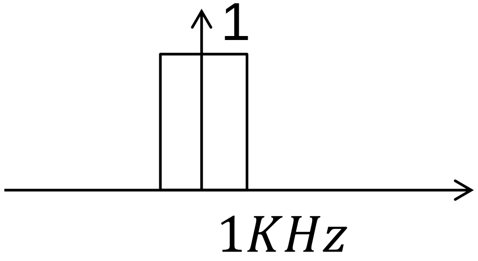


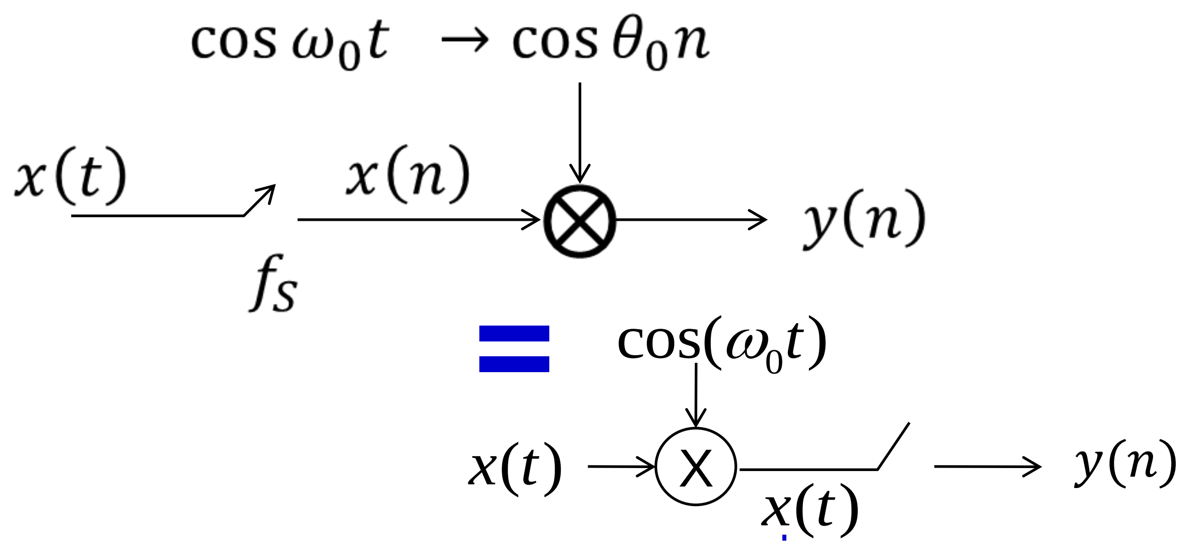
$y(n) = \hat{y}(n)?$

Yes, $\quad$ if $\theta_0 = \omega_0 / f_S$

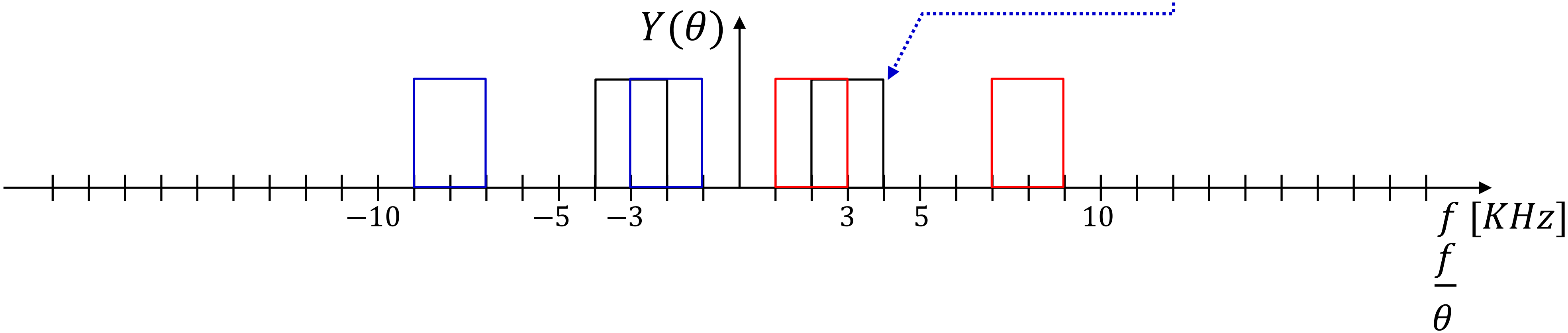
Exercise : FT and DTFT spectra

$x(t) \leftrightarrow X(\omega)$





2) $f_S = 5KHz, f_0 = 3KHz$

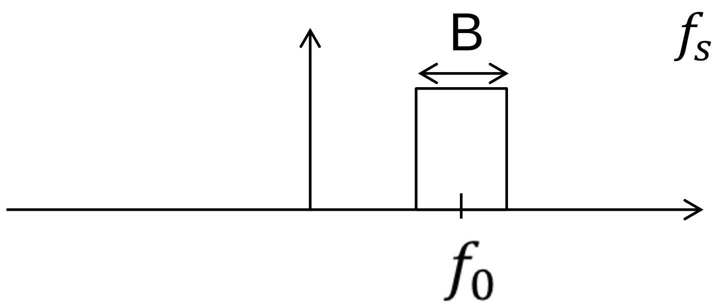


Aliasing occurs

Minimum sampling rate should be ~ $f_{s(minumum)} = 2 \times 4KHz$

Exercise : FT and DTFT spectra

Find minimum sampling rates for the signals with the following spectra



$f_s = B\,??$