Practical Spectral Analysis
: Windows

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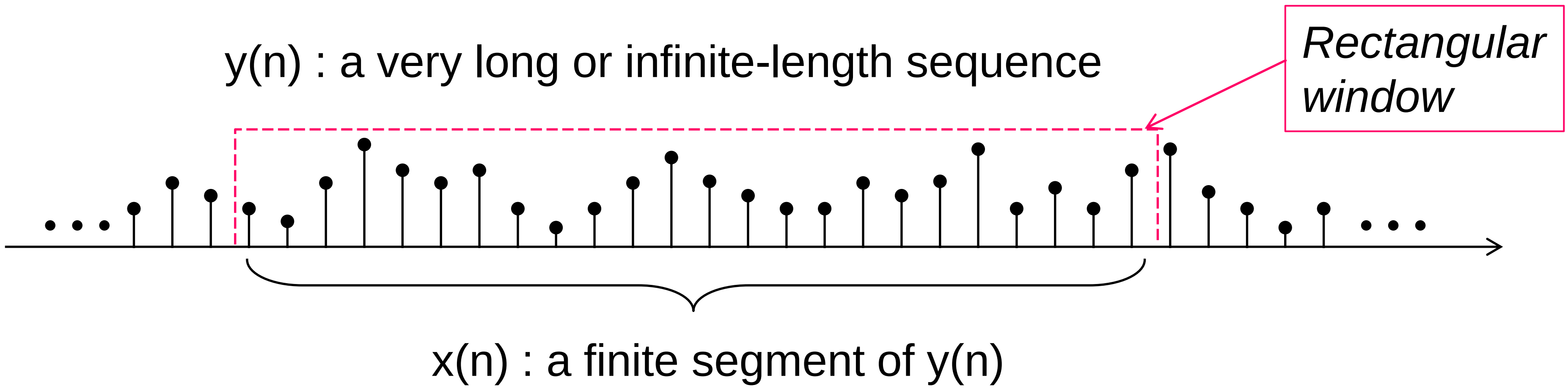
Effect of Rectangular Windowing

Practical spectral analysis issues

- ☞ **One can't take FFT of an infinite-length sequence**
- ☞ **Very often, we should use a small part of a sequence, even when spectral analysis of an infinite-length sequence is attempted.**



Effect of Rectangular Windowing

How to limit the length of a sequence ?

☞ Short-time spectral analysis by Windowing

☞ Rectangular window of length N

$$w_r(n) = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & otherwise \end{cases}$$

❖ Windowed sequence

$$x(n) \ = y(n) w_r(n)$$

1. The two spectra are different: $Y(\theta) \neq X(\theta)$
2. Other way of windowing?

☞ Is a rectangular window the best choice?

Effect of Rectangular Windowing

Example)

$$y(n) = a^n u(n) \quad \longleftrightarrow \quad Y(\Theta) = 1/(1 - ae^{-j\Theta})$$

$$x(n) = y(n)\, w_r(n) \longleftrightarrow X(\Theta) = \sum_{n=0}^{N-1} a^n e^{-j\Theta n}$$

$$= \frac{1 - a^N e^{-j\Theta N}}{1 - ae^{-j\Theta}} = \frac{1}{1 - ae^{-j\Theta}} - \underbrace{\frac{a^N e^{-j\Theta N}}{1 - ae^{-j\Theta}}}_{\text{Difference}}$$

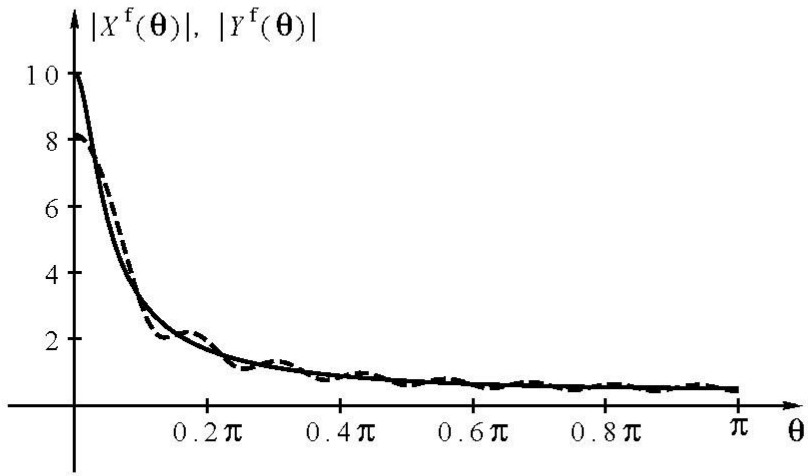


Figure 6.5 The Fourier transform (magnitude) of an exponential sequence. Solid line: unwindowed; dashed line: windowed.

Effect of Rectangular Windowing

Windowed spectrum

- $X(\Theta) = \dfrac{1}{2\pi} Y(\Theta) * W_r(\Theta)$

$W_r(\theta) =$

$$\sum_{n=0}^{N-1} e^{-j\theta n} = \frac{1-e^{-j\theta N}}{1-e^{-j\theta}} = \frac{e^{-j\theta N/2}}{e^{-j\theta/2}} \frac{e^{j\theta N/2}-e^{-j\theta N/2}}{e^{j\theta/2}-e^{-j\theta/2}} = e^{-j\theta(N-1)/2} \frac{\sin(\frac{\theta N}{2})}{\sin(\frac{\theta}{2})}$$

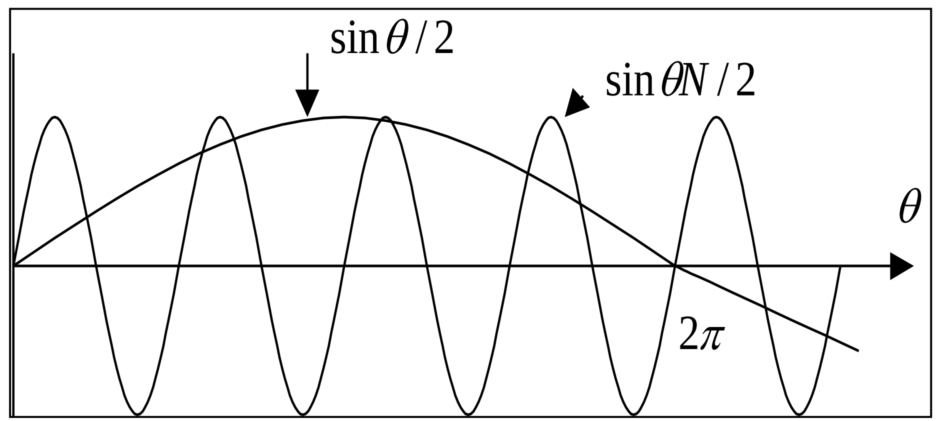
➔ $D(\Theta, N) = \dfrac{\sin(\Theta N/2)}{\sin(\Theta/2)}$

- Zeros of the numerator:

$$\Theta N/2 = \pm n\pi \rightarrow \Theta = \pm 2n\pi/N$$

- Zeros of the denominator:

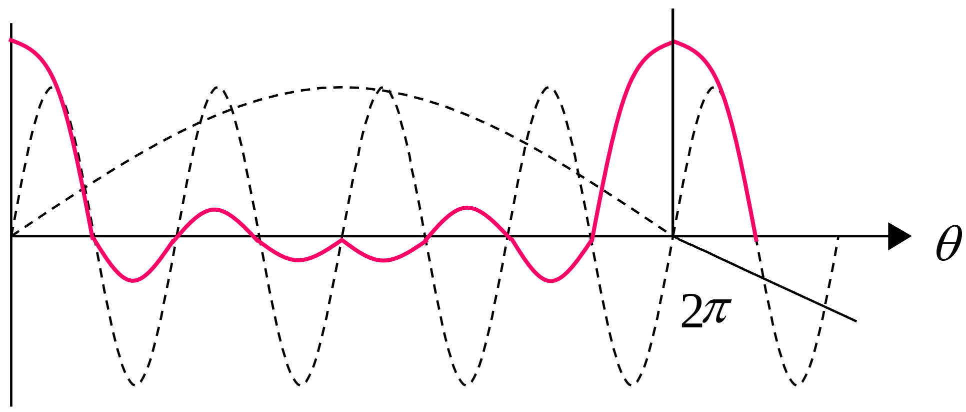
$$\Theta/2 = \pm m\pi \rightarrow \Theta = \pm 2m\pi$$



Effect of Rectangular Windowing

Windowed spectrum – Rectangular window

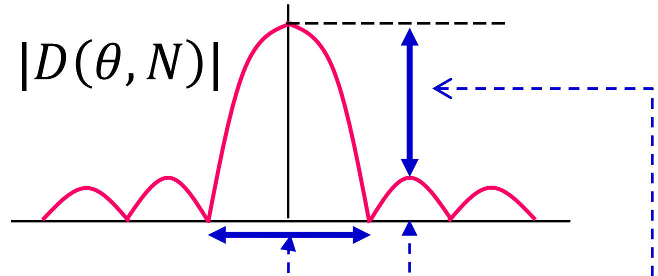
- $W_r(\theta) =$
  $$\sum_{n=0}^{N-1} e^{-j\theta n} = \frac{1-e^{-j\theta N}}{1-e^{-j\theta}} = \frac{e^{-j\theta N/2}}{e^{-j\theta/2}} \frac{e^{j\theta N/2}-e^{-j\theta N/2}}{e^{j\theta/2}-e^{-j\theta/2}} = e^{-j\theta(N-1)/2} \frac{\sin(\frac{\theta N}{2})}{\sin(\frac{\theta}{2})}$$
  - Peak values: $\Theta = \pm 2m\pi$
  - Null points: $\Theta = \pm 2n\pi/N, \quad n = 1, 2, \ldots, N-1$



Effect of Rectangular Windowing

Spectrum of a Rectangular window

$$D(\Theta, N) = \frac{\sin(\Theta N/2)}{\sin(\Theta/2)}$$



- **Mainlobe width**

$$2\frac{2\pi}{N}$$

$$\Theta = 3\pi/N$$

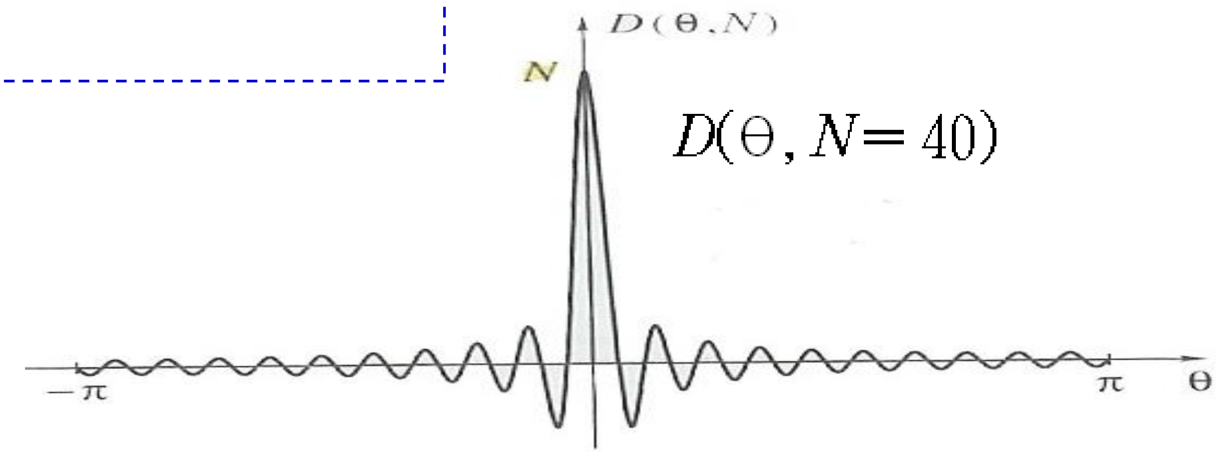
- **Sidelobes**

Local Maxima between zero points:

$$\Theta = (2n+1)\pi/N, \quad n = 1, 2, \ldots, N-1$$

Peak sidelobe: -13.5dB

at $\Theta = 3\pi/N$



Effect of Rectangular Windowing

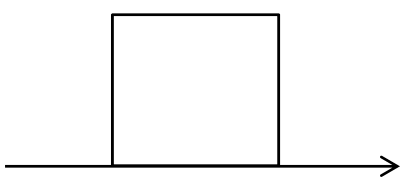
Windowed spectrum

- **Example (Analog case)**

$$y(t) = W \cdot \operatorname{sinc}(Wt) \longleftrightarrow Y(f) = rect(f/W)$$

$$w_r(t) = \frac{1}{D} rect(t/D) \longleftrightarrow W_r(f) = \operatorname{sinc}(Df)$$

$$X(f) = Y(f) * W_r(f)$$



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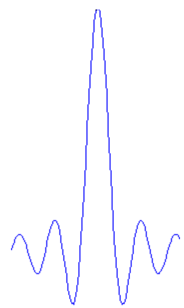
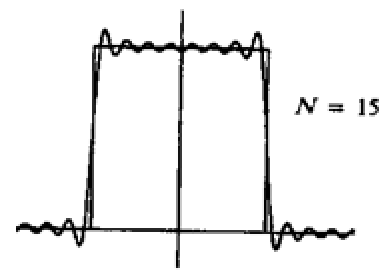




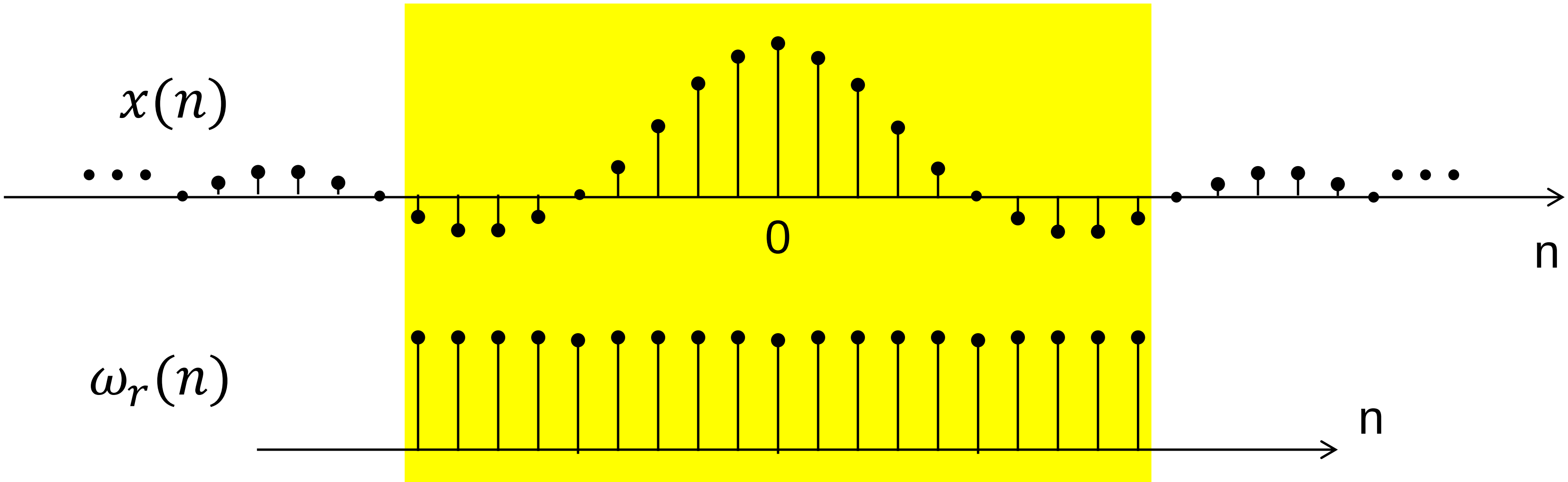
Gibbs Phenomenon !!

Effect of Rectangular Windowing

Windowed spectrum: case 1

$$X(\Theta) = rect(\Theta/2W) \longleftrightarrow x(n) = \frac{\omega_c}{\pi} \frac{\sin(\omega_c n)}{\omega_c n}$$

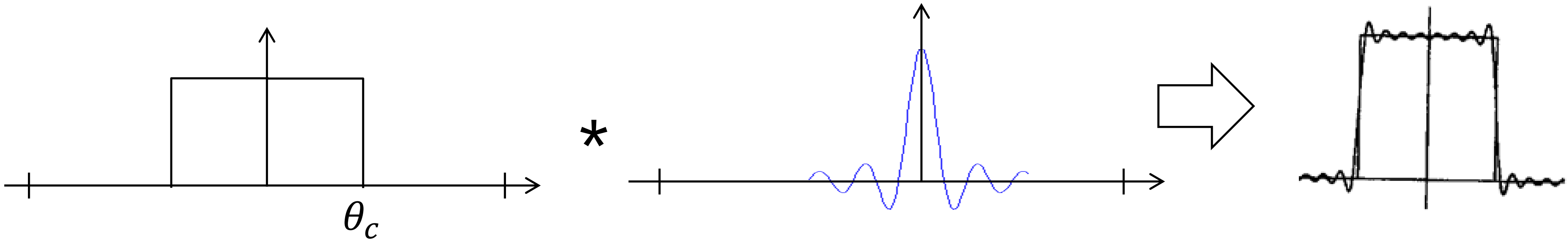
$$x_N(n) = x(n) w_r(n)$$

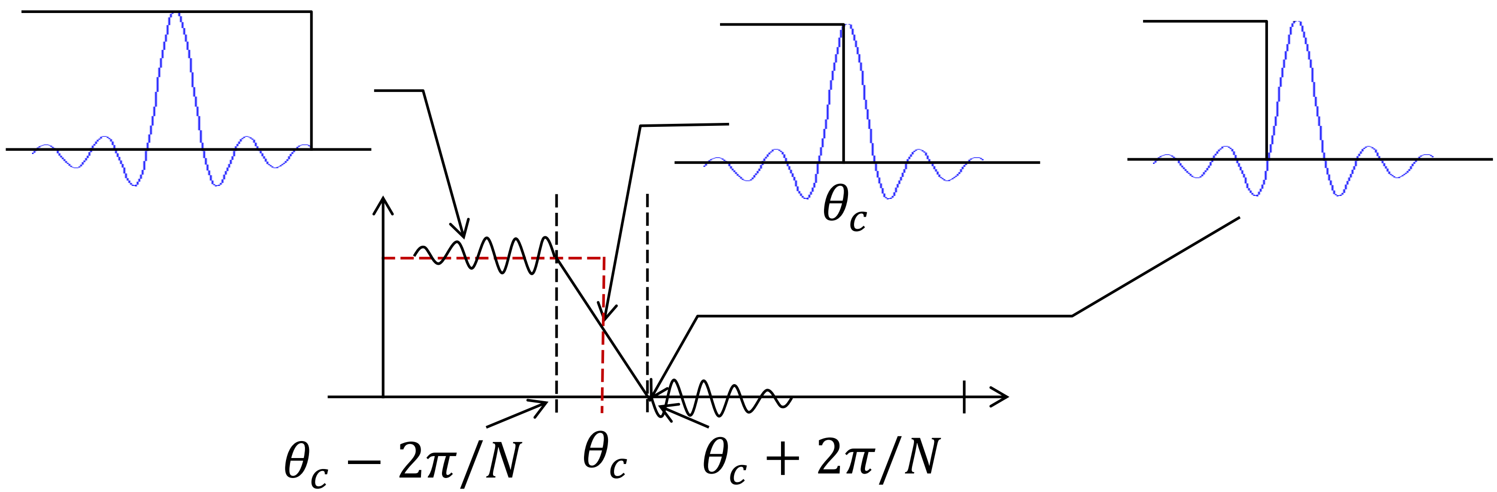


Effect of Rectangular Windowing

Windowed spectrum: case 1

$$X(\Theta) = \frac{1}{2\pi}\, Y(\Theta) * W_r(\Theta)$$

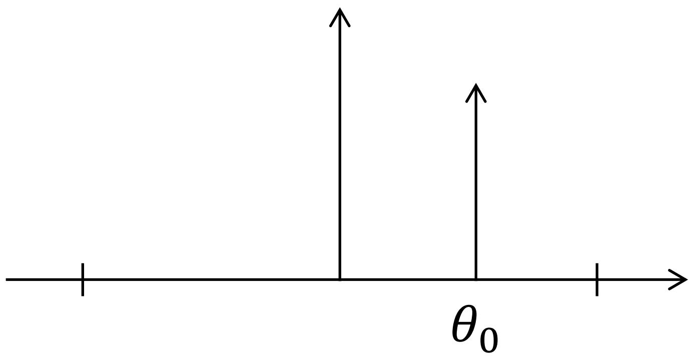




Effect of Rectangular Windowing

Windowed spectrum: case 2

$$y(n) = \exp(j\theta_0 n) \qquad\qquad X(\Theta) = \frac{1}{2\pi}\, Y(\Theta) * W_r(\Theta)$$



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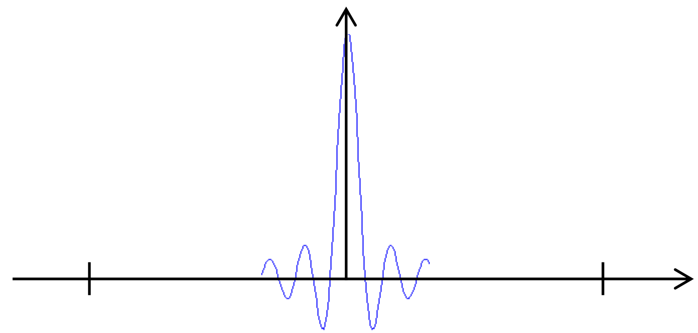
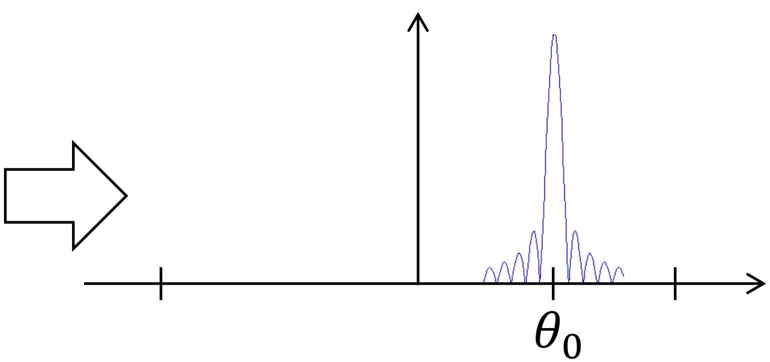
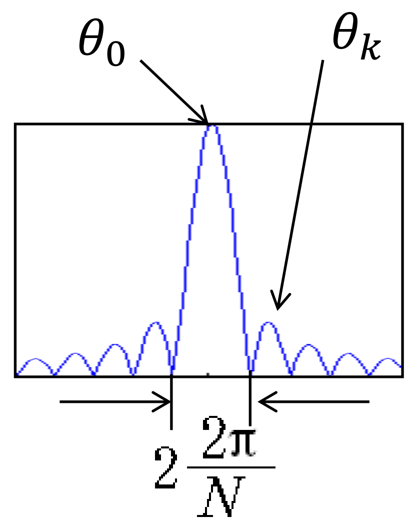


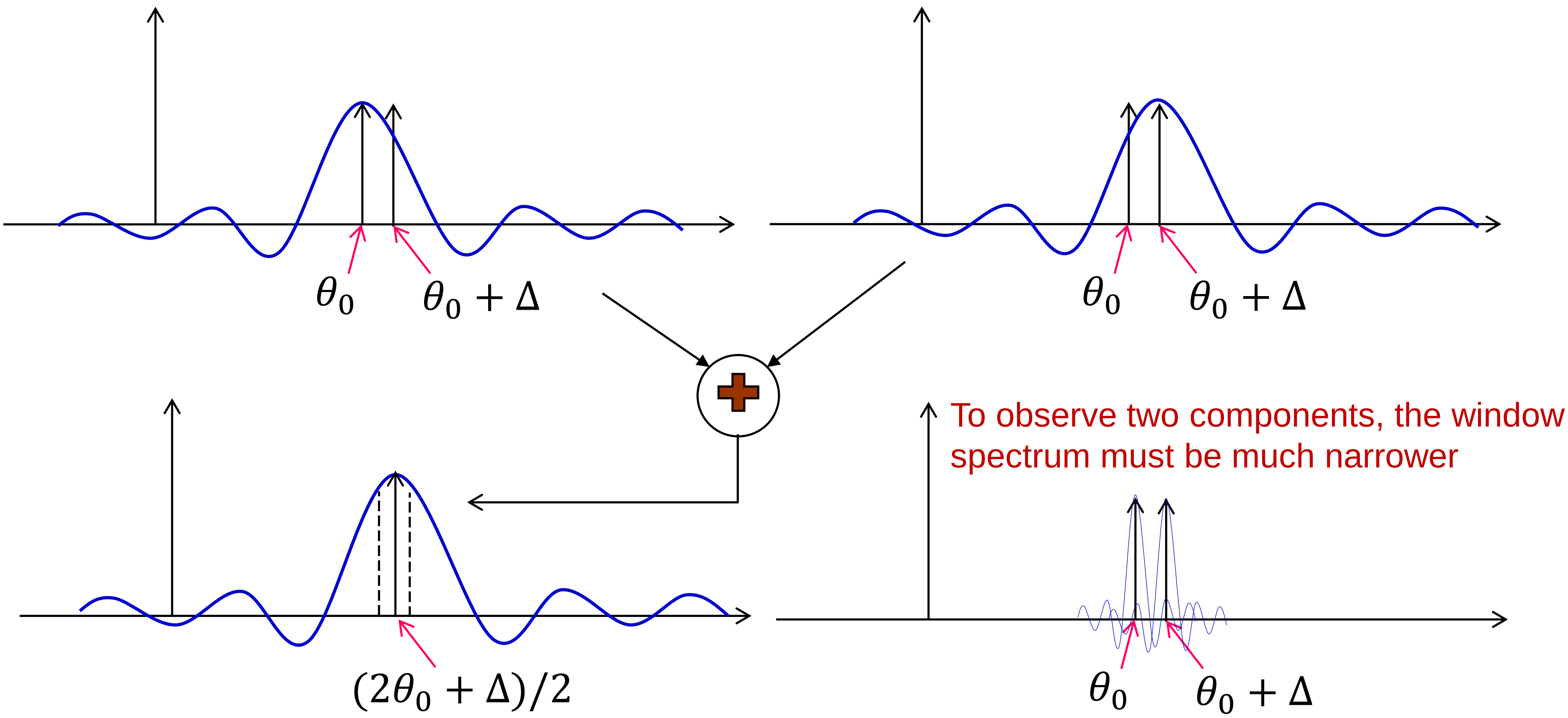




Effect of Rectangular Windowing

Windowed spectrum: case 3

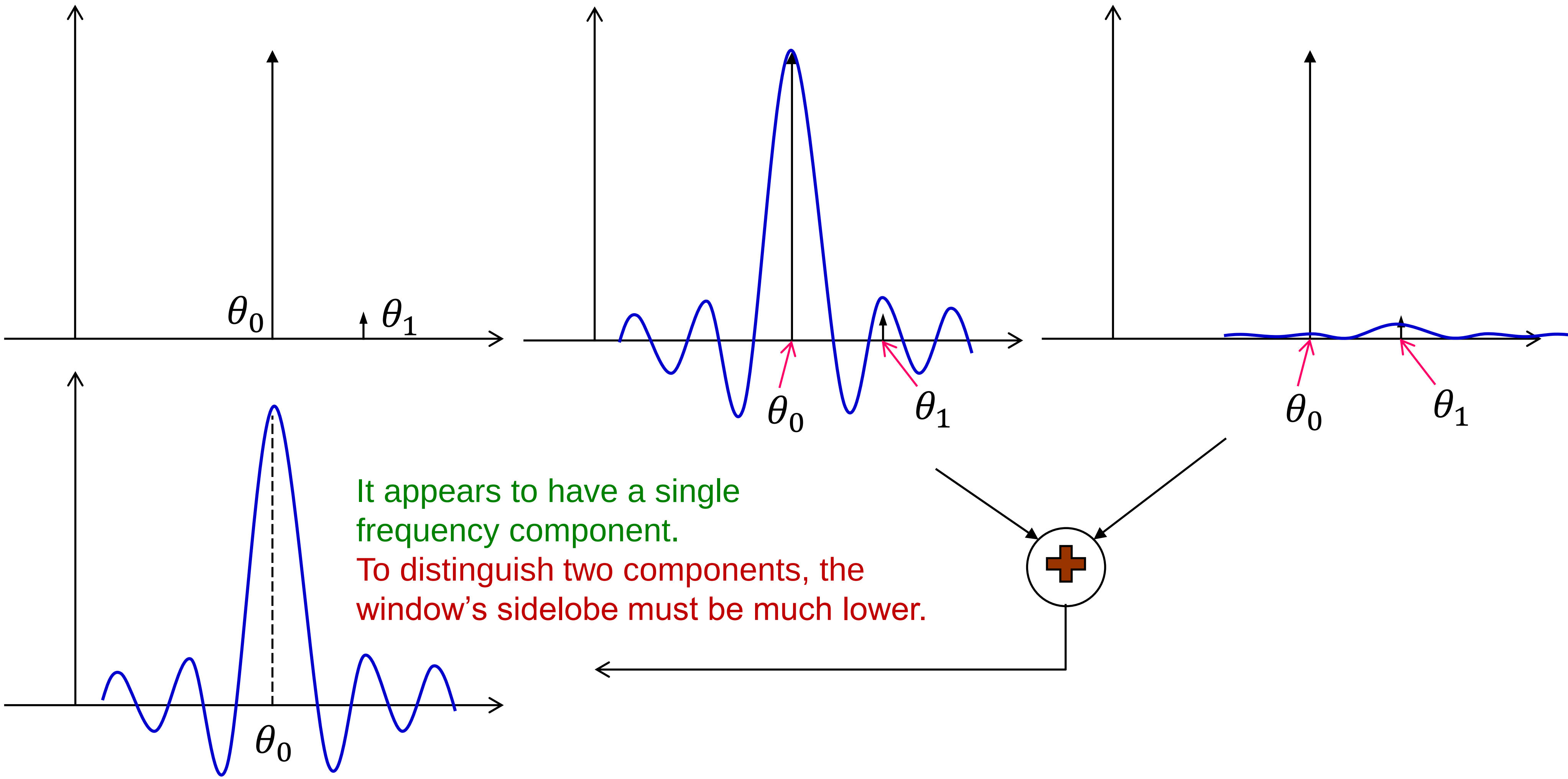
Are there two spectral components or just one component ?



Effect of Rectangular Windowing

Windowed spectrum: case 4

Can the second frequency component be recognized?



It appears to have a single frequency component.

To distinguish two components, the window's sidelobe must be much lower.

Effect of Rectangular Windowing

Windowed spectrum

Mainlobe width of $W_r(f)$

➔ **Resolution of $X(f)$ that is a smeared version of $Y(f)$**

Sidelobe levels of $W_r(f)$

➔ **Sidelobe levels of $X(f)$**

Windowing

Windowed signal

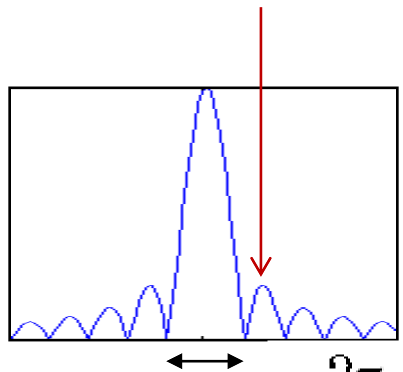
$$x(n) = w(n)y(n)$$

❖ **Desired Window Properties**

- Narrow mainlobe
- Low sidelobes

Example) Mainlobe width: Minimum, $2\frac{2\pi}{N}$

Sidelobe level: -13.5 dB (too high)



$2\frac{2\pi}{N}$

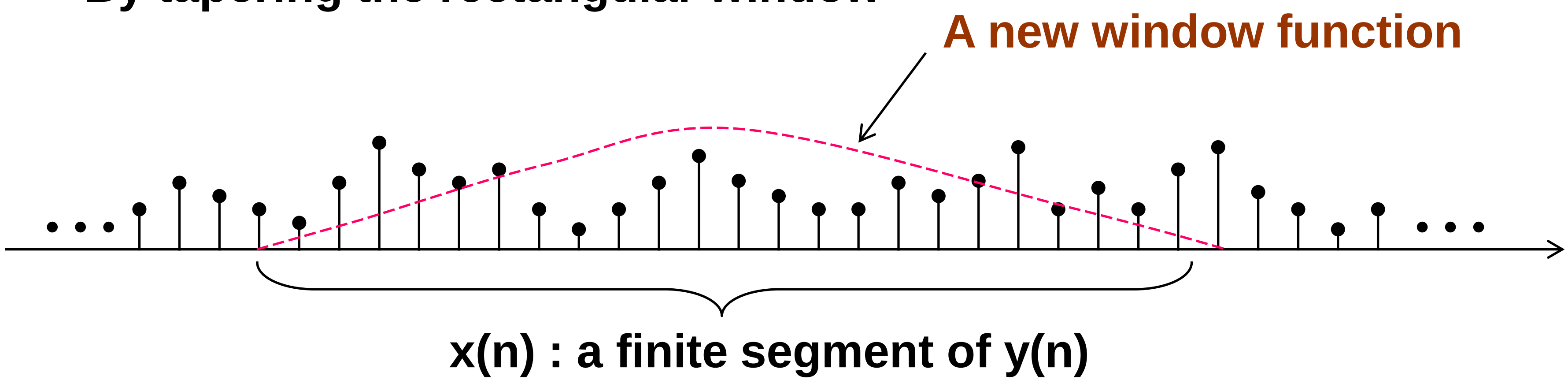
Windowing

Windowed signal

$$x(n) = w(n)y(n)$$

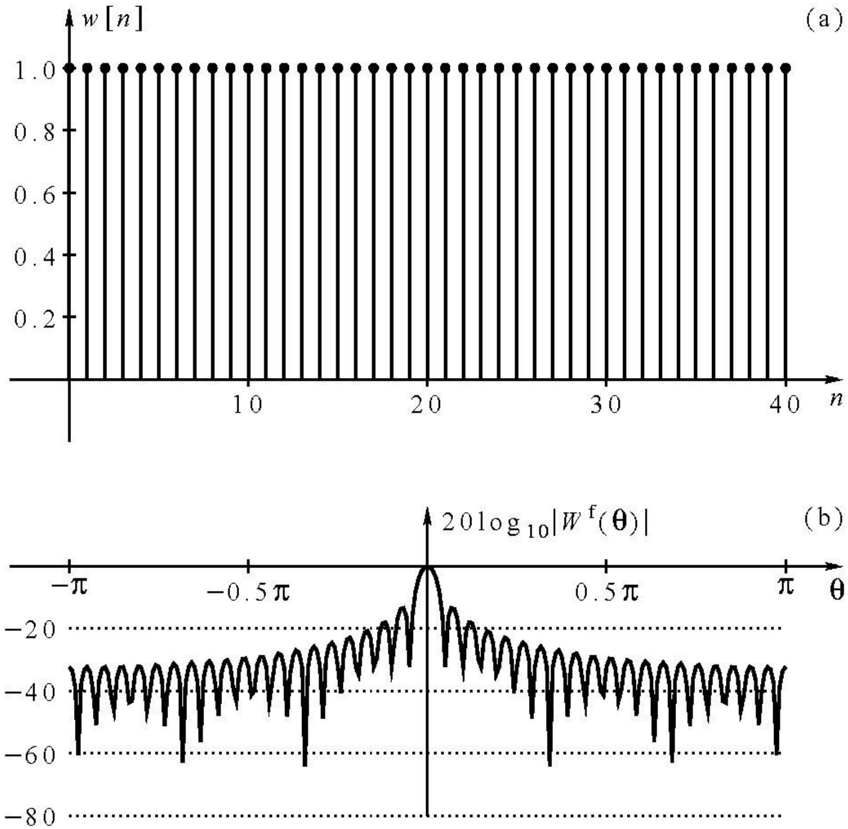
❖ **How to reduce the sidelobe levels?**

By tapering the rectangular window



Common Windows

Rectangular window



For odd N

$$D(\theta, N)e^{-j\theta(N-1)/2}$$

Figure 6.8 A rectangular window, $N = 41$: (a) time-domain plot; (b) frequency-domain magnitude plot.

Common Windows

Bartlett (or Triangular) Window

N=41

(N+1)/2=21

For odd N

$$w_t(n) = \frac{2}{N+1} w_r(n) * w_r(n)$$
$$= 1 - \frac{|2n-N+1|}{N+1}, \quad 0 \le n \le N-1$$

where length of $w_r(n)$ = *(N+1)/2*

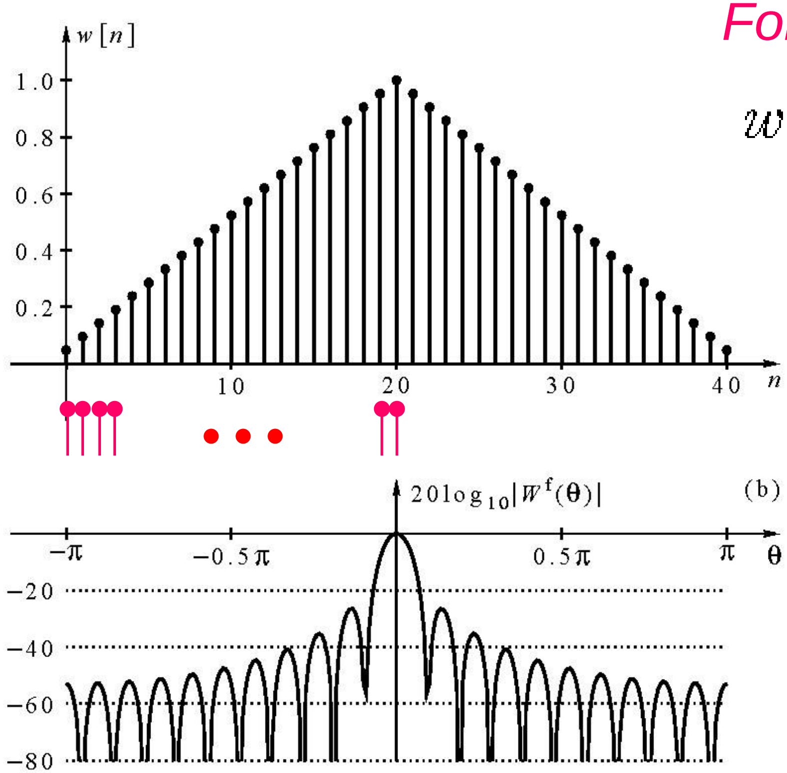


Figure 6.9 Bartlett window, $N = 41$: (a) time-domain plot; (b
plot.

Mainlobe width: $4\frac{2\pi}{N+1} \approx 2\frac{4\pi}{N}$

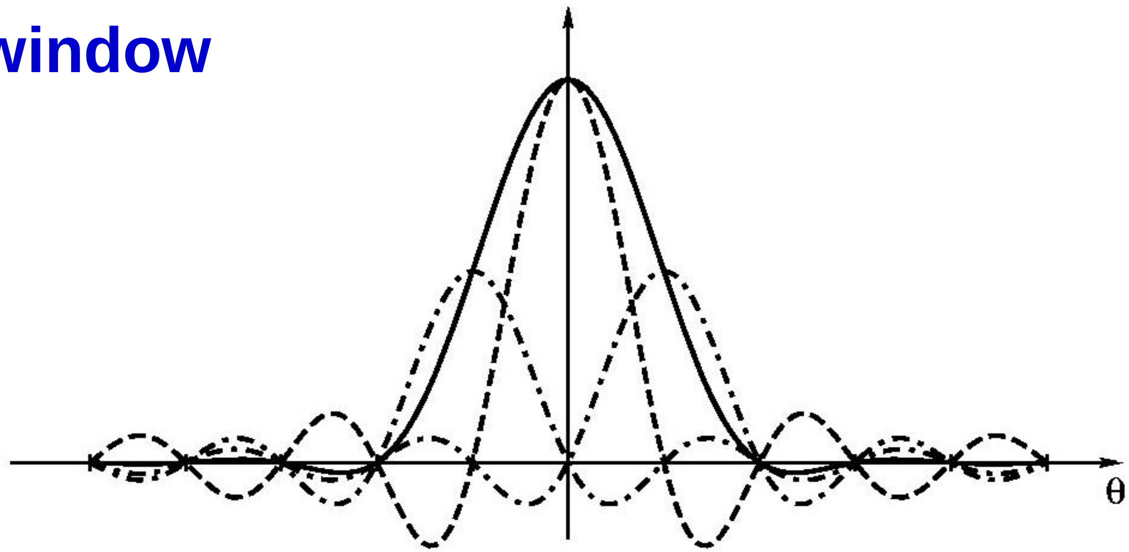
Peak sidelobe level:

$$2 \times (-13.5) = -27dB$$

$$W_t(\Theta) = \frac{2}{N+1} D^2(\Theta, (N+1)/2)\, e^{-j\Theta(N-1)/2}$$
$$= \frac{2\sin^2[\Theta(N+1)/4]}{(N+1)\sin^2(\Theta/2)} e^{-j\Theta(N-1)/2}$$

Common Windows

Hanning window



$$W_{han}(\Theta) = \left[0.5D(\Theta, N) + 0.25D(\Theta - \frac{2\pi}{N-1}, N) + 0.25D(\Theta + \frac{2\pi}{N-1}, N)\right]e^{-j0.5\Theta(N-1)}$$

$$= 0.5W_r(\Theta) - 0.25W_r(\Theta - \frac{2\pi}{N-1}) - 0.25W_r(\Theta + \frac{2\pi}{N-1})$$

$$W_{han}(n) = 0.5 - 0.25\exp(\frac{j2\pi n}{N-1}) - 0.25\exp(-\frac{j2\pi n}{N-1})$$

$$= 0.5\left[1 - \cos(\frac{2\pi n}{N-1})\right], \quad 0 \le n \le N-1 \ .$$

Common Windows

Hanning window

- ❖ **Mainlobe width = $8\pi/N$: Same as that of Barttlet window**
- ❖ **Peak sidelobe = -32 dB. Lower than that of Barttlet window**

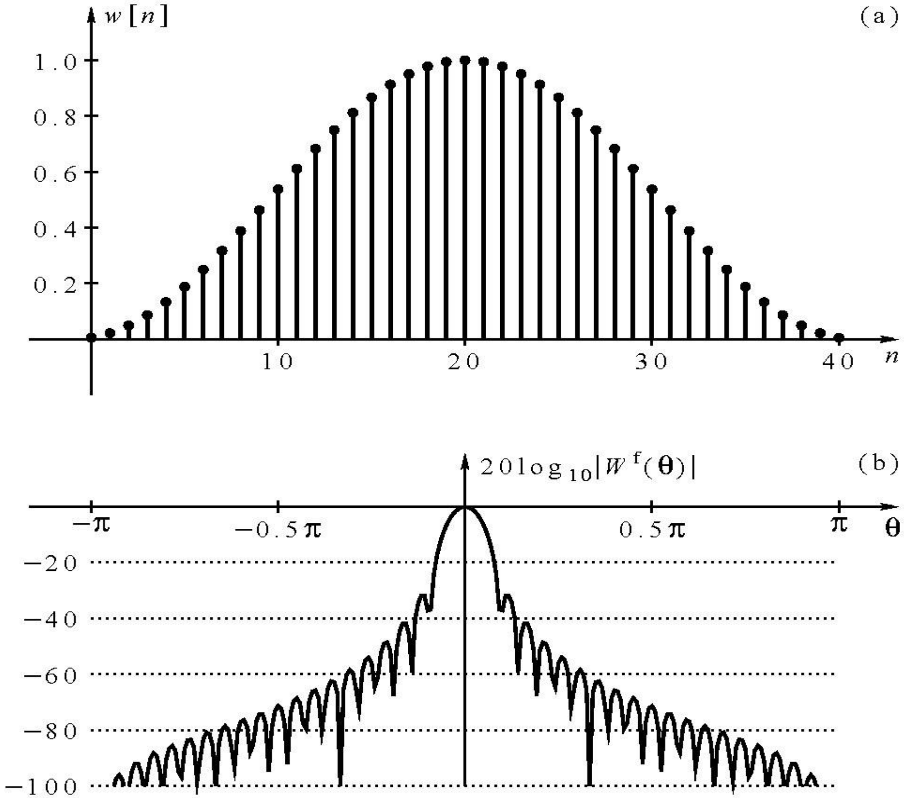


Figure 6.11 Hann window, $N = 41$: (a) time-domain plot; (b) frequency-domain magnitude plot.

Modified Hanning window

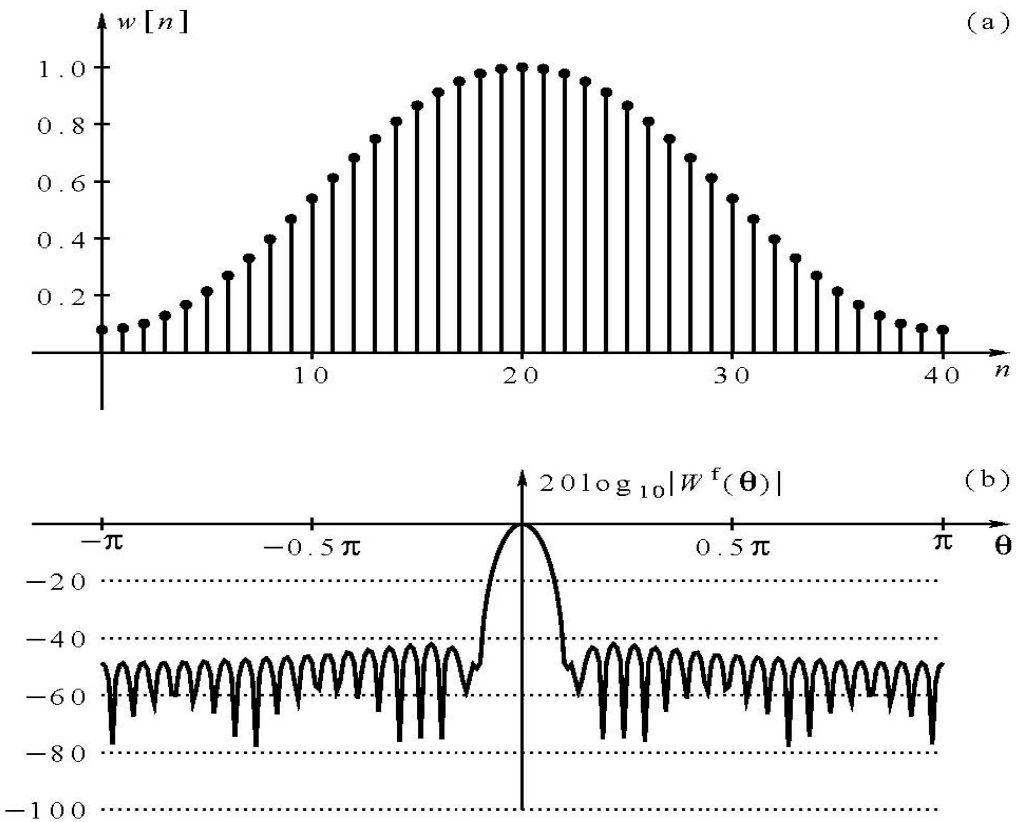
$$w_{hn}(n) = 0.5\left[1 - \cos\left(\frac{2\pi(n+1)}{N+1}\right)\right], \quad 0 \le n \le N-1$$

Common Windows

Hamming window

- ❖ **Mainlobe width =** $8\pi/N$
- ❖ **Sidelobe levels compared with Hanning window?**

$$w_{hm}(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right),\ \ 0 \le n \le N-1,$$



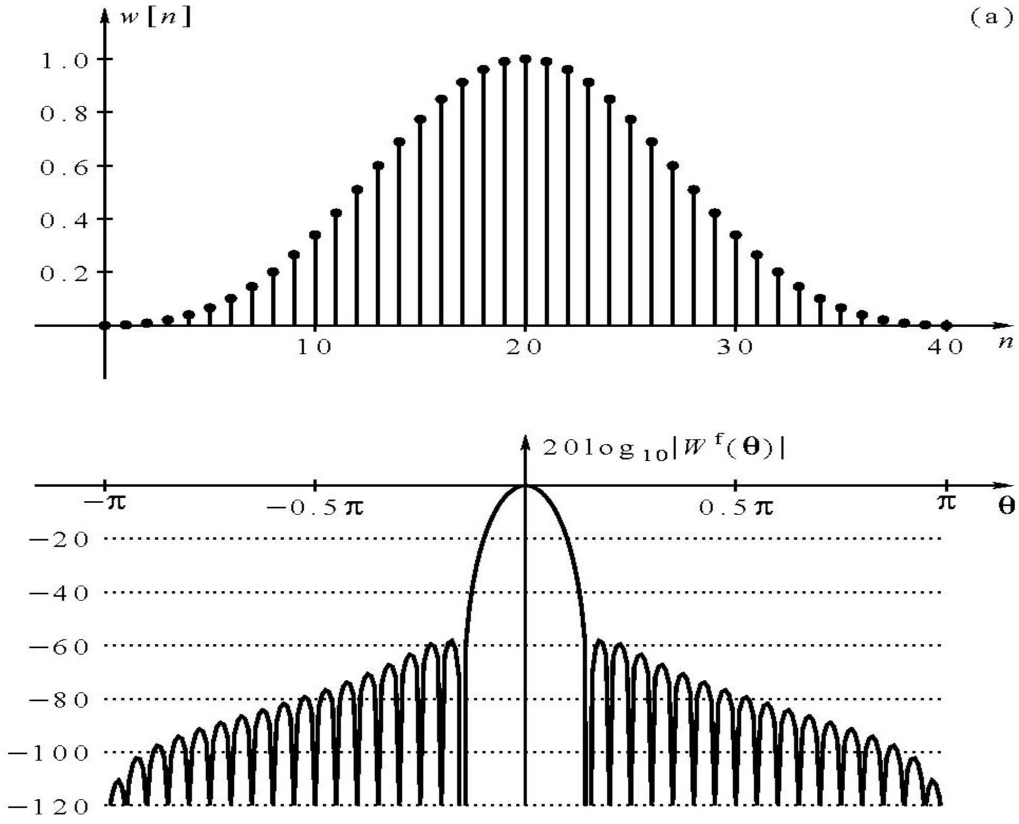
Which one do you think is better, Hanning or Hamming?

Common Windows

Blackman window

- ❖ **Mainlobe width =** $12\pi/N$
- ❖ **Peak sidelobe level = -57dB. Much lower than previous ones.**

$$w_b(n) = 0.42 - 0.5\cos\left(\frac{2\pi n}{N-1}\right) + 0.08\cos\left(\frac{4\pi n}{N-1}\right), \quad 0 \le n \le N-1,$$



Common Windows

Optimum Window

❖ **Optimality criteria**

1) Dolph (or Dolph-Cheshev) criterion

Minimize the mainlobe width for a fixed window length under the constraint that the sidelobes not exceed a given maximum value.

2) Kaiser's criterion

Minimize the mainlobe width for a fixed window length under the constraint that the energy in the sidelobes not exceed a given percentage of the total energy.

Common Windows

Kaiser Window

$$w_k(n) = \frac{I_0\left[\alpha\sqrt{1-\left(\frac{|2n-N+1|}{N-1}\right)^2}\right]}{I_0(\alpha)}, \quad 0 \le n \le N-1$$

$$I_0(x) = \sum_{k=0}^{\infty}\left(\frac{x^k}{2^k k!}\right)^2 :$$ *Modified Bessel function of order zero*

α *determines the mainlobe width and the sidelobe levels.*

$$\alpha\uparrow \;\rightarrow\; (mainlobe\ \ width\uparrow)\ (sidelobe\downarrow)$$

Common Windows

Kaiser Window

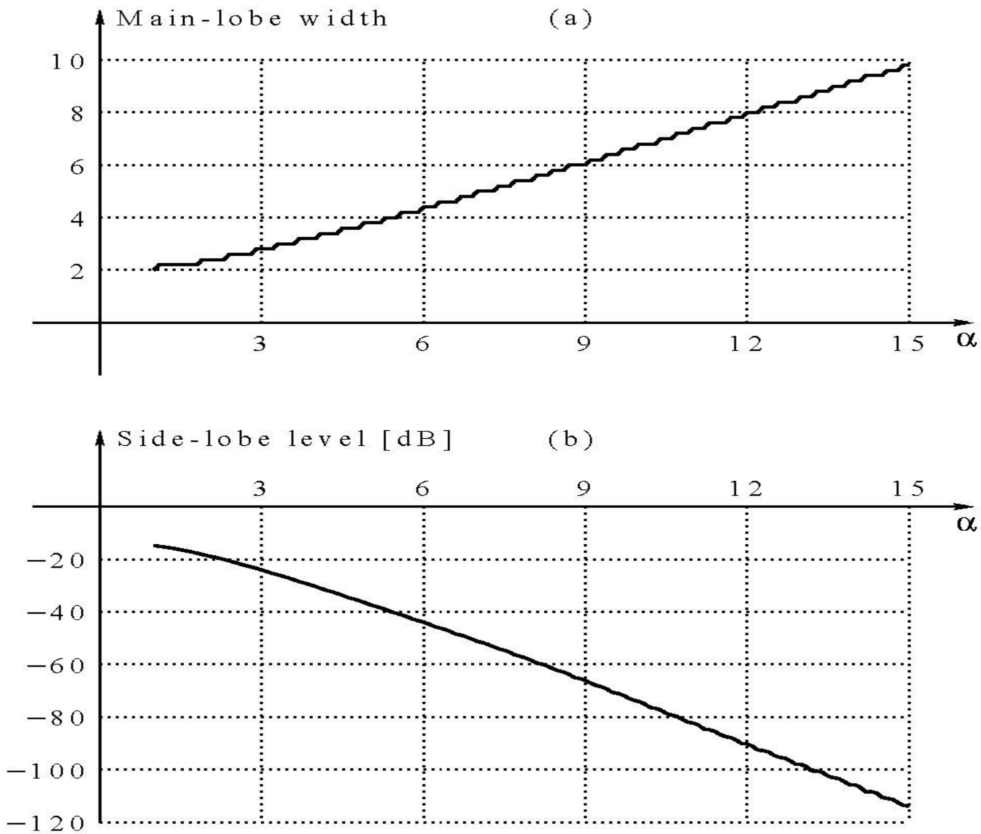


Figure 6.15 Properties of the Kaiser window as a function of the parameter α: (a) main-lobe width, as a multiple of $2\pi/N$; (b) side-lobe level.

Common Windows

Dolph Window

- Equi-ripple window
- Sensitive to coefficient accuracy

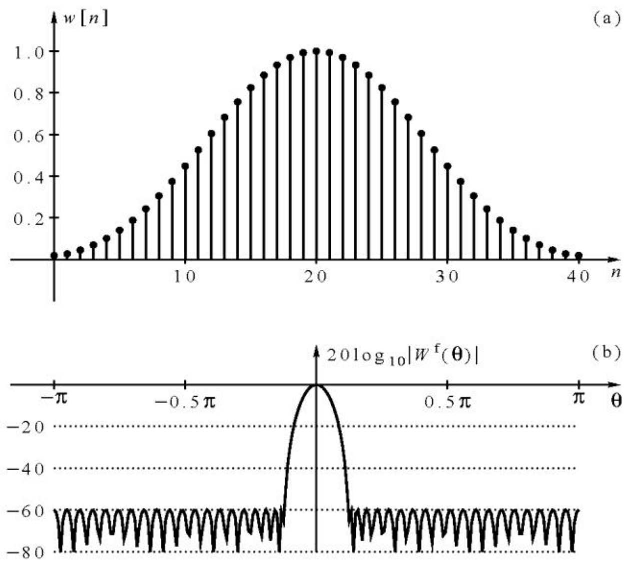
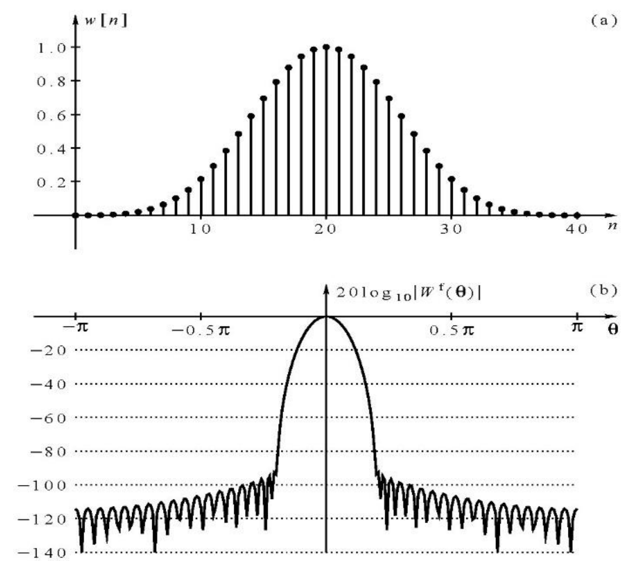


Figure 6.17 Dolph window, $N = 41$, $\alpha = -60$ dB: (a) time-domain plot; (b) frequency-domain magnitude plot.



gure 6.16 Kaiser window, $N = 41$, $\alpha = 12$: (a) time-domain plot; (b) frequency-domain agnitude plot.