Chapter 1

Notes

Addition and multiplication on $\mathbb{C}$

$$(a+bi)+(c+di)=(a+c)+(b+d)i$$

$$(a+bi)(c+di)=(ac-bd)+(ad+bc)i$$

Commutativity

$\alpha+\beta=\beta+\alpha$ $\quad \alpha=a+bi \quad a,b,c,d\in\mathbb{R}$

$\beta=c+di$

$$a+bi+c+di=c+di+a+bi$$

$\Downarrow$ commutativity over $\mathbb{R}$

$$c+bi+a+di$$

$$bi+a \to (a)+(b)i$$

$a=a \quad c=a$

$b=b \quad d=0$

$$c+a+bi+di$$

$ab=ba \quad a,b\in\mathbb{C}$

$$(e+fi)(g+hi)=(g+hi)(e+fi)$$

$eg + fgi + ehi - fh = ge + fgi + ehi - hf$

$= eg + fgi + ehi - fh$

$x, y \in F^n$

prove $x + y = y + x$

$x + y = (x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$

$\forall\, x_j + y_j \in x + y,\ x_j + y_j = y_j + x_j$

$\therefore (x_1 + y_1, \dots, x_n + y_n) = (y_1 + x_1, \dots, y_n + x_n)$

$= y + x$

Arithmetic of fields

$a, b \in \mathbb{F}$

commutativity

$a + b = b + a \qquad ab = ba$

associativity

$(a + b) + c = a + (b + c) \qquad (ab)c = a(bc)$

$a + 0 = a \qquad a \cdot 1 = a$

$a + -a = 0 \qquad a \neq 0,\ \exists b,\ ab = 1$

$c(a+b) = ac + ab$

is $\{0, 1\}$ a field?

prove $a+b = b+a$ $\quad a, b \in \{0, 1\}$

a	b	a+b	b+a
0	0	0	0
0	1	1	1
1	0	1	1
1	1	0	0

✓

Ex 1.A

1 $\quad a, b \in \mathbb{R}$

Find c and d such that

$$\frac{1}{a+bi} = c + di \qquad (a+bi)(a-bi) = a^2 + b^2$$

$$\frac{1-i}{a+bi} + \frac{1}{a+bi}\, i$$

$$c = \frac{1-i}{a+bi} \qquad d = \frac{1}{a+bi}$$

$$c + di = \frac{1-i+i}{a+bi} = \frac{1}{a+bi}$$

rationalizing c $\rightarrow$ $c = \dfrac{(1-i)(a-bi)}{a^2+b^2} = \dfrac{1 - ai - bi - b}{a^2+b^2} = (1-b) + (-a-b)i$

c and d have to be real numbers

$$\frac{1}{a+bi} = c+di$$

$$\frac{a-bi}{a^2+b^2} = c+di$$

$$\frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} i$$

$$c = \frac{a}{a^2+b^2} \quad d = -\frac{b}{a^2+b^2}$$

(3) Two distinct square roots of i

$$\sqrt{49} = 7$$

→ square roots of a number are other numbers which, when squared, equal the original number

$$a^2 = i \qquad b^2 = i$$

$$a = (-1)^{1/4} \qquad b = -(-1)^{1/4}$$

$$\sqrt{i} = z$$

$$z^2 = i$$

$$(a+bi)^2 = i$$

$a^2 + 2abi - b^2 = i$

$\textcircled{\cdot}$ $a^2 - b^2 + 2abi = i$

$a^2 - b^2 = 0$

$2ab = 1$

$ab = \frac{1}{2}$ $\quad a = \pm b$

$\cancel{a^2 - b}$

$\cancel{-b^2 = \frac{1}{2}}$

$a = b$

$a^2 = \frac{1}{2}$

$a = \pm \frac{1}{\sqrt{2}}$

$z_1 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$

$z_2 = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i$