Calculus 2 - HCMUS - April 25th, 2025

Semester 2 : Year 2024 - 2025 - Time Duration: 60 minutes

Remark: Assuming in the following questions, all of the conditions involving differentiability, formulas of composite function derivative, directional derivatives are satisfied.

Question 1. (2pts) Let $z = x\ln\left(x^2 + y^4\right), x = 2s + 3t, z = 5s - 3t$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. Then apply the composite function derivative formula (chain rule) into solving $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

Question 2. (2pts) Let $z = f\left(x; y\right), x = t\cos s, y = t\sin s$. Find $\frac{\partial^2 z}{\partial s \partial t}$

Question 3. (2pts) Given two-variable funtion f defined by $f(x; y) = x^4 - xy + y^3$. Find the derivative of f at point $(1; 2)$ aligning with the x-axis an angle of -60°.

Question 4. (2pts) Supposing the coordinate system assigned upon flat region such that the direction of x-axis and y-axis is respectively East and North, the unit of distance is meter (m). The temperature at point (x, y) is modeled by the formula $T(x; y) = 100e^{-2x^2+3y^2}$ (°Celcius).

(a) At point (1; 2), if heading towards North West, then what is the rate of change of temperature (require implicit measurement), does the value decrease or increase?

(b) At point (1; 2), which direction will cause the temperature to abruptly decrease and how much is the maximum rate of change if following that direction?

Question 5. (2pts) Do the following requests:

(a) Formulate the line equation tangent to the curve line $(C) : x^2 + y^2 + x^4y^4 = 1$ at the point (1; 0)

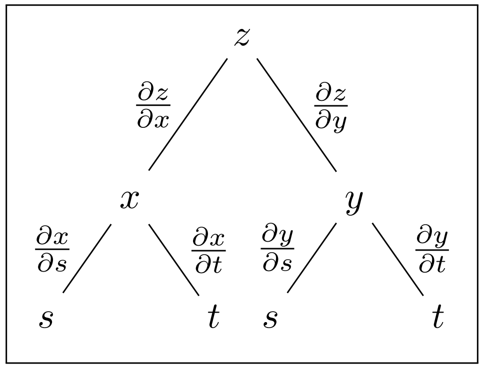
(b) Write the equation of the plane tangent to the curved surface $(S) : x^2y + y^2z - z^2x = 1$ at the point (1; 1; 0).

END

SUGGESTED ANSWER

The Tree Diagram For Multivariable Chain Rule

Tips: To use the tree diagram, determine which derivative you want to take (s or t). If f or z is respective to (s or t), you need to express (**df** or **dz**) in terms of (**ds** or **dt**). Then follow all possible paths that contain the variable you choose, with each line corresponding to an expansion of the "top" quantity until the final "bottom" quantity where variable(s) are held to be constant.



Question 1. $z = x\ln\left(x^2+y^4\right), x = 2s+3t, z = 5s-3t$

$$\frac{\partial z}{\partial x} = \frac{2x^2}{x^2+y^4} + \ln(x^2+y^4) \quad ; \quad \frac{\partial z}{\partial y} = \frac{4y^3x}{x^2+y^4}$$

Observe that the differential process ranges from $z \to s$ and $z \to t$, which aligns with the chain z - t intervened by sub-variables x and y. Hence,

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} = \frac{4x^2+4xy^3}{x^2+y^4} + 2\ln(x^2+y^4)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} = \frac{6x^2-12xy^3}{x^2+y^4} + 3\ln(x^2+y^4)$$

Question 2. $z = f\left(x;y\right), x = t\cos s, y = t\sin s$

$$\begin{aligned}\frac{\partial}{\partial s}\left[\frac{\partial z}{\partial t}\right] = \frac{\partial}{\partial s}\left[\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}\right] &= \frac{\partial}{\partial s}\left[f_x\cdot\cos s + f_y\cdot\sin s\right] \\ &= \frac{\partial f_x}{\partial s}\cdot\cos s - f_x\cdot\sin s + \frac{\partial f_y}{\partial s}\sin s + f_y\cdot\cos s\end{aligned}$$

For each derivatives taken on another functions with respect to x, y, we also do the same by following the tree diagram though requires patience and thorough calculation.

$$\frac{\partial f_x}{\partial s} = \frac{\partial f_x}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f_x}{\partial y}\cdot\frac{\partial y}{\partial s} = t\cdot\cos s\cdot f_{xy} - t\cdot\sin s\cdot f_{xx} \tag{1}$$

$$\frac{\partial f_y}{\partial s} = \frac{\partial f_y}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f_y}{\partial y}\cdot\frac{\partial y}{\partial s} = t\cdot\cos s\cdot f_{yy} - t\cdot\sin s\cdot f_{yx} \tag{2}$$

Replace the unknown derivatives with (1) and (2)

$$\boxed{\frac{\partial^2 z}{\partial s \partial t} = \frac{1}{2}\sin(2s)(f_{yy} - f_{xx}) + f_{xy} \cdot t \cdot \cos(2s) + f_y \cos s - f_x \sin s}$$

Question 3. Directional derivative with an angle of 120 degrees along x-axis. The procedure involves finding unit vector, differential equations with respect to x and y and compiling together with specific point $P(1,2)$.

$$\begin{aligned}\hat{\mathbf{u}} = \cos\left(\frac{\pi}{3}\right)\hat{\imath} + \sin\left(\frac{\pi}{3}\right)\hat{\jmath} = -\frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath} \quad &\Longrightarrow \quad D_{\hat{\mathbf{u}}} f(x,y) = f_x \cdot u_1 + f_y \cdot u_2 \\ &\Longrightarrow \quad D_{\hat{\mathbf{u}}} f(x,y) = -\frac{1}{2} \cdot (4x^3 - y) + \frac{\sqrt{3}}{2} \cdot (3y^2 - x) \\ &\Longrightarrow \quad \boxed{D_{\hat{\mathbf{u}}} f(1,2) = \frac{11\sqrt{3} - 2}{2}}\end{aligned}$$

Question 4. Suppose we head towards North West, the angle between the the axes and direction are respectively 45 degrees, but we consider the second quadrant where x-axis lies in negative values, so the unit vector will be

$$\hat{\mathbf{u}} = \cos\left(\frac{\pi}{2}\right)\hat{\imath} + \sin\left(\frac{\pi}{2}\right)\hat{\jmath} = -\frac{\sqrt{2}}{2}\hat{\imath} + \frac{\sqrt{2}}{2}\hat{\jmath}$$

(a) The process of finding rate of change is similar to previous question, except thar we display under gradient concepts.

$$\nabla T(x,y) = (-400 \cdot x \cdot e^{-2x^2+3y^2})\hat{\imath} + (600 \cdot y \cdot e^{-2x^2+3y^2})\hat{\jmath} \quad \Longrightarrow \quad \nabla T(1,2) = (-400e^{10})\hat{\imath} + (1200e^{10})\hat{\jmath}$$

$$D_{\hat{\mathbf{u}}} \nabla T(1,2) = -\frac{\sqrt{2}}{2} \cdot (-400e^{10}) + \frac{\sqrt{2}}{2} \cdot (1200e^{10}) \approx 24920101.33 \ \ (°\text{C / m})$$

(b) For fastest decline in temperature, the opposite direction to North West (South East) will obtain maximum rate of change. There is another way to identify which quadrant belongs to by calculating the tangent $\tan\theta = \frac{y}{x} = -\frac{1200}{400} = -3$

$$\|\nabla T(1,2)\| = \sqrt{(400e^{10})^2 + (1200e^{10})^2} \approx 27861520.29 \ \ (°\text{C / m})$$

Implicitly, South East direction brings about a rate of change of -27861520.29 (°C / m) (4th quadrant)

Question 5.

$$\nabla f(x,y) = (2x + 4x^3y^4) \cdot \hat{\imath} + (2y + 4y^3x^4) \cdot \hat{\jmath} \Longrightarrow \nabla f(1,0) = 2 \cdot \hat{\imath} + 0 \cdot \hat{\jmath}$$

(a) Since the slope of the tangent curve is undefined (infinity form), i.e the gradient vector is perpendicular to the level curve. Hence, the line equation tangent to the curve is $x = 1$.

(b) Normal vector to a family of level surfaces for $F(x,y,z)$

$$\nabla F(x,y,z) = (2xy - z^2)\hat{\imath} + (x^2 + 2yz)\hat{\jmath} + (y^2 - 2zx)\hat{k} \quad \Longrightarrow \quad \vec{n} = \nabla F(1,1,0) = \langle 2, 1, 1 \rangle$$

Hence, the equation of the plane tangent to the curved surface is $\boxed{(P): 2x + y + z = 3}$

END