Linear Algebra - HCMUS - February 16th, 2023

Semester 1 : Year 2022 - 2023 - Time Estimation : 90 minutes

Question 1. Given the solution space W_1 in $\mathbb{R}^4$ of the following linear system

$$\begin{cases} x_1 + 2x_2 - 2x_3 + 5x_4 = 0 \\ 3x_1 + 6x_2 - x_3 + 5x_4 = 0 \\ 2x_1 + 4x_2 - 3x_3 + 8x_4 = 0 \end{cases}$$

and W_2 is the subset spawned by $\{v_1 = (1, -2, 2, 3), v_2 = (0, 1, -4, -2)\}$

(a) Find the basis of space W_1

(b) Find the dimension of spaces: $W_1 + W_2, W_1 \cap W_2$

Question 2. In $\mathbb{R}^3$ there exists the subspace W consisting the basis $\mathcal{B} = \{u_1 = (1, 2, 2), u_2 = (2, 3, 1)\}$.

(a) Let $u = (a, b, c) \in \mathbb{R}^3$. Find the condition of respectively a, b, c such that $u \in W$. From the evidence, calculate $[u]_{\mathcal{B}}$.

(b) Given $v_1 = (2, 1, -5)$ and $u_2 = (-4, -5, 1)$. Show that $\mathcal{C} = \{v_1, v_2\}$ is a basis of W. Find the matrix transformation from $\mathcal{B} \to \mathcal{C}$.

(c) Find $[u]_{\mathcal{B}}$ given that $[u]_{\mathcal{C}} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

Question 3. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation identified by

$$f(x, y) = (x + y, 2x + y, 3x + 2y)$$

and $v = (3, 1, 4)$. Does $v \in \operatorname{Im}(f)$. How come?

Question 4. Given $u = (2, 1, 3)$ and linear mapping $f : \mathbb{R}^3 \to \mathbb{R}^2$ comprising the matrix represented by bases $\mathcal{B} = \{u_1 = (1, 1, 2), u_2 = (1, 2, 0), u_3 = (1, 2, 1)\}$ and $\mathcal{C} = \{v_1 = (2, 1), v_2 = (1, 2)\}$

$$[f]_{\mathcal{B},\mathcal{C}} = \begin{pmatrix} 1, 0, 1 \\ 2, 1, 1 \end{pmatrix}$$

(a) Find $[u]_{\mathcal{B}}$.

(b) Find $f(u)$.

Question 5. Given $u_1 = (1, 0), u_2 = (1, 1), u_3 = (1, -2), v_1 = (2, 1), v_2 = (1, 1)$ and $v_3 = (4, 2)$. Does there exist a linear mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(u_1) = v_1, f(u_2) = v_2, f(u_3) = v_3$?. Justify the answer.

END

SUGGESTED ANSWER

Question 1.

(a) Express the linear system under the matrix form (whichever styles you prefer: parentheses or square matrices)

$$\begin{bmatrix} 1 & 2 & -2 & 5 \\ 3 & 6 & -1 & 5 \\ 2 & 4 & -3 & 8 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & 0 & -5 & -10 \\ 0 & 0 & 1 & -2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{cases} x_1 = t_1 - 2t_2 \\ x_2 = t_2 \\ x_3 = 2t_1 \\ x_4 = t_1 \end{cases}$$

$$\Longrightarrow \boxed{\vec{x} = \left\{ t_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, t_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}}$$

Check for linear independence of the subset

$$u = \alpha_1 u_1 + \alpha_2 u_2 = \vec{0} \Longrightarrow \begin{cases} \alpha_1 - 2\alpha_2 = 0 \\ \alpha_1 = \alpha_2 = 0 \end{cases}$$

Hence, $\mathcal{B}_1$ is a basis for W_1 with $\dim(W_1) = 2$

(b) Do the similar process of checking independence to the space W_2, which also brings about the only solution exists in nullspace $\alpha_1 = \alpha_2 = 0$. Let us find the dimension of intersection of subspaces. Intuitively we can observe that both bases are scalar multipliers of each other and all unique, so we conclude that $\dim(W_1 \cap W_2) = 0$. For rougher procedure, we might be solving manually the linear systems that would lead us towards zero solution.

$$W_1 \cap W_2 \, \{v \in V | v \in W_1 \vee v \in W_2\}$$

1. W_1:

$$\begin{cases} x = \alpha_1 - 2\alpha_2 \\ y = \alpha_2 \\ z = 2\alpha_1 \\ t = \alpha_1 \end{cases} \iff \begin{cases} x = t - 2y \\ z = 2t \end{cases}$$

2. W_2:

$$\begin{cases} x = \alpha_1 \\ y = \alpha_2 - 2\alpha_1 \\ z = 2\alpha_1 - 4\alpha_2 \\ t = 3\alpha_1 - 2\alpha_2 \end{cases} \Longrightarrow \begin{cases} z = -6x - 4 \\ t = -x - 2y \end{cases}$$

3. W_1 & W_2

$$\begin{cases} x = t - 2y \\ z = 2t \\ z = -6x - 4 \\ t = -x - 2y \end{cases} \Longrightarrow \begin{cases} 2x = -4y \\ -2x - 4y = -6x - 4y \end{cases} \Longrightarrow x = 0 \vee y = 0$$

$$\Longrightarrow \boxed{U \cap W = \{v = (0,0,0,0)\}, \quad \dim(U \cap W) = 0}$$

By properties of dimensions for finite-dimensional vector space W_1, W_2

$$\boxed{\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) = 2 + 2 - 0 = 4}$$

Question 2.

(a) $u = (a, b, c) \in \mathbb{R}^3, u \in W \iff u = \alpha_1 u_1 + \alpha_2 u_2$

$$(a, b, c) = \alpha_1(1, 2, 2) + \alpha_2(2, 3, 1) \Longrightarrow \begin{bmatrix} 1 & 2 & a \\ 2 & 3 & b \\ 2 & 1 & c \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 2 & a \\ 0 & -1 & b-2a \\ 0 & -3 & c-2a \end{bmatrix} \Longrightarrow \frac{2a-c}{3} = \frac{2a-b}{1} \iff \boxed{4a = 3b - c}$$

From the previously solved linear system, we can derive the coordinate of u vector relative to the base $\mathcal{B}$

$$\begin{cases} \alpha_1 = a - 2\alpha_2 \\ \alpha_2 = c - 2\alpha_1 \end{cases} \Longrightarrow \begin{cases} \alpha_1 = 2b - 3a \\ \alpha_2 = 2a - b \end{cases} \Longrightarrow [u]_{\mathcal{B}} = \begin{bmatrix} (u_1)_{\mathcal{B}} \\ (u_2)_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} a + \begin{bmatrix} -1 \\ 2 \end{bmatrix} b$$

(b) Check if $\mathcal{C}$ is a basis for W

$$\begin{cases} (2, 1, -5) = \alpha_1(1, 2, 2) + \alpha_2(2, 3, 1) \\ (-4, -5, 1) = \alpha_1(1, 2, 2) + \alpha_2(2, 3, 1) \end{cases} \iff \left\{ \begin{aligned} &\left[\begin{aligned} \alpha_1 &= 2 \\ \alpha_2 &= -3 \end{aligned} \right. \\ &\left[\begin{aligned} \alpha_1 &= -4 \\ \alpha_2 &= 3 \end{aligned} \right. \end{aligned} \right.$$

Since both vectors can be expressed as linear combination of $\mathcal{B}$ and v1 & v2 are not scalar multipliers, $\mathcal{C}$ is a basis of $\mathcal{B}$. Additionally, the matrix transformation from $\mathcal{B} \to \mathcal{C}$ is:

$$\mathrm{P}_{\mathcal{B}\to\mathcal{C}} = \left([c_1]_{\mathcal{B}}\, [c_2]_{\mathcal{B}}\right) = \begin{pmatrix} 2 & -4 \\ -3 & 3 \end{pmatrix}$$

(c)

$$[u]_{\mathcal{B}} = \mathrm{P}_{\mathcal{B}\to\mathcal{C}}\, [u]_{\mathcal{C}} = \begin{bmatrix} 2 & -4 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -16 \\ 9 \end{bmatrix}$$

Question 3. By aggregating the operations between x and y, we can derive its subsets spawned from the function and verify the linear independence.

$$f(x, y) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} y \Longrightarrow B = \{x = (1, 2, 3), y = (1, 1, 2)\}$$

Check for independence:

$$\alpha_1 x + \alpha_2 y = \vec{0} \iff \begin{cases} \alpha_1 + \alpha_2 = 0 \\ 2\alpha_1 + \alpha_2 = 0 \\ 3\alpha_1 + 2\alpha_2 = 0 \end{cases} \Longleftarrow \alpha_1 = \alpha_2 = 0$$

Therefore, B is the basis for f $\longrightarrow \mathrm{Im}(f) = \{(1, 2, 3), (1, 1, 2)\}$

$$v = \alpha_1 b_1 + \alpha_2 b_2 \Longrightarrow (3, 1, 4) = \alpha_1(1, 2, 3) + \alpha_2(1, 1, 2) \iff \begin{cases} \alpha_1 + \alpha_2 = 3 \\ 2\alpha_1 + \alpha_2 \\ 4 = 3\alpha_1 + 2\alpha_2 \end{cases} \Longrightarrow \boxed{\alpha = \begin{bmatrix} -2 \\ 5 \end{bmatrix}, v \in \mathrm{Im}(f)}$$

Any images can be expressed as linear combination of column bases.

Question 4. The coordinate of vector relative to the basis $\mathcal{B}$ is identified by the linear combination of column vectors.

$$u = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = \alpha_1(1,1,2) + \alpha_2(1,2,0) + \alpha_3(1,2,1) \Longrightarrow \begin{cases} \alpha_1 + \alpha_2 + \alpha_3 = 2 \\ \alpha_1 + \alpha_2 + \alpha_3 = 1 \\ \alpha_1 + \alpha_2 + \alpha_3 = 3 \end{cases}$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & 1 \\ 2 & 0 & 1 & 3 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & -2 & -1 & -1 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 3 & -3 \end{bmatrix} \Longrightarrow \begin{cases} \alpha_1 = 2 \\ \alpha_2 = 1 \\ \alpha_3 = -1 \end{cases} \Longrightarrow \boxed{[u]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}}$$

The coordinate of matrix transformation from B to C, which results in the function $f(u)$ spanned by the basis $\mathcal{C}$

$$[f(u)]_{\mathcal{C}} = [f]_{\mathcal{B},\mathcal{C}} [u]_B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \Longrightarrow f(u) = c_1 v_1 + c_2 v_2 = (2,1) + 5(1,2) = (7,11)$$

Question 5. Notice that a mapping exists if a vector is formed by the linear combination of two vectors. Therefore, we need to prove if the one-to-one function also yield the respective values according to initial conditions, or not yields the same linear combination of v_3.

$$u_3 = \alpha_1 u_1 + \alpha_2 u_2 \Longrightarrow f(u_3) = \alpha_1 f(u_1) + \alpha_2 f(u_2)$$

Find the constants of the vector v_3 spanned by u_1 and u_2

$$\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Longrightarrow \begin{cases} \alpha_1 + \alpha_2 = 1 \\ \alpha_2 = -2 \end{cases} \Longrightarrow \begin{cases} \alpha_1 = 3 \\ \alpha_2 = -2 \end{cases}$$

$$f(u_3) = \alpha_1 f(u_1) + \alpha_2 f(u_2) = 3 \begin{pmatrix} 2 \\ 1 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 4 \\ 2 \end{pmatrix} (v_3)$$

Hence, there doesn't exist a linear mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ that satisfies the conditions.

END