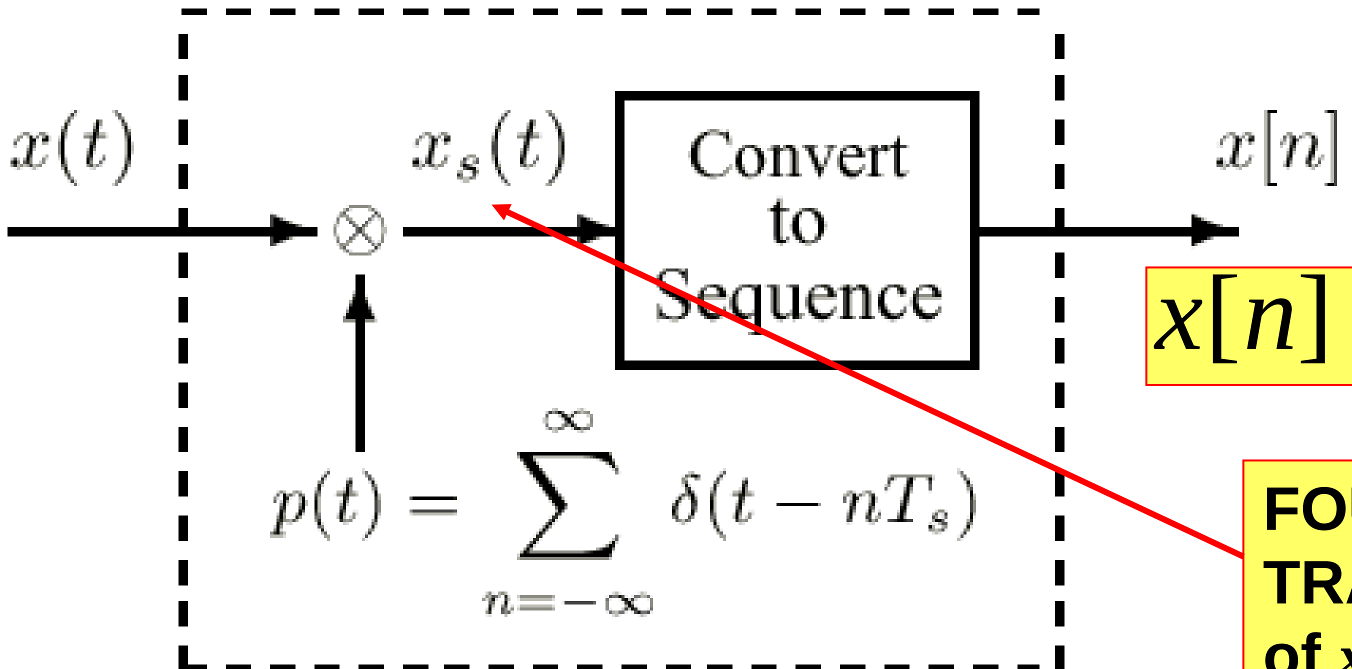
Chapter 12
Filtering, Modulation, and Sampling

Ideal C-to-D Converter

- Mathematical Model for A-to-D

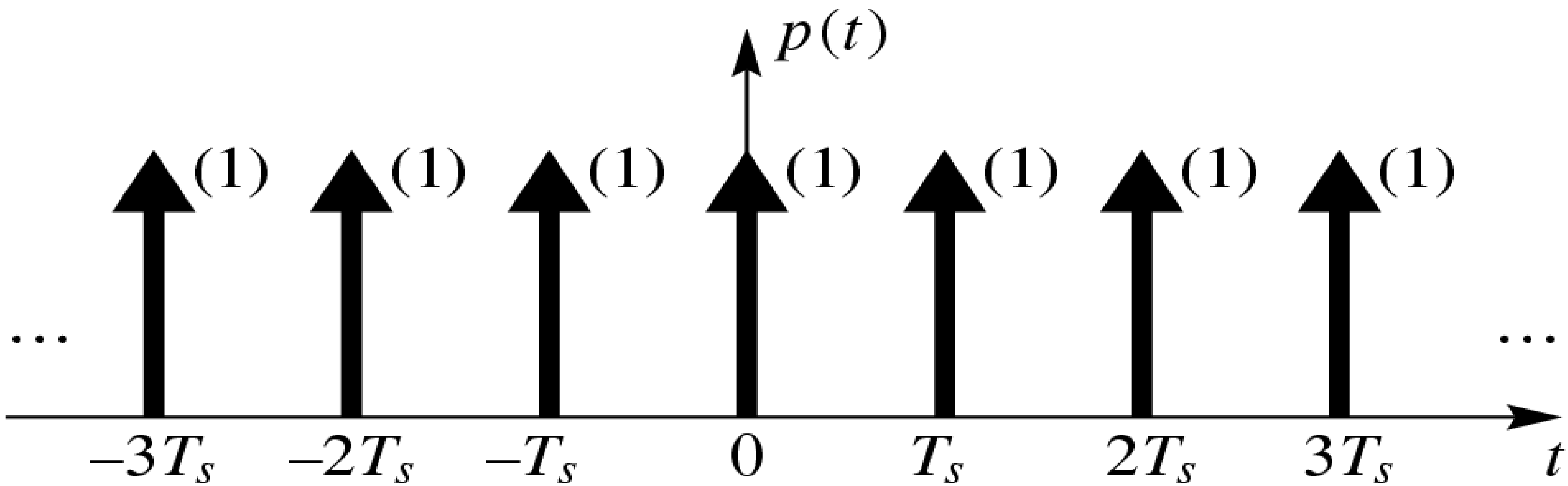


Ideal C-to-D Converter

$$x[n] = x(nT_s)$$

FOURIER TRANSFORM of $x_s(t)$???

Periodic Impulse Train



$$p(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT_s)=\sum_{k=-\infty}^{\infty}a_k e^{jk\omega_s t}$$

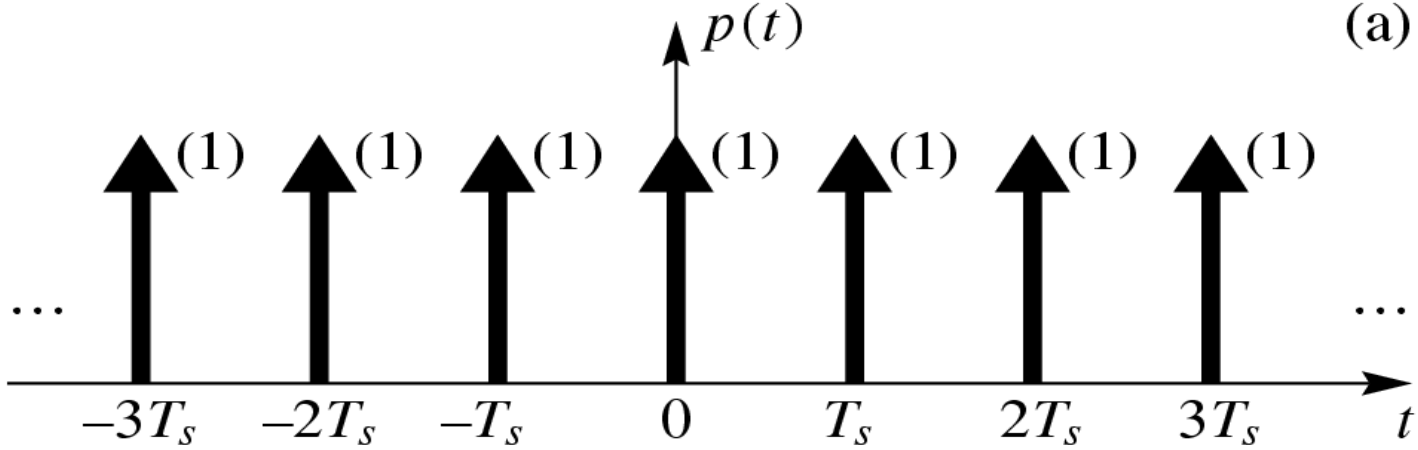
$$\omega_s=\frac{2\pi}{T_s}$$

$$a_k=\frac{1}{T_s}\int_{-T_s/2}^{T_s/2}\delta(t)e^{-jk\omega_s t}dt=\frac{1}{T_s}$$

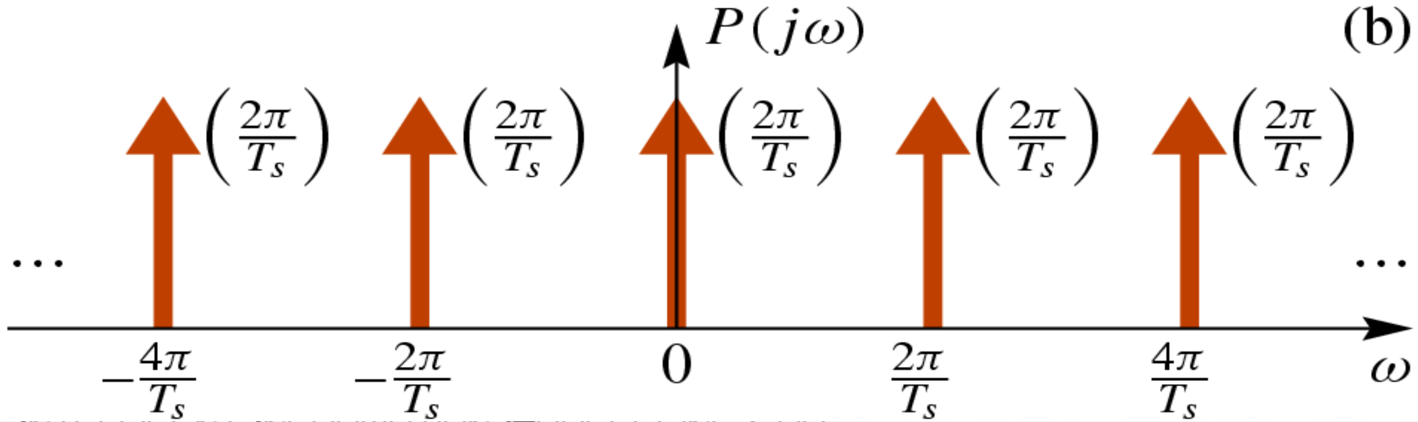
Fourier Series

FT of the Impulse Train

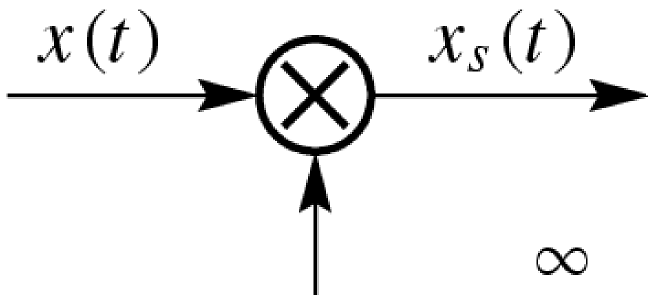
$$p(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT_s) \quad\leftrightarrow\quad P(j\omega)=\sum_{k=-\infty}^{\infty}\frac{2\pi}{T_s}\delta(\omega-k\omega_s)$$



$$\omega_s=\frac{2\pi}{T_s}$$



Impulse Train Sampling

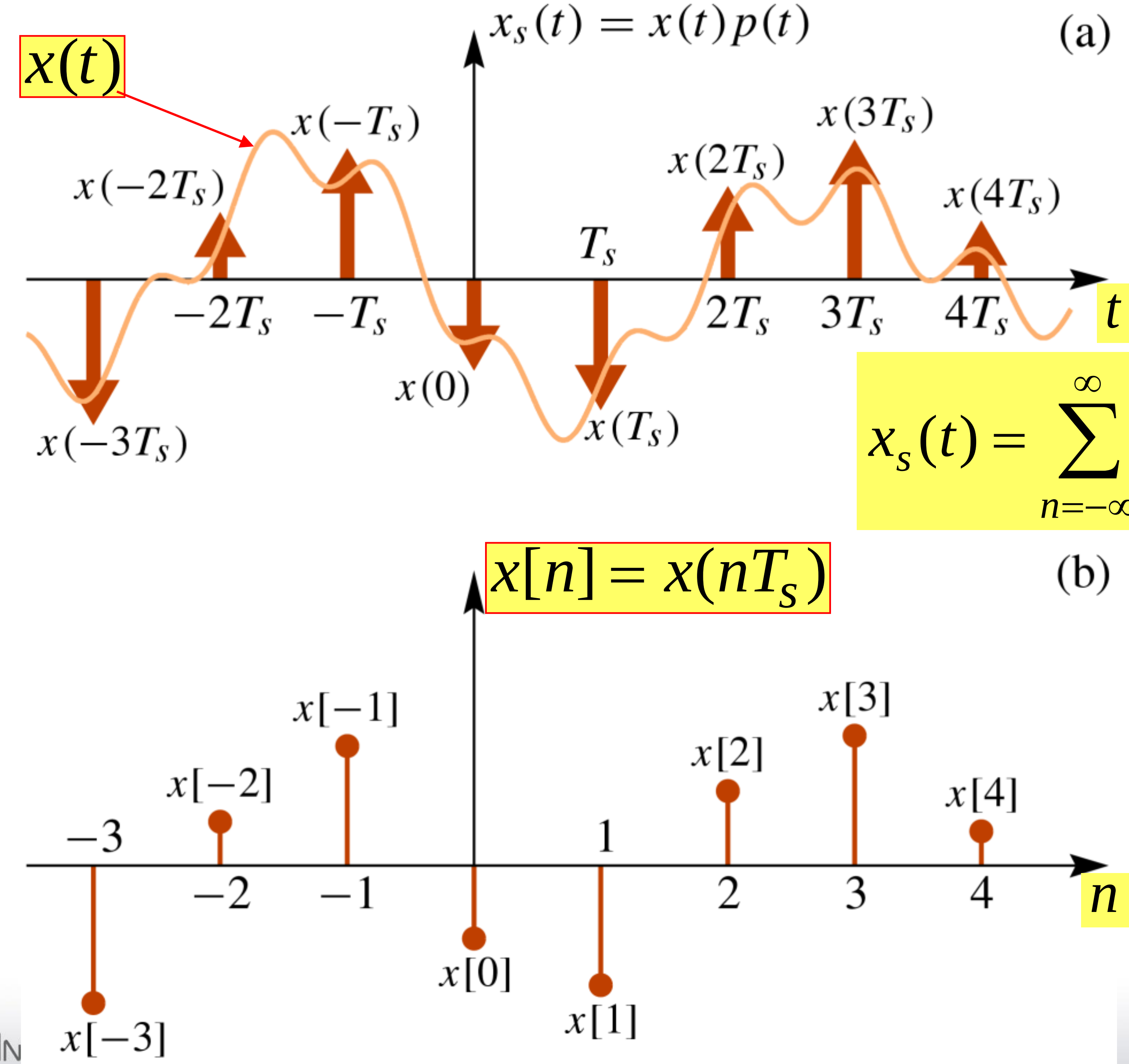


$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} \underline{x(t)\delta(t - nT_s)}$$

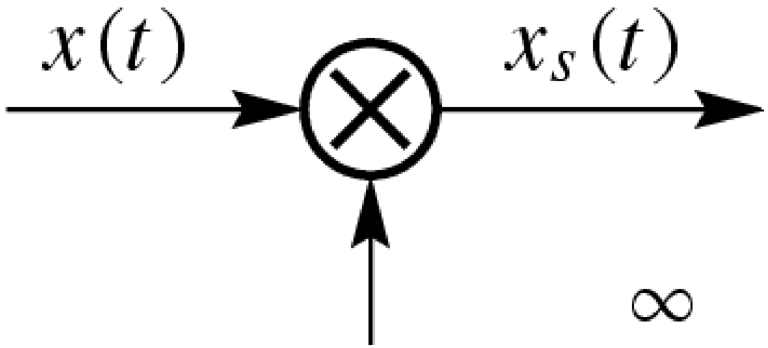
$$\boxed{x_s(t) = \sum_{n=-\infty}^{\infty} \underline{x(nT_s)\delta(t - nT_s)}}$$

Illustration of Sampling



$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

Sampling: Frequency Domain



$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}$$

Frequency-Domain Analysis

$$x_s(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT_s) = \sum_{n=-\infty}^{\infty}x(nT_s)\delta(t-nT_s)$$

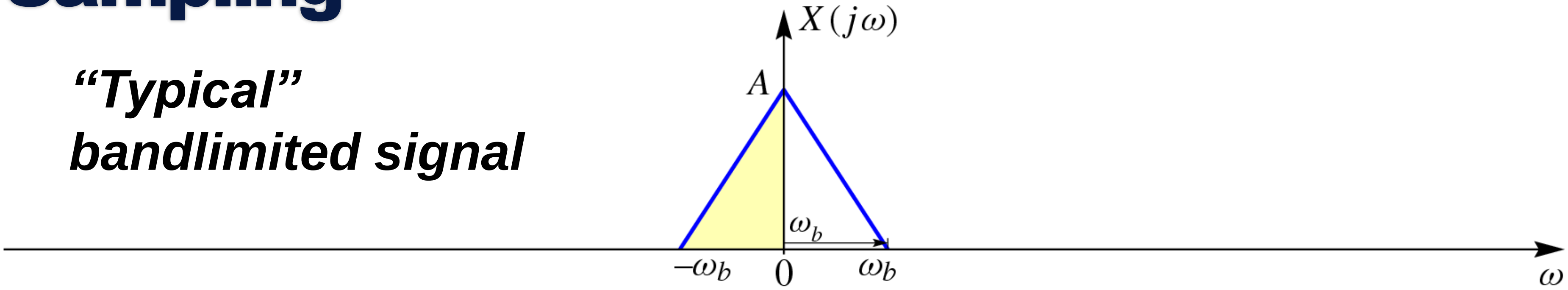
$$x_s(t) = x(t)\sum_{k=-\infty}^{\infty}\frac{1}{T_s}e^{jk\omega_s t} = \frac{1}{T_s}\sum_{k=-\infty}^{\infty}\underline{x(t)e^{jk\omega_s t}}$$

$$\boxed{X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty}\underline{X(j(\omega-k\omega_s))}} \quad \boxed{\omega_s = \frac{2\pi}{T_s}}$$

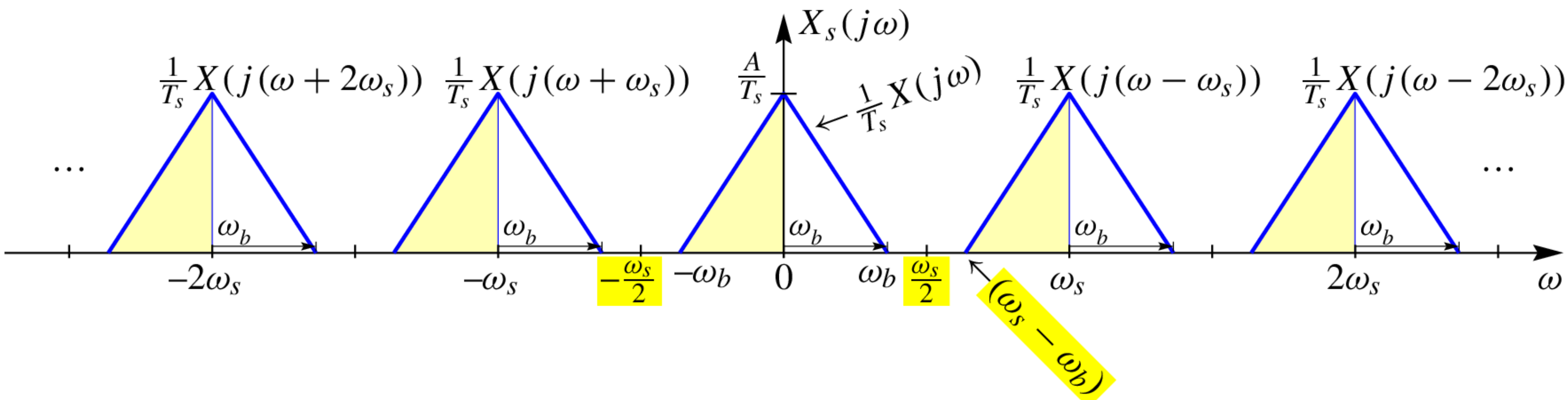
EXPECT FREQUENCY SHIFTING !!!

Frequency-Domain Representation of Sampling

"Typical" bandlimited signal

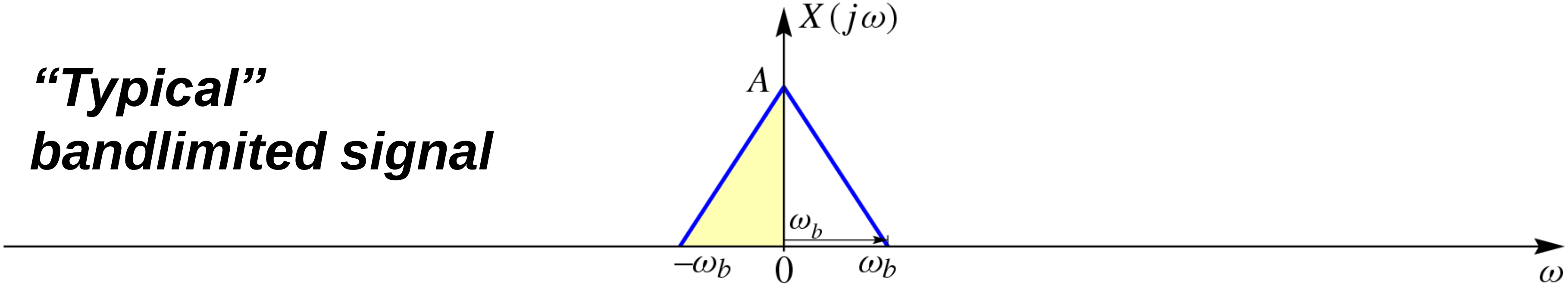


$$X_s(j\omega)=\frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j(\omega-k\omega_s))$$

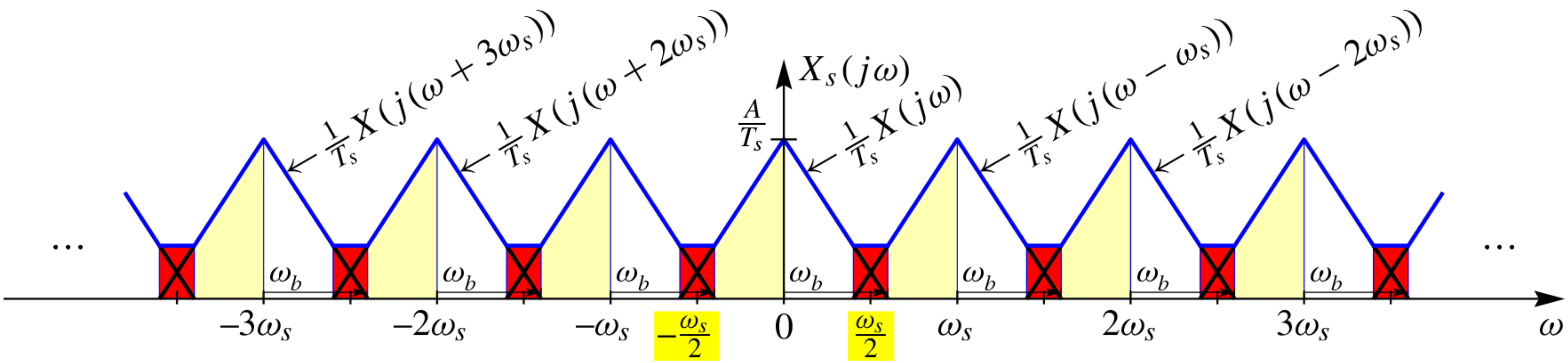


Aliasing Distortion

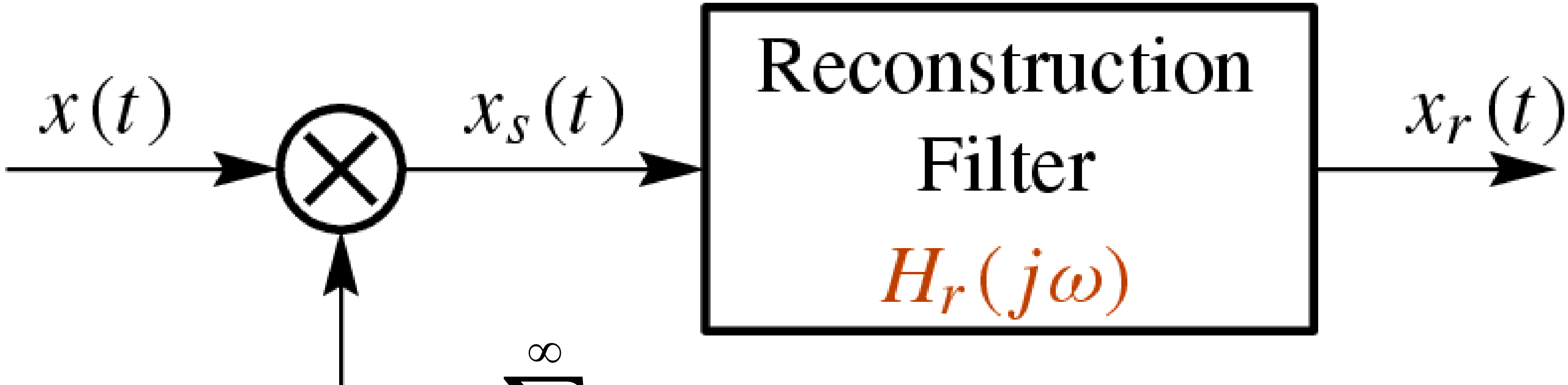
"Typical" bandlimited signal



- If $\omega_s < 2\omega_b$, the copies of $X(j\omega)$ overlap, and we have aliasing distortion.



Reconstruction of $x(t)$



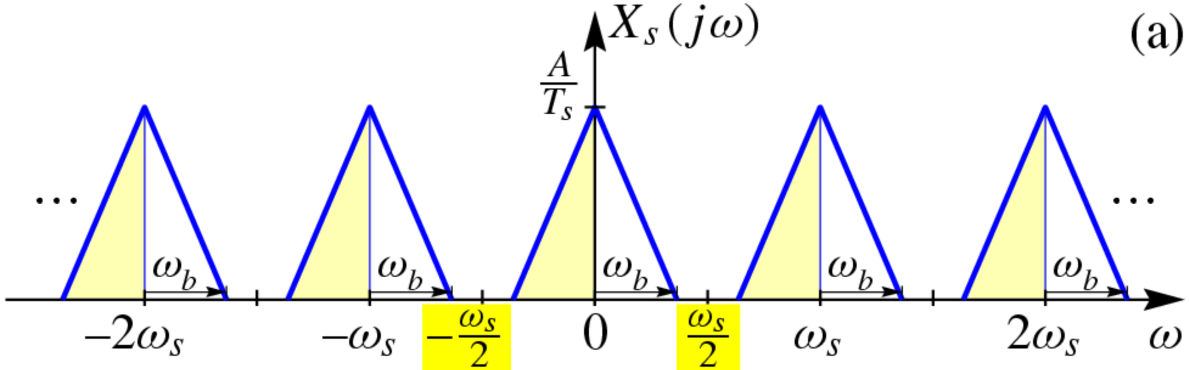
$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

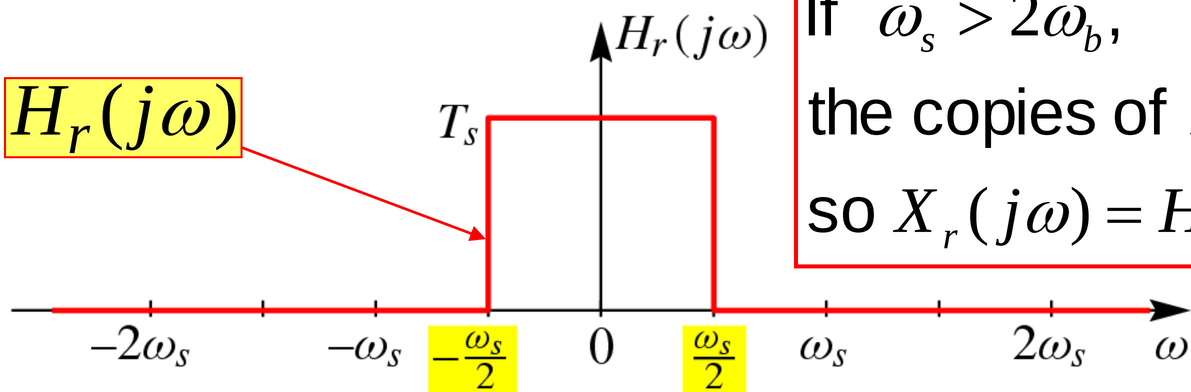
$$\boxed{x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)}$$

$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$$\boxed{X_r(j\omega) = H_r(j\omega) X_s(j\omega)}$$

Reconstruction in Frequency-Domain

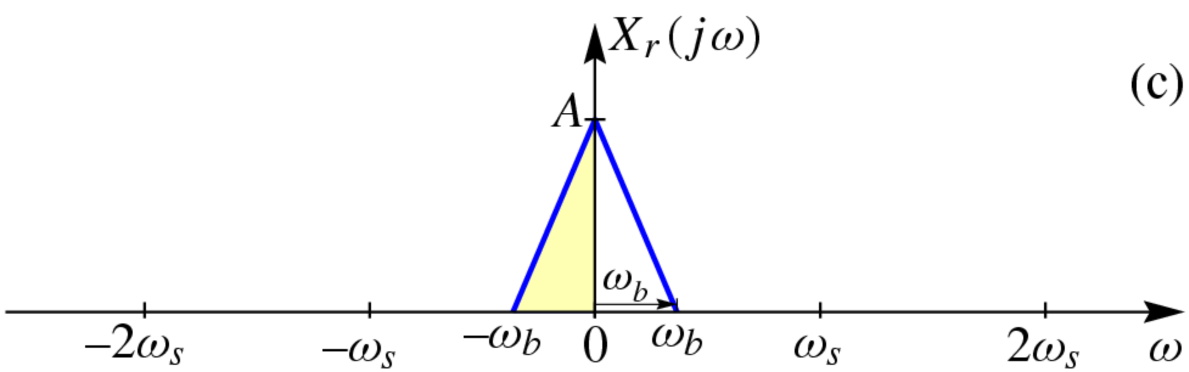




If $\omega_s > 2\omega_b$,

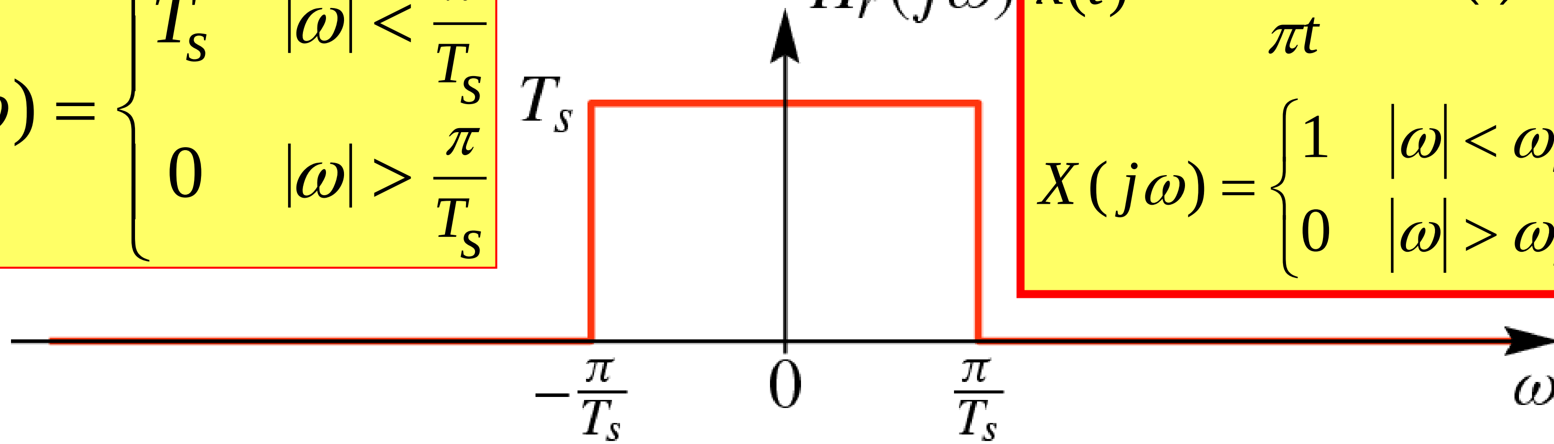
the copies of $X(j\omega)$ do not overlap,

so $X_r(j\omega) = H_r(j\omega)X_s(j\omega) = X(j\omega)$.



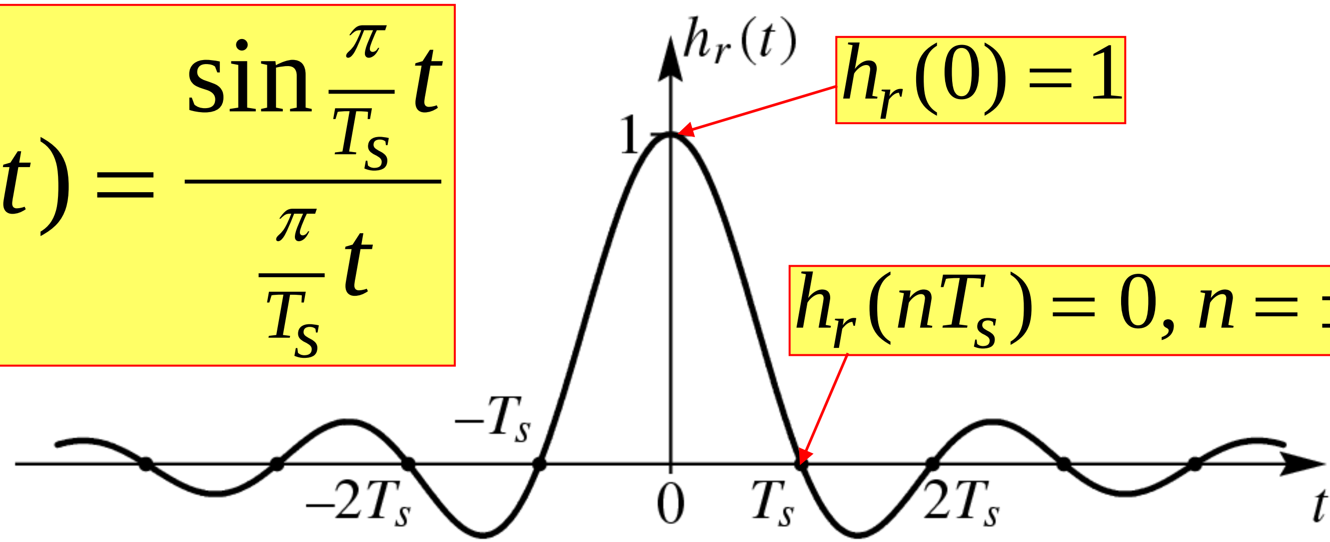
Ideal Reconstruction Filter

$$H_r(j\omega) = \begin{cases} T_s & |\omega| < \dfrac{\pi}{T_s} \\ 0 & |\omega| > \dfrac{\pi}{T_s} \end{cases}$$



$$x(t) = \frac{\sin(\omega_b t)}{\pi t} \Leftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < \omega_b \\ 0 & |\omega| > \omega_b \end{cases}$$

$$h_r(t) = \frac{\sin \frac{\pi}{T_s} t}{\frac{\pi}{T_s} t}$$



$h_r(0) = 1$

$h_r(nT_s) = 0, n = \pm 1, \pm 2, \ldots$

Signal Reconstruction

$$x_r(t) = h_r(t) * x_s(t) = h_r(t) * \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) h_r(t - nT_s)$$

$$\boxed{x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin \frac{\pi}{T_s}(t - nT_s)}{\frac{\pi}{T_s}(t - nT_s)}}$$

Ideal bandlimited interpolation formula

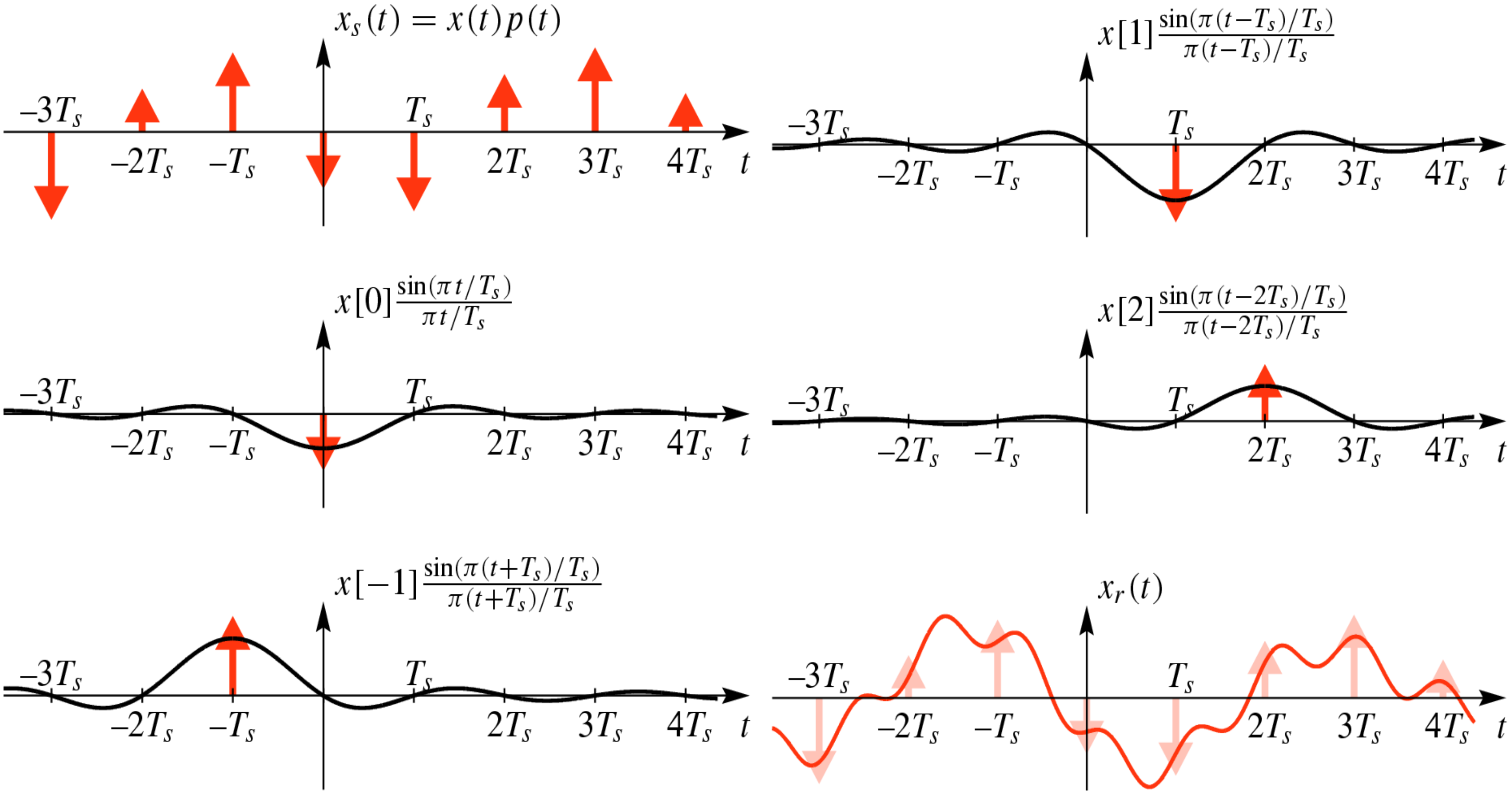
Shannon/Nyquist Sampling Theorem

- **"SINC" interpolation** is the ideal.
  - PERFECT RECONSTRUCTION of BANDLIMITED SIGNALS

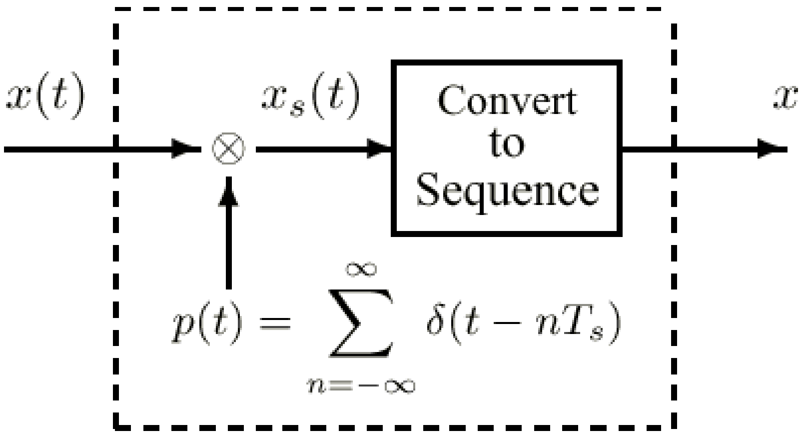
A signal $x(t)$ with bandlimited Fourier transform such that $X(j\omega) = 0$ for $|\omega| \geq \omega_b$ can be reconstructed exactly from samples taken with sampling rate $\omega_s = 2\pi/T_s \geq 2\omega_b$ using the following bandlimited interpolation formula:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin\left[\frac{\pi}{T_s}(t - nT_s)\right]}{\frac{\pi}{T_s}(t - nT_s)}.$$

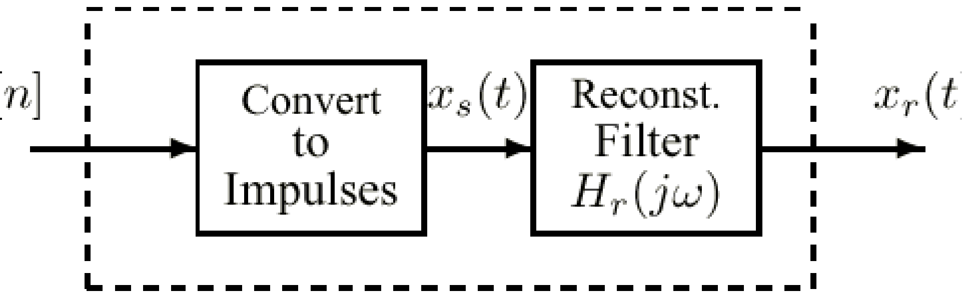
Reconstruction in Time-Domain



Ideal C-to-D and D-to-C



Ideal C-to-D Converter



Ideal D-to-C Converter

$$\boxed{x[n] = x(nT_s)}$$

Ideal Sampler

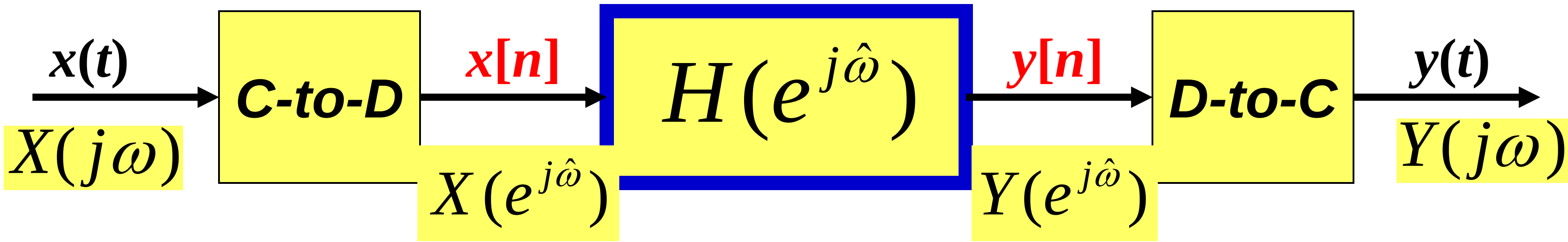
$$\boxed{x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin \frac{\pi}{T_s}(t - nT_s)}{\frac{\pi}{T_s}(t - nT_s)}}$$

Ideal bandlimited interpolator

$$\boxed{X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))}$$

$$\boxed{X_r(j\omega) = H_r(j\omega) X_s(j\omega)}$$

DT Filtering of CT Signals



- If no aliasing occurs in sampling $x(t)$,

$$X(j\omega) = 0 \text{ for } |\omega| > \omega_s / 2$$

- Then, it follows that

$$Y(j\omega) = H_{\text{eff}}(e^{j\omega})X(j\omega)$$

- Overall effective frequency response

$$H_{\text{eff}}(e^{j\omega}) = H(e^{j\omega T_s})$$

$$\hat{\omega} = \omega T_s = \frac{2\pi f}{f_s}$$

EFFECTIVE Frequency Response

- Assume NO Aliasing, then
  - ANALOG FREQ. <--> DIGITAL FREQ.

$$\boxed{\hat{\omega} = \omega T_s = \frac{\omega}{f_s}}$$

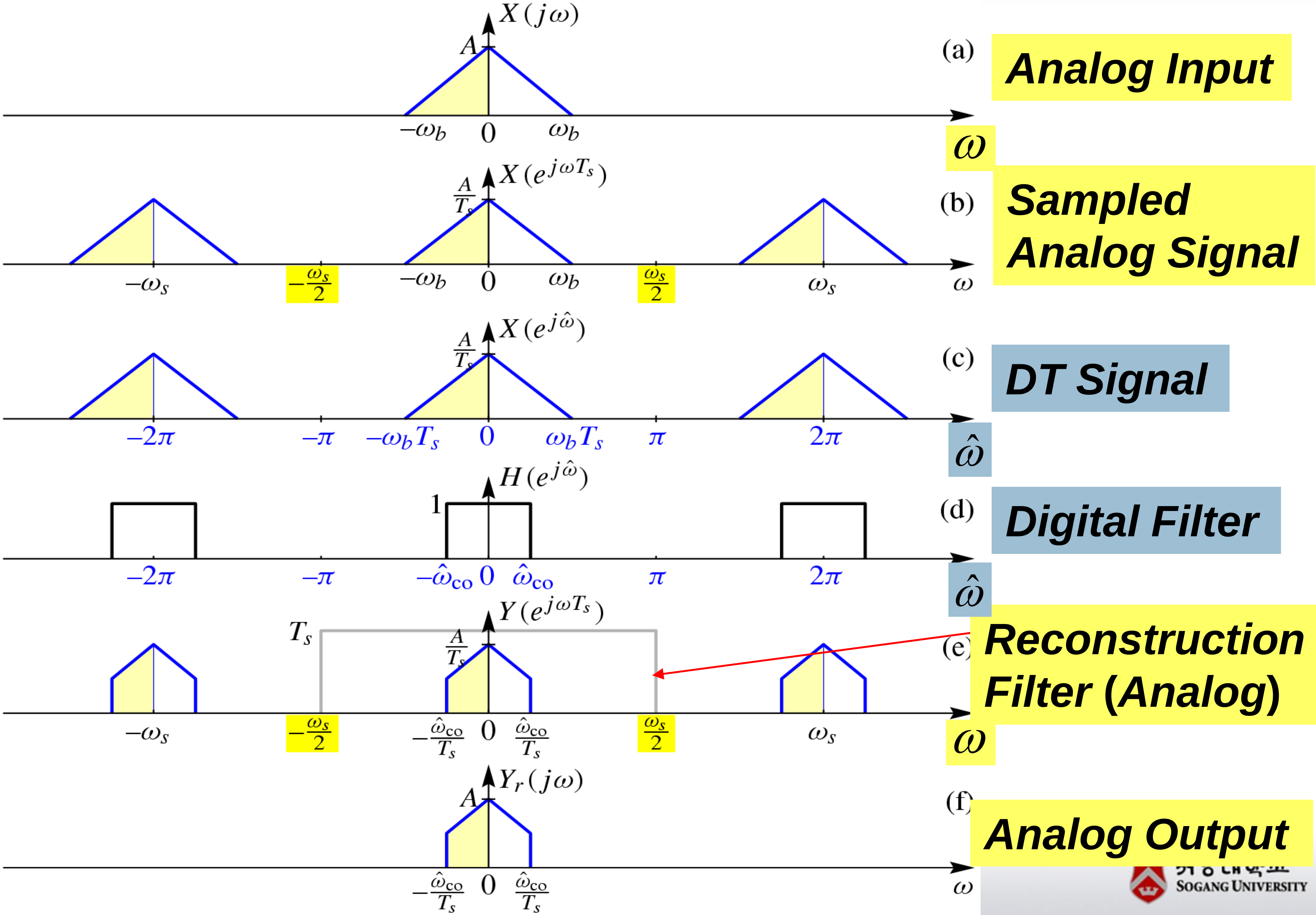
  - So, we can plot:
    - Scaled Frequency Axis

DIGITAL FILTER

$$\boxed{H(e^{j\omega T_s}) \text{ vs. } \omega}$$

ANALOG FREQUENCY

DT Filtering of a CT Signal



H_{eff} for 11-pt Running Averager

- Frequency Response for a Discrete-Time System

$$\boxed{H(e^{j\hat{\omega}}) = \frac{\sin(11\hat{\omega}/2)}{\sin(\hat{\omega}/2)}}$$

$$\boxed{\hat{\omega} = \omega T_s = \frac{\omega}{f_s} = \frac{\omega}{1000}}$$

- Overall Effective Frequency Response

$$\boxed{H_{\text{eff}}(j\omega) = H(j\omega T_s) = \frac{\sin(11\omega/2000)}{\sin(\omega/2000)}}$$

Example

- Given:

$$x(t) \longrightarrow \boxed{\text{A-to-D}} \xrightarrow{x[n]} \boxed{\dfrac{2+2z^{-1}}{1-0.8z^{-1}}} \xrightarrow{y[n]} \boxed{\text{D-to-A}} \xrightarrow{y(t)}$$

- Find the output, *y*(*t*).
  - When $\boxed{x(t) = \cos(2000\pi t)}$

$$\boxed{f_s = 5000 \text{ Hz}}$$

  - Because $\boxed{\omega T_s = 2000\pi / 5000 = 0.4\pi}$

$$\boxed{x[n] = \cos(0.4\pi n)}$$

NO Aliasing

SINUSOIDAL RESPONSE

- $x[n]$ = SINUSOID => $y[n]$ = SINUSOID
- Get MAGNITUDE & PHASE from $H(z)$.

If $x[n] = e^{j\hat{\omega}n}$,

then $y[n] = H(e^{j\hat{\omega}})e^{j\hat{\omega}n}$,

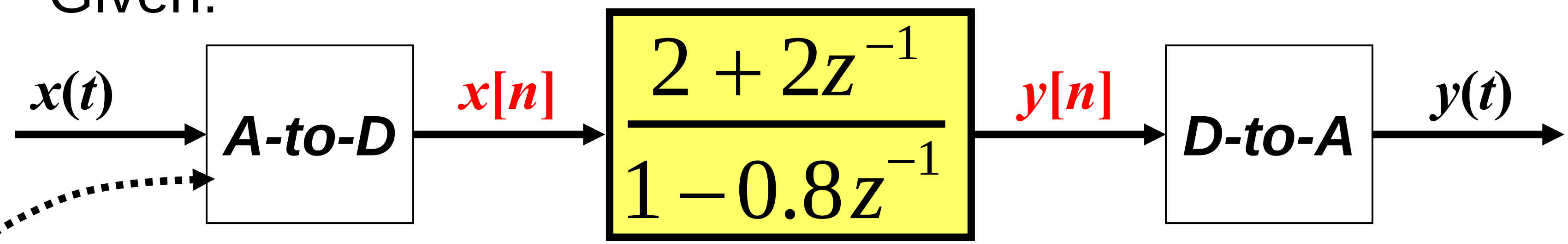
where $H(e^{j\hat{\omega}}) = H(z)\big|_{z=e^{j\hat{\omega}}}$.

- Then $$\boxed{y[n] = M\cos(0.4\pi n + \psi)}$$

$$\boxed{H(e^{j0.4\pi}) = \frac{2+2e^{-j0.4\pi}}{1-0.8e^{-j0.4\pi}} = 3.02e^{-j0.452\pi}}$$

Answer for the Example

- Given:



- When $\boxed{x(t) = \cos(2000\pi t)}$

$$\boxed{f_s = 5000\text{ Hz}}$$

- The output

$$\boxed{y(t) = 3.02\cos(2000\pi t - 0.452\pi)}$$

Thank you