Digital Filter - Introduction

Prof. Tai-kyong Song
Dept. of Electronic Engineering
SOGANG UNIVERSITY

Digital Filter - Introduction

Filtering operations

- **Noise suppression**
- **Enhancement of selected freq. range**
- **Bandwidth limiting**
- **Removal or attenuation of specific frequencies**
- **Special operations**
  - Differentiation, Integration
  - Hilbert transform
  - Other mathematical operations

Digital Filter - Introduction

	Analog filter	Digital filter
Realization	Analog circuits	Digital hardware, software
Noise sensitivity	Poor	Free except for quantization noise
Linearity	Nonlinear	Ideal
Dynamic range	limited	Can be infinite
Accuracy	limited	Can achieve any desired level
Stability	Poor	Ideal
Flexibility	Very limited	Maximum
Repeatability	Imperfect	Ideal
Frequency range	Theoretically infinite (Hardware limitation)	$f_s/2$
Size and Cost	limited	Can achieve any goals
Operation complexity	limited	Any desirable degree

Digital Filter - Introduction

Digital Filter types

- **FIR (Finite Impulse Response) filter: All-zero filter**
- **IIR (Infinite Impulse Response) filter: Pole-zero filter**

FIR vs. IIR

- **FIR : Ideal response (linear phase), stable**
- **IIR :**
  - Better magnitude response (sharper transition and/or lower stopband attenuation than FIR with the same number of parameters: HW efficient)
  - Established filter types and design methods.

Filter Specifications / Classifications

- **Low-pass, High-pass, Band-pass, Band-stop(reject), Multi-band filters**
- **Differentiator, Hilbert transform, etc.**

Digital Filter - Introduction

Filter Specifications

- **Frequency response** $H(\Theta) = |H(\Theta)|e^{j\Psi(\Theta)}$
  - Magnitude function (response): $|H(\Theta)|$
  - Phase response: $\Psi(\Theta)$

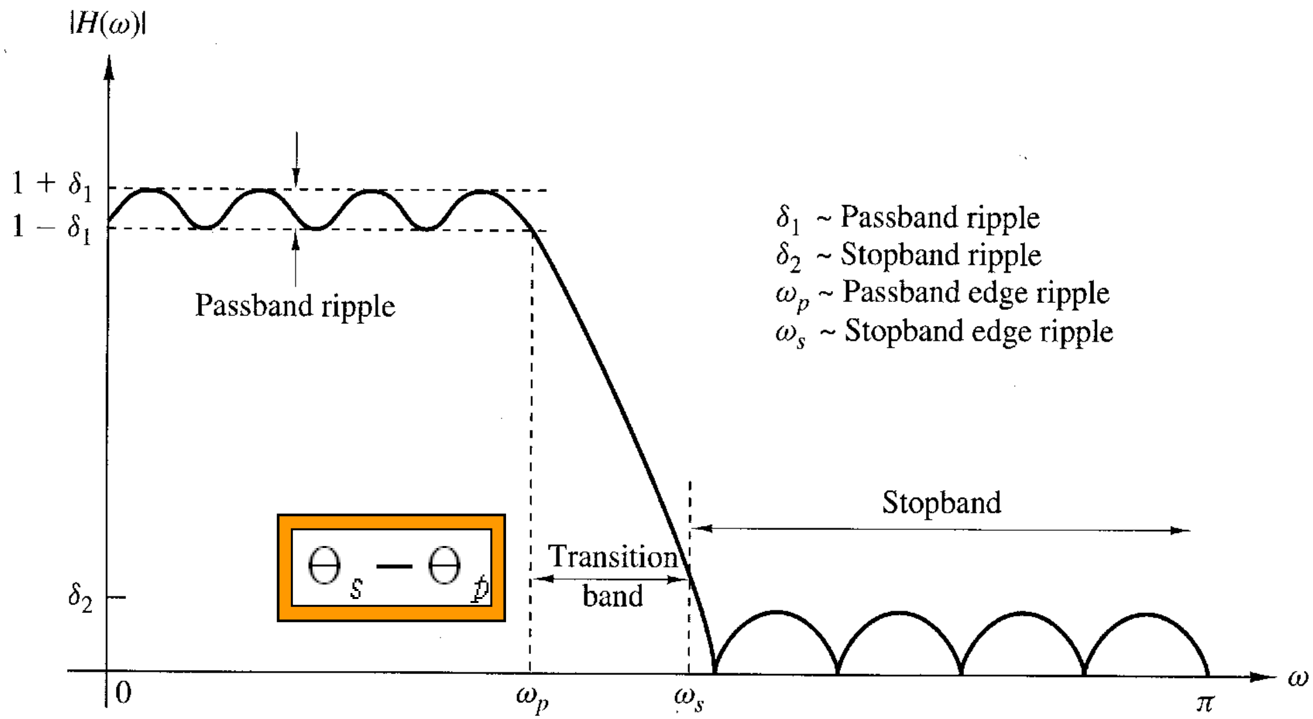


Figure 8.2 Magnitude characteristics of physically realizable filters.

$\boxed{\Theta_s - \Theta_p}$

δ_1: passband ripple

δ_2: stopband ripple

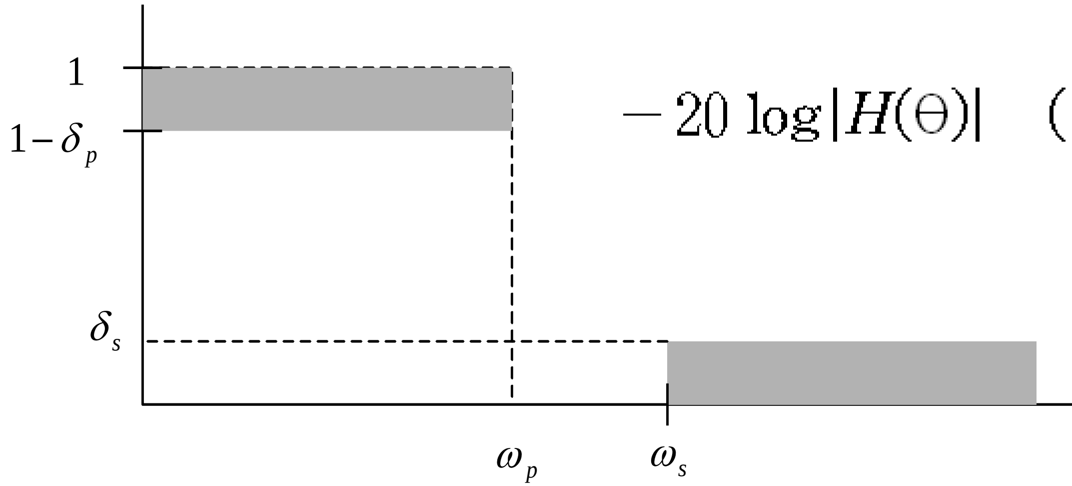
Θ_p: passband edge freq.

Θ_s: stopband edge freq.

bandwidth of the filter = passband width

Filter Specifications

❖ **Other definition (generally used for IIR filters)**



$$-20\log|H(\Theta)| \quad (dB)$$

- Passband ripple : $A_p = -20\log(1-\delta_p) \approx 8.6859\,\delta_p$
- Stopband attenuation : $A_s = -20\log\delta_s$
- (-3dB) cutoff frequency : Θ_{3db}

Digital Filter - Introduction

Filter Specifications

- **LTI filters are preferred.**

$$H(\Theta) = \frac{\sum_{k=0}^{M} b_k e^{-j\Theta k}}{1 + \sum_{k=1}^{N} a_k e^{-j\Theta k}} \qquad y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

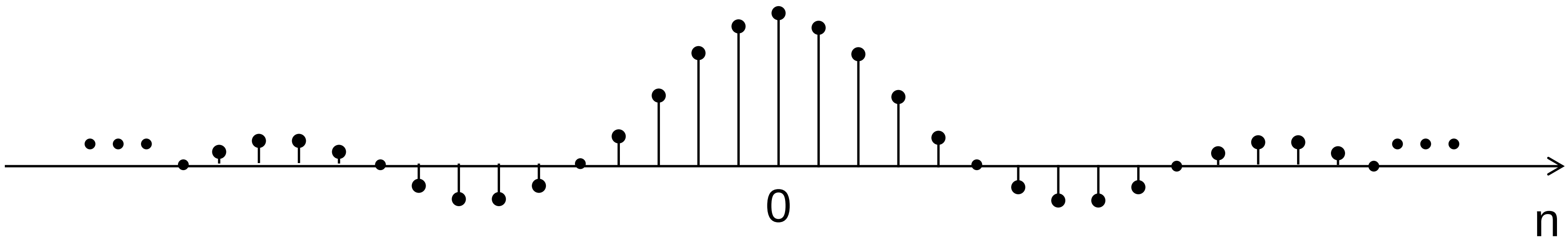
- **Filter spec:** $\Theta_p, \ \Theta_s, \ \delta_1, \ \delta_2, \ M, \ N$

- **A monotone response, either increasing and decreasing**
- **An oscillating, or rippling, response**
- **Equiripple filter**

Digital Filter - Introduction

Ideal lowpass filter

$$H(\theta)=\begin{cases}1, & |\theta|\le\theta_c \\ 0, & \theta_c<\theta\le\pi\end{cases} \qquad h(n)=\begin{cases}\dfrac{\theta_c}{\pi}, & n=0 \\ \dfrac{\theta_c}{\pi}\,\dfrac{\sin\theta_c n}{\theta_c n}, & n\neq 0\end{cases}$$



- ❖ **Zero phase response** $H\left(e^{j\theta}\right)=A(\theta)e^{j\phi(\theta)}$ $\quad A(\theta)=1$ $\quad \boxed{\phi(\theta)=0}$
- ❖ **Noncausal system**
- ❖ **Infinite duration**

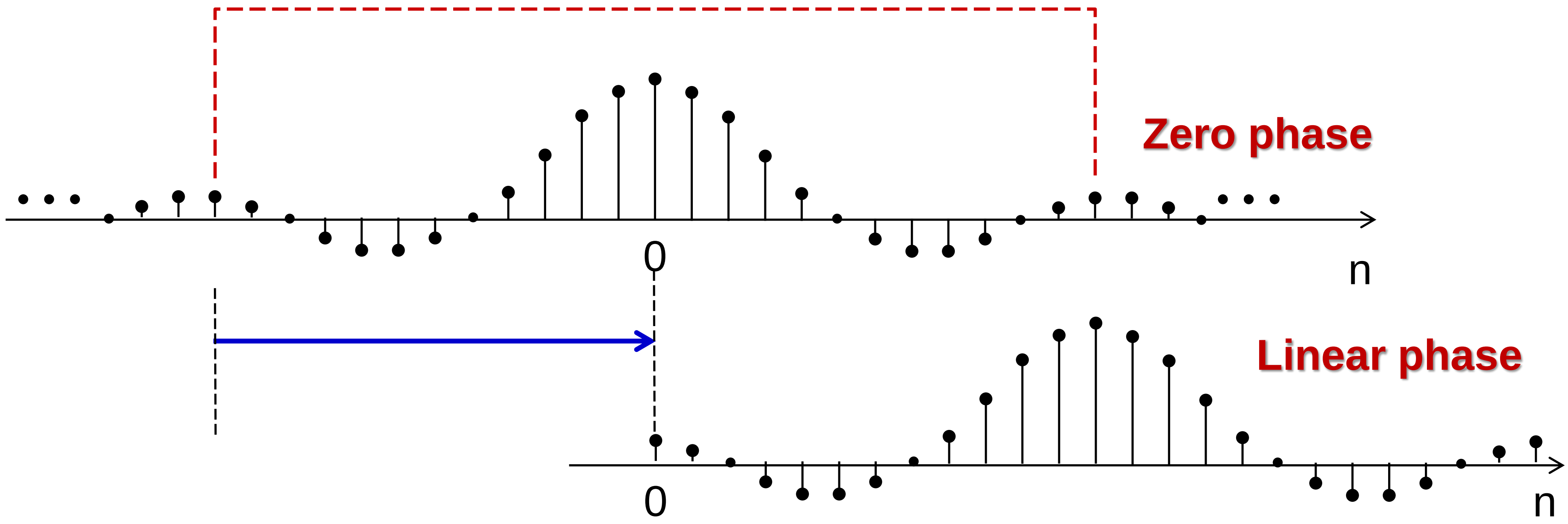
Digital Filter - Introduction

Ideal lowpass filter

- To make it causal, the impulse response is shifted after being truncated.

→ Windowing

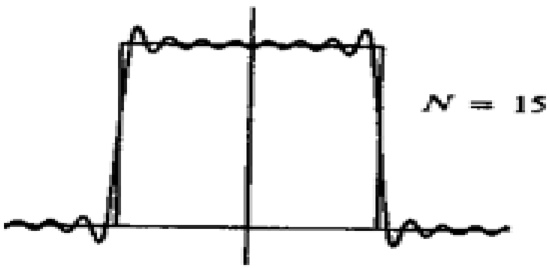
→ Shifting : Linear phase / Constant group delay



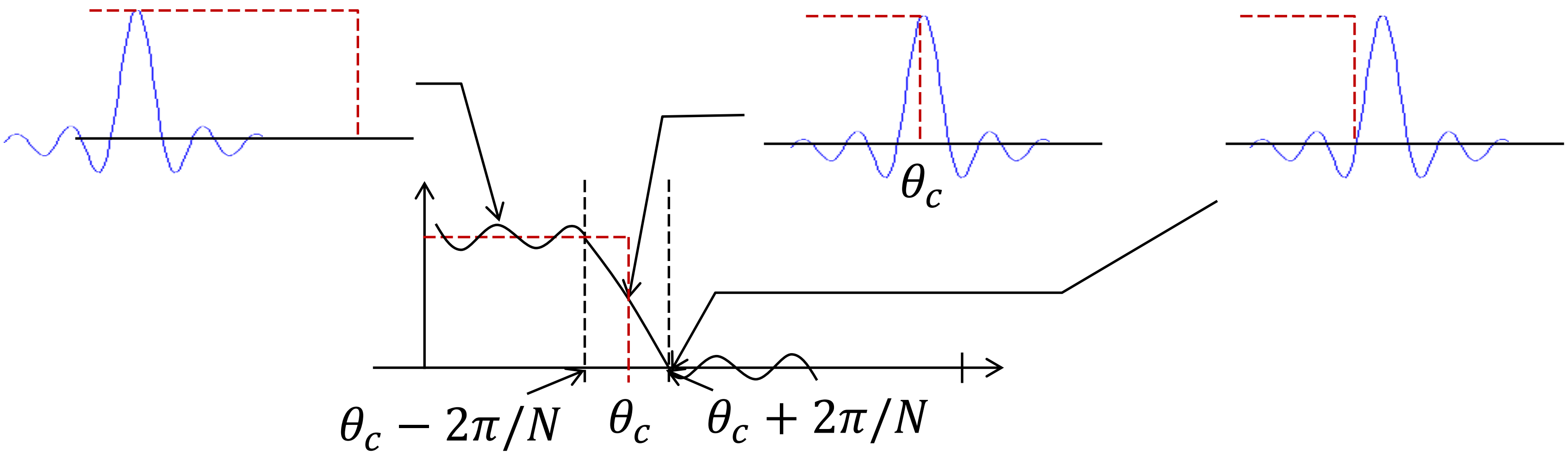
Digital Filter - Introduction

Ideal lowpass filter

- Note that truncation with a rectangular window results in the Gibbs phenomenon.



- Transition band
- Ripples in passband, stopband



Digital Filter - Introduction

FIR filter

- **All-zero filters**

$$y(n) = b_0 x(n) + b_1 x(n-1) + \cdot\ \cdot\ \cdot + b_{M-1} x(n-M+1)$$

$$= \sum_{k=0}^{M-1} b_k x(n-k)$$

$$y(n) = \sum_{k=0}^{M-1} h(k) x(n-k)$$

- **Finite impulse response:** $h(k) = b_k$
- **System function**

$$H(z) = \sum_{k=0}^{M-1} h(k) z^{-k}$$ *: a polynomial of degree M-1 in* z^{-1}

Digital Filter - Introduction

FIR filter - Introduction

- stable
- noncausal $\xrightarrow{delay}$ causal
- Linear phase : constant time or group delay.
  - h(n) = h(M-1-n) : symmetric or
  - h(n) = -h(M-1-n) : antisymmetric
- Used in many applications:
  - speech-processing, data-transmission, image processing, etc.
- Need a long filter (or many taps) for sharp transition.
- Methods for designing an FIR filter
  - window method
  - frequency-sampling method
  - optimal or minimax method

FIR Filter

Ideal Filter Response

Linear phase

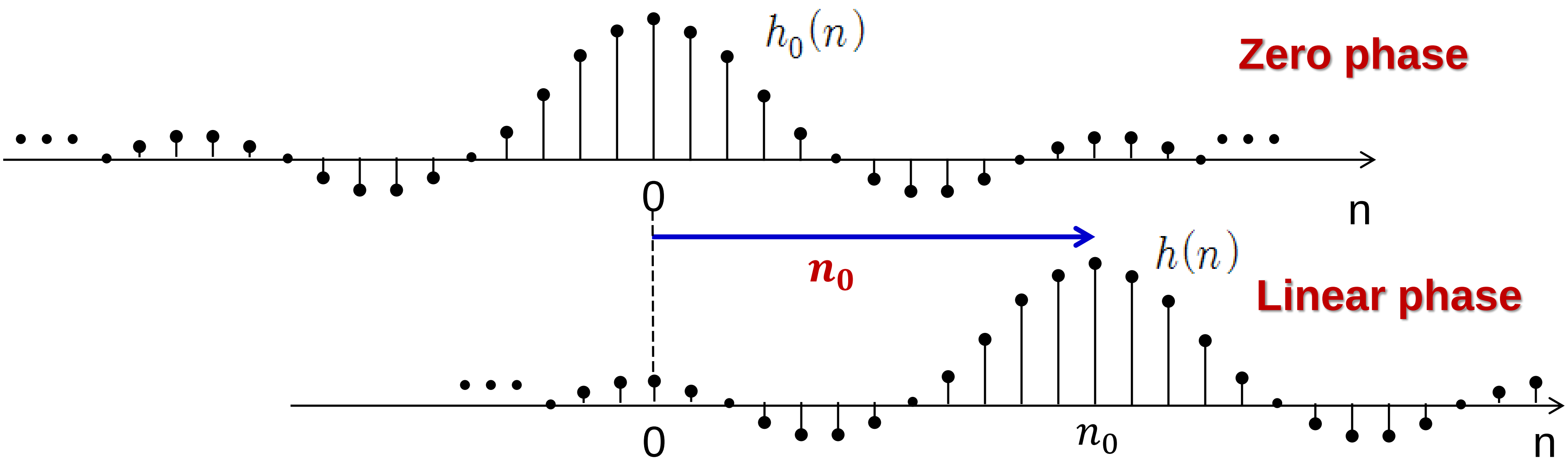
$$H(\Theta) = \begin{cases} Ce^{-j\Theta n_0}, & \Theta_1 < \Theta < \Theta_2 \\ 0, & otherwise \end{cases}$$

$$\boxed{h(n) = h_0(n - n_0)}$$

$$\boxed{H(z) = z^{-n_0} H_0(z)}$$

$$\Rightarrow \quad Y(\Theta) = X(\Theta)H(\Theta) = CX(\Theta)e^{-j\Theta n_0} \qquad \Theta_1 < \Theta < \Theta_2$$

$$y(n) = C\,x(n - n_0)$$: ***Group delay***: n_0



FIR Filter

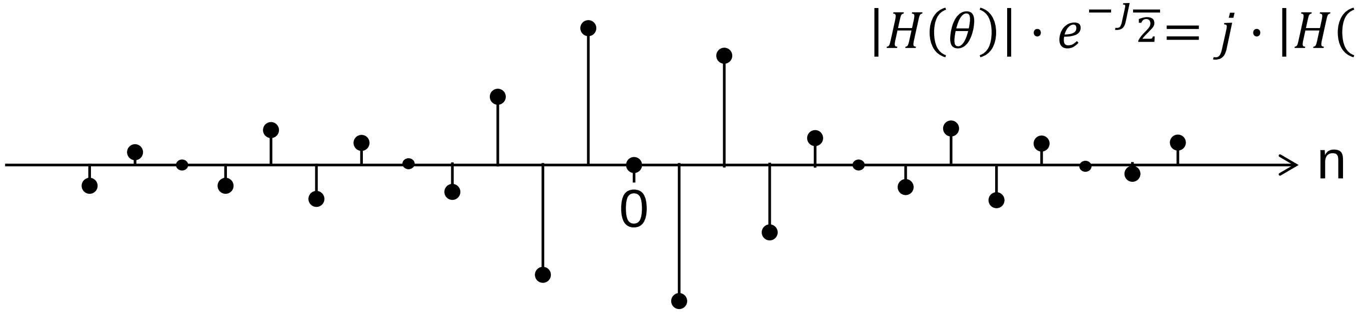
Group delay or envelope delay

$$\tau_g(\Theta) = -\frac{d\Theta(\Theta)}{d\Theta}$$

$$|H(\theta)| \cdot e^{-j(\theta n_0 + costant)}$$

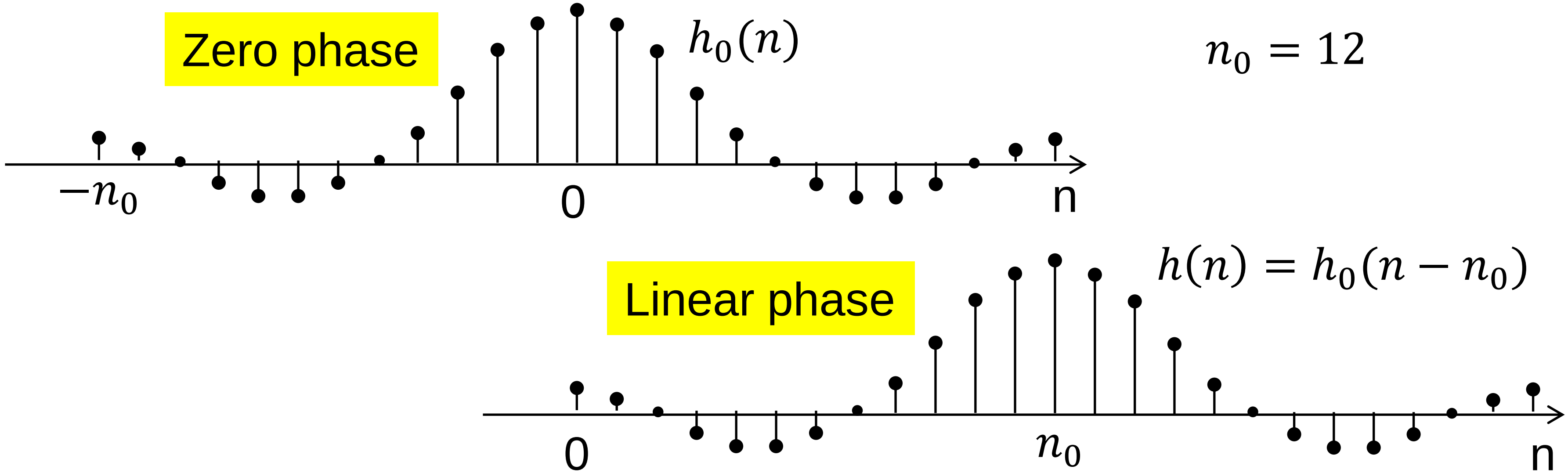
- $\tau_g(\Theta) = n_0$ **in the linear phase systems.**
  - ※ **All the signal components with different frequencies shall be delayed equally.**
  - ※ **No phase distortion, which is of great importance.**
  - ※ **Only FIR systems can have the linear phase characteristic.**
- $\tau_g(\Theta) = 0$ **in the zero phase systems.**
  - ※ **This must be a noncausal system.**

$$|H(\theta)| \cdot e^{-j\frac{\pi}{2}} = j \cdot |H(\theta)|$$



FIR Filter

Linear phase vs Zero phase from group delay viewpoint



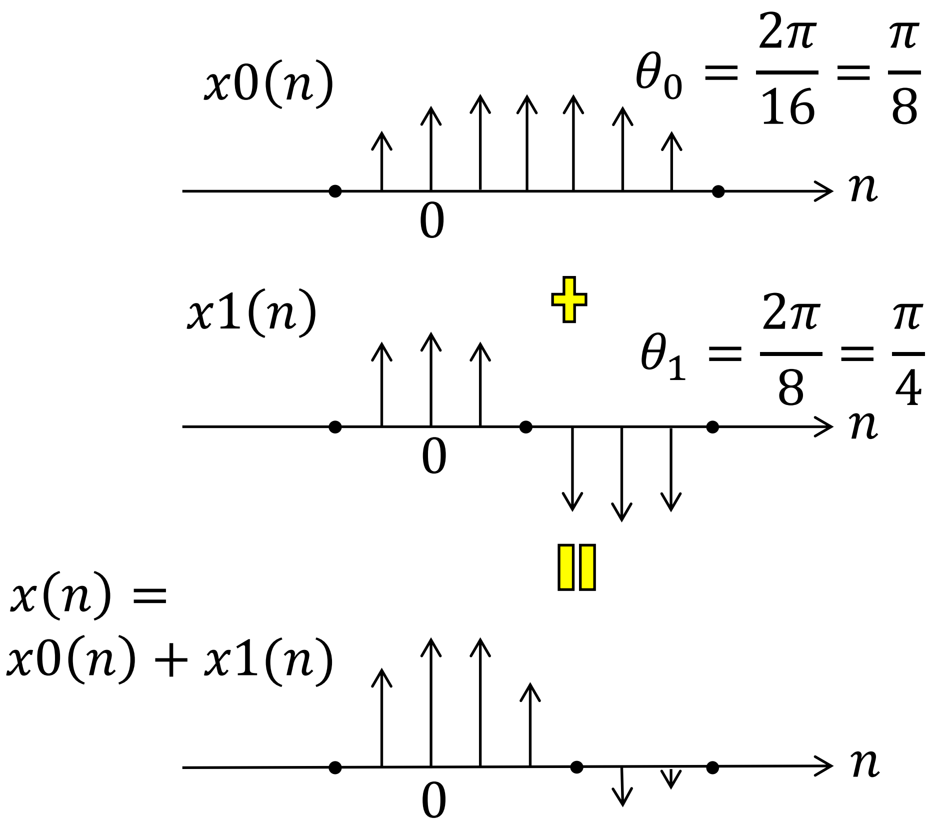
$n_0 = 12$

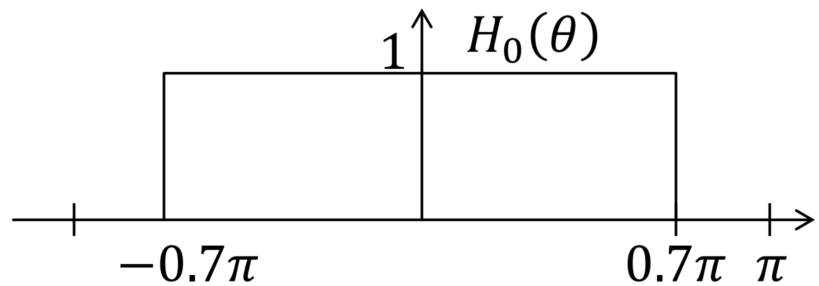
$$y_0(n) = h_0(n) * x(n) = cx(n) \qquad Y_0(\theta) \leftrightarrow c \cdot X(\theta)$$

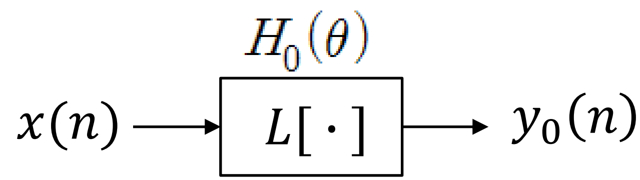
$$\begin{aligned} y(n) &= h(n) * x(n) = h_0(n - n_0) * x(n) = h_0(n) * x(n - n_0) \\ &= y_0(n - n_0) = cx(n - n_0) \qquad Y(\theta) \leftrightarrow ce^{j\theta n_0} \cdot X(\theta) \end{aligned}$$

FIR Filter

Group delay vs. linear phase $x(n) = x0(n) + x1(n)$





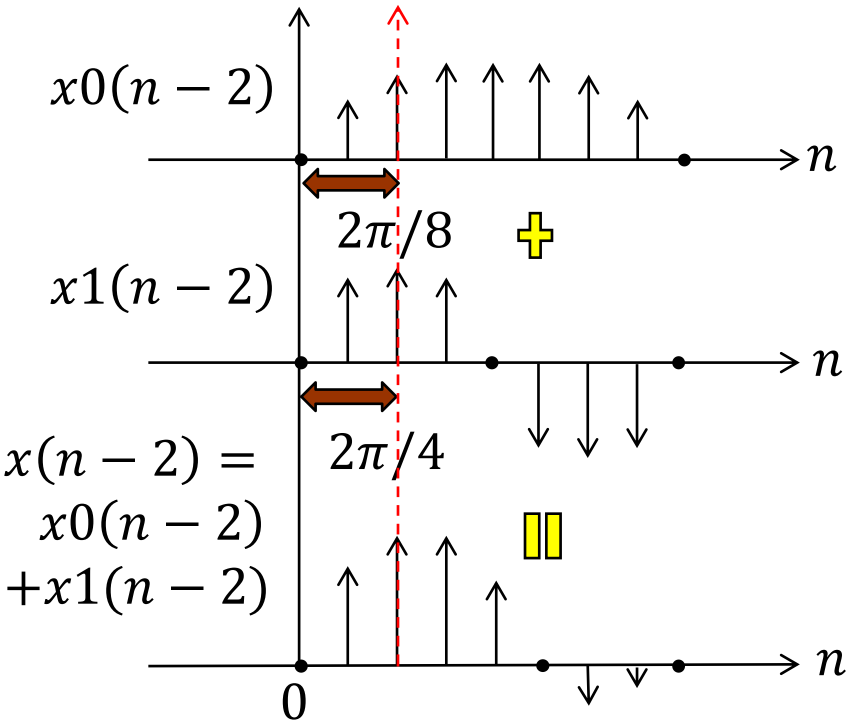


$$y_0(n) = L[x(n)] = L[x0(n)] + L[x1(n)]$$

$$y_0(n) = x(n) = x0(n) + x1(n)$$

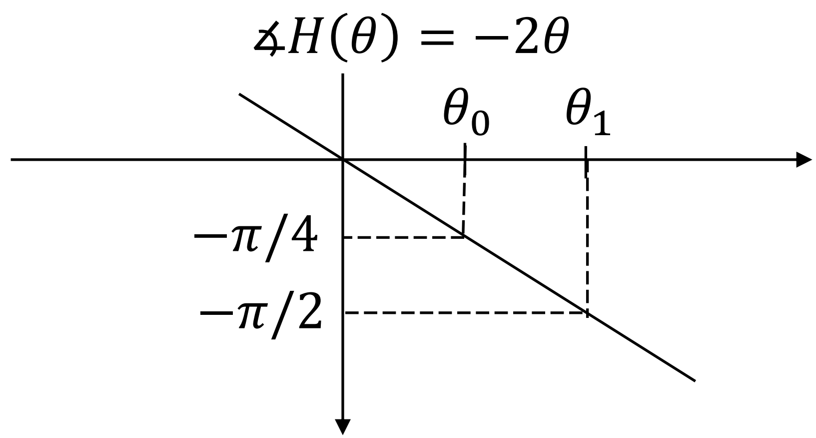
FIR Filter

Group delay vs. linear phase



$$\textcolor{red}{x(n) = x0(n) + x1(n)} \longrightarrow \boxed{H(\theta) = 1 \cdot e^{-j\theta n_0}} \longrightarrow y(n)$$

$n_0 = 2$



$$Y(\theta) = e^{-j\theta 2}X(\theta) = \underline{X0(\theta)}e^{-j\pi/4} + \underline{X1(\theta)}e^{-j\pi/2}$$

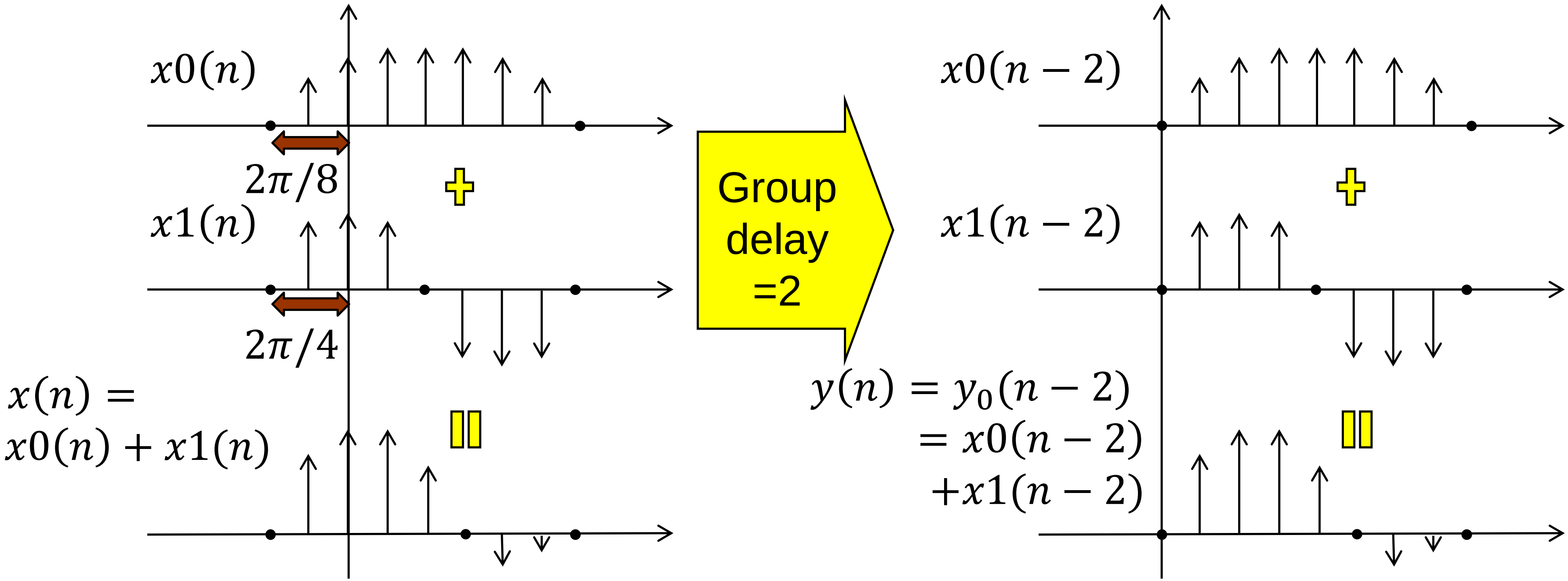
$$y(n) = x(n-2) = x0(n-2) + x1(n-2)$$

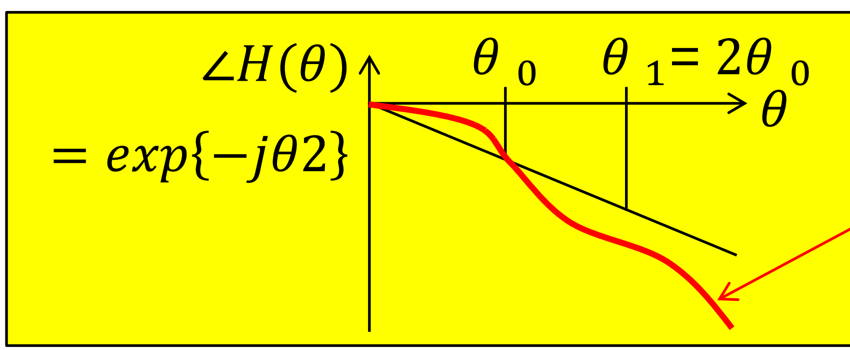
$$\underline{e^{j\frac{\pi}{8}n}}e^{-j\frac{2\pi}{8}} = e^{j\frac{\pi}{8}(n-2)}$$

$$\underline{e^{j\frac{\pi}{4}n}}e^{-j\frac{2\pi}{4}} = e^{j\frac{\pi}{4}(n-2)}$$

FIR Filter

Group delay vs. linear phase $x(n) = x0(n) + x1(n)$

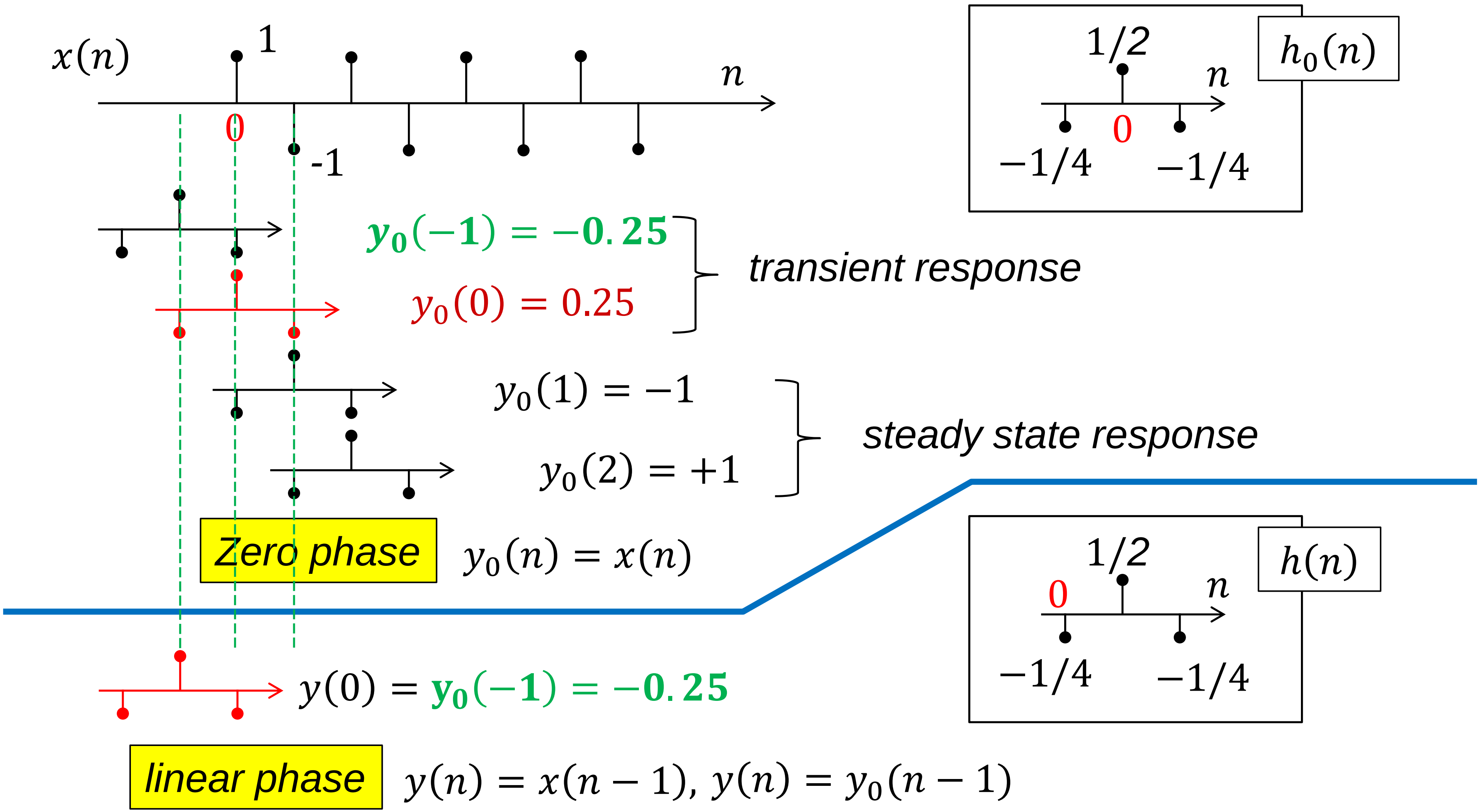




What if phase is nonlinear?

FIR Filter

Group delay vs. linear phase



FIR Filter

Linear Phase FIR filter

$$H(z) = h_0 + h_1 z^{-1} + h_2 z^{-2} + h_3 z^{-3} + h_4 z^{-4} \quad (M = 5,\ \ h_0 = \pm h_4,\ \ h_1 = \pm h_3)$$

$$H(z) = h_0(1 \pm z^{-4}) + h_1(z^{-1} \pm z^{-3}) + h_2 z^{-2}$$

$$= z^{-2}\{h_2 + h_0[z^2 \pm z^{-2}] + h_1[z \pm z^{-1}]\}, \quad (M-1)/2 = 2,\ (M-3)/2 = 1$$

$$= z^{-(M-1)/2}\left\{h_2 + \sum_{n=0}^{(M-3)/2} h_n [z^{(M-1-2n)/}2 \pm z^{-(M-1-2n)}/2]\right\}$$

$$H(e^{jw}) = e^{-jw(M-1)/2}\left\{h_2 + \sum_{n=0}^{(M-3)/2} h_n [e^{jw(M-1-2n)/2} \pm e^{-jw(M-1-2n)/2}]\right\}$$

$$= e^{-jw(M-1)/2}\left\{h_2 + \sum_{n=0}^{(M-3)/2} h_n [e^{jw(M-1-2n)/2} \pm e^{-jw(M-1-2n)/2}]\right\}$$

$$= \begin{cases} e^{-jw(M-1)/2}\left\{h_2 + \sum_{n=0}^{(M-3)/2} 2h_n [\cos(\omega(M-1-2n)/2)\right\} & \text{Even Symmetric} \\ 2je^{(-jw(M-1)/2)}\left\{0 + \sum_{n=0}^{(M-3)/2} h_n [\sin(\omega(M-1-2n)/2)\right\},\ \ h_2 = 0 & \text{Odd Symmetric} \end{cases}$$

FIR Filter

Linear Phase FIR filter

❖ **Linear phase condition**

$$h(n) = \pm h(M-1-n) \qquad n = 0, 1, \ldots, M-1$$

$$\boxed{z^{-(M-1)}H(z^{-1}) = \pm H(z)}$$

$$H(z) = h_0 + h_1 z^{-1} + \cdots + h_{M-2} z^{-(M-2)} + h_{M-1} z^{-(M-1)}$$

$$H(z^{-1}) = h_0 + h_1 z^{1} + \cdots + h_{M-2} z^{(M-2)} + h_{M-1} z^{(M-1)}$$

$$z^{-(M-1)}H(z) = h_0 z^{-(M-1)} + h_1 z^{-(M-2)} + \cdots + h_{M-2} z^{-1} + h_{M-1}$$

$$= h_{M-1} + h_{M-2} z^{-1} + \cdots + h_1 z^{-(M-2)} + h_0 z^{-(M-1)}$$

FIR Filter

Linear Phase FIR filter

- **Zeros of LP FIR filter must occur in reciprocal pairs.**

$$z^{-(M-1)}H(z^{-1}) = \pm H(z)$$

- **If h(n) is real, complex-valued zeros of H(z) must occur in complex-conjugate pairs.**

$$\because H^*(z) = H(z^*)$$

$\Rightarrow$ If z_i is a zero,
then so are $z_i^*, 1/z_i, 1/z_i^*$

$$\boxed{H(z_k) = H(z_k^{-1}) = H(z_k^*) = H(1/z_k^*) = 0}$$

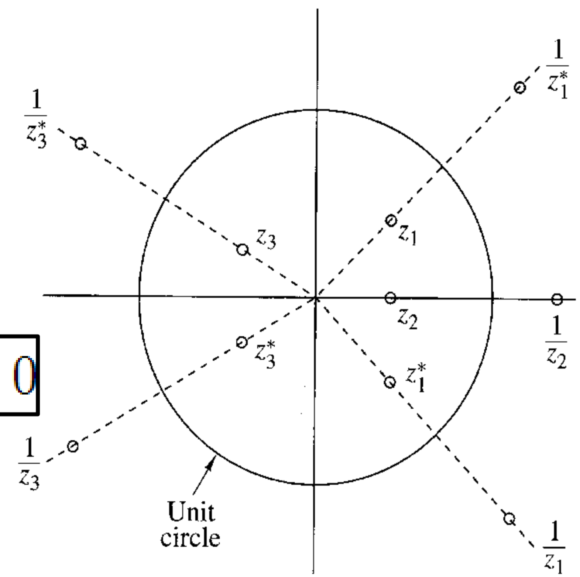


Figure 8.3 Symmetry of zero locations for a linear-phase FIR filter.

FIR Filter

Linear Phase FIR filter

❖ **Frequency response analysis of LP FIR filters**

$$h(n) = h(M-1-n)$$

$$H(\Theta) = H_r(\Theta)\, e^{-j\Theta(M-1)/2}$$

❖ $H_r(\Theta)$ **: Amplitude function that can have negative values.**

$$H_r(\Theta) = h\left(\frac{M-1}{2}\right) + 2\sum_{n=0}^{(M-3)/2} h(n)\cos\Theta\left(\frac{M-1}{2} - n\right) \qquad M \text{ odd}$$

$$H_r(\Theta) = 2\sum_{n=0}^{(M/2)-1} h(n)\cos\Theta\left(\frac{M-1}{2} - n\right) \qquad M \text{ even}$$

$$\psi(\Theta) = \begin{cases} -\Theta\left(\dfrac{M-1}{2}\right), & \text{if } H_r(\Theta) > 0 \\ -\Theta\left(\dfrac{M-1}{2}\right) + \pi, & \text{if } H_r(\Theta) < 0 \end{cases}$$

FIR Filter

Linear Phase FIR filter

❖ **Frequency response analysis of LP FIR filters**

$$h(n) = -h(M-1-n)$$

$$H(\theta) = H_r(\theta)\, e^{j[-\theta(M-1)/2+\pi/2]}$$

$$H_r(\theta) = 2\sum_{n=0}^{(M-3)/2} h(n)\sin\theta\left(\frac{M-1}{2}-n\right) \qquad M \ odd$$

$$H_r(\theta) = 2\sum_{n=0}^{(M/2)-1} h(n)\sin\theta\left(\frac{M-1}{2}-n\right) \qquad M \ even$$

$$\Theta(\theta) = \begin{cases} \dfrac{\pi}{2} - \theta\left(\dfrac{M-1}{2}\right), & \text{if } H_r(\theta) > 0 \\ \dfrac{3\pi}{2} - \theta\left(\dfrac{M-1}{2}\right), & \text{if } H_r(\theta) < 0 \end{cases}$$