Frequency domain analysis and discrete-time signals and systems

Continuous time LTI System

$$x(t) \longrightarrow \boxed{L[\bullet]} \longrightarrow y(t)$$

$$\begin{aligned}
y(t) &= L[x(t)] = L[\int_{-\infty}^{\infty} x(\lambda)\delta(t-\lambda)d\lambda \\
&= \int_{-\infty}^{\infty} x(\lambda)L[\delta(t-\lambda)]d\lambda \quad : \quad superposition \\
&= \int_{-\infty}^{\infty} x(\lambda)h(t,\lambda)d\lambda = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda \quad : \quad Time-invariant \\
&= x(t) * h(t)
\end{aligned}$$

Impulse response of a system $L[\bullet] : h(t)$

$$y(t) = h(t) * x(t) = x(t) * h(t)$$

Continuous time LTI System

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$e^{j\omega t} \longrightarrow H(\omega)e^{j\omega t}$$

Note:

1) $Y(\omega) = H(\omega)X(\omega)$

2) $h(t)$: Impulse response

3) $H(\omega)$: Frequency response

4) Impulse response vs. frequency response

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t} d\omega$$

$$L[\delta(t)] = L\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t} d\omega\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} L[e^{j\omega t}]d\omega] = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{j\omega t}]d\omega = h(t)$$

Continuous time LTI System

Magnitude and phase response

Input *System*

$$x(t) = A\cos(\omega_0 t + \phi_0) \longrightarrow H(\omega) = |H(\omega)|e^{j\psi(\omega)}$$

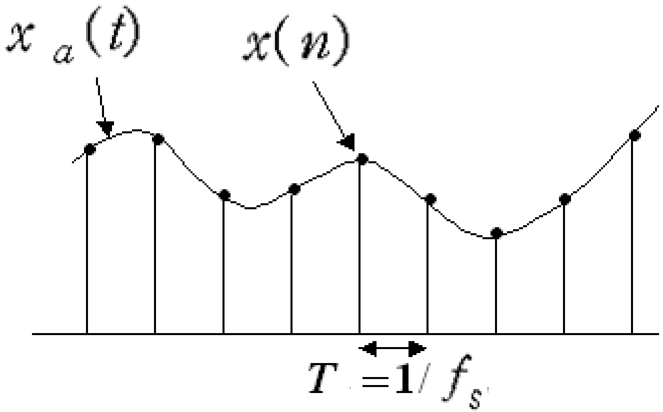
$$x(t) = \frac{A(\omega_0)}{2}\{e^{j(\omega_0 t + \phi_0)} + e^{-j(\omega_0 t + \phi_0)}\}$$

$$\begin{aligned}
y(t) &= A(\omega_0)\{H(\omega_0)e^{j\omega_0 t}e^{j\phi_0} + H(-\omega_0)e^{-j\omega_0 t}e^{-j\phi_0}\}/2 \\
&= A(\omega_0)\{H(\omega_0)e^{j\omega_0 t}e^{j\phi_0} + H^*(\omega_0)e^{-j\omega_0 t}e^{-j\phi_0}\}/2 \quad \textbf{if } h(t) \textit{ is real} \\
&= A(\omega_0)\cdot Re[H(\omega_0)e^{j\omega_0 t}e^{j\phi_0}] \\
&= A(\omega_0)\cdot Re[|H(\omega_0)|e^{j\omega_0 t}e^{j\phi_0}e^{j\psi(\omega_0)}] \\
&= A(\omega_0)\cdot |H(\omega_0)|\cos(\omega_0 t + \phi_0 + \psi(\omega_0))
\end{aligned}$$

Discrete-Time signals

Introduction: CT and DT sinusoidal signals

$$x_a(t) = A\cos(2\pi f t + \phi)$$



- **Sampling interval:** T

$$x(n) = x_a(nT), \qquad -\infty < n < \infty$$

- **Sampling frequency :** $f_s = 1/T$
- **Sampling instants:** $t = nT = \dfrac{n}{f_s}$
- **Sampled signal / DT signal**

$$x_a(nT) \equiv x(n) = A\cos(2\pi f nT + \phi) = A\cos\left(\frac{2\pi n f}{f_s} + \phi\right)$$

$$= A\cos(2\pi n \underline{f} + \phi)$$

$$= A\cos(\theta n + \phi)$$

$$\boxed{\underline{f} = \frac{f}{f_s} \qquad \theta = \omega T}$$

Discrete-Time signals

Introduction: Representation of frequencies

- **relative or normalized frequency :** $\underline{f} = \dfrac{f}{f_s}$ $\quad \Theta = \omega T$

- **Two ways of representation:**

	type 1	type 2
CT(analog) frequency:	f	F
CT angular frequency:	$\omega = 2\pi f$	$\Omega = 2\pi F$
Digital(normalized) freq:	$\underline{f}$	f
Digital angular freq:	$\theta = 2\pi \underline{f}$	$\omega = 2\pi f$

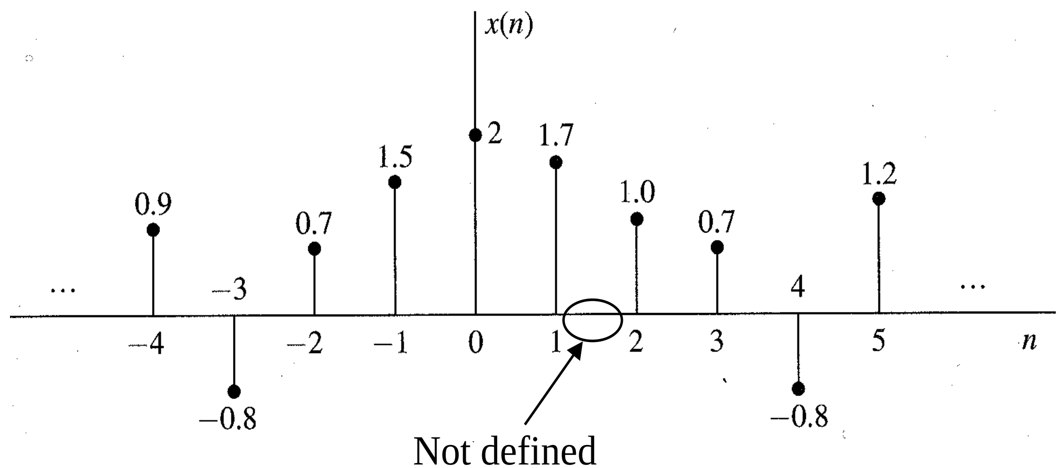
Discrete-Time signals

Definition

- $\{x(n)\}$: Discrete-time sequence, digital sequence
- $x(n)$: n-th sample. Also used to represent a sequence, $\{x(n)\}$, i.e., defined only for integer values of n

Representation

- Graphical Representation



- Functional representation

$$x(n)=\begin{cases} 1, & \text{for } n=1,3 \\ 4, & \text{for } n=2 \\ 0, & \text{elsewhere} \end{cases} \qquad (2.1.1)$$

Discrete-Time signals

Representation

❖ Tabular representation

n	$\cdots$	-2	-1	0	1	2	3	4	5	$\cdots$
$x(n)$	$\cdots$	0	0	0	1	4	1	0	0	$\cdots$

❖ Sequence representation

$$x(n)=\{3,-1,\underset{\uparrow}{-2},5,0,4,-1\} \qquad (2.1.4)$$

$$x(n)=\{\underset{\uparrow}{0},1,4,2\} \qquad (2.1.5)$$

❖ Most general representation

$$x(n)=\sum_{k=-\infty}^{\infty}x(k)\cdot\delta(n-k)=\sum_{k=-\infty}^{\infty}x(n-k)\cdot\delta(k)$$

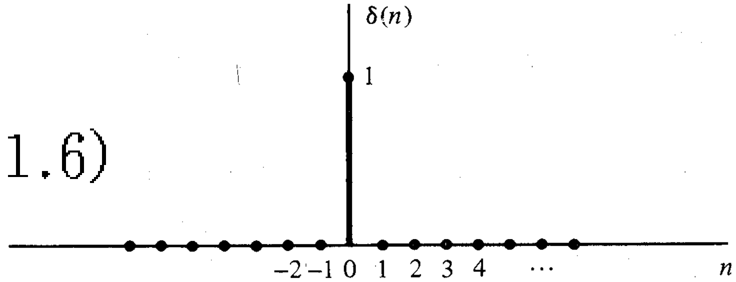
$$Ex)\ \delta(n-1)+4\delta(n-2)+\delta(n-3)$$

Discrete-Time signals

Elementary DT signals

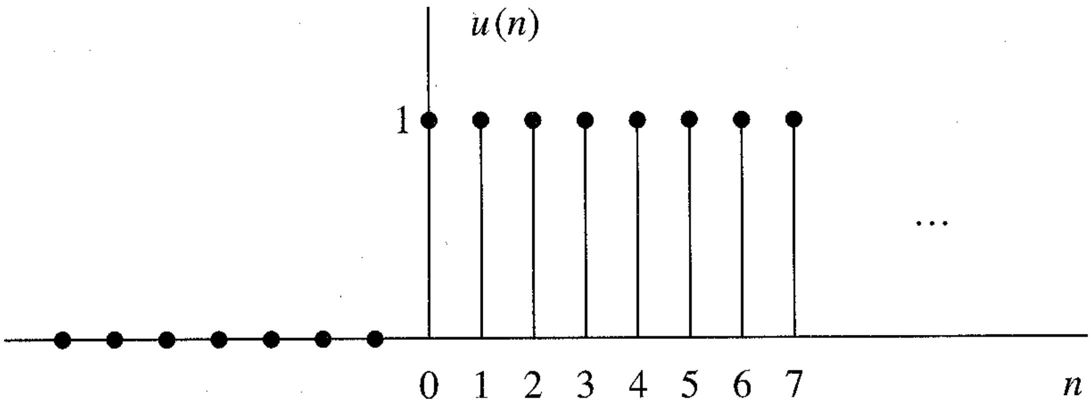
❖ **Unit impulse or Unit sample sequence**

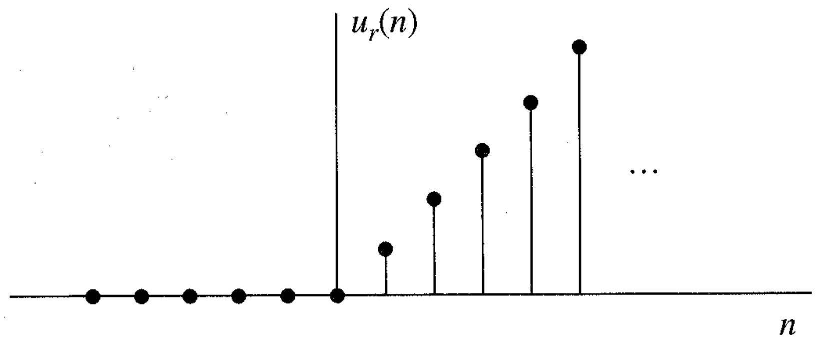
$$\delta(n) \equiv \begin{cases} 1, & \text{for } n = 0 \\ 0, & \text{for } n \neq 0 \end{cases} \qquad (2.1.6)$$



❖ **Unit step and ramp signals**

$$u(n) \equiv \begin{cases} 1, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases} \qquad u_r(n) \equiv \begin{cases} n, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases} \qquad (2.1.8)$$





Discrete-Time signals

Elementary DT signals

- **Exponential signal:** $x(n) = a^n \quad \text{for } all\ n \qquad (2.1.9)$
- **Complex exponential**

$$x(n) = r^n e^{j\Theta n} = r^n(\cos\Theta n + j\sin\Theta n) \qquad (2.1.10)$$

$$x_R \equiv r^n \cos\Theta n \qquad x_I(n) \equiv r^n \sin\Theta n \qquad (2.1.12)$$

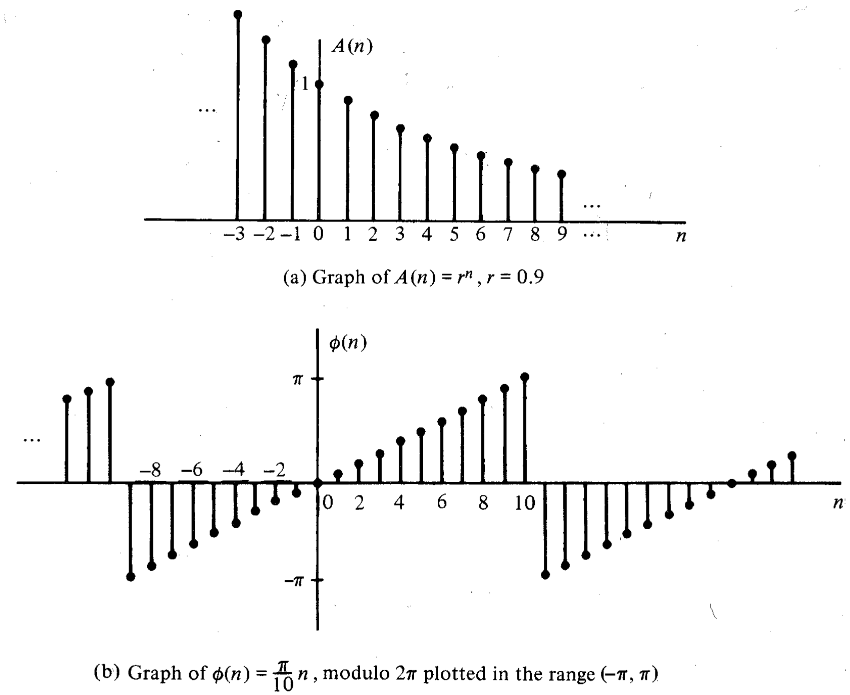


Figure 2.7 Graph of amplitude and phase function of a complex-valued exponential signal: (a) graph of $A(n) = r^n$, $4 = 0.9$; (b) graph of $\phi(n) = (\pi/10)n$, modulo 2π plotted in the range $(-\pi, \pi]$.

$$x(n) = A(n)\exp(j\Theta n)$$

$$|x(n)| = A(n) \equiv r^n$$

$$\angle x(n) = \Phi(n) \equiv \Theta n$$

Note: Phase is defined over the interval

$$[-\pi,\ \pi] \ or\ [o,\ 2\pi]$$

Discrete-Time signals

Elementary DT signals

❖ DT sinusoidal signals

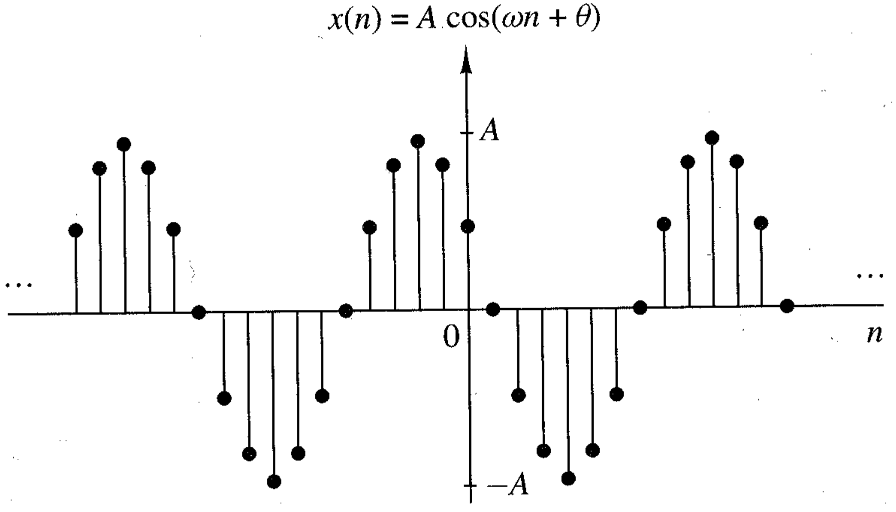
$$x(n) = A\cos(\Theta n + \phi), \quad -\infty < n < +\infty$$

$\Theta = 2\pi \underline{f}$: radians / sample ; cycles / sample

$\underline{f}$: $(DIGITAL\ \ frequencies)$

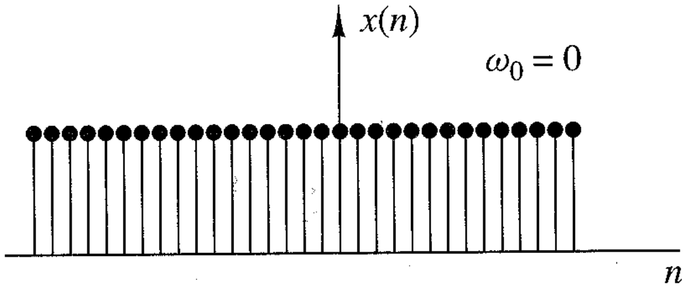
Example) $\Theta = \pi/6 = 2\pi/12 \qquad \phi = \pi/3 = 2\pi/6 \qquad \underline{f} = 1/12$

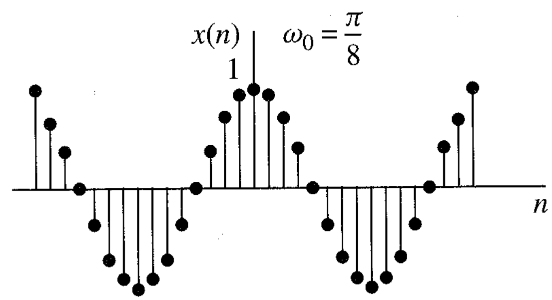
$\Rightarrow$ 12 samples / period, advanced by 2 samples

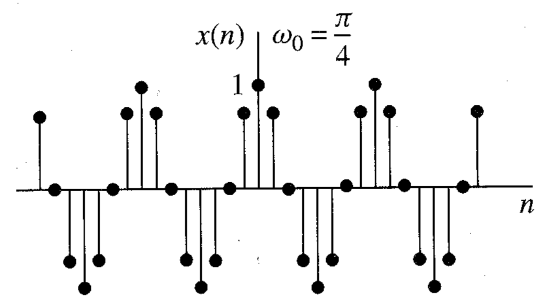


Discrete-Time signals

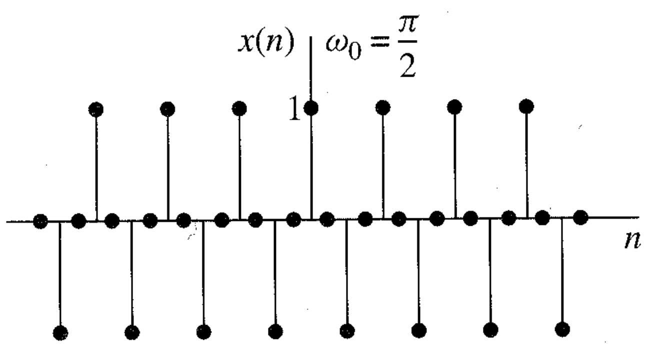
Example) $x(n)\cos(\Theta n) \quad \Theta = \omega_0$

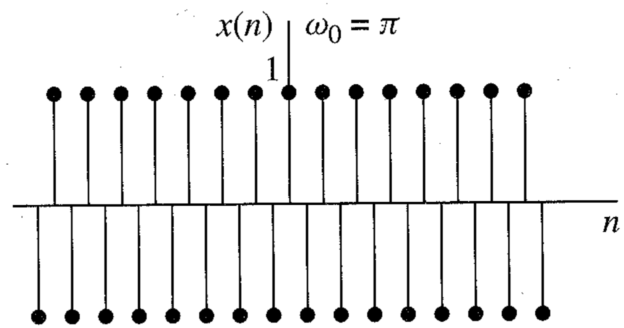






f :





f :

Discrete-Time signals

Fundamental frequency range

$$\boxed{-\frac{1}{2} < f < \frac{1}{2}, \quad -\pi < \theta < \pi} \qquad -\infty < f < \infty, \quad -\infty < \omega < \infty$$

$$-\frac{1}{2T} = -\frac{f_s}{2} \le f \le \frac{f_s}{2} = \frac{1}{2T} \qquad -\frac{\pi}{T} = -\pi f_s \le \omega = 2\pi f \le \pi f_s = \frac{\pi}{T}$$

Highest frequency signal that can be sampled without ambiguity

$$f_{max} = \frac{f_s}{2} = \frac{1}{2T} \qquad (1.4.11)$$

$$\omega_{max} = \pi f_s = \frac{\pi}{T}$$

Folding frequency : $f_N = f_s/2$

Discrete-Time signals

Aliasing in sampling process

$$x_a(t) = A\cos(2\pi f_0 t + \Theta) \qquad (1.4.14)$$

$$f_0 = f_\Delta + f_s/2, \quad 0 < f_\Delta < f_s/2 \Rightarrow \pi < \omega_0 < 2\pi$$

$$x(n) = A\cos(2\pi f_0 nT + \phi) = A\cos(2\pi n f_0/f_s + \phi) = A\cos(2\pi \underline{f_0} n + \phi)$$

$$= A\cos(2\pi n(f_\Delta + f_s/2)/f_s + \phi) = A\cos(2\pi(\underline{f_\Delta} + 1/2)n + \phi)$$

$$= A\cos(2\pi(\underline{f_\Delta} - 1/2)n + \phi)$$

$$\begin{aligned} &2\pi \underline{f_\Delta} n + \phi + \pi n \\ &= 2\pi \underline{f_\Delta} n + \phi + \pi n - 2\pi n \\ &= 2\pi \underline{f_\Delta} n + \phi - \pi n \end{aligned}$$

where $\underline{f_0} = f_0/f_s, \quad \underline{f_\Delta} = f_\Delta/f_s$

$$\boxed{\frac{f_s}{2} + f_\Delta = -\frac{f_s}{2} + f_\Delta}$$

Discrete-Time signals

Aliasing in sampling process

$$x(n) \equiv x_a(nT) = A\cos\left(2\pi \frac{f_0 + k f_s}{f_s} n + \phi\right)$$

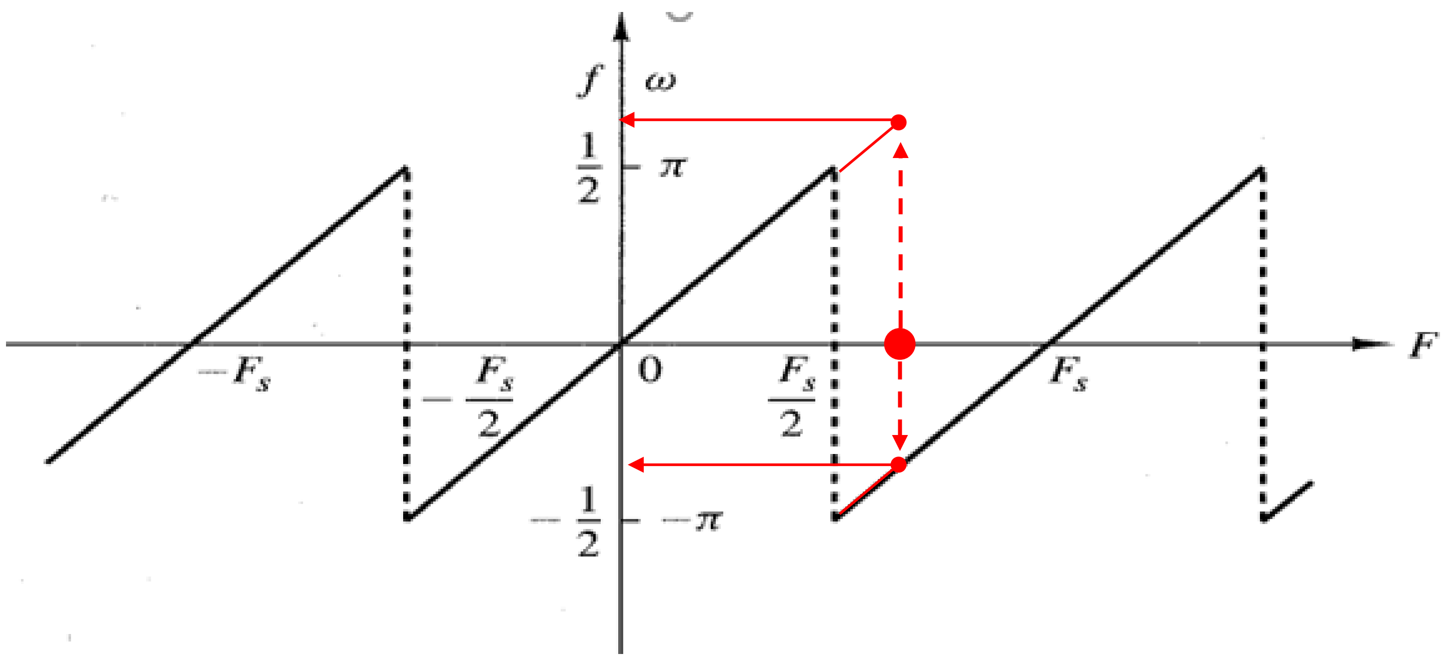
$$= A\cos(2\pi n\, f_0 / f_s + \phi + 2\pi kn)$$

$$= A\cos(2\pi\, \underline{f_0}\, n + \phi)$$

$$k f_s + f_\Delta \Rightarrow \underline{f_\Delta}$$

$$\frac{f_s}{2} + f_\Delta \Rightarrow -\frac{1}{2} + \underline{f_\Delta}$$

$$-\frac{f_s}{2} - f_\Delta \Rightarrow \frac{1}{2} - \underline{f_\Delta}$$



Discrete-Time signals

Classification of DT signals

- **Energy signals**

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty$$

- **Power signals:**
  - Average power of a signal:

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 \qquad (2.1.16)$$

  - Signal energy over the finite interval $-N \leq n \leq N$

$$E_N = \sum_{n=-N}^{N} |x(n)|^2 \qquad (2.1.17)$$

$$\boxed{E = \lim_{N \to \infty} E_N} \qquad \boxed{P = \lim_{N \to \infty} \frac{1}{2N+1} E_N}$$

  - If $E < \infty$, then $P = 0$
  - If E is infinite, P is either finite or infinite
  - •If $P < \infty$, it is a power signal.

Discrete-Time signals

Classification of DT signals

- **Periodic signals**

$$x(n+N)=x(n) \quad \text{for} \quad all \ n \qquad (2.1.20)$$

- **Power of periodic signals -- periodic signals are power signals.**

$$P=\frac{1}{N}\sum_{n=0}^{N-1}|x(n)|^{2}<\infty$$

- **Symmetric and anti-symmetric signals**

$$x(-n)=x(n) \qquad\qquad x(-n)=-x(n)$$

- **Any signal can be expressed as the sum of even and odd signal components**

$$x(n)=x_{e}(n)+x_{0}(n) \qquad (2.1.28)$$

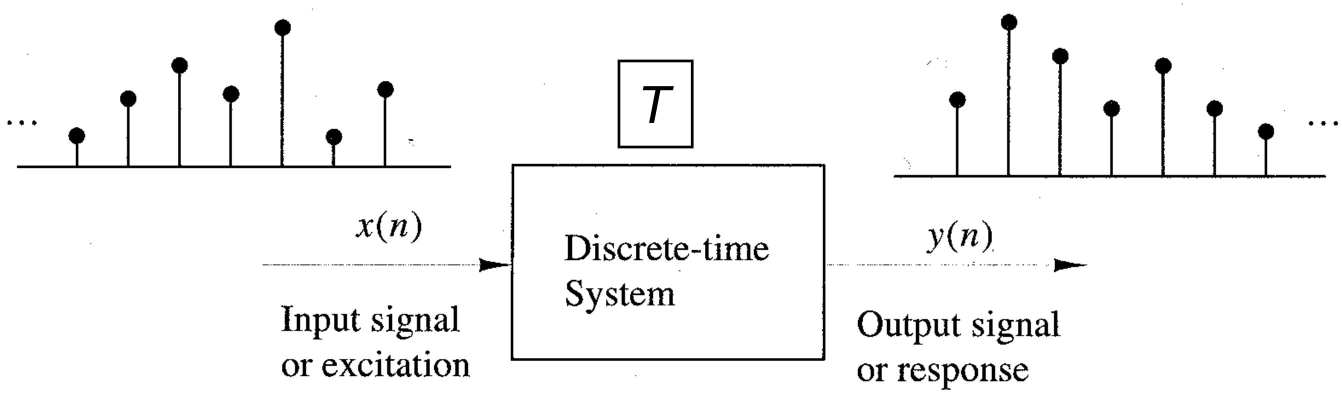
$$x_{e}(n)=\frac{1}{2}[x(n)+x(-n)] \qquad x_{o}=\frac{1}{2}[x(n)-x(-n)]$$

Discrete-Time Systems

Definition

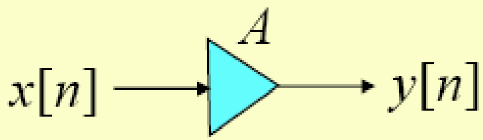
A DT system is a device or algorithm that operates on a DT signal, called the input or excitation, according to some well-defined rule, to produce the other DT signal called the output or the response of the system.

$$y(n) \equiv \mathrm{T}[x(n)]$$



Discrete-Time Systems

Multiplication operation → multiplier



$$y[n] = A \cdot x[n]$$

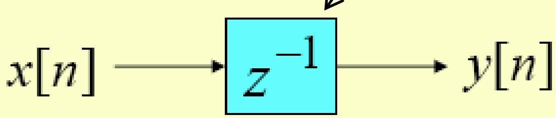
Time-shifting

$$y[n] = x[n - N]$$

- ❖ **Delaying if N > 0**
- ❖ **Advance if N < 0**

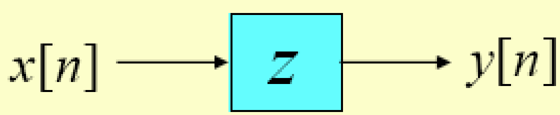
Memory : register

Unit delay



$$y[n] = x[n-1]$$

Unit advance



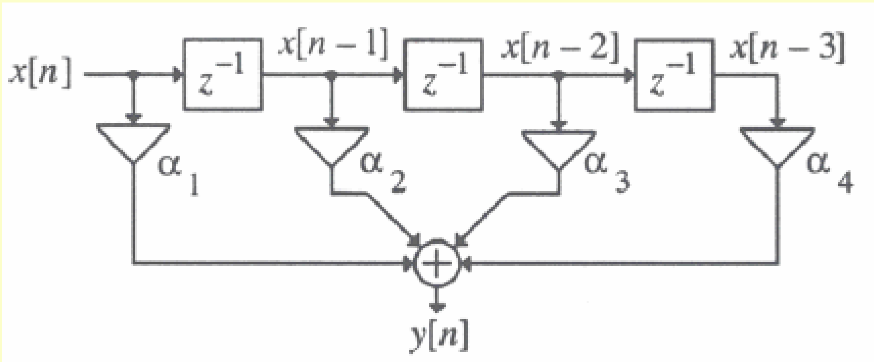
$$y[n] = x[n+1]$$

Time reversal (folding)

$$y[n] = x[-n]$$

Discrete-Time Systems

DT system: Example



$$y[n] = \alpha_1 x[n] + \alpha_2 x[n-1] + \alpha_3 x[n-2] + \alpha_4 x[n-3]$$

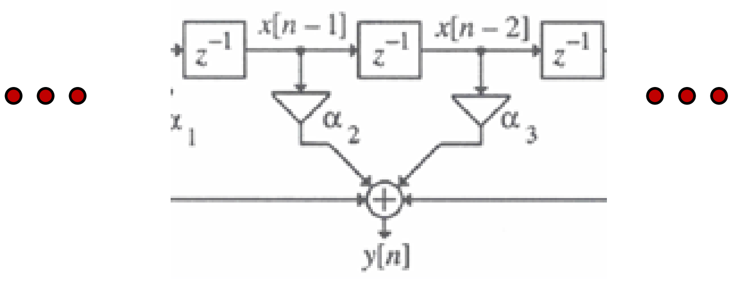
We can design any system using only these basic operational elements.

Discrete-Time Systems

❖ Example: Accumulator

$$y(n) = \sum_{k=-\infty}^{n} x(k) \qquad\qquad y(n) = \sum_{k=0}^{n} x(k)$$

How many memories and adders are needed to implement the accumulator using the architecture below?



Practical approach

$$y(n) = \sum_{k=-\infty}^{n} x(k) = \sum_{k=-\infty}^{n-1} x(k) + x(n)$$

$$= y(n-1) + x(n)$$

Discrete-Time Systems

❖ **Example: Accumulator**

Can be calculated iteratively as

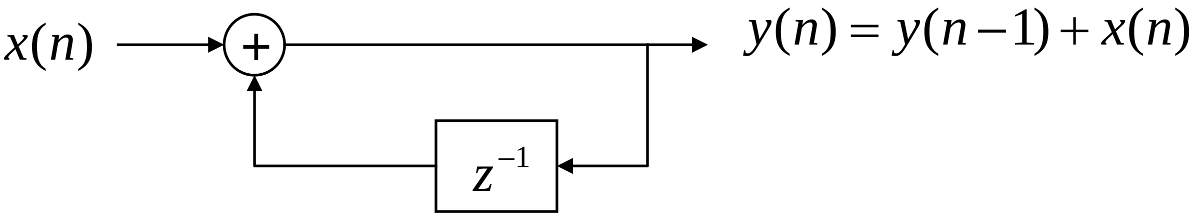
$$y(n_0) = y(n_0 - 1) + x(n_0) \qquad y(0) = y(-1) + x(0)$$

$$y(n_0 + 1) = y(n_0) + x(n_0 + 1)$$

$n \geq n_0$ **= 0 (*typically*)**

We need to know the initial condition $y(n_0)$.

If $y(n_0) = 0$, the system is said to be initially relaxed.



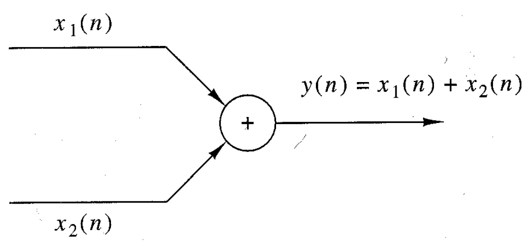
Discrete-Time Systems

DT LTI system: n-th order system

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

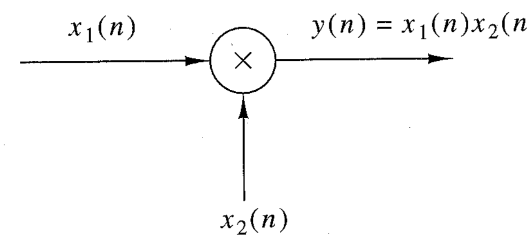
Block Diagram

adder

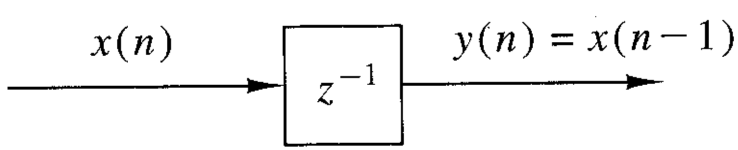


Constant multiplier

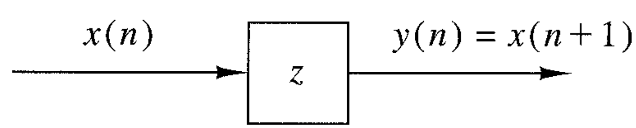
Signal multiplier



Unit delay element

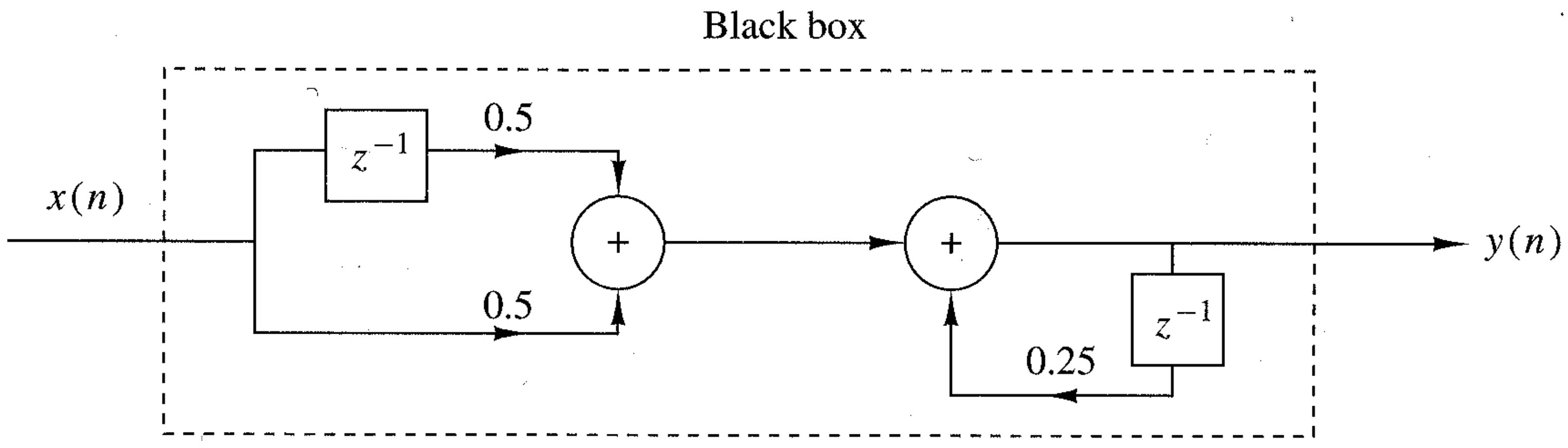


Unit advance element



Discrete-Time Systems

Example) $$y(n) = \frac{1}{4}x(n-1) + \frac{1}{2}x(n) + \frac{1}{2}y(n-1)$$



(a)

Discrete-Time Systems

Classification of DT systems

- **Static or memoryless system if it requires no delay or memory devices.**

  That is, the output is determined by the input sample at the same time.

- **Dynamic system is one that is not static.**

- **Dynamic system has memory of duration N if the output is completely determined by the input samples in the interval from $n-N$ to n ($N \geq 0$)**

- **Finite memory system vs. infinite memory system ($N = \infty$)**

Discrete-Time Systems

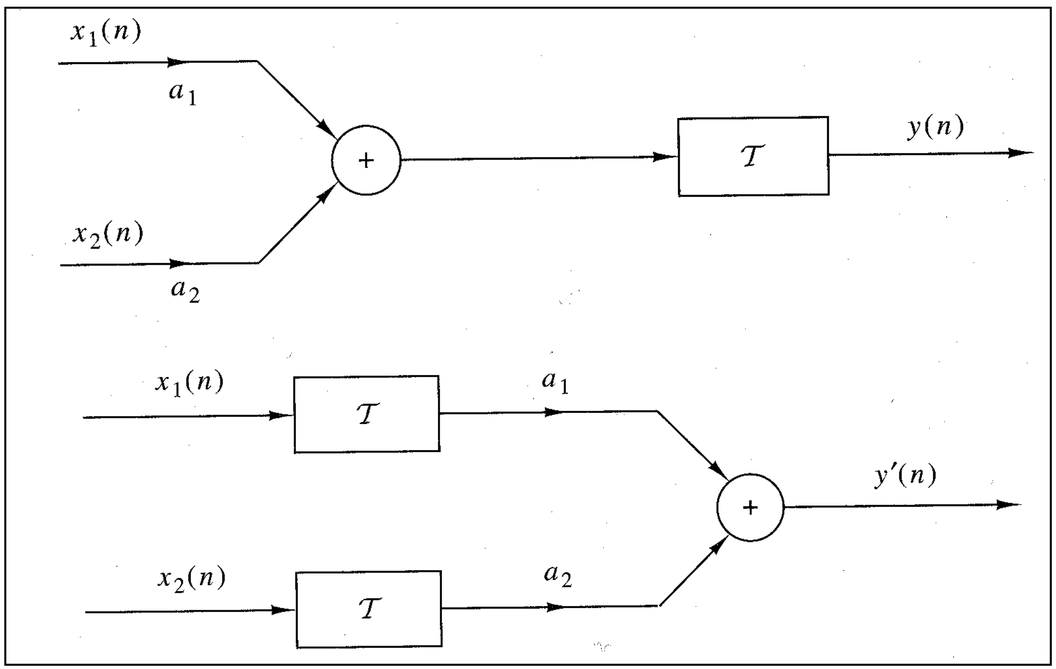
LTI system

❖ **Time Invariant : A relaxed system T is time invariant or shift invariant if and only if**

$$x(n) \xrightarrow{T} y(n) \Rightarrow x(n-k) \xrightarrow{T} y(n-k)$$

❖ **Linear system: A system is linear if and only if**

$$T[a_1 x_1(n) + a_2 x_2(n)] = a_1 T[x_1(n)] + a_2 T[x_2(n)]$$



Discrete-Time Systems

LTI system

❖ Relaxed, linear system produces a zero output for a zero input.

$$\mathrm{T}[a_1 x_1(n)] = a_1 \mathrm{T}[x_1(n)] = a_1 y_1(n)$$

$$x_1(n) = 0 \rightarrow y(n) = 0$$

❖ A system producing nonzero output with a zero input,

- the system may be either non-relaxed or nonlinear

❖ Nonlinear system

A relaxed system that does not satisfy the superposition principle.

Discrete-Time Systems

Causal vs. Noncausal systems

- **Causal, if** $y(n) \propto x(k), k \leqslant n$
- **Noncausal, if not**

Stable vs. Unstable systems

- **An arbitrary relaxed system is said to be bounded input-bounded output (BIBO) stable if and only if every bounded input produces a bounded output.**
- $|x(n)| \leq M_y < \infty \qquad |y(n)| \leq M_y < \infty \quad for\ all\ n$

Discrete-Time Systems

Analysis of DT systems

- **Direct solution of the Input-Output equation for a linear system, generally having**

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

- **Difference equation for the $N-th$ order system**

where a_k, b_k : constant parameters independent of $x(n)$, $y(n)$

- **Indirect method : Use z-transform**

$$\frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

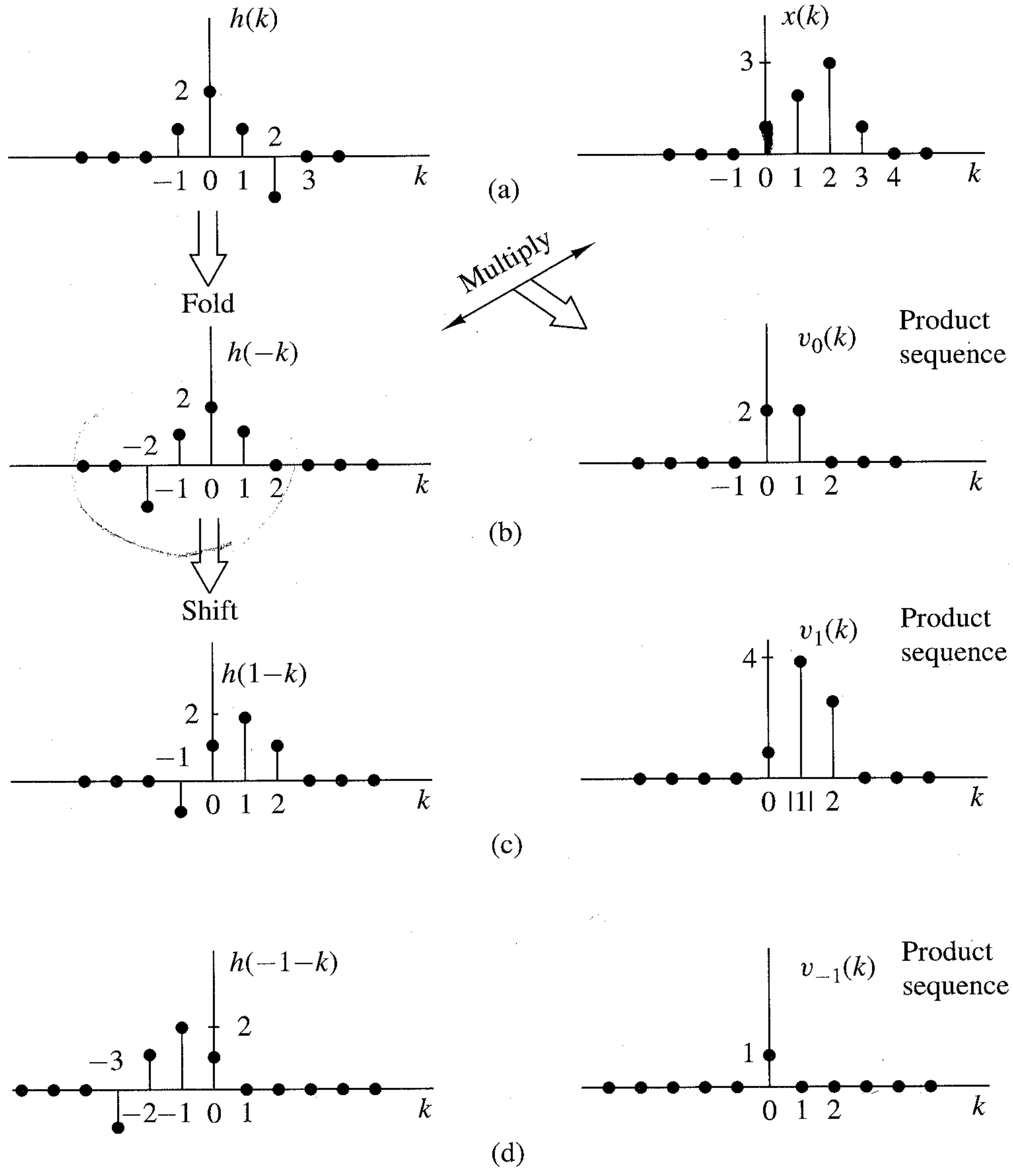
Discrete-Time Systems

Analysis of DT systems

- ❖ **Impulse response approach: Convolution Sum**

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

$$y(n_0) = \sum_{k=-\infty}^{\infty} x(k)h(n_0-k)$$



Discrete-Time Systems

Properties of LTI system

❖ **Commutative law**

$$y(n)=x(n)*h(n)\equiv\sum_{k=-\infty}^{\infty}x(k)h(n-k)$$

$$y(n)=h(n)*x(n)\equiv\sum_{k=-\infty}^{\infty}h(k)x(n-k)$$

$$\Leftrightarrow\ x(n)*h(n)=h(n)*x(n)$$

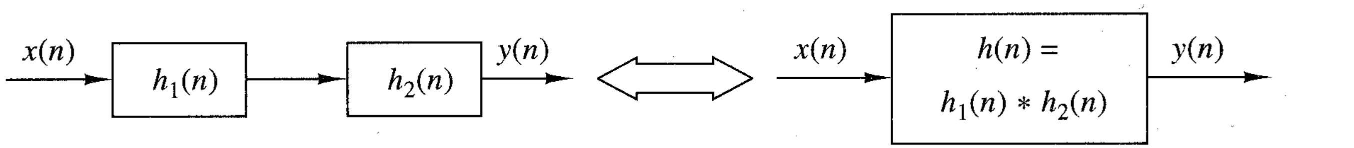
❖ **Associative law**

$$[x(n)*h_1(n)]*h_2(n)=x(n)*[h_1(n)*h_2(n)]$$

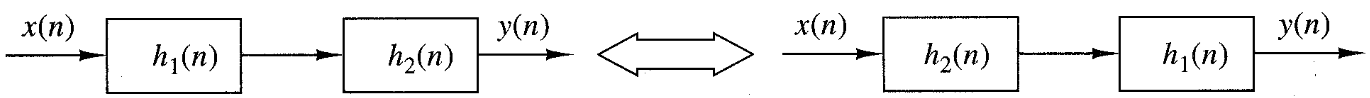
$$y(n)=y_1(n)*h_2(n)=[x(n)*h_1(n)]*h_2(n)$$

$$y(n)=h(n)*x(n),\quad h(n)=h_1(n)*h_2(n)$$

$$h(n)=h_1(n)*h_2(n)=h_2(n)*h_1(n)$$



(a)



(b)

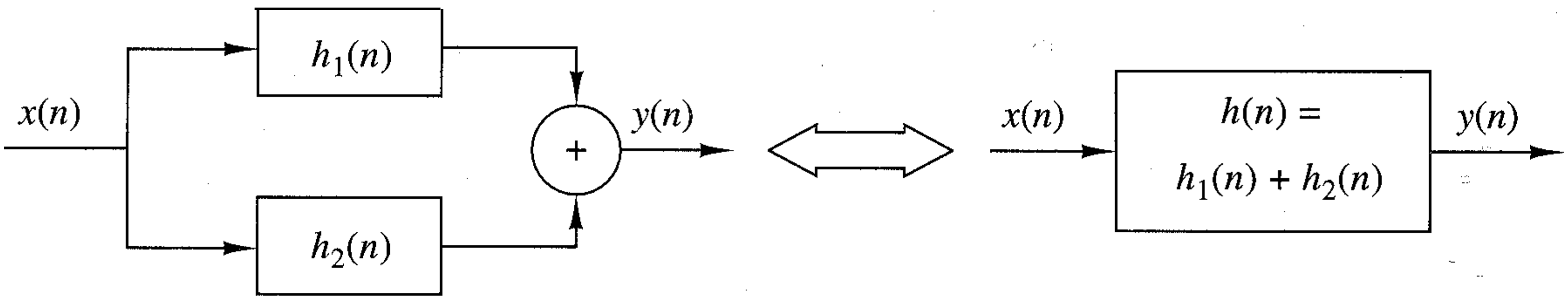
Discrete-Time Systems

Properties of LTI system

❖ Distributive law

$$x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$$

$$\Rightarrow \quad h(n) = h_1(n) + h_2(n)$$



Discrete-Time Systems

Causal LTI system

- Condition for a LTI system to be causal

$$y(n_o) = \sum_{k=-\infty}^{\infty} h(k)x(n_o - k) = \underbrace{\sum_{k=0}^{\infty} h(k)x(n_o - k)}_{\text{present and past,}} + \underbrace{\sum_{k=-\infty}^{-1} h(k)x(n_o - k)}_{\text{future}} \quad \text{SAMPLES}$$

$$= \sum_{k=0}^{\infty} h(k)x(n_o - k) \quad \text{if} \quad h(n) = 0, -\infty \le n \le -1$$

Theorem) An LTI system is causal iff its impulse response, h(n), is zero for n < 0.

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n - k) = \sum_{k=-\infty}^{n} x(k)h(n - k)$$

- Convolution formula for causal input
  - x(n) is called a causal sequence if x(n) = 0 for n < 0.
  - LTI system with a causal input

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n - k) = \sum_{k=0}^{n} h(k)x(n - k)$$

Discrete-Time Systems

Causal LTI system

- Stability of LTI systems

  Theorem) A LTI system is (BIBO) stable if its impulse response is absolutely summable .

  Proof) $y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$

  $$|\,y(n)\,| = \left|\sum_{k=-\infty}^{\infty} h(k)x(n-k)\right| \le \sum_{k=-\infty}^{\infty} |h(k)||x(n-k)| \le M_x \sum_{k=-\infty}^{\infty} |h(k)|$$

  If $\sum_k |h(k)| \le C < \infty$, then $|y(n)| \le M_y < \infty$. Q.E.D.

$$S_h \equiv \sum_{k=-\infty}^{\infty} |h(k)| < \infty \quad \textit{implies} \quad h(n) \to 0 \quad as \;\; n \to \infty$$

$\Rightarrow$ *Any excitation at the input to the system, which is of finite duration, produces an output that is "transient" in nature; that is, its amplitude decays with time and dies out eventually, when the system is stable.*

Discrete-Time Systems

Analysis of DT systems

❖ **Example:** $y(n) = ay(n-1) + x(n)$

$$y(0) = ay(-1) + x(0)$$
$$y(1) = ay(0) + x(1) = a^2y(-1) + ax(0) + x(1)$$
$$y(2) = ay(1) + x(2) = a^3y(-1) + a^2x(0) + ax(1) + x(2)$$
$$\vdots$$
$$y(n) = a^{n+1}y(-1) + \sum_{k=0}^{n} a^k x(n-k), \quad n \geqq 0$$

Discrete-Time Systems

Analysis of DT systems

❖ **Example:** $y(n) = ay(n-1) + x(n) \qquad x(n) = Ae^{j\Theta n}u(n)$

$$y(n) = a^{n+1}y(-1) + \sum_{k=0}^{n} a^k x(n-k) \qquad n \geq 0$$

$$y(n) = a^{n+1}y(-1) + A\sum_{k=0}^{n} a^k e^{j\Theta(n-k)} = a^{n+1}y(-1) + A\left[\sum_{k=0}^{n}(ae^{-j\Theta})^k\right]e^{j\Theta n}$$

$$= a^{n+1}y(-1) + A\frac{1-a^{n+1}e^{-j\Theta(n+1)}}{1-ae^{-j\Theta}}e^{j\Theta n} \qquad n \geq 0$$

$$= \underbrace{a^{n+1}y(-1) + \frac{Aa^{n+1}e^{-j\Theta(n+1)}}{1-ae^{-j\Theta}}e^{j\Theta n}}_{\text{Transient Response}} + \underbrace{\frac{A}{1-ae^{-j\Theta}}e^{j\Theta n}}_{\text{Stable Response}} \qquad n \geq 0$$

Discrete-Time Systems

Analysis of DT systems

❖ **Example:** $y(n) = ay(n-1) + x(n)$ $\quad x(n) = Ae^{j\Theta n}u(n)$

$$y_{ss}(n) = \lim_{n\to\infty} y(n) = \frac{A}{1-ae^{-j\Theta}} e^{j\Theta n} = AH(\Theta)e^{j\Theta n}$$

$$y_{tr} = \underbrace{a^{n+1}y(-1)}_{\text{Zero-input response}} + \underbrace{\frac{Aa^{n+1}e^{-j\Theta(n+1)}}{1-ae^{-j\Theta}} e^{j\Theta n}}_{\text{Transient response by the input exponential}} \qquad n \geq 0$$

$$\to 0 \;\; as \;\; n \to \infty$$

Discrete-Time Systems

FIR and IIR systems

- FIR system: A system having a finite-duration impulse response.
- IIR system: A system having a infinite-duration impulse response.
- Causal LTI FIR system: $h(n) = 0$, n < 0 and n ≥ M.

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$$ → Needs a finite memory of length M samples.

- Causal LTI IIR system: $y(n) = \sum_{k=0}^{\infty} h(k)x(n-k)$

Question) How to implement IIR systems ?

They have infinite-duration impulse responses.

Needs infinite-length memory and multipliers ?

→ Recursive system