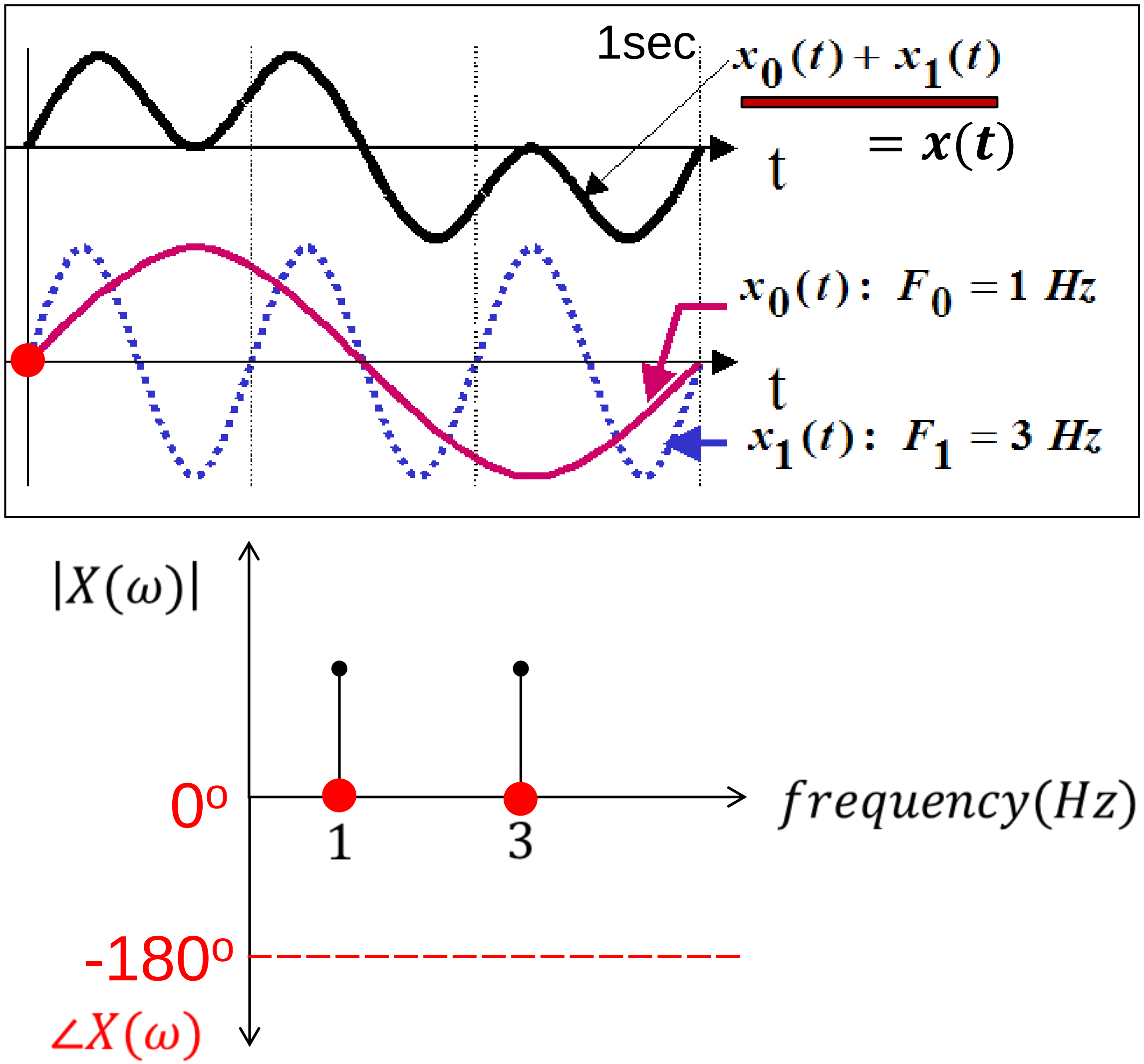
Frequency domain analysis

CT signals and Systems

FT of (real or complex) CT signal x(t):



CT signals and Systems

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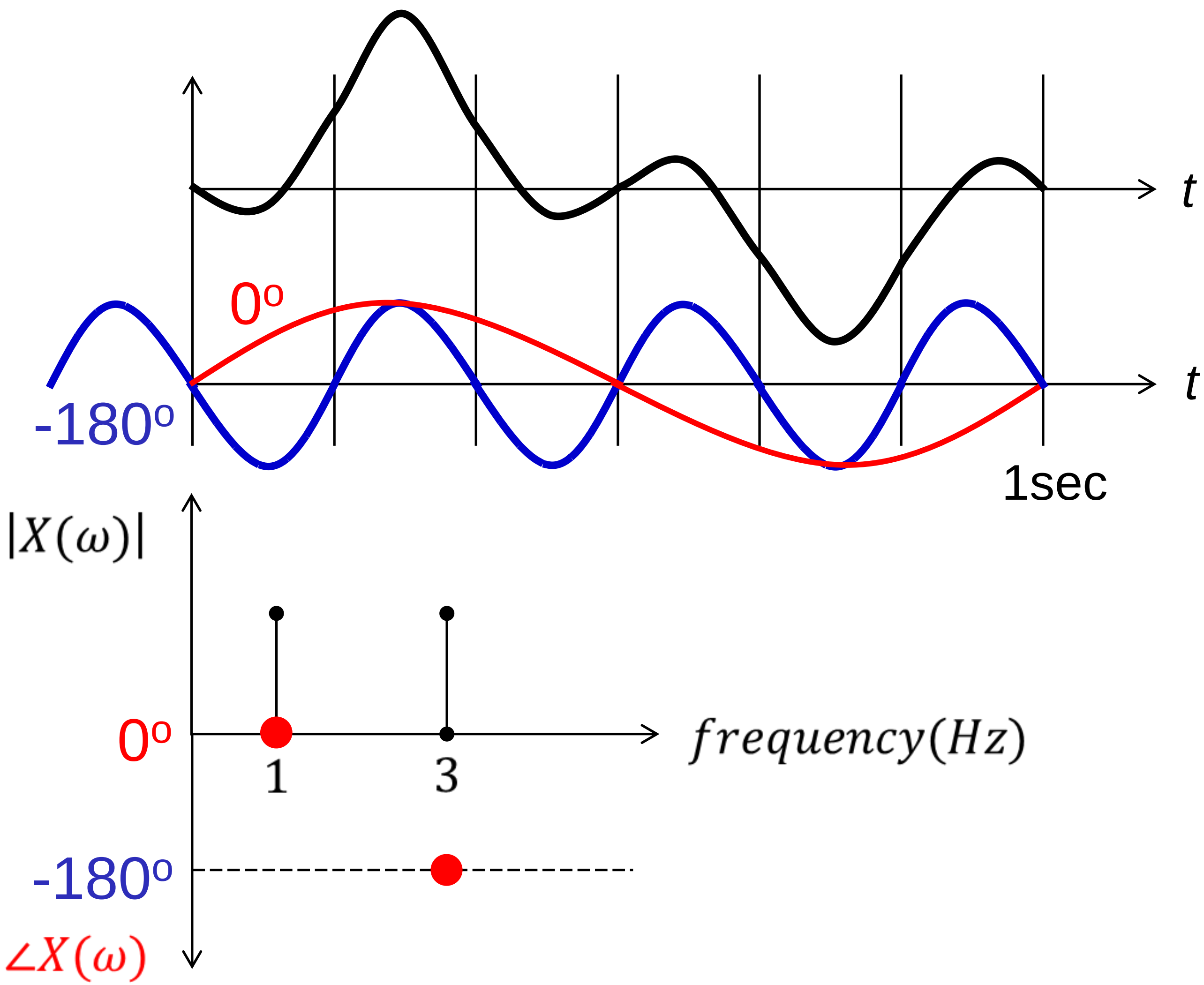


CT signals and Systems

CTFT and Analog Frequency

$$x(t) \overset{FT}{\Leftrightarrow} X(\omega)$$

- $\omega = 2\pi f$: **Angular frequency [radians/second]**
- f : **Frequency in Hertz [Hz]**

- **Continuous-time sinusoidal signal**

$$x(t) = A\cos(\omega t + \phi), \qquad -\infty < \phi < +\infty$$

$$A: Amplitude, \qquad \phi: Phase$$

$x(t)$ is periodic with the fundamental period $T = 1/f$

$$\Rightarrow \; x(t) = x(t + T)$$

CT signals and Systems

CTFT and Analog Frequency

❖ Complex exponential signals

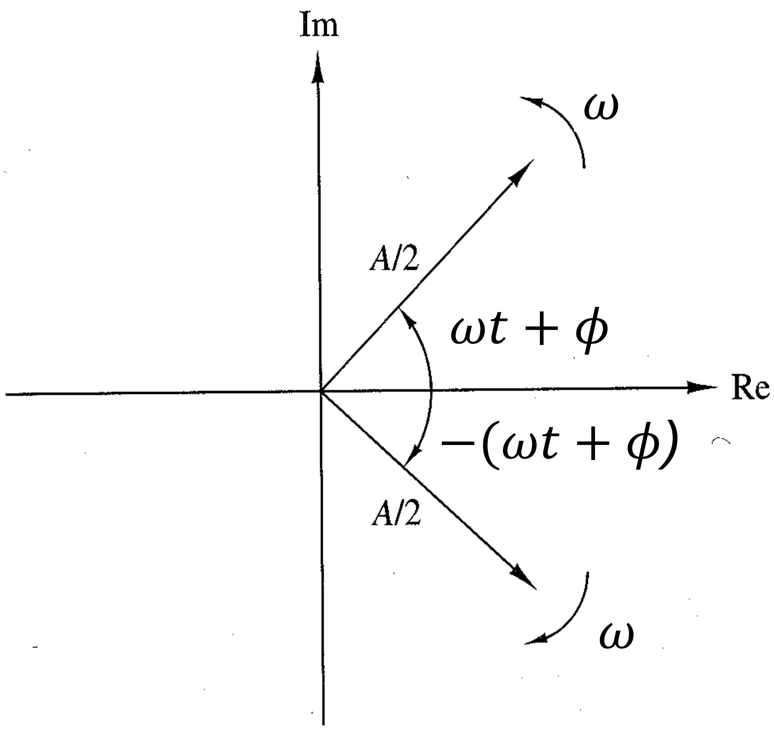
$$x(t) = Ae^{j(\omega t+\phi)}$$

$$x(t) = A\cos(\omega t+\phi) = \frac{A}{2}\left[e^{j(\omega t+\phi)} + e^{-j(\omega t+\phi)}\right]$$

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$$

$$\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$$

$$\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}$$



Frequency is a positive quantity.
But, we often define and use positive and negative frequencies for analysis purpose only.

Figure 1.11 Representation of a cosine function by a pair of complex-conjugate exponentials (phasors).

CT signals and Systems

FT of (real or complex) CT signal x(t)

$$X^F(\omega) = X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt = X(f)$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X^F(\omega) e^{j\omega t} d\omega = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t} df$$

$$x(t) \overset{FT}{\Longleftrightarrow} X(\omega) \qquad \omega = 2\pi f$$

CT signals and Systems

FT of (real or complex) CT signal x(t)

❖ **A sufficient condition for the existence of the FT**

1) x(t) has a finite number of discontinuities
2) x(t) has a finite number of maxima and minima
3) x(t) is absolutely integrable, i.e.,

$$\left| \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \right| \leq \int_{-\infty}^{\infty} |x(t)| dt < \infty$$

CTFT examples

Basic signals

$$\boxed{\delta(t) \leftrightarrow \int_{-\infty}^{\infty} \delta(t) e^{-j\Omega t} dt = 1 \qquad \Omega = 2\pi F}$$

$\omega \longrightarrow \Omega$

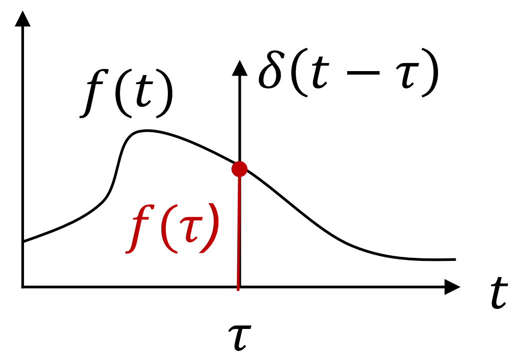
$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

$$\int_{-\infty}^{\infty} \delta(t) f(t) dt = f(0)$$

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\pm j\Omega t} d\Omega = \int_{-\infty}^{\infty} e^{\pm j2\pi F t} dF$$

$$\int_{-\infty}^{\infty} \delta(t-\tau) f(t) dt = f(\tau)$$

$$e^{j\Omega_0 t} \leftrightarrow \int_{-\infty}^{\infty} e^{j\Omega_0 t} e^{-j\Omega t} dt = \int_{-\infty}^{\infty} e^{-j(\Omega - \Omega_0)t} dt = 2\pi\delta(\Omega - \Omega_0)$$



CTFT examples

Basic signals

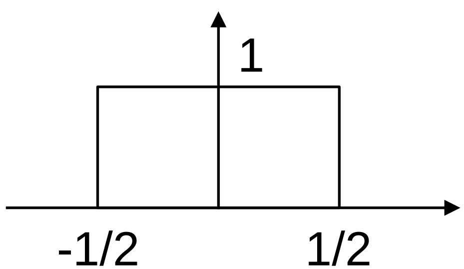
$$rect(t) = \Pi(t) \longleftrightarrow \operatorname{sinc}(f) = \operatorname{sinc}(\omega/2\pi)$$

$$\Pi(t/T) \longleftrightarrow T \cdot \operatorname{sinc}(fT) = T \cdot \operatorname{sinc}(\omega T/2\pi)$$

$$\operatorname{sinc}(t) \longleftrightarrow rect(\omega/2\pi) = rect(f)$$

$$\exp(-\pi t^2) \longleftrightarrow \exp(-\pi f^2)$$

$$\int_{-\infty}^{\infty} \Pi(t) e^{-j\omega t} dt = \int_{-1/2}^{1/2} e^{-j\omega t} dt = \frac{1}{-j\omega}\left\{e^{-j\omega/2} - e^{+j\omega/2}\right\}$$



$$= \frac{2}{\omega}\frac{e^{j\omega/2} - e^{-j\omega/2}}{2j} = \frac{1}{\pi f} sin(\pi f) = \operatorname{sinc} f$$

CTFT Properties

Linearity: $z(t) = ax(t) + by(t) \Leftrightarrow z(\omega) = aX(\omega) + bY(\omega)$

Time shift: $x(t-\tau) \longleftrightarrow e^{-i\omega\tau}X(\omega) \qquad x(t+\tau) \longleftrightarrow e^{+i\omega\tau}X(\omega)$

Frequency shift (modulation): $e^{j\omega_c t}x(t) \leftrightarrow X(\omega - \omega_c)$

Time and frequency scale: $x(at) \leftrightarrow \frac{1}{|a|}X(\frac{\omega}{a}) \implies x(-t) \leftrightarrow X(-\omega)$

$$\Rightarrow \Pi(t/T) \longleftrightarrow T \cdot \operatorname{sinc}(fT) = T \cdot \operatorname{sinc}(\omega T/2\pi)$$

Conjugation: $x^*(t) \leftrightarrow X^*(-\omega)$ or $\overline{x}(t) \leftrightarrow \overline{X}(-\omega)$

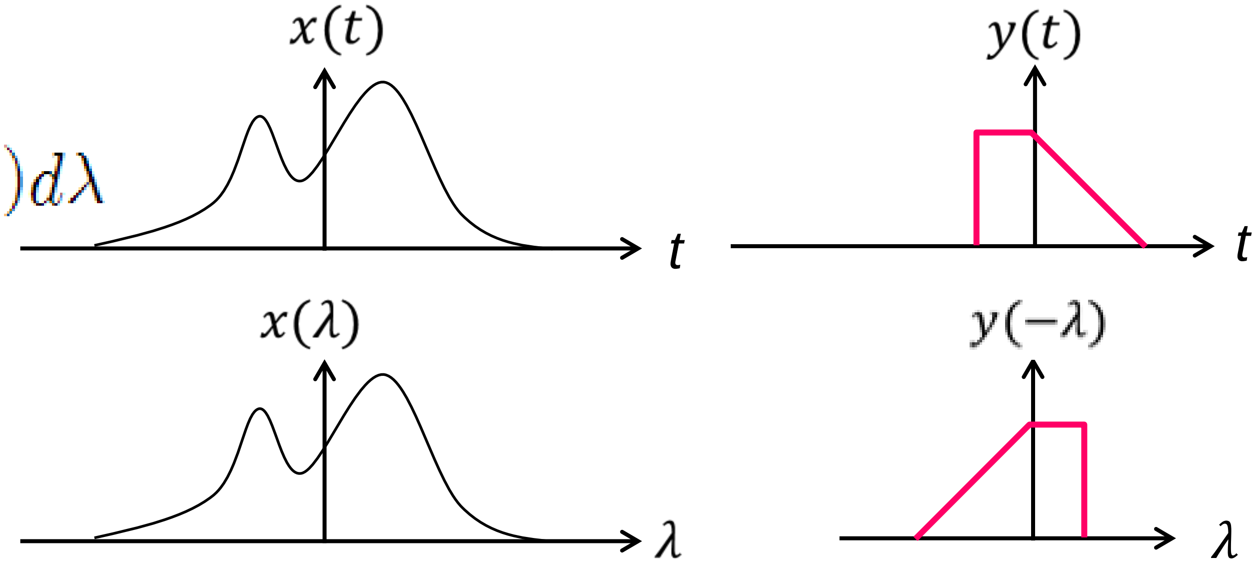
$x^*(-t) \leftrightarrow X^*(\omega)$

Time-domain convolution: $\{x * y\}(t) = x(t) * y(t)$

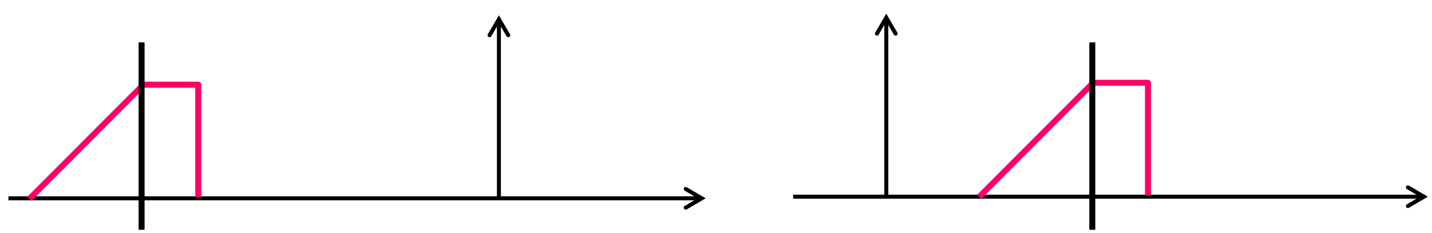
$$x(t) * y(t) = \int_{-\infty}^{+\infty} x(\lambda)y(t-\lambda)d\lambda \longleftrightarrow X(\omega)Y(\omega)$$

Convolution

$$x(t) * y(t)$$
$$= \int_{-\infty}^{+\infty} x(\lambda) y(t-\lambda) d\lambda$$



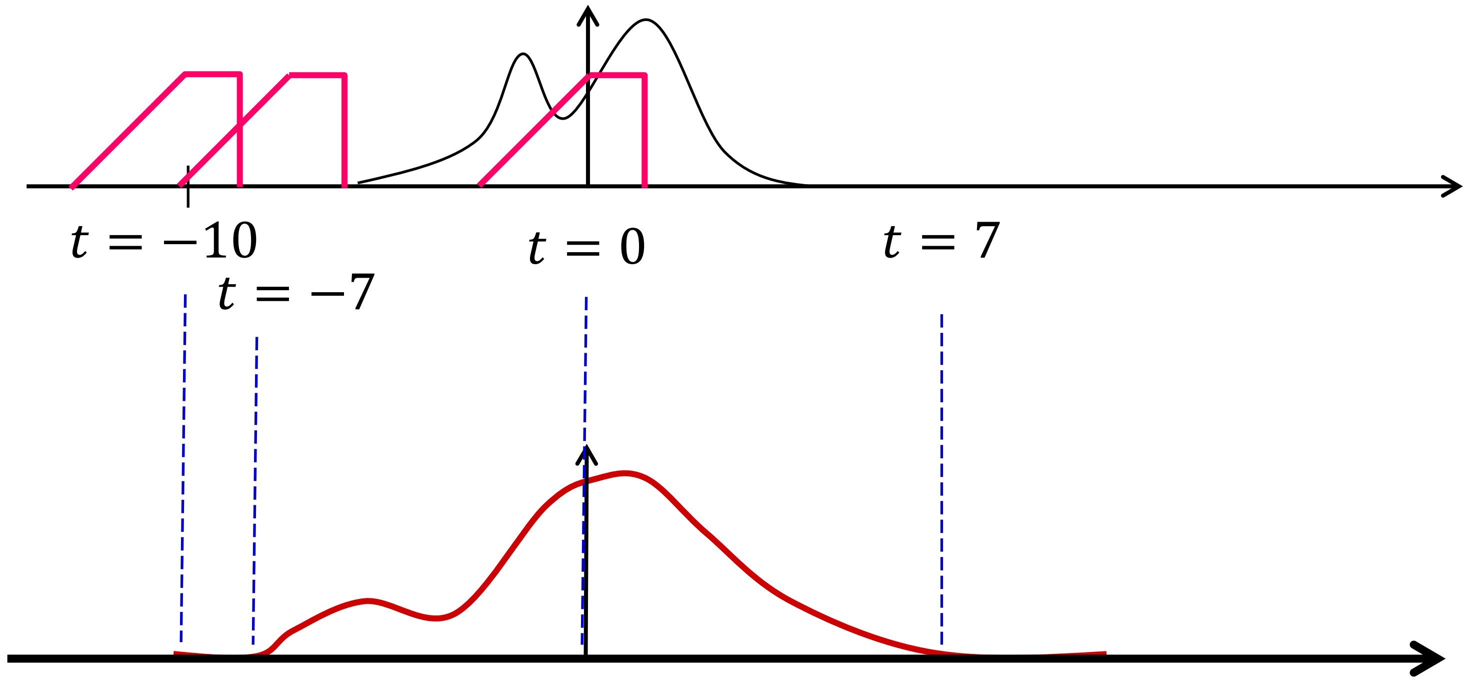
$$\boxed{y(t-\lambda) = y(-(\lambda - t))}$$



t < 0 *(ex. t = -10sec)*

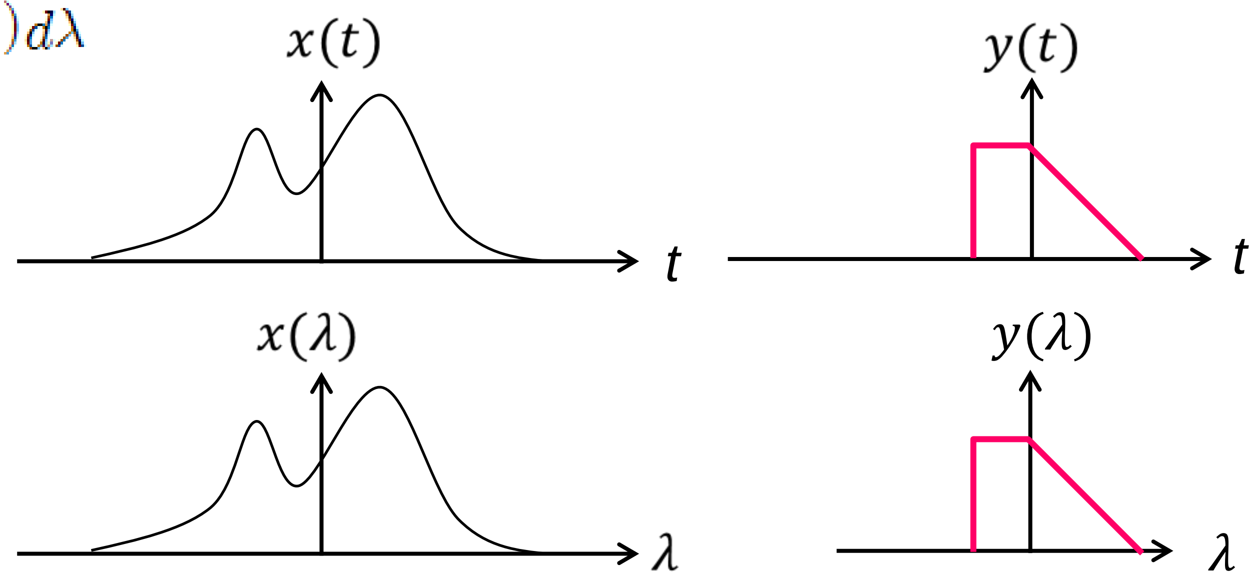
t > 0 *(ex. t = +5sec)*

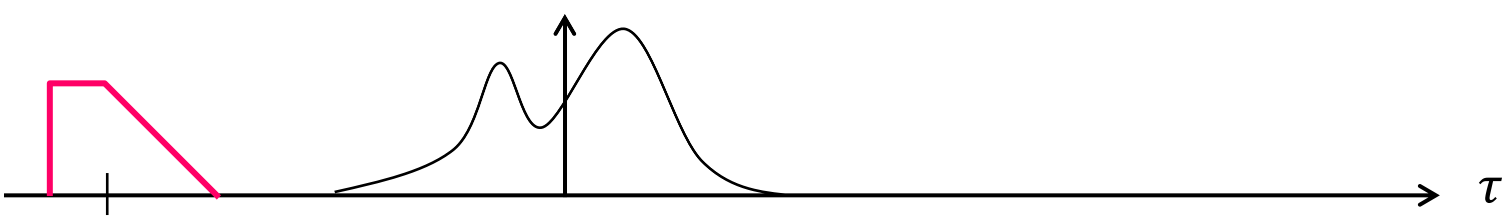
Convolution



Correlation

$$R_{xy}(t) = \int_{-\infty}^{+\infty} x(\lambda)\, y(\lambda - t)\, d\lambda$$





CTFT Properties

Crosscorrelation:

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t) y^*(t-\tau)\,dt$$

$$= \int_{-\infty}^{\infty} x(\lambda) y^*(-(\tau-\lambda))\,d\lambda = x(\tau) * y^*(-\tau) \quad \Leftrightarrow \quad X(\omega) Y^*(\omega$$

Autocorrelation:

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t) x^*(t-\tau)\,dt = x(\tau) * x^*(-\tau) \quad \Leftrightarrow \quad X(\omega) X^*(\omega) = |X(\omega)|^2$$

CTFT Properties

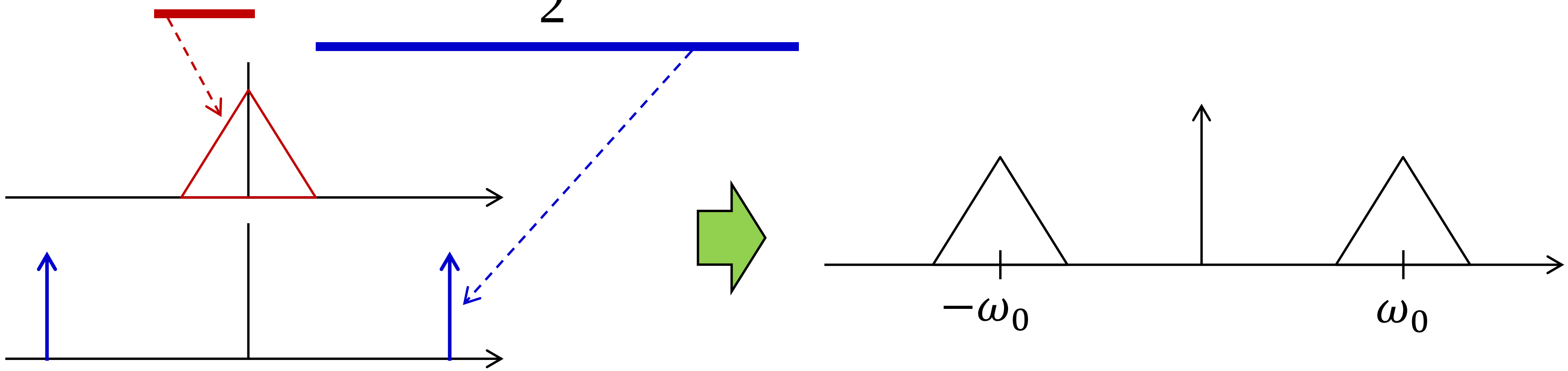
Multiplication: $x(t)y(t) \leftrightarrow \dfrac{1}{2\pi} X(\omega) * Y(\omega)$

$$x(t) = a(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}\{A(\omega-\omega_0) + A(\omega+\omega_0)\}$$

$$x(t) = a(t)\cos(\omega_0 t) = \frac{1}{2} a(t) \times \{\exp(j\omega_0 t) + \exp(-j\omega_0 t)\}$$

$$X(\omega) = FT[a(t)\cos(\omega_0 t)] = \frac{1}{2\pi} A(\omega) * FT[\frac{\exp(j\omega_0 t) + \exp(-j\omega_0 t)}{2}]$$

$$= A(\omega) * \frac{\delta(\omega-\omega_0) + \delta(\omega+\omega_0)}{2}$$



CTFT Properties

Parseval's theorem:

$$\int_{-\infty}^{\infty} x(t) y^*(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) Y^*(\omega) d\omega$$

$$\Rightarrow \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

CTFT Properties

Symmetry: If x(t) is real, then $X^*(\omega) = X(-\omega)$ ⬅ $\boxed{x^*(t) \leftrightarrow X^*(-\omega)}$

$\boxed{x(t) = x^*(t)}$ $X(\omega) = X^*(-\omega)$

Magnitude $|X(\omega)|$, $Re[X(\omega)]$: (even) symmetric

Phase $\measuredangle X(\omega)$, $Im[X(\omega)]$: odd or anti- symmetric

⇨ Magnitude response : $M(\omega) = |X(\omega)|$

Phase response : $\varphi(\omega) = \measuredangle X(\omega)$

$$X(\omega) = M(\omega)e^{j\varphi(\omega)} = M(\omega)\cdot \cos\varphi(\omega) + j\,M(\omega)\cdot \sin\varphi(\omega)$$
$$= X_R(\omega) + jX_I(\omega)$$

$$X^*(-\omega) = M(-\omega)e^{-j\varphi(-\omega)} = M(-\omega)\cdot \cos\varphi(-\omega) - j\,M(-\omega)\cdot \sin\varphi(-\omega)$$
$$= X_R(-\omega) - jX_I(-\omega)$$