Introduction to Digital Signal Processing

Z-Transform

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One-sided Z-Transform

Definition $$X^{+}(z) \equiv \sum_{n=0}^{\infty} x(n) z^{-n} \qquad \boldsymbol{x(n)} \xleftrightarrow{\boldsymbol{z^{+}}} \boldsymbol{X(z)}$$

does not contain information about the signal for negative time (n < 0)

$$\delta(n-k) \longleftrightarrow z^{-k} \text{ if } k>0$$

$$\delta(n-k) \longleftrightarrow 0 \text{ if } k<0$$

Unique only for causal signals

$$X\{x(n)u(n)\} = X^{+}\{x(n)\}$$

ROC is always the exterior of a circle.

One-sided Z-Transform

Properties

Shifting properties

- Time delay

$$x(n-k) \xleftrightarrow{z^+} z^{-k}\left[X^+(z)+\sum_{n=1}^{k} x(-n)z^{n}\right] \quad k>0$$

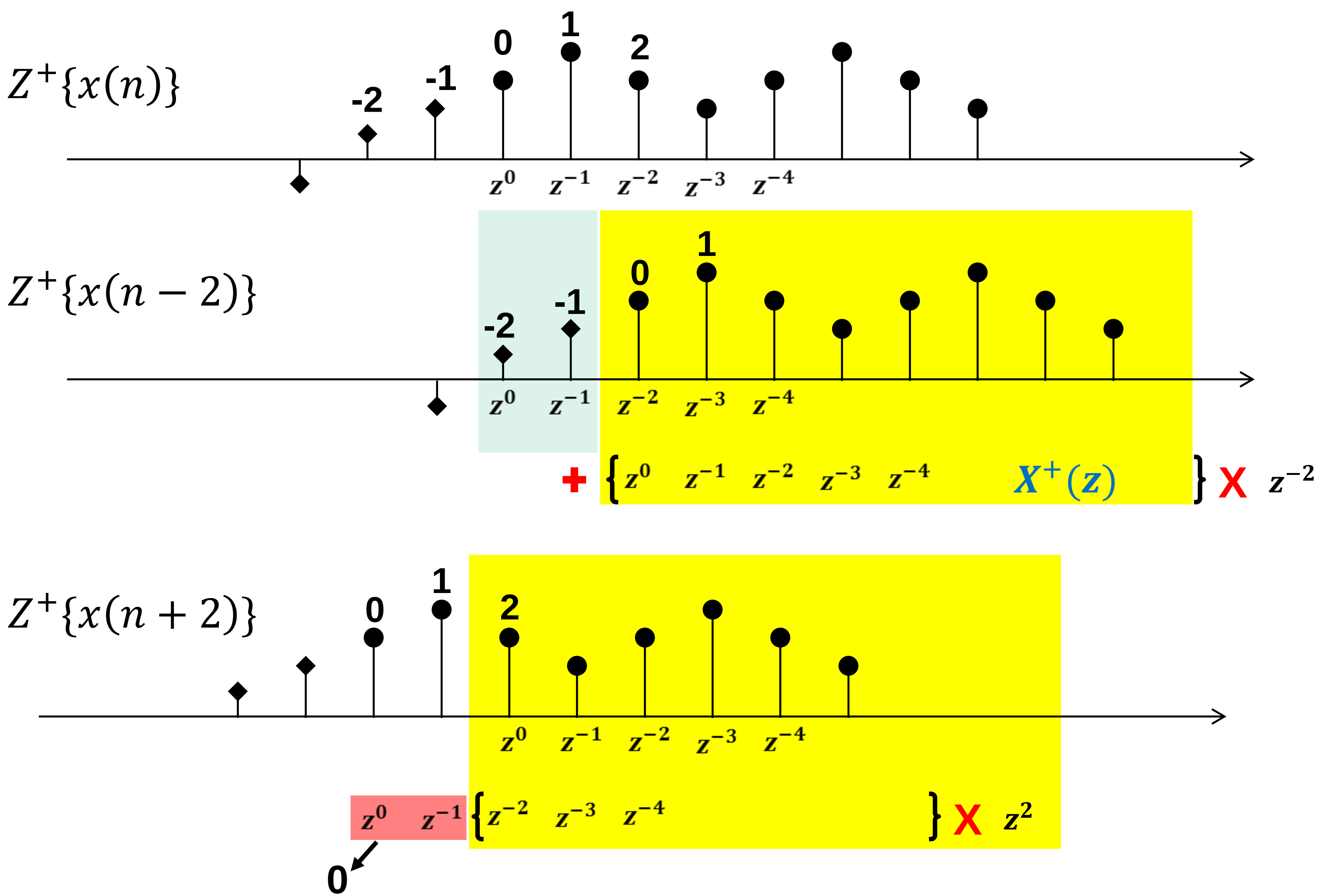
$$Z^+\{x(n-k)\} = \left[x(-k)+x(-k+1)z^{-1}+\cdots+x(-1)z^{-k+1}\right]+z^{-k}X^+(z)$$

In case $x(n)$ is causal, $x(n-k) \xleftrightarrow{z^+} z^{-k}X^+(z)$

- Time advance

$$x(n+k) \xleftrightarrow{z^+} z^{k}\left[X^+(z)-\sum_{n=0}^{k-1} x(n)z^{-n}\right] \quad k>0$$

One-sided Z-Transform



One-sided Z-Transform

Final value theorem

$$\lim_{n\to\infty} x(n) = \lim_{z\to 1}(z-1)X^{+}(z)$$

$$Z[x(n)] = X(z) = \sum_{n=0}^{\infty} x(n)z^{-n}$$

$$X^{+}(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots$$

$$Z[x(n+1)] = \sum_{n=0}^{\infty} x(n)z^{-n} = zX(z) - zx(0)$$

$$= x(1) + x(2)z^{-1} + \cdots = zX(z) - zx(0)$$

$$Z[x(n+1)] - Z[x(n)] = zX(z) - zx(0) - X(z)$$

$$(\mathrm{z}-1)X(z) - zx(0) = [x(1)-x(0)]z^{0} + [x(2)-x(1)]z^{-1} + [x(3)-x(2)]z^{-2} + \ldots$$

$$\lim_{\mathrm{z}\to 1}[(\mathrm{z}-1)X(z)] - x(0) = x(\infty) - x(0)$$

One-sided Z-Transform

Solving difference equation using the one-sided z-transform

$$y(n) = a y(n-1) + x(n) \quad |a| < 1 \qquad y(-1) = 1, \;\; x(n) = u(n)$$

$$Y^+(z) = a[z^{-1} Y^+(z) + y(-1)] + X^+(z)$$

$$\Rightarrow Y^+(z) = \frac{a}{1 - az^{-1}} y(-1) + \frac{1}{(1 - az^{-1})(1 - z^{-1})}$$

$$= \frac{a}{1 - az^{-1}} y(-1) + \frac{\frac{1}{1 - a^{-1}}}{1 - az^{-1}} + \frac{\frac{1}{1 - a}}{1 - z^{-1}}$$

$$= \frac{a}{1 - az^{-1}} y(-1) - \frac{\frac{a}{1 - a}}{1 - az^{-1}} + \frac{\frac{1}{1 - a}}{1 - z^{-1}}$$

$$y(n) = a^{n+1} y(-1) u(n) + \frac{1 - a^{n+1}}{1 - a} u(n)$$

Analysis of LTI systems in the z-domain

System transfer function

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{m=0}^{M} b_m x(n-m) \quad (a_0 = 1)$$

Taking the z-transform of both sides gives

$$Z\left\{\sum_{k=0}^{N} a_k y(n-k)\right\} = Z\left\{\sum_{m=0}^{M} b_m x(n-m)\right\} \Leftrightarrow \sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{m=0}^{M} b_m z^{-m} X(z)$$

$$\rightarrow \; Y(z) = \frac{\sum_{m=0}^{M} b_m z^{-m}}{\sum_{k=0}^{N} a_k z^{-k}} X(z)$$

$$\text{System Transfer Function} = H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{m=0}^{M} b_m z^{-m}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Analysis of LTI systems in the z-domain

Note:

If $x(n) = \delta(n)$, X(z) = 1 and Y(z) = H(z).

Hence, $H(z) = Z\{h(n)\} = \sum_{n=-\infty}^{\infty} h(n)z^{-n} \rightarrow H(e^{j\omega}) = \left\{ Z\{h(n)\} = \sum_{n=-\infty}^{\infty} h(n)z^{-n} \right\}_{z=e^{j\omega}}$

Example) $y(n) - ay(n-1) = x(n) = \delta(n)$

$$\rightarrow Y(z) = \frac{1}{1-az^{-1}} = H(z) \rightarrow h(n) = a^n u(n) \text{ (see chap 2)}$$

$$x(n) \rightarrow \boxed{h(n)} \rightarrow y(n) \qquad X(z) \rightarrow \boxed{H(z)} \rightarrow Y(z)$$

$y(n) = \sum_{k=0}^{n} h(k)x(n-k)$: for a causal input and a system

$H(z) = H_1(z)H_2(z) = H_2(z)H_1(z)$

Analysis of LTI systems in the z-domain

Stability

Stability theorem) A causal LTI system having transfer function H(z) is stable if and only if H(z) contains no poles on and outside the unit circle |z| = 1

Proof) For a causal stable system, it should be satisfied that

$$|H(z)| = \left|\sum_{k=0}^{\infty} h(k) z^{-k}\right| \leq \sum_{k=0}^{\infty} |h(k)||z|^{-k} < \infty$$

$$\textit{which requires } |z| \geq 1.$$

This means that there can exist no poles on and outside the unit circle.

System function vs. frequency response

System function with a rational function of z

$$H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=0}^{N} a_k z^{-k}} = b_0 \frac{\prod_{k=1}^{M} (1 - z_k z^{-1})}{\prod_{k=1}^{N} (1 - p_k z^{-1})}$$

Frequency response

$$H(\omega) = \frac{B(\omega)}{A(\omega)} = \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{1 + \sum_{k=1}^{N} a_k e^{-j\omega k}} = b_0 \frac{\prod_{k=1}^{M} (1 - z_k e^{-j\omega})}{\prod_{k=1}^{N} (1 - p_k e^{-j\omega})}$$

Magnitude squared of $H(\omega)$

- $|H(\omega)|^2 = H(\omega) H^*(\omega)$

$$H^*(\omega) = b_0 \frac{\prod_{k=1}^{M} (1 - z_k^* e^{j\omega})}{\prod_{k=1}^{N} (1 - p_k^* e^{j\omega})} = b_0 \left. \frac{\prod_{k=1}^{M} (1 - z_k^* z)}{\prod_{k=1}^{N} (1 - p_k^* z)} \right|_{z = e^{j\omega}} \equiv H^*(1/z^*) \Big|_{z = e^{j\omega}}$$

System function vs. frequency response

- For real $h(n)$
  - Complex poles and zeros occur in complex-conjugate pairs.
  - $H^*(1/z^*) = H(z^{-1}), \quad H^*(\omega) = H(-\omega)$
  - $|H(\omega)|^2 = H(\omega)H^*(\omega) = H(\omega)H(-\omega) = H(z)H(z^{-1})|_{z=e^{j\omega}}$

Graphical analysis of the Frequency response function

$$H(e^{j\omega}) = b_0 \frac{\prod_{k=1}^{M}(1 - z_k e^{-j\omega})}{\prod_{k=1}^{N}(1 - p_k e^{-j\omega})} = b_0 e^{j\omega(N-M)} \frac{\prod_{k=1}^{M}(e^{j\omega} - z_k)}{\prod_{k=1}^{N}(e^{j\omega} - p_k)}$$

$$\longrightarrow e^{j\omega} - z_k = V_k e^{j\theta_k(\omega)}$$

$$\longrightarrow e^{j\omega} - p_k = U_k e^{j\phi_k(\omega)}$$

$$V_k(\omega) = |e^{j\omega} - z_k| \qquad U_k(\omega) = |e^{j\omega} - p_k|$$

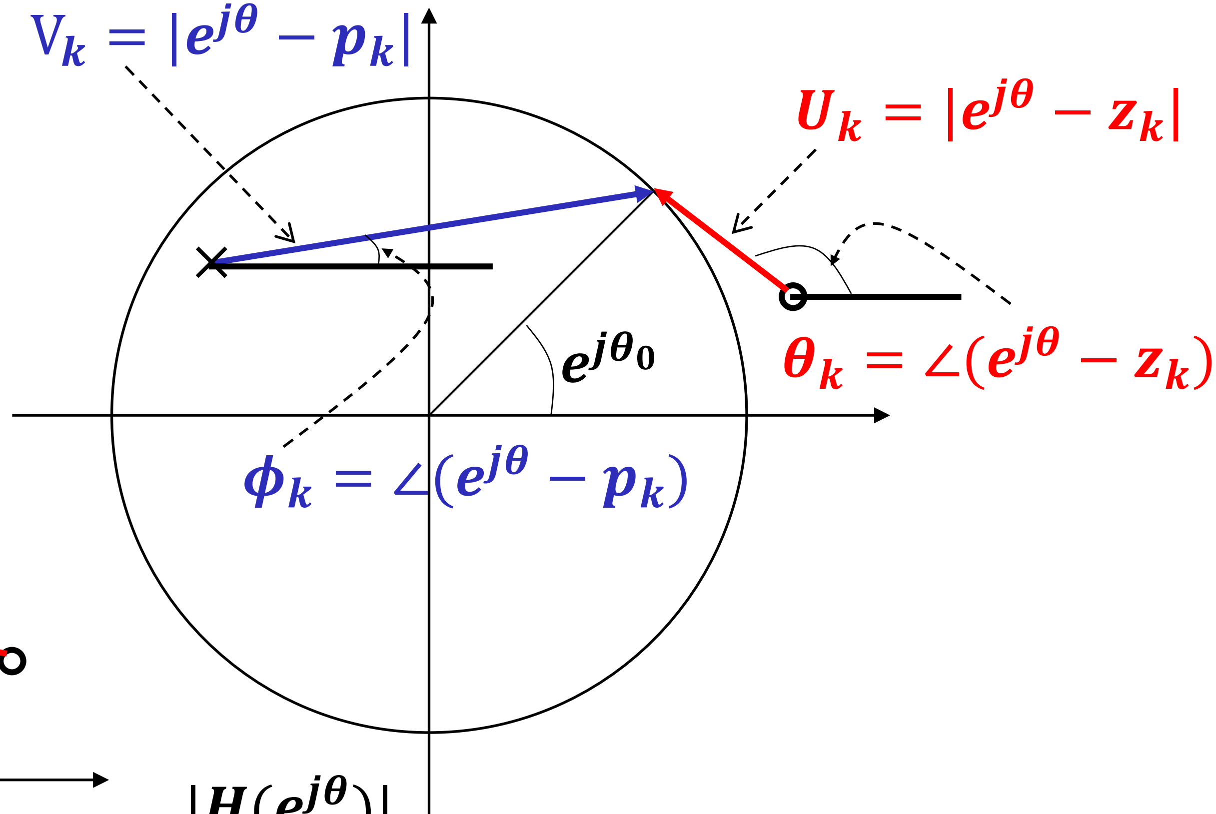
$$|H(\omega)| = |b_0| \frac{V_1(\omega)\, V_2(\omega) \cdots V_M(\omega)}{U_1(\omega)\, U_2(\omega) \cdots U_N(\omega)}$$

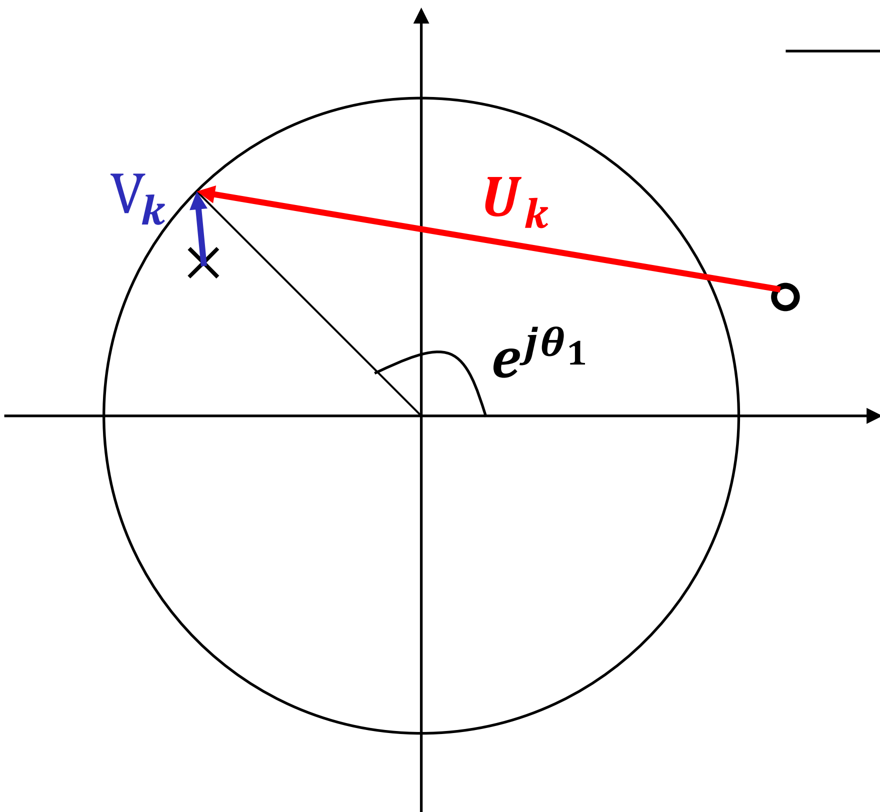
$$\theta_k(\omega) = \angle(e^{j\omega} - z_k) \qquad \phi_k(\omega) = \angle(e^{j\omega} - p_k)$$

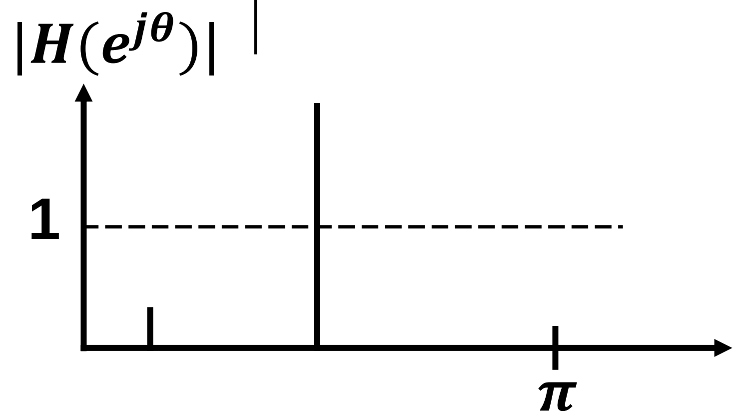
$$\angle H(\omega) = \angle b_0 + \omega(N-M) + [\theta_1(\omega) + \theta_2(\omega) + \cdots + \theta_M(\omega)] - [\phi_1(\omega) + \phi_2(\omega) + \cdots + \phi_N(\omega)]$$

Graphical analysis of the Frequency response function

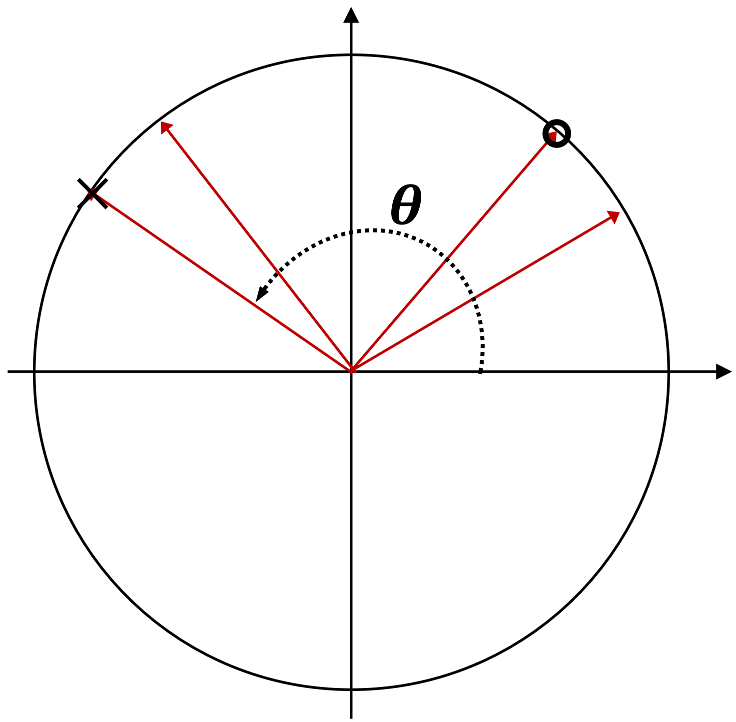
$$H(e^{j\theta}) = \frac{e^{j\theta} - z_k}{e^{j\theta} - p_k}$$





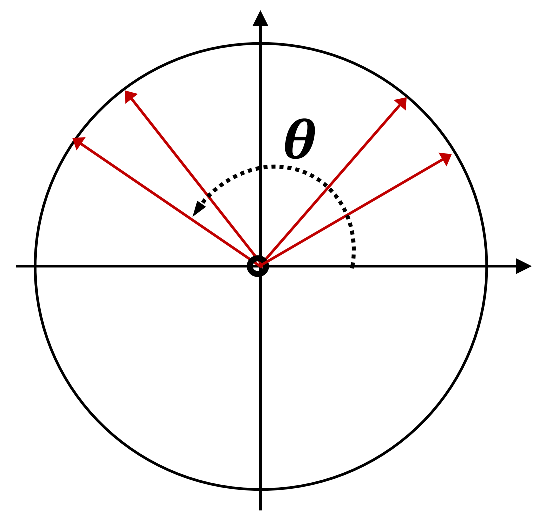


Graphical analysis of the Frequency response function



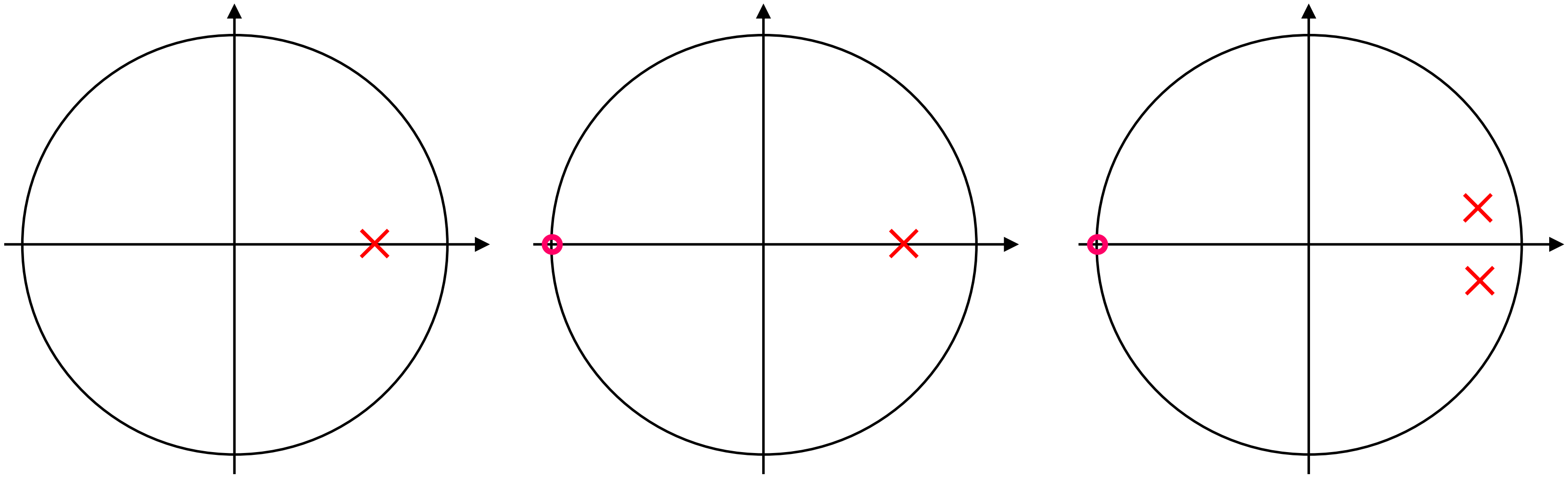
- Zeros on the unit circle
  - $|H(\omega)| = 0$
  - $\omega = \measuredangle z_k$
- Poles on the unit circle
  - $|H(\omega)| = \infty$
  - $\omega = \measuredangle p_k$

- Zeros and poles at the origin
  act only as time delays : can be used for causality !!

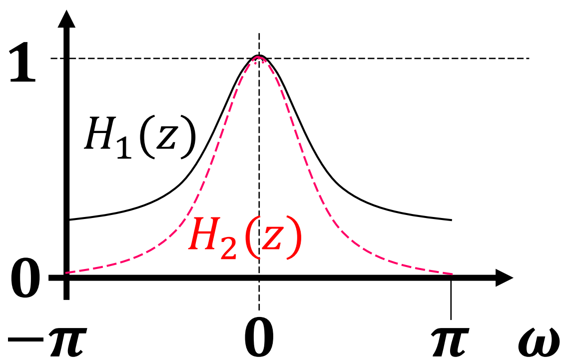


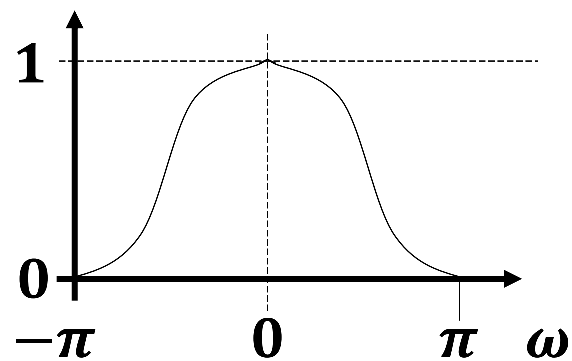
Graphical analysis of the Frequency response function

Pole-zero placement in digital filter design: Lowpass case

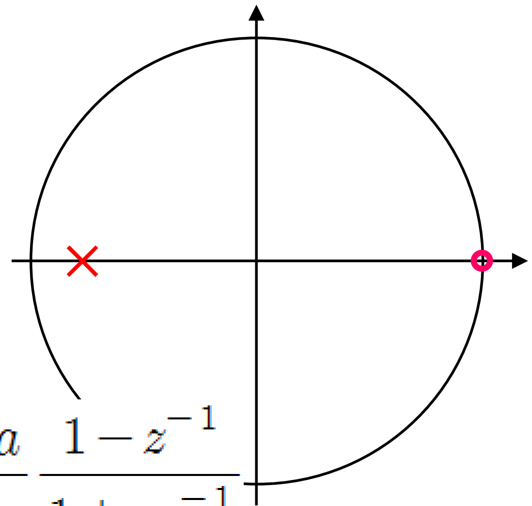


$$H_1(z) = \frac{b_0}{1 - az^{-1}} = \frac{1-a}{1 - az^{-1}} \qquad H_2(z) = \frac{1-a}{2}\frac{1+z^{-1}}{1 - az^{-1}}$$

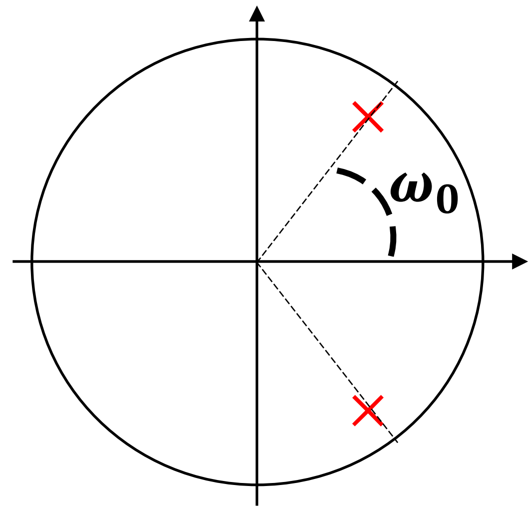


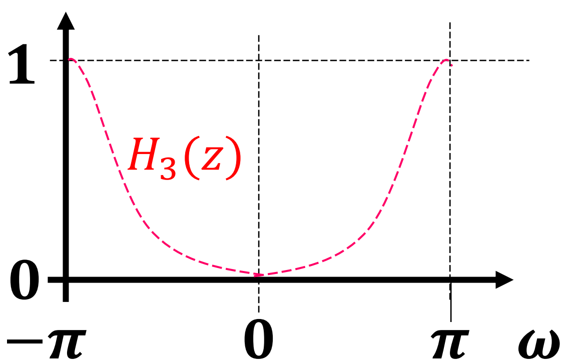


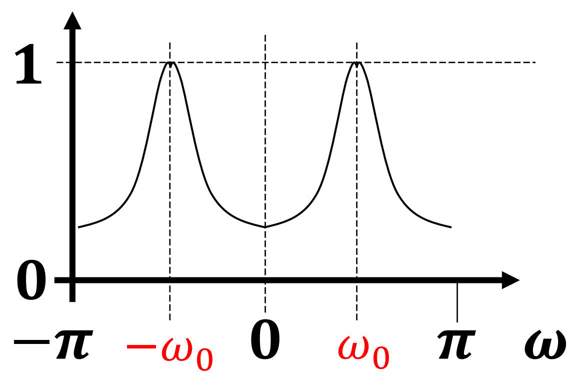
Pole-zero placement in digital filter design: Highpass and bandpass



$$H_3(z) = \frac{1-a}{2}\,\frac{1-z^{-1}}{1+az^{-1}}$$





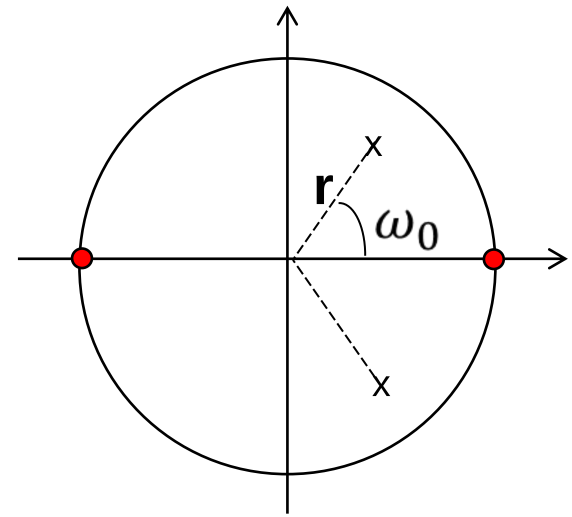


Examples of basic DSP blocks

Digital resonator

$$H(z) = \frac{b_0}{(1 - re^{j\omega_0}z^{-1})(1 - re^{-j\omega_0}z^{-1})},$$

$$p_{1,2} = re^{\pm j\omega_0} \qquad 0 < r < 1$$

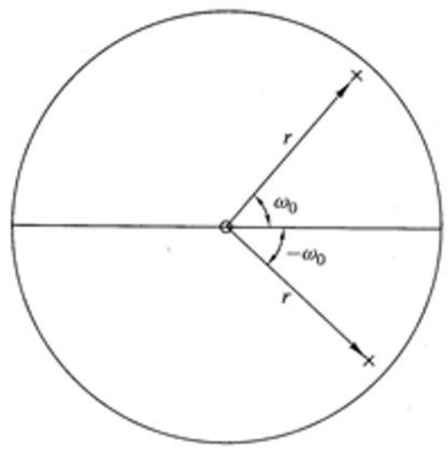


How does the following function improve the previous one ?

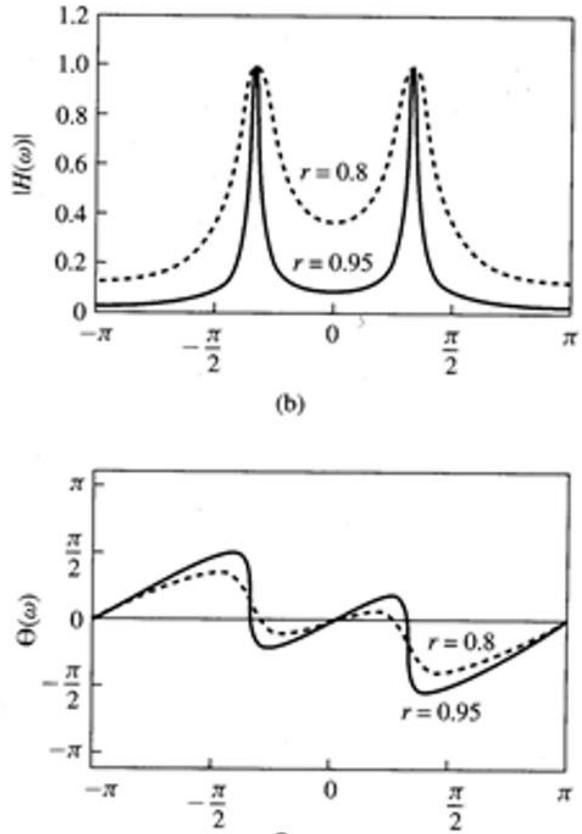
$$H(z) = G\frac{(1 - z^{-1})(1 + z^{-1})}{(1 - re^{j\omega_0}z^{-1})(1 - re^{-j\omega_0}z^{-1})}$$

$$= \frac{1 - z^{-2}}{1 - (2r\cos\omega_0)z^{-1} + r^2 z^{-2}}$$

Examples of basic DSP blocks



(a)



Examples of basic DSP blocks

Notch Filter

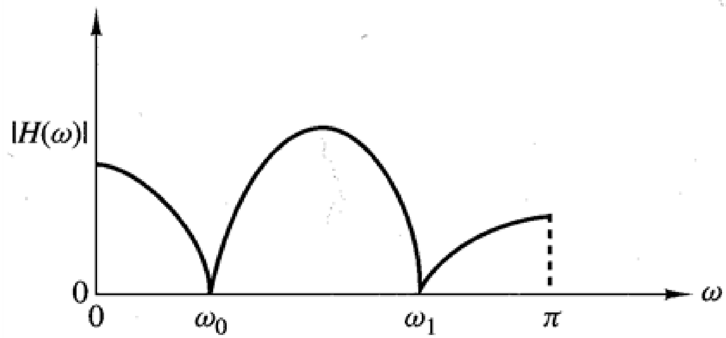


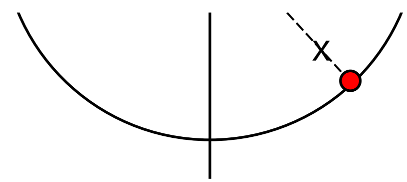
Figure 4.50 Frequency response characteristic of a notch filter.

- $H(z) = b_0(1 - e^{j\omega_0}z^{-1})(1 - e^{-j\omega_0}z^{-1}) = b_0(1 - 2\cos\omega_0 z^{-1} + z^{-2})$

  $z_{1,2} = e^{\pm j\omega_0}$

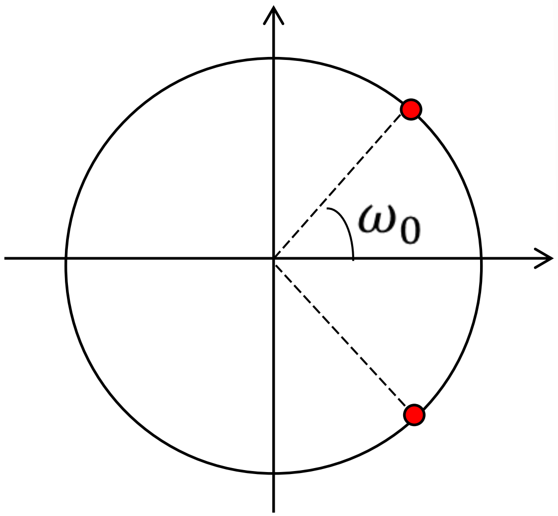
- $H(z) = b_0 \dfrac{1 - 2\cos\omega_0 z^{-1} + z^{-2}}{1 - (2r\cos\omega_0)z^{-1} + r^2 z^{-2}}$

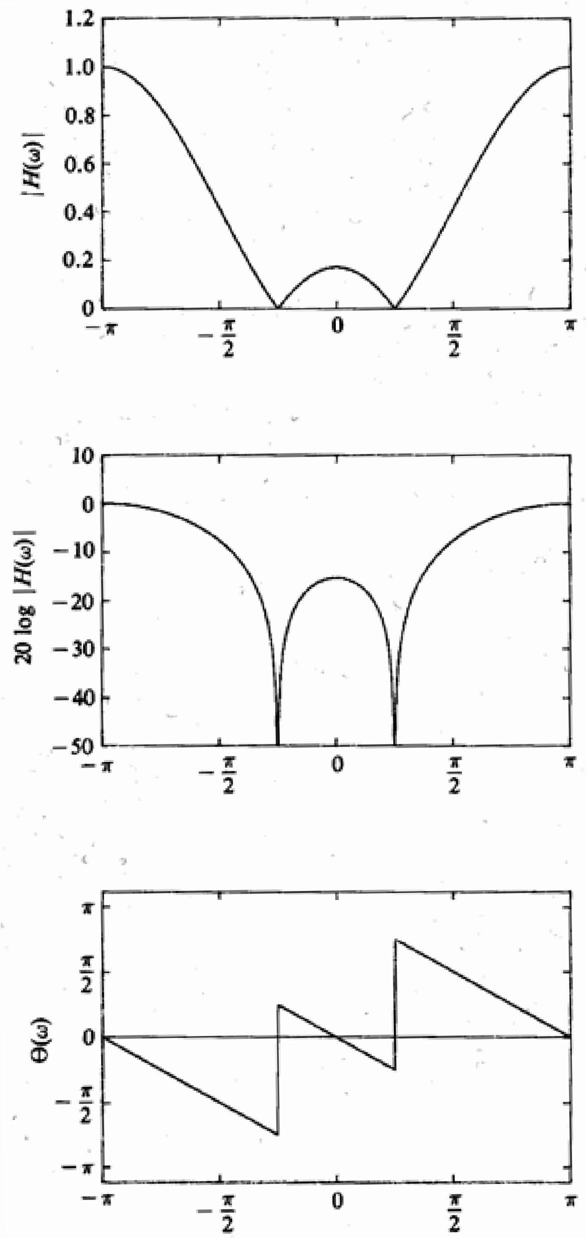
  $p_{1,2} = re^{\pm j\omega_0}$



Examples of basic DSP blocks

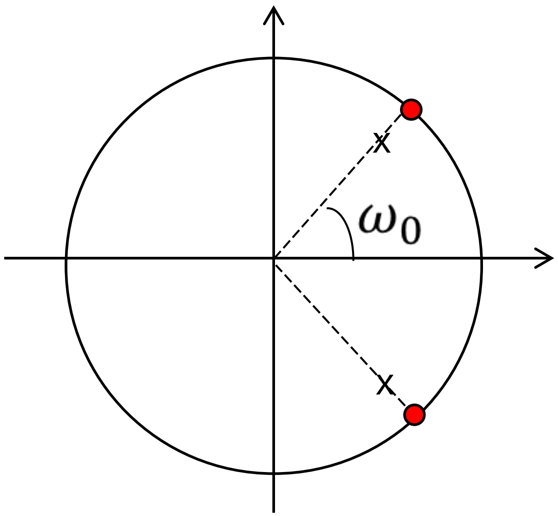
Notch Filter

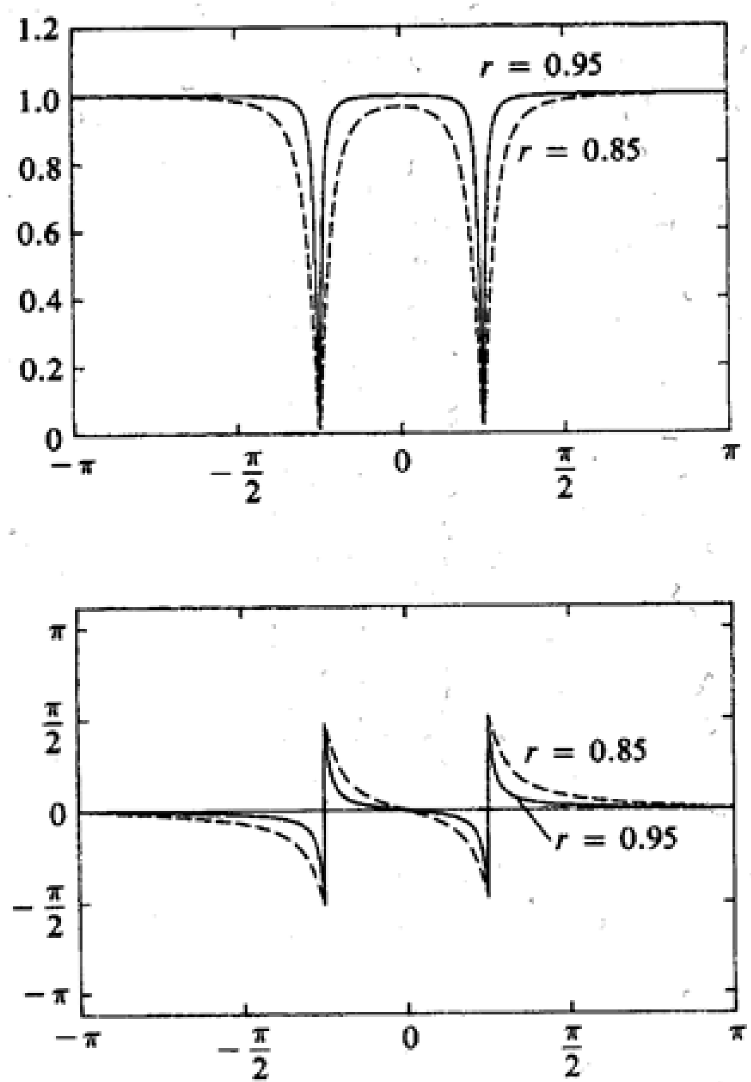


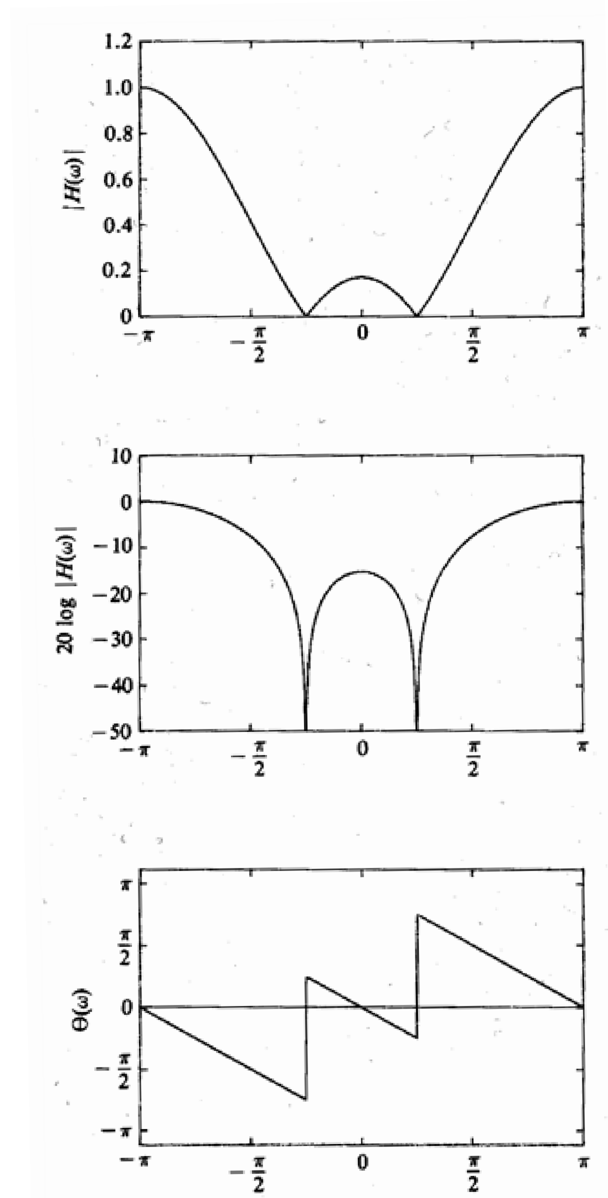


Examples of basic DSP blocks

Notch Filter







Examples of basic DSP blocks

Comb Filter

- Filters in which the nulls occur periodically across the frequency band Frequently used in many applications where harmonic interferences should be eliminated.
- Example: Moving average filter

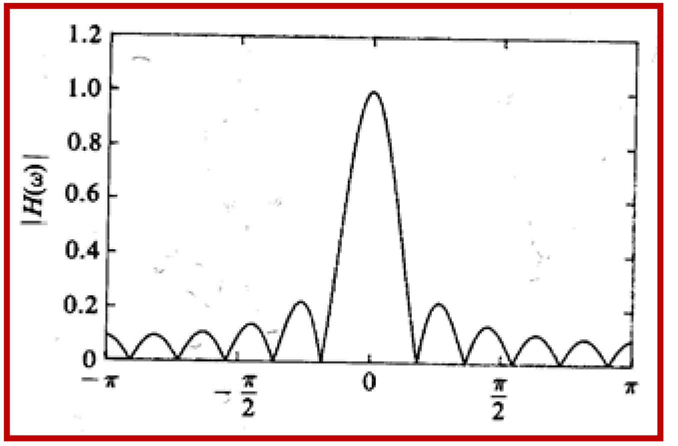
$$y(n) = \frac{1}{M+1}\sum_{k=0}^{M} x(n-k)$$

$$H(z) = \frac{1}{M+1}\sum_{k=0}^{M} z^{-k} = \frac{1}{M+1}\frac{[1-z^{-(M+1)}]}{(1-z^{-1})}$$

$$H(z) = \frac{e^{-j\omega M/2}}{M+1}\frac{\sin\omega\left(\frac{M+1}{2}\right)}{\sin(\omega/2)},$$

$$z_k = e^{j2\pi k/(M+1)} \qquad k = 1, 2, 3, \ldots, M$$

M=10

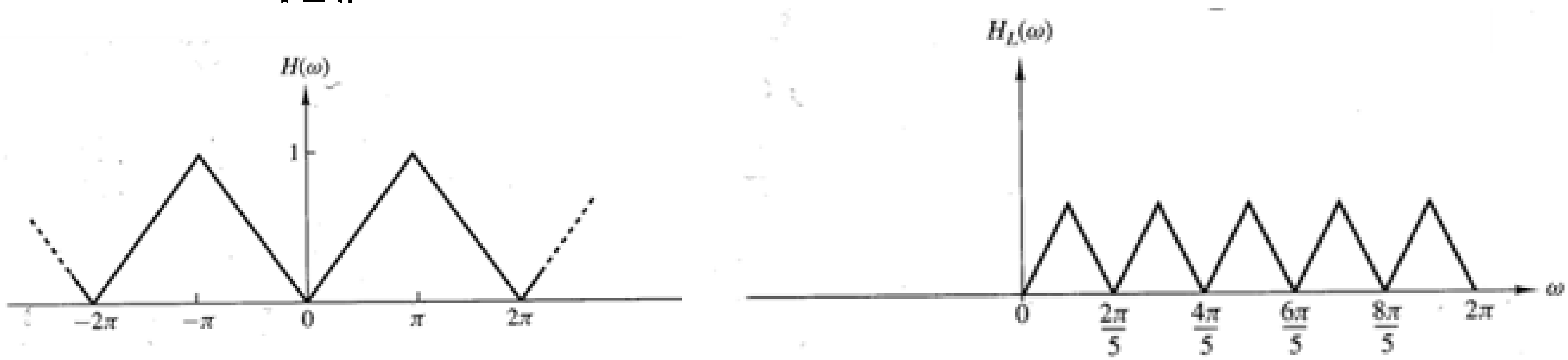


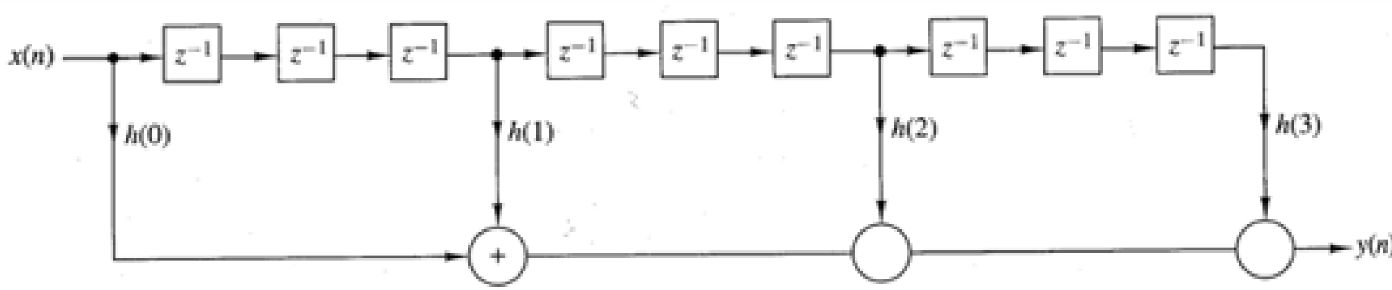
Examples of basic DSP blocks

- Comb filters designed by modifying FIR filter

$$H(z) = \sum_{k=0}^{M} h(k) z^{-k} \Rightarrow H_L(z) = \sum_{k=0}^{M} h(k) z^{-kL}$$

$$\Rightarrow H_L(\omega) = \sum_{k=0}^{M} h(k) e^{-jkL\omega} = H(L\omega)$$





Examples of basic DSP blocks

Ex) Replace z by z^L in the moving average filter

$$H_L(z) = \frac{1}{M+1}\frac{1-z^{-L(M+1)}}{1-z^{-L}}$$

$$H_L(\omega) = \frac{1}{M+1}\frac{\sin[\omega L(M+1)/2]}{\sin(\omega L/2)} e^{-j\omega LM/2}, \qquad z_k = e^{j2\pi k/L(M+1)}$$

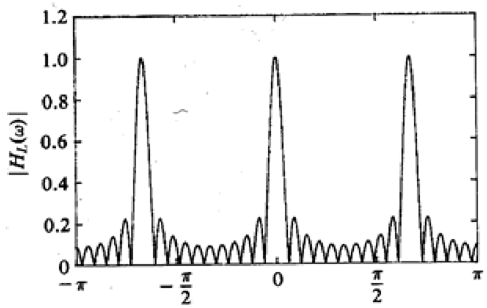


Figure 4.56 Magnitude response characteristic for a comb filter given by (4.5.40), with $L = 3$ and $M = 10$.

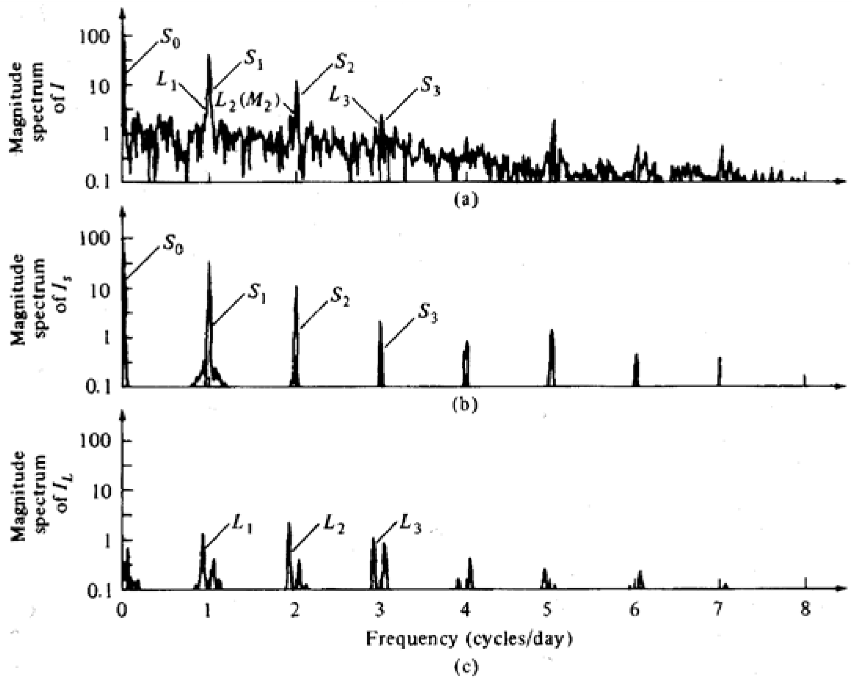


Figure 4.57 (a) Spectrum of unfiltered electron content data; (b) spectrum of output of solar filter; (c) spectrum of output of lunar filter. [From paper by Bernhardt et al. (1976). Reprinted with permission of the American Geophysical Union.]

Examples of basic DSP blocks

All-pass filters

- Flat magnitude response / arbitrary phase response

$$|H(\omega)| = 1 \qquad -\pi \le \omega \le \pi$$

ex) $H(z) = z^{-k}$: useless

- Useful form

$$H(z) = \frac{a_N + a_{N_1} z_{-1} + \cdots + a_1 z^{-N+1} + z^{-N}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} = \frac{\sum_{k=0}^{N} a_k z^{-N+k}}{\sum_{k=0}^{N} a_k z^{-k}} \qquad a_0 = 1$$

$$A(z) = 1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}$$

$$A(z^{-1}) = 1 + a_1 z + a_2 z^2 + \cdots + a_N z^N$$

$$z^{-N} A(z^{-1}) = a_N + a_{N-1} z^{-1} + \cdots + a_1 z^{-(N-1)} + 1$$

Examples of basic DSP blocks

All-pass filters

- Flat magnitude response / arbitrary phase response

$$|H(\omega)| = 1 \qquad -\pi \le \omega \le \pi$$

ex) $H(z) = z^{-k}$: useless

- Useful form

$$H(z) = \frac{a_N + a_{N_1} z_{-1} + \cdots + a_1 z^{-N+1} + z^{-N}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} = \frac{\sum_{k=0}^{N} a_k z^{-N+k}}{\sum_{k=0}^{N} a_k z^{-k}} \qquad a_0 = 1$$

- General form with real coefficients

$$H(z) = z^{-N} \frac{A(z^{-1})}{A(z)} \quad \text{where} \quad A(z) = \sum_{k=0}^{N} a_k z^{-k} \qquad a_0 = 1$$

$$\Rightarrow \quad |H(\omega)|^2 = H(z)H(z^{-1})|_{z=e^{j\omega}} = 1$$

$$H_{ap}(z) = \prod_{k=1}^{N_R} \frac{z^{-1} - \alpha_k}{1 - \alpha_k z^{-1}} \prod_{k=1}^{N_c} \frac{(z^{-1} - \beta_k)(z^{-1} - \beta_k^*)}{(1 - \beta_k z^{-1})(1 - \beta_k^* z^{-1})}$$

Examples of basic DSP blocks

Pole-zero patterns of All-pass filters

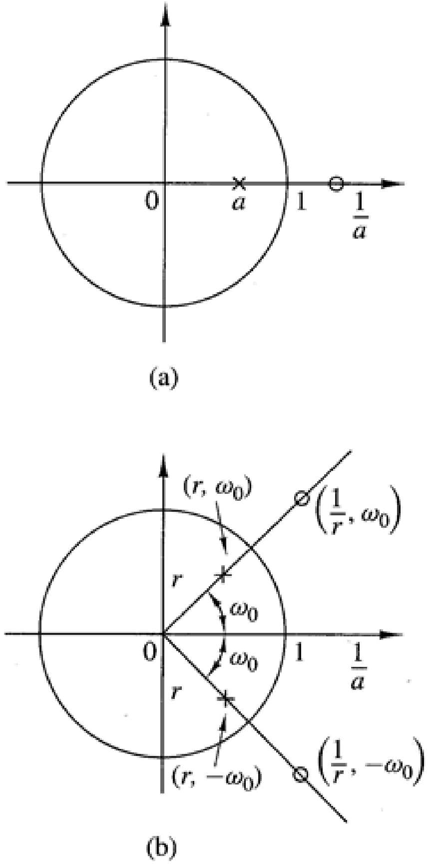


Figure 4.58 Pole–zero patterns of (a) a first-order and (b) a second-order all-pass filter.

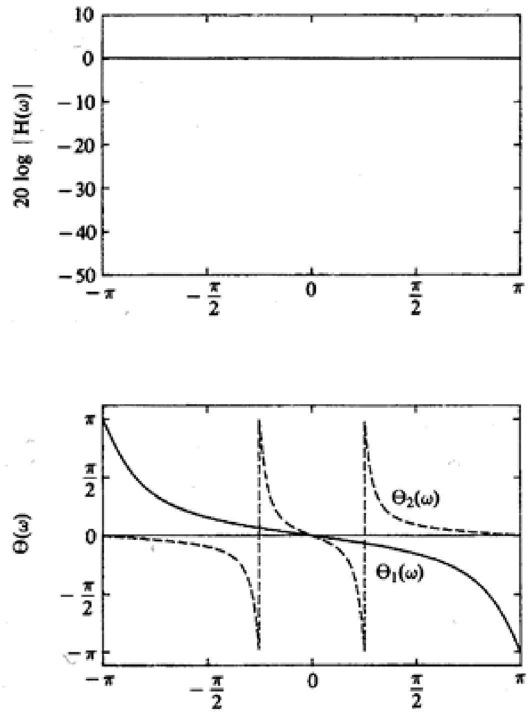


Figure 4.59 Frequency response characteristics of an all-pass filter with system functions (1) $H(z) = (0.6 + z^{-1})/(1 + 0.6z^{-1})$, (2) $H(z) = (r^2 - 2r\cos\omega_0 z^{-1} + z^{-2})/(1 - 2r\cos\omega_0 z^{-1} + r^2 z^{-2})$, $r = 0.9$, $\omega_0 = \pi/4$.

Examples of basic DSP blocks

- Can be used for phase equalization

$$H_{ap}(\omega) = \frac{e^{-j\omega} - re^{-j\Theta}}{1 - re^{j\Theta}e^{-j\omega}}$$

$$\Theta_{ap}(\omega) = -\omega - 2\tan^{-1}\frac{r\sin(\omega - \Theta)}{1 - r\cos(\omega - \Theta)}$$

$$\tau_z(\omega) = \frac{d\Theta_{ap}(\omega)}{d\omega} = \frac{1 - r^2}{1 + r^2 - 2r\cos(\omega - \Theta)}$$