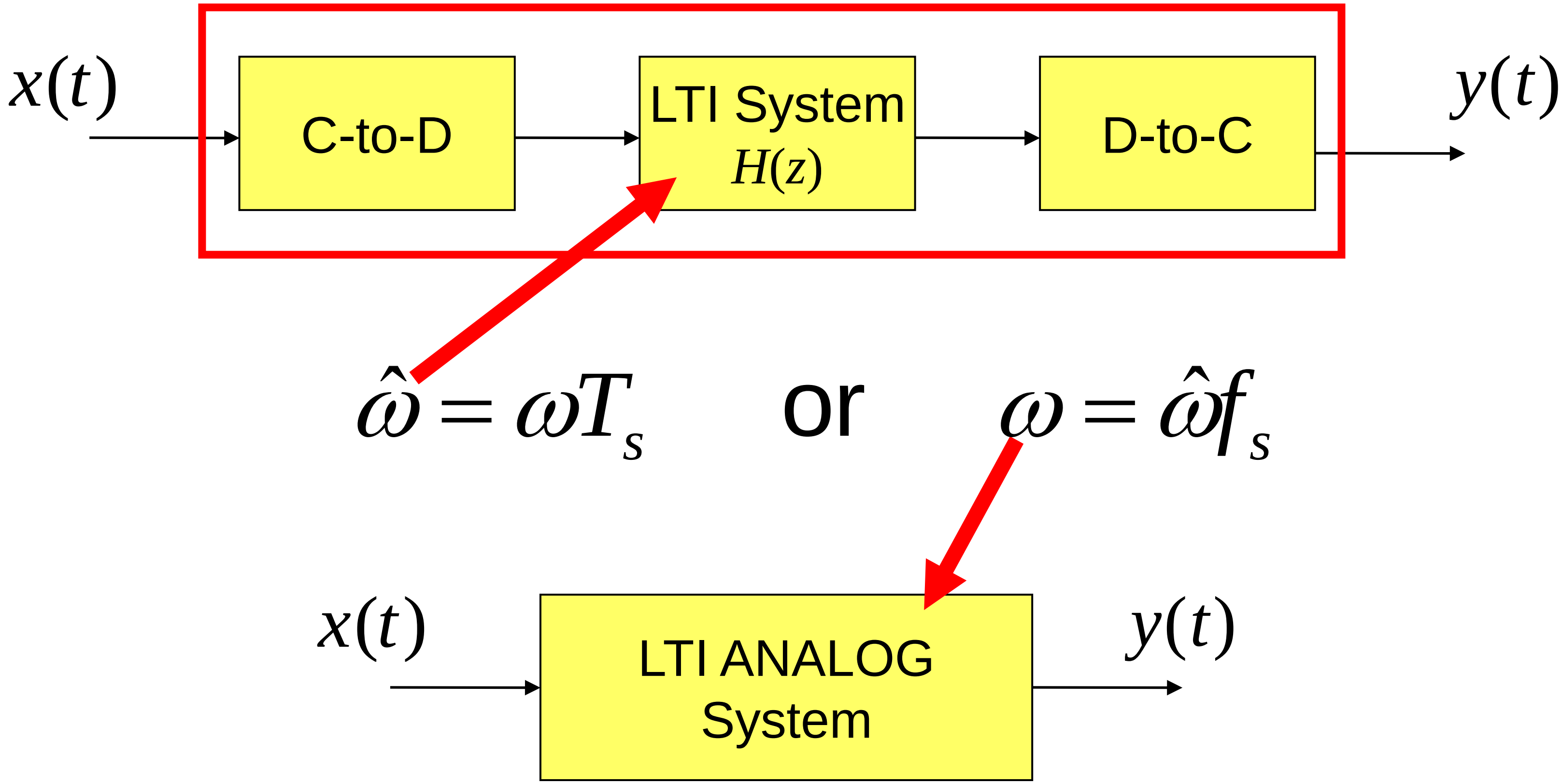
Chapter 9
Continuous-Time Signals and LTI Systems

D-T Filtering of C-T Signals



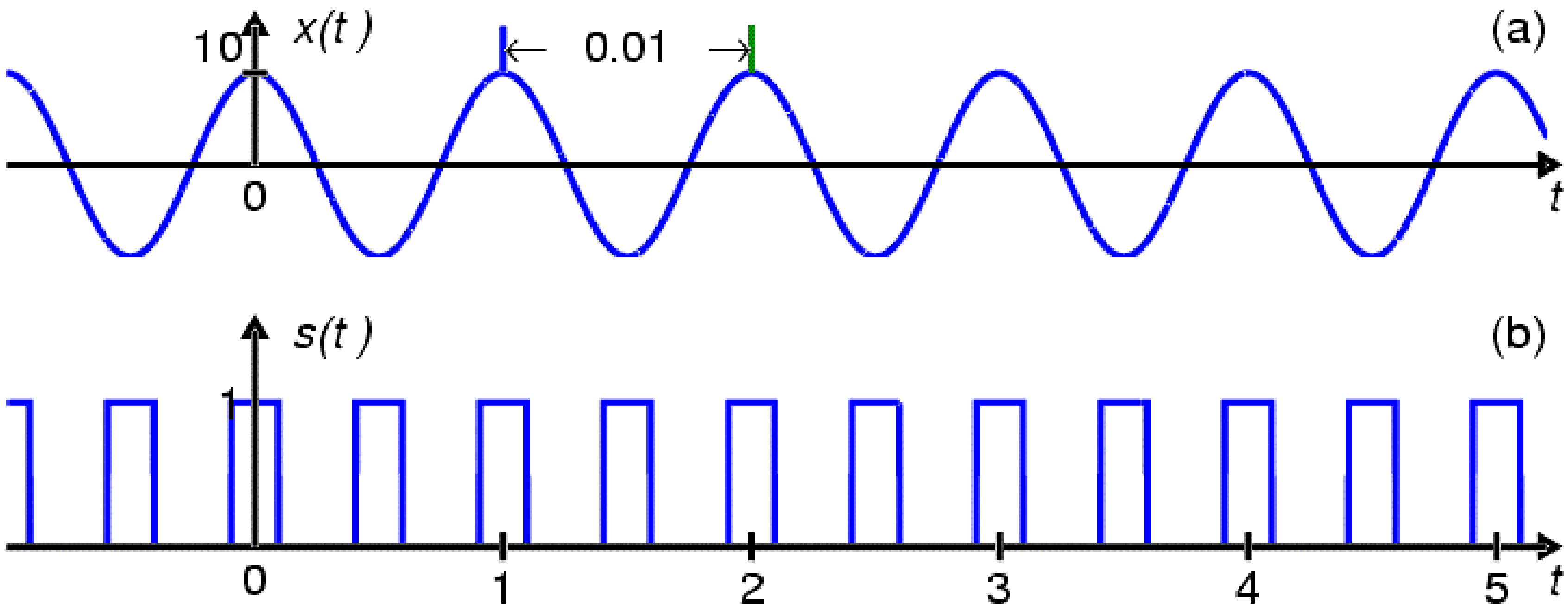
ANALOG SIGNALS $x(t)$

- INFINITE LENGTH
  - PERIODIC SIGNALS (t = time in secs)
    - SINUSOIDS
  - ONE-SIDED, e.g., for $t>0$
    - UNIT STEP: $u(t)$
- FINITE LENGTH
  - FINITE-LENGTH PULSE
- IMPULSE SIGNAL: $\delta(t)$
- DISCRETE-TIME Signal
  - $x[n]$ is list of numbers.

CT Signals: PERIODIC

$$x(t) = 10\cos(200\pi t)$$

Sinusoidal signal



INFINITE DURATION

Square Wave

CT Signals: ONE-SIDED

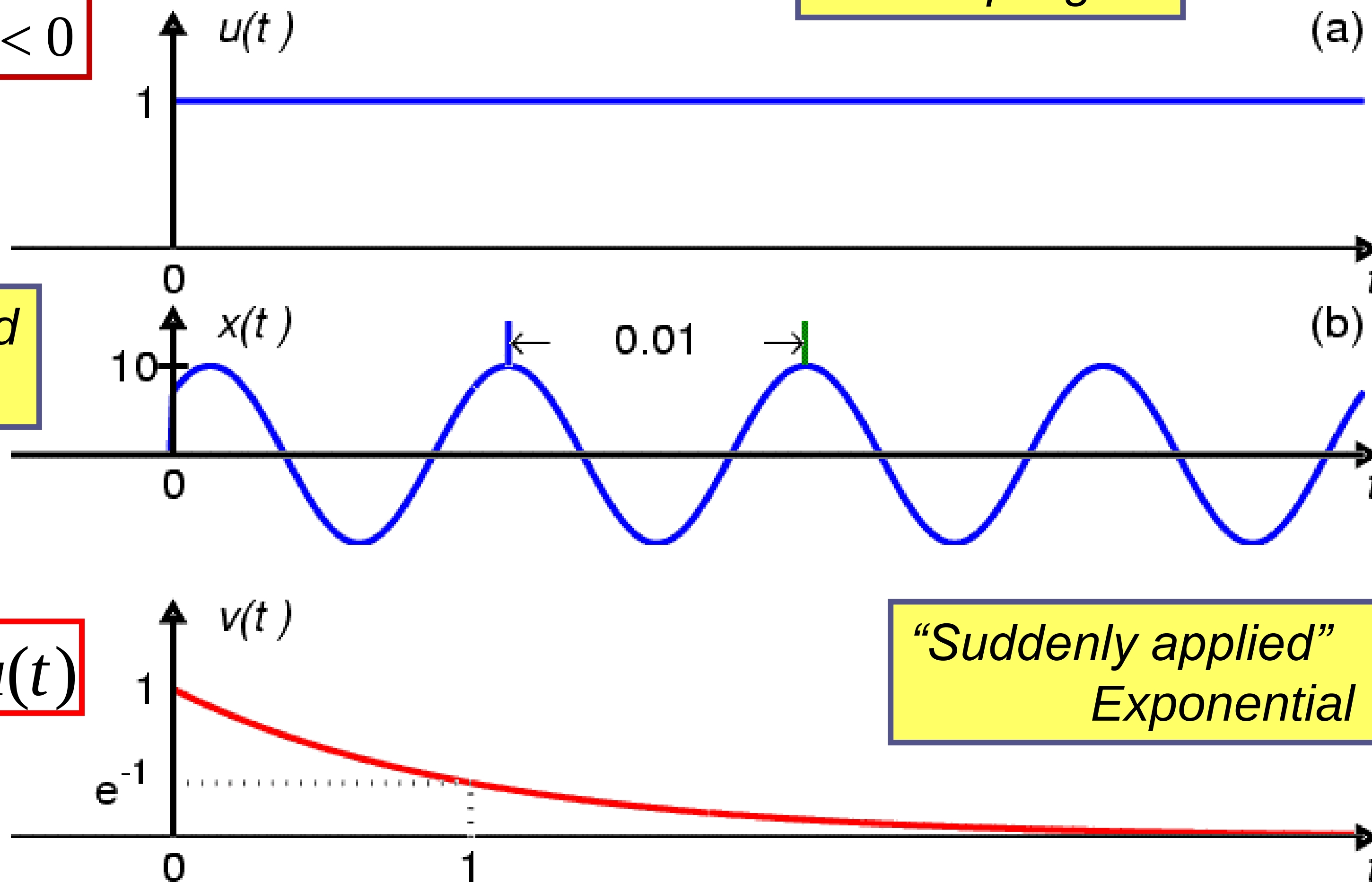
$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

Unit step signal

One-Sided Sinusoid

$$v(t) = e^{-t}u(t)$$

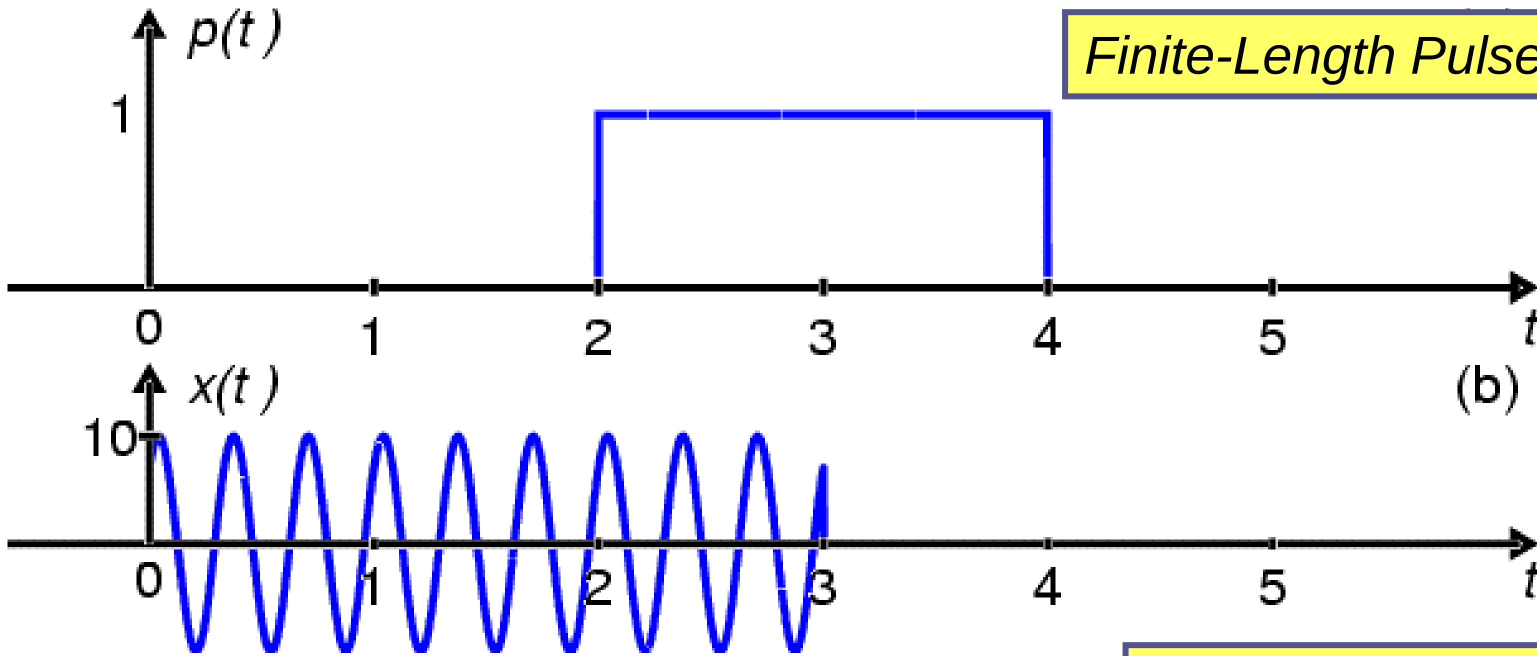
"Suddenly applied" Exponential



CT Signals: FINITE LENGTH

$$\boxed{p(t) = u(t-2) - u(t-4)}$$

Finite-Length Pulse signal

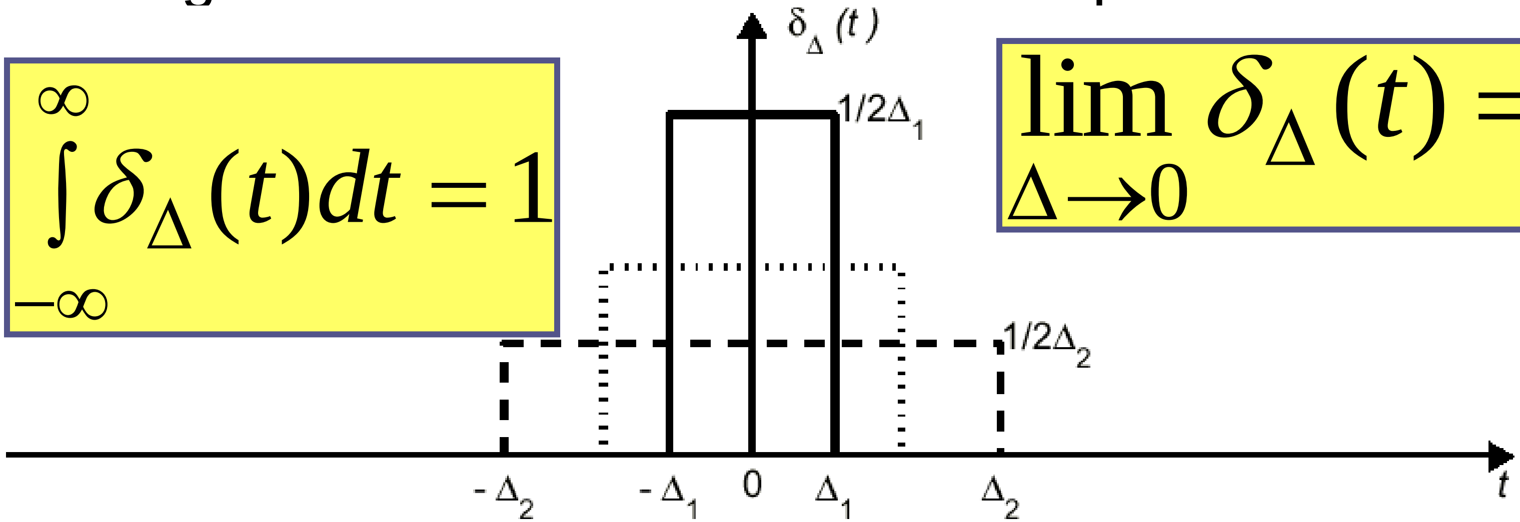


Sinusoid multiplied by a finite-length pulse

What is an Impulse?

- A signal that is concentrated at one point.

$$\int_{-\infty}^{\infty} \delta_{\Delta}(t)dt = 1$$



$$\lim_{\Delta \to 0} \delta_{\Delta}(t) = \delta(t)$$

- One "**INTUITIVE**" definition is:

$$\delta(t) = 0, \quad t \neq 0$$

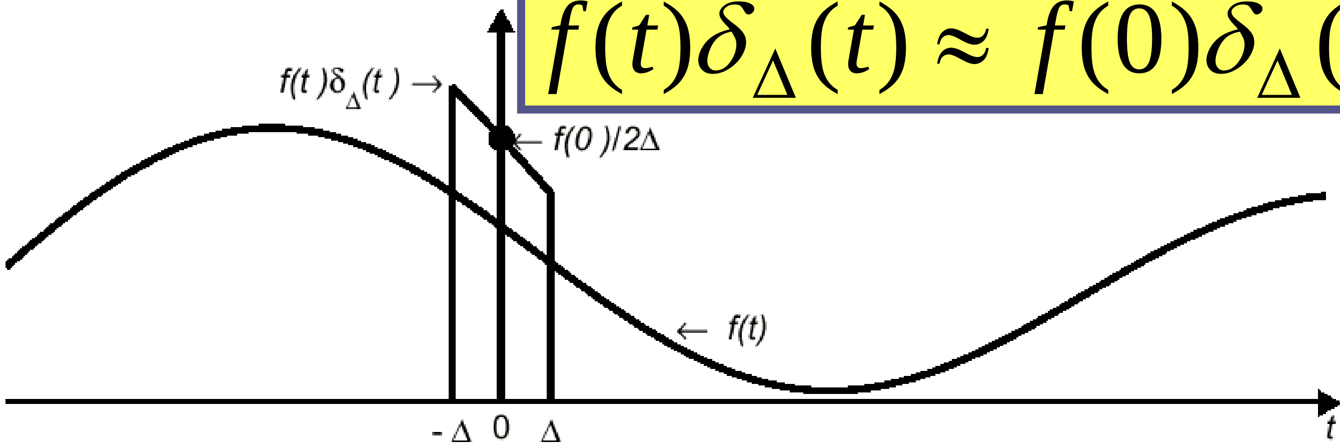
Concentrated at t=0

$$\int_{-\infty}^{\infty} \delta(\tau)d\tau = 1$$

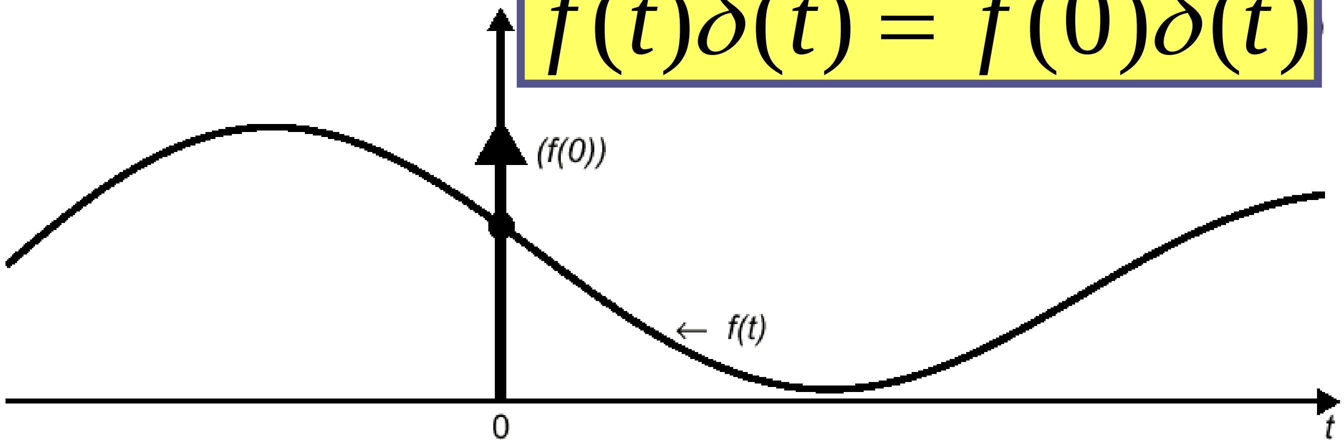
Unit area

Sampling Property

$$f(t)\delta_\Delta(t) \approx f(0)\delta_\Delta(t)$$

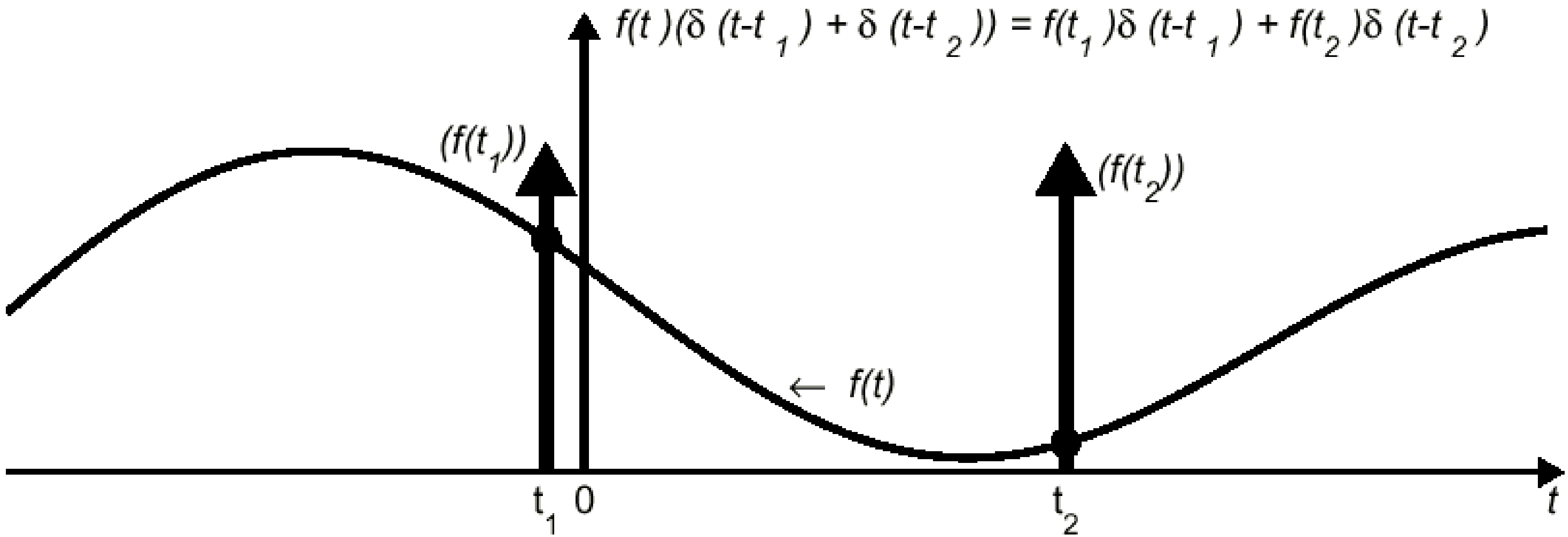


$$f(t)\delta(t) = f(0)\delta(t)$$



General Sampling Property

$$f(t)\delta(t-t_0) = f(t_0)\delta(t-t_0)$$



Properties of the Impulse

$$\delta(t-t_0)=0, \quad t \neq t_0$$

Concentrated at one time

$$\int_{-\infty}^{\infty} \delta(t-t_0)dt = 1$$

Unit area

$$f(t)\delta(t-t_0) = f(t_0)\delta(t-t_0)$$

Sampling Property

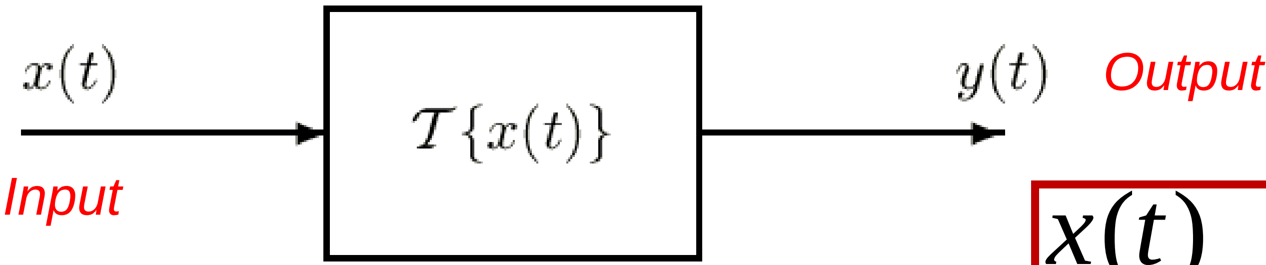
$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = f(t_0)$$

Extract one value of f(t).

$$\frac{du(t)}{dt} = \delta(t)$$

Derivative of unit step

Continuous-Time Systems



$$\boxed{x(t) \quad \mapsto \quad y(t)}$$

- Examples:
  - Delay
    $$y(t) = x(t - t_d)$$
  - Modulator
    $$y(t) = [A + x(t)]\cos \omega_c t$$
  - Integrator
    $$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

CT BUILDING BLOCKS

- INTEGRATOR
- DIFFERENTIATOR
- DELAY by t_o
- MODULATOR (e.g., AM Radio)
- MULTIPLIER & ADDER

Ideal Delay

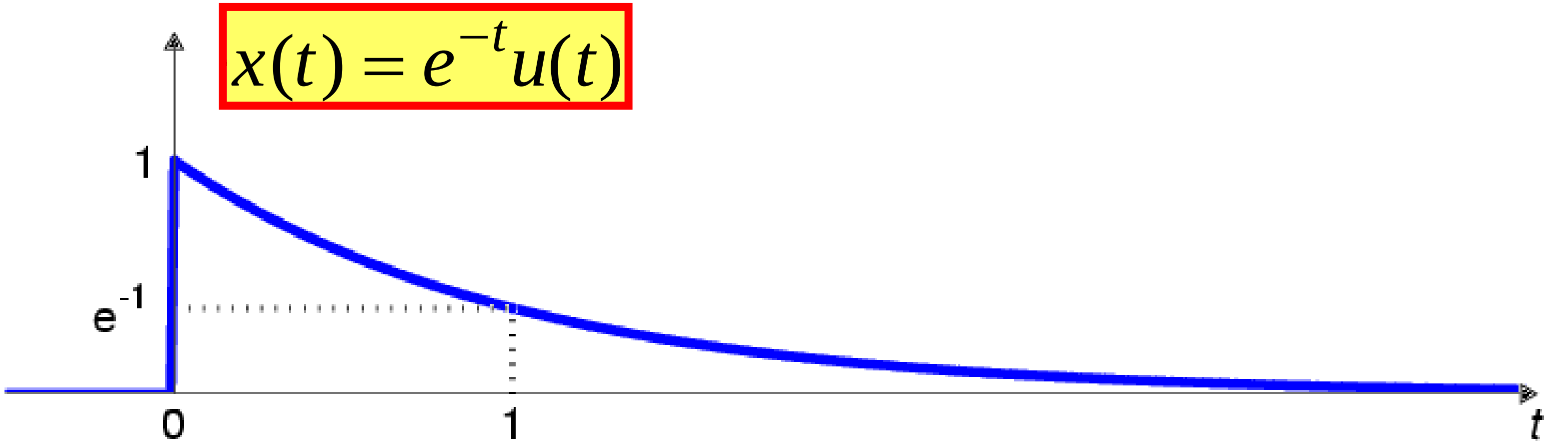
- Mathematical Definition:

$$y(t) = x(t - t_d)$$

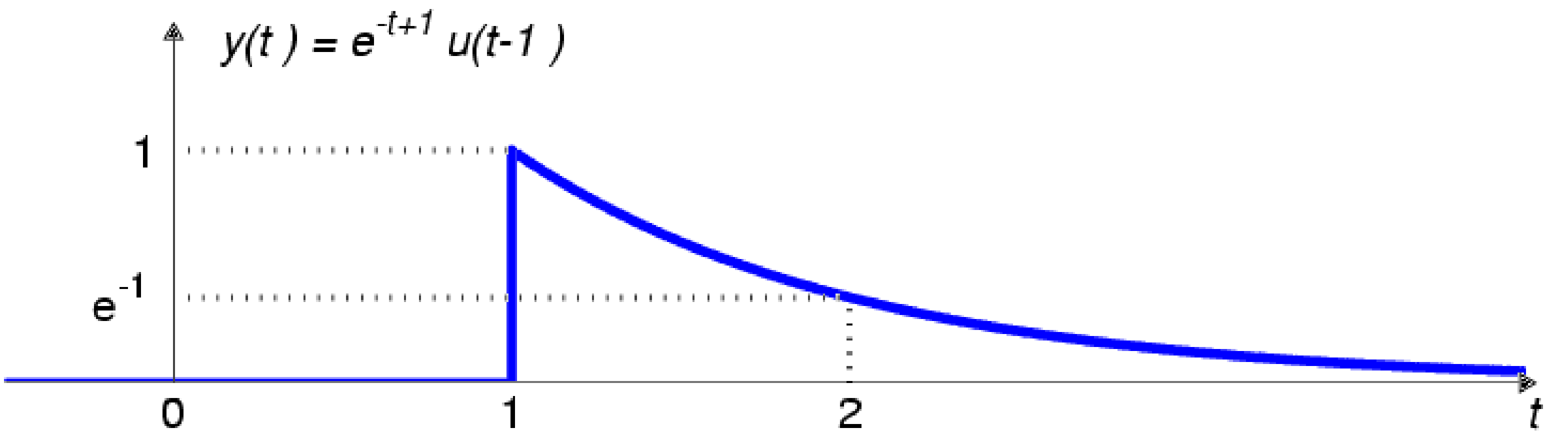
- To find the IMPULSE RESPONSE, $h(t)$, let $x(t)$ be an impulse, so

$$h(t) = \delta(t - t_d)$$

Output of an Ideal Delay of 1 sec



$$y(t) = x(t-1) = e^{-(t-1)}u(t-1)$$



Integrator (1)

- Mathematical Definition:

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

Running Integral

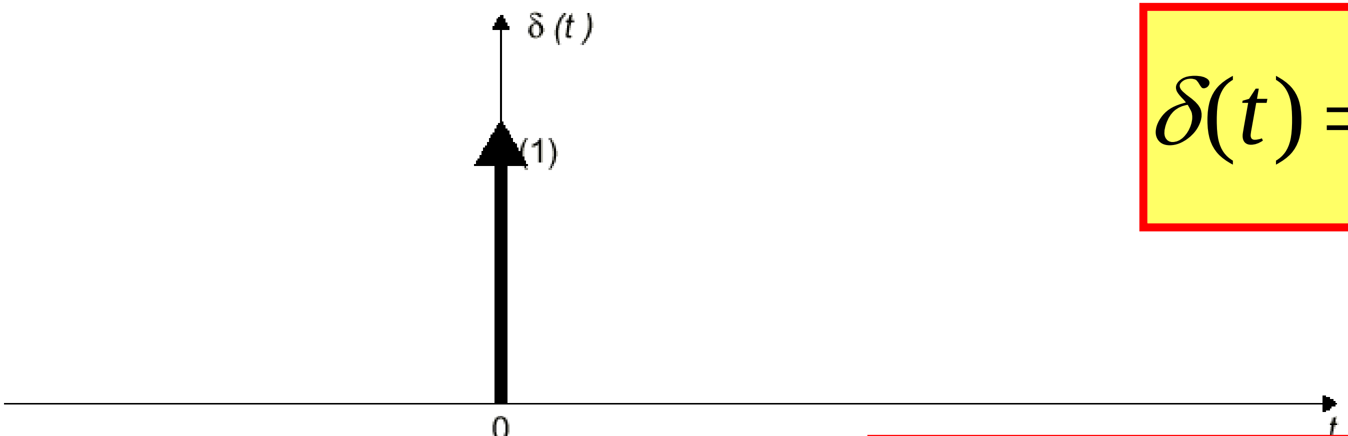
- To find the IMPULSE RESPONSE, *h*(*t*), let *x*(*t*) be an impulse, so

$$h(t) = \int_{-\infty}^{t} \delta(\tau)d\tau = u(t)$$

  - If $t<0$, we get zero.
  - If $t>0$, we get one.
    - Thus we have $h(t) = u(t)$ for the integrator.

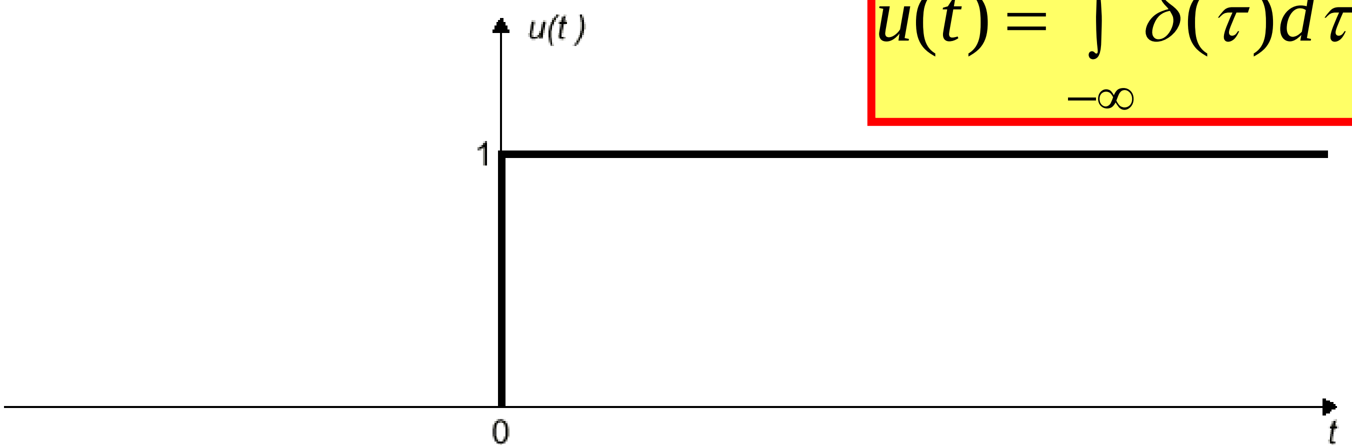
Integrator (2)

- Graphical Representation

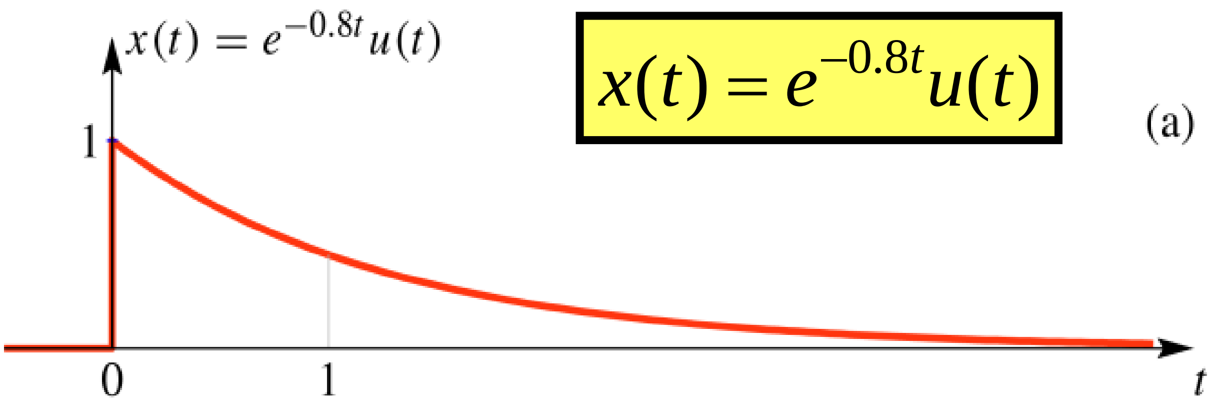


$$\delta(t) = \frac{du(t)}{dt}$$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

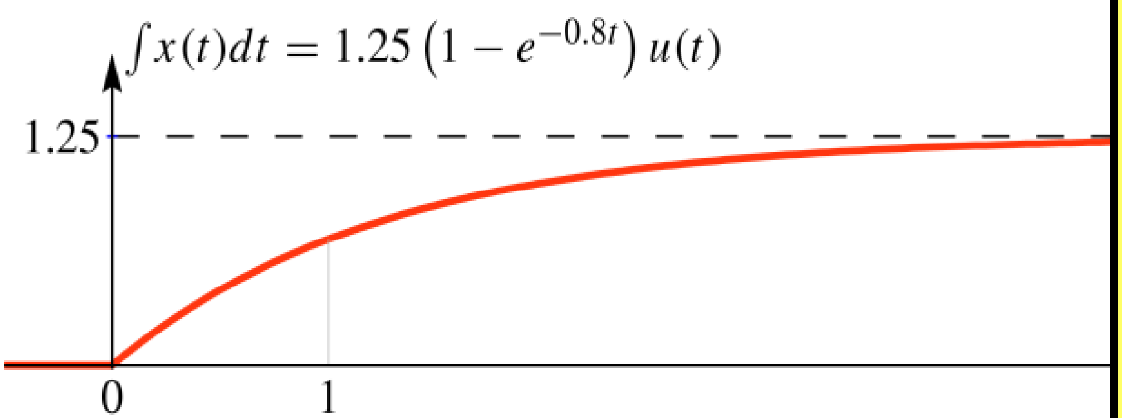


Output of an Integrator



$$\boxed{x(t) = e^{-0.8t}u(t)}$$

(a) $$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$
$$= x(t) * u(t)$$



$$y(t) = \int_{-\infty}^{t} e^{-0.8\tau}u(\tau)d\tau$$
$$= \begin{cases} 0 & t < 0 \\ \int_{-\infty}^{t} e^{-0.8\tau}u(\tau)d\tau & t \geq 0 \end{cases}$$
$$= 1.25(1 - e^{-0.8t})u(t)$$

Differentiator

- Mathematical Definition:

$$y(t) = \frac{dx(t)}{dt}$$

- To find *h*(*t*), let *x*(*t*) be an impulse, so

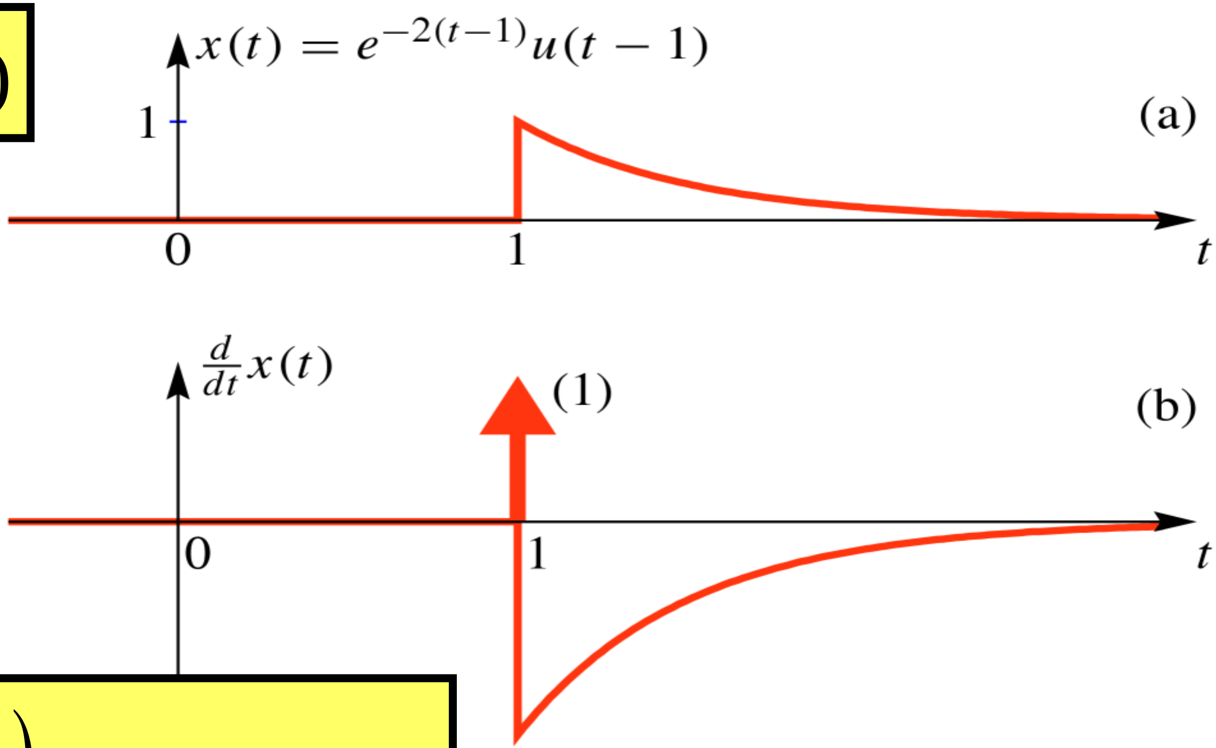
$$h(t) = \frac{d\delta(t)}{dt} = \delta^{(1)}(t)$$

Doublet

Differentiator Output

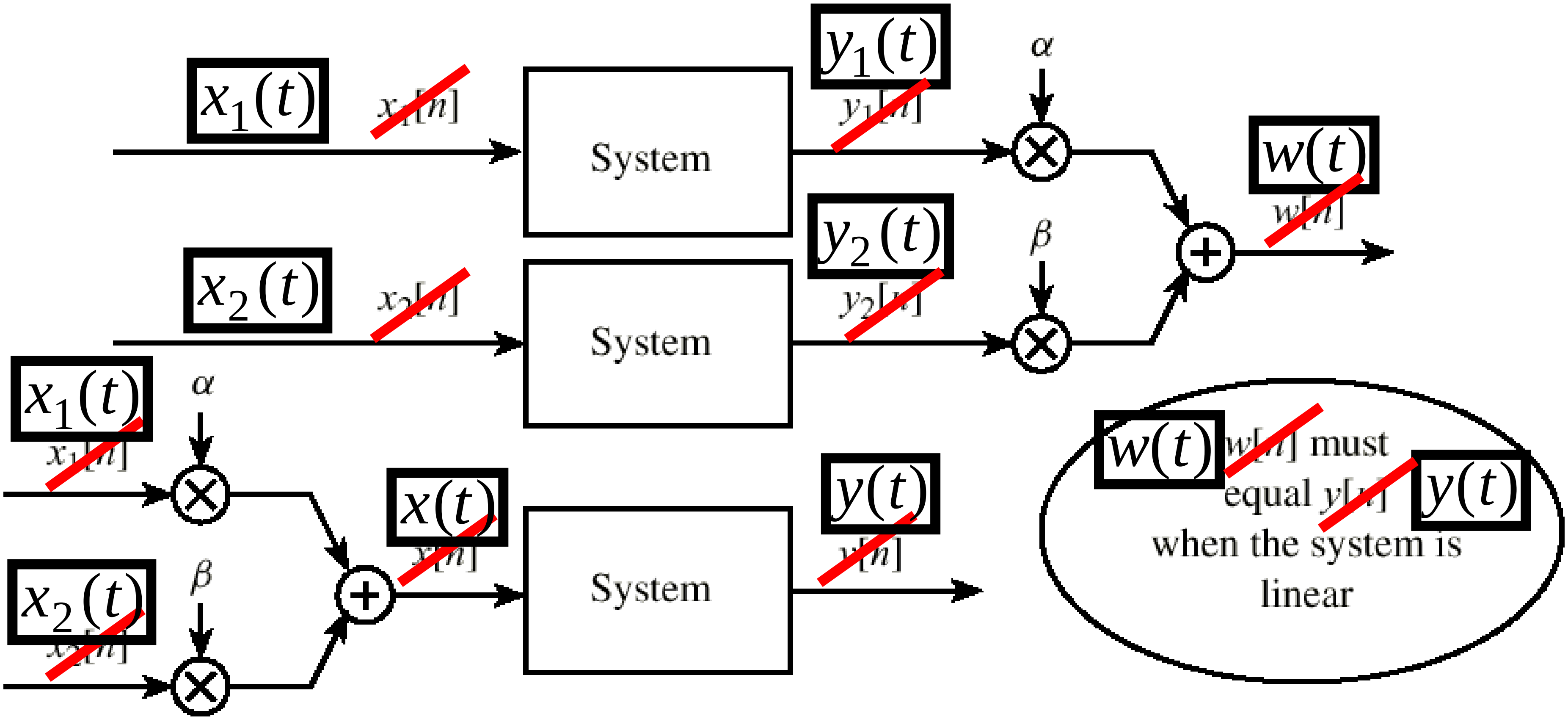
$$\boxed{x(t) = e^{-2(t-1)}u(t-1)}$$

$$y(t) = \frac{dx(t)}{dt}$$

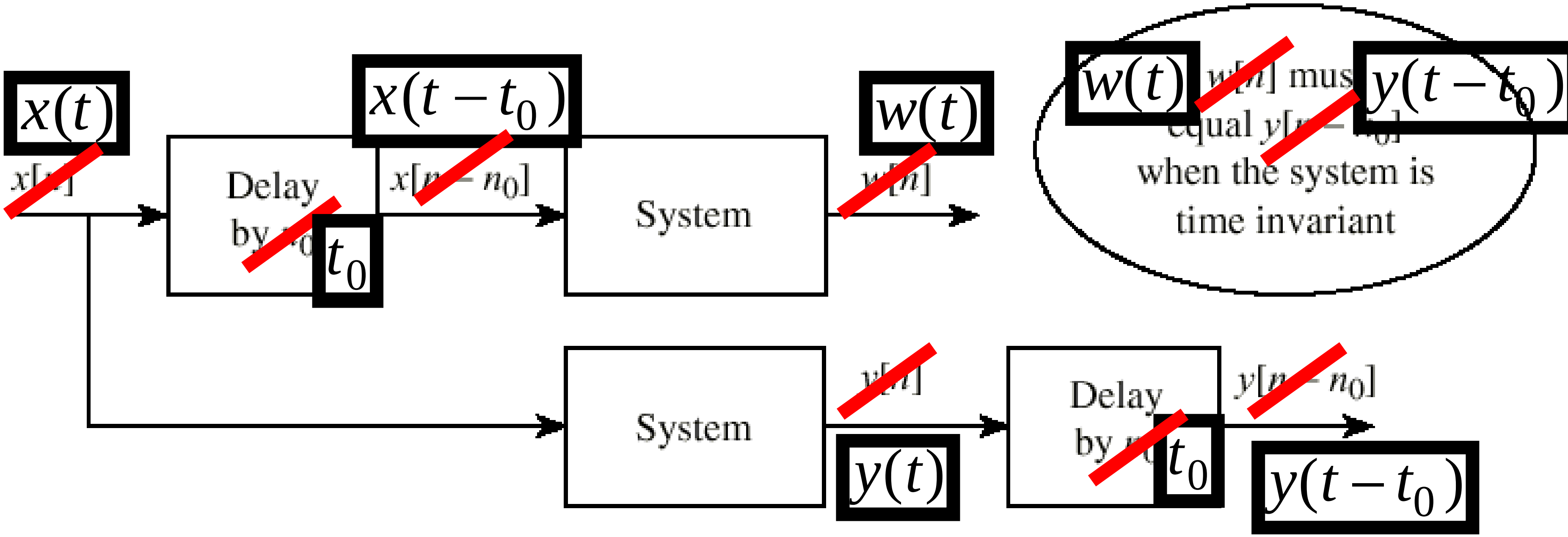


$$\begin{aligned} y(t) &= \frac{d}{dt}\left(e^{-2(t-1)}u(t-1)\right) \\ &= -2e^{-2(t-1)}u(t-1) + e^{-2(t-1)}\delta(t-1) \\ &= -2e^{-2(t-1)}u(t-1) + 1\delta(t-1) \end{aligned}$$

Testing for Linearity



Testing for Time-Invariance



Ideal Delay

$$y(t) = x(t - t_d)$$

- Linear

$$ax_1(t - t_d) + bx_2(t - t_d) = ay_1(t) + by_2(t)$$

- Time-Invariant

$$w(t) = x((t - t_0) - t_d)$$

$$y(t - t_0) = x((t - t_d) - t_0) = x((t - t_0) - t_d)$$

Integrator

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

- Linear

$$\int_{-\infty}^{t} [ax_1(\tau) + bx_2(\tau)]d\tau = ay_1(t) + by_2(t)$$

- Time-Invariant

$$w(t) = \int_{-\infty}^{t} x(\tau - t_0)d\tau \qquad \text{let} \quad \sigma = \tau - t_0$$

$$\Rightarrow \quad w(t) = \int_{-\infty}^{t-t_0} x(\sigma)d\sigma = y(t - t_0)$$

Modulator

$$y(t) = [A + x(t)]\cos\omega_c t$$

- **Not** linear

$$[A + ax_1(t) + bx_2(t)]\cos\omega_c t$$
$$\neq a[A + x_1(t)]\cos\omega_c t + b\,[A + x_2(t)]\cos\omega_c t$$

- **Not** time-invariant

$$w(t) = [A + x(t - t_0)]\cos\omega_c t \neq y(t - t_0)$$

Linear and Time-Invariant (LTI) Systems

- If a continuous-time system is both linear and time-invariant, then the output $y(t)$ is related to the input $x(t)$ by a **convolution integral**

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

where $h(t)$ is the **impulse response** of the system.

Convolution is Linear.

- Substitute $x(t)=ax_1(t)+bx_2(t)$.

$$y(t)=\int_{-\infty}^{\infty}[ax_1(\tau)+bx_2(\tau)]h(t-\tau)d\tau$$

$$=a\int_{-\infty}^{\infty}x_1(\tau)h(t-\tau)d\tau+b\int_{-\infty}^{\infty}x_2(\tau)h(t-\tau)d\tau$$

$$=ay_1(t)+by_2(t)$$

Therefore, convolution is linear.

Convolution is Time-Invariant.

- Substitute $x(t\text{-}t_0)$.

$$w(t) = \int_{-\infty}^{\infty} h(\tau)x((t-\tau)-t_o)d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)x((t-t_o)-\tau)d\tau$$

$$= y(t-t_o)$$

Convolution is Commutative.

$$h(t)*x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

let $\sigma = t-\tau$ and $d\sigma = -d\tau$

$$\boxed{h(t)*x(t)} = -\int_{\infty}^{-\infty} h(t-\sigma)x(\sigma)d\sigma$$

$$= \int_{-\infty}^{\infty} x(\sigma)h(t-\sigma)d\sigma = \boxed{x(t)*h(t)}$$

Convolution is Associative, and Distributive over Addition.

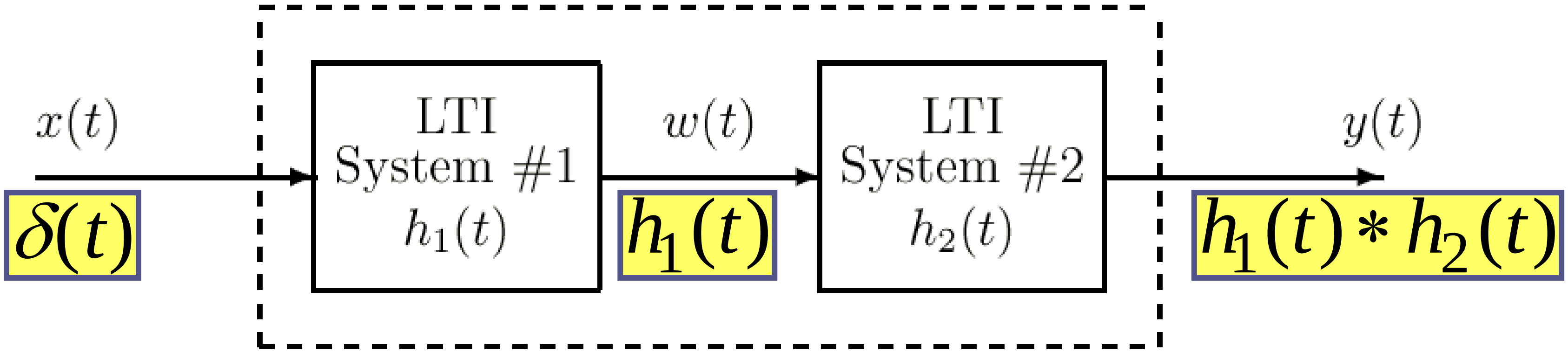
- Associativity

$$[x(t)*h_1(t)]*h_2(t)=x(t)*[h_1(t)*h_2(t)]$$

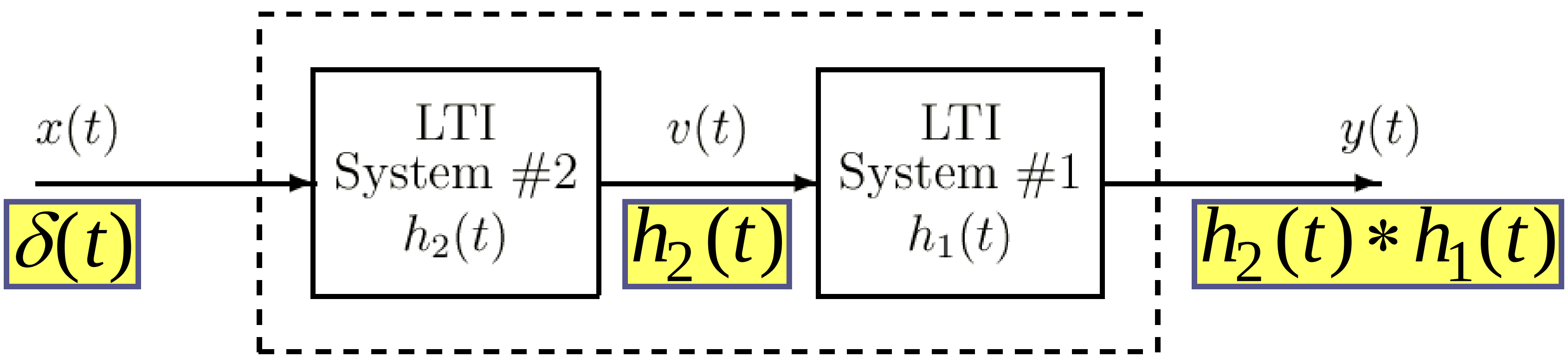
- Distributivity over addition

$$x(t)*[h_1(t)+h_2(t)]=x(t)*h_1(t)+x(t)*h_2(t)$$

Cascade of LTI Systems

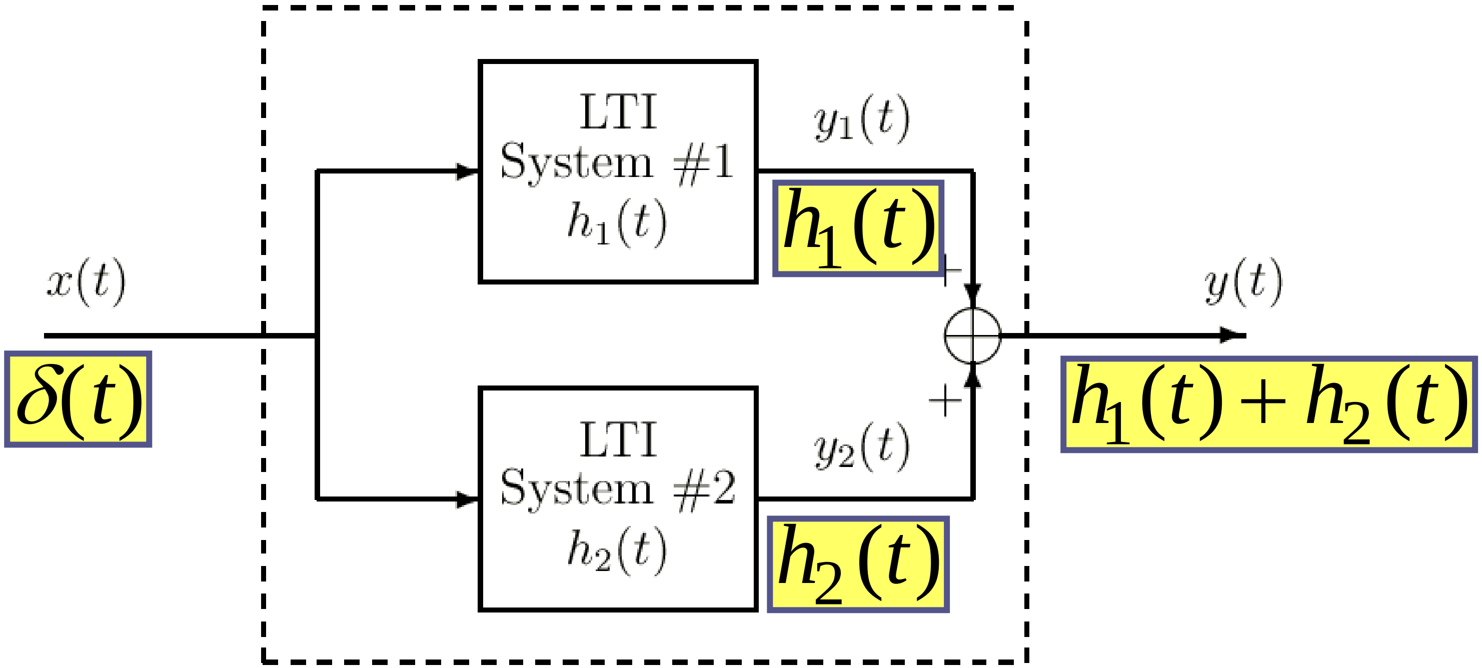


$$h(t) = h_1(t) * h_2(t) = h_2(t) * h_1(t)$$

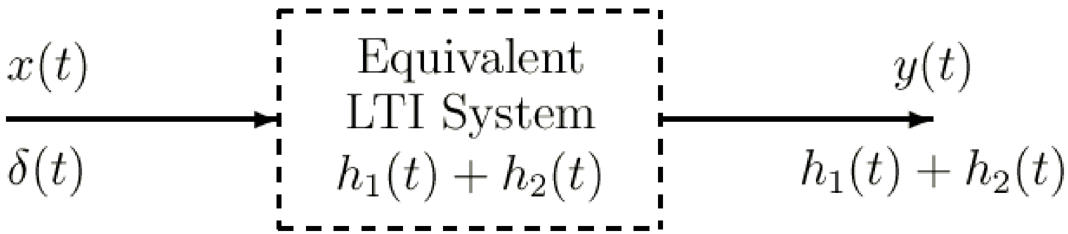


(b)

Parallel LTI Systems



$$h(t) = h_1(t) + h_2(t)$$



(b)

Sampling and Convolution

- Sampling property

$$\boxed{f(t)\delta(t-t_0) = f(t_0)\delta(t-t_0)}$$

- Convolution identity

$$\boxed{x(t) * \delta(t) = x(t)}$$

Impulse Responses of Basic LTI Systems

- Integrator $h(t) = \int_{-\infty}^{t} \delta(\tau) d\tau = u(t)$

$$y(t) = \int_{-\infty}^{t} x(\tau) d\tau = x(t) * u(t) = x^{(-1)}(t)$$

- Differentiator $h(t) = \dfrac{d\delta(t)}{dt} = \delta^{(1)}(t)$

$$y(t) = \frac{dx(t)}{dt} = x(t) * \delta^{(1)}(t) = x^{(1)}(t)$$

- Ideal delay $h(t) = \delta(t - t_d)$

$$y(t) = x(t - t_d) = x(t) * \delta(t - t_d) = x(t - t_d)$$

Convolution of Impulses, etc.

- Convolution of two impulses

$$\delta(t-t_1)*\delta(t-t_2)=\delta(t-t_1-t_2)$$

- Convolution of step and shifted impulse

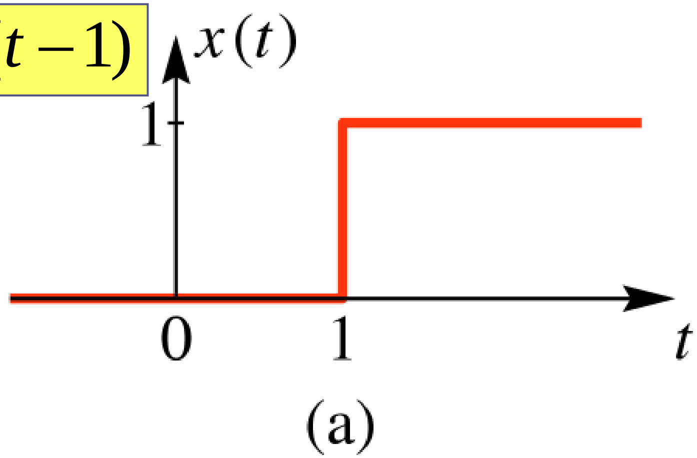
$$u(t)*\delta(t-t_0)=u(t-t_0)$$

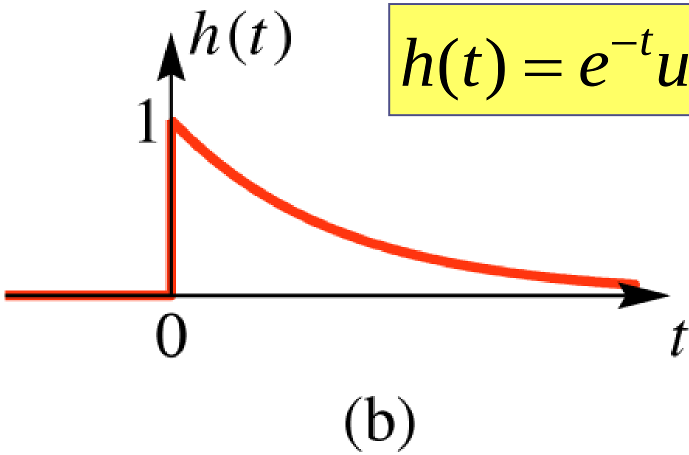
- Convolution of step and derivative of impulse

$$u(t)*\delta^{(1)}(t)=\delta(t)$$

Evaluating a Convolution

$x(t) = u(t-1)$

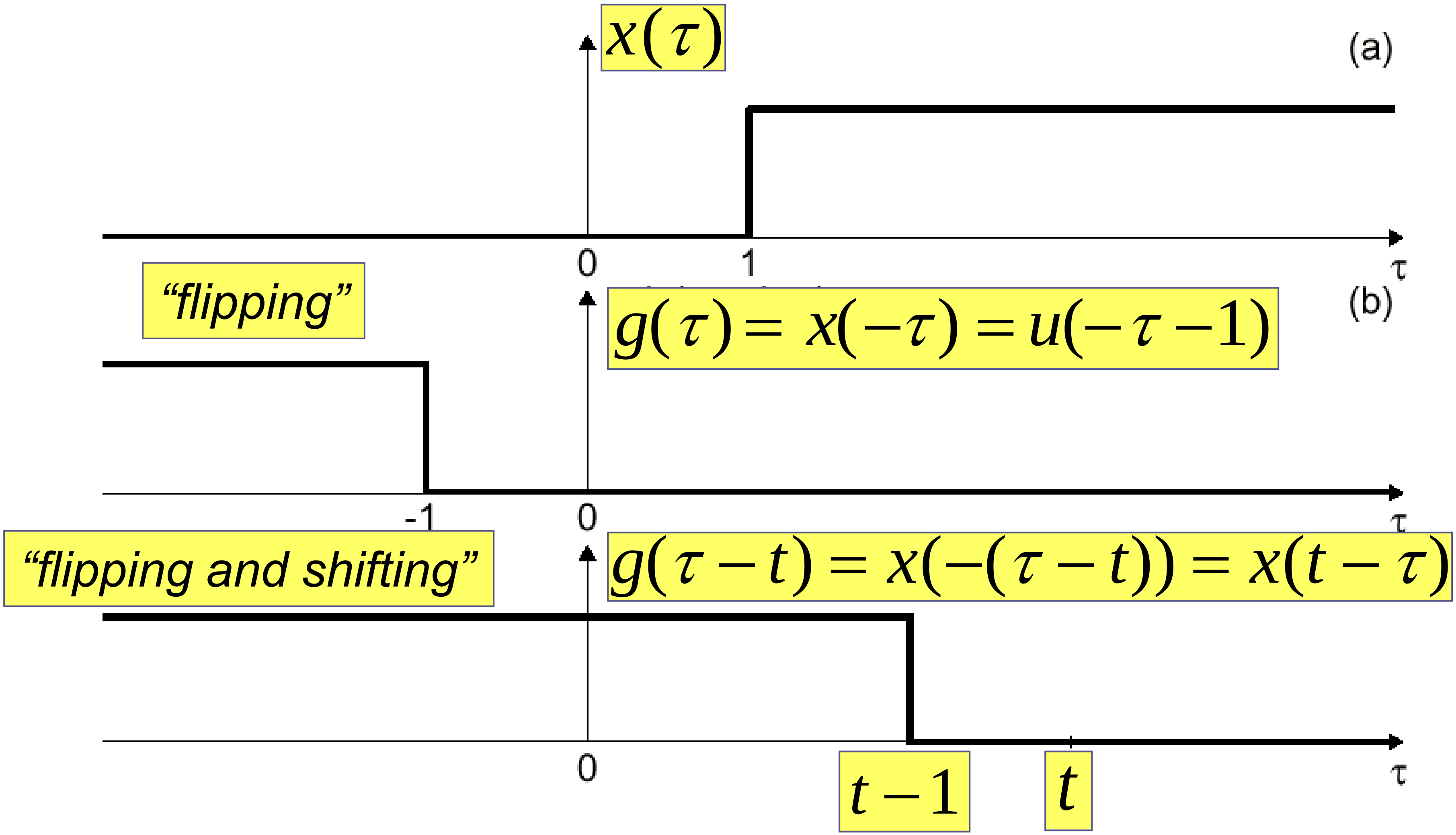




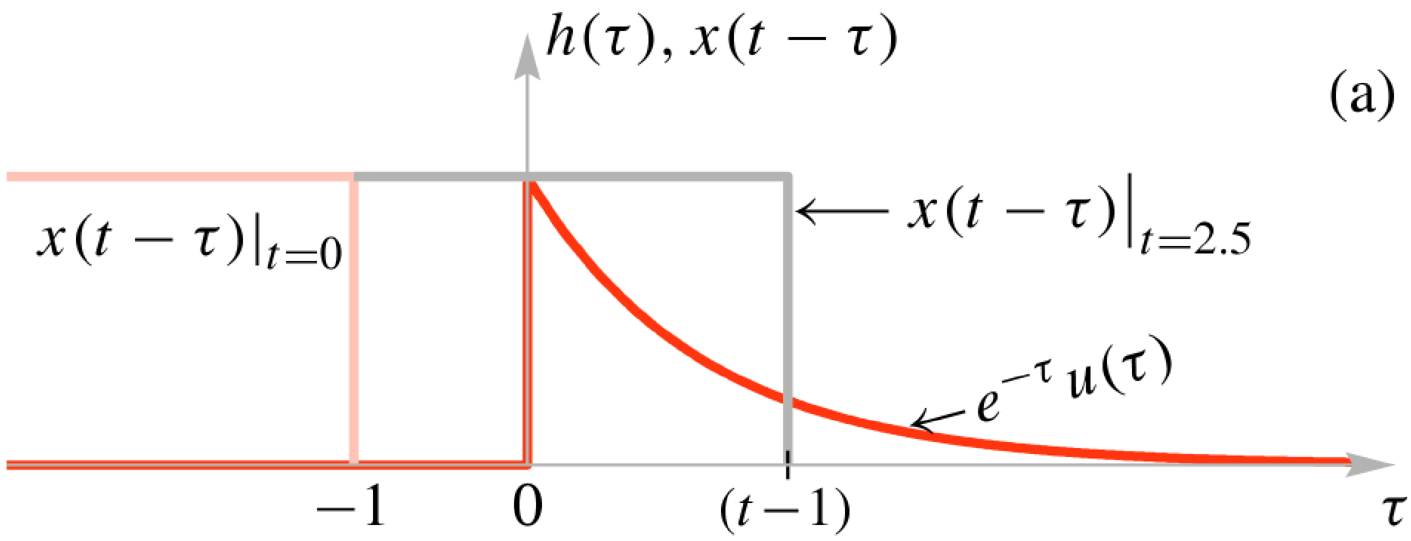
$h(t) = e^{-t}u(t)$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = h(t) * x(t)$$

"Flipping and Shifting"



Evaluating the Integral



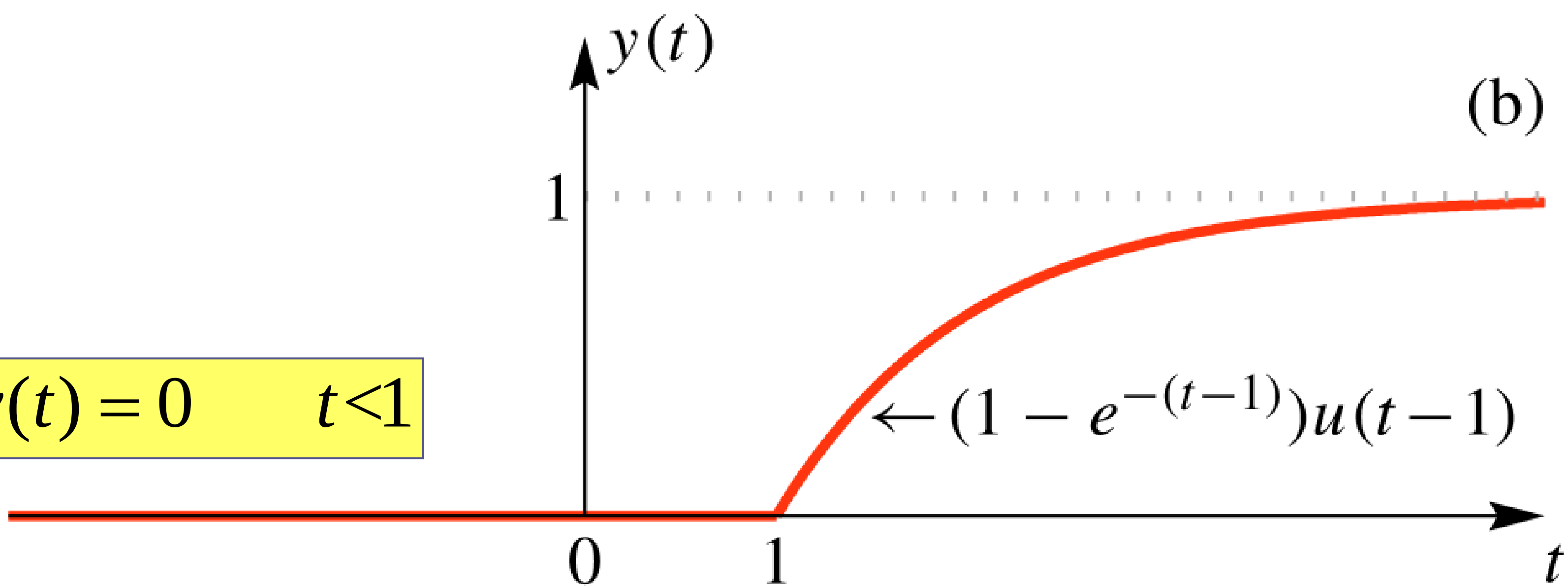
$$y(t) = \begin{cases} 0 & t-1<0 \\ \displaystyle\int_0^{t-1} e^{-\tau}\,d\tau & t-1>0 \end{cases}$$

Solution

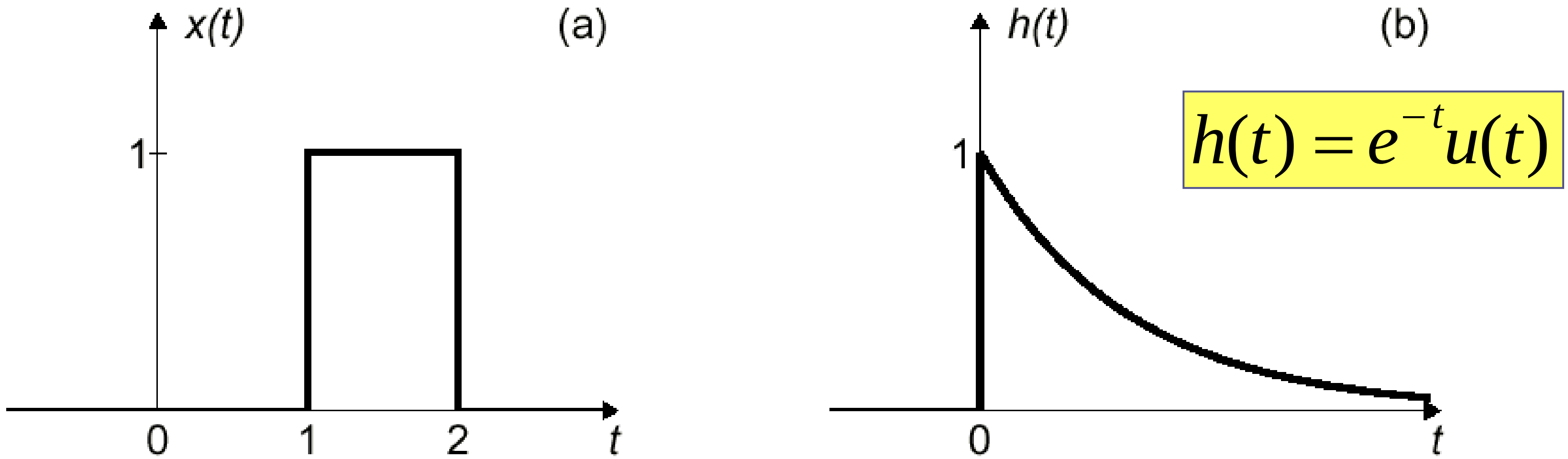
$$y(t) = \int_0^{t-1} e^{-\tau} d\tau = -e^{-\tau}\Big|_0^{t-1}$$

$$= 1 - e^{-(t-1)} \qquad t \geq 1$$

$$y(t) = 0 \qquad t < 1$$



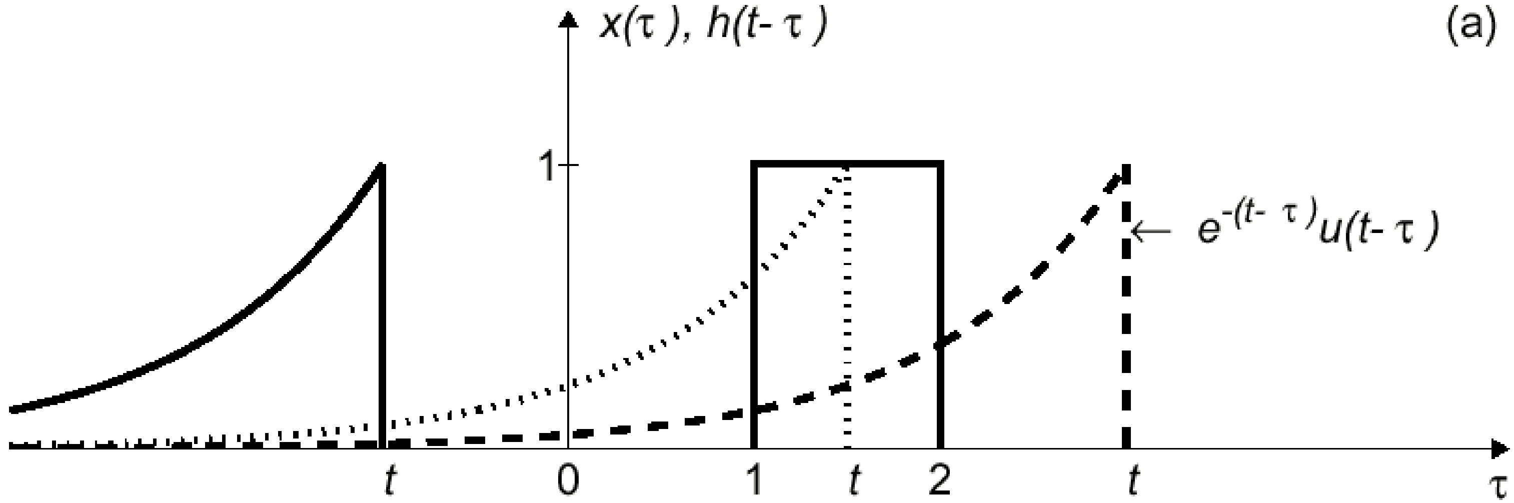
Another Convolution Example



$$h(t) = e^{-t}u(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

Evaluating the Integral

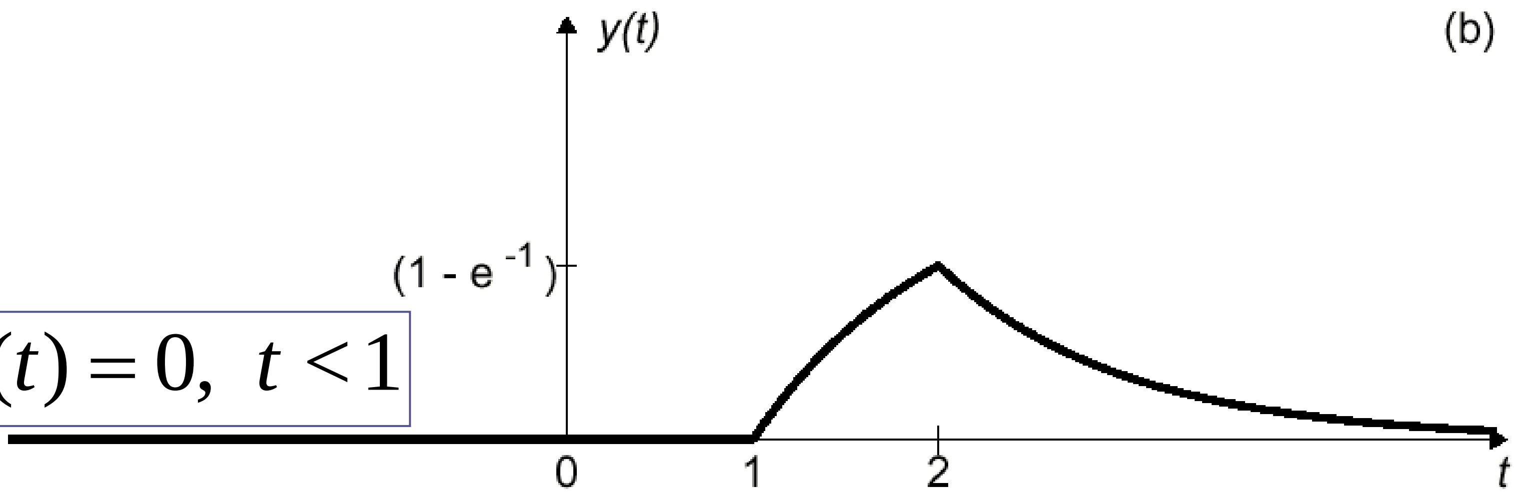


$$
y(t) = \begin{cases} 0, & t < 1 \\ \int_1^t e^{-(t-\tau)} d\tau, & 1 \le t \le 2 \\ \int_1^2 e^{-(t-\tau)} d\tau, & 2 \le t \end{cases}
$$

Solution

$$y(t)=\begin{cases}\displaystyle\int_1^t e^{-(t-\tau)}d\tau = e^{-(t-\tau)}\Big|_1^t = 1 - e^{-(t-1)}, & 1\le t\le 2\\ \displaystyle\int_1^2 e^{-(t-\tau)}d\tau = e^{-(t-\tau)}\Big|_1^2 = e^{-(t-2)} - e^{-(t-1)}, & 2\le t\end{cases}$$

$$\boxed{y(t)=0,\ \ t<1}$$



General Convolution Example

$$x(t) = e^{-at}u(t)$$

$$h(t) = e^{-bt}u(t), \quad b \neq a$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)e^{-b(t-\tau)}u(t-\tau)d\tau = \begin{cases} e^{-bt}\int_{0}^{t} e^{-a\tau}e^{b\tau}d\tau & t>0 \\ 0 & t<0 \end{cases}$$

$$= \begin{cases} \dfrac{e^{-at}-e^{-bt}}{-a+b} & t>0 \\ 0 & t<0 \end{cases} = \frac{e^{-at}-e^{-bt}}{b-a}u(t)$$

Special Case: $u(t)$

$$x(t) = e^{-at}u(t), \quad a \neq 0$$

$$h(t) = u(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

$$= \frac{1}{a}(1 - e^{-at})u(t)$$

$$\text{if} \quad a = 2$$

$$y(t) = \frac{1}{2}(1 - e^{-2t})u(t)$$

Convolve Unit Steps.

$$x(t) = u(t)$$

$$h(t) = u(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} u(\tau)u(t-\tau)d\tau = \begin{cases} \int_0^t 1 d\tau & t > 0 \\ 0 & t < 0 \end{cases}$$

$$= \begin{cases} t & t > 0 \\ 0 & t < 0 \end{cases} = tu(t)$$

Unit Ramp

Stability & Causality

- A system is stable if and only if every bounded input produces a bounded output.
- A continuous-time *LTI system* is stable if and only if

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty$$

- A system is causal if and only if $y(t_0)$ depends only on $x(\tau)$ for $\tau \leq t_0$.
- A continuous-time *LTI system* is causal if and only if

$$h(t) = 0 \text{ for } t < 0$$

Thank you

- Homework
  - P-8.5, 12(a,b), 16, 17, 19, 20, 22
  - P-9.2(b,d), 3(d), 5, 8, 10, 12, 16, 17, 22, 25
- Reading assignment
  - ~ Chapter 10