Вывод градиента MSPE.

1) $$MSPE(w) = \frac{1}{m}\sum_{i=1}^{m}\left(\frac{y_i - \hat{y}_i}{y_i}\right)^2 = \frac{1}{m}\sum_{i=1}^{m}\left(\frac{y_i - \langle x_i w\rangle}{y_i}\right)^{\boxed{2}} \ (*)$$

$\hat{y} = \langle x_i, w\rangle$

$$= \frac{1}{m}\left[\mathbb{1}_{m\times 1} - X./y \cdot w\right]^T \times \left[\mathbb{1}_{m\times 1} - X./y \cdot w\right]$$

2) Получим матричную форму:

Опр.:

$$\vec{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \rightsquigarrow |a|^2 = \underset{\langle \bar{a}, \bar{a}\rangle}{a^T a} = \sum_{i=1}^{m} a_i^2$$

$$a_i = \frac{y_i - \langle x_i w\rangle}{y_i} = 1 - \frac{\langle x_i, w\rangle}{y_i} \qquad X$$

$$\vec{a} = \underbrace{\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}}_{\mathbb{1}_{m\times 1}} - \underbrace{\begin{pmatrix} \frac{x_{11}}{y_1} & \frac{x_{12}}{y_1} & \cdots & \frac{x_{1n}}{y_1} \\ \vdots & & & \\ \frac{x_{m1}}{y_m} & & \cdots & \frac{x_{mn}}{y_m} \end{pmatrix}_{m\times n}}_{X./y} \times \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}_{n\times 1} = \vec{b}_{m\times 1}$$

4) Градиент: $Q(w) = \frac{1}{m}\sum_{i=1}^{m}\left[\frac{y_i - \langle x_i w\rangle}{y_i}\right]^{\boxed{2}}$ $\circledast$

$$\vec{\nabla}_w Q(w) = ?$$

$$m \cdot \frac{\partial}{\partial w_j} Q(w) = \frac{\partial}{\partial w_i}\sum_{i=1}^{m}\left(\frac{y_i - x_1^i w_1 - x_2^i w_2 - \dots x_j^i w_j - \dots - x_n^i w_n}{y_i}\right)^2 =$$

$$= \sum_{i=1}^{m}\frac{\partial}{\partial w_j}\left(\frac{y_i - x_1^i w_1 - x_2^i w_2 - \dots x_j^i w_j - \dots - x_n^i w_n}{y_i}\right)^2 =$$

$$\frac{\partial}{\partial w_j}\left[f^2(w_j)\right] = 2\cdot f(w_j)\cdot f'(w_j)$$

$$= \sum_{i=1}^{m} 2\cdot\left[\frac{y_i - \langle x_i w\rangle}{y_i}\right]\cdot\left(-\frac{x_j^i}{\boxed{y_i}}\right)$$

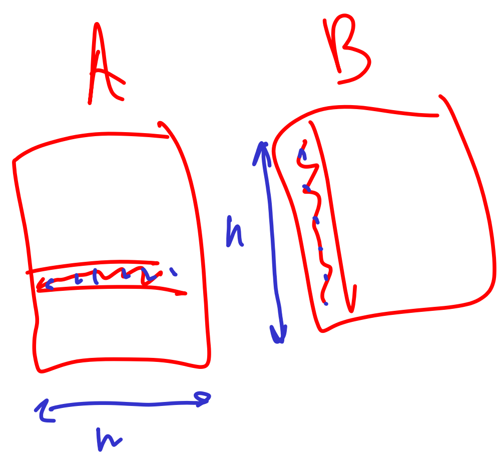
$$\hookrightarrow \frac{\partial}{\partial w_j} Q(w) = -\frac{2}{m}\sum_{i=1}^{m} x_j^i \cdot \frac{y_i - \langle x_i w\rangle}{y_i^2}$$

$$x_{ij} \in X$$

Скалярное произв.

$$a^i \in A \qquad b_1 \in B$$

$$\sum_{k=1}^{n} a_{ik}\cdot b_{k1} = ab$$



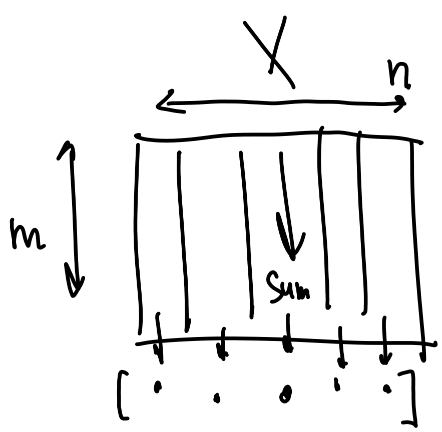
$$-\frac{2}{m}\sum_{i=1}^{m} x_{ij}\frac{y_i-\langle x_i w\rangle}{y_i^2} = -\frac{2}{m}\sum_{i=1}^{m} x_{ij}\frac{y_i}{y_i^2} + \frac{2}{m}\sum_{i=1}^{m} x_{ij}\cdot\frac{\langle x_i w\rangle}{y_i^2}$$

$$-\frac{2}{m}\sum_{i=1}^{m} \frac{x_{ij}}{y_i^2}$$

$$\frac{\partial}{\partial w_i} Q(w) = \frac{2}{m}\left[\underbrace{\sum_{i=1}^{m} x_{ij}\frac{\langle x_i w\rangle}{y_i^2}} - \sum_{i=1}^{m} \frac{x_{ij}}{y_i}\right] = \begin{pmatrix}\vdots\\ \bullet_i\\ \vdots\end{pmatrix}$$

$$\frac{\partial}{\partial w} Q(w) = \frac{2}{m}\left[\underbrace{\underbrace{(X./y^2)^T}_{n\times m}\cdot\underbrace{\underbrace{X}_{m\times n}\underbrace{w}_{n\times 1}}_{m\times 1}}_{n\times 1} - \underbrace{\frac{X^T\cdot\underbrace{\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}}_{m\times 1}}{y_i}}_{n\times 1}\right]$$

⊛ grad!!



$$\underset{n\times m}{X^T}\cdot\underset{m\times 1}{\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}} = \underset{n\times 1}{\cdot}$$

Сумма по i строке X : $\langle x^i ; \begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}\rangle$

Сумма по $i^{ый}$ столбцу X^T

$$\frac{\partial}{\partial w} Q(w) = \frac{2}{m}\left[(X./y)^T \cdot Xw - X^T \cdot \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{m\times 1} \right] \quad ⊛ \text{ grad!!}$$

Сравним с MSE:

$$\mathcal{L} = MSE(w) = \frac{1}{m}\sum_{i=1}^{m} \left(y_i - \langle x_i w\rangle\right)^2$$

$$\frac{\partial}{\partial w}\mathcal{L} = -\frac{2}{m} X^T (y - Xw) \quad ✓$$

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$$(AB - AC) = A(B - C)$$

$$y = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$⊛ \xrightarrow{?} ✓ \quad \rightsquigarrow \frac{2}{m}\left[X^T \cdot Xw - X^T \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \right] = \frac{2}{m} X^T \left(Xw - \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \right)$$

$$✓ \rightsquigarrow -\frac{2}{m} X^T \left(\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} - Xw \right) = -\frac{2}{m} X^T \left(\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} - Xw \right)$$

Как $X./y$?

$$\frac{x_{11}}{y_{11}} \quad \frac{x_{12}}{y_{11}} \quad \cdots \quad \frac{x_{1n}}{y_{11}}$$

$$\begin{pmatrix} y_{11} \\ y_{21} \\ \vdots \\ y_{n1} \end{pmatrix}$$