Basics of Computational Image Analysis

With examples in Matlab andC++

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Chapter 1

Foundations

In the development of image analysis there are many theorems that depend on real and complex analysis. This section presents the required material but does not prove any theorems or explain them in detail. For a deeper understanding of these concepts see the cited references.

1.1 Topology and Measures

These come from Rudin [2]. Measure theory provides the notion of length, area and volume of sets. This is necessary for the formal definitions of integrals.

Definition 1.1.1. If $\mathfrak{M}$ is a σ-algegra in X, then X is a *measurable space*. The members of $\mathfrak{M}$ are the *measurable sets* in X.

Definition 1.1.2. If X is a measurable space, Y is a topological space and $f : X \to Y$ then f is *measurable* if $f^{-1}(V)$ is measurable in X for every open set V in Y.

Definition 1.1.3. A function μ defined on a σ-algebra $\mathfrak{M}$ is *countably additive* means that if $\{A_i\}$ is a disjoint countable collection of members of $\mathfrak{M}$ then

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i).$$

Definition 1.1.4. A *positive measure* (measure) is a function μ, defined on a σ-algebra $\mathfrak{M}$, whose range is in $[0, \infty]$ and which is countably additive.

Definition 1.1.5. A *measure space* is a measurable space which has a positive measure defined on the σ-algebra of its measurable sets.

1.2 $\mathbf{L}^p$ Spaces

A $\mathbf{L^p}$ space is a normed vector space(?).

Definition 1.2.1. Let $(X, \mathfrak{M}, \mu)$ be a measure space and $0 < p < \infty$. The $\mathbf{L}^p$-norm of a function $f : X \to \mathbb{R}$ is defined as:

$$\|f\|_p = \left(\int_X |f|^p d\mu \right)^{1/p}$$

Definition 1.2.2 ($\mathbf{L}^p$ Space)**.** [1]
Let $1 \leq p < \infty$ and $f : X \to \mathbb{R}$ then the set of functions:

$$\mathbf{L}^p = \{f : \mathbf{L^p} \int_X |f|^p d\mu < \infty\}$$

is a vector space called the $\mathbf{L}^p$ *space*.

Chapter 2

The Convolution and Correlation

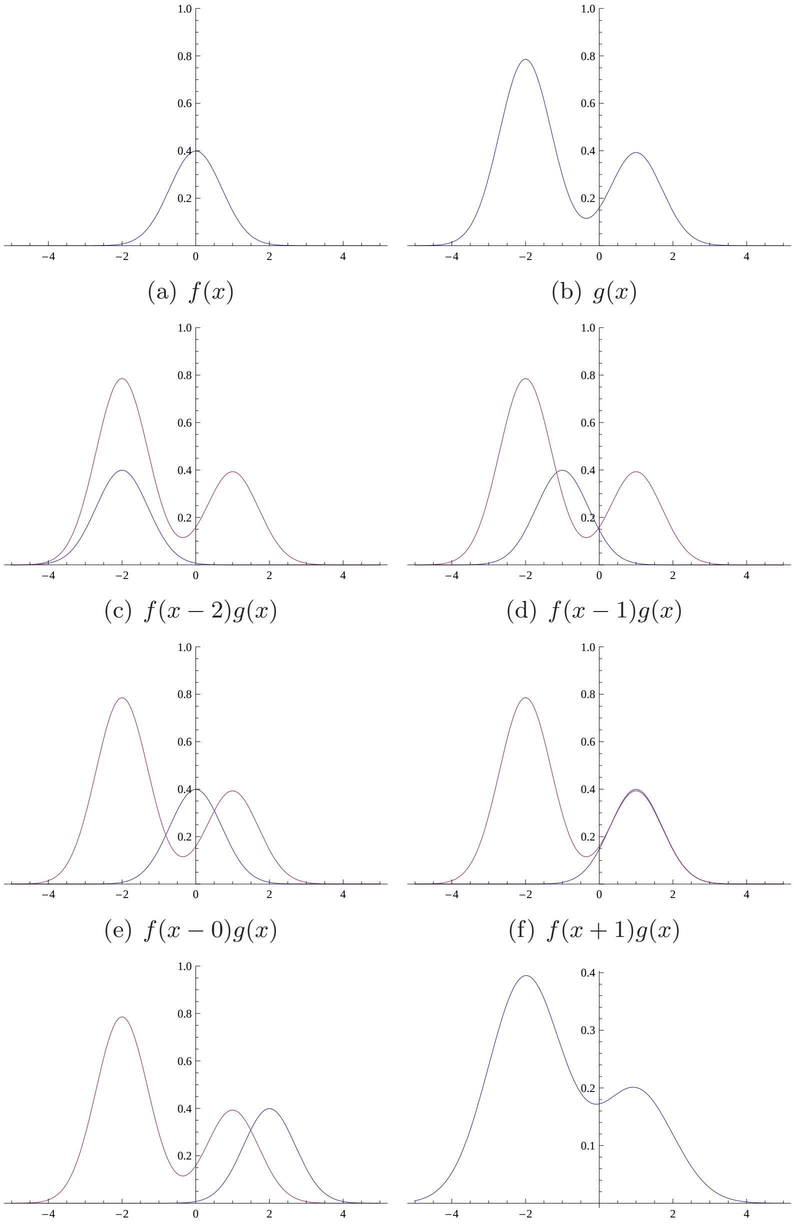
One of the most important concepts in image analysis is convolution. The convolution is a transform that takes two functions and returns a different function. This new function has been created by essentially smearing one of the functions over the other combining their magnitude at every point. Figure 2.1 shows how this is done.

2.1 One-Dimensional Convolution

Definition 2.1.1. Let f and g belong to $L^1(\mathbb{R})$. The *convolution*, $f * g$ defined as:

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x - \tau) g(\tau) d\tau \tag{2.1.1}$$

In the context of image analysis we are dealing with real-valued functions that are generally bounded (by intensity such as 0 to 255). So these functions behave nicely and therefore we can assume the covolution exists. For more information on the existance of the convolution see Yeh [3].



(a) $f(x)$ (b) $g(x)$

(c) $f(x-2)g(x)$ (d) $f(x-1)g(x)$

(e) $f(x-0)g(x)$ (f) $f(x+1)g(x)$

(g) $f(x+2)g(x)$ (h) $f*g$

Figure 2.1: One-dimensional convolution example

The convolution has many properties that are useful.

Proposition 2.1.2. *The convolution has the following properties (prove).*

a $f * g = g * f$ *(commutative)*

b $f * (g * h) = (f * g) * h$ *(associative)*

c $f * (g + h) = (f * g) + (f * h)$ *(distributive)*

d $a(f * g) = (af * g) = (f * ag)s$ *for* $a \in \mathbb{R}$ *(associativity with scalar multiplication)*

Proof.

a Substitute $\kappa = x - \tau$, $d\kappa = -d\tau$ into the defintion of convolution so $(f * g)(x) = -\int_{\infty}^{-\infty} f(\kappa)g(x-\kappa)d\kappa$. Switching the limits of integration and the order of f and g we have $(f * g)(x) = \int_{-\infty}^{\infty} g(x-\kappa)f(\kappa)d\kappa$. Rename the variable of integration κ back to τ (this rename doesn't change anything) and we get $(f * g)(x) = \int_{-\infty}^{\infty} g(x-\tau)f(\tau)d\tau = (g * f)(x)$

b Using the defintion

$$\begin{aligned} (f * (g * h))(x) &= \int_{-\infty}^{\infty} f(x-\tau)(g * h)(\tau)d\tau \\ &= \int_{-\infty}^{\infty} f(x-\tau)\left[\int_{-\infty}^{\infty} g(\tau-\kappa)h(\kappa)d\kappa\right]d\tau \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\tau)g(\tau-\kappa)h(\kappa)d\kappa d\tau. \end{aligned}$$

By Fubini's theorem we can switch the order of integration so

$$\begin{aligned} (f * (g * h))(x) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\tau)g(\tau-\kappa)h(\kappa)d\tau d\kappa \\ &= \int_{-\infty}^{\infty} f(x-\tau)\left[\int_{-\infty}^{\infty} g(\tau-\kappa)h(\kappa)d\tau\right]d\kappa. \end{aligned}$$

Since we are working in a translation invarient metric we have

$$\begin{aligned} (f * (g * h))(x) &= \int_{-\infty}^{\infty} f(x-\tau)\int_{-\infty}^{\infty}[g((\tau+\kappa)-\kappa)h(x-(\tau+\kappa))d\tau]d\kappa \\ &= \int_{-\infty}^{\infty} f(x-\tau)\int_{-\infty}^{\infty}[g(\tau)h((x-\kappa)-\tau))d\tau]d\kappa \\ &= \int_{-\infty}^{\infty} f(x-\tau)(g * h)(x-\kappa)d\kappa \\ &= ((f * g) * h)(x). \end{aligned}$$

c

$$\begin{aligned} (f * (g + h))(x) &= \int_{-\infty}^{\infty} f(x-\tau)(g(\tau)+h(\tau))d\tau \\ &= \int_{-\infty}^{\infty} f(x-\tau)g(\tau)+f(x-\tau)h(\tau))d\tau \\ &= \int_{-\infty}^{\infty} f(x-\tau)g(\tau)+\int_{-\infty}^{\infty} f(x-\tau)h(\tau))d\tau \\ &= (f * g)(x) + (f * h)(x). \end{aligned}$$

d

$$\begin{array}{ll} a(f*g)(x) & = a\int_{-\infty}^{\infty} f(x-\tau)g(\tau)d\tau \\ & = \int_{-\infty}^{\infty} af(x-\tau)g(\tau)d\tau \\ & = (af*h)(x). \end{array}$$

The other equality is proved exactly the same way.

□

Differentiation.

$$convolution!derivative1D\frac{d}{dx}(f*g) = (\frac{df}{dx}*g)(x) = (f*\frac{dg}{dx}g)(x)$$

2.2 Two-Dimensional Convolution

In two dimensions:

Definition 2.2.1. Let f and g belong to $L^2\mathbb{R}^{\nvDash}$. The convolution, $(f*g)$ defined as:

$$(f*g)(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\tau, y-\kappa)g(\tau,\kappa)d\tau d\kappa \tag{2.2.1}$$

Differentiation in two variables.

$$convolution!derivative2D\frac{\partial}{\partial x}(f*g) = \frac{\partial f}{\partial x}*g = f*\frac{\partial g}{\partial x}$$

$$\frac{\partial}{\partial y}(f*g) = \frac{\partial f}{\partial y}*g = f*\frac{\partial g}{\partial y}$$

2.3 Discrete Convolution

We will only examine the two dimensional case.

Definition 2.3.1. Let f and g be functions ...?

$$(f\otimes g)(x,y) = \sum_{i=0}^{M-1}\sum_{j=0}^{N-1} f(i-x, j-y)g(i,j).$$

2.4 Convolution Implementation

```
void LinearConvolution(double X[],double Y[], double Z[], int lenx, int leny)
{
  double *zptr,s,*xp,*yp;
  int lenz;
  int i,n,n_lo,n_hi;

  lenz=lenx+leny-1;
  zptr=Z;

  for (i=0;i<lenz;i++) {
    s=0.0;
    n_lo=0>(i-leny+1)?0:i-leny+1;
    n_hi=lenx-1<i?lenx-1:i;
    xp=X+n_lo;
    yp=Y+i-n_lo;
    for (n=n_lo;n<=n_hi;n++) {
      s+=*xp * *yp;
      xp++;
      yp--;
      }
    *zptr=s;
    zptr++;
    }
}
```

2.5 One-Dimensional Correlation

Correlation

Definition 2.5.1.

Appendix A

Test appendix

Test test

Bibliography

[1] F. Jones. *Lebesgue Integration on Euclidean Space*. Jones & Bartlett Publishers, 2000. `http://books.google.com/books?hl=en&id=3U7tresTD1AC`.

[2] W. Rudin. *Real and Complex Analysis*. McGraw Hill, 1974.

[3] J. Yeh. *Real Analysis: Theory of Measure and Integration*. World Scientific, 2006.

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