Lecture 3:
Discrete Fourier Transform

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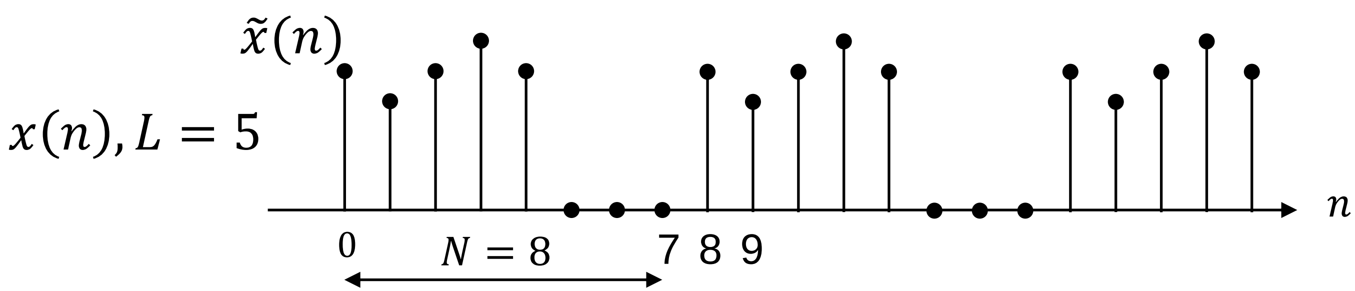
DFT

Background

- Consider $x[n] \longleftrightarrow X(\Theta)$ defined for all $\Theta \in [-\pi, \pi]$
- Do we have to know $X(\Theta)$ for all $\Theta \in [-\pi, \pi]$ to recover $x[n]$?

- For a finite-duration sequence $x(n),\ 0 \leqq n \leqq L-1$

$$\tilde{x}(n) = \sum_{k=-\infty}^{\infty} x(n-kN) \quad where \quad N \geqq L \qquad \boxed{\tilde{x}(n) = x(n \mod N)}$$



$$\Rightarrow \tilde{X}(\theta) = cX(\theta_k) = X(k),\ \theta_k = \frac{2\pi}{N}k,\ \ where\ \ 0 \leqq k \leqq N-1$$

Note) *Both* $x(n)$ *and* $X(k)$ *are discrete*

DFT

Background

For $x(n)$ with a finite duration L, i.e., $x(n)$, $0 \leqq n \leqq L-1$

$$X(e^{j\theta}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\theta n}$$

If we evaluate it at $\theta = \frac{2\pi k}{N}$, $N \geq L$, *to obtain*

$$X(e^{j\theta})|_{\theta = k \cdot (2\pi/N)} = X(e^{2\pi k/N}) = X(\frac{2\pi}{N} k)$$

$$\Rightarrow \quad X(e^{2\pi k/N}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi kn/N}, \qquad k = 0, 1, \ldots, N-1$$

$$X(e^{2\pi k/N}) = \cdots + \underbrace{\sum_{n=-N}^{-1} x(n) e^{-j2\pi kn/N}}_{(l=-1)} + \underbrace{\sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}}_{(l=0)} + \underbrace{\sum_{n=N}^{2N-1} x(n) e^{-j2\pi kn/N}}_{(l=1)} + \cdots$$

$$= \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n) e^{-j2\pi kn/N}$$

DFT

By the change of variable, $n = n - lN$

$$X(e^{2\pi k/N}) = \sum_{n=0}^{N-1}\left[\sum_{l=-\infty}^{\infty} x(n-lN)e^{-j2\pi k(n-lN)/N}\right]$$

$$= \sum_{n=0}^{N-1}\left[\sum_{l=-\infty}^{\infty} x(n-lN)e^{-j2\pi kn/N}\right] \text{for } k = 0,1,\ldots,N-1$$

$$\tilde{x}(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$$

Again, repeated sequence of $x(n)$ *yield sampled spectrum* $X(k)$

When $N \geq L$

$$X(k) = X(e^{j(2\pi/N)k}) = \sum_{n=0}^{N-1} x(n)e^{j(2\pi/N)kn}, \quad for \;\; 0 \leqq k \leqq N-1$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j(2\pi/N)kn}, \quad for \;\; 0 \leqq n \leqq N-1 \quad where \;\; X(k) = X(e^{j2\pi/N)k})$$

Remember: $X(k)$ is the DTFT of $\tilde{x}(n)$

DFT

Definition

❖N-point DFT

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

$$0 \leqq k, n \leqq N-1$$

Where

$$W_N = e^{j2\pi/N} \Rightarrow W_N^0 = W_N^N = 1, \;\; W_N^n = W_N^{n \pm mN}$$

$$X(k) = DTFT\left[\, \tilde{x}(n) = \sum_{m=-\infty}^{\infty} x(n+mN) \,\right]$$

$$X(k) = X(k+N)$$

DFT

Definition

- $W_N = e^{j2\pi/N}$

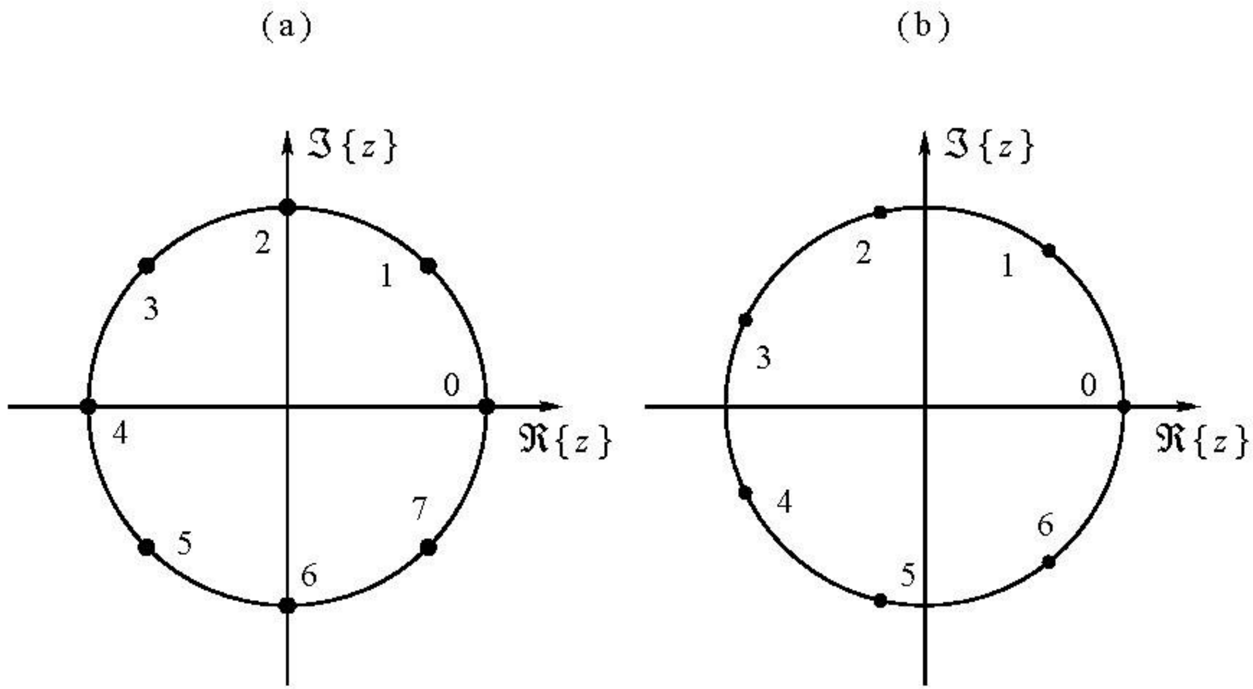
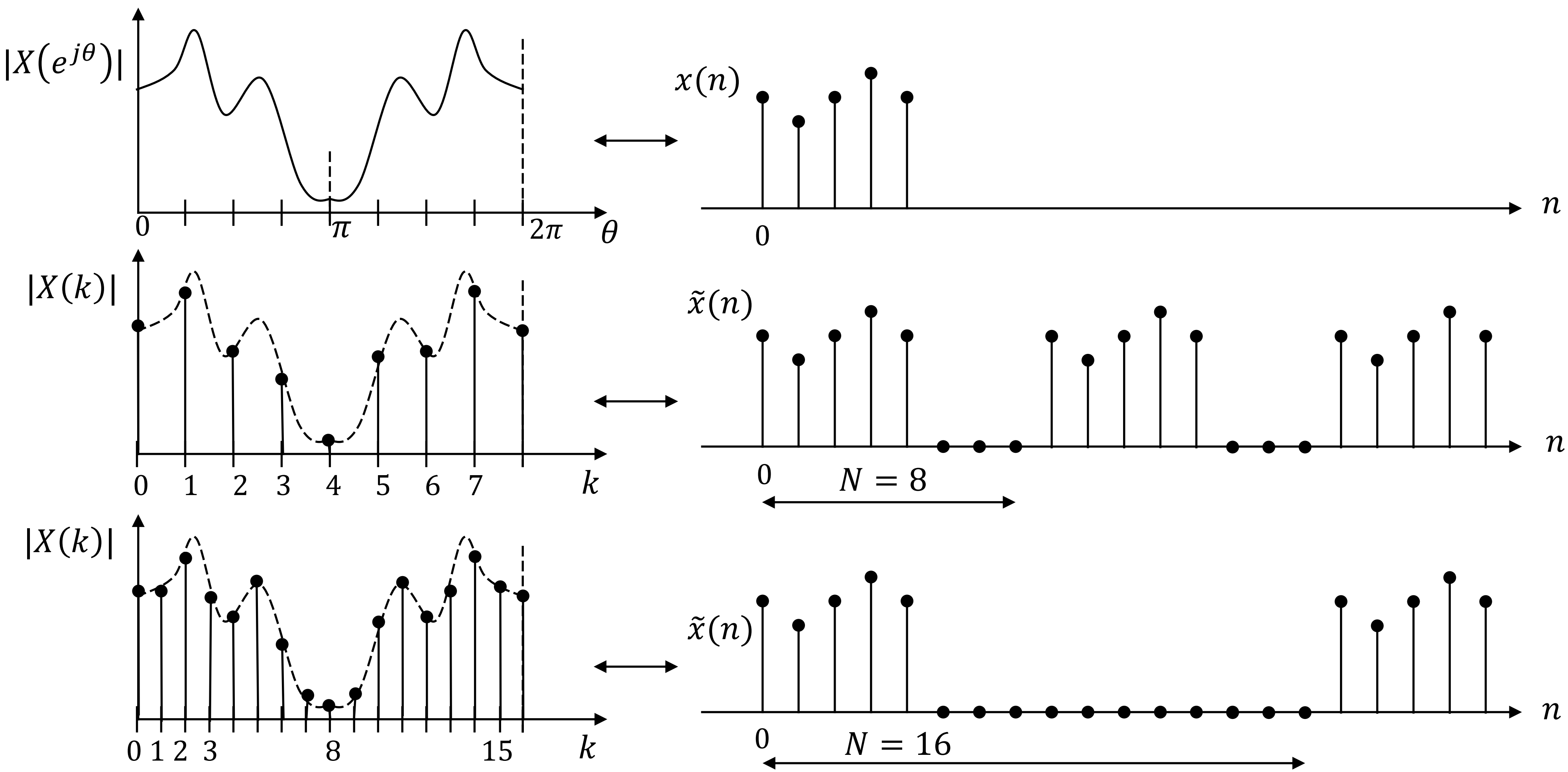


Figure 4.2 The sequence W_N^n in the complex plane: (a) even N; (b) odd N. The numbers indicate the values of n.

DFT

Definition

❖N-point DFT



DFT

Some properties of DFT and its kernel function

$$\sum_{n=0}^{N-1} W_N^{kn} = N\delta[k \mod N] = N\delta(k - mN) = \begin{cases} N, & (k \mod N) = 0 \\ 0, & (k \mod N) \neq 0 \end{cases}$$

Proof)

$$k = mN \qquad \sum_{n=0}^{N-1} W_N^{kn} = \sum_{n=0}^{N-1} W_N^{nmN} = \sum_{n=0}^{N-1} 1 = N$$

$$k \neq mN \qquad \sum_{n=0}^{N-1} W_N^{kn} = \frac{W_N^{Nk} - 1}{W_N^{k} - 1} = \frac{1 - 1}{W_N^{k} - 1} = 0.$$

Examples)

1) $$x[n] = \begin{cases} 1, & n = 0 \\ 0, & 1 \leq k \leq N-1 \end{cases} \longleftrightarrow X[k] = 1, \quad 0 \leq k \leq N-1$$

2) $$x[n] = 1, \quad 0 \leq n \leq N-1$$

$$\longleftrightarrow X[k] = \sum_{n=0}^{N-1} W_N^{-kn} = N \cdot \delta(k \mod N) = \begin{cases} N, & k = 0 \\ 0, & 1 \leqq k \leqq N-1 \end{cases}$$

DFT

Inverse DFT

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, W_N^{kn} \qquad n, k \le N-1$$

$$\frac{1}{N}\sum_{k=0}^{N-1} X[k]\, W_N^{kn} = \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{m=0}^{N-1} x[m]\, W_N^{-km}\right] W_N^{kn} = \frac{1}{N}\sum_{m=0}^{N-1} x[m]\left[\sum_{k=0}^{N-1} W_N^{(n-m)k}\right]$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} x[m] N\delta[(n-m) \bmod N] = x[n]$$

DFT

Things to know about DFT

- **We use DFT to examine frequency spectrum of a finite length, L, discrete sequence (digital sequence)**
- **When taking its N-DFT, we assume that the sequence is periodic**
- **Hence, DFT length N must be chosen larger than L to avoid aliasing in time domain**
- **Frequency aliasing occurs if x(n), or x(t), is not band-limited**

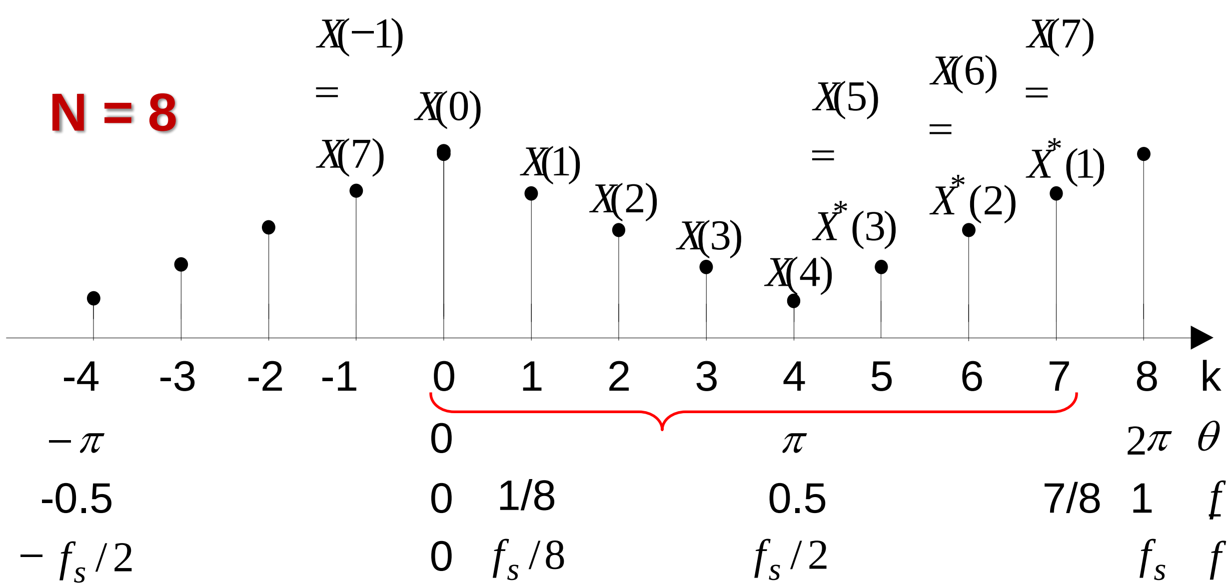
DFT

Things to know about DFT

- **Analog frequency:** $k \to f_k = k \cdot \delta_f = k \cdot \dfrac{f_s}{N}$, $\delta_f = f_s / N$
- **Digital frequency:**

$$k \to \underline{f_k} = f_k' = \frac{k}{N} \;, \quad \theta_k = k \cdot \delta_\theta = 2\pi \cdot k/N \;, \quad \delta_\theta = 2\pi/N$$

$$X(k) = X(k+N) \Rightarrow X(-k) = X(N-k) = X^*(k)$$



Matrix Representation of DFT

DFT Matrix of dimension N

$$F_N = \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ W_N^0 & W_N^{-2} & W_N^{-4} & \cdots & W_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \cdots & \\ W_N^0 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \cdots & W_N^{-(N-1)^2} \end{bmatrix} \qquad X[k] = \sum_{n=0}^{N-1} x[n] W_N^{-kn}$$

$$x_N = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} \qquad X_N = \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix}$$

Matrix-form representation of DFT

$$X_N = F_N \cdot x_N$$

$$x_N = N^{-1} F_N^* X_N$$

Matrix Representation of DFT

$$X[k] = \sum_{n=0}^{N-1} W_N^{-nk} x[n]$$

$Ex)\ case\ for\ N = 4$

$$X[k] = W_4^{-0k}x[0] + W_4^{-1k}x[1] + W_4^{-2k}x[2] + W_4^{-3k}x[3]$$

$For\ X[k = 1]$

$$\begin{pmatrix} \boldsymbol{X[0]} \\ \boldsymbol{X[1]} \\ \boldsymbol{X[2]} \\ \boldsymbol{X[3]} \end{pmatrix} = \begin{pmatrix} \boldsymbol{W_4^{-0\cdot 0}} & \boldsymbol{W_4^{-1\cdot 0}} & \boldsymbol{W_4^{-2\cdot 0}} & \boldsymbol{W_4^{-3\cdot 0}} \\ \boldsymbol{W_4^{-0\cdot 1}} & \boldsymbol{W_4^{-1\cdot 1}} & \boldsymbol{W_4^{-2\cdot 1}} & \boldsymbol{W_4^{-3\cdot 1}} \\ \boldsymbol{W_4^{-0\cdot 2}} & \boldsymbol{W_4^{-1\cdot 2}} & \boldsymbol{W_4^{-2\cdot 2}} & \boldsymbol{W_4^{-3\cdot 2}} \\ \boldsymbol{W_4^{-0\cdot 3}} & \boldsymbol{W_4^{-1\cdot 3}} & \boldsymbol{W_4^{-2\cdot 3}} & \boldsymbol{W_4^{-3\cdot 3}} \end{pmatrix} \begin{pmatrix} \boldsymbol{x[0]} \\ \boldsymbol{x[1]} \\ \boldsymbol{x[2]} \\ \boldsymbol{x[3]} \end{pmatrix}$$

Matrix Representation of DFT

Properties of DFT matrix

- $W_N^{-nk} = 1 \quad \text{for} \quad n = 0 \ \text{and/or} \ k = 0$
- $W_N^{-kn} = W_N^{-nk} \qquad [W_N^{nm}]^* = W_N^{-nm}$
- $F_N \cdot F_N^{T*} = F_N \cdot F_N^{+} = N I_N$

$$[F_N \cdot F_N^{+}]_{kl} = \sum_{n=0}^{N-1} W_N^{-kn} \{ W_N^{-nl} \}^{*T} = \sum_{n=0}^{N-1} W_N^{-kn} W_N^{\ln} = \sum_{n=0}^{N-1} W_N^{(l-k)n} = N\delta[(l-k) \bmod N]$$

- $N^{-1/2} F_N$ is unitary: A symmetric unitary matrix: Normalized DFT matrix

Note: A complex $N \times N$ matrix is called a unitary matrix if $Q_N \cdot Q_N^{T*} = I_N$

If it is real, it is called an orthonormal matrix.

Matrix Representation of DFT

Properties of DFT matrix

- DFT basis vectors (orthogonal)

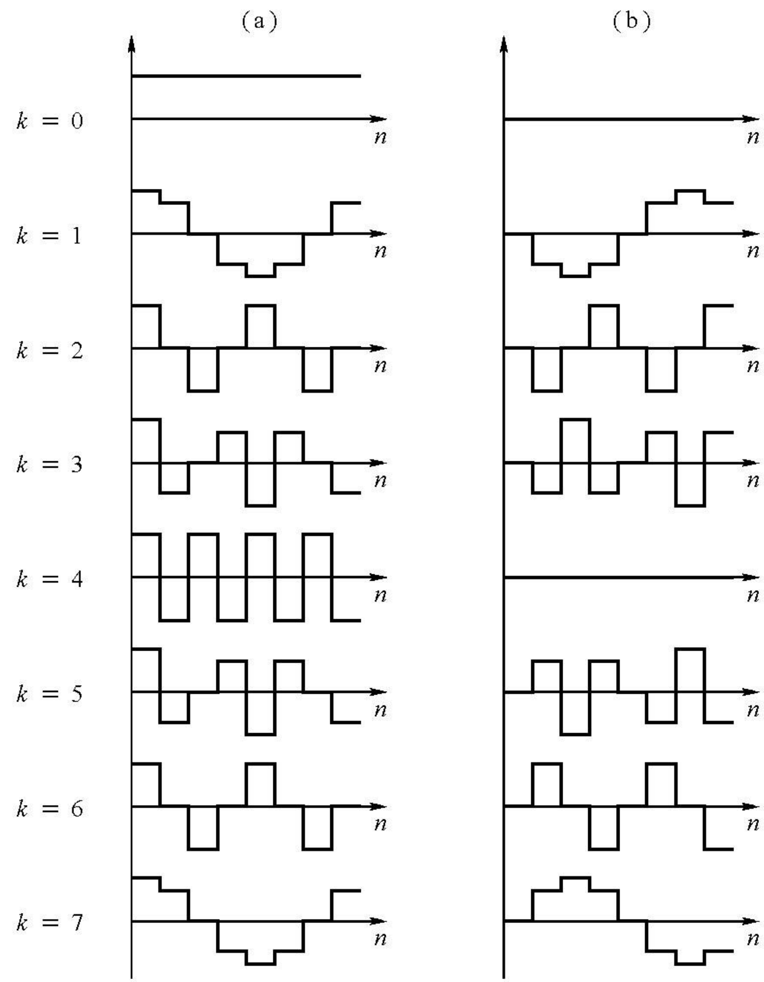


Figure 4.5 The DFT basis vectors for $N = 8$: (a) real part; (b) imaginary part.

Properties of the DFT

Linearity and Periodicity

$$z(n) = ax(n) + by(n) \Leftrightarrow Z(k) = aX(k) + bY(k)$$

$$X(k) = X(k+N)$$

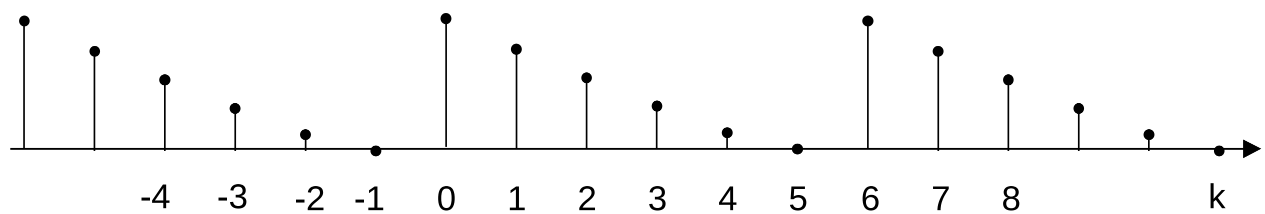
(Circular) Time Shift

$$\tilde{y}(n) = y(n) = \tilde{x}(n-m) = x[(n-m) \bmod N] \Leftrightarrow Y(k) = W_N^{-km} X(k)$$

(pf) See pp. 102 or can be proven graphically.

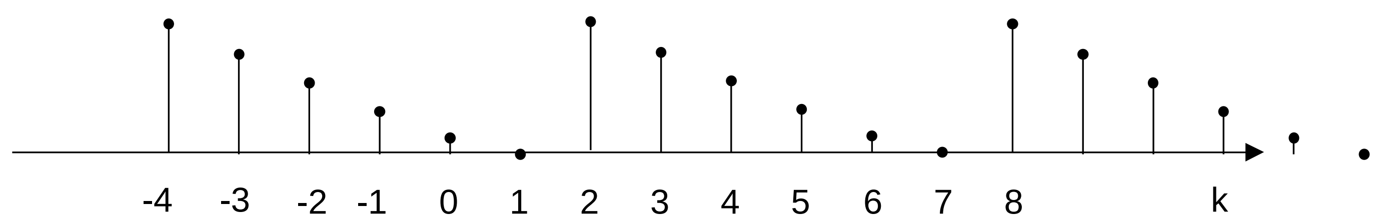
Note: As far as DFT is concerned, any shift is considered to be circular !!

$$\tilde{x}(n) = x(n \bmod N)$$



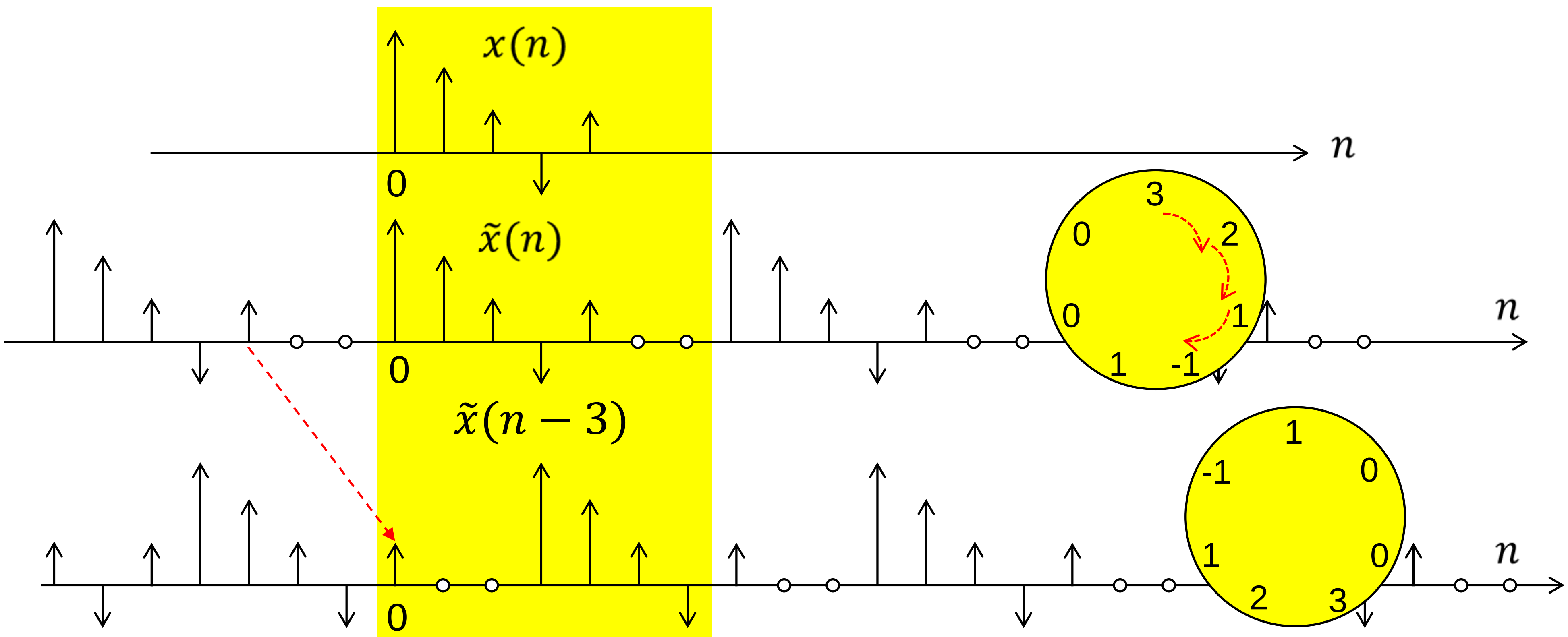
$$x(n-2)$$

$$\tilde{x}(n-2) = x((n-2) \bmod N)$$



Properties of the DFT

7-Point Circular Shift



Properties of the DFT

(Circular) frequency shift

$$W_N^{mn} x(n) \Leftrightarrow X(k-m) = X[(k-m) \mod N]$$

Complex conjugate

$$x^*(n) \Leftrightarrow X^*[(N-k) \mod N] = X^*(N-k)$$

$$DFT[x^*(n)] = \sum_{n=0}^{N-1} x^*(n) W_N^{-nk} = \left\{ \sum_{n=0}^{N-1} x(n) W_N^{nk} \right\}^* = X^*(-k) = X^*(N-k)$$

Symmetry properties

- If $x(n)$ is real-valued, then $X(k) = X^*(-k) = X^*(N-k)$ *or* $X^*(k) = X(-k) = X(N-k)$
  - $Re[X(k)], \quad |X(k)|$: Even symmetric
  - $Im[X(k)], \quad \measuredangle X(k)$: Odd symmetric

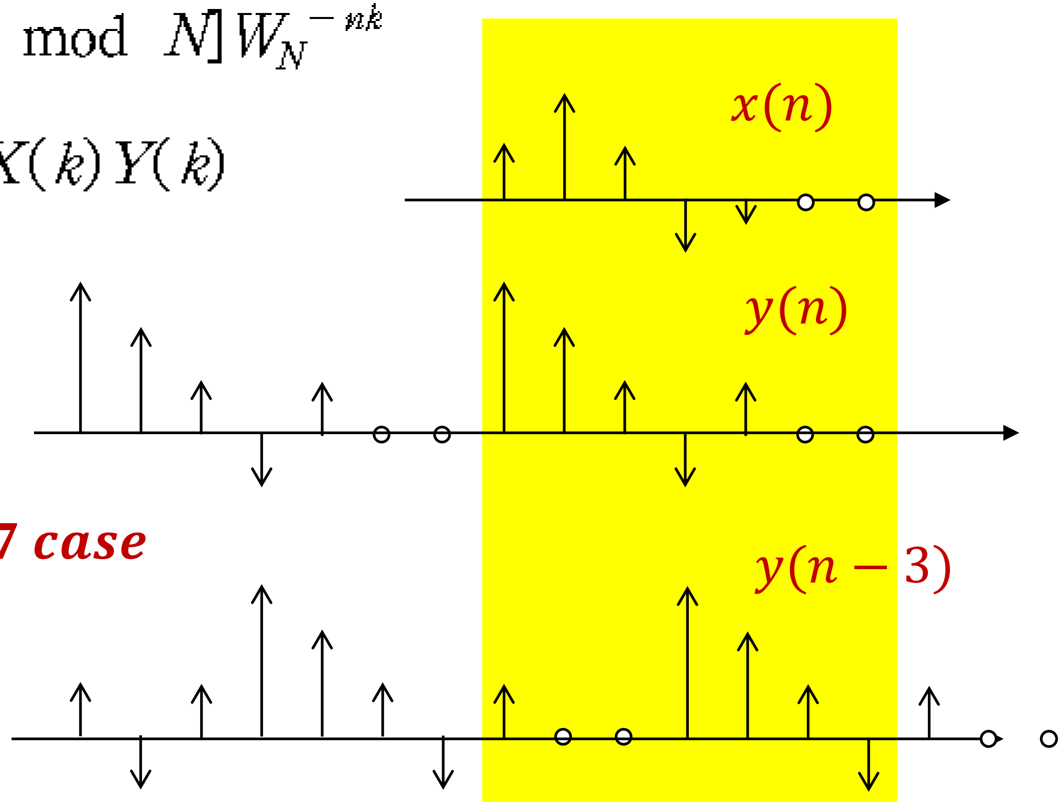
Properties of the DFT

Circular convolution in time domain

$$z(n) = x(n) * y(n) \Leftrightarrow Z(k) = X(k)Y(k) \quad \boxed{z(n) = x(n) \circledast y(n)}$$

$$\begin{aligned} Z(k) &= \sum_{n=0}^{N-1} z(n) W_N^{-nk} = \sum_{n=0}^{N-1} \left\{ \sum_{m=0}^{N-1} x(m) y[(n-m) \mod N] \right\} \cdot W_N^{-nk} \\ &= \sum_{m=0}^{N-1} x(m) \sum_{n=0}^{N-1} y[(n-m) \mod N] W_N^{-nk} \\ &= \sum_{m=0}^{N-1} x(m) W_N^{-mk} Y(k) = X(k)Y(k) \end{aligned}$$

Refer to 16 page.



Properties of the DFT

Multiplication in time domain

$$z(n) = x(n)y(n) \Leftrightarrow Z(k) = \frac{1}{N} X(k) \circledast Y(k)$$

Circular correlation

$$x(n) \circledast y^*(-n) \Leftrightarrow X(k)Y^*(k)$$

Parseval's theorem

$$\sum_{n=0}^{N-1} x(n)y^*(n) \Leftrightarrow \frac{1}{N} \sum_{k=0}^{N-1} X(k)Y^*(k)$$

$$\sum_{n=0}^{N-1} |x(n)|^2 \Leftrightarrow \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

Properties of the DFT

Zero padding

$$x(n) \;\; vs. \;\; x_a(n) \qquad x_a(n) = \begin{cases} x(n), & 0 \le n \le N-1 \\ 0, & N \le n \le M-1 \end{cases}$$

- $X(k) \;\; vs. \;\; X_a(k)$

$$X(k): \;\; k = 0,1,2,\ldots,N-1 \qquad k \rightarrow f_k = k \cdot \delta_f = k \cdot \frac{f_s}{N}, \qquad \delta_f = f_s/N$$

$$\Theta_k = k \cdot \delta_\Theta = 2\pi \cdot k/N, \qquad \delta_\Theta = 2\pi/N$$

$$X_a(k): \;\; k = 0,1,2,\ldots,M-1 \quad k \rightarrow f_k = k \cdot \underline{\delta_f} = k \cdot \frac{f_s}{M}, \qquad \underline{\delta_f} = f_s/M$$

$$\Theta_k' = k \cdot \underline{\delta_\Theta} = 2\pi \cdot k/M, \qquad \underline{\delta_\Theta} = 2\pi/M$$

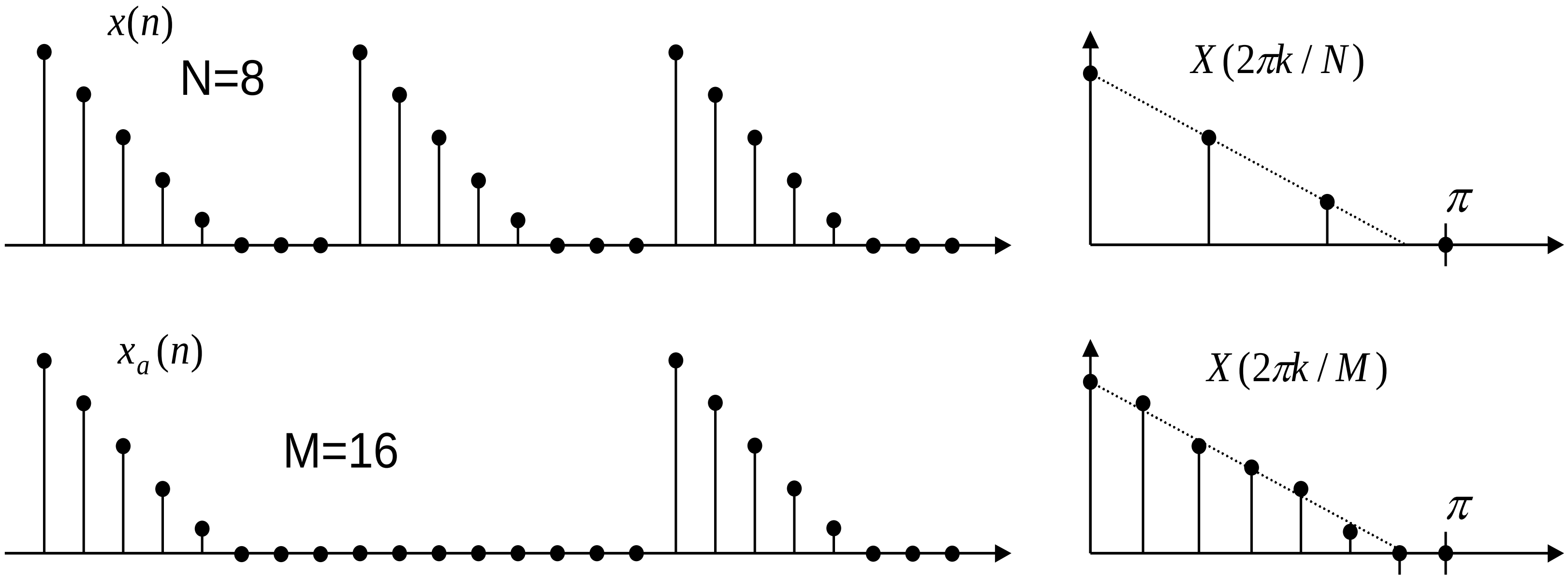
$$X_a(k) = \sum_{n=0}^{M-1} x_a(n) \exp(-j2\pi kn/M) = \sum_{n=0}^{N-1} x(n) \exp(-j2\pi kn/M) = X(\Theta_k')$$

where $\Theta_k' = 2\pi k/M, \;\; k = 0,1,2,\ldots,M-1$

$$\boxed{\underline{\delta_f} = \frac{N}{M}\delta_f \qquad \Theta_k' = \frac{N}{M}\Theta_k}$$

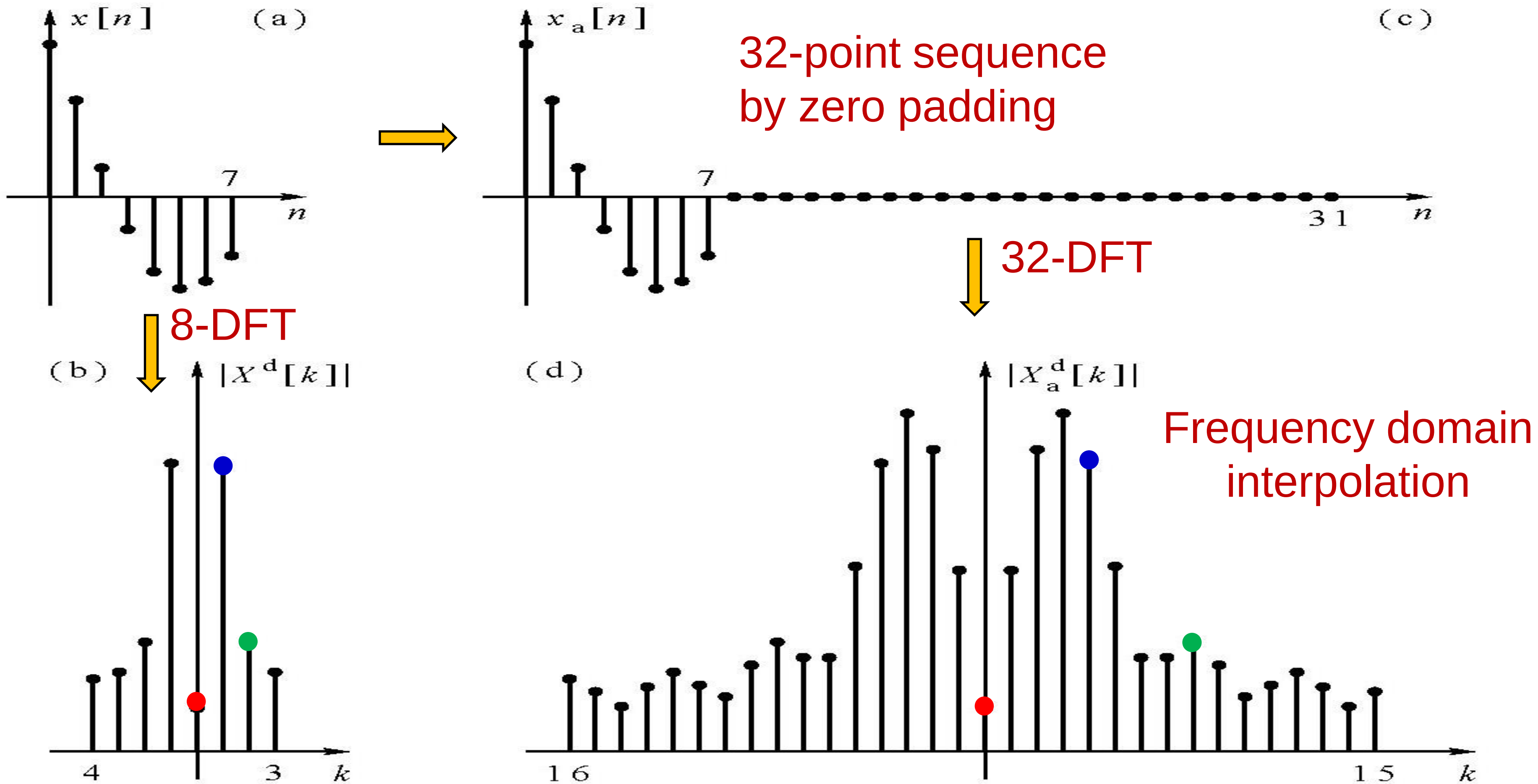
Properties of the DFT

Zero padding in time domain improves frequency resolution



Properties of the DFT

Zero padding in time domain improves frequency resolution



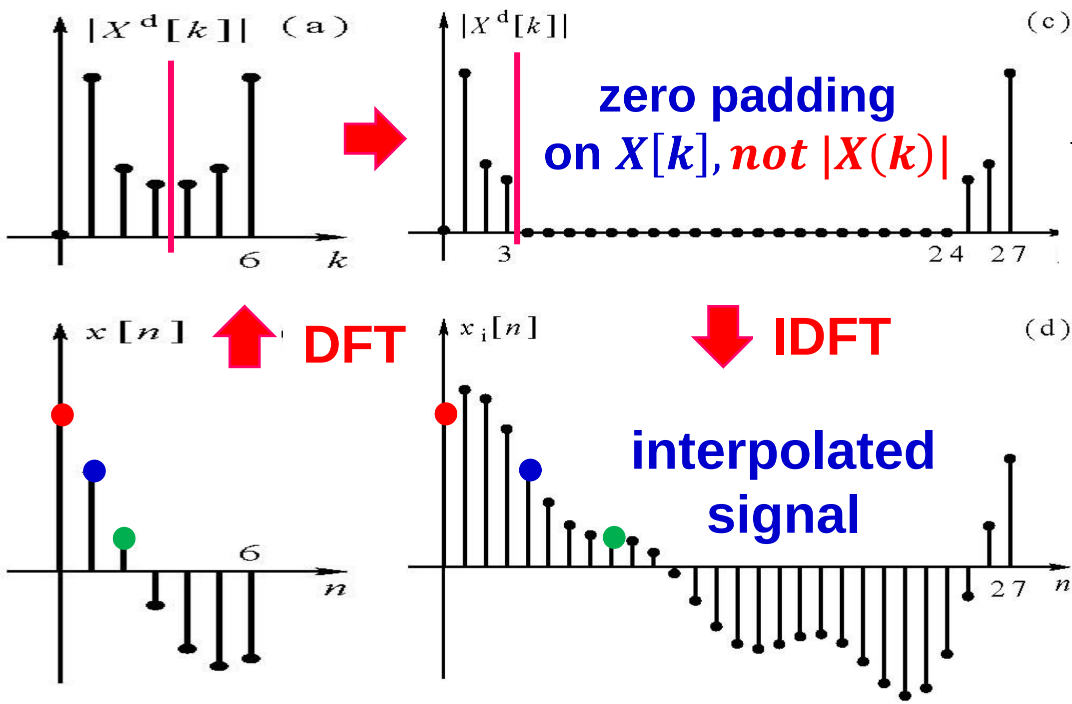
Properties of the DFT

Zero padding in frequency domain (time domain interpolation)

$$X_i(k) = \begin{cases} LX(k), & 0 \le k \le (N-1)/2 \\ LX(k-M+N), & M-(N-1)/2 \le k \le M-1 \\ 0 & \text{otherwise}. \end{cases}$$ *for odd N*

L : Interpolation factor, $M = LN$

Zero padding in freq. domain → Time domain interpolation



$$x_i(n) = \frac{1}{M}\sum_{k=0}^{M-1} X_i(k)\, W_N^{nk}, \quad 0 \le n \le M-1$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} x(m) \frac{\sin[\pi(n-mL)/L]}{\sin[\pi(n-mL)/M]}$$

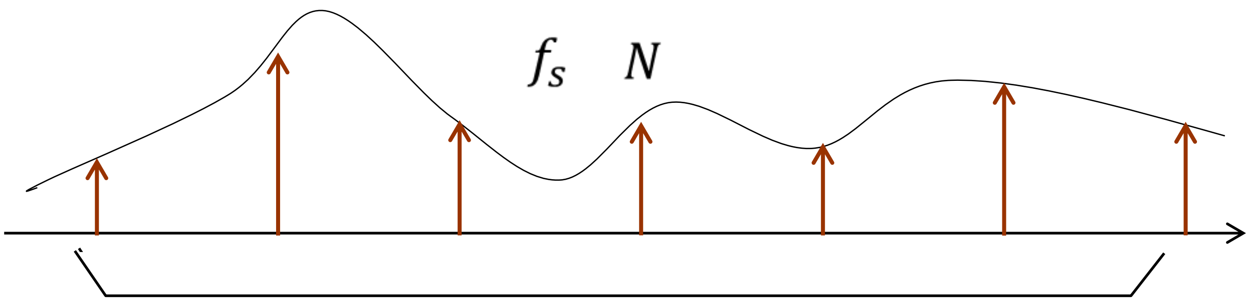
$$x_i(kL) = x(k)$$

: *L*-fold interpolation.

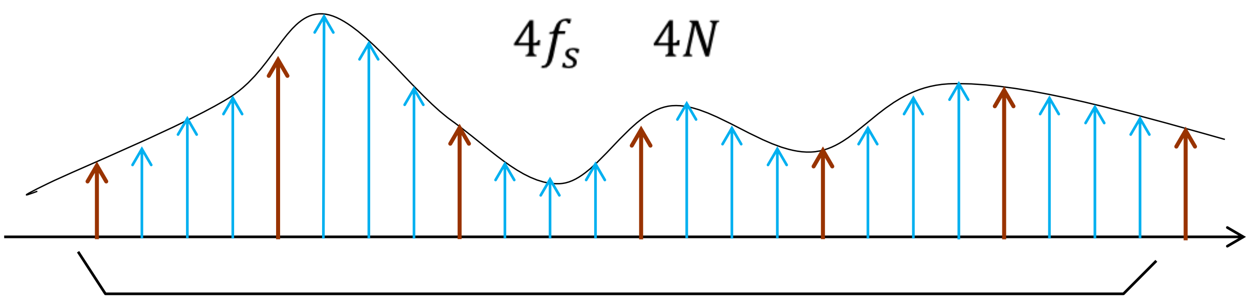
Properties of the DFT

Zero padding in frequency domain (time domain interpolation)

How are the DFT spectra of $\boldsymbol{x}(\boldsymbol{t})$ different when sampled at f_s and $4f_s$

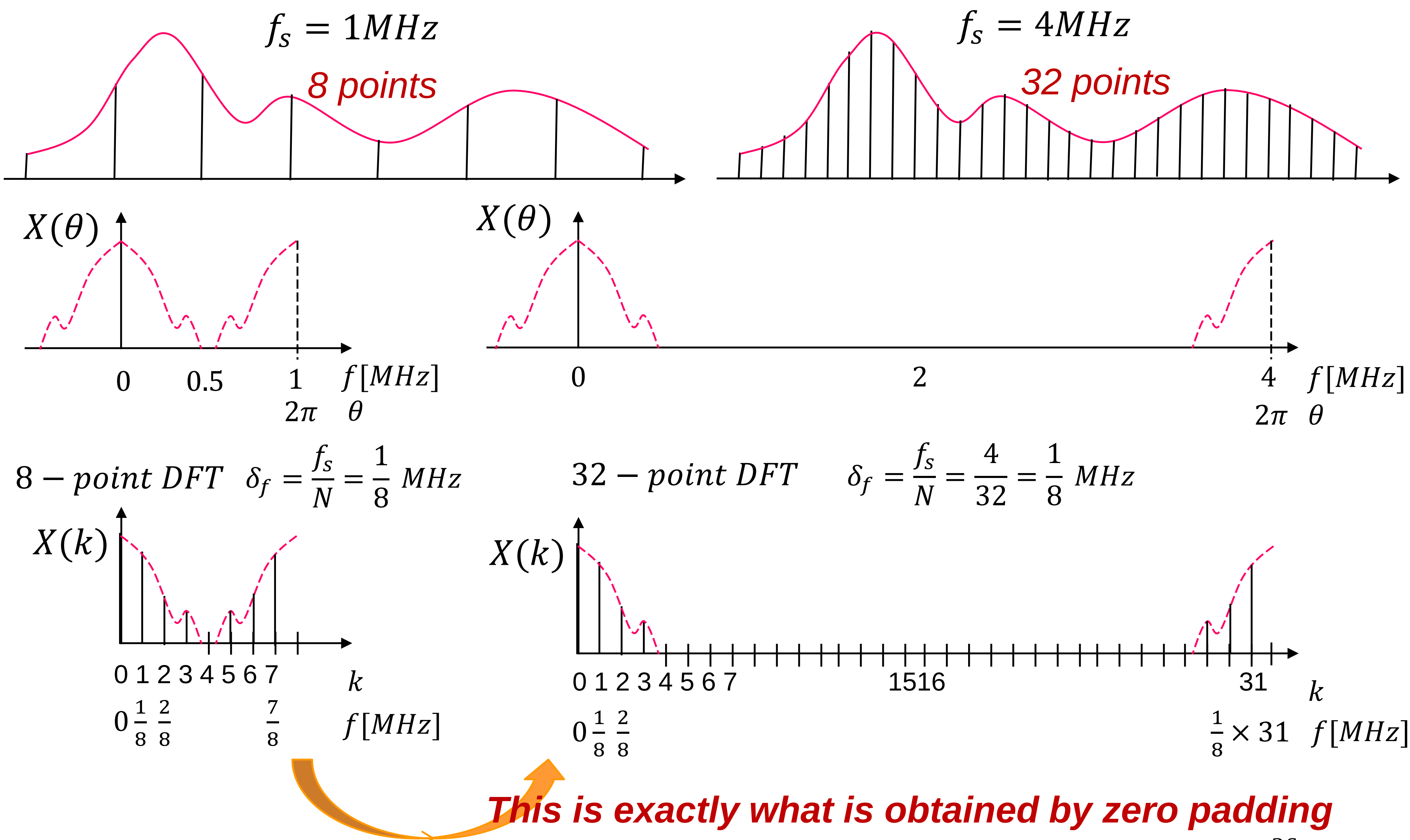


$$\delta_f = f_s/N$$



$$\delta'_f = \frac{4f_s}{4N} = \frac{f_s}{N} = \delta_f$$

Properties of the DFT



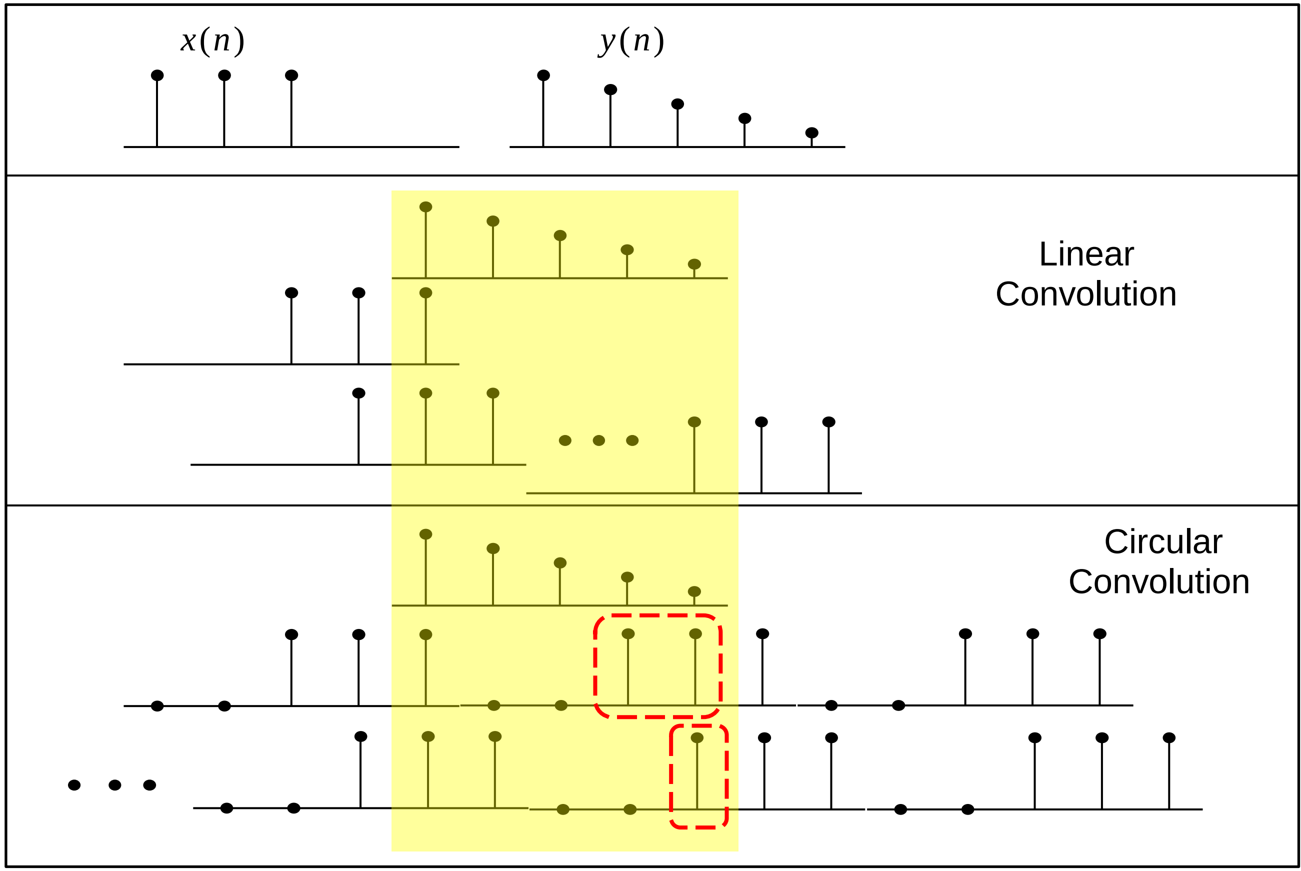
This is exactly what is obtained by zero padding

Circular Convolution

Definition

- x(n) and y(n): Length N (*n=0, 1, 2, …, N-1*)
- Circular (or periodic) convolution

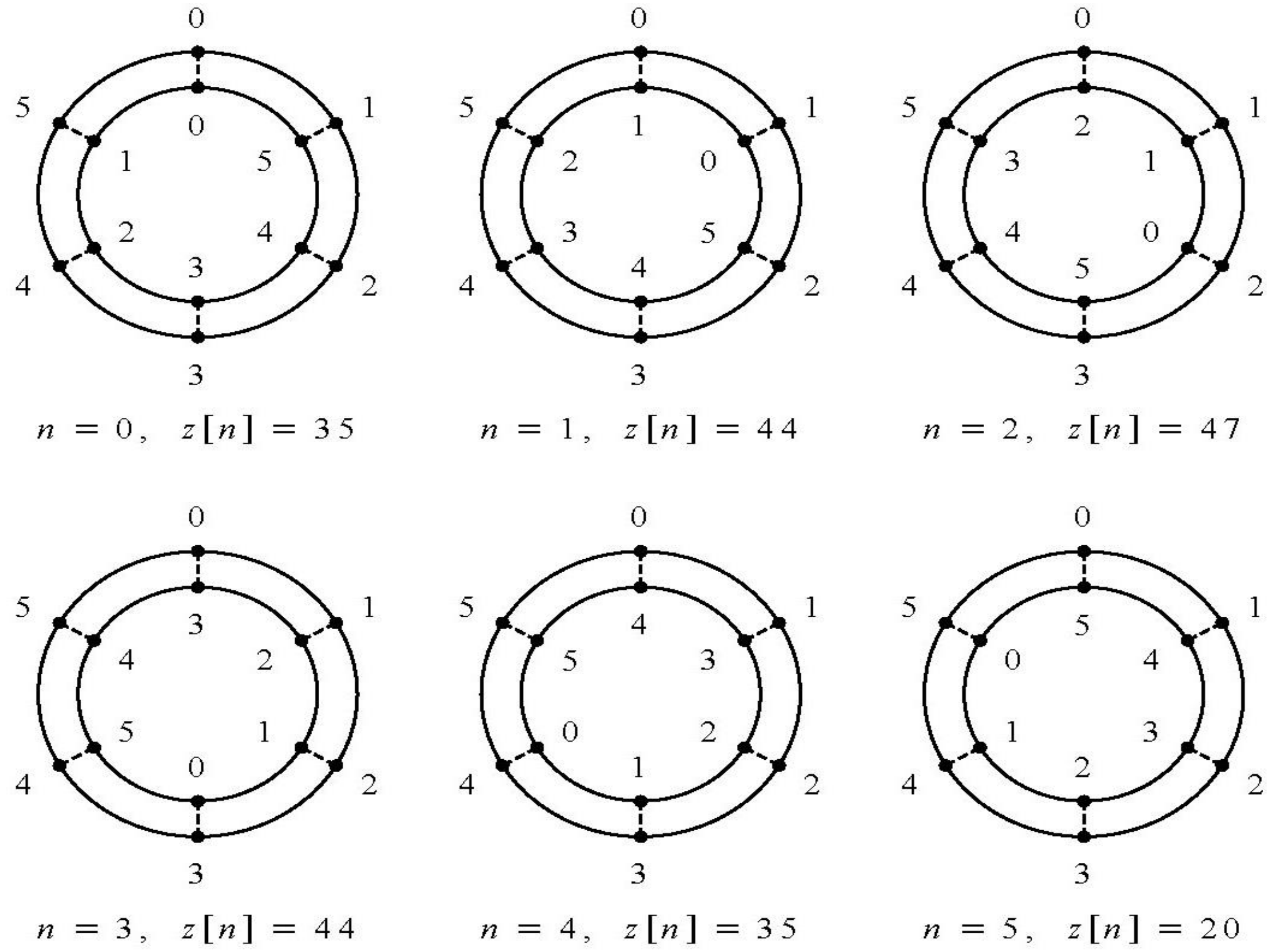
$$z(n) = x(n) \circledast y(n) = \sum_{m=0}^{N-1} x[m]\, y[(n-m) \bmod N]$$



Generally, linear and circular correlations give different results

Circular Convolution

Graphical representation



$$x(n) \circledast y(n) = y(n) \circledast x(n), \quad (x(n) \circledast y(n)) \circledast z(n) = x(n) \circledast (y(n) \circledast z(n)).$$

Linear Convolution via Circular Convolution

Case 1: Two finite sequences

- **Linear convolution** $x_3(n) = x_1(n) * x_2(n)$ **where**

$$\begin{cases} x_1(n):\ \textit{Length}\ N_1\ \ (n = 0, 1, \ldots\ldots, N_1 - 1) \\ x_2(n):\ \textit{Length}\ N_2\ \ (n = 0, 1, \ldots\ldots, N_2 - 1) \end{cases}$$

$$\Rightarrow\ \textit{Length of}\ x_3(n) = N_1 + N_2 - 1$$

- **Circular convolution** $\tilde{x}_3(n) = \tilde{x}_1(n) \circledast \tilde{x}_2(n)$ **where**

$$\tilde{x}_3(n) = x_3(n)\ \ \text{for}\ \ 0 \le n \le N - 1\ \ (N \ge N_1 + N_2 - 1)$$

$$\tilde{x}_1(n) = \begin{cases} x_1(n), 0 \le n \le N_1 - 1 \\ 0, \quad N_1 \le n \le N - 1 \end{cases} \qquad \tilde{x}_2(n) = \begin{cases} x_2(n), 0 \le n \le N_2 - 1 \\ 0, \quad N_2 \le n \le N - 1 \end{cases}$$

$$x_3(n) = \tilde{x}_3(n)\ \ (0 \le n \le N - 1)$$

$$= IDFT\{X_1(k) \cdot X_2(k)\}$$

L. conv
=
C. conv

$x_1(n), x_2(n)$

↓

N-DFT
$X_1(k), X_2(k)$

↓

$X_3(k) = X_1(k) \times X_2(k)$

↓

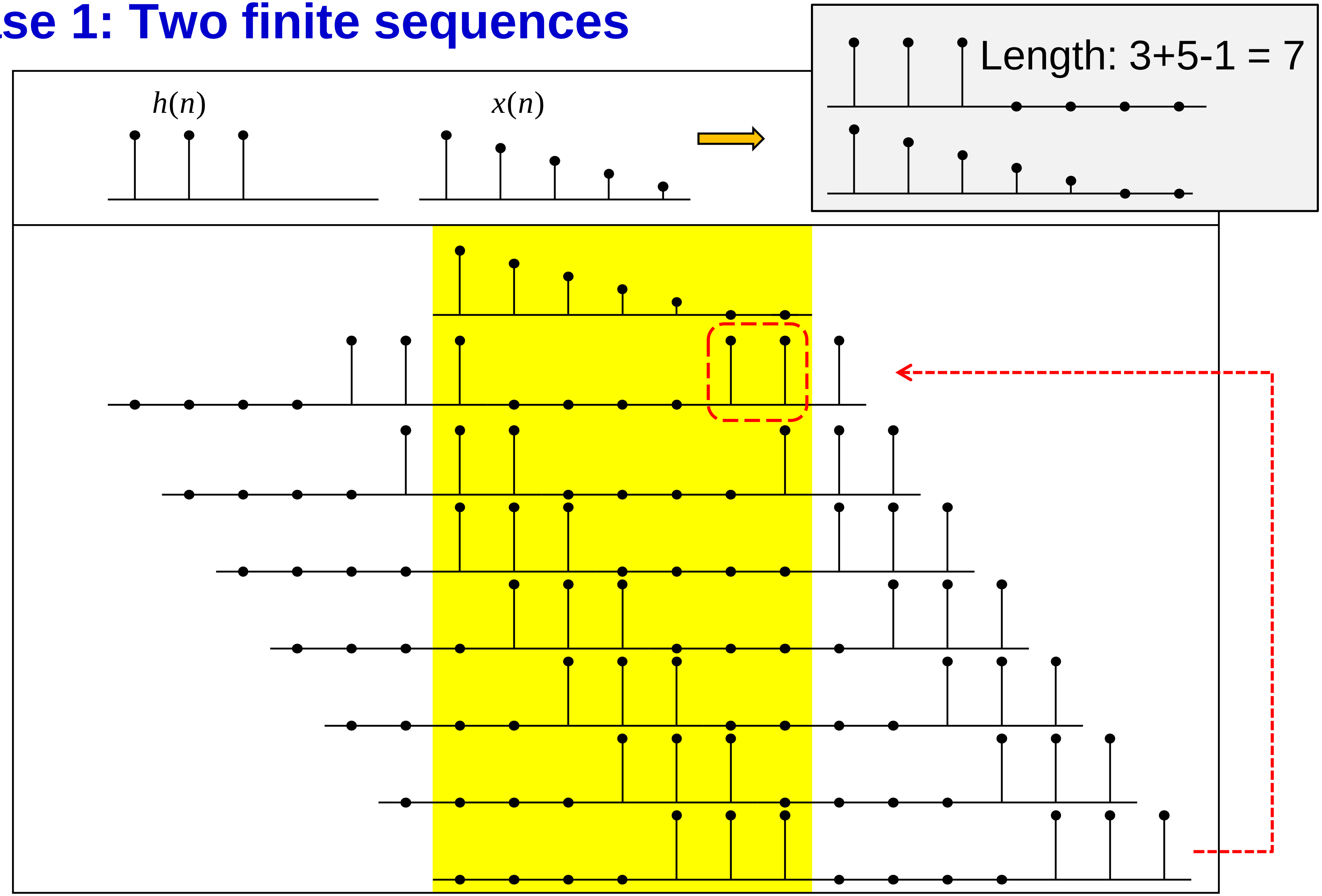
$\tilde{x}_3(n) = IDFT[X_3(k)]$

↓

$x_3(n) = \tilde{x}_3(n)$
$0 \le n \le N_1 + N_2 - 1$

Linear Convolution via Circular Convolution

Case 1: Two finite sequences



Linear Convolution via Circular Convolution

Example

$$a = [1\ \ 1\ \ 1\ \ 1];\ \ a(n) = \{\overset{\downarrow}{1},1,1,1\}$$

$$b = [-0.5\ \ 1\ \ -0.5];\ \ b(n) = \{-0.5,\overset{\downarrow}{1},-0.5\}$$

$$c_{ab} = conv(a,b) = a * b = \underline{\{\text{-}0.5\ \ 0.5\ \ 0\ \ \ 0\ \ \ 0.5\ \ \ \text{-}0.5\}}$$

$$cc_{ab4} = cconv(a,b,4) = a \circledast b = \{0\ \ 0\ \ \ 0\ \ 0\}$$

$$cc_{ab5} = cconv(a,b,5) = a \circledast b = \{-1\ \ 0.5\ \ 0\ \ 0\ \ 0.5\}$$

$$cc_{ab6} = cconv(a,b,6) = a \circledast b = \underline{\{-0.5\ \ 0.5\ \ 0\ \ 0\ \ 0.5\ \ \ -0.5\}}$$

$$cc_{ab7} = cconv(a,b,7) = a \circledast b = \{\underline{-0.5\ \ 0.5\ \ 0\ \ 0\ \ 0.5\ \ \ -0.5}\ \ 0\}$$

$$a_{dft4} = fft(a,4) = \{4\ \ 0\ \ 0\ \ 0\}$$

$$b_{dft4} = fft(b,4) = \{0\ \ \ -j\ \ \ -2\ \ +j\}$$

$$c_{ab4} = ifft(a_{dft4}\ .*\ b_{dft4}, 4) = \{0,\qquad 0,\qquad 0,\qquad 0\}$$

$$c_{ab6} = \underline{ifft(fft(a,6)\ .*\ \ fft(b,6),6)} = \underline{\{-0.5\ \ 0.5\ \ 0\ \ 0\ \ 0.5\ \ \ -0.5\}}$$

Linear Convolution via Circular Convolution

Case 2: One finite sequence and one infinite sequence

❖ Overlap-add method

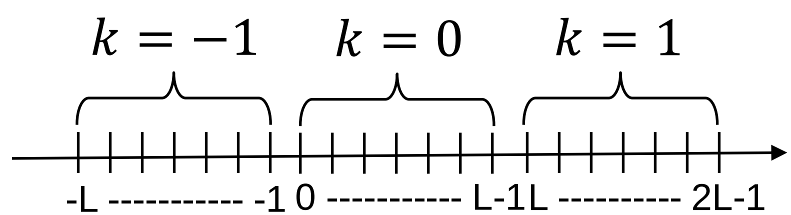
$$y(n) = h(n) * x(n) \quad Len[h(n)] = M, \quad Len[x(n)] = \infty \ \ or\ very\ long$$

$$\Rightarrow \ \ Block\ \ filtering\ \ technique$$

- Decompose $x(n)$ into a sum of sections

$$x_k(n) = \begin{cases} x(n), & kL \le n \le (k+1)L - 1 \\ 0\ , & otherwise \end{cases}$$

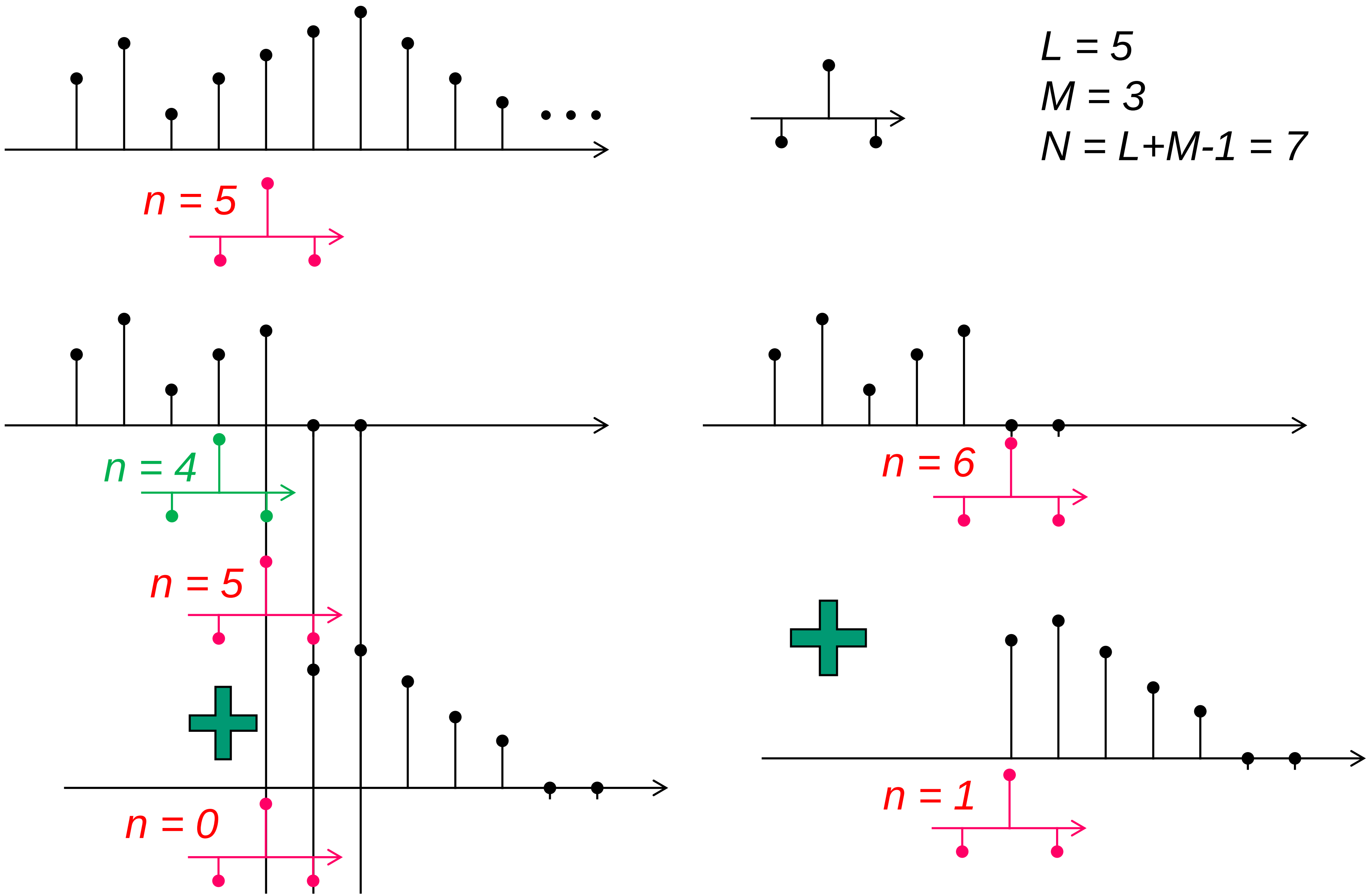
$$\Rightarrow \ \ x(n) = \sum_{k=0}^{\infty} x_k(n)$$



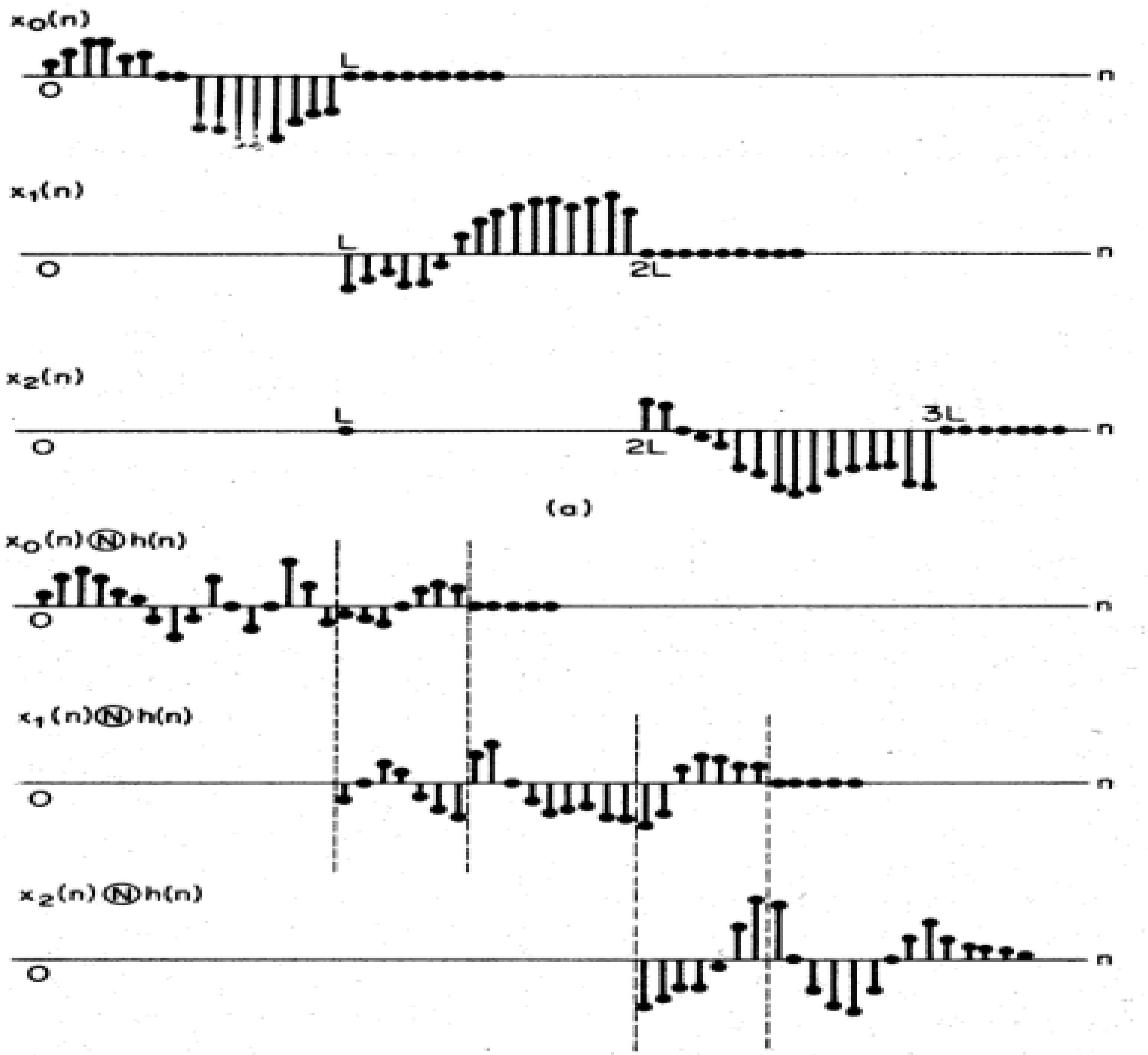
$$y(n) = \sum_{k=0}^{\infty} y_k(n) = \sum_{k=0}^{\infty} x_k(n) * h(n)$$

$y_k(n)$ can be obtained using DFT $\quad (N = L + M - 1)$

Linear Convolution via Circular Convolution



Linear Convolution via Circular Convolution



Linear Convolution via Circular Convolution

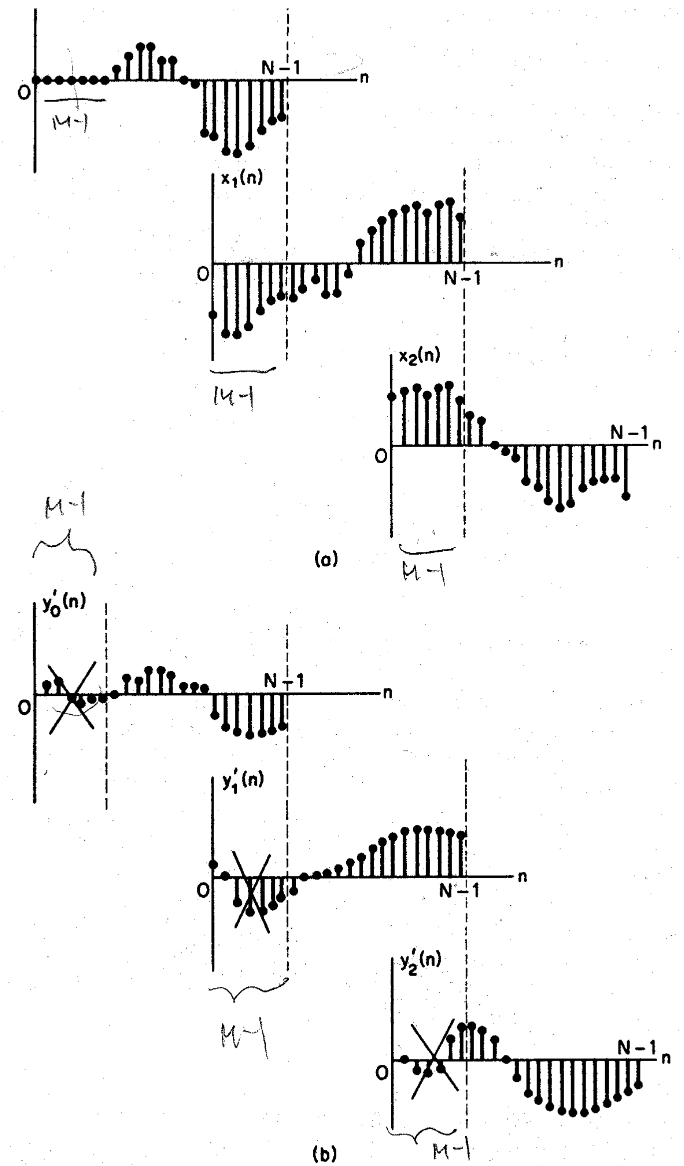
Case 2: One finite sequence and one infinite sequence

❖ **Overlap-save method**

$$x_k(n) = x(n + k(N - M + 1)) \quad 0 \le n \le N - 1$$

$$y(n) = \sum_{k=0}^{\infty} y_k(n - k(N + M - 1))$$

where $$y_k(n) = \begin{cases} y_k'(n), & M - 1 \le n \le N - 1 \\ 0\ , & otherwise \end{cases}$$



Discrete Cosine Transform (DCT)

Backgrounds

- ❖ **Discrete-time version of the Fourier Cosine Series**
  - Real valued transform for real signals
  - Linear transform: $\underline{X}_N = C_N \cdot \underline{x}_N$
  - Extend the signal symmetrically around the origin in **4 different ways.**

Type-I DCT

- ❖ **N samples => 2N-2 samples**

$$x_1(n) = \begin{cases} \sqrt{2}\,x(n), & n = 0, N-1 \\ x(n) & 1 \le n \le N-2 \\ x(2N-n-2) & N \le n \le 2N-3 \end{cases}$$