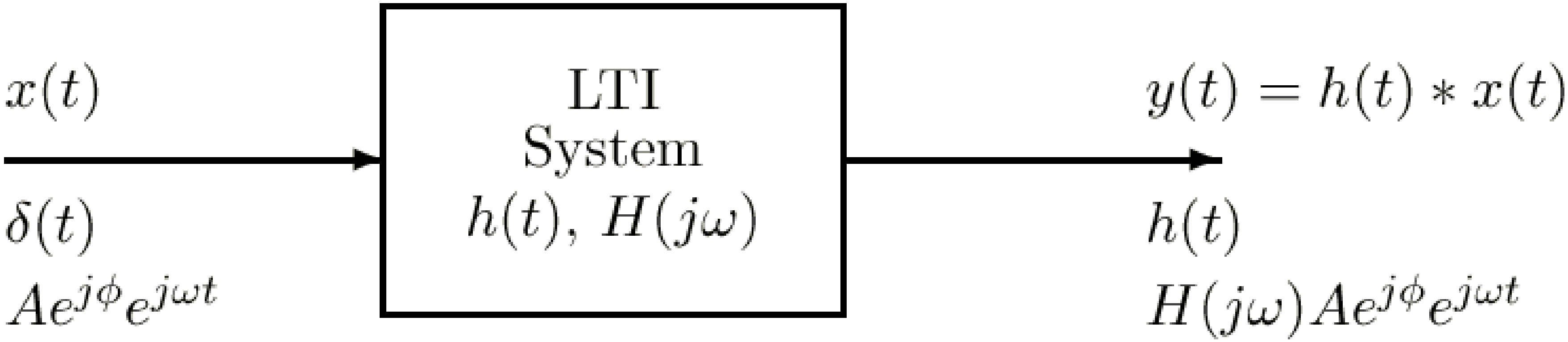
Chapter 10
Frequency Response

LTI Systems



- An impulse response characterizes an LTI system

$$\boxed{y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau}$$

- Response to a complex exponential gives frequency response $H(j\omega)$.

Why Frequency Response?

- SUPERPOSITION (Linearity)
  - Make $x(t)$ a weighted sum of signals.
  - Then $y(t)$ is also a sum of the corresponding outputs with the same weights.
- Use SINUSOIDS.
  - Make $x(t)$ a weighted sum of sinusoids.
    - **Fourier Analysis**
      - **Any $x(t)$ = weighted sum of sinusoids**
  - Then $y(t)$ is also a sum of sinusoids with different magnitudes and phases.
- LTI SYSTEMS
  - Frequency Response $H(j\omega)$ changes each sinusoidal component.

Complex Exponential Input

$$x(t) = Ae^{j\varphi}e^{j\omega t} \mapsto y(t) = \boxed{H(j\omega)}Ae^{j\varphi}e^{j\omega t}$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)Ae^{j\varphi}e^{j\omega(t-\tau)}d\tau$$

$$y(t) = \left(\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau\right)Ae^{j\varphi}e^{j\omega t}$$

$$\boxed{H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau}$$

Frequency Response

When does $H(j\omega)$ Exist?

- When is $|H(j\omega)| < \infty$?

$$|H(j\omega)| = \left| \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau \right| \le \int_{-\infty}^{\infty} |h(\tau)| \left| e^{-j\omega\tau} \right| d\tau$$

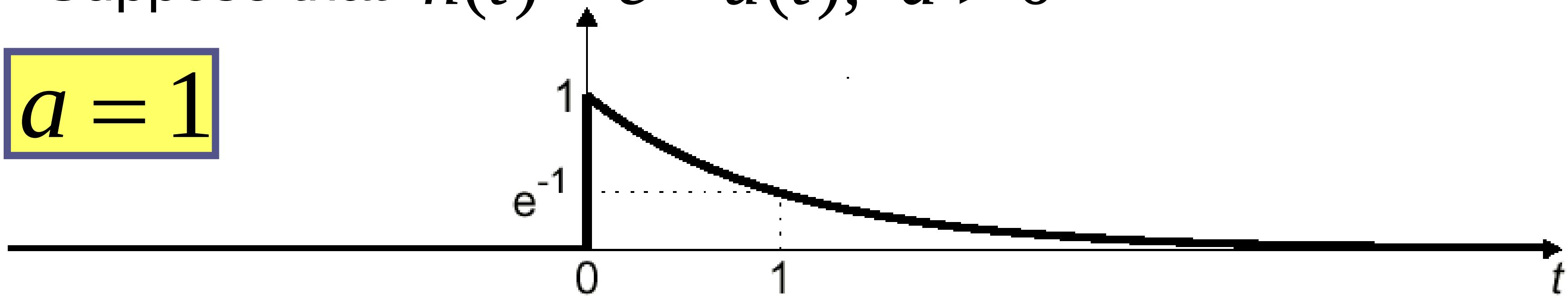
Sufficient condition $\boxed{|H(j\omega)| \le \int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty}$

- Thus the frequency response exists if the LTI system is a **stable** system.

Example: Frequency Response

- Suppose that $h(t) = e^{-at}u(t),\ a > 0$

$\boxed{a = 1}$



$$H(j\omega) = \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)e^{-j\omega\tau}d\tau = \int_{0}^{\infty} e^{-(a+j\omega)\tau}d\tau$$

$\boxed{a > 0}$

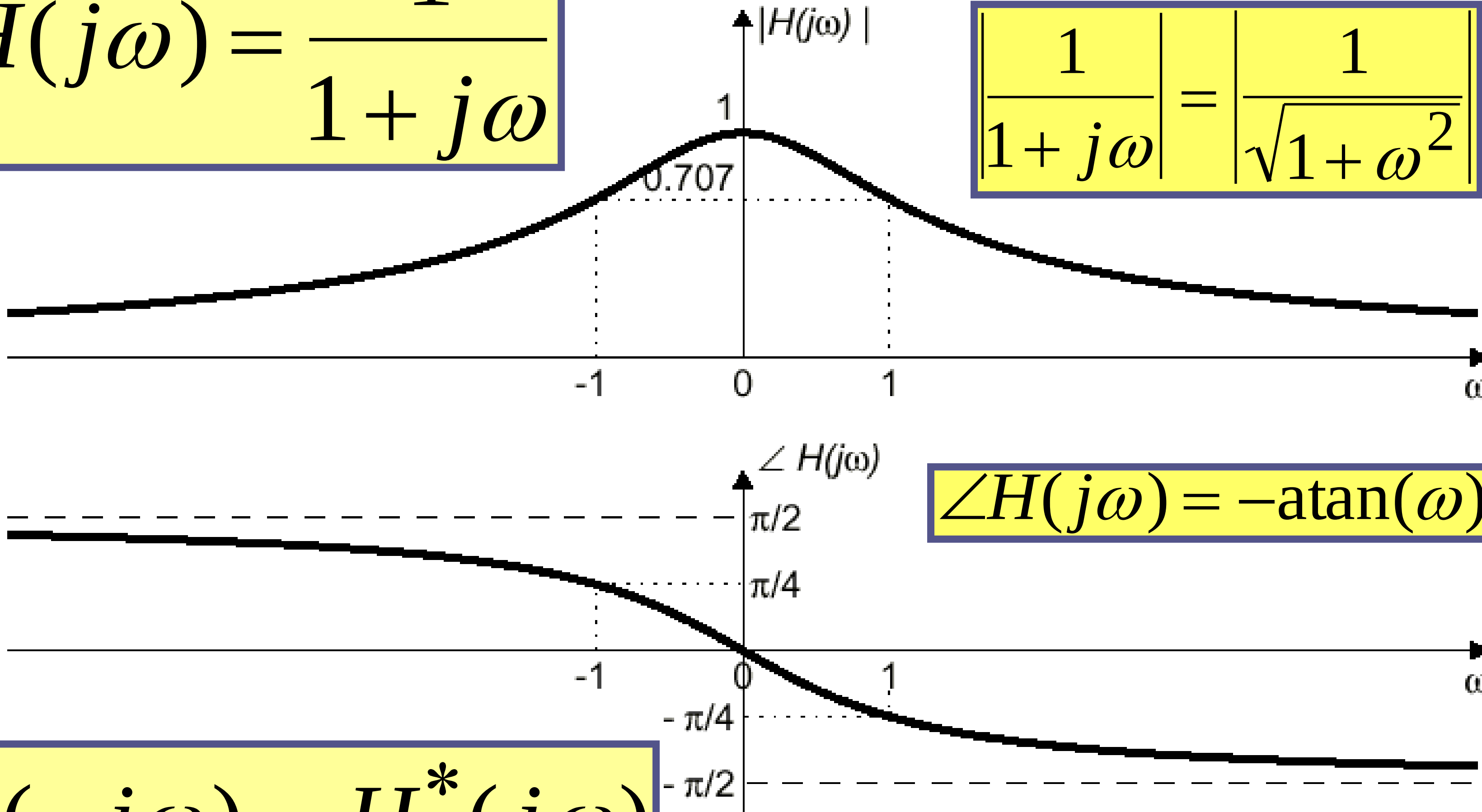
$$H(j\omega) = \left.\frac{e^{-(a+j\omega)\tau}}{-(a+j\omega)}\right|_0^{\infty} = \left.\frac{e^{-a\tau}e^{-j\omega\tau}}{-(a+j\omega)}\right|_0^{\infty} = \frac{1}{a+j\omega}$$

$$\boxed{h(t) = e^{-at}u(t) \Leftrightarrow H(j\omega) = \frac{1}{a+j\omega}}$$

Example: Magnitude and Phase Plots

$$H(j\omega) = \frac{1}{1 + j\omega}$$

$$\left|\frac{1}{1+j\omega}\right| = \left|\frac{1}{\sqrt{1+\omega^2}}\right|$$



$$\angle H(j\omega) = -\mathrm{atan}(\omega)$$

$$H(-j\omega) = H^*(j\omega)$$

Frequency Response of an Integrator

- Impulse Response
  - $h(t) = u(t)$
- NOT a Stable System
  - Frequency response $H(j\omega)$ does NOT exist.

$$h(t) = e^{-at}u(t) \Leftrightarrow H(j\omega) = \frac{1}{a + j\omega} \to \frac{1}{j\omega}\ ?$$

Need another term.

"Leaky" Integrator (*a* is small.)
Cannot build a perfect Integral.

$$\boxed{a \to 0}$$

Ideal Delay:

$$\boxed{y(t) = x(t - t_d)}$$

$$H(j\omega) = \int_{-\infty}^{\infty} \delta(\tau - t_d) e^{-j\omega\tau} d\tau = e^{-j\omega t_d}$$

$$\boxed{H(j\omega) = e^{-j\omega t_d}}$$ linear phase

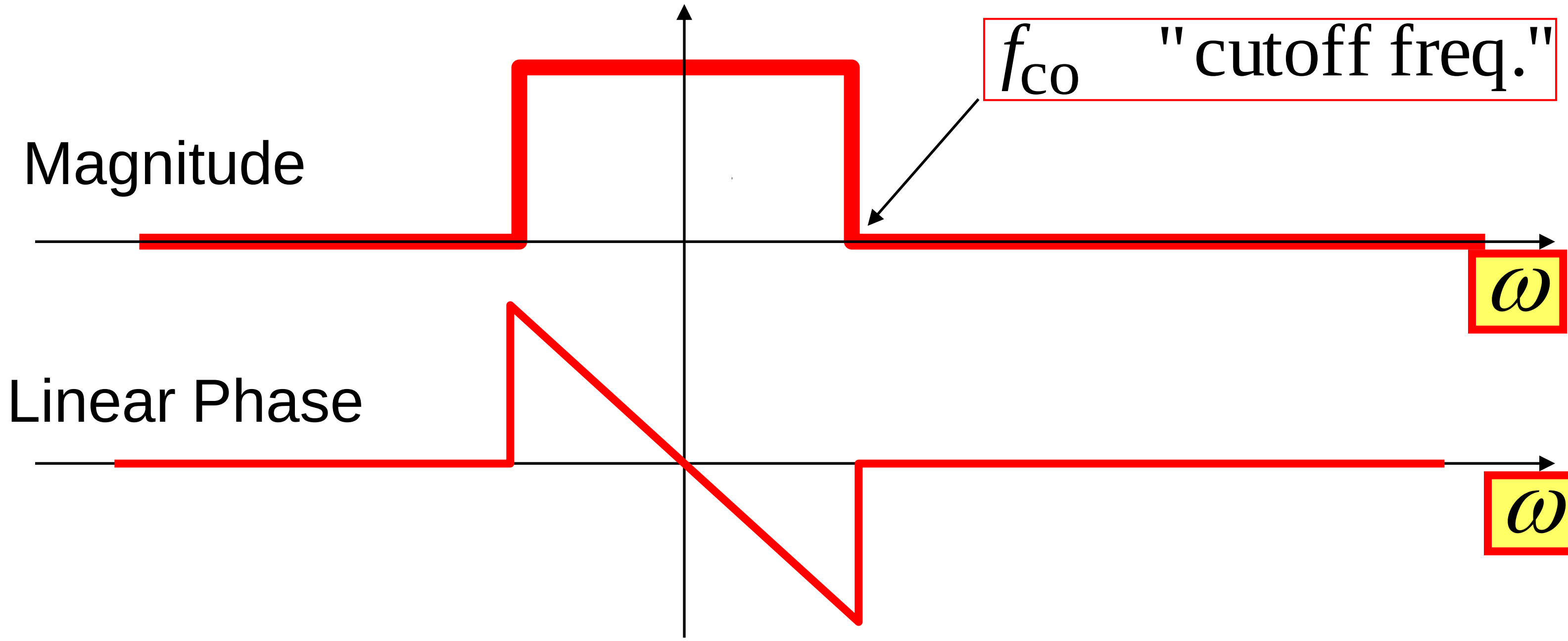
$$x(t) = e^{j\omega t} \mapsto$$

$$\boxed{H(j\omega)}$$

$$y(t) = e^{j\omega(t - t_d)} = \left(e^{-j\omega t_d}\right) e^{j\omega t}$$

Ideal Lowpass Filter with Delay

$$H_{LP}(j\omega) = \begin{cases} e^{-j\omega t_d} & |\omega| < \omega_{co} \\ 0 & |\omega| > \omega_{co} \end{cases}$$



Example: Ideal Lowpass Filter with Delay

$$H_{LP}(j\omega) = \begin{cases} e^{-j3\omega} & |\omega| < 2 \\ 0 & |\omega| > 2 \end{cases}$$

$$x(t) = 10e^{j\pi/3}e^{j1.5t} \mapsto \quad y(t) = H(j1.5)10e^{j\pi/3}e^{j1.5t}$$

$$y(t) = \left(e^{-j4.5}\right)10e^{j\pi/3}e^{j1.5t} = 10e^{j\pi/3}e^{j1.5(t-3)}$$

Symmetry of $H(j\omega)$

- When $h(t)$ is real-valued,

$$H^*(j\omega) = \left(\int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt \right)^*$$

$$= \int_{-\infty}^{\infty} h^*(t) e^{+j\omega t} dt$$

$$= \int_{-\infty}^{\infty} h(t) e^{-j(-\omega)t} dt = H(-j\omega)$$

Conjugate symmetry

Cosine Input

$$x(t) = A\cos(\omega_0 t + \phi) = \frac{A}{2} e^{j\phi} e^{j\omega_0 t} + \frac{A}{2} e^{-j\phi} e^{-j\omega_0 t}$$

$$y(t) = H(j\omega_0)\frac{A}{2} e^{j\phi} e^{j\omega_0 t} + H(-j\omega_0)\frac{A}{2} e^{-j\phi} e^{-j\omega_0 t}$$

Since $H(-j\omega_0) = H^*(j\omega_0)$

$$\boxed{y(t) = A\left|H(j\omega_0)\right|\cos(\omega_0 t + \phi + \angle H(j\omega_0))}$$

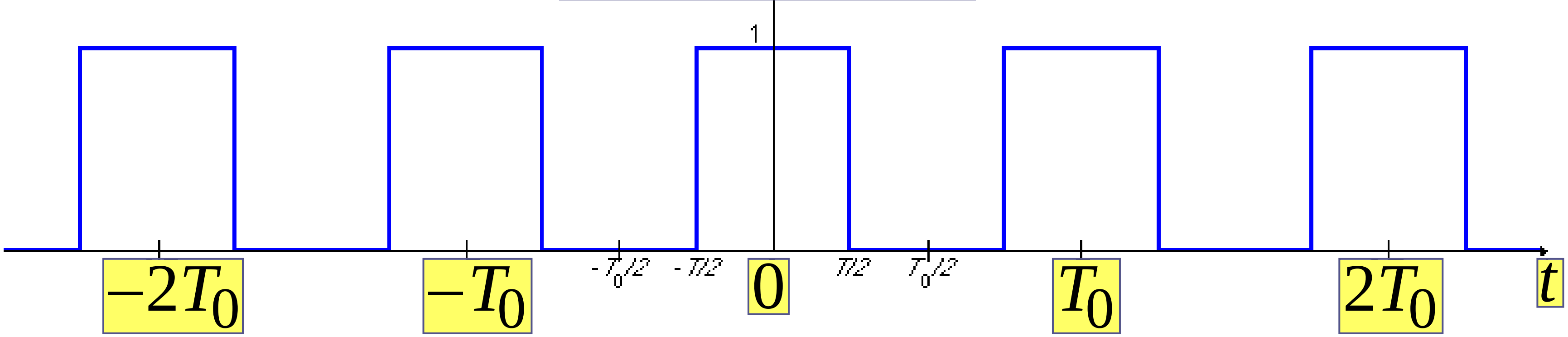
Review Fourier Series.

- ANALYSIS
  - Get representation from the signal.
  - Works for PERIODIC Signals.
- Fourier Coefficients
  - INTEGRAL over one period

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 kt}dt$$

General Periodic Signals

$$x(t) = x(t + T_0)$$



$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt}$$

Fourier Synthesis

Fundamental Freq.

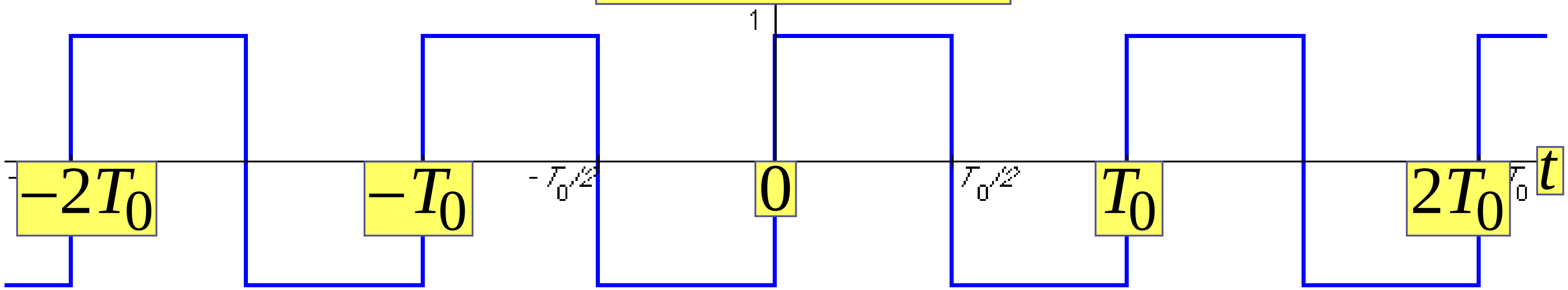
$$\omega_0 = 2\pi / T_0 = 2\pi f_0$$

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j\omega_0 kt} dt$$

Fourier Analysis

Square Wave Signal

$$x(t) = x(t + T_0)$$



$$a_k = \frac{1}{T_0}\int_0^{T_0/2} (1)e^{-j\omega_0 kt}dt + \frac{1}{T_0}\int_{T_0/2}^{T_0} (-1)e^{-j\omega_0 kt}dt$$

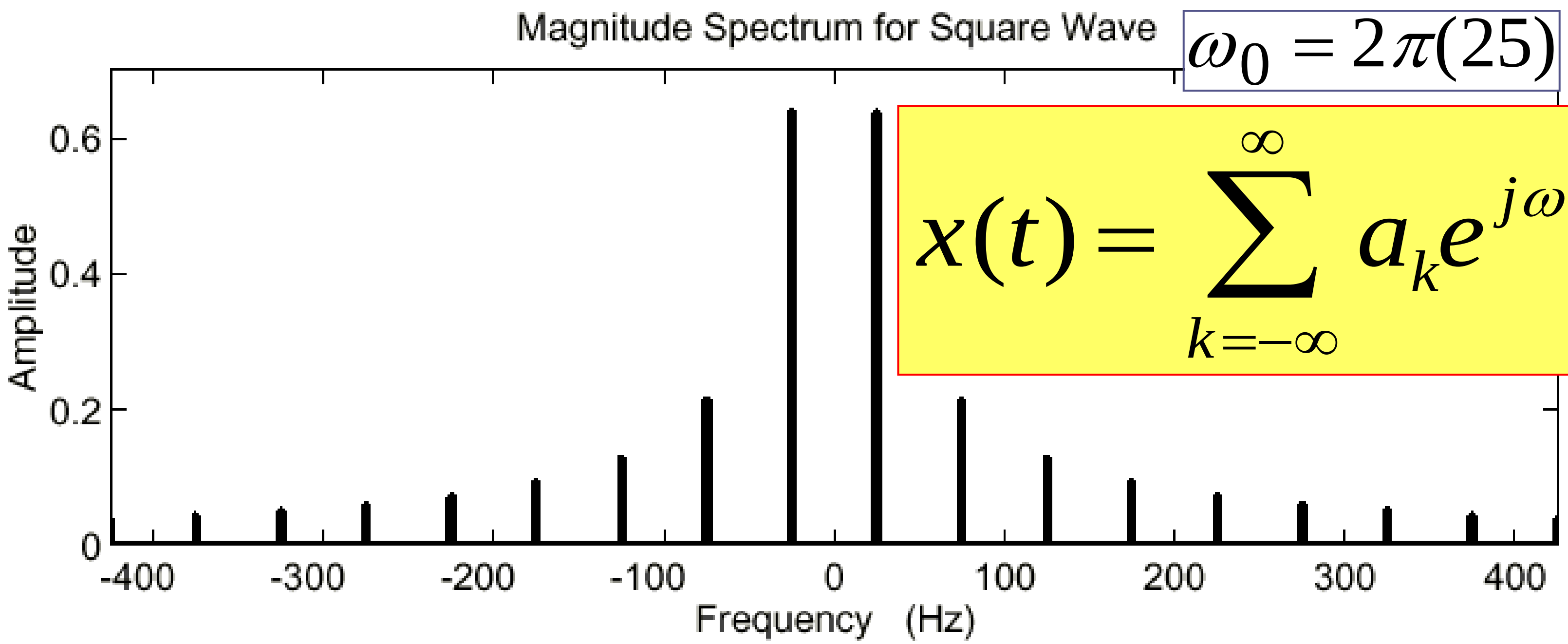
$$a_k = \left.\frac{e^{-j\omega_0 kt}}{-j\omega_0 kT_0}\right|_0^{T_0/2} - \left.\frac{e^{-j\omega_0 kt}}{-j\omega_0 kT_0}\right|_{T_0/2}^{T_0} = \frac{1-e^{-j\pi k}}{j\pi k}$$

Spectrum from Fourier Series

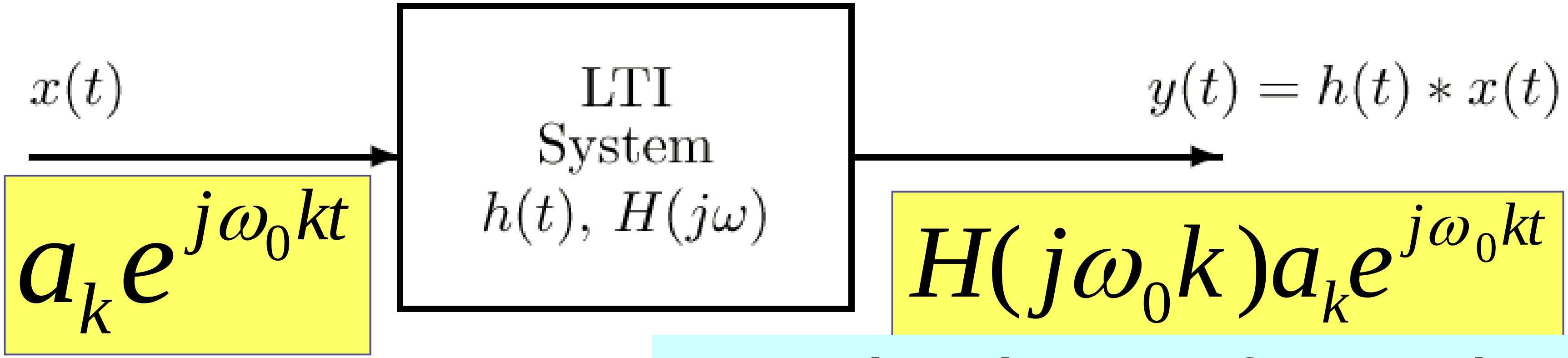
$$a_k = \frac{1 - e^{-j\pi k}}{j\pi k} = \begin{cases} \dfrac{2}{j\pi k} & k = \pm 1, \pm 3, \ldots \\ 0 & k = 0, \pm 2, \pm 4, \ldots \end{cases}$$

$$\omega_0 = 2\pi(25)$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k t}$$



LTI Systems with Periodic Inputs



Output has the *same* frequencies.

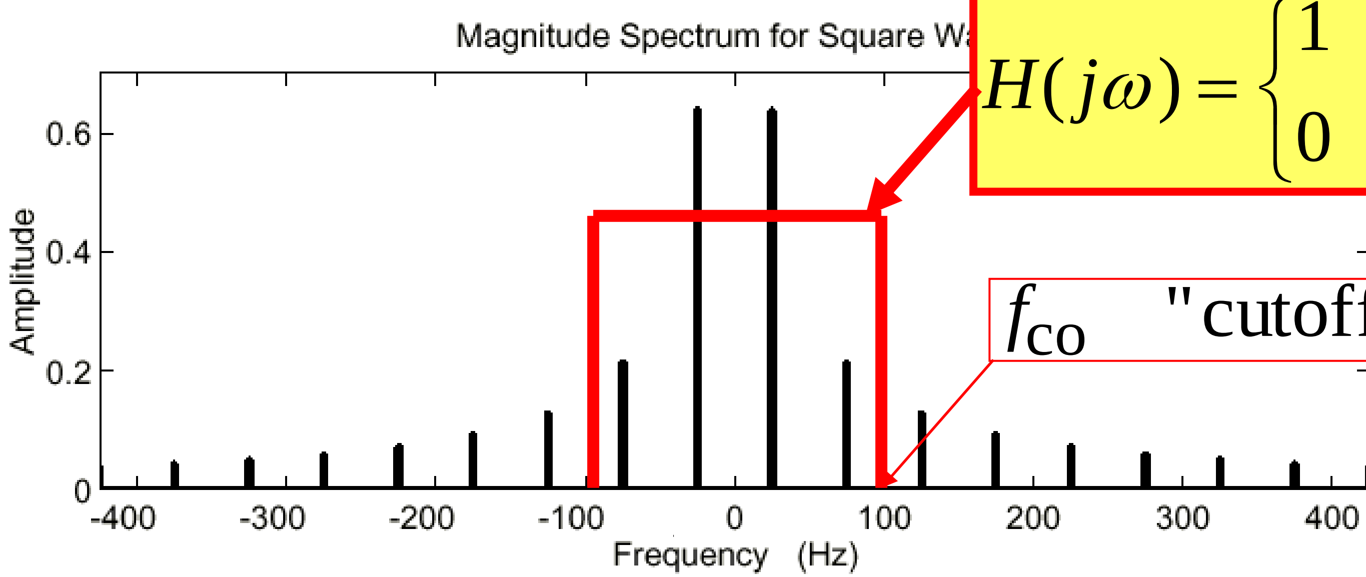
- By superposition,

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(j\omega_0 k) e^{j\omega_0 kt} = \sum_{k=-\infty}^{\infty} b_k e^{j\omega_0 kt}$$

$$\boxed{b_k = a_k H(j\omega_0 k)}$$

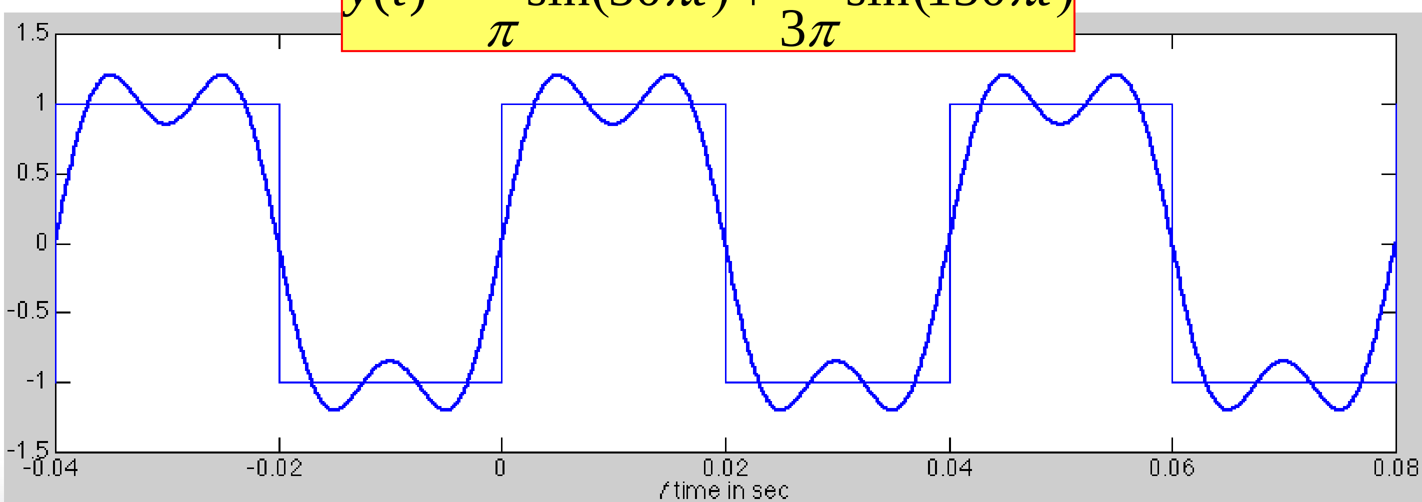
Ideal Lowpass Filter (100 Hz)

$$H(j\omega) = \begin{cases} 1 & |\omega| < \omega_{co} \\ 0 & |\omega| > \omega_{co} \end{cases}$$



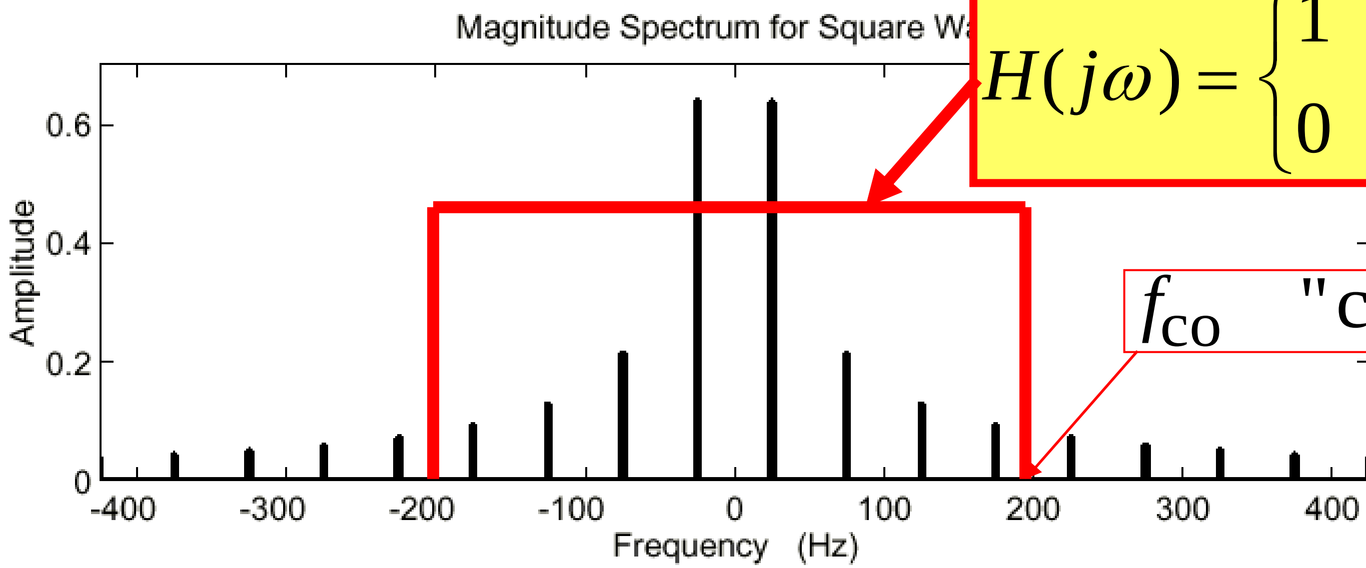
f_{co} "cutoff freq."

$$y(t) = \frac{4}{\pi}\sin(50\pi t) + \frac{4}{3\pi}\sin(150\pi t)$$



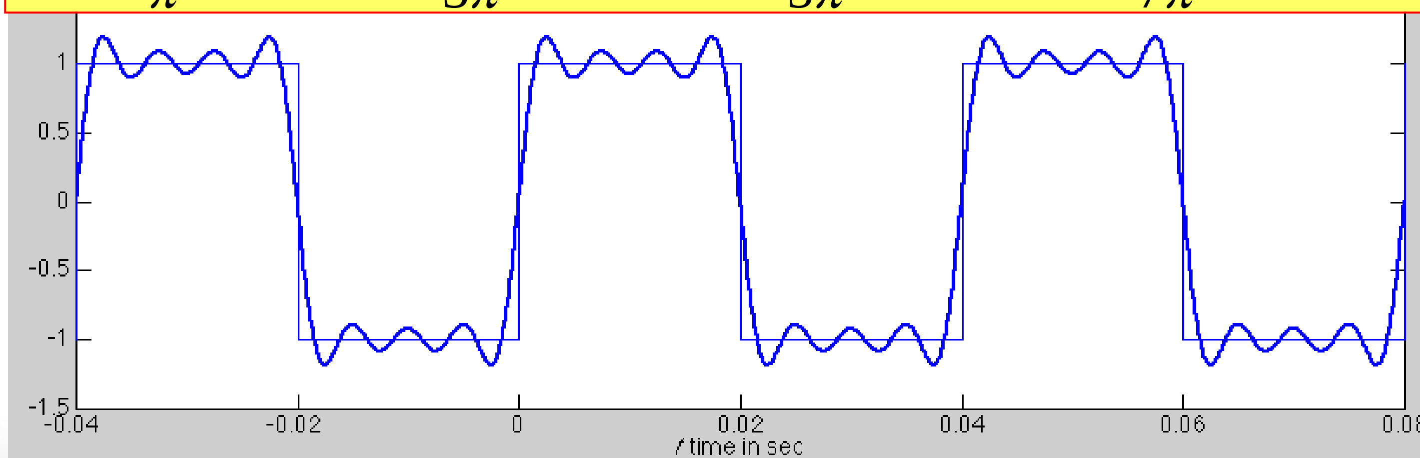
Ideal Lowpass Filter (200 Hz)

$$H(j\omega)=\begin{cases}1 & |\omega|<\omega_{co}\\ 0 & |\omega|>\omega_{co}\end{cases}$$



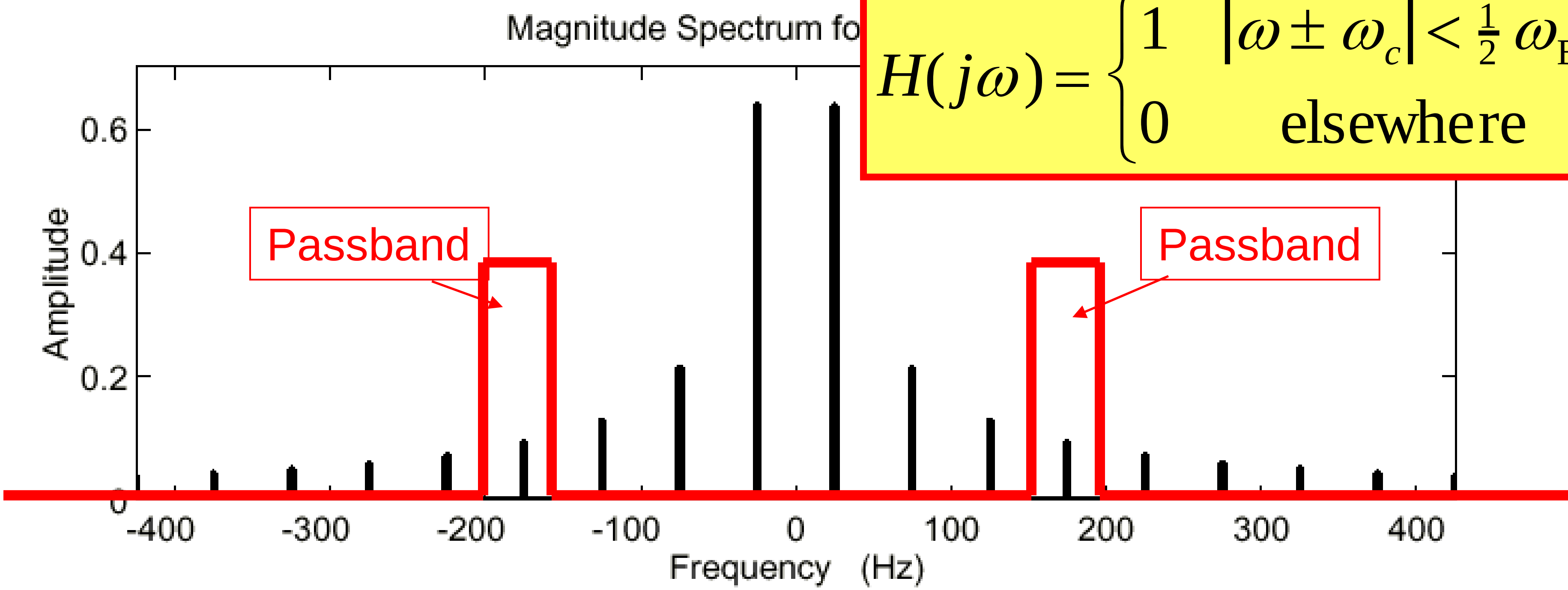
f_{co} "cutoff"

$$y(t)=\frac{4}{\pi}\sin(50\pi t)+\frac{4}{3\pi}\sin(150\pi t)+\frac{4}{5\pi}\sin(250\pi t)+\frac{4}{7\pi}\sin(350\pi t)$$



Ideal Bandpass Filter

$$H(j\omega) = \begin{cases} 1 & |\omega \pm \omega_c| < \frac{1}{2}\omega_B \\ 0 & \text{elsewhere} \end{cases}$$



Ouput signal

$$y(t) = \frac{2}{j7\pi} e^{j2\pi(175)t} - \frac{2}{j7\pi} e^{-j2\pi(175)t} = \frac{4}{7\pi}\cos\left(2\pi(175)t - \frac{1}{2}\pi\right)$$

Example

$$\boxed{H(j\omega) = e^{-j\omega t_d}}$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt} \mapsto y(t) = \sum_{k=-\infty}^{\infty} b_k e^{j\omega_0 kt}$$

$$\boxed{b_k = a_k H(j\omega_0 k) = a_k e^{-j\omega_0 k t_d}}$$

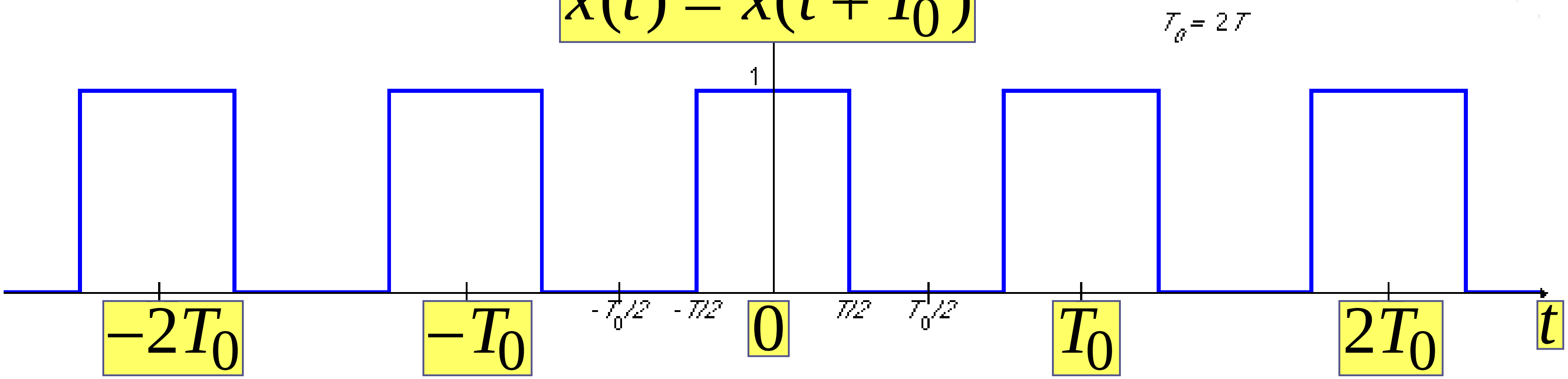
$$y(t) = \sum_{k=-\infty}^{\infty} a_k e^{-j\omega_0 k t_d} e^{j\omega_0 kt} = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k(t-t_d)}$$

$$\boxed{\therefore \quad y(t) = x(t-t_d)}$$

Chapter 11
Continuous-Time Fourier Transform

Fourier Series: Periodic *x*(*t*)

$$x(t) = x(t + T_0)$$



$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt}$$

Fourier Synthesis

Fundamental Freq.

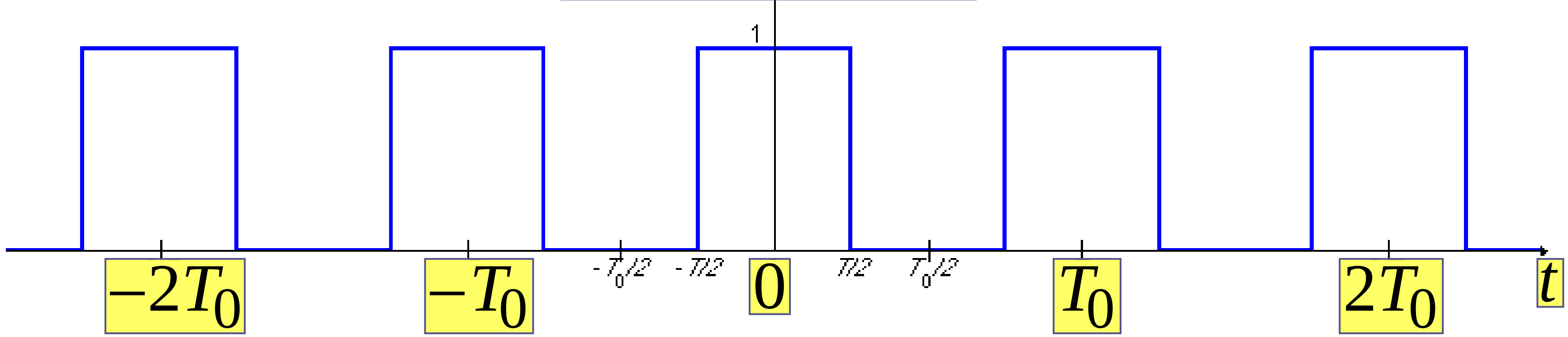
$$\omega_0 = 2\pi / T_0 = 2\pi f_0$$

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j\omega_0 kt} dt$$

Fourier Analysis

Square Wave Signal

$$x(t) = x(t + T_0)$$

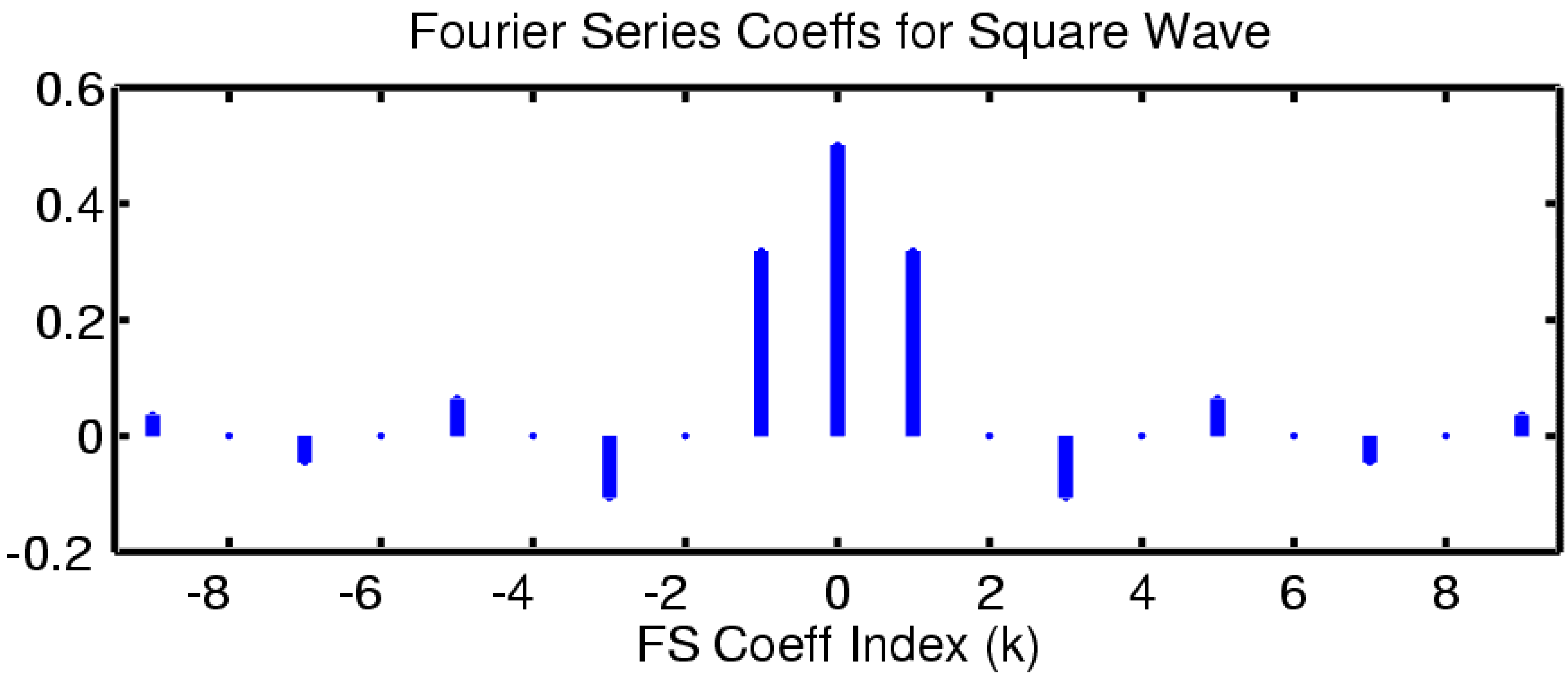


$$a_k = \frac{1}{T_0} \int_{-T_0/4}^{T_0/4} (1) e^{-j\omega_0 k t} dt$$

$$a_k = \left. \frac{e^{-j\omega_0 kt}}{-j\omega_0 k T_0} \right|_{-T_0/4}^{T_0/4} = \frac{e^{-j\pi k/2} - e^{j\pi k/2}}{-j2\pi k} = \frac{\sin(\pi k / 2)}{\pi k}$$

Spectrum from Fourier Series

$$a_k = \frac{\sin(\pi k/2)}{\pi k} = \begin{cases} \neq 0 & k = 0, \pm 1, \pm 3, \ldots \\ 0 & k = \pm 2, \pm 4, \ldots \end{cases}$$



What if $x(t)$ is not periodic?

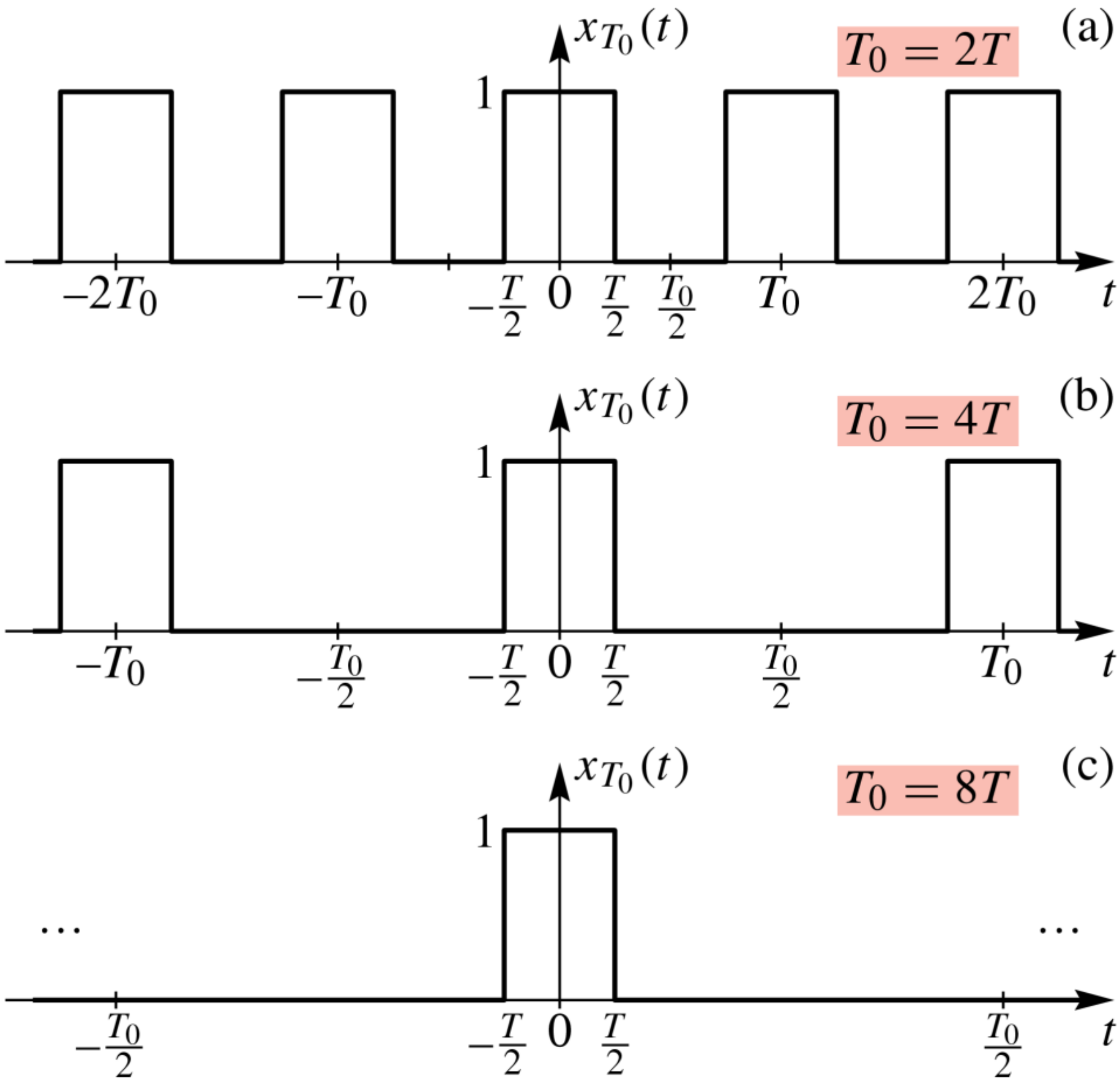
- Sum of Sinusoids?
  - Non-harmonically related sinusoids would not be periodic, but would probably be non-zero for all *t.*
- Fourier transform
  - Gives a "sum" (actually an **integral**) that involves **ALL** frequencies.
  - Can represent signals that are identically zero for negative *t* (right-sided signals).

Limiting Behavior of Waveform

T_0=2T

T_0=4T

T_0=8T



Fourier Series Representation

$$(a_k T_0) = \int_{-T_0/2}^{T_0/2} x_{T_0}(t) e^{-j\omega_0 k t} dt = \int_{-T/2}^{T/2} (1) e^{-j\omega_0 k t} dt$$

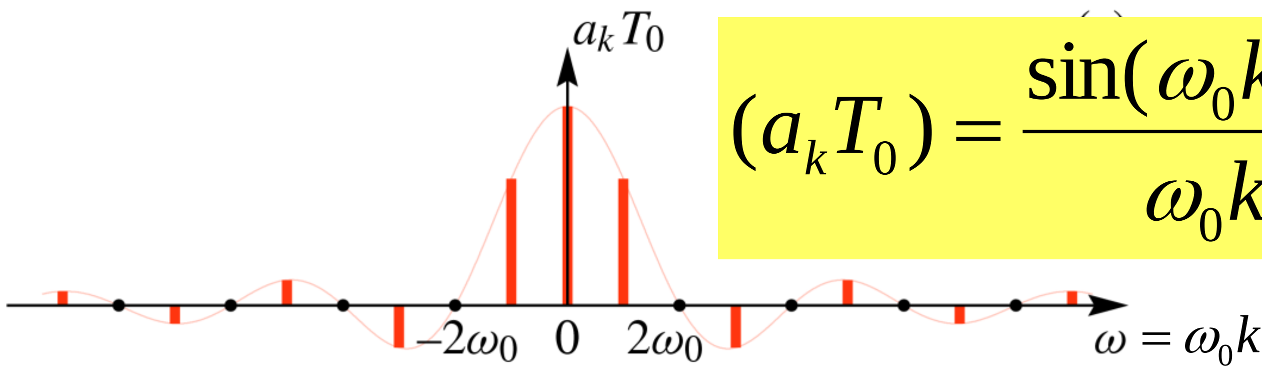
$$= -\frac{1}{j\omega_0 k} e^{-j\omega_0 k t} \Big|_{-T/2}^{T/2}$$

$$= -\frac{e^{-j\omega_0 k T/2} - e^{j\omega_0 k T/2}}{j\omega_0 k}$$

$$= \frac{\sin(\omega_0 k T/2)}{\omega_0 k/2}$$

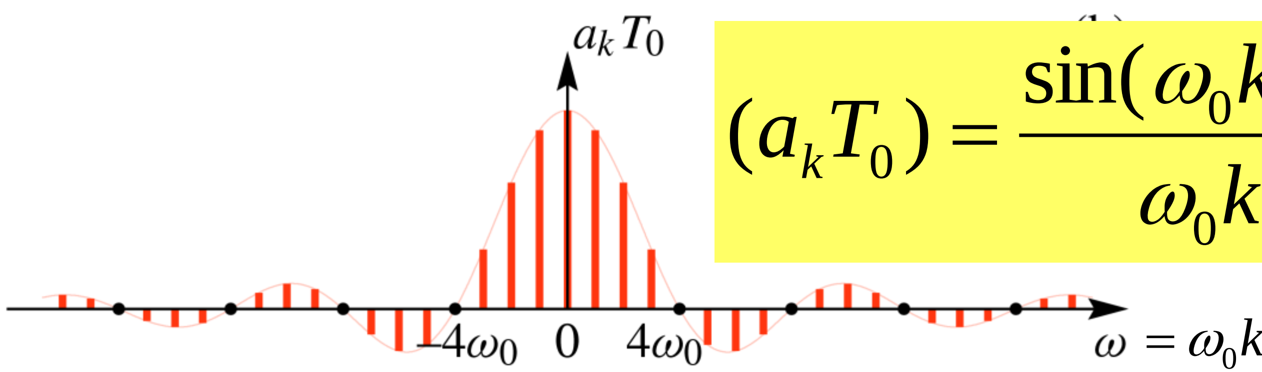
Limiting Behavior of Spectrum

T_0=2T



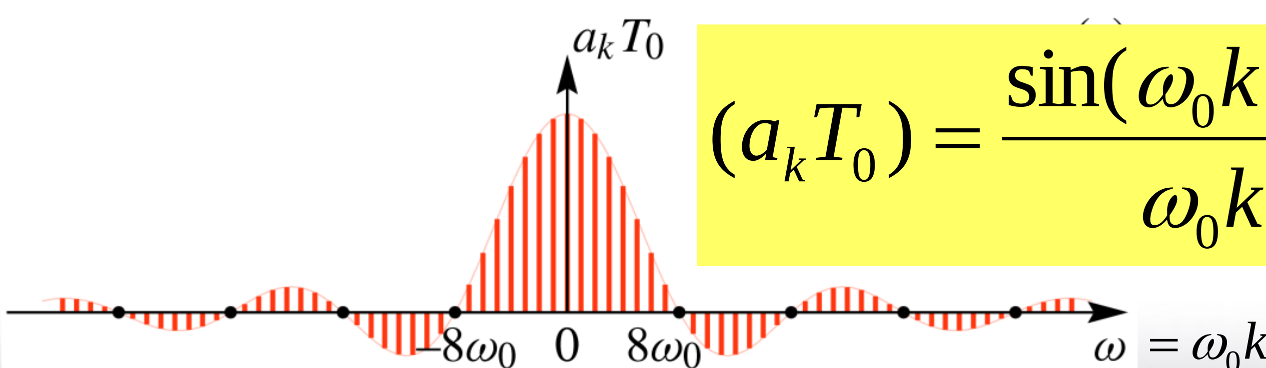
$$(a_k T_0) = \frac{\sin(\omega_0 k T_0 / 4)}{\omega_0 k / 2}$$

T_0=4T



$$(a_k T_0) = \frac{\sin(\omega_0 k T_0 / 8)}{\omega_0 k / 2}$$

T_0=8T



$$(a_k T_0) = \frac{\sin(\omega_0 k T_0 / 16)}{\omega_0 k / 2}$$

Fourier Series in the LIMIT (long period)

$$T_0 a_k = \int_{-T_0/2}^{T_0/2} x_{T_0}(t) e^{-j\omega_0 k t} dt \mapsto X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier Analysis

$$\lim_{T_0 \to \infty} T_0 a_k = X(j\omega)$$

$$\lim_{T_0 \to \infty} \omega_0 k = \omega$$

$$\lim_{T_0 \to \infty} \frac{2\pi}{T_0} = d\omega$$

$$x_{T_0}(t) = \tfrac{1}{2\pi} \sum_{k=-\infty}^{\infty} (T_0 a_k) e^{j\omega_0 k t} \left(\tfrac{2\pi}{T_0}\right) \mapsto x(t) = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Fourier Synthesis

Definition: Fourier Transform

- For non-periodic signals

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier Analysis
↓
Forward CT Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Fourier Synthesis
↓
Inverse CT Fourier Transform

$x(t)$ → ***Time-domain representation***

$X(j\omega)$ → ***Frequency-domain representation***

WHY use the Fourier transform?

- Manipulate the **"Frequency Spectrum"**.
- Analog Communication Systems
  - AM(Amplitude Modulation), FM
- Ideal Filters: mostly BPFs
- Frequency Shifters
  - aka Modulators, Mixers, or Multipliers: $x(t)p(t)$

Existence of the Fourier Transform

- When is $|X(j\omega)| < \infty$?

$$|X(j\omega)| = \left| \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right| \le \int_{-\infty}^{\infty} |x(t)| dt$$

Sufficient condition $\int_{-\infty}^{\infty} |x(t)| dt < \infty$ Absolutely integrable

- Not a necessary condition!!!
  - Some functions do not satisfy the sufficient condition.
    - Allow impulse signals to get a Fourier transform representation.

Example 1: Forward CT Fourier Transform

- Right-sided real exponential signal

$$\boxed{x(t) = e^{-at}u(t)} \quad \boxed{a > 0}$$

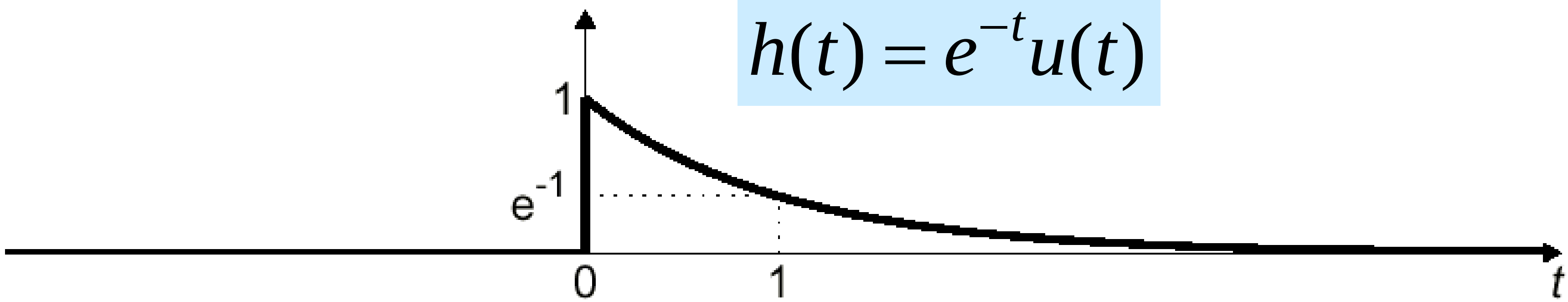
$$X(j\omega) = \int_0^{\infty} e^{-at} e^{-j\omega t} dt = \int_0^{\infty} e^{-(a+j\omega)t} dt$$

$$X(j\omega) = -\frac{e^{-at} e^{-j\omega t}}{a + j\omega}\Bigg|_0^{\infty} = \frac{1}{a + j\omega}$$

$$\boxed{X(j\omega) = \frac{1}{a + j\omega}}$$

Frequency Response

- Fourier Transform of $h(t)$ is the Frequency Response.

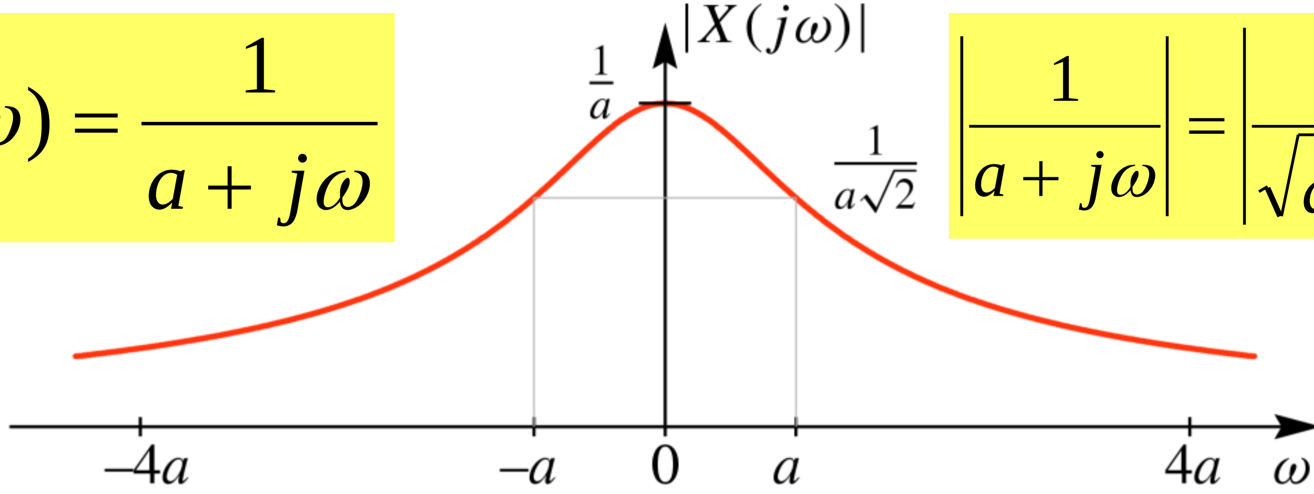


$$h(t) = e^{-t}u(t)$$

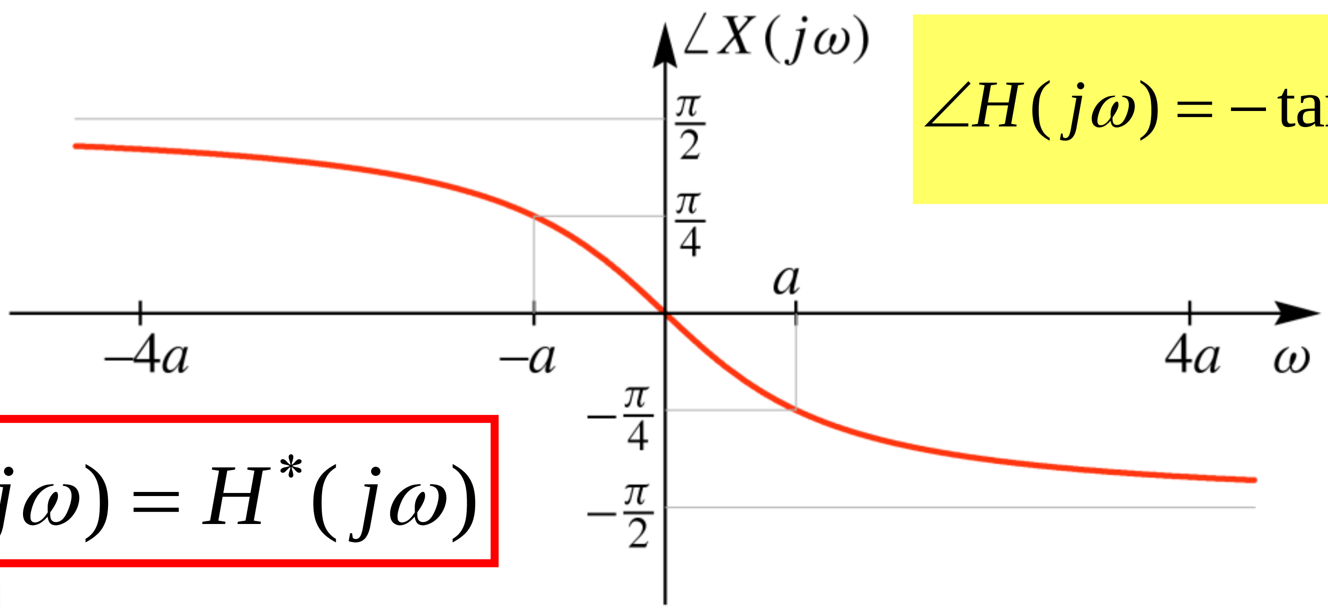
$$h(t) = e^{-t}u(t) \Leftrightarrow H(j\omega) = \frac{1}{1+j\omega}$$

Magnitude and Phase Plots

$$H(j\omega) = \frac{1}{a + j\omega}$$



$$\left|\frac{1}{a + j\omega}\right| = \left|\frac{1}{\sqrt{a^2 + \omega^2}}\right|$$



$$\angle H(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

$$\boxed{H(-j\omega) = H^*(j\omega)}$$

Example 2: Forward CT Fourier Transform

- Rectangular pulse signal

$$\boxed{x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases}}$$

$$X(j\omega) = \int_{-T/2}^{T/2} (1) e^{-j\omega t} dt = \int_{-T/2}^{T/2} e^{-j\omega t} dt$$

$$X(j\omega) = \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{-T/2}^{T/2} = \frac{e^{-j\omega T/2} - e^{j\omega T/2}}{-j\omega}$$

$$\boxed{X(j\omega) = \frac{\sin(\omega T/2)}{(\omega/2)}}$$

Sinc Function

- Formal definition

$$\mathrm{sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}$$

- Fourier transform of the rectangular pulse signal

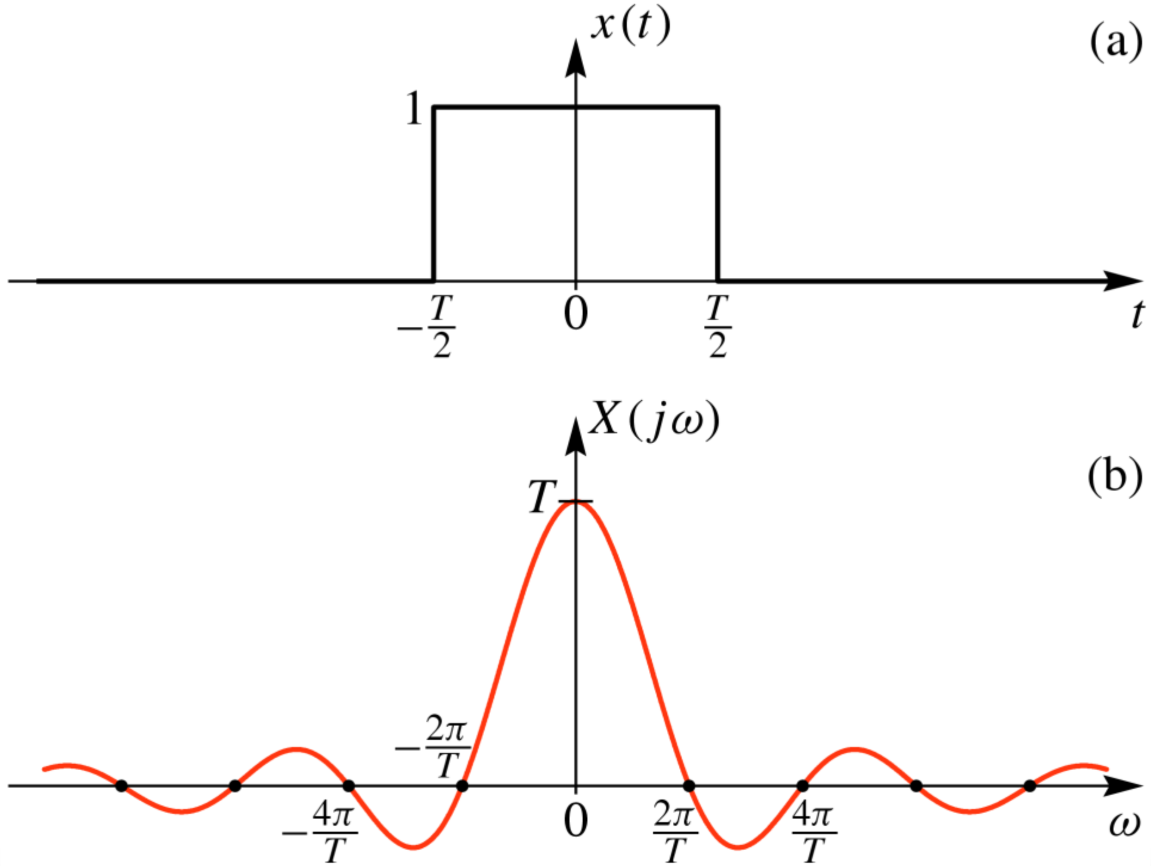
$$X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2} = T\mathrm{sinc}(\omega T/2\pi)$$

- Important property (using L'Hôpital's rule)

$$\mathrm{sinc}(0) = 1$$

Fourier Transform Pair (1)

$$x(t)=\begin{cases}1 & |t|<T/2\\ 0 & |t|>T/2\end{cases} \Leftrightarrow X(j\omega)=\frac{\sin(\omega T/2)}{(\omega/2)}$$



Example 3: Inverse CT Fourier Transform

- Bandlimited signal

$$\boxed{X(j\omega) = \begin{cases} 1 & |\omega| < \omega_b \\ 0 & |\omega| > \omega_b \end{cases}}$$

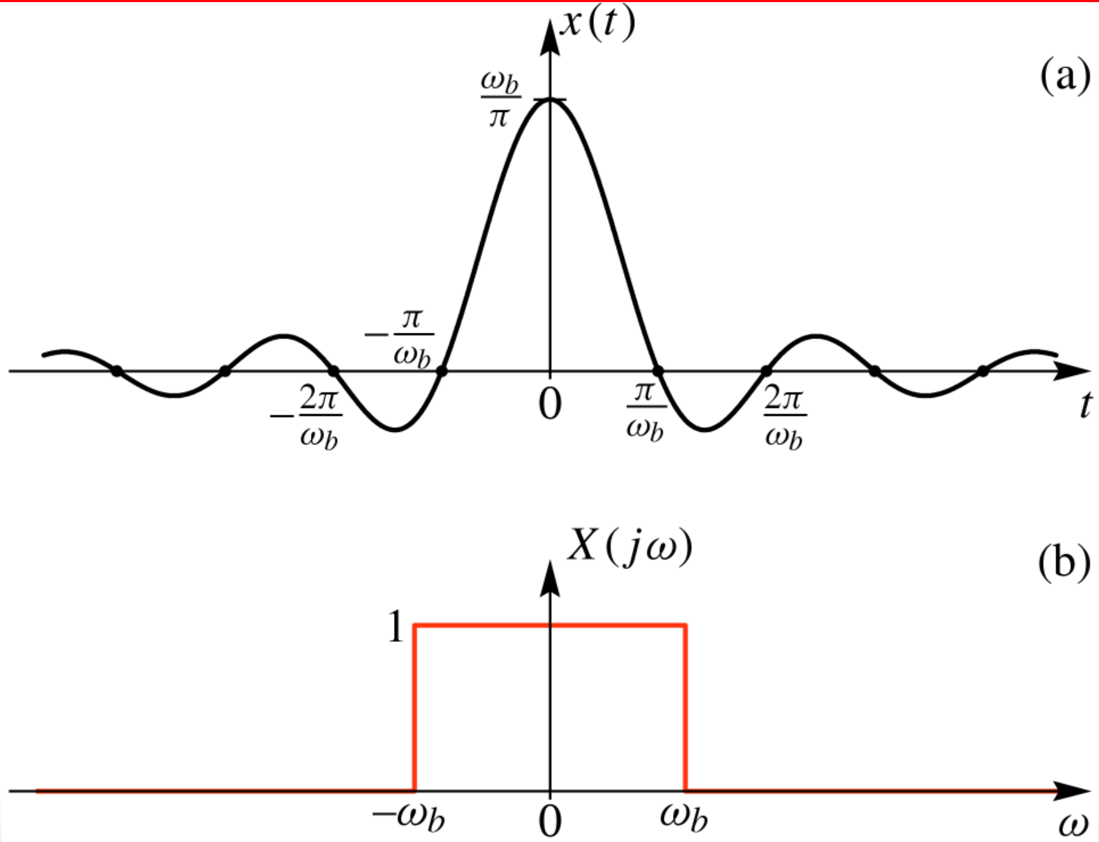
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\omega_b}^{\omega_b} 1e^{j\omega t}d\omega$$

$$x(t) = \frac{1}{2\pi}\frac{e^{j\omega t}}{jt}\bigg|_{-\omega_b}^{\omega_b} = \frac{1}{2\pi}\frac{e^{j\omega_b t} - e^{-j\omega_b t}}{jt}$$

$$\boxed{x(t) = \frac{\sin(\omega_b t)}{\pi t}}$$

Fourier Transform Pair (2)

$$x(t) = \frac{\sin(\omega_b t)}{\pi t} \quad \Leftrightarrow \quad X(j\omega) = \begin{cases} 1 & |\omega| < \omega_b \\ 0 & |\omega| > \omega_b \end{cases}$$



Example 4: Forward CT Fourier Transform

- Impulse in time

$$\boxed{x(t) = \delta(t - t_0)}$$

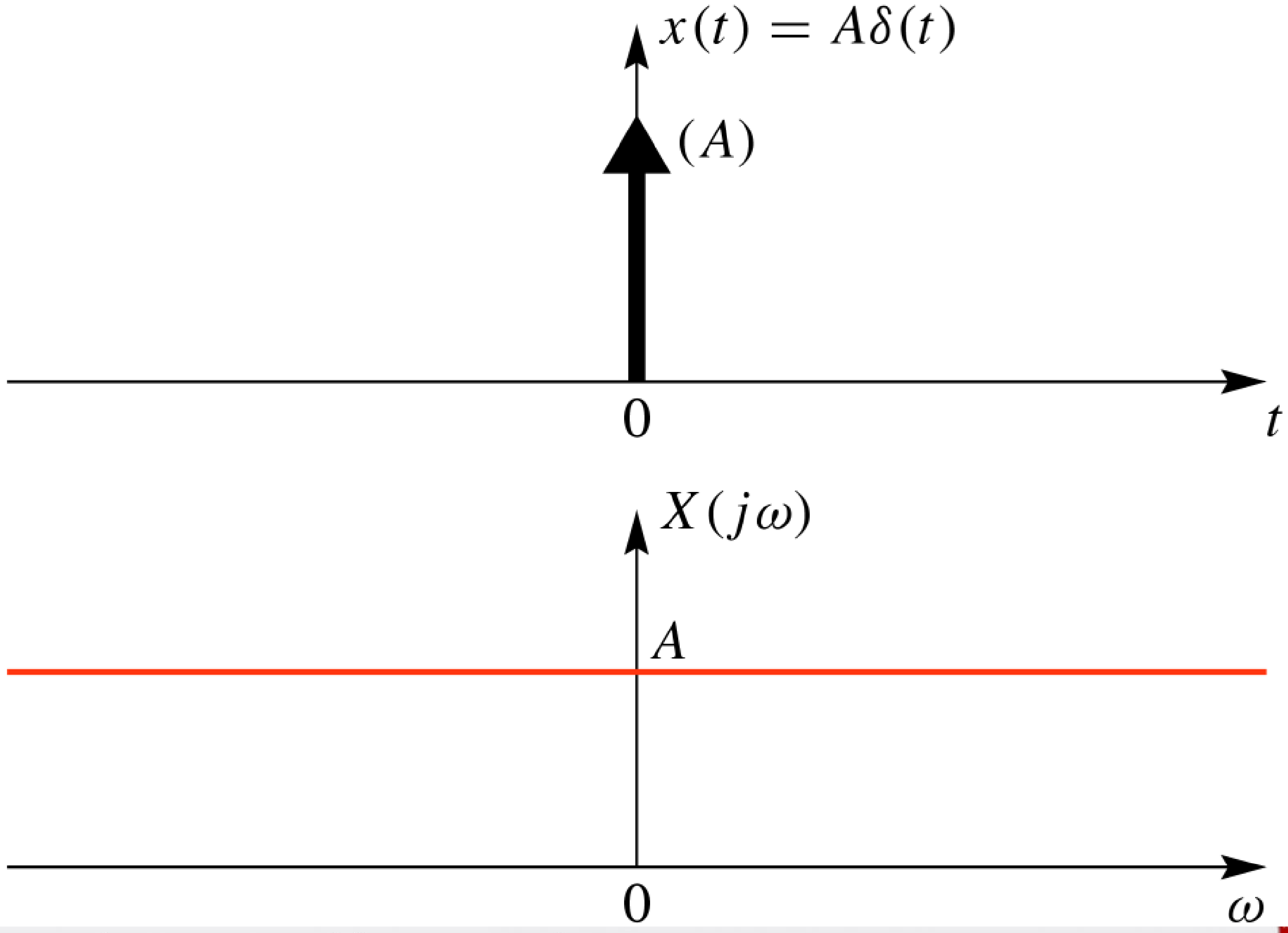
$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

Shifting Property of the Impulse

$$\boxed{X(j\omega) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j\omega t} dt = e^{-j\omega t_0}}$$

Fourier Transform Pair (3)

$$\boxed{x(t) = \delta(t) \Leftrightarrow X(j\omega) = 1}$$



Example 5: Inverse CT Fourier Transform

- Impulse in frequency

$$\boxed{X(j\omega) = 2\pi\delta(\omega - \omega_0)}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}d\omega = e^{j\omega_0 t}$$

$$\boxed{x(t) = e^{j\omega_0 t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)}$$

$$\boxed{x(t) = 1 \Leftrightarrow X(j\omega) = 2\pi\delta(\omega)}$$

$$\boxed{\begin{aligned} &x(t) = \cos(\omega_0 t) \Leftrightarrow \\ &\qquad X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0) \end{aligned}}$$

Fourier Transform Pair (4)

$$x(t) = \cos(\omega_0 t) \Leftrightarrow$$
$$X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

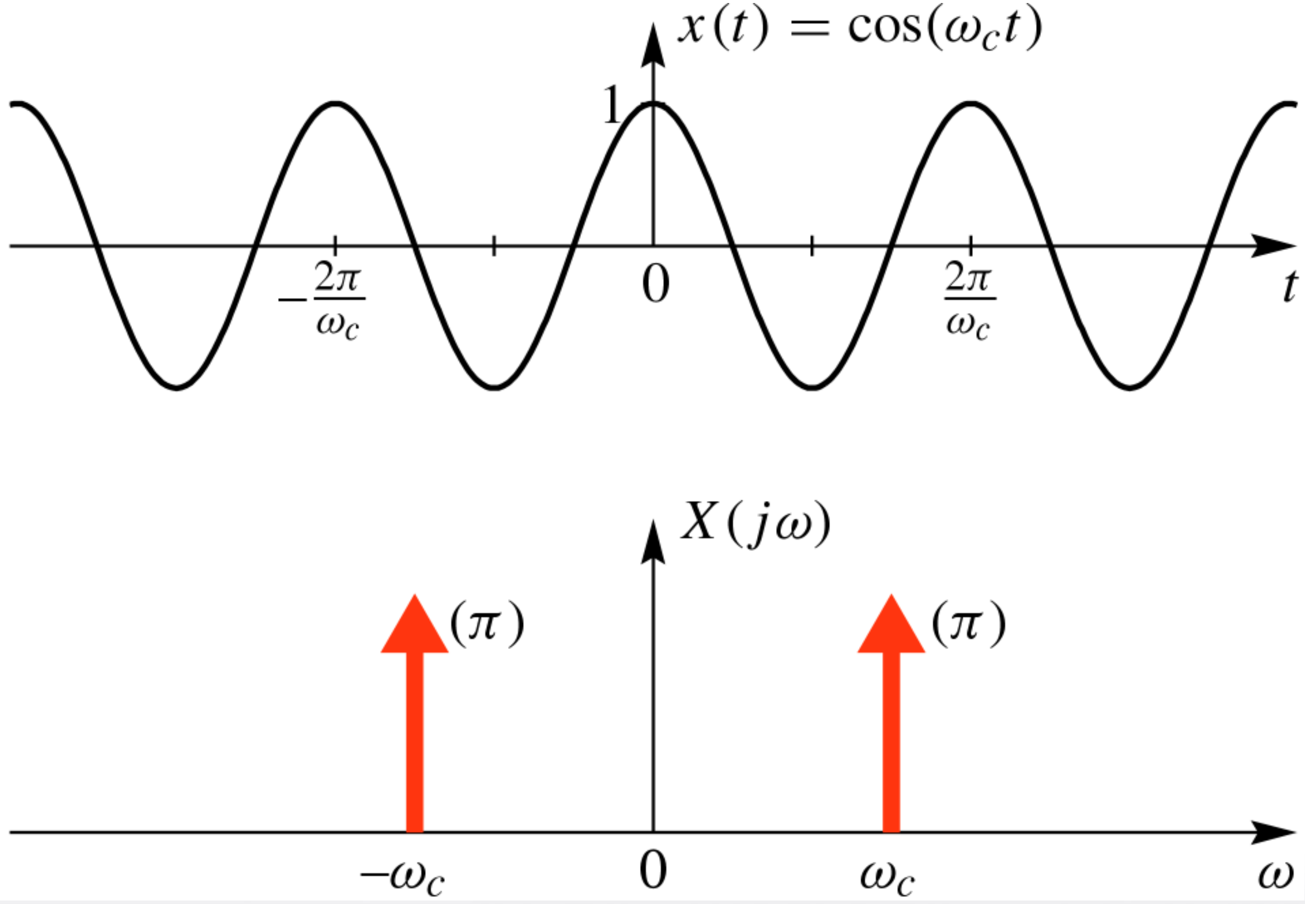


Table of Fourier Transforms

$$x(t) = e^{-at}u(t) \Leftrightarrow X(j\omega) = \frac{1}{a + j\omega}$$

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \quad \Leftrightarrow \quad X(j\omega) = \frac{\sin(\omega T/2)}{(\omega/2)}$$

$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)} \quad \Leftrightarrow \quad X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$

$$x(t) = \delta(t - t_0) \Leftrightarrow X(j\omega) = e^{-j\omega t_0}$$

$$x(t) = e^{j\omega_0 t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

Fourier Transform of a General Periodic Signal

- If $x(t)$ is periodic with period T_0 ,

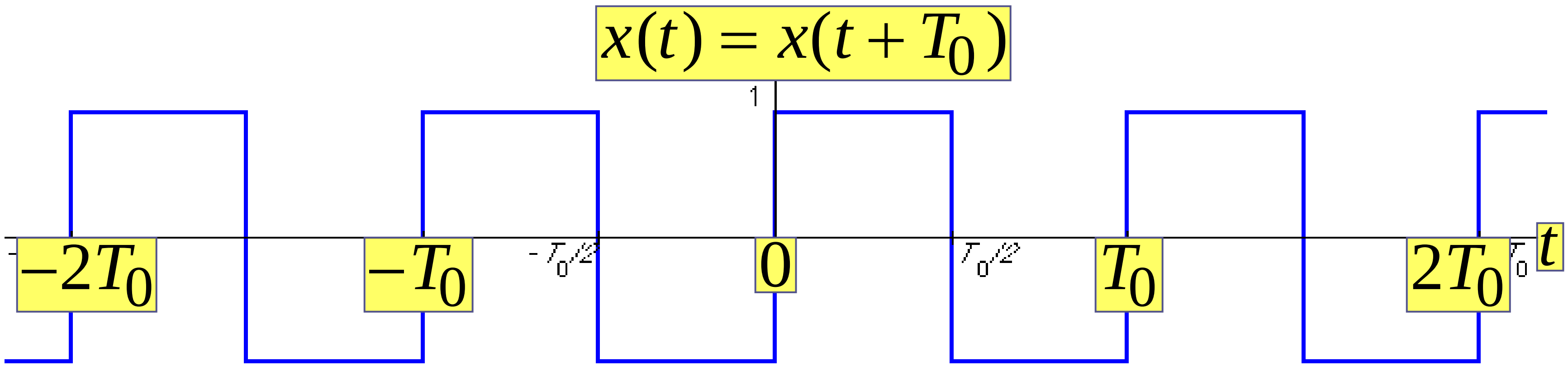
$$x(t)=\sum_{k=-\infty}^{\infty}a_k e^{jk\omega_0 t} \qquad a_k=\frac{1}{T_0}\int_0^{T_0}x(t)e^{-jk\omega_0 t}dt$$

Since $e^{jk\omega_0 t}\Leftrightarrow 2\pi\delta(\omega-k\omega_0)$,

$$X(j\omega)=\int_{-\infty}^{\infty}\sum_{k=-\infty}^{\infty}a_k e^{jk\omega_0 t}e^{-j\omega t}dt$$

$$=\sum_{k=-\infty}^{\infty}a_k\int_{-\infty}^{\infty}e^{jk\omega_0 t}e^{-j\omega t}dt=\sum_{k=-\infty}^{\infty}2\pi a_k\delta(\omega-k\omega_0)$$

Square Wave Signal

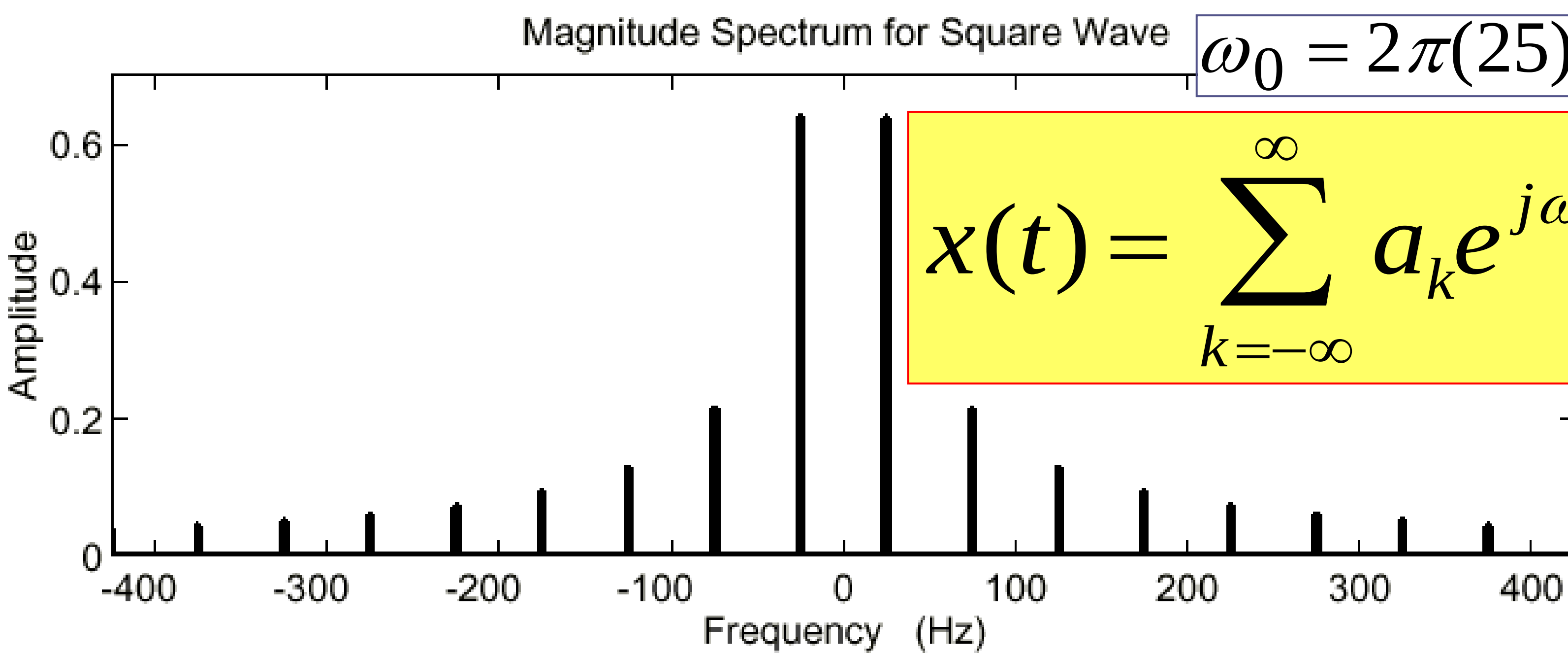


$$a_k = \frac{1}{T_0}\int_0^{T_0/2} (1)e^{-j\omega_0 kt}dt + \frac{1}{T_0}\int_{T_0/2}^{T_0} (-1)e^{-j\omega_0 kt}dt$$

$$a_k = \left.\frac{e^{-j\omega_0 kt}}{-j\omega_0 kT_0}\right|_0^{T_0/2} - \left.\frac{e^{-j\omega_0 kt}}{-j\omega_0 kT_0}\right|_{T_0/2}^{T_0} = \frac{1 - e^{-j\pi k}}{j\pi k}$$

Spectrum from Fourier Series

$$a_k = \frac{1 - e^{-j\pi k}}{j\pi k} = \begin{cases} \dfrac{2}{j\pi k} & k = \pm 1, \pm 3, \ldots \\ 0 & k = 0, \pm 2, \pm 4, \ldots \end{cases}$$

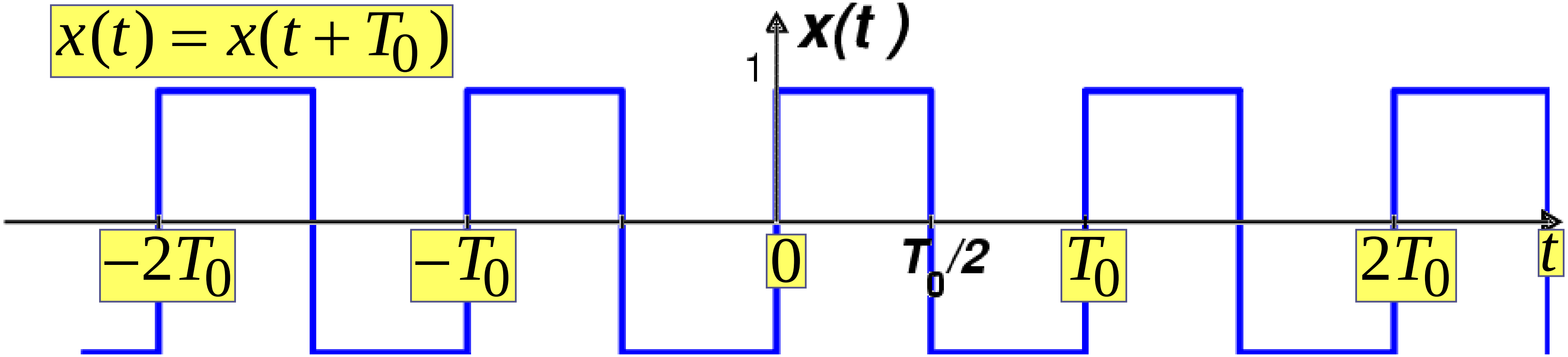


$$\omega_0 = 2\pi(25)$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k t}$$

Fourier Transform of the Square Wave

$$x(t) = x(t+T_0)$$



$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

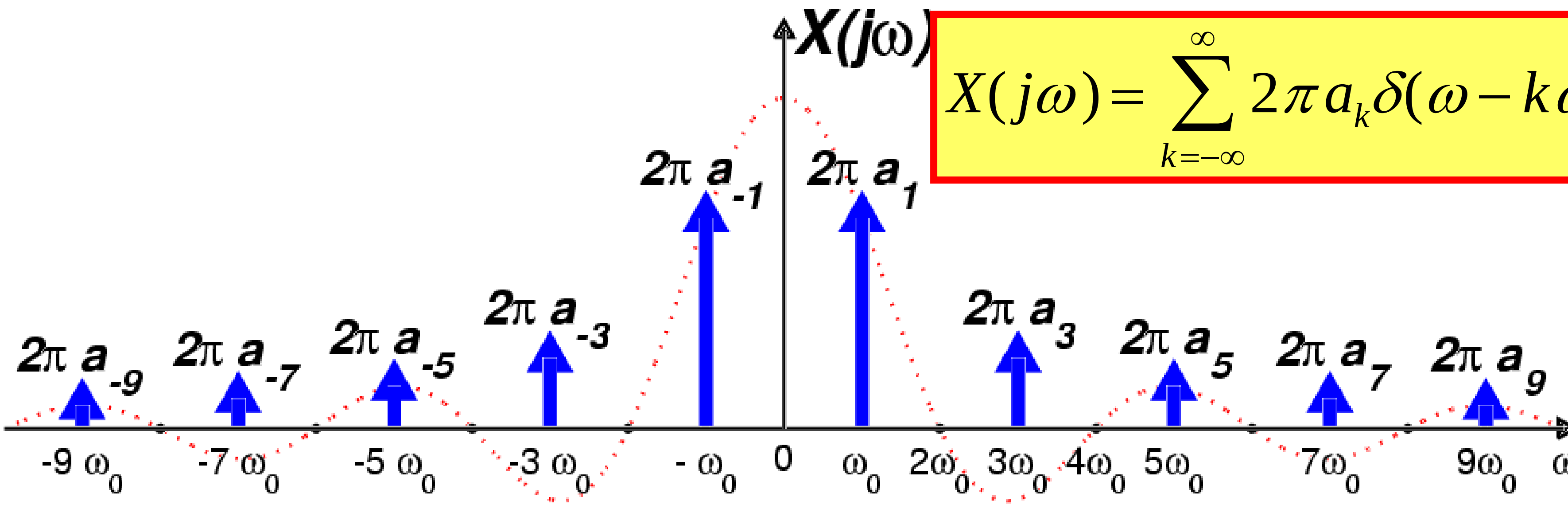


Table of Easy FT Properties

Linearity Property

$$ax_1(t)+bx_2(t) \Leftrightarrow aX_1(j\omega)+bX_2(j\omega)$$

Delay Property

$$x(t-t_d) \Leftrightarrow e^{-j\omega t_d} X(j\omega)$$

Scaling

$$x(at) \Leftrightarrow \frac{1}{|a|} X(j(\tfrac{\omega}{a}))$$

Scaling Property (1)

$$\boxed{x(at) \Leftrightarrow \frac{1}{|a|} X(j\frac{\omega}{a})}$$

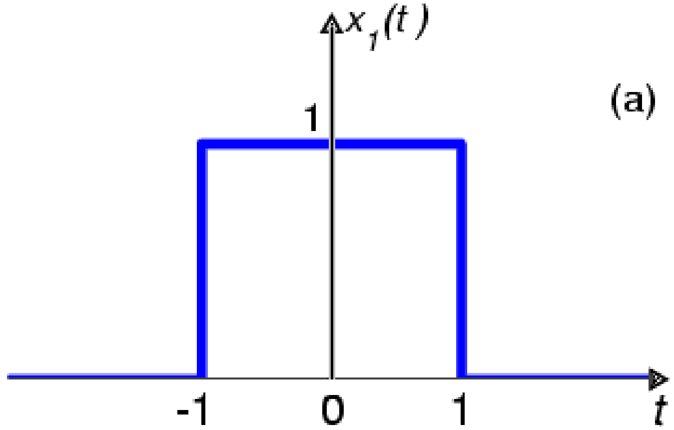
$$\int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(\lambda) e^{-j\omega(\lambda/a)} \frac{d\lambda}{|a|}$$

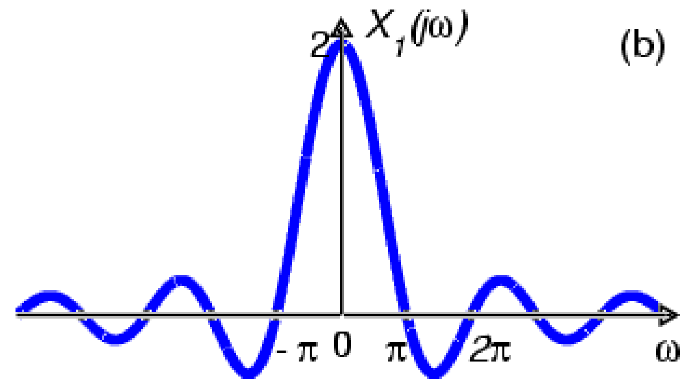
$$= \frac{1}{|a|} X(j\frac{\omega}{a})$$

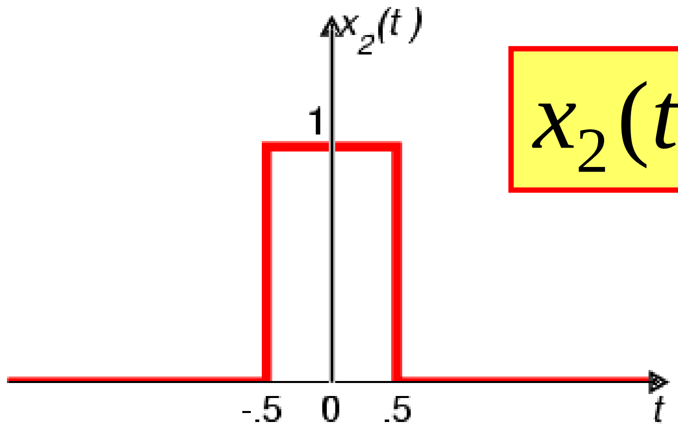
$x(2t)$ shrinks; $\frac{1}{2} X(j\frac{\omega}{2})$ expands.

Scaling Property (2)

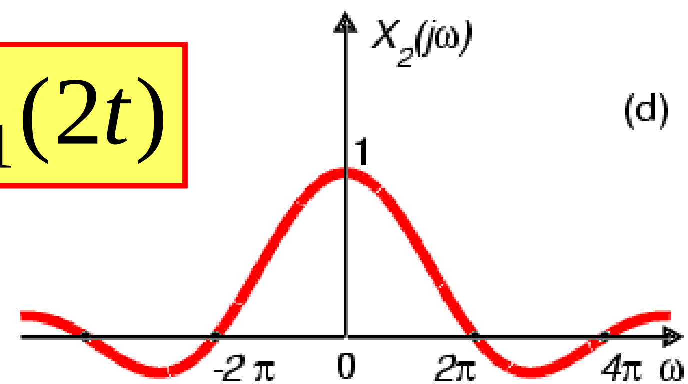
$$x(at) \Leftrightarrow \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)$$







$$x_2(t) = x_1(2t)$$



Uncertainty Principle

- Try to make $x(t)$ shorter.
  - Then $X(j\omega)$ will get wider.
  - Narrow pulses have wide bandwidth.

- Try to make $X(j\omega)$ narrower.
  - Then $x(t)$ will have longer duration.

- **Cannot simultaneously reduce time duration and bandwidth**.

Delay Property

$$\boxed{x(t-t_d) \Leftrightarrow e^{-j\omega t_d} X(j\omega)}$$

$$\int_{-\infty}^{\infty} x(t-t_d)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega(\tau+t_d)}d\tau$$

$$= e^{-j\omega t_d} X(j\omega)$$

For example, $e^{-a(t-5)}u(t-5) \Leftrightarrow \dfrac{e^{-j\omega 5}}{a+j\omega}$

Significant FT Properties

Convolution Property

$$x(t)*h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

Multiplication Property

$$x(t)p(t) \Leftrightarrow \frac{1}{2\pi}X(j\omega)*P(j\omega)$$

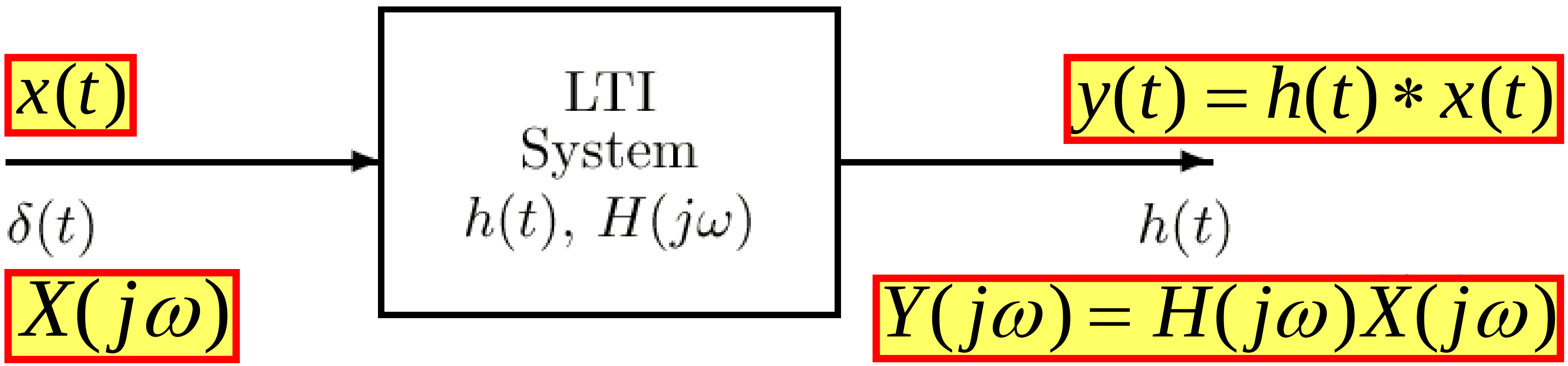
$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega-\omega_0))$$

Differentiation Property

$$\frac{dx(t)}{dt} \Leftrightarrow (j\omega)X(j\omega)$$

$$\frac{d^k x(t)}{dt^k} \Leftrightarrow (j\omega)^k X(j\omega)$$

Convolution Property (1)



- Convolution in the time-domain

$$\boxed{y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau}$$

corresponds to **<u>MULTIPLICATION</u>** in the frequency-domain.

$$\boxed{Y(j\omega) = H(j\omega)X(j\omega)}$$

Convolution Property (2)

$$Y(j\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau \right) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} x(\tau) \left(\int_{-\infty}^{\infty} h(t-\tau) e^{-j\omega t} dt \right) d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau) \left(\int_{-\infty}^{\infty} h(\sigma) e^{-j\omega\sigma} dt \right) e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau) H(j\omega) e^{-j\omega\tau} d\tau = H(j\omega) \int_{-\infty}^{\infty} x(\tau) e^{-j\omega\tau} d\tau$$

$$= H(j\omega) X(j\omega)$$

Convolution Example

- Bandlimited **Input** Signal
  - "sinc" function

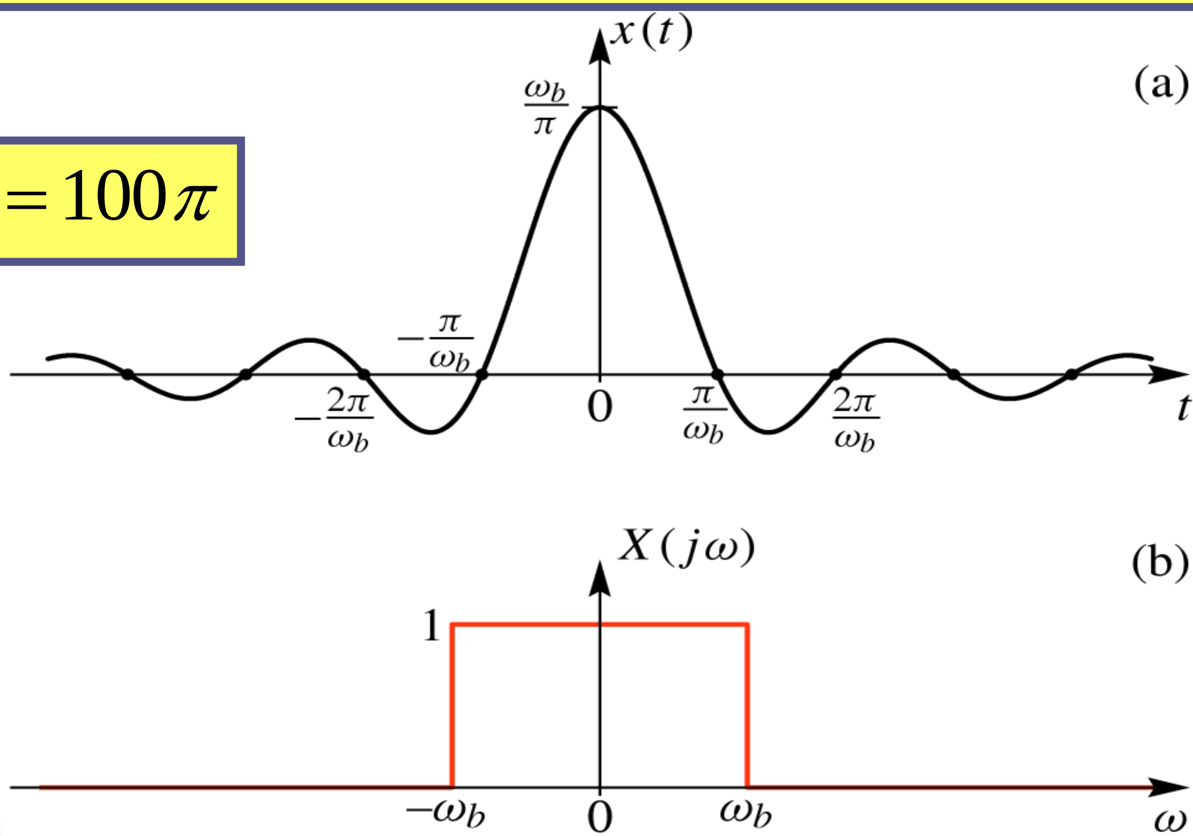
$$\text{sinc}(\theta) = \frac{\sin(\pi\theta)}{\pi\theta}$$

- Ideal LPF (Lowpass Filter)
  - ***h*(*t*)** is a "sinc".
- **Output** is Bandlimited.
  - Convolve "sincs".

Ideally Bandlimited Signal

$$x(t) = \frac{\sin(100\pi t)}{\pi t} \Leftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < 100\pi \\ 0 & |\omega| > 100\pi \end{cases}$$

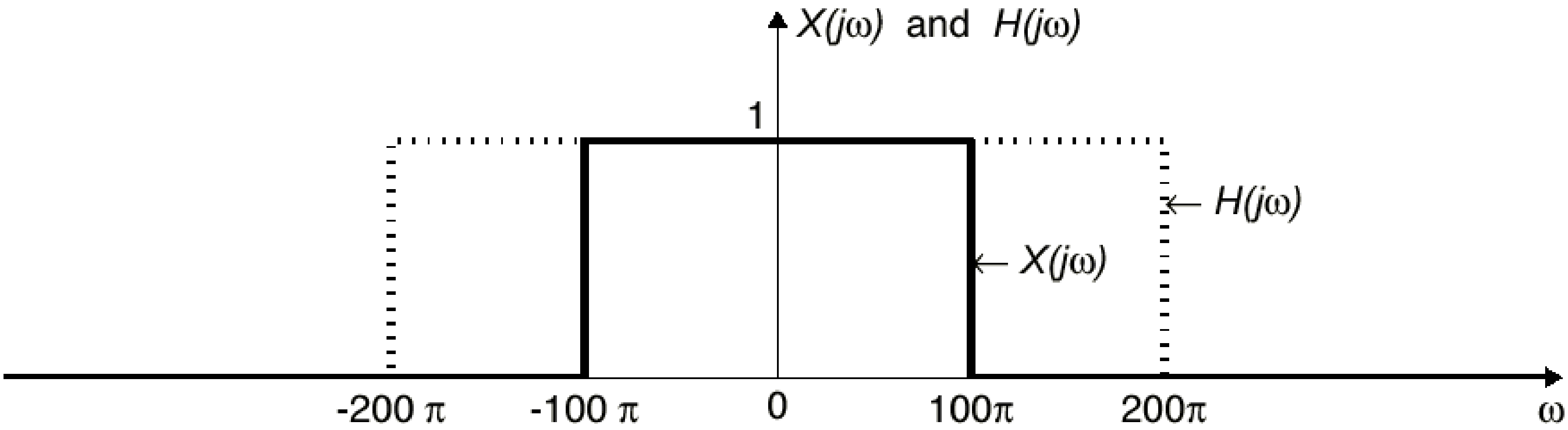
$$\omega_b = 100\pi$$



Convolution Example

$$\boxed{x(t) * h(t) \Leftrightarrow H(j\omega)X(j\omega)}$$

$$\frac{\sin(100\pi t)}{\pi t} * \frac{\sin(200\pi t)}{\pi t} = \frac{\sin(100\pi t)}{\pi t}$$

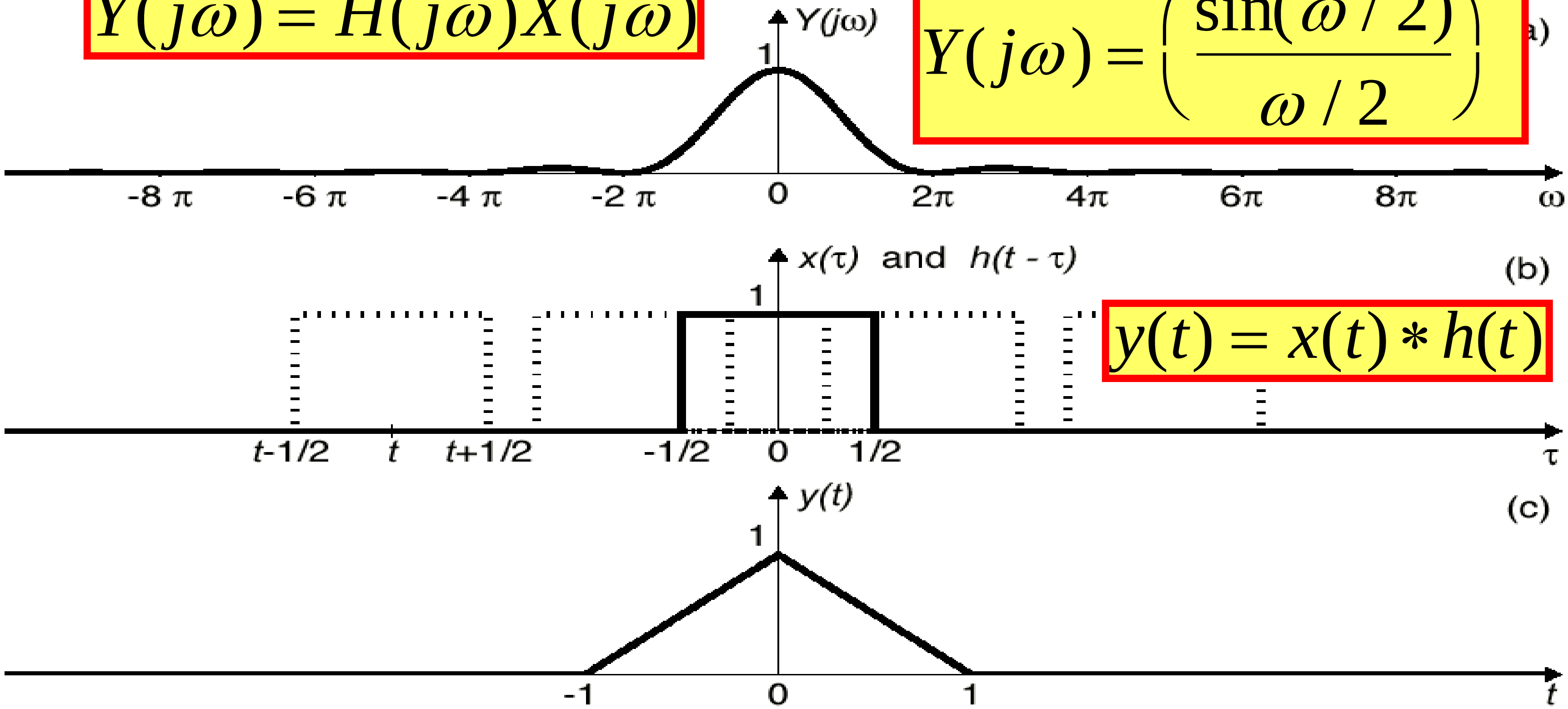


Convolution Example 2

$$Y(j\omega) = H(j\omega)X(j\omega)$$

$$Y(j\omega) = \left(\frac{\sin(\omega/2)}{\omega/2} \right)^2$$

$$y(t) = x(t) * h(t)$$



$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \Leftrightarrow \quad X(j\omega) = \frac{\sin(\omega T/2)}{(\omega/2)}$$

Partial fraction expansions (1)

$$x(t) = \delta(t) - e^{-t}u(t) \qquad h(t) = e^{-2t}u(t)$$

$$X(j\omega) = 1 - \frac{1}{1+j\omega} = \frac{j\omega}{1+j\omega} \qquad H(j\omega) = \frac{1}{2+j\omega}$$

$$Y(j\omega) = X(j\omega)H(j\omega) = \frac{j\omega}{(1+j\omega)(2+j\omega)} = \frac{A}{1+j\omega} + \frac{B}{2+j\omega}$$

- Using partial fraction expansions

$$(1+j\omega)Y(j\omega)\Big|_{j\omega=-1} = \frac{j\omega}{2+j\omega}\Bigg|_{j\omega=-1} = A + \frac{B(1+j\omega)}{2+j\omega}\Bigg|_{j\omega=-1} = A$$

$$Y(j\omega) = -\frac{1}{1+j\omega} + \frac{2}{2+j\omega}$$

$$y(t) = -e^{-t}u(t) + 2e^{-2t}u(t)$$

Partial fraction expansions (2)

- Long division

$$Y(j\omega)=\frac{1-\omega^2+j\omega}{(1+j\omega)(2+j\omega)}=\frac{1+j\omega+(j\omega)^2}{2+3j\omega+(j\omega)^2}$$

$$=1-\frac{1+2j\omega}{(1+j\omega)(2+j\omega)}=1+\frac{A}{1+j\omega}+\frac{B}{2+j\omega}$$

- Using partial fraction expansions

$$-\left.\frac{1+2j\omega}{2+j\omega}\right|_{j\omega=-1}=A+\left.\frac{B(1+j\omega)}{2+j\omega}\right|_{j\omega=-1}=A$$

$$Y(j\omega)=1+\frac{1}{1+j\omega}-\frac{3}{2+j\omega}$$

$$y(t)=\delta(t)+e^{-t}u(t)-3e^{-2t}u(t)$$

Cosine Input to LTI System

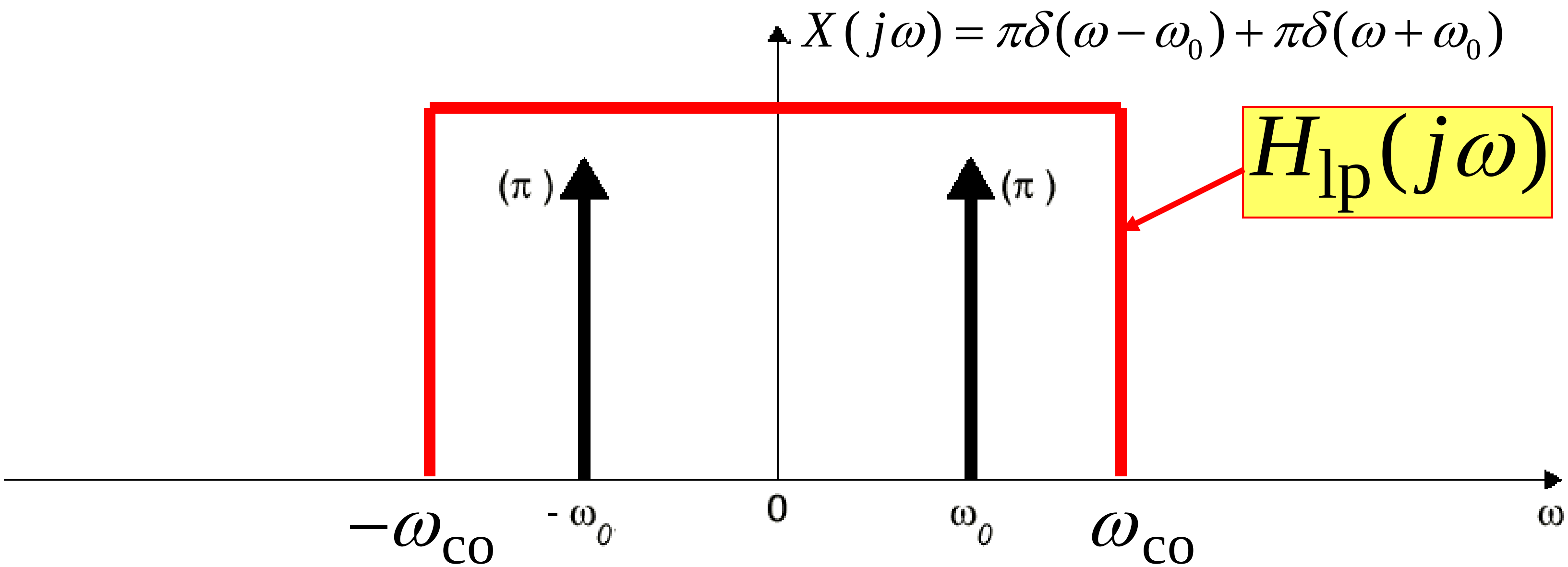
$$x(t) = \cos(\omega_0 t) \Leftrightarrow$$
$$X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

$$Y(j\omega) = H(j\omega)X(j\omega)$$
$$= H(j\omega)[\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)]$$
$$= H(j\omega_0)\pi\delta(\omega - \omega_0) + H(-j\omega_0)\pi\delta(\omega + \omega_0)$$

$$x(t) = e^{j\omega_0 t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

$$\begin{aligned} y(t) &= H(j\omega_0)\tfrac{1}{2}e^{j\omega_0 t} + H(-j\omega_0)\tfrac{1}{2}e^{-j\omega_0 t} \\ &= H(j\omega_0)\tfrac{1}{2}e^{j\omega_0 t} + H^*(j\omega_0)\tfrac{1}{2}e^{-j\omega_0 t} \\ &= |H(j\omega_0)|\cos(\omega_0 t + \angle H(j\omega_0)) \end{aligned}$$

Ideal Lowpass Filter (1)



$$y(t) = x(t) \qquad \text{if } \omega_0 < \omega_{co}$$

$$y(t) = 0 \qquad \text{if } \omega_0 > \omega_{co}$$

Ideal Lowpass Filter (2)

$$X(j\omega)=\sum_{k=-\infty}^{\infty}2\pi a_k\delta(\omega-k\omega_0)$$

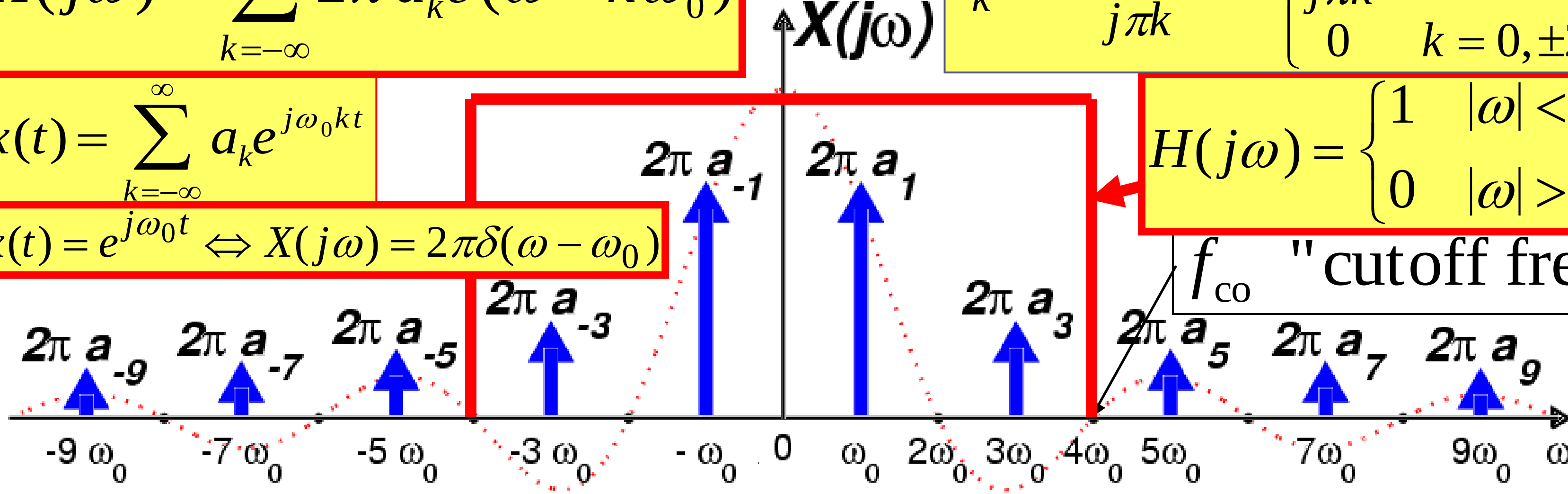
$$a_k=\frac{1-e^{-j\pi k}}{j\pi k}=\begin{cases}\dfrac{2}{j\pi k} & k=\pm1,\pm3,\dots\\ 0 & k=0,\pm2,\pm4,\dots\end{cases}$$

$$x(t)=\sum_{k=-\infty}^{\infty}a_k e^{j\omega_0 kt}$$

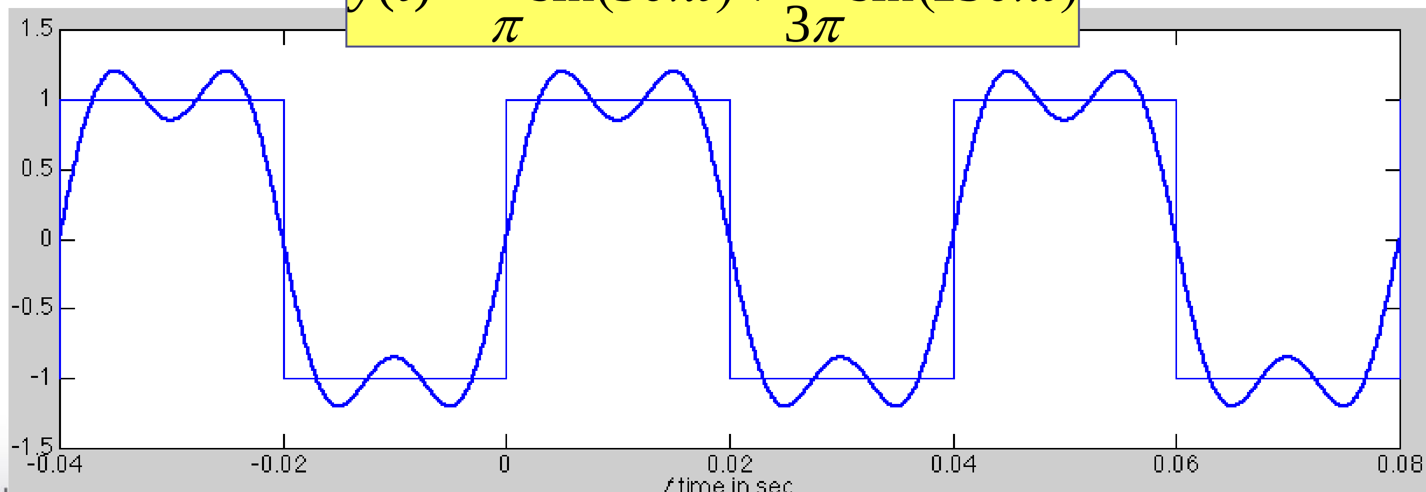
$$x(t)=e^{j\omega_0 t}\Leftrightarrow X(j\omega)=2\pi\delta(\omega-\omega_0)$$

$$H(j\omega)=\begin{cases}1 & |\omega|<\omega_{co}\\ 0 & |\omega|>\omega_{co}\end{cases}$$

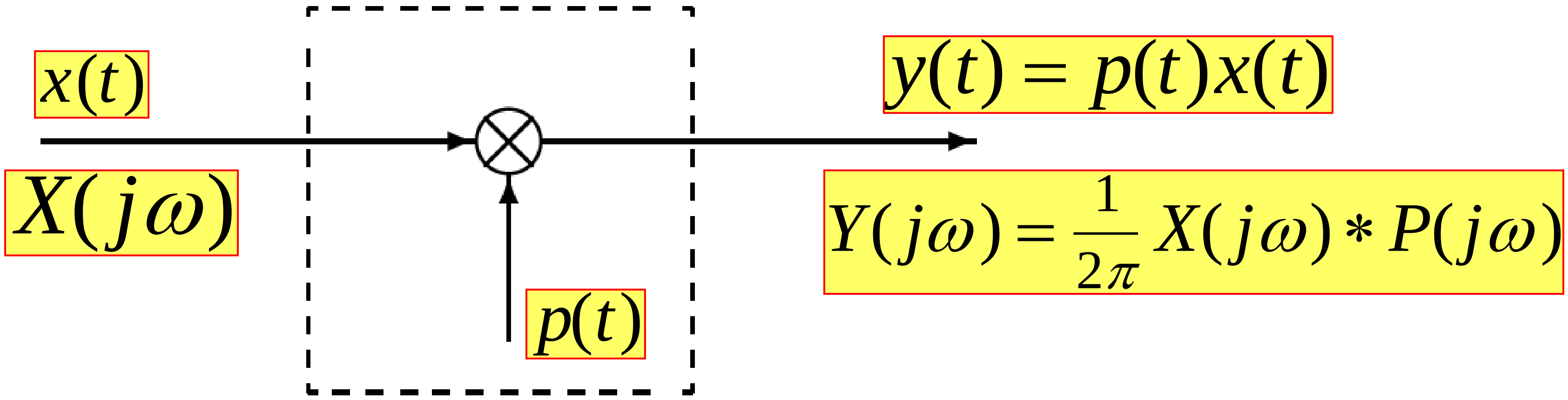
f_{co} "cutoff freq."



$$y(t)=\frac{4}{\pi}\sin(50\pi t)+\frac{4}{3\pi}\sin(150\pi t)$$



Signal Multiplier (Modulator)



- Multiplication in the time-domain corresponds to convolution in the frequency-domain.

$$Y(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\theta) P(j(\omega - \theta)) d\theta$$

Multiplication Property

$$Y(j\omega)=\int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt=\int_{-\infty}^{\infty} p(t)x(t)e^{-j\omega t}dt$$

$$=\int_{-\infty}^{\infty} p(t)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\lambda)e^{j\lambda t}d\lambda\right)e^{-j\omega t}dt$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\lambda)\left(\int_{-\infty}^{\infty} p(t)e^{-j(\omega-\lambda)t}dt\right)d\lambda$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\lambda)P(j(\omega-\lambda))d\lambda$$

$$=\frac{1}{2\pi}X(j\omega)*P(j\omega)$$

Frequency Shifting Property (1)

$$\boxed{x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega-\omega_0))}$$

$$\int_{-\infty}^{\infty} x(t)e^{j\omega_0 t}e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}dt = X(j(\omega-\omega_0))$$

$$Y(j\omega) = \frac{1}{2\pi}X(j\omega) * 2\pi\delta(\omega-\omega_0) = X(j(\omega-\omega_0))$$

$$\boxed{y(t) = \frac{\sin 7t}{\pi t}e^{j\omega_0 t} \Leftrightarrow Y(j\omega) = \begin{cases} 1 & \omega_0 - 7 < \omega < \omega_0 + 7 \\ 0 & elsewhere \end{cases}}$$

Frequency Shifting Property (2)

$$\boxed{y(t) = x(t)p(t) \Leftrightarrow Y(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)}$$

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \quad \Leftrightarrow \quad X(j\omega) = \frac{\sin(\omega T/2)}{(\omega/2)}$$

$$p(t) = \cos(\omega_0 t) \Leftrightarrow P(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

$$y(t) = x(t)\cos(\omega_0 t) \Leftrightarrow$$

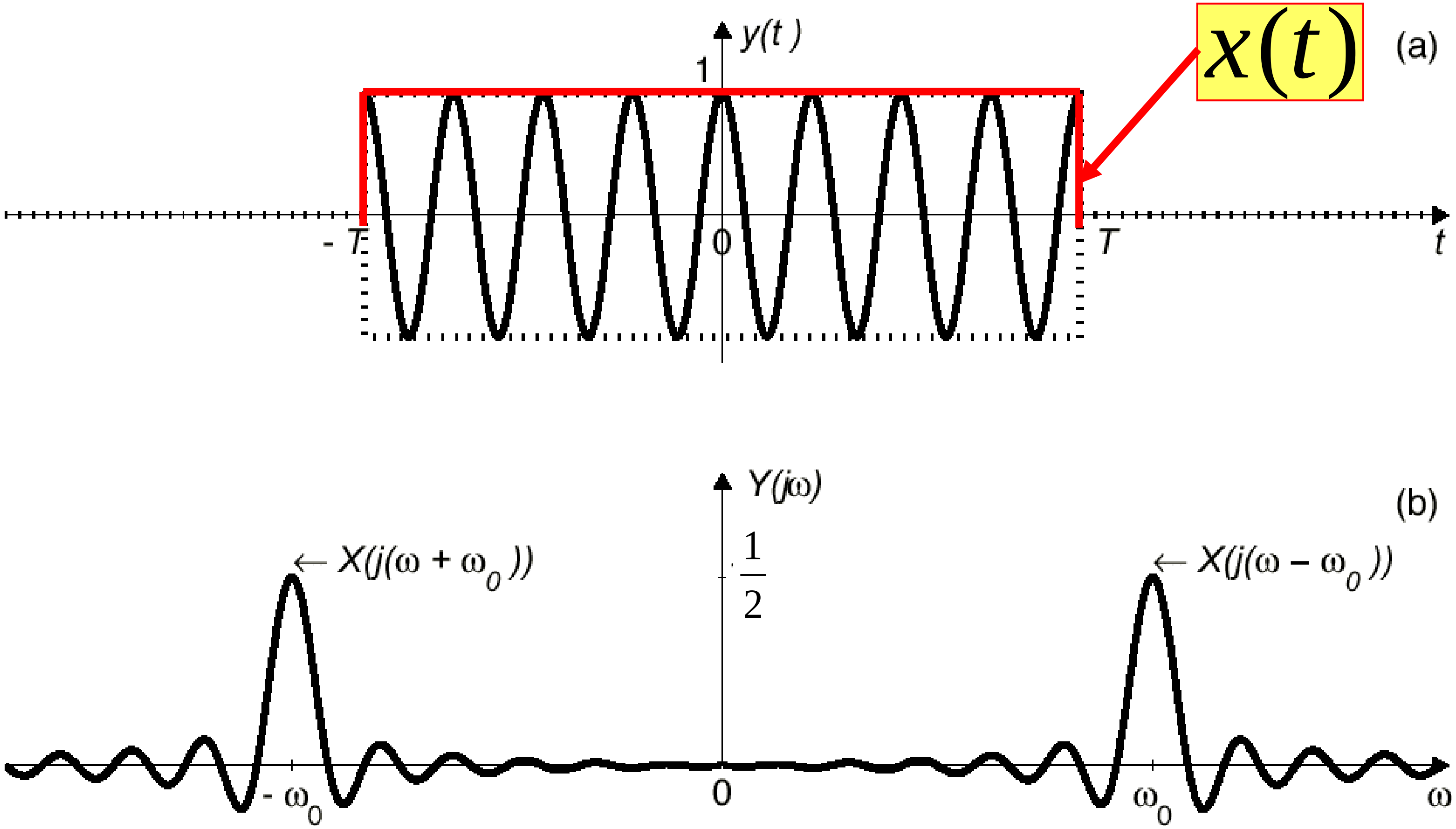
$$Y(j\omega) = \frac{1}{2\pi} X(j\omega) * [\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)]$$

$$\boxed{Y(j\omega) = \frac{1}{2} X(j(\omega - \omega_0)) + \frac{1}{2} X(j(\omega + \omega_0))}$$

Frequency Shifting Property (3)

$$y(t) = x(t)\cos(\omega_0 t) \Leftrightarrow$$

$$Y(j\omega) = \frac{1}{2}X(j(\omega-\omega_0)) + \frac{1}{2}X(j(\omega+\omega_0))$$



Differentiation Property

$$\frac{dx(t)}{dt} = \frac{d}{dt}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega\right)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} (j\omega)X(j\omega)e^{j\omega t}d\omega$$

Multiply by jω.

$$\frac{d}{dt}\left(e^{-at}u(t)\right) = -ae^{-at}u(t) + e^{-at}\delta(t)$$

$$= \delta(t) - ae^{-at}u(t)$$

$$\Leftrightarrow \frac{j\omega}{a+j\omega}$$

Differential Equations

- If a system is described by a differential equation,

$$\frac{d^2y(t)}{dt^2}+2\frac{dy(t)}{dt}=\frac{dx(t)}{dt}+3x(t)$$

$$(j\omega)^2Y(j\omega)+2(j\omega)Y(j\omega)=(j\omega)X(j\omega)+3X(j\omega)$$

$$H(j\omega)=\frac{Y(j\omega)}{X(j\omega)}=\frac{(j\omega)+3}{(j\omega)^2+2(j\omega)}$$

- For a higher-order differential equation,

$$\sum_{k=0}^{N}a_k\frac{d^ky(t)}{dt^k}=\sum_{n=0}^{M}b_n\frac{d^nx(t)}{dt^n}$$

$$H(j\omega)=\sum_{n=0}^{M}b_n(j\omega)^n\Big/\sum_{k=0}^{N}a_k(j\omega)^k$$

Strategy for using the FT

- Develop a set of known Fourier transform pairs.
  - Refer to Table 11-2.
- Develop a set of "theorems" or properties of the Fourier transform.
  - Refer to Table 11-3.
- Develop skill in formulating the problem in either the time-domain or the frequency-domain, *which ever leads to the simplest solution*.

FT of the Impulse Train

- The periodic impulse train is

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$\boxed{\omega_0 = 2\pi / T_0}$$

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \delta(t) e^{-j\omega_0 kt} dt = \frac{1}{T_0} \quad \text{for all } k$$

$$\boxed{\therefore \quad P(j\omega) = \left(\frac{2\pi}{T_0}\right) \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0)}$$

Thank you

- Homework
  - P-10.2, 3, 5, 7, 8, 9
  - P-11.2(b,c), 3(b,d), 4(a,b), 6(b,c), 7(b), 8(c,d), 9, 11, 13, 14, 17
- Reading assignment
  - ~ Chapter 12