Chapter 8
IIR Filters

SECOND-ORDER FILTERS

- Two FEEDBACK TERMS

$$y[n] = a_1 y[n-1] + a_2 y[n-2]$$
$$+ b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

$$\boxed{H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}}}$$

Example: SECOND-ORDER FILTERS

$$\boxed{\frac{1-0.45z^{-1}}{1-0.9z^{-1}+0.81z^{-2}}}$$

$$\boxed{y[n]=0.9y[n-1]-0.81y[n-2]+x[n]-0.45x[n-1]}$$

```
aa = [ 1, -0.9, 0.81 ];
bb = [ 1, -0.45 ];
hh = filter( bb, aa, xx );
HH = freqz( bb, aa, [-pi:pi/100:pi] );
```

Implementation of second-order IIR filters (1)

- Difference equation

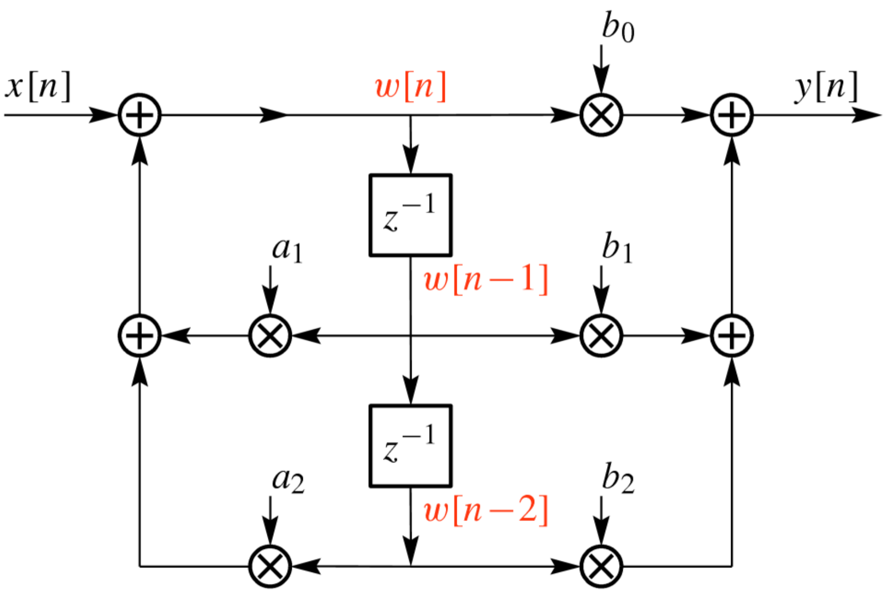
$$y[n] = a_1 y[n-1] + a_2 y[n-2] + b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

- System function

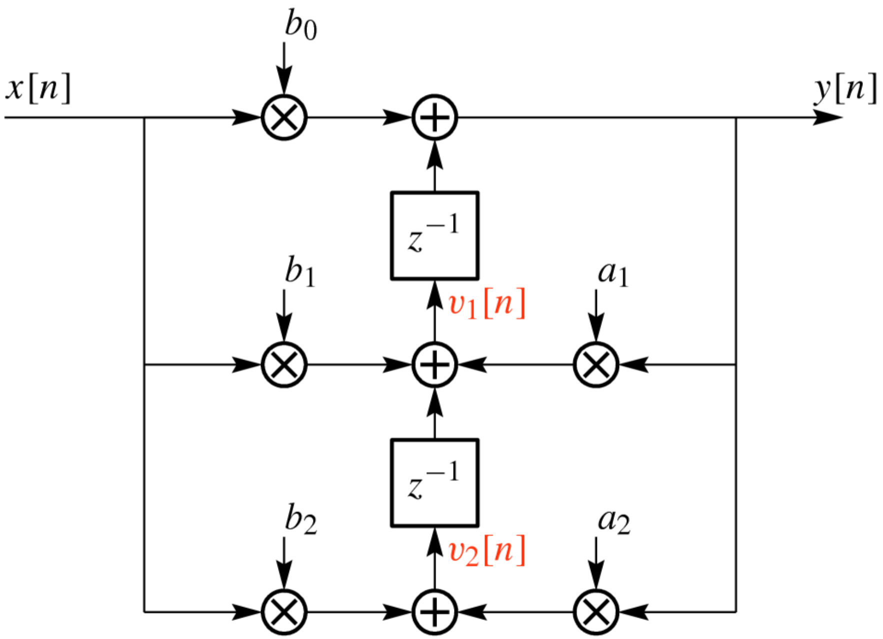
$$H(z) = \frac{1}{1 - a_1 z^{-1} - a_2 z^{-2}} (b_0 + b_1 z^{-1} + b_2 z^{-2}) = \frac{1}{A(z)} B(z)$$

Implementation of second-order IIR filters (2)

Direct Form II Structure



Transposed Direct Form II Structure



POLES OF SECOND-ORDER FILTERS

- The Denominator is QUADRATIC.
  - 2 Poles: REAL
    or COMPLEX CONJUGATES

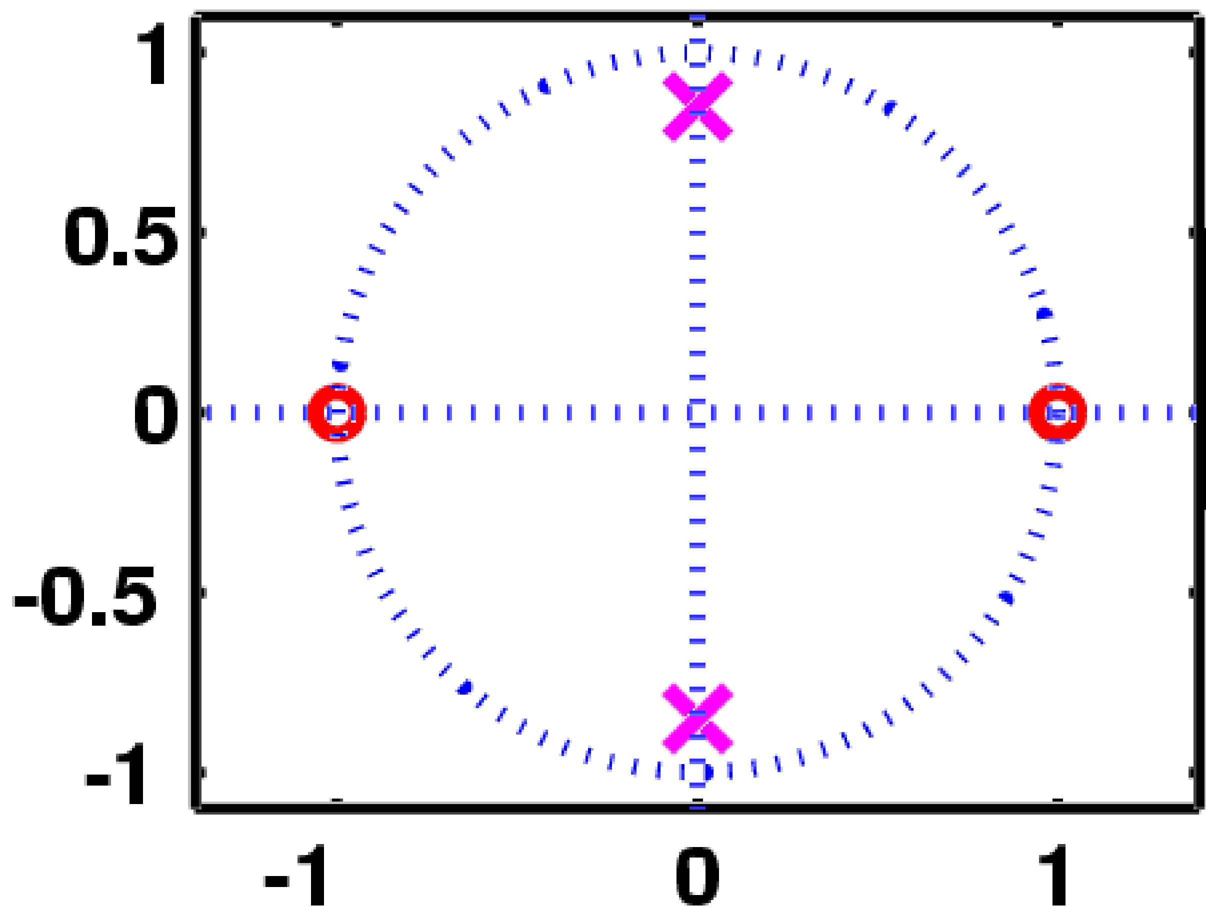
$$\boxed{\frac{a_1 \pm \sqrt{a_1^2 + 4a_2}}{2}}$$

$$\boxed{H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2}} = \frac{b_0 z^2 + b_1 z + b_2}{z^2 - a_1 z - a_2}}$$

PROPERTY OF REAL POLYNOMIALS

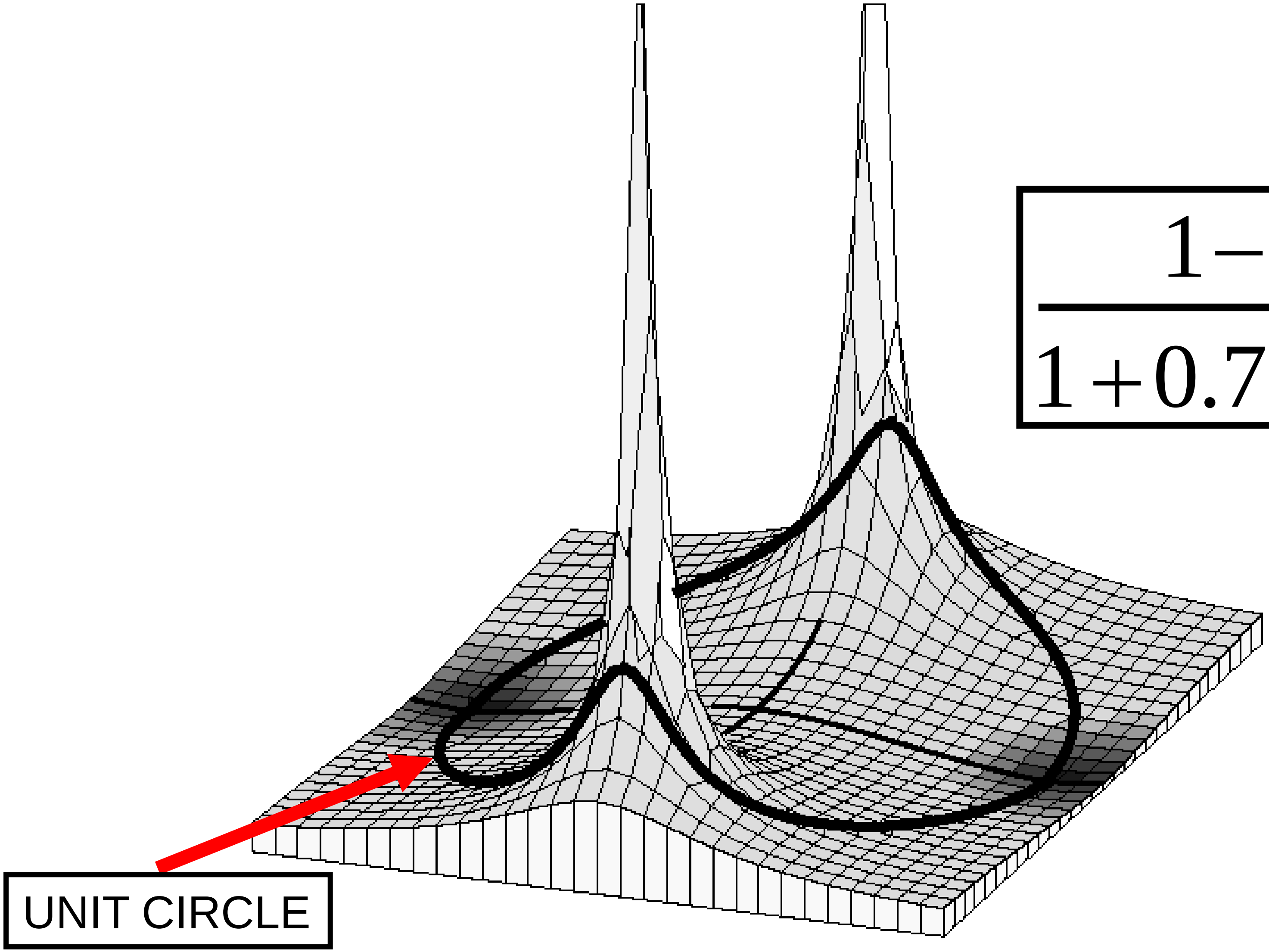
A polynomial of degree N has N roots. If all the coefficients of the polynomial are real, the roots either must be real, or must occur in complex conjugate pairs.

Complex POLE-ZERO PLOT



$$\frac{1-z^{-2}}{1+0.7225z^{-2}}$$

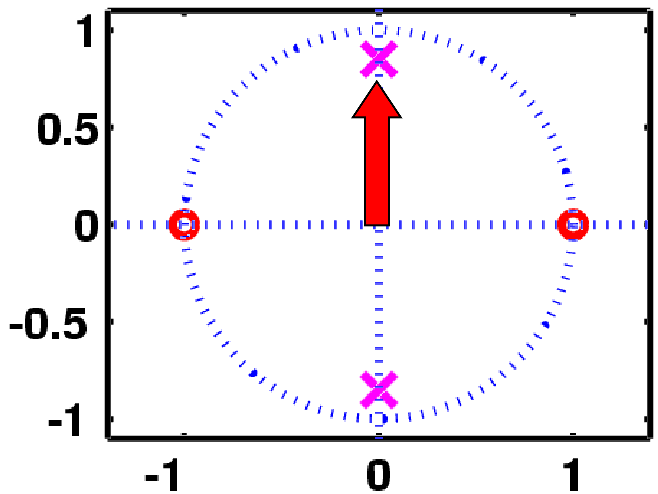
3-D VIEW



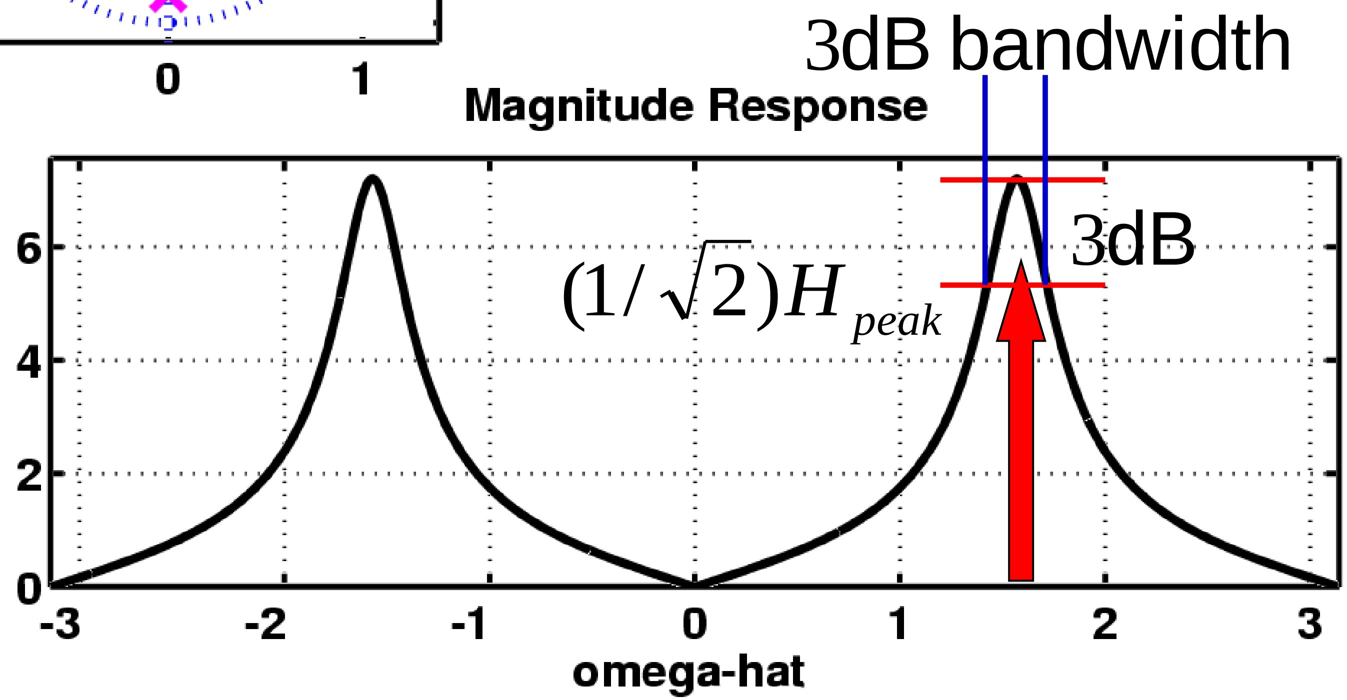
$$\frac{1 - z^{-2}}{1 + 0.7225 z^{-2}}$$

The poles are at $z = 0.85e^{\pm j\pi/2}$ and the zeros at $z = \pm 1$.

FREQ. RESPONSE from POLES & ZEROS



$$H(e^{j\hat{\omega}}) = \frac{1 - e^{-j2\hat{\omega}}}{1 + 0.7225 e^{-j2\hat{\omega}}}$$



TWO COMPLEX POLES

- Find the Impulse Response ?
  - Can OSCILLATE vs. *n*.
  - "RESONANCE"

$$\boxed{(p_k)^n = (re^{j\theta})^n = r^n e^{jn\theta}}$$

- Find the **FREQUENCY RESPONSE.**
  - Depends on the Pole Location.
  - Close to the Unit Circle?
    - Correspond to a high peak.
    - Make the **BANDPASS FILTER**.

$$\text{pole} = re^{j\theta}$$

$$r \to 1?$$

2nd ORDER z-transform PAIR (1)

$$h[n] = Ar^n \cos(\theta n + \varphi)u[n] = Ar^n \tfrac{1}{2}(e^{j(\theta n+\varphi)} + e^{-j(\theta n+\varphi)})u[n]$$

$$H(z) = A\frac{1}{2}\left(\frac{e^{j\varphi}}{1 - re^{j\theta}z^{-1}} + \frac{e^{-j\varphi}}{1 - re^{-j\theta}z^{-1}}\right)$$

$$H(z) = A\frac{1}{2}\left(\frac{e^{j\varphi} - re^{-j(\theta-\varphi)}z^{-1} + e^{-j\varphi} - re^{j(\theta-\varphi)}z^{-1}}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}}\right)$$

$$H(z) = A\frac{\cos\varphi - r\cos(\theta - \varphi)z^{-1}}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}}$$

2nd ORDER z-transform PAIR (2)

$$h[n] = Ar^n \cos(\theta n + \varphi)u[n]$$

$$H(z) = A\frac{\cos\varphi - r\cos(\theta - \varphi)z^{-1}}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}}$$

$$h[n] = r^n \cos(\theta n)u[n]$$

$$H(z) = \frac{1 - r\cos\theta z^{-1}}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}}$$

2nd ORDER EXAMPLE

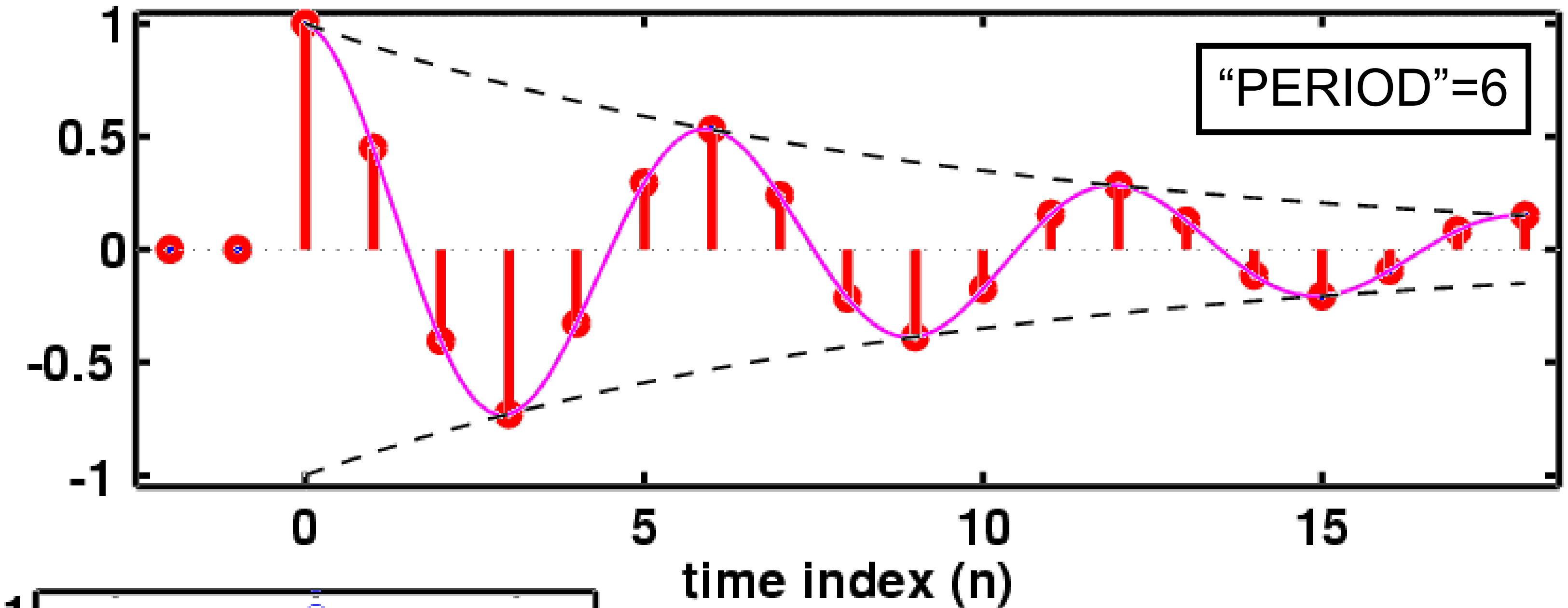
$$h[n] = (0.9)^n \cos(\tfrac{\pi}{3} n) u[n] = (0.9)^n \tfrac{1}{2}(e^{j\pi n/3} + e^{-j\pi n/3}) u[n]$$

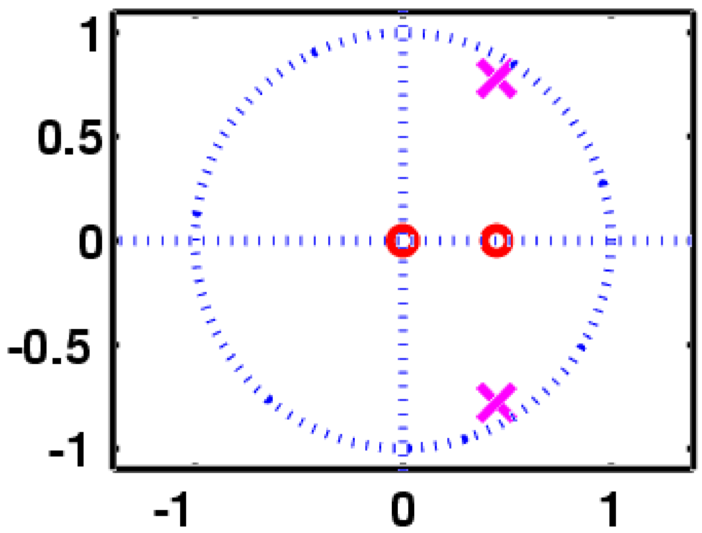
$$H(z) = \frac{0.5}{1 - 0.9 e^{j\pi/3} z^{-1}} + \frac{0.5}{1 - 0.9 e^{-j\pi/3} z^{-1}}$$

$$H(z) = \frac{1 - 0.9\cos(\frac{\pi}{3}) z^{-1}}{(1 - 0.9 e^{j\pi/3} z^{-1})(1 - 0.9 e^{-j\pi/3} z^{-1})}$$

$$H(z) = \frac{1 - 0.45 z^{-1}}{1 - 0.9 z^{-1} + 0.81 z^{-2}}$$

$h[n]$ Decays & Oscillates.

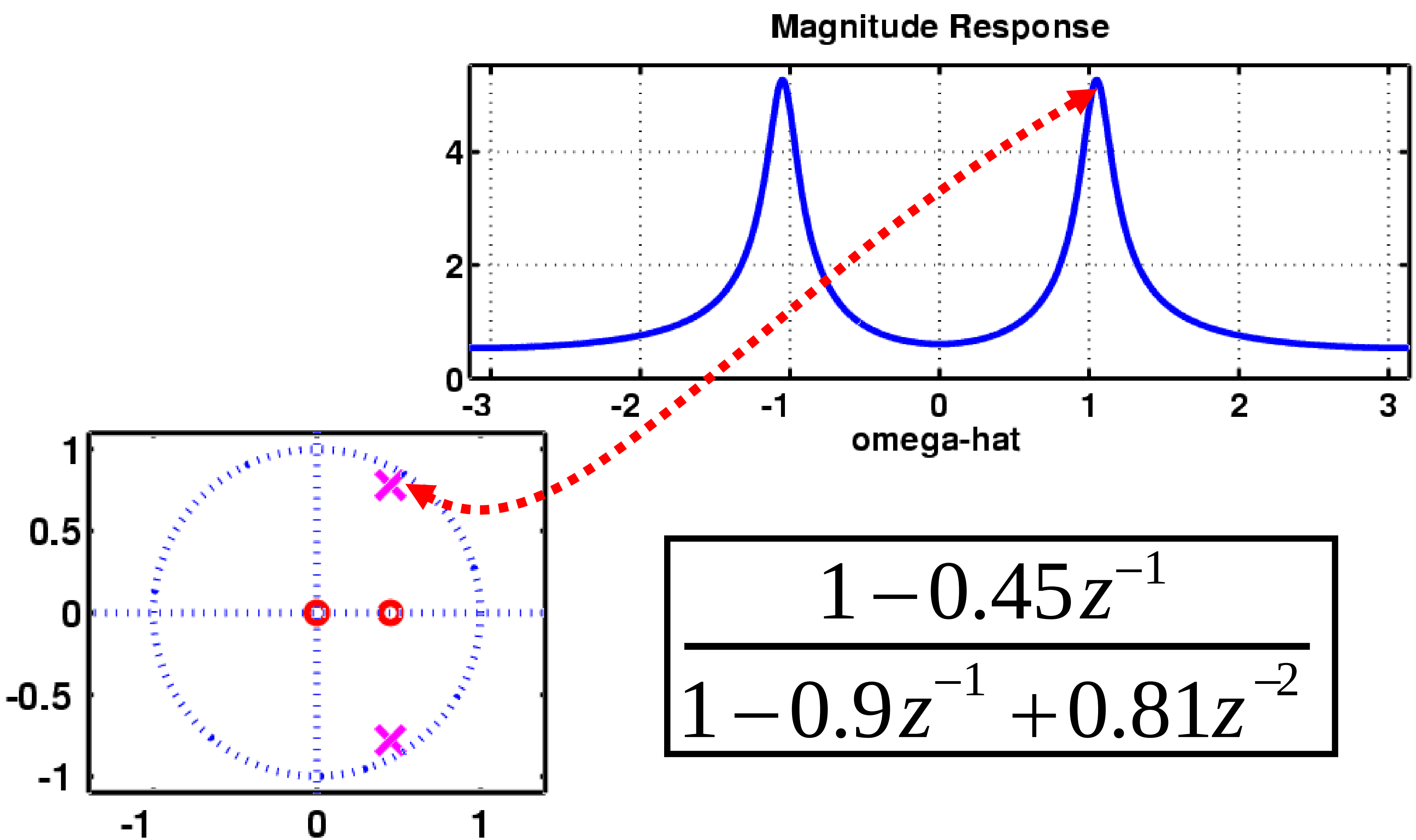




$$h[n] = (0.9)^n \cos\left(\tfrac{\pi}{3} n\right) u[n]$$

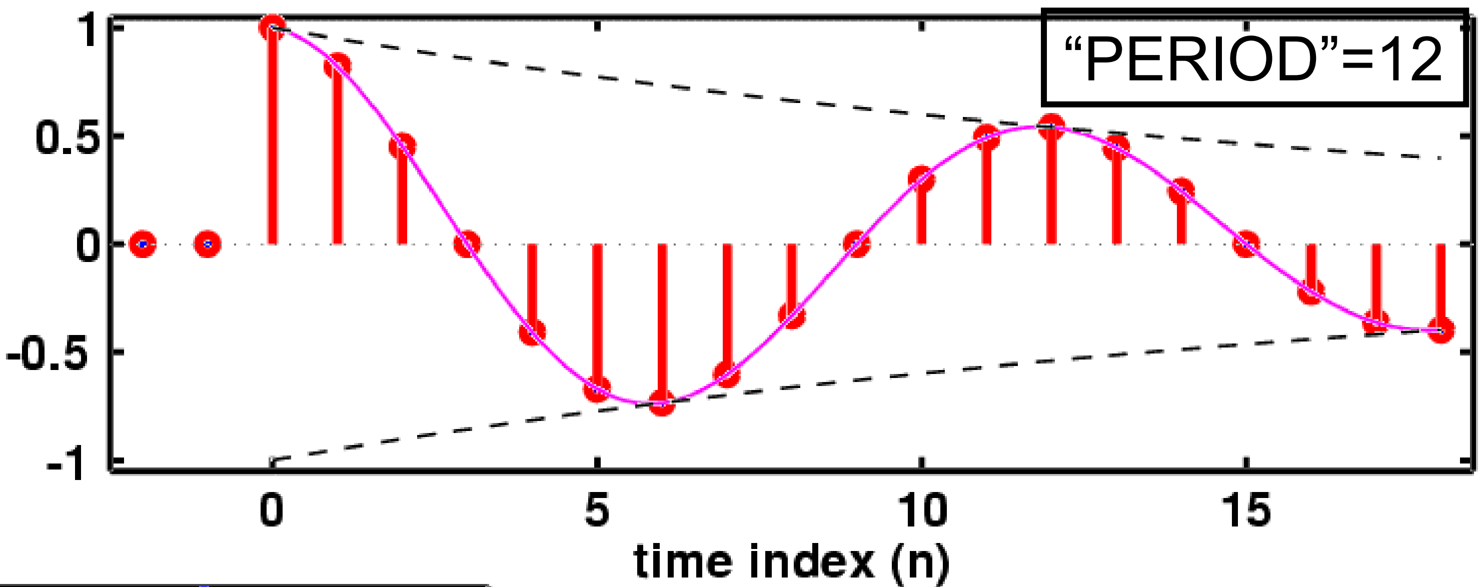
$$\frac{1 - 0.45 z^{-1}}{1 - 0.9 z^{-1} + 0.81 z^{-2}}$$

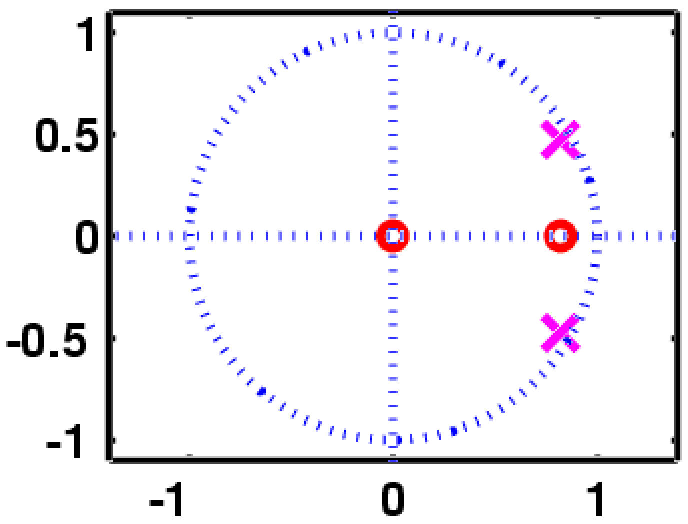
FREQ. RESPONSE from POLES & ZEROS



$$\frac{1-0.45z^{-1}}{1-0.9z^{-1}+0.81z^{-2}}$$

$h[n]$ Decays & Oscillates.

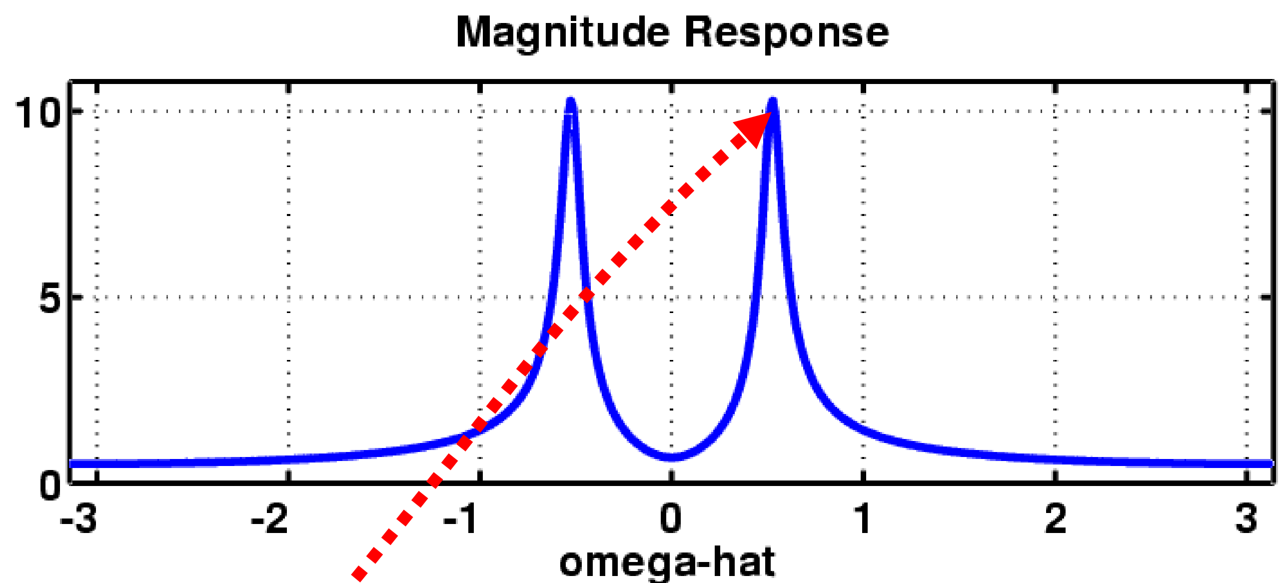


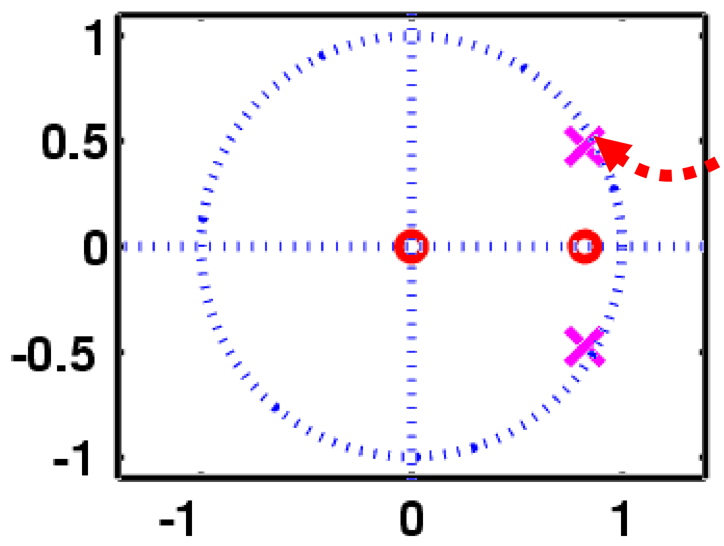


$$h[n] = (0.95)^n \cos(\tfrac{\pi}{6} n) u[n]$$

$$\frac{1 - 0.8227 z^{-1}}{1 - 1.6454 z^{-1} + 0.9025 z^{-2}}$$

FREQ. RESPONSE from POLES & ZEROS





$$\frac{1-0.8227z^{-1}}{1-1.6454z^{-1}+0.9025z^{-2}}$$

CALCULATE the RESPONSE.

$$x[n] = e^{j\hat{\omega}_0 n}u[n]$$

$$\boxed{H(z)}$$

$$X(z) = \frac{1}{1 - e^{j\hat{\omega}_0}z^{-1}}$$

$$Y(z) = H(z)X(z) = \left(\frac{b_0}{1 - a_1 z^{-1}}\right)\left(\frac{1}{1 - e^{j\hat{\omega}_0}z^{-1}}\right)$$

Use the *z*-Transform Method
and the PARTIAL FRACTION EXPANSION.

GENERAL INVERSE z-TRANSFORM

PROCEDURE FOR INVERSE z-TRANSFORMATION ($M < N$)

1. Factor the denominator polynomial of $H(z)$ and express the pole factors in the form $(1 - p_k z^{-1})$ for $k = 1, 2, \ldots, N$.
2. Make a partial fraction expansion of $H(z)$ into a sum of terms of the form

$$H(z) = \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}} \quad \text{where} \quad A_k = H(z)(1 - p_k z^{-1})\Big|_{z=p_k}$$

3. Write down the answer as

$$h[n] = \sum_{k=1}^{N} A_k (p_k)^n u[n]$$

SPLIT $Y(z)$. (1)

Partial Fraction Expansion

$$Y(z)=\left(\frac{b_0}{1-a_1z^{-1}}\right)\left(\frac{1}{1-e^{j\hat{\omega}_0}z^{-1}}\right)=\frac{A}{1-a_1z^{-1}}+\frac{B}{1-e^{j\hat{\omega}_0}z^{-1}}$$

$$Y(z)(1-a_1z^{-1})\Big|_{z=a_1}=A+\frac{B(1-a_1z^{-1})}{1-e^{j\hat{\omega}_0}z^{-1}}\Bigg|_{z=a_1}=A$$

$$A=Y(z)(1-a_1z^{-1})\Big|_{z=a_1}=\frac{b_0}{1-e^{j\hat{\omega}_0}z^{-1}}\Bigg|_{z=a_1}=\frac{b_0}{1-e^{j\hat{\omega}_0}a_1^{-1}}$$

$$B=Y(z)(1-e^{j\hat{\omega}_0}z^{-1})\Big|_{z=e^{j\hat{\omega}_0}}=\frac{b_0}{1-a_1z^{-1}}\Bigg|_{z=e^{j\hat{\omega}_0}}=\frac{b_0}{1-a_1e^{-j\hat{\omega}_0}}$$

SPLIT $Y(z)$. (2)

$$Y(z) = H(z)X(z) = \left(\frac{b_0}{1-a_1 z^{-1}}\right)\left(\frac{1}{1-e^{j\hat{\omega}_0} z^{-1}}\right)$$

$$Y(z) = \frac{A}{1-a_1 z^{-1}} + \frac{B}{1-e^{j\hat{\omega}_0} z^{-1}}$$

$$= \frac{(A+B)+(-a_1 B - e^{j\hat{\omega}_0} A)z^{-1}}{(1-a_1 z^{-1})(1-e^{j\hat{\omega}_0} z^{-1})}$$

$$\Rightarrow (A+B) = b_0 \text{ and } (-a_1 B - e^{j\hat{\omega}_0} A) = 0$$

INVERT *Y*(*z*) to *y*[*n*].

- Use the *z*-Transform Table.

$$Y(z) = \frac{\left(\frac{b_0 a_1}{a_1 - e^{j\hat{\omega}_0}}\right)}{1 - a_1 z^{-1}} + \frac{\left(\frac{b_0}{1 - a_1 e^{-j\hat{\omega}_0}}\right)}{1 - e^{j\hat{\omega}_0} z^{-1}}$$

$$y[n] = \left(\frac{b_0 a_1}{a_1 - e^{j\hat{\omega}_0}}\right)(a_1)^n u[n] + \left(\frac{b_0}{1 - a_1 e^{-j\hat{\omega}_0}}\right) e^{j\hat{\omega}_0 n} u[n]$$

TWO PARTS of $y[n]$

- **TRANSIENT Component**
  - Acts like $(a_1)^n$.
  - Dies out ?
    - If $|a_1|<1$

$$\boxed{\left(\frac{b_0 a_1}{a_1 - e^{j\hat{\omega}_0}}\right) a_1^n u[n]}$$

- **STEADY-STATE Component**
  - Depends on the input.
  - e.g., Sinusoidal

$$\boxed{\left(\frac{b_0}{1 - a_1 e^{-j\hat{\omega}_0}}\right) e^{j\hat{\omega}_0 n} u[n]}$$

STEADY-STATE COMPONENT

- When the Transient component dies out…
- Limit as "*n*" approaches infinity.
- Use the Frequency Response to get the Magnitude & the Phase for a sinusoid.

$$y[n] \rightarrow y_{ss}[n] = \left(\frac{b_0}{1 - a_1 e^{-j\hat{\omega}_0}} \right) e^{j\hat{\omega}_0 n}$$

$$= H(e^{j\hat{\omega}_0}) e^{j\hat{\omega}_0 n}$$

THREE INPUTS

- Given:

$$\longrightarrow \boxed{H(z) = \frac{5}{1+0.8z^{-1}}} \longrightarrow$$

- Find the output, *y*[*n*].
  - When

$$x[n] = \cos(0.2\pi n)$$

$$x[n] = u[n]$$

$$x[n] = \cos(0.2\pi n)u[n]$$

SINUSOID RESPONSE

- Given: $$H(z) = \frac{5}{1+0.8z^{-1}}$$
- The input: $$x[n] = \cos(0.2\pi n)$$
- Then $y[n]$ $$y[n] = M\cos(0.2\pi n + \psi)$$

$$H(e^{j0.2\pi}) = \frac{5}{1+0.8e^{-j0.2\pi}} = 2.92e^{j0.28}$$

Step Response (1)

- Given: $$\boxed{H(z) = \frac{5}{1+0.8z^{-1}}}$$
- The input: $$\boxed{x[n] = u[n]}$$
  $$\boxed{X(z) = \frac{1}{1-z^{-1}}}$$
- Then $y[n]$

Step Response (2)

$$Y(z) = H(z)X(z) = \left(\frac{5}{1+.8z^{-1}}\right)\left(\frac{1}{1-z^{-1}}\right)$$

$$Y(z) = \frac{\frac{20}{9}}{1+.8z^{-1}} + \frac{\frac{25}{9}}{1-z^{-1}}$$

$$y[n] = \frac{20}{9}(-.8)^n u[n] + \frac{25}{9}u[n]$$

Transient ***Steady-State***

$$y[n] \to \frac{25}{9} \quad \text{as} \quad n \to \infty$$

SINUSOID Starting at *n*=0 (1)

- Given: $$H(z)=\frac{5}{1+0.8z^{-1}}$$

- The input: $$x[n]=\cos(0.2\pi n)u[n]$$ $$=\Re\{e^{j0.2\pi n}u[n]\}$$

- Then *y*[*n*] $$y[n]=\Re\{5(-0.8)^n u[n]*e^{j0.2\pi n}u[n]\}$$

SINUSOID Starting at n=0 (2)

$$y[n] = \Re\{5(-0.8)^n u[n] * e^{j0.2\pi n} u[n]\}$$

$$Y(z) = H(z)X(z) = \frac{5}{1+0.8z^{-1}} \frac{1}{1-e^{j0.2\pi} z^{-1}}$$

$$Y(z) = \frac{\frac{5}{1+1.25e^{j0.2\pi}}}{1+0.8z^{-1}} + \frac{\frac{5}{1+0.8e^{-j0.2\pi}}}{1-e^{j0.2\pi} z^{-1}}$$

$$= \frac{2.19 - j0.8}{1+0.8z^{-1}} + \frac{2.81 + j0.8}{1-e^{j0.2\pi} z^{-1}}$$

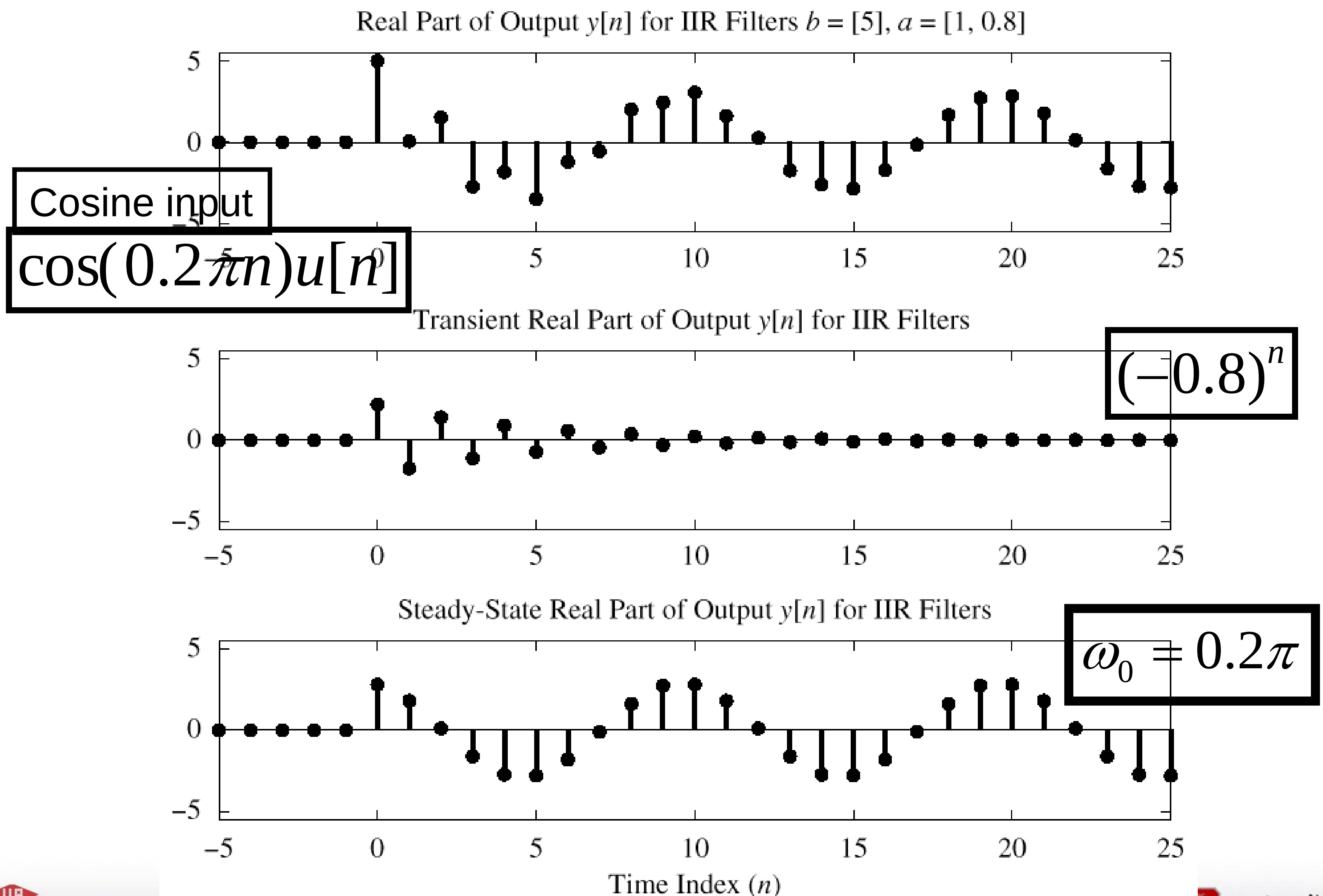
$H(e^{j0.2\pi})$

$$y[n] = \Re\{(2.19 - j0.8)(-0.8)^n u[n] + 2.92 e^{j0.28} e^{j0.2\pi n} u[n]\}$$

$$y[n] = 2.19(-0.8)^n u[n] + 2.92\cos(0.2\pi n + 0.28) u[n]$$

Transient **Steady-State**

SINUSOID Starting at n=0 (3)



Stability (1)

- When Does the TRANSIENT DIE OUT ?

STEADY-STATE RESPONSE AND STABILITY

A stable system is one that does not "blow up." This intuitive statement can be formalized by saying that the output of a stable system can always be bounded ($|y[n]| < M_y$) whenever the input is bounded ($|x[n]| < M_x$).[3]

$$y[n] = a_1 y[n-1] + b_0 x[n]$$

$$H(z) = \frac{b_0}{1 - a_1 z^{-1}}$$

$$h[n] = b_0 a_1^n u[n]$$

$$\boxed{\text{need } |a_1| < 1}$$

- If $|a| \geq 1$, the impulse response will not die out.
  - May produce unbounded outputs even if the input is bounded.

Stability (2)

- Nec. & suff. condition: $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$

$$h[n] = b(a)^n u[n] \Leftrightarrow H(z) = \frac{b}{1 - az^{-1}}$$

$$\sum_{n=0}^{\infty} |b||a|^n < \infty \text{ if } |a| < 1 \Rightarrow$$

Poles must be inside the unit circle.

STABILITY CONDITION

- ALL POLES LIE INSIDE the UNIT CIRCLE.
- UNSTABLE EXAMPLE:

POLE @ z=1.1

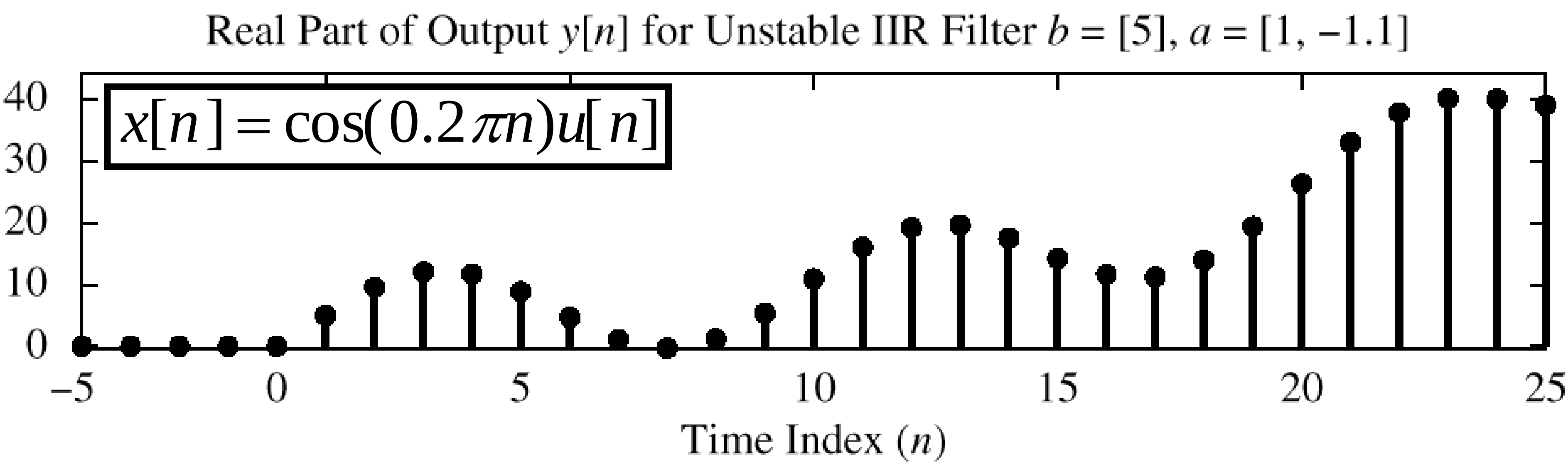


Figure 8.15 Illustration of an unstable IIR system. Pole is at $z = 1.1$.

Magnitude of the frequency response

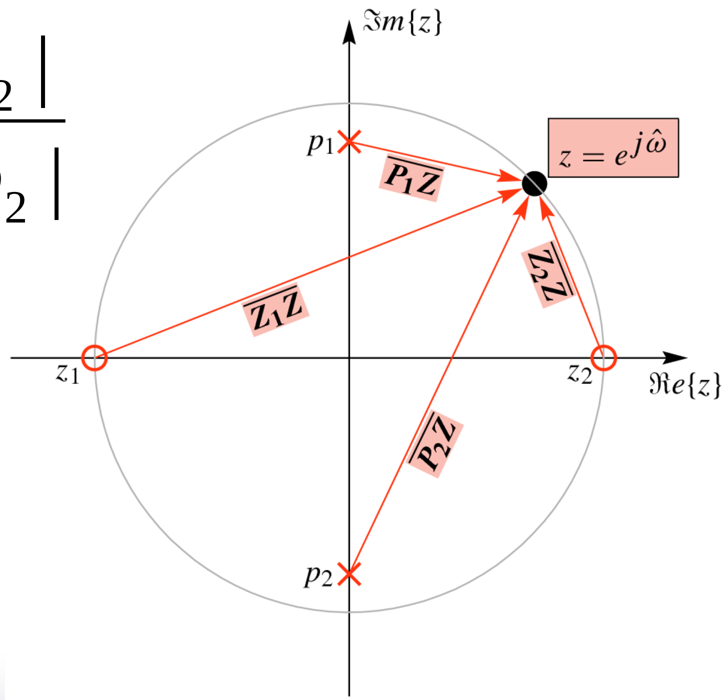
- System function

$$H(z) = G\frac{(z-z_1)(z-z_2)}{(z-p_1)(z-p_2)}$$

- Magnitude of the frequency response

$$\left|H(e^{j\hat{\omega}})\right| = G\frac{|e^{j\hat{\omega}}-z_1||e^{j\hat{\omega}}-z_2|}{|e^{j\hat{\omega}}-p_1||e^{j\hat{\omega}}-p_2|}$$

$$= G\frac{\overline{Z_1Z}\cdot\overline{Z_2Z}}{\overline{P_1Z}\cdot\overline{P_2Z}}$$



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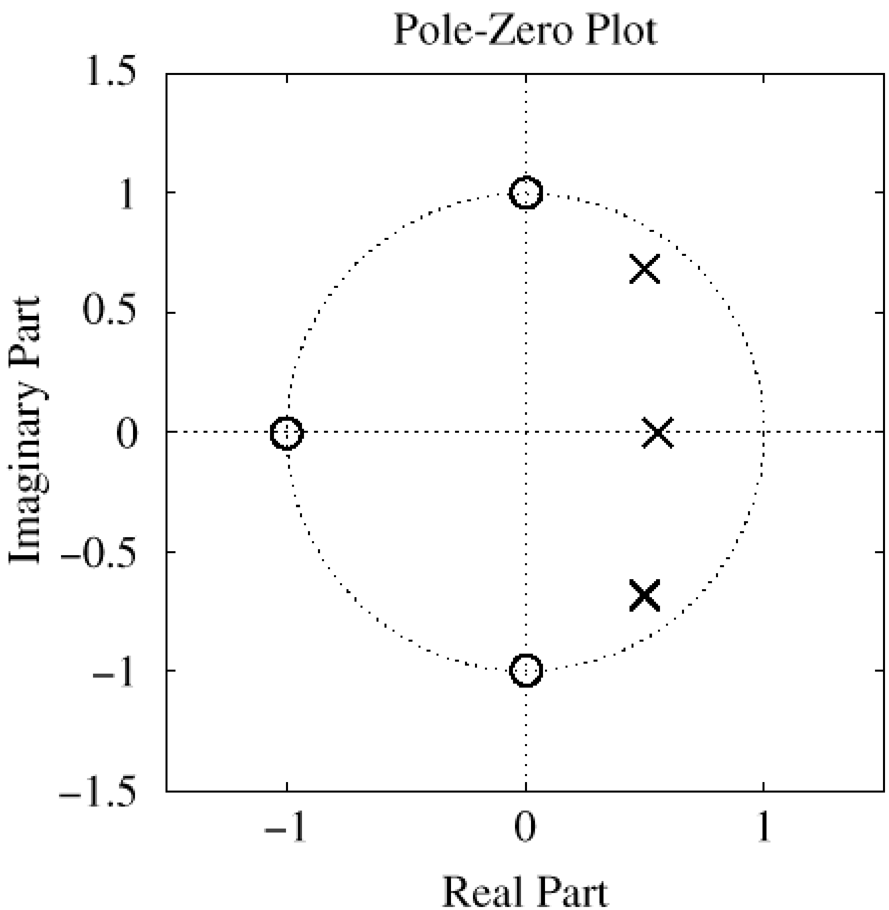
IIR Elliptic LPF (N=3) (1)

- Higher-order IIR filters
  - Realize frequency responses with flatter passbands and stopbands and sharper transition regions.
- Matlab functions to design filters with prescribed frequency-selective characteristics
  - `butter`, `cheby1`, `cheby2`, and `ellip`
- Lowpass elliptic filter
  - Coefficients were obtained using the Matlab function `ellip`.

IIR Elliptic LPF (N=3) (2)

- `ellip`(3,1,30,0.3)

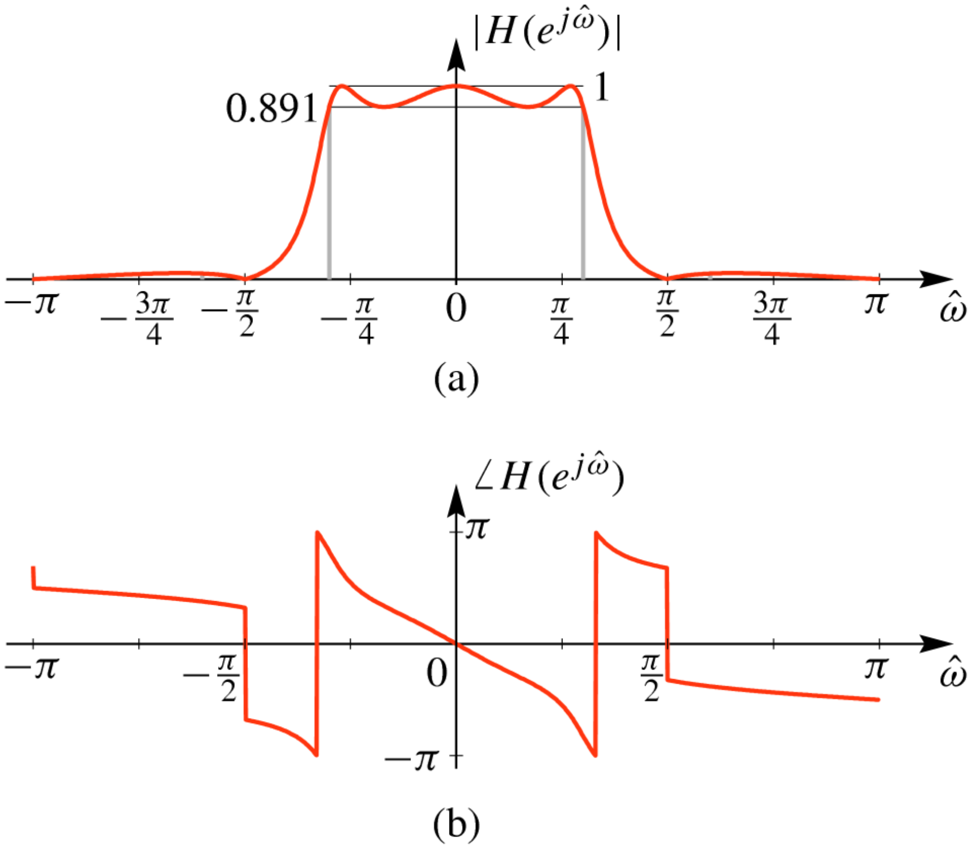
  (order, ripple(dB), att.(dB), bandwidth)

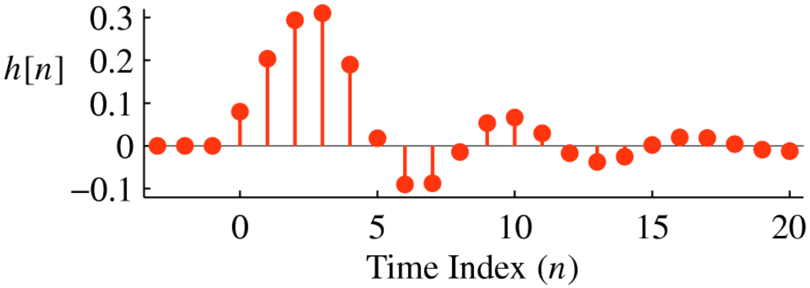


3 POLES

IIR Elliptic LPF (*N*=3) (3)

- Passband: $|\hat{\omega}| \leq 2\pi(0.15)$





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Thank you

- Homework
  - P-8.5, 12(a,b), 16, 17, 19, 20, 22
- Reading assignment
  - ~ Section 9.5