z-Transform

Prof. Tai-kyong Song
Dept. of Electronic Engineering
SOGANG UNIVERSITY

Z-Transform

- Definition

$$X(z) \equiv \sum_{n=-\infty}^{\infty} x(n) z^{-n} \qquad X(z) = Z\{x(n)\}$$

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz \qquad x(n) \leftrightarrow X(z)$$

- Z-transform is an infinite power-series that exists for values of z in a certain region, called Region of Convergence (ROC)
- Important tool for the analysis of DT signals and DT systems

Z-Transform

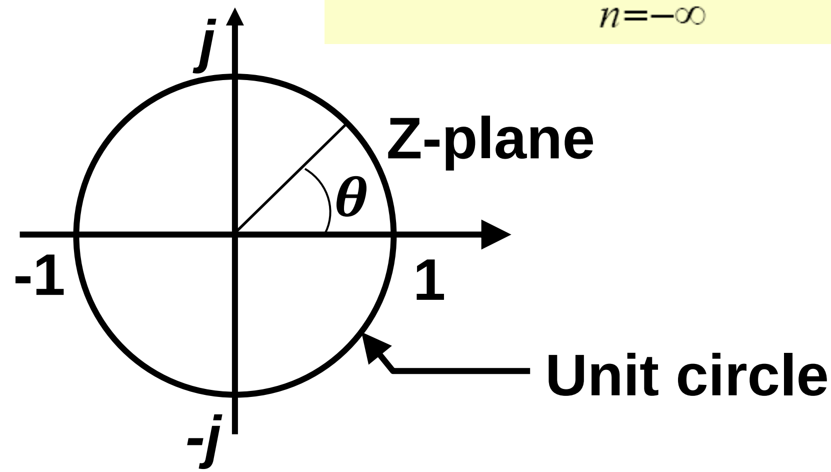
DTFT vs. Z-transform

$$X(z) \equiv \sum_{n=-\infty}^{\infty} x(n) z^{-n} \xrightarrow{z = r e^{j\omega}} X(z) = \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-j\omega n}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$X\left(e^{j\theta}\right) = X(z)\big|_{z=e^{j\theta}}$$

or $X(\theta)$



Existence (Convergence) of Z-transform

$$|X(z)| \le \sum_{n=-\infty}^{\infty} |x(n) r^{-n} e^{-j\omega n}| = \sum_{n=-\infty}^{\infty} |x(n) r^{-n}| < \infty$$

Z-Transform

- Finding ROC

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

  - 1st sum (of anti-causal part) converges where $r < r_1$

$$n \to -n$$

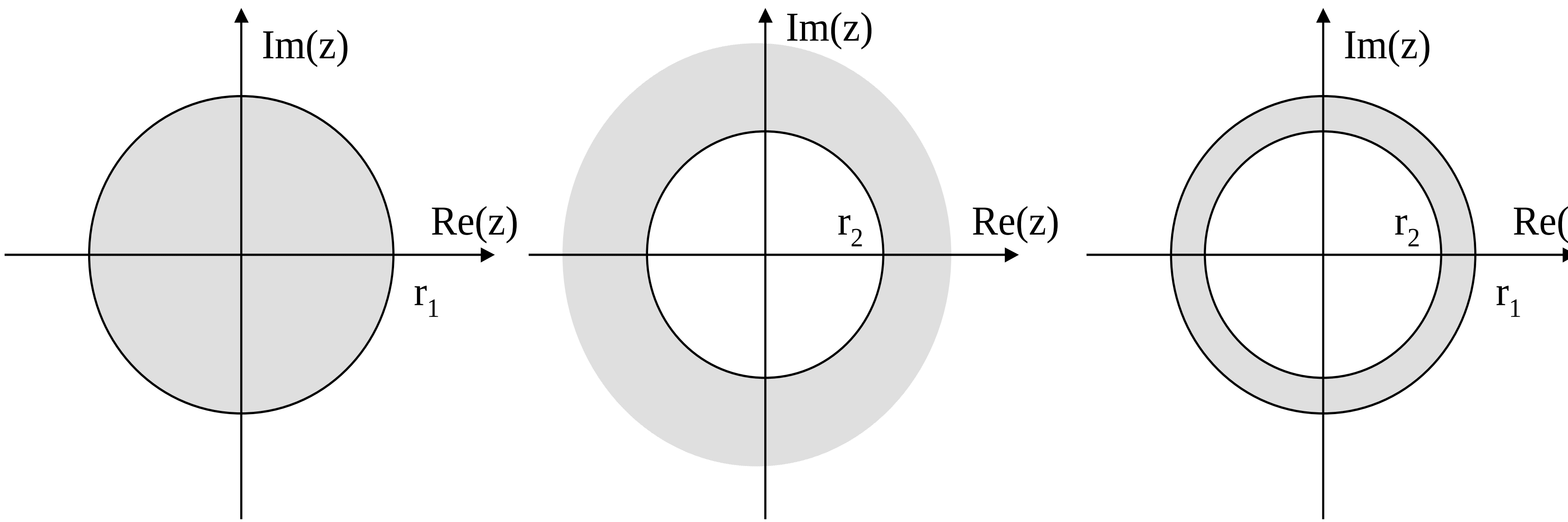
$$|X(z)| \leq \sum_{n=-\infty}^{-1} |x(n) r^{-n}| + \sum_{n=0}^{\infty} \left| \frac{x(n)}{r^n} \right| \leq \sum_{n=1}^{\infty} |x(-n) r^n| + \sum_{n=0}^{\infty} \left| \frac{x(n)}{r^n} \right|$$

  - 2nd sum (of causal part) converges where $r > r_2$

Z-Transform

- ROC of is generally specified as the annular region in the z-plane

$$r_2 < r < r_1$$



$$Note: \ If \ \ r_2 > r_1, \ X(z) \ does \ not \ exist.$$

Z-Transform

❖ Examples

$$X_1(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5} \quad ROC:\ z \neq 0$$

$$X_2(z) = z^2 + 2z + 5 + 7z^{-1} + z^{-3} \quad ROC:\ z \neq 0 \quad z \neq \infty$$

$$X(z) = 1, \quad (\delta(n) \longleftrightarrow 1) \quad ROC:\ entire\ z-plane$$

$$X_3(z) = z^{-k} \ (\longleftrightarrow \ \delta(n-k)\ ) \quad ROC:\ z \neq 0$$

$$k > 0$$

$$X_4(z) = z^{k} \ (\longleftrightarrow \ \delta(n+k)\ ) \quad ROC:\ z \neq \infty$$

❖ ROC of a finite-duration signal: entire z-plane except possibly at z = 0 and/or ∞

Z-Transform

$$Ex)\ x(n) = a^n u(n)$$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (a z^{-1})^n$$

For $|az^{-1}| < 1, \quad that\ is, |z| > |a|, \quad X(z) = \dfrac{1}{1 - az^{-1}}$

$$x(n) = a^n u(n) \longleftrightarrow X(z) = \frac{1}{1 - az^{-1}} \qquad ROC: |z| > |a|$$

$$Ex)\ x(n) = -a^n u(-n-1) \longleftrightarrow X(z) = \frac{1}{1 - az^{-1}} \qquad ROC: |z| < |a|$$

$$X(z) = -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=1}^{\infty} a^{-n} z^n = -\frac{a^{-1}z}{1 - a^{-1}z} = \frac{1}{1 - az^{-1}}$$

- It is obvious that a DT signal is uniquely determined by its z-transform and ROC.

Z-Transform

Characteristic families of Signals with their corresponding ROC

$$X(z) \equiv \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

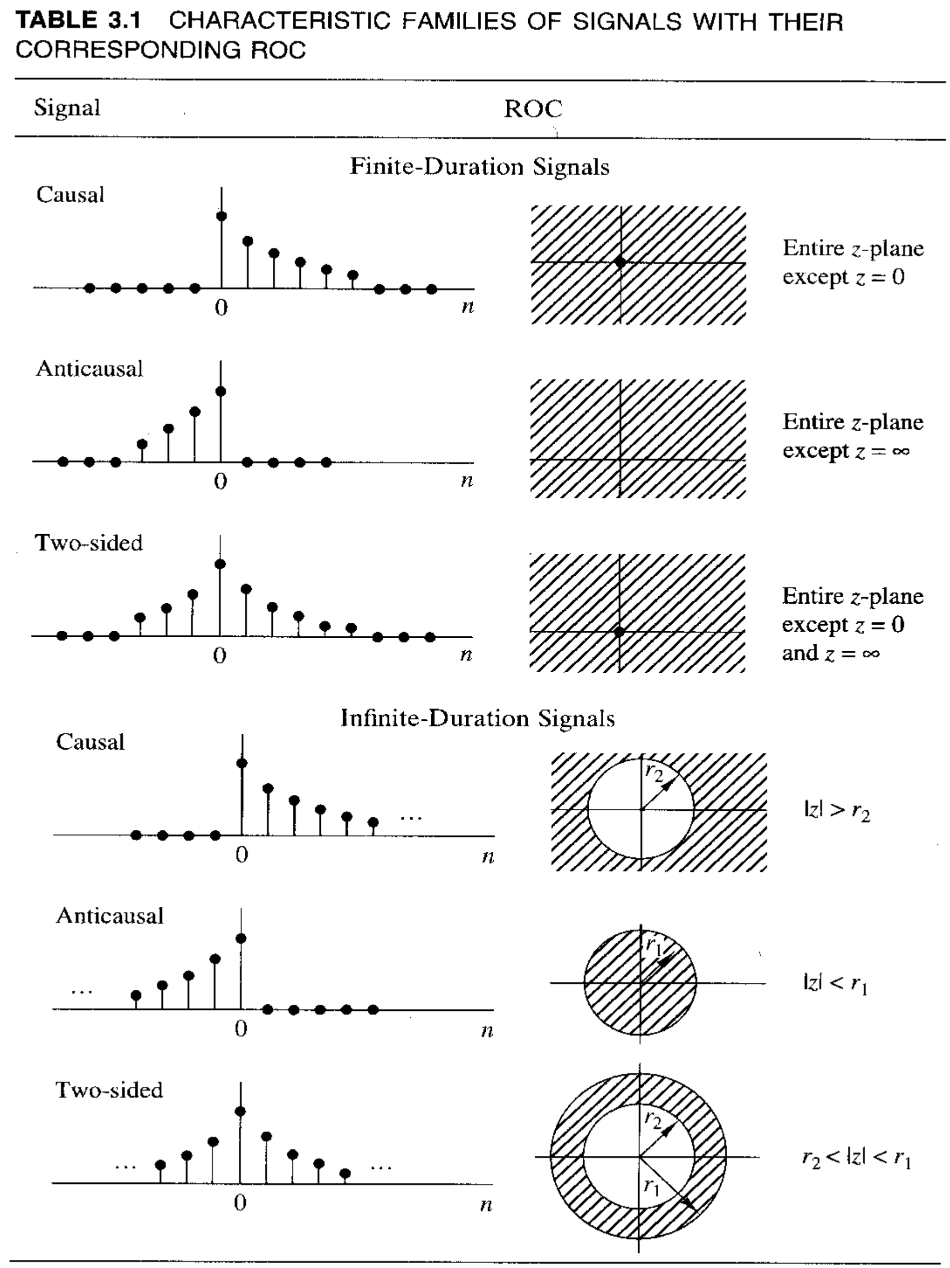


TABLE 3.1 CHARACTERISTIC FAMILIES OF SIGNALS WITH THEIR CORRESPONDING ROC

Signal	ROC
Finite-Duration Signals	
Causal	Entire z-plane except $z = 0$
Anticausal	Entire z-plane except $z = \infty$
Two-sided	Entire z-plane except $z = 0$ and $z = \infty$
Infinite-Duration Signals	
Causal	$\lvert z \rvert > r_2$
Anticausal	$\lvert z \rvert < r_1$
Two-sided	$r_2 < \lvert z \rvert < r_1$

Z-Transform

- ROC of a signals
  - Causal signal : $r > |r_1|$
  - Anti-causal signal : $r < |r_2|$
  - Finite-duration : Entire z-plane except possibly at $z = 0$ and/or ∞
- Classifications
  - Bilateral (two-sided) z-transform : $X(z) \equiv \sum_{n=-\infty}^{\infty} x(n) z^{-n}$
  - Unilateral (one-sided) z-transform : $X^{+}(z) \equiv \sum_{n=0}^{\infty} x(n) z^{-n}$

Z-Transform: Properties

Property	Time Domain	z-Domain	ROC
Notation	$x(n)$ $x_1(n)$ $x_2(n)$	$X(z)$ $X_1(z)$ $X_2(z)$	ROC: $r_2 < \|z\| < r_1$ ROC_1 ROC_2
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$	At least the intersection of ROC_1 and ROC_2
Time shifting	$x(n-k)$	$z^{-k}X(z)$	That of $X(z)$, except $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$
Scaling in the z-domain	$a^n x(n)$	$X(a^{-1}z)$	$\|a\|r_2 < \|z\| < \|a\|r_1$
Time reversal	$x(-n)$	$X(z^{-1})$	$\dfrac{1}{r_1} < \|z\| < \dfrac{1}{r_2}$
Conjugation	$x^*(n)$	$X^*(z^*)$	ROC
Real part	$\text{Re}\{x(n)\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Includes ROC
Imaginary part	$\text{Im}\{x(n)\}$	$\frac{1}{2}[X(z) - X^*(z^*)]$	Includes ROC
Differentiation in the z-domain	$nx(n)$	$-z\dfrac{dX(z)}{dz}$	$r_2 < \|z\| < r_1$
Convolution	$x_1(n) * x_2(n)$	$X_1(z)X_2(z)$	At least, the intersection of ROC_1 and ROC_2
Correlation	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$R_{x_1x_2}(z) = X_1(z)X_2(z^{-1})$	At least, the intersection of ROC of $X_1(z)$ and $X_2(z^{-1})$
Initial value theorem	If $x(n)$ causal	$x(0) = \lim\limits_{z\to\infty} X(z)$	
Multiplication	$x_1(n)x_2(n)$	$\dfrac{1}{2\pi j}\oint_C X_1(v)X_2\left(\dfrac{z}{v}\right)v^{-1}dv$	At least $r_{1l}r_{2l} < \|z\| < r_{1u}r_{2u}$
Parseval's relation	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \dfrac{1}{2\pi j}\oint_C X_1(v)X_2^*(1/v^*)v^{-1}dv$		

Z-Transform: Properties

Linearity

$$x(n) = a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{z} X(z) = a_1 X_1(z) + a_2 X_2(z)$$

Ex) $x(n) = \cos\omega_0 n \cdot u(n)$

$$x(n) = \frac{1}{2} e^{j\omega_0 n} u(n) + \frac{1}{2} e^{-j\omega_0 n} u(n)$$

$$X(z) = \frac{1}{2} Z\{e^{j\omega_0 n} u(n)\} + \frac{1}{2} Z\{e^{-j\omega_0 n} u(n)\}$$

$$e^{j\omega_0 n} u(n) \longleftrightarrow \frac{1}{1 - e^{j\omega_0} z^{-1}} \quad (|z| > 1)$$

$$e^{-j\omega_0 n} u(n) \longleftrightarrow \frac{1}{1 - e^{-j\omega_0} z^{-1}} \quad (|z| > 1)$$

$$=> \ (\cos w_0 n) u(n) \xLeftrightarrow{z} \frac{1 - z^{-1}\cos w_0}{1 - 2z^{-1}\cos w_0 + z^{-2}} \qquad \text{ROC : } |z| > 1$$

$$=> \ (\sin w_0 n) u(n) \xLeftrightarrow{z} \frac{z^{-1}\sin w_0}{1 - 2z^{-1}\cos w_0 + z^{-2}}$$

Z-Transform: Properties

Time shifting: $x(n-k) \overset{z}{\Longleftrightarrow} z^{-k}X(z)$

Scaling in the z-domain:

If $x(n) \longleftrightarrow X(z)$ $ROC: r_1 < |z| < r_2$

$$a^n x(n) \overset{z}{\Longleftrightarrow} X(a^{-1}z) \quad ROC: |a|r_1 < |z| < |a|r_2$$

Note: Let's rewrite as $a = r_0 e^{j\omega_0}$ and $z = re^{j\omega}$.

then, $z \to a^{-1}z = \dfrac{r}{r_0} e^{j(\omega-\omega_0)}$: Frequency shifting

Ex) $$a^n(\cos w_0 n)u(n) \overset{z}{\Longleftrightarrow} \frac{1-az^{-1}\cos w_0}{1-2az^{-1}\cos w_0 + a^2 z^{-2}} \quad |z| > |a|$$

$$a^n(\sin w_0 n)u(n) \overset{z}{\Longleftrightarrow} \frac{az^{-1}\sin w_0}{1-2az^{-1}\cos w_0 + a^2 z^{-2}} \quad |z| > |a|$$

Z-Transform: Properties

Time reversal: $x(-n) \overset{z}{\Longleftrightarrow} X(z^{-1}) \qquad \frac{1}{r_2} < |z| < \frac{1}{r_1}$

Differentiation in the z-domain: $nx(n) \overset{z}{\Longleftrightarrow} -z\frac{dX(z)}{dz}$

Ex) $a^n u(n) \longleftrightarrow 1/(1-az^{-1}) \qquad ROC: \ |z| > |a|$

$$na^n u(n) \overset{z}{\Longleftrightarrow} X(z) = -z\frac{dX_1(z)}{dz} = \frac{az^{-1}}{(1-az^{-1})^2} \qquad |z| > |a|$$

$$nu(n) \overset{z}{\Longleftrightarrow} \frac{z^{-1}}{(1-z^{-1})^2} \qquad |z| > 1$$

Z-Transform: Properties

Convolution:

$$x(n) = x_1(n) * x_2(n) \xLeftrightarrow{z} X(z) = X_1(z)X_2(z)$$

Correlation:

$$r_{x_1x_2}(l) = \sum_{n=-\infty}^{\infty} x_1(n)x_2(n-l) \xLeftrightarrow{z} R_{x_1x_2}(z) = X_1(z)X_2(z^{-1})$$

Multiplication:

$$x(n) = x_1(n)x_2(n) \xLeftrightarrow{z} X(z) = \frac{1}{2\pi j}\oint_C X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}dv$$

Parseval's relation:

$$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi j}\oint_C X_1(v)X_2^*\left(\frac{1}{v^*}\right)v^{-1}dv$$

$$x^*(n) \longleftrightarrow X^*(z^*)$$

Z-Transform: Properties

Initial value theorem for causal x(n):

$$x(0) = \lim_{z\to\infty} X(z) \qquad X(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots$$

Final value theorem

$$\lim_{n\to\infty} x(n) = \lim_{z\to 1} (z-1)X^{+}(z)$$

$$Z[x(n)] = X(z) = \sum_{n=0}^{\infty} x(n) z^{-n} \qquad Z[x(n+1)] = \sum_{n=0}^{\infty} x(n+1)z^{-n} = zX(z) - zx(0)$$

$$Z[x(n+1)] - Z[x(n)] = zX(z) - zx(0) - X(z)$$

$$(\mathrm{z}-1)X(z) - zx(0) = [x(1) - x(0)]z^{0} + [x(2) - x(1)]z^{-1} + [x(3) - x(2)]z^{-2} + \ldots$$

$$\lim_{\mathrm{z}\to 1} [(\mathrm{z}-1)X(z)] - x(0) = x(\infty) - x(0)$$

Z-Transform: Common z-transform pairs

	Signal, $x(n)$	z-Transform, $X(z)$	ROC
1	$\delta(n)$	1	All z
2	$u(n)$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
3	$a^n u(n)$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
4	$na^n u(n)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
5	$-a^n u(-n-1)$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
6	$-na^n u(-n-1)$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
7	$(\cos\omega_0 n)u(n)$	$\dfrac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
8	$(\sin\omega_0 n)u(n)$	$\dfrac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$	$\lvert z\rvert > 1$
9	$(a^n\cos\omega_0 n)u(n)$	$\dfrac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
10	$(a^n\sin\omega_0 n)u(n)$	$\dfrac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

Rational z-Transforms

LTI system described by a constant coefficient difference equation has a rational transfer function of z^{-1}

$$G(z)=\frac{B(z)}{A(z)}=\frac{b_0+b_1z^{-1}+\dots+b_{M-1}z^{-(M-1)}+b_Mz^{-M}}{a_0+a_1z^{-1}+\dots+a_{N-1}z^{-(N-1)}+a_Nz^{-N}}$$

$$G(z)=\frac{z^{-M}z^{M}}{z^{-N}z^{N}}\cdot\frac{b_0+b_1z^{-1}+\dots+b_{M-1}z^{-(M-1)}+b_Mz^{-M}}{a_0+a_1z^{-1}+\dots+a_{N-1}z^{-(N-1)}+a_Nz^{-N}}$$

$$=z^{N-M}\cdot\frac{b_0z^{M}+b_1z^{M-1}+\dots+b_{M-1}z+b_M}{a_0z^{N}+a_1z^{N-1}+\dots+a_{N-1}z+a_N}$$

- Degree of B(z) = M
- Degree of A(z) = N

Rational z-Transforms

Poles and Zeros

zero

$$G(z) = \frac{b_0 \Pi_{i=1}^{M}(1 - z_i z^{-1})}{a_0 \Pi_{k=1}^{N}(1 - p_k z^{-1})} = z^{(N-M)} \frac{b_0 \Pi_{k=1}^{M}(z - z_i)}{a_0 \Pi_{i=1}^{N}(z - p_k)}$$

pole

- G(z) has M finite zeros and N finite poles
- If $N > M$, there are $N - M$ additional zeros at $z = 0$
- If $M > N$, there are $M - N$ additional poles at $z = 0$

Including trivial poles and zeros at z=0 or ∞, the number of poles and zeros are equal

Rational z-Transforms

Some related MATLAB functions

$$\boldsymbol{H(z)=\frac{B(z)}{A(z)}=\frac{b_0+b_1z^{-1}+\cdots+b_{m-1}z^{-(m-1)}+b_mz^{-m}}{a_0+a_1z^{-1}+\cdots+a_{n-1}z^{-(n-1)}+b_nz^{-n}}}$$

- [z, p, k] = tf2zpk(b, a)

  finds the zeros, poles, and gains of a discrete-time transfer function

$$H(z)=\frac{Z(z)}{P(z)}=k\frac{(z-z_1)(z-z_2)\cdots(z-z_m)}{(z-p_1)(z-p_2)\cdots(z-p_n)}$$

Rational z-Transforms

Some related MATLAB functions

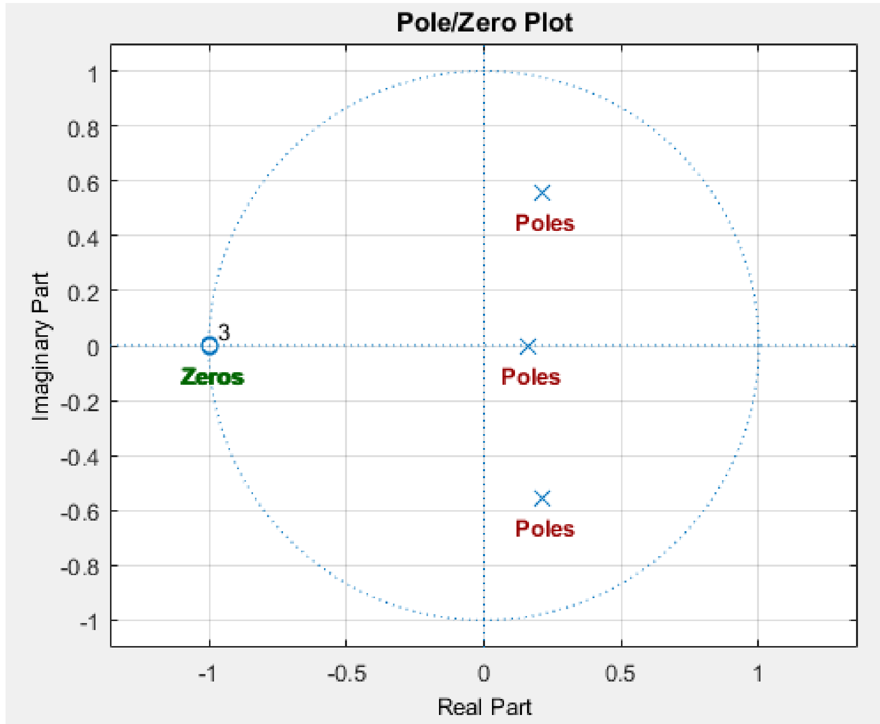
- [z, p, k] = tf2zpk(b, a)

$$H(z) = \frac{Z(z)}{P(z)} = k\,\frac{(z - z_1)(z - z_2)\cdots(z - z_m)}{(z - p_1)(z - p_2)\cdots(z - p_n)}$$

zplane(z, p)

grid

Title('Pole/Zero Plot')



Rational z-Transforms

Some related MATLAB functions

$$H(z) = \frac{1 - 0.5z^{-1}}{1 + 1.8z^{-1} + 0.6z^{-2} - 0.2z^{-3}}$$

```
n = [1 -0.5];

d = [1 1.8 0.6  -0.2];

[z, p, k] = tf2zpk(n, d);

[num, den] = zp2tf(z, p, k);
```

```
z =   0.5
p =  -1.0000
     -1.0000
      0.2000
k =   1
```

```
num = [0   0     1    -0.5]
den  = [1  1.8  0.6  -0.2 ]
```

$$H(z) = k\,\frac{(z - z_1)(z - z_2)\cdots(z - z_n)}{(z - p_1)(z - p_2)\cdots(z - p_m)}$$

Rational z-Transforms

$$x(n) = a^n u(n), \;\; a > 0$$

$$X(z) = \sum_{n=0}^{M-1} a^n z^{-n} = \sum_{n=0}^{M-1} (az^{-1})^n = \frac{1-(az^{-1})^M}{1-az^{-1}} = \frac{z^M - a^M}{z^{M-1}(z-a)}$$

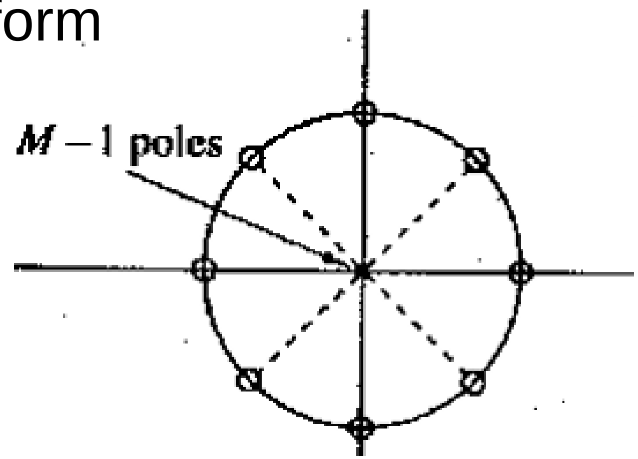
$$z^M - a^M = 0 \;\Rightarrow\; z_k^M = a^M e^{j2\pi k} \;\Rightarrow\; z_k = a e^{j2\pi k/M}, \;\; k = 0, 1, 2, \;\; \ldots \;\; , M-1$$

$$X(z) = \frac{\underline{(z-z_0)}(z-z_1)(z-z_2)\cdots(z-z_{M-1})}{z^{M-1}\underline{(z-a)}} \qquad (z-a) \;:\; \mathit{Common\ factor}$$

$$= \frac{(z-z_1)(z-z_2)\cdots(z-z_{M-1})}{z^{M-1}}$$: Irreducible form

Zeros: $z_k = a e^{j2\pi k/M}, \;\; k = 1, 2, \;\; \ldots \;\; , M-1$

Poles: M-1 poles at zero



Rational z-Transforms

- Determining X(z) from a pole-zero plot.

$$X(z) = G\frac{(z-z_1)(z-z_2)}{(z-p_1)(z-p_2)} = G\frac{z(z-r\cos\omega_0)}{(z-re^{j\omega_0})(z-re^{-j\omega_0})} \quad \text{ROC: } |z|>r$$

$$\Leftrightarrow X(z) = G\frac{1-rz^{-1}\cos\omega_0}{1-2rz^{-1}\cos\omega_0 + r^2 z^{-2}} \quad \text{ROC: } |z|>r$$

From the z-transfor pair table,

$$x(n) = G(r^n \cos\omega_0 n)u(n)$$

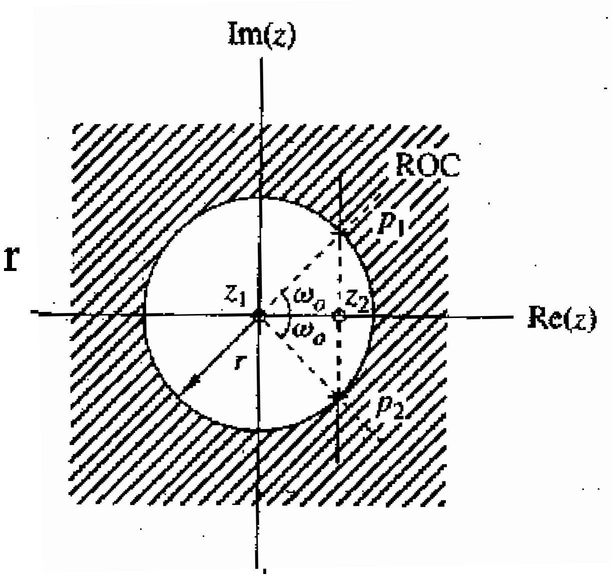


Figure 3.9 Pole-zero pattern for Example 3.3.3.

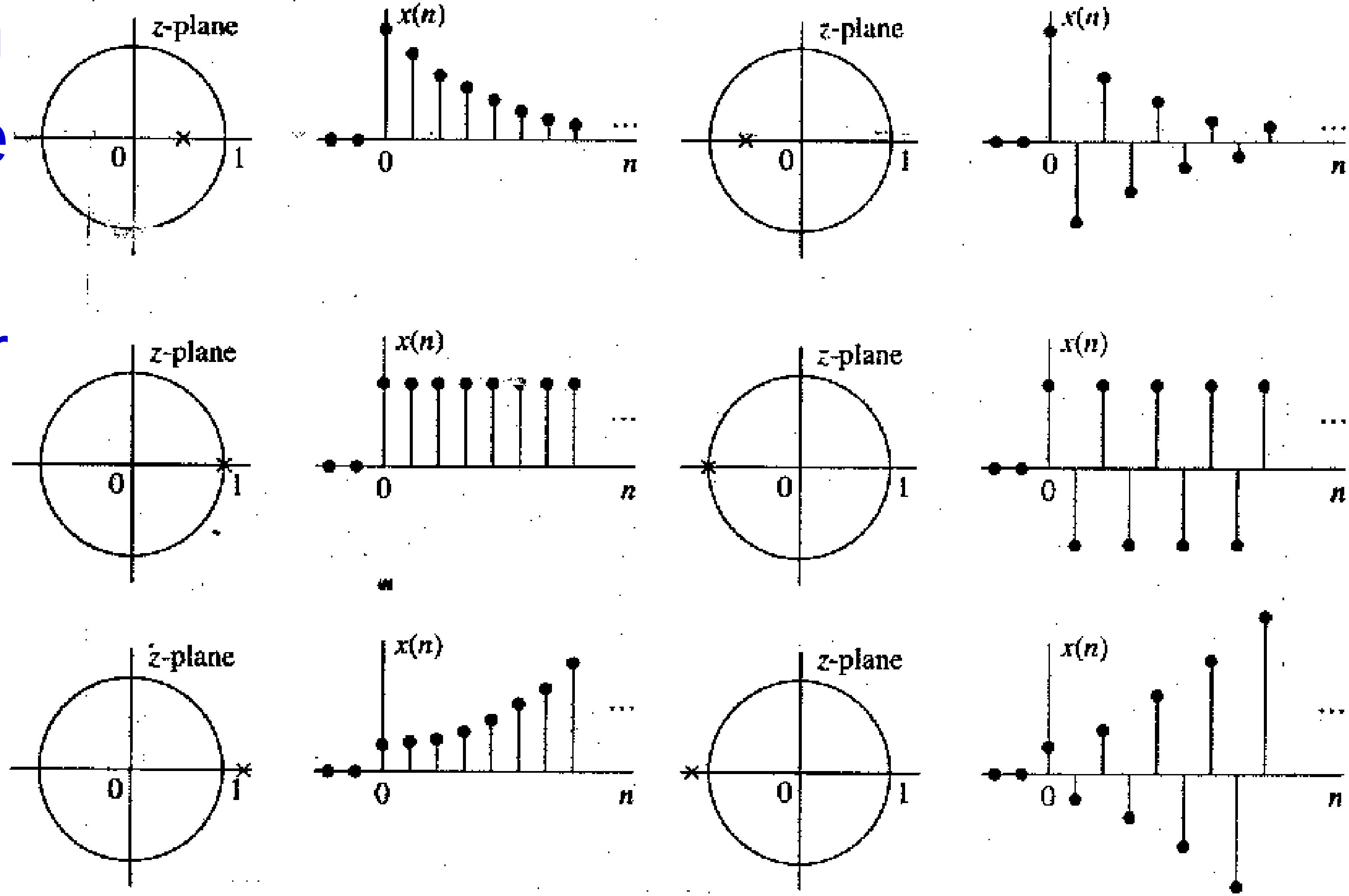
Note) In general, if a polynomial has real coeffs, its roots are either real or occurs in complex-conjugate pairs.

Rational z-Transforms

Pole Location and time domain behavior

$$x(n) = a^n u(n) \stackrel{z}{\longleftrightarrow} X(z) = 1/(1 - az^{-1}) \quad \text{ROC: } |z| > |a|$$

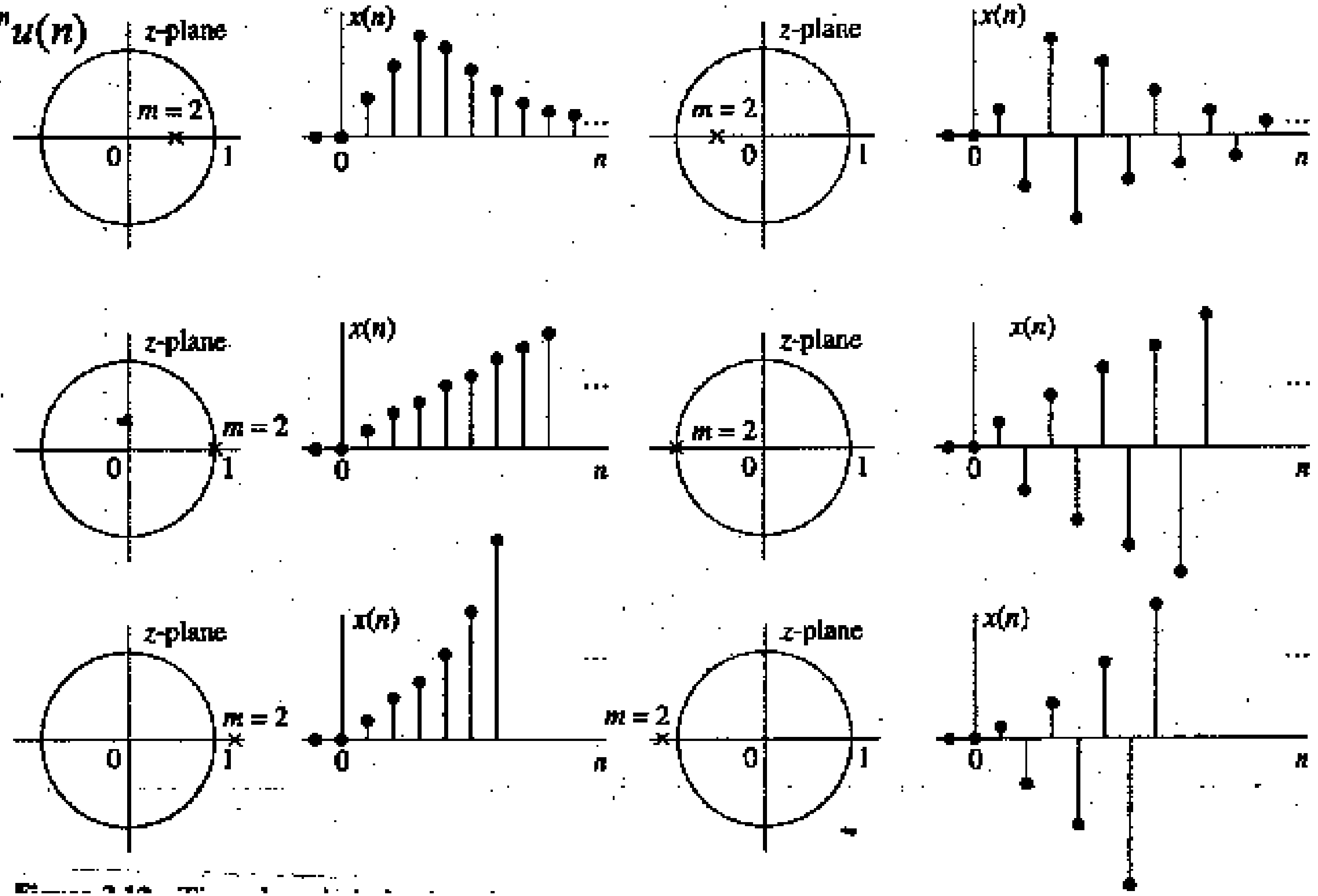
which has one zero at z = 0 and one pole at z = a.



Rational z-Transforms

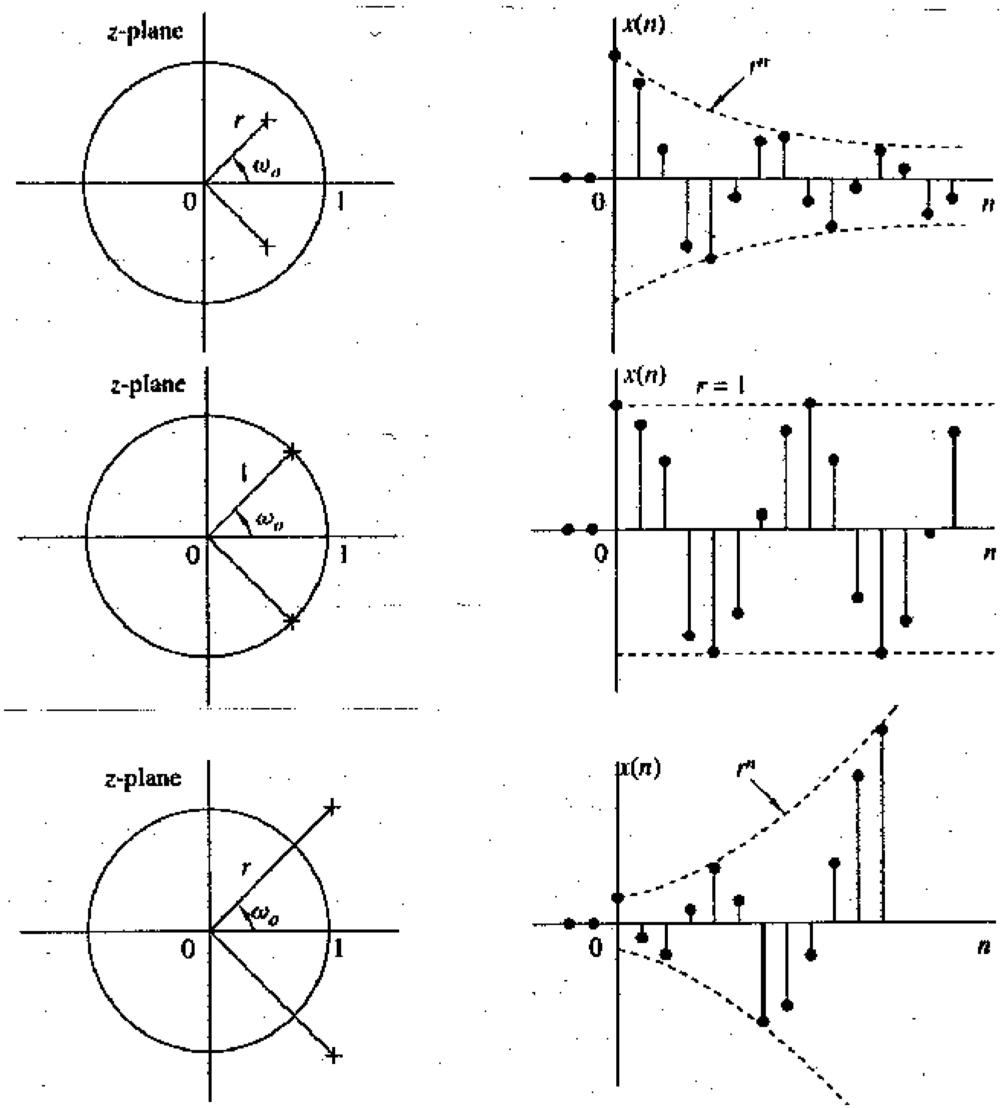
A real causal signal with a double pole has the form

$$x(n) = na^n u(n)$$



Rational z-Transforms

- A real causal signal with a pair of complex-conjugate poles:



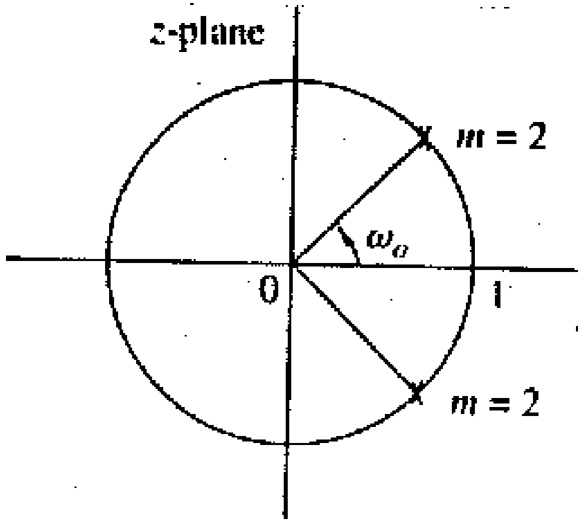
$x(n) = a^n \sin \omega_0 n \cdot u(n)$ or

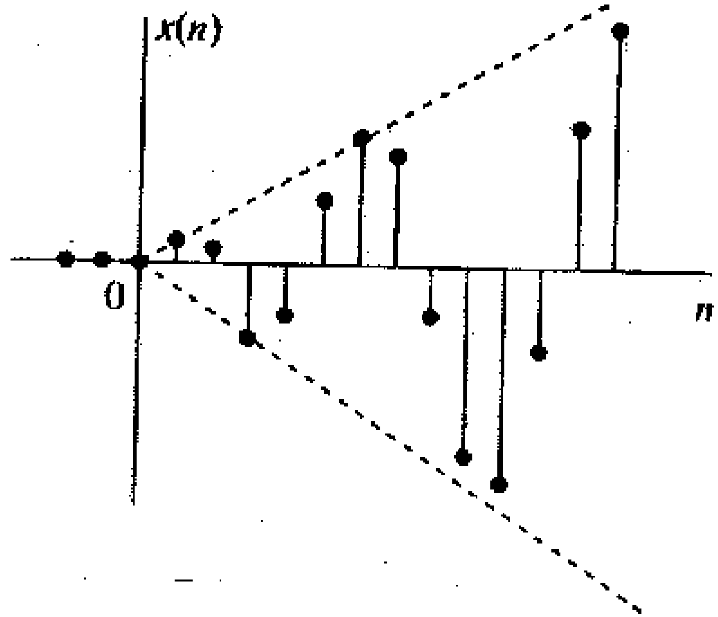
$x(n) = a^n \cos \omega_0 n \cdot u(n)$

Note) Location of zeros only affects the signal phase in this case.

Rational z-Transforms

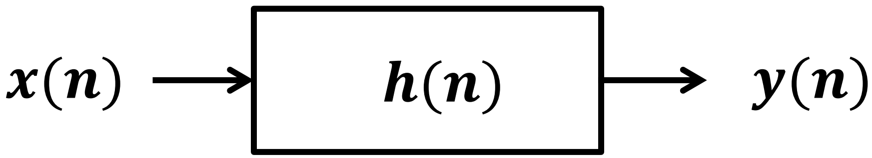
A real causal signal with a double pair of complex-conjugate poles:





Rational z-Transforms

The system function of a LTI system



$$y(n) = h(n) * x(n) \xleftrightarrow{z} Y(z) = H(z)X(z)$$

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

$$\xleftrightarrow{z} Y(z) = -\sum_{k=1}^{N} a_k z^{-k} Y(z) + \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$Y(z)\left(1 + \sum_{k=1}^{N} a_k z^{-k}\right) = X(z)\left(\sum_{k=0}^{M} b_k z^{-k}\right)$$

Rational transfer function

$$\boxed{H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}}$$

Rational z-Transforms

The system function of a LTI system

- All zero system (FIR, Non-recursive)

$$H(z) = \sum_{k=0}^{M} b_k z^{-k} = \frac{1}{z^M} \sum_{k=0}^{M} b_k z^{M-k}$$

- All pole system (IIR, recursive)

$$H(z) = \frac{b_0}{1 + \sum_{k=1}^{N} a_k \cdot z^{-k}} = \frac{b_0}{\sum_{k=0}^{N} a_k \cdot z^{-k}} \qquad a_0 \equiv 1$$

- Pole-zero system (IIR, recursive)

Rational z-Transforms

Determining the impulse response from difference equation

$$Ex)\ y(n) = \frac{1}{2} y(n-1) + 2x(n)$$

$$\rightarrow Y(z) = \frac{1}{2} z^{-1} Y(z) + 2X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2}{1-(1/2)\,z^{-1}}$$

From the table,

$$\rightarrow h(n) = 2\left(\frac{1}{2}\right)^{n} u(n)$$

No information on the initial condition and transient response

Inverse z-Transform

- Formula

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz \qquad X(z) = \sum_{k=-\infty}^{\infty} x(k) z^{-k}$$

$$\frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz = \frac{1}{2\pi j} \oint_C \sum x(k) z^{-k+n-1} dz = \sum \frac{x(k)}{2\pi j} \oint_C z^{-k+n-1} dz$$

$$= \sum \begin{bmatrix} Residues of & X(z) z^{n-1} \\ at the poles & inside C \end{bmatrix} = \begin{cases} x(n), & k = n \\ 0, & otherwise \end{cases}$$

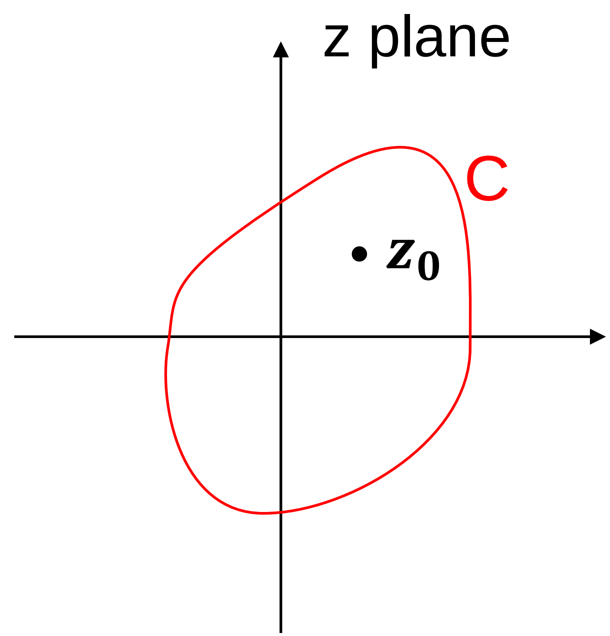
- Methods
  - Direct evaluation by contour integration
  - Series expansion method
  - Partial-fraction expansion

Inverse z-Transform

❖ Contour Integration

Cauchy residue theorem)

Let $f(z)$ be a function of the complex variable z and C be a closed path in the z-plane. If the (k+1)-order derivative of $f(z)$ exists on and inside the contour C and **if $f(z)$ has no poles at z_0**, then



$$\frac{1}{2\pi j}\oint_C \frac{f(z)}{(z-z_0)^k}dz = \begin{cases} \frac{1}{(k-1)!}\left.\frac{d^{k-1}f(z)}{dz^{k-1}}\right|_{z=z_0} & if\ \ z_0\ \ is\ \ inside\ \ \ C \\ 0, & if\ \ z_0\ \ is\ \ outside\ \ \ C \end{cases}$$

Inverse z-Transform

In general, if $X(z)z^{n-1}$ is a rational function of z, it may be expressed as

$$X(z)z^{n-1} = \frac{f(z)}{(z-z_0)^m} \qquad \boxed{x(n) = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz}$$

where $X(z)z^{n-1}$ has m poles at $z = z_0$ and
$f(z)$ has no poles at $z = z_0$

Note) **The residues of** $X(z)z^{n-1}$ at $z = z_0$ is given by

$$Res\left[X(z)z^{n-1} \ \ at \ \ z = z_0\right] = \frac{1}{(m-1)!}\frac{d^{m-1}f(z)}{dz^{m-1}}\Bigg|_{z=z_0}$$

Note) **First-order pole**: $m = 1$ at $z = z_0$

$$Res\left[X(z)z^{n-1} \ \ at \ \ z = z_0\right] = f(z_0)$$

Inverse z-Transform

Example) $X(z) = 1/(1 - az^{-1}) \qquad |z| > |a|$

$$x(n) = \frac{1}{2\pi j}\oint_C \frac{z^{n-1}}{1 - az^{-1}}dz = \frac{1}{2\pi j}\oint_C \frac{\overbrace{z^n}^{f(z)}}{z - \underbrace{a}_{z_0}}dz$$

$$where \;\; C \;\; is \;\; a \;\; circle \;\; of \;\; a \;\; radius \quad |z| > |a|$$

$$For \;\; n \geqq 0 \rightarrow one \;\; pole \;\; at \;\; z = a \rightarrow x(n) = a^n, \;\; n \geqq 0$$

Inverse z-Transform

- Power series expansion method
  - Expand X(z) into a power series of the form

$$\boldsymbol{X(z) = \sum_{n=-\infty}^{\infty} c_n z^{-n} \; in\, the\, given\, ROC}$$

$$\rightarrow \boldsymbol{x(n) = c_n}$$

  - When X(z) is rational, the expansion can be done by long division

    ➔ closed-form solution is not possible

Inverse z-Transform

- Power series expansion method
  - The function impz can be used to find the inverse of a rational z-transform G(z)
  - impz computes the coefficients of the power series expansion of G(z)
  - The number of coefficients can either be user specified or determined automatically

[h, t] = impz(num, den) or impz(num, den, N) or
impz(num, den, N, Fs)

Inverse z-Transform

❖ Power series expansion method

$$\frac{1}{1-1.5z^{-1}+0.5z^{-2}}$$

$$(a)\quad ROC:\ |z|>1 \rightarrow \ Causal\ \ signal$$

$$\rightarrow Find\ a\ power\ series\ expansion\ in\ negative\ powers\ of\ z$$
$$by\ long\ division$$

$$\begin{array}{r|l} & 1+\frac{3}{2}z^{-1}+\frac{7}{4}z^{-2}+\cdots \\ \hline 1-1.5z^{-1}+0.5z^{-2} & 1 \end{array}$$

```
n=[1]
d=[1 -1.5 0.5]
[h, t] = impz(n, d, 7)
h = { 1.0   1.5   1.75   1.875  1.9375  1.9688  1.9844 }
```

Inverse z-Transform

❖ Power series expansion method

(b) $ROC:\ |z|<1 \rightarrow$ *Noncausal signal*

$\rightarrow$ *Find a power series expansion in positive powers of z by long division*

$$
\begin{array}{r|l}
 & 2z^2+6z^3+14z^4+\cdots \\ \hline
\frac{1}{2}z^{-2}-\frac{3}{2}z^{-1}+1 & 1 \\
 & 1-3z+2z^2 \\ \cline{2-2}
 & \quad 3z-2z^2 \\
 & \qquad \vdots
\end{array}
$$

Inverse z-Transform

- Partial-fraction expansion method
  - A rational $G(z)$ can be expressed as

$$G(z) = \frac{P(z)}{D(z)} = \frac{\sum_{i=0}^{M} p_i z^{-i}}{\sum_{i=0}^{N} d_i z^{-i}}$$

  - If $M \geq N$ then $G(z)$ can be re-expressed as

$$G(z) = \sum_{l=0}^{M-N} c_l z^{-l} + \boxed{\frac{P_1(z)}{D(z)}} \rightarrow \textbf{Proper function}$$

$$where\ degree\ of\ P_1(z) < N$$

Inverse z-Transform

- Partial-fraction expansion method: **Example**

$$G(Z)=\frac{2+0.8z^{-1}+0.5z^{-2}+0.3z^{-3}}{1+0.8z^{-1}+0.2z^{-2}}$$

Long division

$$=-3.5+1.5z^{-1}+\frac{5.5+2.1z^{-1}}{1+0.8z^{-1}+0.2z^{-2}}$$

Partial fraction expansion

$$=-3.5+1.5z^{-1}+\frac{r_1}{1-p_1z^{-1}}+\frac{r_2}{1-p_2z^{-1}}$$

$\Rightarrow$ $g(n)=-3.5\delta(n)+1.5\delta(n-1)+(r_1p_1^n+r_2p_2^n)u(n)$

Note) For a real rational function, complex poles must occur in conjugate pairs. And, $If\ \ p_1=p_2^*,\ \ then\ \ r_1=r_2^*$

Inverse z-Transform

- Partial-fraction expansion method
  - Simple poles: Generally, the rational z-transform of interest $X(z)$ is a proper fraction with **"simple poles"**

$$X(Z) = \frac{N(z)}{D(z)} = \frac{N(z)}{(1-p_1 z^{-1})(1-p_2 z^{-1})\cdots(1-p_N z^{-1})}$$

$$p_1 \neq p_2 \neq \cdots \neq p_N \ : \ \mathit{Simple\ \ Poles}$$

$$\Rightarrow \quad X(z) = \sum_{i=1}^{N}\left(\frac{r_i}{1-p_i z^{-1}}\right)$$

residues

Inverse z-Transform

- Partial-fraction expansion method
  - The constants r_l (residues) are given by

$$r_l = (1 - p_l z^{-1}) X(z) \Big|_{z = p_l}$$

0 when $z = p_l$

$$(1 - p_l z^{-1}) X(z) = (1 - p_l z^{-1}) \sum_{i=1}^{N} \left(\frac{r_i}{1 - p_i z^{-1}} \right) = r_l + (1 - p_l z^{-1}) \sum_{i \neq l} \left(\frac{r_i}{1 - p_i z^{-1}} \right)$$

  - Each term of the sum in partial-fraction expansion has an ROC by $|z| > |p_l|$ and, thus

$$x(n) = (r_1 p_1^n + r_2 p_2^n + \cdots + r_N p_N^n) u(n)$$

Inverse z-Transform

- Partial-fraction expansion method
  - Example

$$H(z) = \frac{z(z+2)}{(z-0.2)(z+0.6)} = \frac{1+2z^{-1}}{(1-0.2z^{-1})(1+0.6z^{-1})}$$

$$H(z) = \frac{r_1}{1-0.2z^{-1}} + \frac{r_2}{1+0.6z^{-1}}$$

$$r_1 = (1-0.2z^{-1})H(z)|_{z=0.2} = \frac{1+2z^{-1}}{1+0.6z^{-1}}|_{z=0.2} = 2.75$$

$$r_2 = (1+0.6z^{-1})H(z)|_{z=-0.6} = \frac{1+2z^{-1}}{1-0.2z^{-1}}|_{z=-0.6} =-1.75$$

➔ $h[n] = 2.75(0.2)^n u(n) - 1.75(-0.6)^n u(n)$

Inverse z-Transform

Partial-fraction expansion method

- Multiple poles

$$G(Z) = \frac{N(z)}{(1-p_1z^{-1})(1-p_2z^{-1})\underline{(1-\nu z^{-1})^L}\cdots(1-p_Nz^{-1})}$$

- Let the pole at $\boldsymbol{z = \nu}$ be of multiplicity L and the remaining $\mathbf{N} - \boldsymbol{L}$ poles be simple and at $\boldsymbol{z = p_l, 1 \leq l \leq N - L}$

Inverse z-Transform

Partial-fraction expansion method

- Then the partial-fraction expansion of G(z) is of the form

$$G(z) = \sum_{l=0}^{M-N} \eta_l z^{-l} + \sum_{i=1}^{N-L} \frac{r_i}{1 - p_i z^{-1}} + \sum_{i=1}^{L} \frac{\gamma_i}{(1 - \nu z^{-1})^i}$$

$$\gamma_i = \frac{1}{(L-i)!(-\nu)^{L-i}} \frac{d^{L-i}}{d(z^{-1})^{L-i}} \left[(1 - \nu z^{-1})^L G(z) \right]_{z=\nu}, \quad 1 \le i \le L$$

- The residues for simple poles are calculated as before

Inverse z-Transform

- Partial-fraction expansion method
  - Using MATLAB: [r,p,k]= residuez(num,den)

    develops the partial-fraction expansion of a rational z-transform with numerator and denominator coefficients given by vectors num and den
  - Vector r contains the residues
  - Vector p contains the poles
  - Vector k contains the constants η_ℓ

    [num,den]=residuez(r,p,k) converts a z-transform expressed in a partial-fraction expansion form to its rational form

Inverse z-Transform

$$G(z) = \frac{2 + 0.8z^{-1} + 0.5z^{-2} + 0.3z^{-3}}{1 + 0.8z^{-1} + 0.2z^{-2}}$$

$$G(z) = -3.5 + 1.5z^{-1} + \frac{5.5 + 2.1z^{-1}}{1 + 0.8z^{-1} + 0.2z^{-2}}$$

```
n = [2 0.8 0.5 0.3];
d = [1 0.8 0.2];
[r, p, k]= residuez(n, d);
```

$\Rightarrow$

r = 2.75 + 0.25i   p : -0.4 + 0.2i   k : -3.5

2.75 - 0.25i   -0.4 - 0.2i   1.5

$$G(z) = -3.5 + 1.5z^{-1} + \frac{r_0}{1 - p_0 z^{-1}} + \frac{r^*{}_0}{1 - p^*{}_0 z^{-1}}$$

$$g(n) = -3.5\delta(n) + 1.5\delta(n-1) + \underline{[r_0(p_0)^n + r^*{}_0\,(p^*{}_0)^n]u(n)}$$

$$r_0 = |r_0|e^{j\alpha_0} \quad and \quad p_0 = |p_0|e^{j\beta_0} \qquad \Downarrow$$

$$\boxed{2|r_0| \cdot |p_0|^n \cdot cos(\beta_0 n + \alpha_0)}$$

Rational z-Transforms

$$H(z)=\frac{1}{1-0.9z^{-1}-0.81z^{-2}+0.729z^{-3}}=\frac{1}{(1-0.9z^{-1})^2(1+0.9z^{-1})},\quad |z|>0.9$$

```
n =1;
d = [1 -0.9 -0.81 0.729];
[r, p, k]= residuez(n, d);
```

r:	0.25	0.5	0.25		
p:	0.9	0.9	-0.9	K:	[ ]

$$H(z)=\boxed{\frac{0.25}{1-0.9z^{-1}}}+\boxed{\frac{0.5}{(1-0.9z^{-1})^2}}+\boxed{\frac{0.25}{1+0.9z^{-1}}}$$

$$H(z)=\frac{0.25}{1-0.9z^{-1}}+\frac{0.5}{0.9}z\frac{(0.9z^{-1})}{(1-0.9z^{-1})^2}+\frac{0.25}{1+0.9z^{-1}}$$

$$x(n)=0.25(0.9)^n u(n)+\frac{5}{9}(n+1)(0.9)^{n+1}u(n+1)+0.25(-0.9)^n u(n)$$

Time shifting $\quad x(n-k) \quad z^{-k}X(z)$

$$na^n u(n) \overset{z}{\Longleftrightarrow} X(z)=-z\frac{dX_1(z)}{dz}=\frac{az^{-1}}{(1-az^{-1})^2}\qquad |z|>|a|$$

Decomposition of rational z-transforms

Rational z-transform

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = b_0 \frac{\prod_{k=1}^{M} (1 - z_k z^{-1})}{\prod_{k=1}^{N} (1 - p_k z^{-1})}$$

- $M \geq N$: Improper ➧ $X(z) = \sum_{k=0}^{M-N} c_k z^{-k} + X_{pr}(z)$
- Decomposition with real coefficients

  complex conjugate poles are combined as

$$\frac{A}{1 - pz^{-1}} + \frac{A^*}{1 - p^* z^{-1}} = \frac{A - Ap^* z^{-1} + A^* - A^* pz^{-1}}{1 - pz^{-1} - p^* z^{-1} + pp^* z^{-2}}$$

$$= \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

$$\boxed{\begin{array}{ll} b_0 = 2Re(A), & a_1 = -2Re(p) \\ b_1 = -2Re(Ap*), & a_2 = |p|^2 \end{array}}$$

Decomposition of rational z-transforms

General result

$$X(z)=\sum_{k=0}^{M-N} c_k z^{-k}+\sum_{k=1}^{K_1}\frac{b_k}{1+a_k z^{-1}}+\sum_{k=1}^{K_2}\frac{b_{0k}+b_{1k}z^{-1}}{1+a_{1k}z^{-1}+a_{2k}z^{-2}}$$

where $K_1+2K_2=N$

- If M=N, $c_0=const$
- If M < N, no FIR terms

Note: Higher order terms should exist when there are multiple poles.

Decomposition of rational z-transforms

- Decomposition into products of simple terms

$$\frac{(1-z_k z^{-1})(1-z_k^* z^{-1})}{(1-p_k z^{-1})(1-p_k^* z^{-1})} = \frac{1+b_{1k} z^{-1}+b_{2k} z^{-2}}{1+a_{1k} z^{-1}+a_{2k} z^{-2}}$$

where

$$b_{1k} = -2\,Re(z_k), \qquad a_{1k} = -2\,Re(p_k)$$

$$b_{2k} = |z_k|^2, \qquad a_{2k} = |p_k|^2$$

$$\Rightarrow \quad X(z) = b_0 \prod_{k=1}^{K_1} \frac{1+b_k z^{-1}}{1+a_k z^{-1}} \prod_{k=1}^{K_2} \frac{1+b_{1k} z^{-1}+b_{2k} z^{-2}}{1+a_{1k} z^{-1}+a_{2k} z^{-2}}$$

where $N = K_1 + 2K_2$