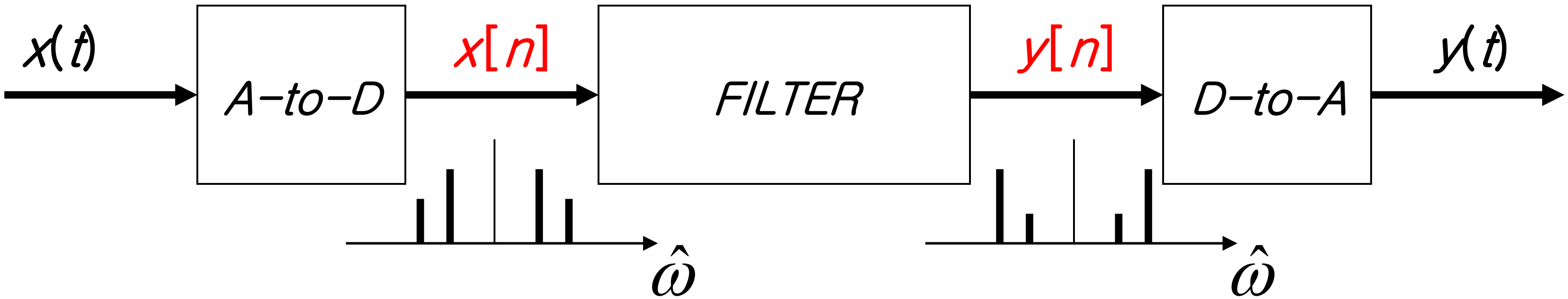
Chapter 6
Frequency Response of FIR Filters

DOMAINS: Time & Frequency

- **Time Domain: "*n*" = time**
  - $x[n]$ discrete-time signal
  - $x(t)$ continuous-time signal
- **Frequency Domain (sum of sinusoids)**
  - Spectrum vs. f (Hz)
  - Spectrum vs. $\hat{\omega}$
    - ANALOG vs. DIGITAL
- Move back and forth QUICKLY.

DIGITAL "FILTERING"



- CONCENTRATE on the **SPECTRUM**.
- SINUSOIDAL INPUT
  - INPUT $x[n]$ = SUM of SINUSOIDS
  - Then, OUTPUT $y[n]$ = SUM of SINUSOIDS

SINUSOIDAL RESPONSE

- INPUT: $x[n]$ = SINUSOID
- OUTPUT: $y[n]$ will also be a SINUSOID.
  - Different amplitude and phase
  - **SAME** Frequency

- AMPLITUDE & PHASE CHANGE
  - Called the **FREQUENCY RESPONSE**.

COMPLEX EXPONENTIAL

$x[n]$ is the input signal—a complex exponential.

$$x[n] = Ae^{j\varphi}e^{j\hat{\omega}n} \qquad -\infty < n < \infty$$

FIR DIFFERENCE EQUATION

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] = \sum_{k=0}^{M} h[k]x[n-k]$$

$$y[n] = \sum_{k=0}^{M} b_k Ae^{j\varphi}e^{j\hat{\omega}(n-k)} = \left(\sum_{k=0}^{M} b_k e^{j\hat{\omega}(-k)}\right)Ae^{j\varphi}e^{j\hat{\omega}n}$$

$$= H(\hat{\omega})Ae^{j\varphi}e^{j\hat{\omega}n}$$

FREQUENCY REPONSE

- Notation:

$$H(e^{j\hat{\omega}}) \text{ in place of } H(\hat{\omega})$$

- At each frequency, we can DEFINE

$$\boxed{H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}} \longleftarrow \boxed{\text{FREQUENCY RESPONSE}}$$

- Complex-valued formula
  - Has MAGNITUDE and PHASE vs. frequency.

EXAMPLE 6.1 (1)

$$\{b_k\} = \{1,\ 2,\ 1\}$$

$$\begin{aligned} H(e^{j\hat{\omega}}) &= 1 + 2e^{-j\hat{\omega}} + e^{-j2\hat{\omega}} \\ &= e^{-j\hat{\omega}}(e^{j\hat{\omega}} + 2 + e^{-j\hat{\omega}}) \\ &= e^{-j\hat{\omega}}(2 + 2\cos\hat{\omega}) \end{aligned}$$

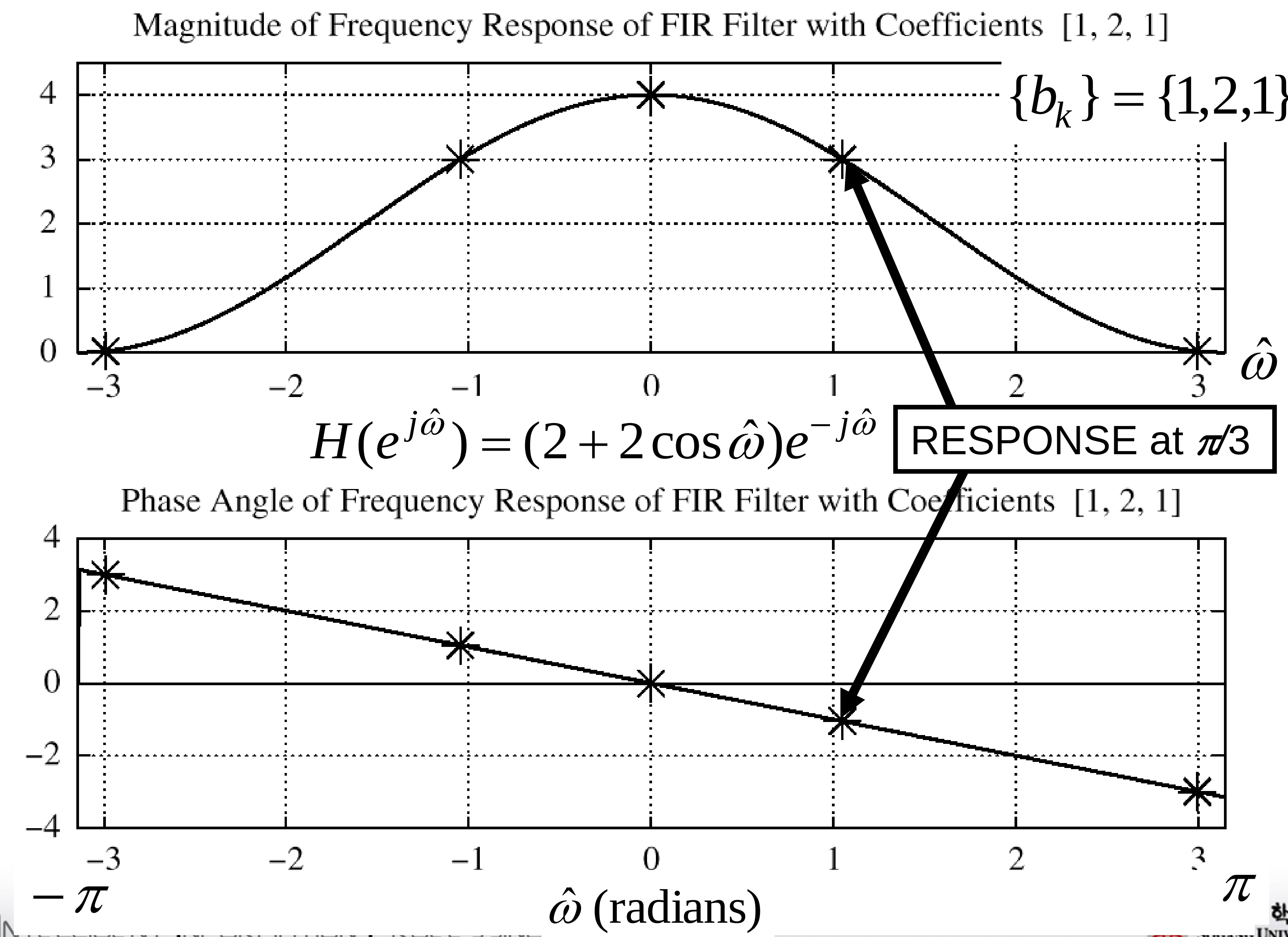
EXPLOIT SYMMETRY.

Since $(2 + 2\cos\hat{\omega}) \geq 0$,

Magnitude is $\left|H(e^{j\hat{\omega}})\right| = (2 + 2\cos\hat{\omega})$,

and Phase is $\angle H(e^{j\hat{\omega}}) = -\hat{\omega}$.

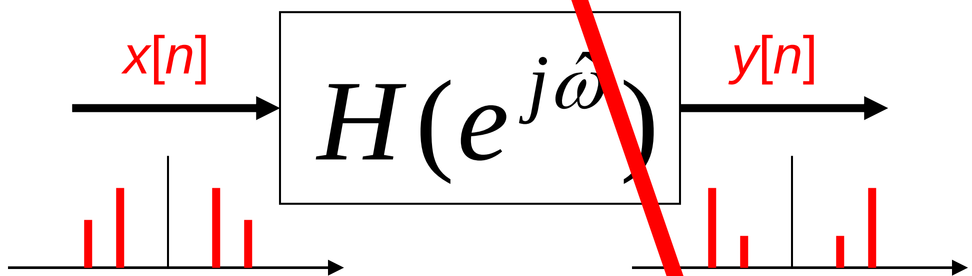
EXAMPLE 6.1 (2)


Magnitude of Frequency Response of FIR Filter with Coefficients [1, 2, 1]
$\{b_k\} = \{1,2,1\}$
$\hat{\omega}$
$H(e^{j\hat{\omega}}) = (2 + 2\cos\hat{\omega})e^{-j\hat{\omega}}$
RESPONSE at $\pi/3$
Phase Angle of Frequency Response of FIR Filter with Coefficients [1, 2, 1]
$-\pi$
π
$\hat{\omega}$ (radians)

EXAMPLE 6.2

Find $y[n]$ when $H(e^{j\hat{\omega}}) = (2 + 2\cos\hat{\omega})e^{-j\hat{\omega}}$

and $x[n] = 2e^{j\pi/4}e^{j(\pi/3)n}$.



Evaluate $H(e^{j\hat{\omega}})$ at $\hat{\omega} = \pi/3$.

$$H(e^{j\hat{\omega}}) = 3e^{-j\pi/3} \quad @\,\hat{\omega} = \pi/3$$

$$y[n] = \left(3e^{-j\pi/3}\right) \times 2e^{j\pi/4}e^{j(\pi/3)n} = 6e^{-j\pi/12}e^{j(\pi/3)n}$$

EXAMPLE: COSINE INPUT (1)

Find $y[n]$ when $H(e^{j\hat{\omega}}) = (2 + 2\cos\hat{\omega})e^{-j\hat{\omega}}$

and $x[n] = 2\cos(\frac{\pi}{3}n + \frac{\pi}{4})$.

$$2\cos(\tfrac{\pi}{3}n + \tfrac{\pi}{4}) = e^{j(\pi n/3+\pi/4)} + e^{-j(\pi n/3+\pi/4)}$$

$$\Rightarrow x[n] = x_1[n] + x_2[n]$$

Use Linearity.

$$y_1[n] = H(e^{j\pi/3})e^{j(\pi n/3+\pi/4)}$$

$$y_2[n] = H(e^{-j\pi/3})e^{-j(\pi n/3+\pi/4)}$$

$$\Rightarrow y[n] = y_1[n] + y_2[n]$$

EXAMPLE: COSINE INPUT (2)

$$H(e^{j\hat{\omega}}) = (2 + 2\cos\hat{\omega})e^{-j\hat{\omega}}$$

$$2\cos(\tfrac{\pi}{3}n + \tfrac{\pi}{4}) = e^{j(\pi n/3 + \pi/4)} + e^{-j(\pi n/3 + \pi/4)}$$

$$\Rightarrow x[n] = x_1[n] + x_2[n]$$

$$y_1[n] = H(e^{j\pi/3})e^{j(\pi n/3+\pi/4)} = 3e^{-j(\pi/3)}e^{j(\pi n/3+\pi/4)}$$

$$y_2[n] = H(e^{-j\pi/3})e^{-j(\pi n/3+\pi/4)} = 3e^{j(\pi/3)}e^{-j(\pi n/3+\pi/4)}$$

$$y[n] = 3e^{j(\pi n/3 - \pi/12)} + 3e^{-j(\pi n/3 - \pi/12)}$$

$$\Rightarrow y[n] = 6\cos(\tfrac{\pi}{3}n - \tfrac{\pi}{12})$$

MATLAB: FREQUENCY RESPONSE

- DENSE GRID **(ww)** from $-\pi$ to $+\pi$
  - `ww = -pi:(pi/100):pi;`
- `HH = freqz(bb,1,ww)`
  - VECTOR **bb** contains Filter Coefficients.
- FILTER COEFFICIENTS $\{b_k\}$

$$\boxed{H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}}$$

LTI SYSTEMS

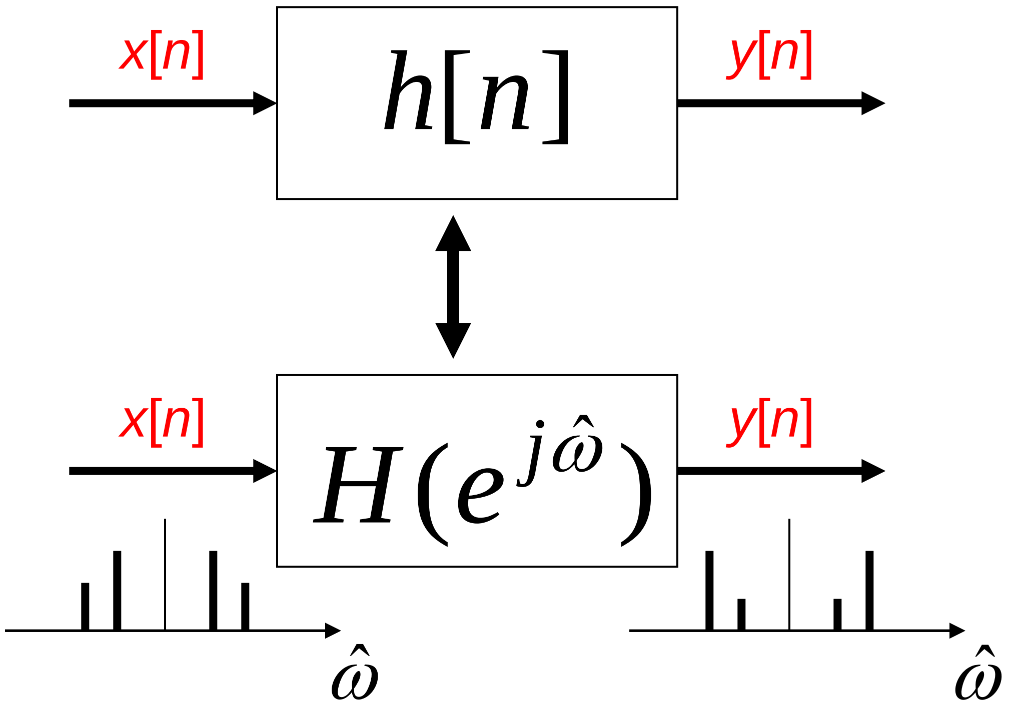
- LTI: Linear & Time-Invariant
- COMPLETELY CHARACTERIZED by:
  - **FREQUENCY RESPONSE**, or
  - IMPULSE RESPONSE *h*[*n*]
- **Sinusoid IN -----> Sinusoid OUT**
  - **At the SAME Frequency**
- Time & Frequency Relation
  - Get Frequency Response from *h*[*n*].
  - Here is the FIR case:

IMPULSE RESPONSE

$$\boxed{H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} b_k e^{-j\hat{\omega}k} = \sum_{k=0}^{M} h[k] e^{-j\hat{\omega}k}}$$

BLOCK DIAGRAMS

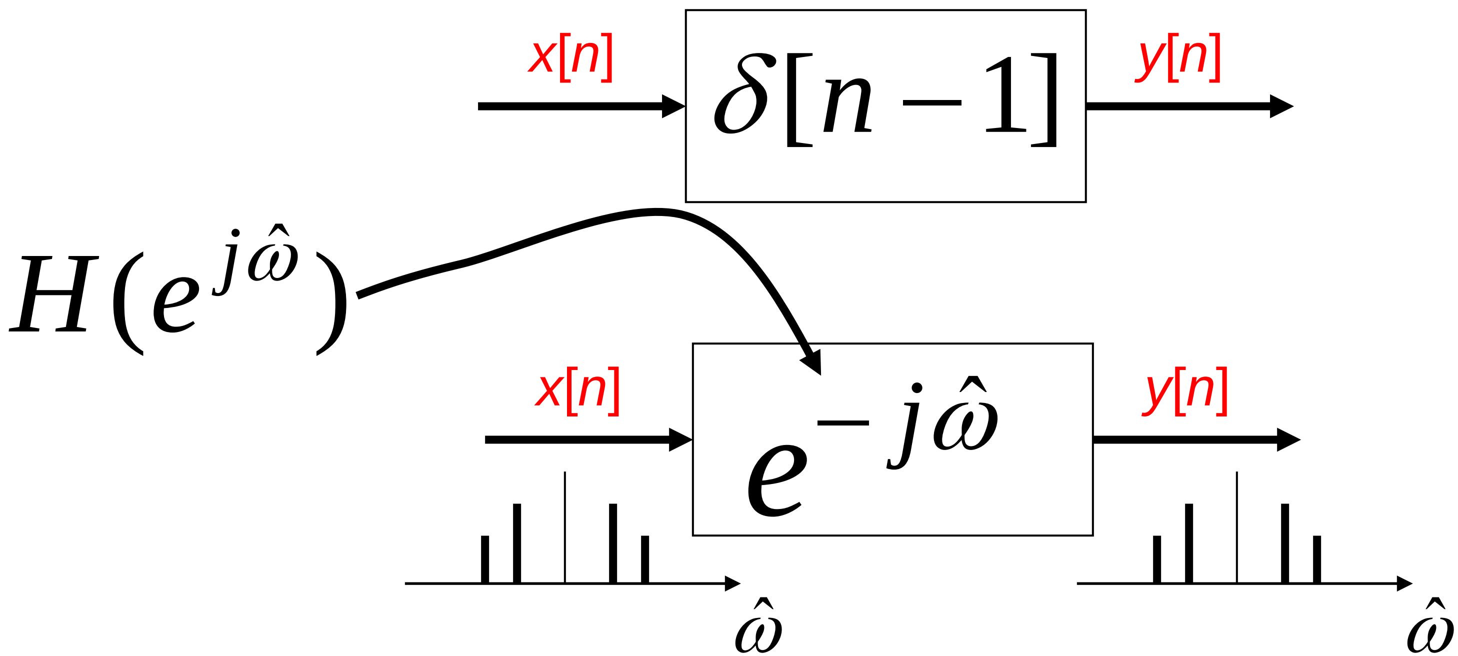
- Equivalent Representations



UNIT-DELAY SYSTEM

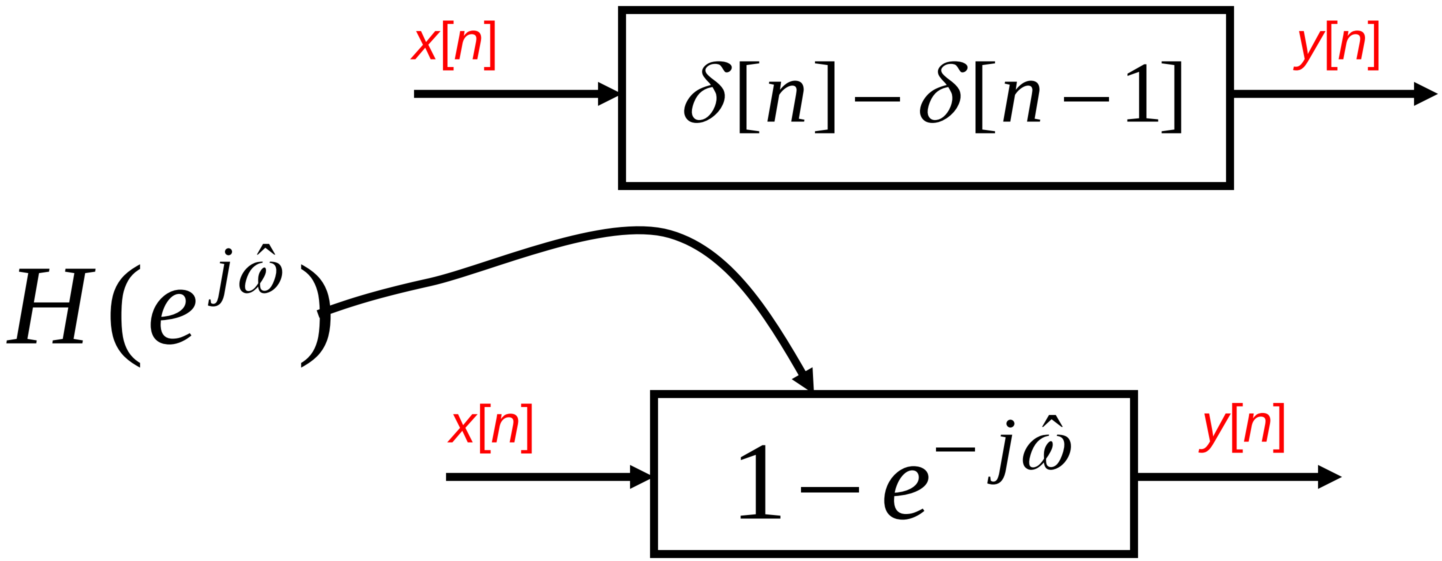
Find $h[n]$ and $H(e^{j\hat{\omega}})$ for $y[n] = x[n-1]$.

$$\boxed{\{b_k\} = \{0,\ 1\}}$$



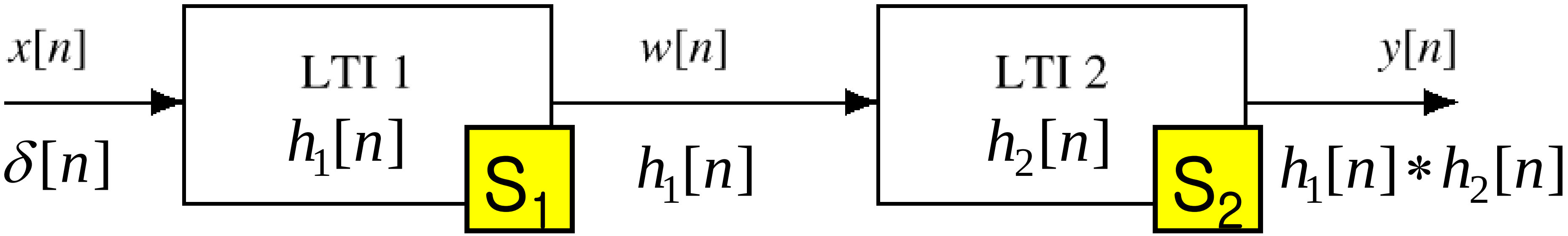
FIRST DIFFERENCE SYSTEM

Find $h[n]$ and $H(e^{j\hat{\omega}})$ for the Difference Equation : $y[n] = x[n] - x[n-1]$.



CASCADE SYSTEMS

- Does the order of S_1 & S_2 matter?
  - NO, LTI SYSTEMS can be rearranged !!!
  - WHAT ARE THE FILTER COEFFS? $\{b_k\}$
  - WHAT is the overall FREQUENCY RESPONSE?



CASCADE EQUIVALENT

- MULTIPLY the Frequency Responses.

$$x[n] \longrightarrow \boxed{H_1(e^{j\hat{\omega}})} \longrightarrow \boxed{H_2(e^{j\hat{\omega}})} \longrightarrow y[n]$$

$$x[n] \longrightarrow \boxed{H(e^{j\hat{\omega}})} \longrightarrow y[n]$$

EQUIVALENT SYSTEM

$$\boxed{H(e^{j\hat{\omega}}) = H_1(e^{j\hat{\omega}})H_2(e^{j\hat{\omega}})}$$

TIME & FREQUENCY

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] = \sum_{k=0}^{M} h[k]x[n-k]$$

FIR DIFFERENCE EQUATION is usually represented in the TIME-DOMAIN.

$$H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} h[k]e^{-j\hat{\omega}k}$$

Periodic with 2π

$$H(e^{j\hat{\omega}}) = h[0] + h[1]e^{-j\hat{\omega}} + h[2]e^{-j2\hat{\omega}} + h[3]e^{-j3\hat{\omega}} + \cdots$$

GENERAL DELAY PROPERTY

Find $h[n]$ and $H(e^{j\hat{\omega}})$ for $y[n] = x[n - n_d]$.

$$h[n] = \delta[n - n_d]$$

$$\boxed{H(e^{j\hat{\omega}}) = \sum_{k=0}^{M} \delta[k - n_d] e^{-j\hat{\omega}k} = e^{-j\hat{\omega}n_d}}$$

ONLY ONE non-ZERO TERM for $k = n_d$

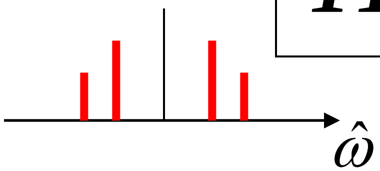
FREQ DOMAIN –> TIME ?? (1)

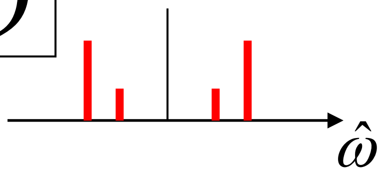
Start with $H(e^{j\hat{\omega}})$ and find $h[n]$ or b_k.

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] \qquad h[n] = ?$$

$$\boxed{H(e^{j\hat{\omega}}) = 7e^{-j2\hat{\omega}}\cos(\hat{\omega})}$$

$$x[n] \longrightarrow \boxed{H(e^{j\hat{\omega}})} \longrightarrow y[n]$$





FREQ DOMAIN –> TIME ?? (2)

$$H(e^{j\hat{\omega}}) = 7e^{-j2\hat{\omega}}\cos(\hat{\omega})$$

EULER's Formula

$$= 7e^{-j2\hat{\omega}}(0.5e^{j\hat{\omega}} + 0.5e^{-j\hat{\omega}})$$

$$= (3.5e^{-j\hat{\omega}} + 3.5e^{-j3\hat{\omega}})$$

$$h[n] = 3.5\delta[n-1] + 3.5\delta[n-3]$$

$$b_k = \{\, 0, 3.5, 0, 3.5 \,\}$$

FREQUENCY RESPONSE

- SINUSOIDAL INPUT SIGNAL
  - The OUTPUT has the SAME FREQUENCY.
  - DIFFERENT Amplitude and Phase
- FREQUENCY RESPONSE of FIR filters
  - MAGNITUDE vs. Frequency
  - PHASE vs. Frequency
  - PLOTTING

MAG

PHASE

$$H(e^{j\hat{\omega}}) = \underbrace{\left|H(e^{j\hat{\omega}})\right|}_{\text{MAG}} e^{\overbrace{j\angle H(e^{j\hat{\omega}})}^{\text{PHASE}}}$$

SINUSOID thru FIR (1)

$$x[n] = X_0 + \sum_{k=1}^{N} \left(\frac{X_k}{2} e^{j\hat{\omega}_k n} + \frac{X_k^*}{2} e^{-j\hat{\omega}_k n} \right)$$

$$= X_0 + \sum_{k=1}^{N} |X_k| \cos(\hat{\omega}_k n + \angle X_k)$$

MULTIPLY MAGS.

ADD PHASES.

if $\mathcal{H}(-\hat{\omega}) = \mathcal{H}^*(\hat{\omega})$, the corresponding output is

$$y[n] = \mathcal{H}(0) X_0 + \sum_{k=1}^{N} \left(\mathcal{H}(\hat{\omega}_k) \frac{X_k}{2} e^{j\hat{\omega}_k n} + \mathcal{H}(-\hat{\omega}_k) \frac{X_k^*}{2} e^{-j\hat{\omega}_k n} \right)$$

$$= \mathcal{H}(0) X_0 + \sum_{k=1}^{N} \boxed{|\mathcal{H}(\hat{\omega}_k)|} |X_k| \cos\left(\hat{\omega}_k n + \angle X_k + \boxed{\angle \mathcal{H}(\hat{\omega}_k)}\right)$$

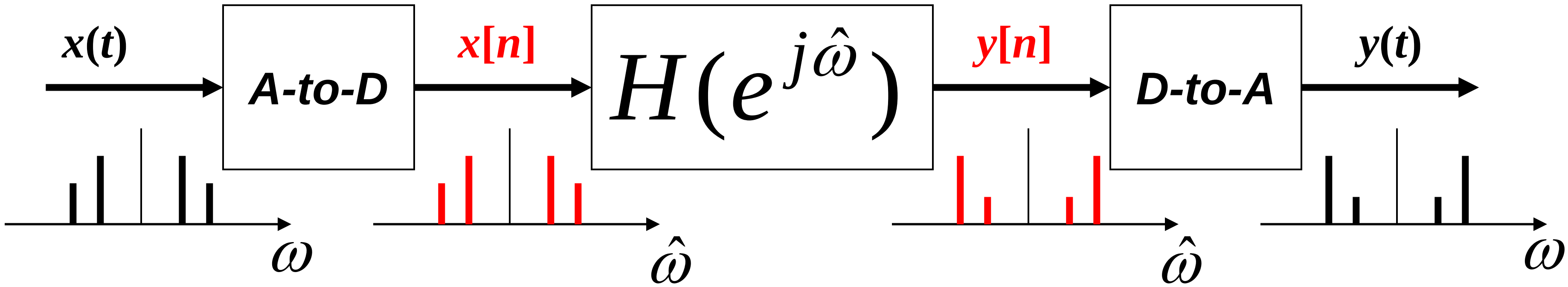
SINUSOID thru FIR (2)

- If $H^*(e^{j\hat{\omega}}) = H(e^{-j\hat{\omega}})$
  - Conjugate symmetry
- Multiply the Magnitudes.
- Add the Phases.

$$x[n] = A\cos(\hat{\omega}_1 n + \phi)$$

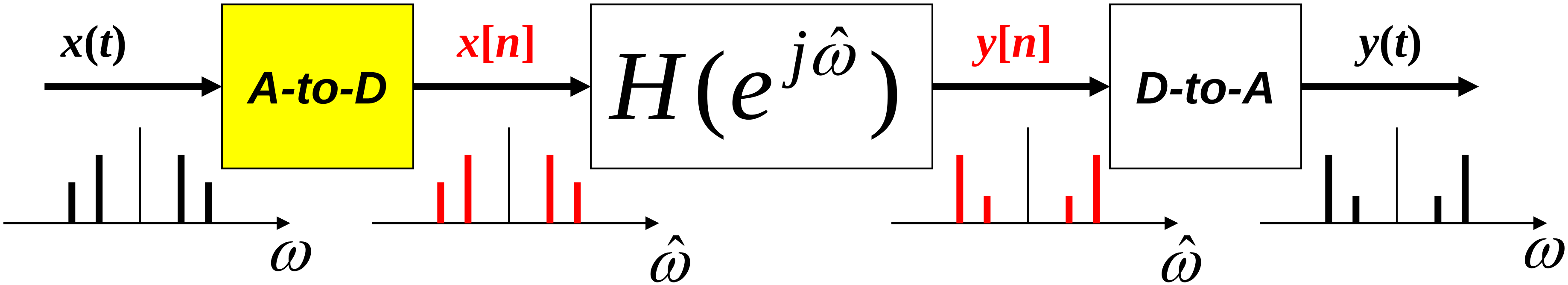
$$\Rightarrow y[n] = A\left|H(e^{j\hat{\omega}_1})\right|\cos(\hat{\omega}_1 n + \phi + \angle H(e^{j\hat{\omega}_1}))$$

DIGITAL "FILTERING"



- SPECTRUM of *x*(*t*) (SUM of SINUSOIDS) $\boxed{\omega}$
- SPECTRUM of *x*[*n*] $\boxed{\hat{\omega}}$
  - Is ALIASING a PROBLEM ?
- SPECTRUM of *y*[*n*]
- Then, OUTPUT *y*(*t*) = SUM of SINUSOIDS $\boxed{\omega}$

A-D FREQUENCY SCALING



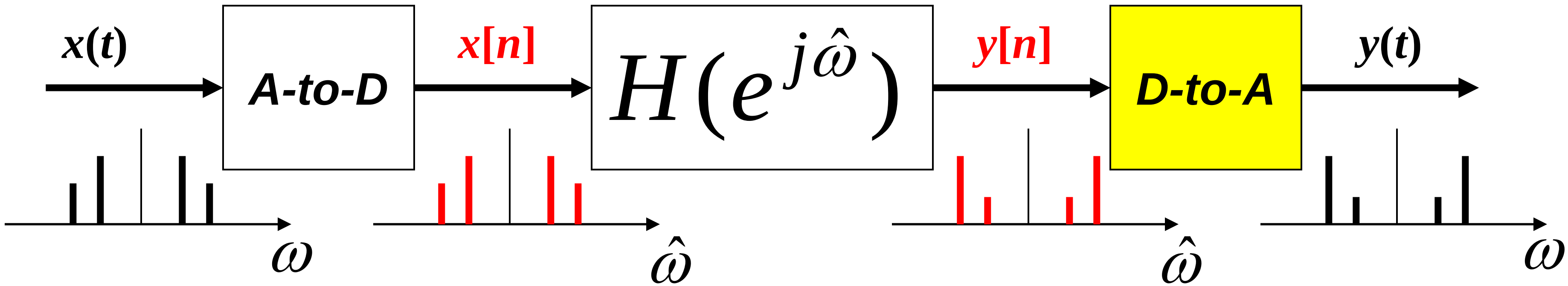
- TIME SAMPLING:
  - IF NO ALIASING:

$$\boxed{t = nT_s}$$

  - FREQUENCY SCALING

$$\boxed{\hat{\omega} = \omega T_s = \frac{\omega}{f_s}}$$

D-A FREQUENCY SCALING



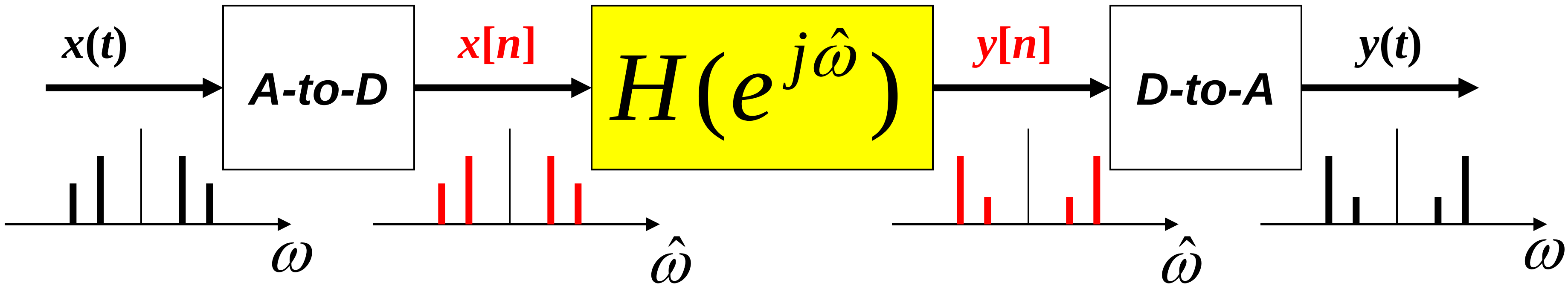
- TIME SAMPLING

$$\boxed{t = nT_s \Rightarrow n \leftarrow t f_s}$$

- RECONSTRUCT up to $0.5f_s$
  - FREQUENCY SCALING

$$\boxed{\omega = \hat{\omega} f_s}$$

TRACK the FREQUENCIES



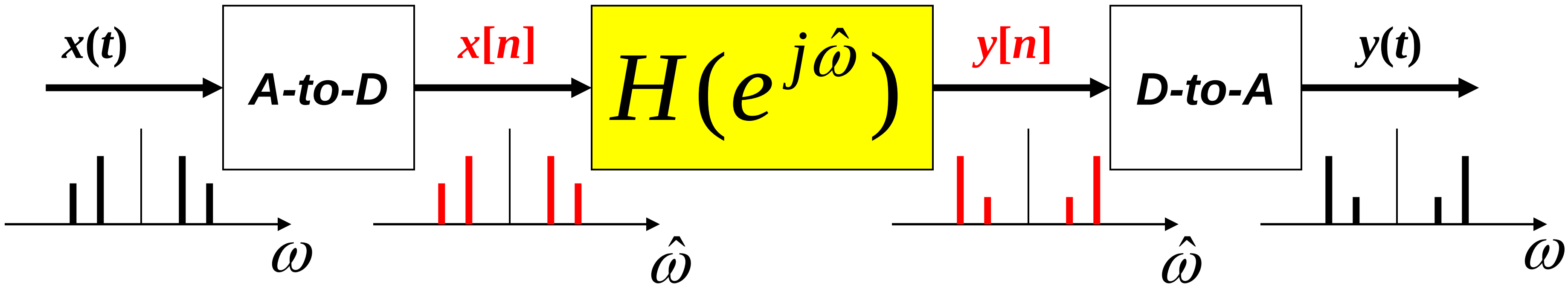
$$\boxed{x(t) = \cos(2\pi(25)t) + \cos(2\pi(250)t - \tfrac{1}{2}\pi)}$$

$x(t)$	$x[n]$		$y[n]$	$y(t)$
250 Hz	0.5π	$H(e^{j0.5\pi})$	0.5π	250 Hz
25 Hz	$.05\pi$	$H(e^{j0.05\pi})$	$.05\pi$	25 Hz

F_s = 1000 Hz

NO new freqs

Ex: 11-pt AVERAGER (1)



$$y[n] = \sum_{k=0}^{10} \frac{1}{11} x[n-k]$$

$$\boxed{H(e^{j\hat{\omega}}) = \frac{\sin(\frac{11}{2}\hat{\omega})}{11\sin(\frac{1}{2}\hat{\omega})} e^{-j5\hat{\omega}}}$$

EVALUATE Frequency Response

$$x(t) = \cos(2\pi(25)t) + \sin(2\pi(250)t)$$

evaluating at 25 and 250 Hz.

$f_s = 1000$

$$\boxed{H(e^{j2\pi(25)/1000})} = \frac{\sin(\pi(25)(11)/1000)}{11\sin(\pi(25)/1000)} e^{-j2\pi(25)(5)/1000}$$

$$= 0.8811 e^{-j\pi/4}$$

MAG. SCALE

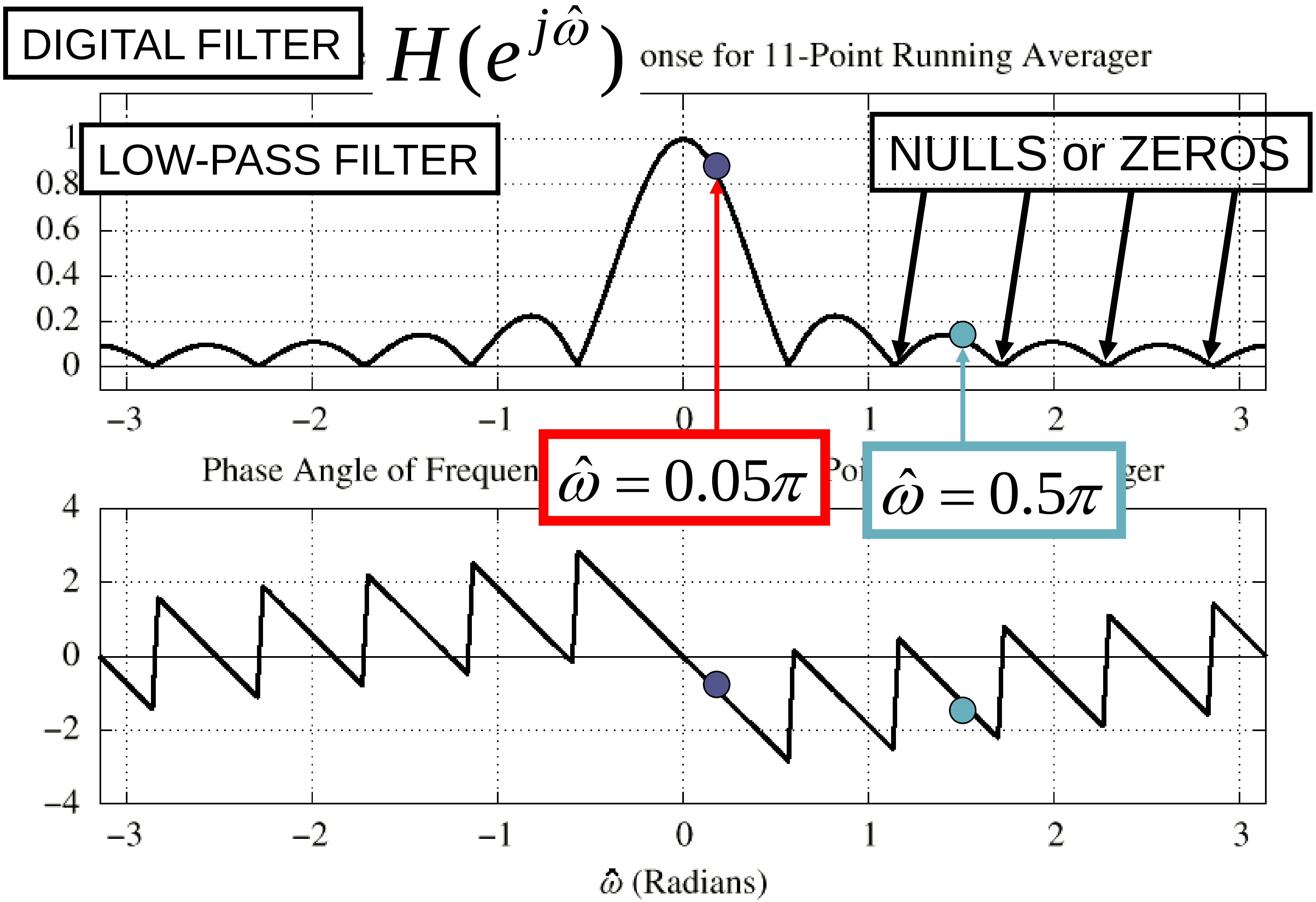
PHASE CHANGE

$$\boxed{H(e^{j2\pi(250)/1000})} = \frac{\sin(\pi(250)(11)/1000)}{11\sin(\pi(250)/1000)} e^{-j2\pi(250)(5)/1000}$$

$$= 0.0909 e^{-j\pi/2}$$

$$y(t) = \underline{0.8811}\cos(2\pi(25)t \underline{- \pi/4}) + \underline{0.0909}\sin(2\pi(250)t \underline{- \pi/2})$$

Ex: 11-pt AVERAGER (2)

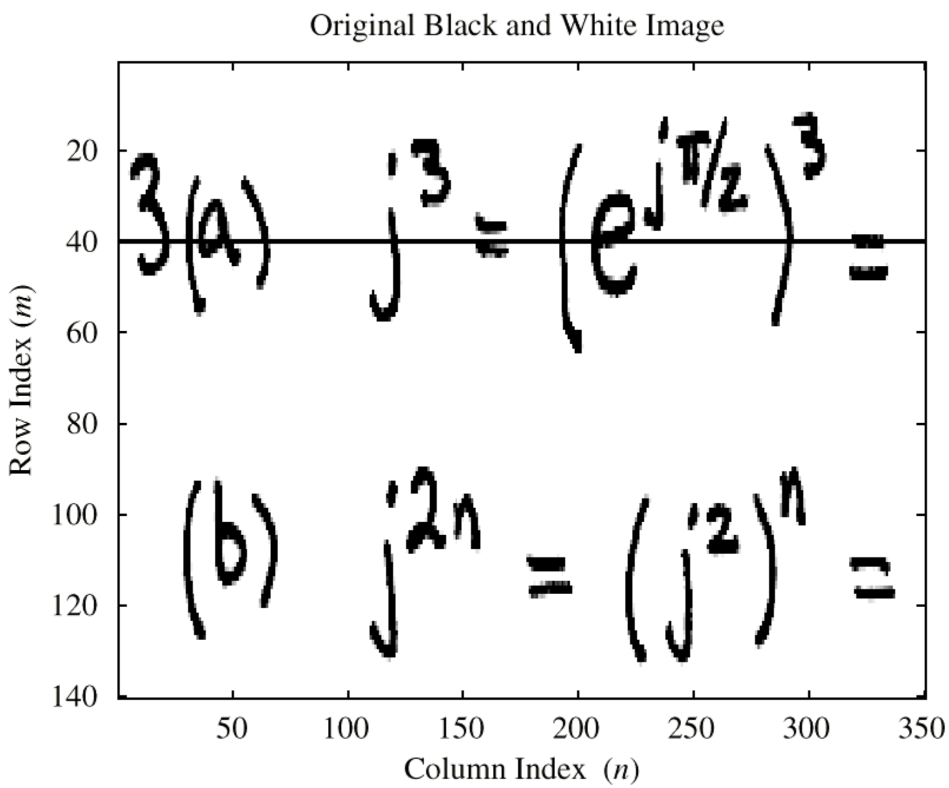


FILTER TYPES

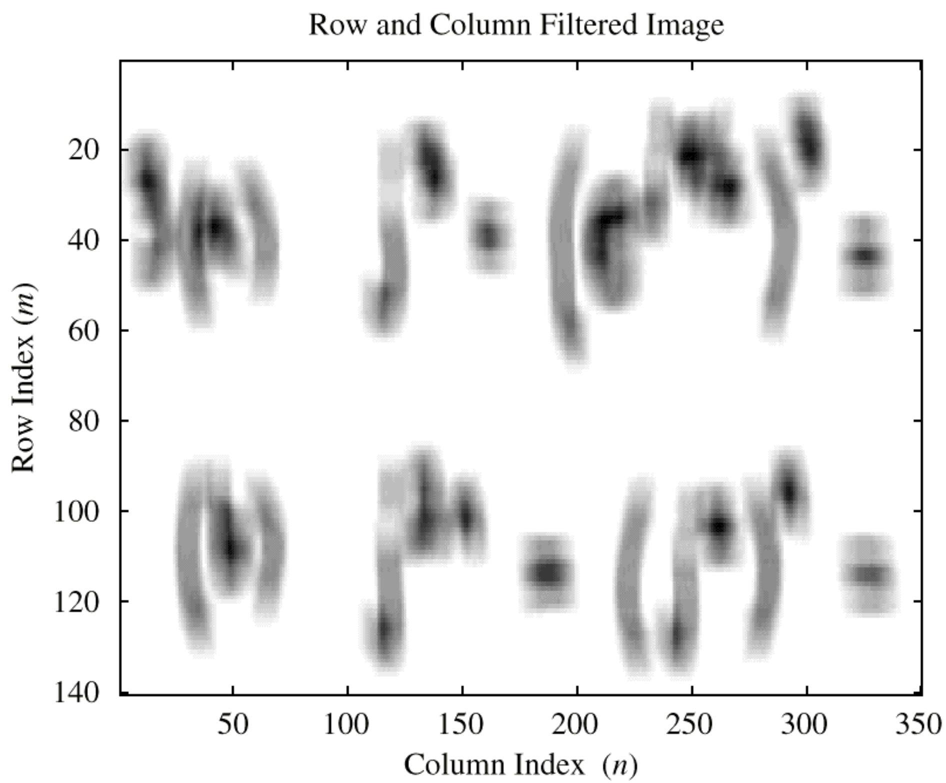
- LOW-PASS FILTER (**LPF**)
  - BLURRING
  - ATTENUATES HIGH FREQUENCIES.
- HIGH-PASS FILTER (**HPF**)
  - SHARPENING for IMAGES
  - BOOSTS THE HIGH FREQUENCIES.
  - REMOVES DC.
- BAND-PASS FILTER (**BPF**)

B & W IMAGE

Original image



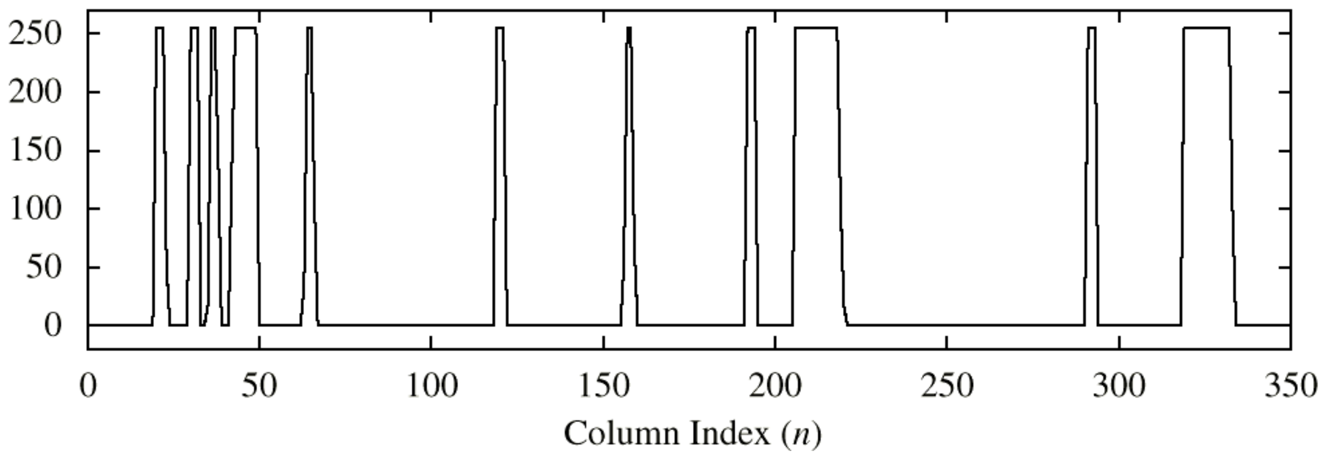
Filtered image
(LPF: BLUR)

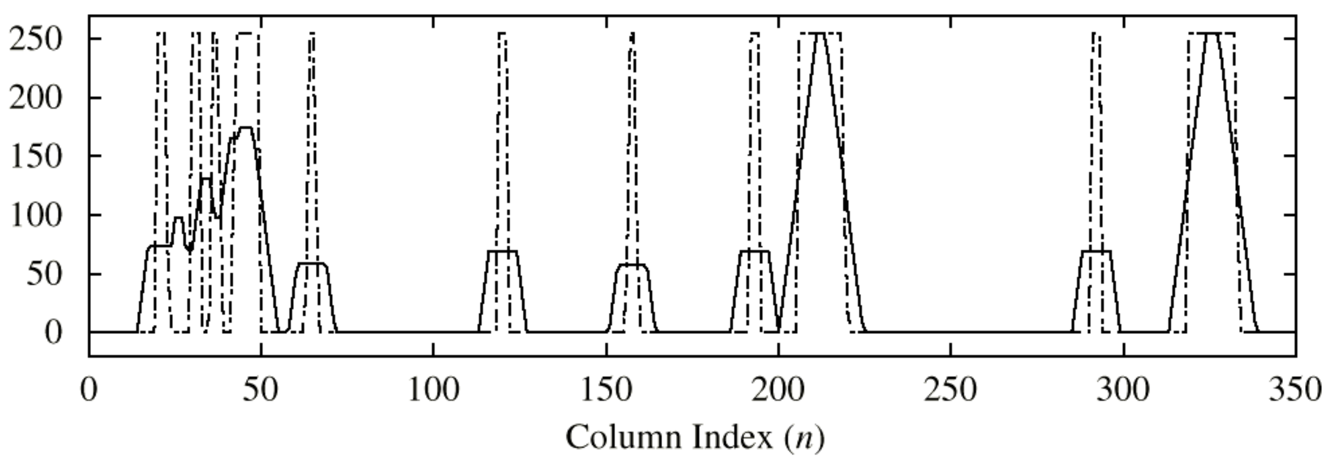


ROW of the IMAGES

BLACK = 255

WHITE = 0





ADJUSTED DELAY by 5 samples

EFFECTIVE Frequency Response

- Assume NO Aliasing, then
  - ANALOG FREQ. <--> DIGITAL FREQ.

$$\boxed{\hat{\omega} = \omega T_s = \frac{\omega}{f_s}}$$

DIGITAL FILTER

  - So, we can plot:
  - Scaled Freq. Axis

$$\boxed{H(\omega T_s) \text{ vs. } \omega}$$

ANALOG FREQUENCY

Thank you

- Homework
  - P-6.1, 3, 5, 8, 10, 12, 13, 15, 18, & 20
- Reading assignment
  - ~ Section 7.5