Discrete Mathematics (2024 Program) - HCMUS - January 8th, 2025
Semester 1 : Year 2024 - 2025 - Time Duration: 90 minutes

Question 1 (1.5 points)

Given $a_0 = 4, a_1 = 24$ and $a_{n+2} = 6a_{n+1} - 9a_n - (4n - 17)2^n$. Calculate a_n in accordance to n ($n \geq 0$).

Question 2 (3.25 points = 1.25pts + 1.25pts + 0.75pts)

Given $m = 43615880, n = -22198176, a = 36567$ and $b = 6886$.

a) Analyze elements m and n so as to find $d = (m, n), e = [m, n]$ and a minimalist form of $\frac{m}{n}$

b) Use Extended Euclidean Algorithm to find $r, s, u, v \in \mathbb{Z}$ satisfying $\mathbf{1} = ra + sb$ and $\frac{1}{ab} = \frac{u}{a} + \frac{v}{b}$

c) Describe the integer divisors of m and calculate the possible number of those in m?

Question 3 (3.25 points = 1.25pts + 1pt + 1pt)

a) Given a binary relation $\mathcal{R}$ on $S = \{1, 2, 3\}$ defined by $\forall x, y \in S,\ x\,\hat{\mathcal{R}}\,y \iff (x - y)^2 \leq 1$. List all sets $H = \{(x, y) \in S^2 |\ x\,\hat{\mathcal{R}}\,y\}$. Evaluate the following properties: Reflexive, Symmetric, Anti-symmetric and Transitive of relation $\mathcal{R}$.

b) Upon $T = \{1, 2, 4, 5, 7, 10, 12, 24, 30\}$, giving ordinal relation Ω identified by $\forall x, y \in T,\ x\,\hat{\Omega}\,y \iff y = x$ or containing **even integer k** such that y = **kx** (k depends on x and y). Draw Hasse diagram of (T, Ω) and find minimal - maximal - least and greatest elements (if possible) of (T, Ω).

c) Apply b) part of **Question 2** in solving the equation $\overline{6886} \cdot \overline{y} = \overline{238}$ in $\mathbb{Z}_{36567}$. Derive the solution of the equation $\overline{6886 \times 6} \cdot \overline{x} = \overline{238 \times 6}$ in $\mathbb{Z}_n$ with $\mathbf{n}$ = 36567 x 6

Question 4 (2 points = 1pt + 1pt) Given boolean function f following by boolean variables x, y, z, t identified by

$$f(x, y, z, t) = x\,\overline{y}\,z\,t \vee \overline{x}\,\overline{z}\,t \vee x\,y\,z\,\overline{t} \vee \overline{x}\,y\,\overline{t} \vee x\,\overline{y}\,z\,\overline{t} \vee \overline{x}\,z\,t$$

a) Draw the Karnaugh map for f and identify its largest implicants (prime implicants).

b) Find the minimal expressions for f (i.e, the minimized Boolean expressions).

END

SUGGESTED ANSWER

Question 1. The general solution for non-homogeneous recurrence relation:

$$\boxed{a_n = a_n^{(h)} + a_n^{(p)}}$$

Rewrite the equation such that it has quadratic form, plus free solution

$$a_{n+2} = 6a_{n+1} - 9a_n - (4n-17)^2 \iff a_n - 6a_{n-1} + 9a_{n-2} = -(4n-8-17)2^{n-2}$$
$$= (-4n+25) \cdot 2^{n-2}$$

The characteristic equation is $K^2 - 6K + 9 = 0 \;\; \rightarrow \;\; K = 3(\text{Even}) \;\; \therefore \;\; \boxed{a_n^{(h)} = (A_0 + A_n) \cdot 3^n}$

Let $(B_0 + B_n n) \cdot 2^{n-2}$ be the particular solution of given recurrence relation

$$\implies (B_0 + B_n) \cdot 2^{n-2} - 6(B_0 + B_n(n-1)) \cdot 2^{n-3} + 9(B_0 + B_n(n-2)) \cdot 2^{n-4} = (-4n+25) \cdot 2^{n-2}$$
$$\implies (B_0 + B_n) - 3(B_0 + B_n n - B_n) + \frac{9}{4}(B_0 + B_n n - 2B_n) = (-4n+25)$$
$$\implies \frac{1}{4}B_0 + \frac{1}{4}B_n n - \frac{3}{2}B_n = (-4n+25)$$

Compare coefficients of particular variables to solve B_0, B_n

$$\begin{cases} \frac{1}{4}B_0 - \frac{3}{2}B_n = 25 \\ \frac{1}{4}B_n = -4 \end{cases} \implies \begin{cases} B_0 = 4 \\ B_n = -16 \end{cases} \implies \boxed{a_n^{(p)} = (4-16n) \cdot 2^{n-2}}$$

$$\therefore \;\; a_n = a_n^{(h)} + a_n^{(p)} = (A_0 + A_n n) \cdot 3^n + (4-16n) \cdot 2^{n-2} = (A_0 + A_n n) \cdot 3^n + (1-4n) \cdot 2^n$$

Since $a_0 = 4, \;\; a_1 = 4$, plug in the initial values and solve the linear recurrence relation

$$\begin{cases} a_0 = 4 \\ a_1 = 24 \end{cases} \implies \begin{cases} 4 = A_0 + 1 \\ 24 = 3(A_0 + A_n) + 2(1-4) \end{cases} \iff \begin{cases} A_0 = 3 \\ A_n = 7 \end{cases}$$

$$\boxed{a_n = (3+7n) \cdot 3^n + (1-4n) \cdot 2^n}$$

Question 2.

(a) To analyze the prime elements, divide the number with ordinal prime numbers (2, 3, 5, 7, ...). If succeed to divide, it's counted as an element, otherwise the value you attempt is not a prime number.

$$m = 2^3 \cdot 5 \cdot 7^3 \cdot 11 \cdot 17^2, \;\; n = 2^5 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 13$$

Consequently, the greatest common divisor and least common multiple of m and n are:

$$d = (m,n) = 2^3 \cdot 7^2 \cdot 11 = 4312, \;\; e = [m,n] = 2^5 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 13 \cdot 17^2 = 22453450200$$

$$\implies \frac{m}{n} = \frac{2^3 \cdot 5 \cdot 7^3 \cdot 11 \cdot 17^2}{2^5 \cdot 3^2 \cdot 7^2 \cdot 11^2 \cdot 13} = \frac{5 \cdot 7 \cdot 17^2}{2^2 \cdot 3^2 \cdot 11 \cdot 13} = \frac{10115}{5148}$$

(b) Given the satisfied parametric equation $1 = ra + sb$, implying the linear combination (or) greatest common divisor of a and b: $d = (a,b)$. Use Extended Euclidean Algorithm to find r, s. Eventually we can also find the latter parametric one within a few exchanges

$$
\begin{aligned}
a = \ & 36567 = 5 \cdot 6886 + 2137 \\
& 6886 = 3 \cdot 2137 + 475 \\
& 2137 = 4 \cdot 475 + 237 \\
& 475 = 2 \cdot 237 + 1 \\
& 237 = 1 \cdot 237 + 0 \Longrightarrow \boxed{d(a,b) = 237}
\end{aligned}
$$

Back-substituion to find the linear combination of 1

$$
\begin{aligned}
1 = 475 - 2 \cdot 237 &= (6886 - 3 \cdot 2137) - 2 \cdot (2137 - 4 \cdot 475) \\
&= 6886 - 5 \cdot 2137 + 8 \cdot 6886 - 24 \cdot 2137 \\
&= 9 \cdot 6886 - 29 \cdot (36567 - 5 \cdot 6886) \\
&= 9 \cdot 6886 - 29 \cdot 36567 + 145 \cdot 6886 \\
&= -29 \cdot 36567 + 154 \cdot 6886 \Longrightarrow \boxed{r = -29 \ \& \ s = 154}
\end{aligned}
$$

Divide the linear equation by ab, thereby it also mirrors the same values as r, s

$$
1 = ra + sb \Longrightarrow \frac{l}{ab} = \frac{r}{b} + \frac{s}{a} = \frac{v}{b} + \frac{u}{a} \Longrightarrow \begin{cases} v = r = -29 \\ u = s = 154 \end{cases}
$$

(c) Given a positive integer which has the form of cumulative prime numbers

$$
\boxed{n = p_1^{e_1} p_2^{e_2} \ \dots \ p_k^{e_k}}
$$

Number of possible divisors (or) factors of an integer is

$$
\boxed{\tau(n) = (e_1 + 1) \ \dots \ (e_k + 1)}
$$

Therefore, the possible divisors for m integer following previous analysis will be $(3+1) \cdot (1+1) \cdot (3+1) \cdot (1+1) \cdot (2+1) = 192$ since we are also including 0th power of each prime numbers.

Question 3.

(a) We are told to find all sets that satisfy $H = \{(x,y) \in S^2 | \ x \, \hat{\mathcal{R}} \, y\}$ with $S = \{1,2,3\}$. Firstly, perform the Cartesian product of two identical sets, then apply the constraint of binary relation to find genuine H combination and evaluate relation properties.

$$
\begin{aligned}
S \times S = \{1,2,3\} \times \{1,2,3\} &= \{(1,1),(2,2),(3,3),(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)\} \\
&\Longrightarrow H = \{(1,1),(2,2),(3,3),(1,2),(2,1),(3,2),(2,3)\}
\end{aligned}
$$

Check the following properties: Reflexive, Symmetric, Anti-symmetric, Transitive

1) Reflexive: $\Big\{x \in S, (x,x) \in H \ \mid \ x\hat{\mathcal{R}}x\Big\}$, R satisfies selected sets in S: $R \in \{(1,1),(2,2),(3,3)\}$ $\therefore$ 'R' is reflexive.

2) Symmetric: $\Big\{x, y \in S, (x,y) \in H \ \mid \ x\hat{\mathcal{R}}y = y\hat{\mathcal{R}}x\Big\}$. Observe that if $(x,y) \in R$, $(1,2) \to (2,1) \ \& \ (2,3) \to (3,2)$. $\therefore$ 'R' is symmetric.

3) Anti-symmetric: $\Big\{x, y \in S, (x,y) \in H \ \mid \ x\hat{\mathcal{R}}y = y\hat{\mathcal{R}}x\Big\}$. If $(x,y) \ \& \ (y,x) \in H \to x = y$. But $(1,2) \in$

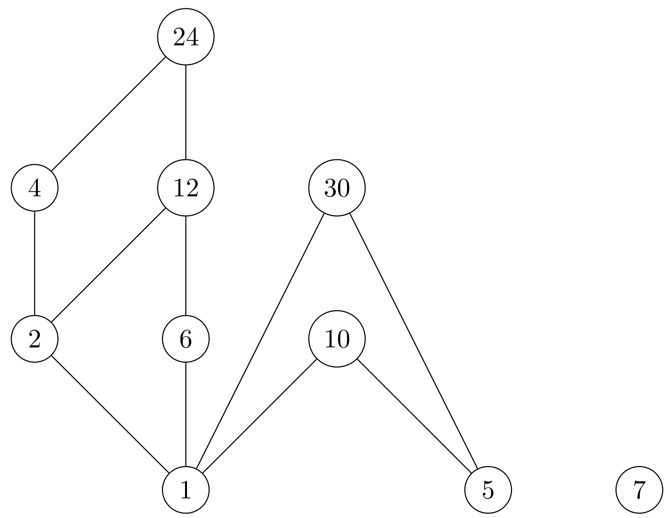
H & $(2,1) \in H \to 1 \neq 2$, similar for (3, 2) and (2, 3). $\therefore$ 'R' is not anti-symmetric.

4) Transitive: $\left\{x, y \in S, (x,y) \in H \mid x\hat{\mathcal{R}}y = y\hat{\mathcal{R}}x\right\}$. If $(x,y),(y,z) \in R \to (x,z) \in R$. But $(1,2) \to (2,3) \notin H$ & $(3,2) \to (2,1) \notin H \therefore$ 'R' is not transitive.

$\therefore$ 'R' is a **tolerance relation** (symmetric and reflexive)

(b) $\forall x, y \in T,\ x\Omega y \iff y = x \vee y = kx\ \ (k \mod 2 = 0)$. List all sets that meet up the constraint, then draw the Hasse diagram.

$$T = \{(1,2),(1,6),(1,10),(1,30),(2,4),(4,24),(2,12),(5,10),(5,30),(6,12),(12,24)\}$$



From the diagram above, minimal and maximal elements are respectively 1 (not smaller than any other set) - 24, 30, 7 (not larger than any other set, whereas 7 is incomparable), and least element is 1 (unique) but there doesn't exist greatest element since 24, 30 and 7 are multiple greater elements (not unique).

(c) Since $1 = 154 \cdot 6886 - 29 \cdot 36567$ from previous question 2b has the form of $1 = xp + nq$

$$\implies \bar{x} \cdot \bar{p} = \bar{1} \iff \overline{6886} \times \overline{154} = 1 \iff \overline{6886}^{-1} = \overline{154}$$

From the given congruence modulo relation, we derive the solution by plug in the snippet.

$$\overline{6886} \times \overline{y} = \overline{238} \iff \overline{y} = \overline{6886}^{-1} \times \overline{238} = \overline{154 \times 238} = \overline{36652} \in \mathbb{Z}_{36567}$$

Or simply written as 85 modulo 36567 as gcd(36652, 36567) = 1

$$\overline{y} = \overline{36652} \in \mathbb{Z}_{36567} \quad \iff \quad \overline{y} = \overline{85} \in \mathbb{Z}_{36567} \quad \iff \quad y \equiv 85 \pmod{36567}$$

The unique solution for the initial equation is $\boxed{\overline{y} = \overline{85 + 36567k},\ \ \forall k \in \mathbb{Z}}$

Deriving from above conclusion, we investigate the scaled modulo congruence relation

$$\overline{6886 \times 6} \cdot \overline{x} = \overline{238 \times 6} \in \mathbb{Z}_{36567 \times 6} \implies (6886 \times 6) \times x \equiv (238 \times 6) \pmod{36567 \times 6}$$

Since $\gcd(36567 \cdot 6, 6886 \cdot 6) = 6$. $\therefore \overline{6886 \times 6}$ is not invertible. Hence, simplify the modulo equation by 6.

$$\begin{aligned} \implies 6886 \times x = 238 \pmod{36567} &\iff \overline{6886} \times \overline{x} = \overline{238} \in \mathbb{Z}_{36567} \\ &\iff \overline{x} = \overline{6886}^{-1} \times \overline{238} = \overline{154 \times 238} = \overline{36652} \\ &\iff \overline{x} = \overline{85} \in \mathbb{Z}_{36567} \implies \boxed{\overline{x} = \overline{85 + 36567k}, \;\; 0 \leq k < 6} \end{aligned}$$

The unique root set of scaled modulus congruence in $Z_{36567\times 6}$ is $\left\{\overline{85}, \overline{36652}, \overline{73219}, \overline{109786}, \overline{146353}, \overline{182920}\right\}$

Question 4. In order to draw Karnaugh map, it's important to first identify minterms (1s cells) by converting to decimal index. There may also be possible cells that are covered besides main ones (logical implicants without variables).

- $x\bar{y}zt \to$ Binary Index: $1011 \to$ Decimal Index: $2^3 + 2^1 + 2^0 = 11$
- $\bar{x}\bar{z}t \to$ Binary Index: $0101, 0001 \to$ Decimal Index: $5, 1$
- $xyz\bar{t} \to$ Binary Index: $1110 \to$ Decimal Index: $2^3 + 2^2 + 2^1 = 13$
- $\bar{x}y\bar{t} \to$ Binary Index: $0110, 0100 \to$ Decimal Index: $6, 4$
- $x\bar{y}z\bar{t} \to$ Binary Index: $1010 \to$ Decimal Index: 10
- $\bar{x}zt \to$ Binary Index: $0111, 0011 \to$ Decimal Index: $7, 3$

Once all minterms is established, draw the Karnaugh map by filling in respective cells:

$xy \backslash zt$	00	01	11	10
00	0	1	1	0
01	1	1	1	1
11	0	0	0	1
10	0	0	1	1

Larger implicants are covered in groups of 2s, 4s, 8s, ... In this case, it can be easily seen that cell 1, 3, 5, 7 are formed into group of 4s, alongside row of 4 - 5 - 7 - 6 and row 10 - 11 as group of 2s. We ignore the case of column 6 - 14 - 10 as it's not in any valid groups, thus treating cell 14 as singleterm. Therefore, the larger implicants of initial boolean function is $\boxed{\bar{x}t, \bar{x}y, x\bar{y}t, xyz\bar{t}}$

Since there is only one minimal expression for boolean function, the optimized function will be:

$$\boxed{f(x, y, z, t) = \bar{x}t \vee \bar{x}y \vee x\bar{y}t \vee xyz\bar{t}}$$

END