HCMIU - Probability - Final-term Test

Semester 2 - Year: 2024 - 2025 - Duration : 90 minutes

Date Modified : Monday, Sep. 22^{nd}, 2025

INSTRUCTIONS:

- Each student is allowed one doubled-sized sheet of reference material (size A4 of similar). All other documents and electronic devices are forbidden, except scientific calculators.
- There is a total of 6 (six) questions.

Question 1. (10 points) A consignment of 100 half-inch nuts was released from the factory. The diameters of 10 randomly taken nuts were measured. Calculate the sample mean and sample standard deviation using the data in the table.

Number of nut	1	2	3	4	5	6	7	8	9	10
Diameter, inch	0.4983	0.5030	0.4982	0.4986	0.5010	0.5009	0.4984	0.4993	0.5014	0.5010

Question 2. (15 points) The life in hours of a 60-watt light bulb is known to be normally distributed with a standard deviation of 52 hours. In a sample of 80 bulbs, the mean lifetime was 1217 hours. Find a 95% confidence interval for the mean lifetime of this type of light bulb.

Question 3. (15 points) A random sample of 400 electronic components manufactured by a certain process are tested, are 30 are found to be defective. Let p represent the proportion of components manufactured by this process that are defective. Find a 95% confidence interval for p.

Question 4. (15 points) Collect 5-year statistics of the average amount of wheat crop (tons) harvested from $1km^2$ per year, the results are as follows: $560, 525, 496, 543, 499$. Test the hypothesis that the mean wheat crop is 550 tons per year with significant level of 5%.

Question 5. (15 points) In a random sample of 500 handwritten zip code digits, 464 were read correctly by an optical character recognition (OCR) system operated by the U.S. Portal Service (USPS). USPS would like to know whether the rate is at least 90% correct. Do the data provide evidence that the rate is at least 90% at level of significance 5%.

Question 6. (30 points) Consider a Markov chain $(X_n)_{n\geq 0}$ with state space $\mathbb{S} = \{1, 2, 3\}$ and transition matrix

$$P = \begin{array}{cc} & \begin{array}{ccc} & \text{To} & \\ 1 & 2 & 3 \end{array} \\ & \begin{bmatrix} 0 & 0.2 & 0.8 \\ 0.2 & 0.5 & 0.3 \\ 0.1 & 0.4 & 0.5 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \ \text{From} \end{array}$$

(a) Evaluate $P(X_4 = 2 | X_2 = 1)$ and $P(X_4 = 2, X_2 = 1 | X_0 = 3)$

(b) Suppose that $P(X_0 = 1) = P(X_0 = 2) = 0.4, P(X_0 = 3) = 0.2$. Compute $P(X_3 = 1)$ and $P(X_5 = 3, X_3 = 1)$

(c) Find the stationary distribution of the Markov chain.

END

SUGGESTED ANSWER

Question 1. Needless to say,

- $$\overline{x} = \frac{x_1 + ... + x_{10}}{10} = \frac{0.4983 + ... + 0.5010}{10} \approx 0.50001$$
- $$S = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n-1}} \approx 1.67 \cdot 10^{-3}$$

Question 2. 95% confident interval for the mean lifetime of type of light bulb

$$\overline{x} \in (\overline{x} - \text{ME}, \overline{x} + \text{ME}) \approx \left(1217 - 1.96 \cdot \frac{52}{\sqrt{80}}.1217 + 1.96 \cdot \frac{52}{\sqrt{80}}\right) \approx (1205.6, 1228.4)\,, \quad ME = z_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}$$

Question 3. 95% confident interval for p

$$\hat{p} \in (p - \text{ME}, p + \text{ME}) \approx (0.049, 0.101), \quad \text{ME} = z_{\frac{\alpha}{2}} \cdot \frac{p(1-p)}{n}$$

Question 4. Since population variance is unknown, use two-sided t-test with n - 1 degree of freedom. Let μ be the average amount of wheat crop (tons) harvested from 1km^2 per year.

- $H_0 : \mu = 550$
- $H_1 : \mu \neq 550$

$$\Longrightarrow t_{\text{obs}} = \frac{\overline{x} - \mu}{\frac{S}{\sqrt{n}}} = \frac{524.6 - 550}{\frac{39.7}{\sqrt{5}}} \approx -1.43 > -t_{0.025,4} \approx -2.776$$

The null hypothesis is not rejected, so the mean of wheat crop is 550 tons per year with significant level of 5%.

Question 5. Let P_0 be the likelihood of the reading correction.

- $H_0 : P_0 \geq 90\%$
- $H_1 : P_1 < 90\%$

At level of significance 5%, use z-test statistic for proportion.

$$z_{\text{obs}} = \frac{\hat{p} - P_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{0.928 - 0.9}{\sqrt{\frac{0.9 \cdot 0.1}{500}}} \approx 2.09$$

Since $z_{\text{obs}} > z_{\frac{\alpha}{2}} = 1.64$, reject H_0, thereby the data does not provide enough evidence to prove that the sale is atleast 90%

Question 6. Find the Markov matrix after transitioning into 2, 3 phases. In (b) question, combine with prior concepts from law of total probability.

$$P^2 = \begin{array}{c} \text{To} \\ \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.12 & 0.42 & 0.46 \\ 0.13 & 0.41 & 0.46 \\ 0.13 & 0.42 & 0.45 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ 1 \\ 2 \\ 3 \end{array} \text{ From}\quad , \quad P^3 = \begin{array}{c} \text{To} \\ \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.13 & 0.418 & 0.452 \\ 0.128 & 0.415 & 0.457 \\ 0.129 & 0.416 & 0.455 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ 1 \\ 2 \\ 3 \end{array} \text{ From}$$

(a)

$$P(X_4 = 2|X_2 = 1) = r_{12}^{(2)} = 0.42$$

$$P(X_4 = 2, X_2 = 1|X_0 = 3) = P(X_4 = 2|X_2 = 1)P(X_2 = 1|X_0 = 3) = r_{12}^{(2)} \cdot r_{31}^{(2)} = 0.42 \cdot 0.13 = 0.0546$$

(b)

$$P(X_3 = 1) = P(X_3 = 1|X_0 = 1)P(X_0 = 1) + P(X_3 = 1|X_0 = 2)P(X_0 = 2) + P(X_2 = 1|X_0 = 3)P(X_0 = 3)$$

$$\Longrightarrow P(X_3 = 1) = 0.13 \cdot 0.4 + 0.128 \cdot 0.4 + 0.2 \cdot 0.129 \approx 0.129$$

$$P(X_5 = 3, X_3 = 1) = P(X_3 = 1)P(X_5 = 3|X_3 = 1) = 0.129 \cdot r_{13}^{(2)} \approx 0.05934$$

(c) Stationary distribution of Markov chain to find out the steady state after an amount of time

$$\begin{cases} \pi P = \pi \\ \sum_{i=1}^{n} \pi_i = 1 \end{cases} \longrightarrow \begin{cases} 0\pi_1 + 0.2\pi_2 + 0.1\pi_3 = \pi_1 \\ 0.2\pi_1 + 0.5\pi_2 + 0.4\pi_3 = \pi_2 \\ 0.8\pi_1 + 0.3\pi_2 + 0.5\pi_3 = \pi_3 \\ \pi_1 + \pi_2 + \pi_3 = 1 \end{cases} \Longrightarrow \boxed{\pi = \begin{bmatrix} \pi_1 & \pi_2 & \pi_3 \end{bmatrix} = \begin{bmatrix} \frac{42}{101} & \frac{46}{101} & \frac{13}{101} \end{bmatrix}}$$

END