HCMIU - Linear Algebra - Mid-term Test
Semester X - Year: Unknown - Duration : 120 minutes
Date Modified : Saturday, July 26th, 2025

INSTRUCTIONS: Each student is allowed a scientific calculator and a maximum of two double-sided sheets of reference material (size A4 or similar), stapled together and marked with their name and ID. All other documents and electronic devices are forbidden.

Part A: True / False Questions. *Each question carries 3 points - Fill your answer in the answer sheet below.*

ANSWER SHEET FOR PART A

1. (T) (F)	4. (T) (F)	7. (T) (F)	10. (T) (F)	13. (T) (F)
2. (T) (F)	5. (T) (F)	8. (T) (F)	11. (T) (F)	14. (T) (F)
3. (T) (F)	6. (T) (F)	9. (T) (F)	12. (T) (F)	15. (T) (F)

1. If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a one-to-one linear transformation, then T is also onto.
2. Every matrix transformation is a linear transformation.
3. If A and B are symmetric, then AB^2A is symmetric.
4. $X = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & -1 \\ 3 & -1 & 2 \end{bmatrix}$ corresponding to eigenvalue $\lambda = 2$
5. If A is a matrix for a linear transformation, then A is invertible.
6. If V is a set that contains the 0-vector, and such that whenever u and v are in V, then $u + v$ is in V, then V is a vector space.
7. If A is orthogonal, then A^T is orthogonal.
8. Any two vector spaces of dimension six are isomorphic.
9. Suppose Q is orthonormal and A is an arbitrary matrix of the same size. Then QAQ^T is similar to A.
10. There is a 5×5 matrix of rank 4 and determinant 1.
11. If A and B are symmetric invertible matrices, then ABA^{-1} is also symmetric and invertible.
12. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^n = \begin{bmatrix} a^n & b^n \\ c^n & d^n \end{bmatrix}$.
13. The matrix $\begin{bmatrix} 0 & 1 & -b \\ 0 & b & 1 \\ 1 & 0 & 0 \end{bmatrix}$ has rank 3 for every real value of b.
14. Suppose that $\vec{v}_1$ and $\vec{v}_2$ form a basis for V, a subspace of $\mathbb{R}^4$; and $\vec{w}_1$, $\vec{w}_2$ form a basis for W, a subspace of $\mathbb{R}^5$. If a mapping $T : V \to W$ is defined $T(\vec{v}_1 + \vec{v}_2) = a\vec{w}_1 + \vec{w}_2$, then T is an isomorphism.
15. If A is 5×5 matrix with two eigenvalues, one eigenspace is 2 dimensional, and the other is 3 dimensional, then A is diagonalizable.

Part B: Show your work in details and indicate answers clearly.

1. *(10 points)* Find the matrix for $T(\vec{w}) = proj_{\tilde{\mathbf{v}}}(\tilde{\mathbf{w}})$ where $\vec{v} = \begin{bmatrix} 1 & -2 & 3 \end{bmatrix}^T$.
2. *(20 points)* Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be defined as

$$T\vec{x} = \begin{bmatrix} 3 & 2 & 1 & 8 \\ 2 & 2 & -2 & 6 \\ 1 & 1 & -1 & 3 \end{bmatrix} \vec{x}$$

   Find the dimension and a basis for *Im(T)*, as well as the dimension and a basis for *Ker(T)*.

3. *(30 points)* Let $A = \begin{bmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{bmatrix}$.
   (a) Find eigenvalues and eigenvectors of A
   (b) Write A as $A = PDP^T$ where D is a diagonal matrix and P is an orthogonal matrix and find A^{2023}
   (c) Find the solution of the differential equation

$$\begin{cases} x'(t) = 13x(t) + y(t) + 4z(t) \\ y'(t) = x(t) + 13y(t) + 4z(t) \\ z'(t) = 4x(t) + 4y(t) + 10z(t) \end{cases}$$

   with the initial condition $x(0) = 1, y(0) = 0, z(0) = -1$.

END

SUGGESTED ANSWER

Part A: True / False Questions

1. True since the matrix T satisfies the linearity condition (one-to-one and onto).
$$T(u+v) = T(u) + T(v), \quad T(cu) = c \cdot T(u)$$
2. True. Same domain and codomain in finite-dimensional space ($\mathbb{R}^n \to \mathbb{R}^n$)
3. False unless identical matrices, also the multiplication works under the dot product between rows and columns.
4. False. The actual eigenvalues are respectively $\lambda \approx -3.09, 4.16, 1.47$, which can be later used to find eigenvectors $(A - \lambda \cdot I)\,\vec{x} = 0$
5. False. A is invertible if $N(A) = \{0\}$ or $\det(A) \neq 0$
6. False. $V = \{0_v\} \to v + 0 = v$, which satisfies not only the additivity $(u, v) \to u + v \in (V \to V)$, but also the homogeneity $(c, v) \to c \cdot v$.
7. True. $A^T A = AA^T = I \to (A^T A)^T = AA^T = I$
8. True if and only if they lie on the same field V - W, and their dimension is codomain dim(V) = dim(W) = 6. Hence, there exists a linear isomorphic (linear & bijective) $T : V \to W$.
9. True. By the definition of similarity $Q^T = Q^{-1} \implies QAQ^T = QAQ^{-1} = I$.
10. False. A 5 by 5 matrix with the rank (number of linearly independent columns / rows) is 4 will produce 0 determinant, resulting from singular property.
11. False. $(ABA^{-1})^T = (A^{-1})^T B^T A^T = (A^{-1})^T BA$.
12. False. $A = \begin{bmatrix} a, b \\ c, d \end{bmatrix} \implies A^n = A \cdot A \cdot A \cdot ... \cdot A$, then perform the dot product operation.
13. True. Since the matrix is invertible with $\det(A) = 1 + b^2 \neq 0$.
14. True. $v_1, v_2 \in V$ and $v_1, v_2 \in W \implies$ dim(V) = dim(W) = 2, followed by the linearity condition. Hence, T is an isomorphism.
15. True. $A_{5x5}, E_2 \in \mathbb{R}^2, E_3 \in \mathbb{R}^{\nvDash} \implies \dim(E_2) + \dim(E_3) = 5$. Hence, A is diagonalizable.

Part B:

Question 1:

$$T(\vec{w}) = \text{proj}_{\vec{v}}(\vec{w}) = \frac{vv^T}{v^T v} = \frac{1}{14}\begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 3 & -6 & 9 \end{bmatrix}$$

Question 2: The image of a linear transformation $\vec{x} \to T\vec{x}$ spans the column vectors of T. Hence, the image is a linear space, whereas the kernel spans the nullspace (consisting of free solutions) in the column vectors of T. In other words,

$$\text{Ker}(T) \approxeq \text{Null}(T), \quad \text{Im}(T) \approxeq \text{Col}(T)$$

$$T = \begin{bmatrix} 3 & 2 & 1 & 8 \\ 2 & 2 & -2 & 6 \\ 1 & 1 & -1 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & 2 & 1 & 8 \\ 0 & \frac{2}{3} & \frac{-8}{3} & \frac{2}{3} \\ 0 & \frac{1}{3} & -\frac{4}{3} & \frac{1}{3} \end{bmatrix} \longrightarrow \begin{bmatrix} 3 & 2 & 1 & 8 \\ 0 & 2 & -8 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Find the basis of the kernel, the set of all $\vec{x} \in \mathbb{R}^4$ such that $A\vec{x} = \vec{0}$

$$\begin{cases} 3x_1 + 2x_2 - x_3 + 8x_4 = 0 \\ 0x_1 + 2x_2 - 8x_3 + 2x_4 = 0 \end{cases} \longrightarrow \begin{cases} x_1 = -3x_3 - 2x_4 \\ x_2 = 4x_3 - x_4 \end{cases} \longrightarrow \begin{cases} x_1 = -3t_1 - 2t_2 \\ x_2 = 4t_1 - t_2 \\ x_3 = t_1 \\ x_4 = t_2 \end{cases}$$

Hence the subset for matrix T, spawned by the t vectors is:

$$x = \{u_1 = (-3, 4, 1, 0), u_2 = (-2, -1, 0, 1)\}$$

By checking for independence, the subset is proved to be the basis for Ker(T):

$$\boxed{B = \left\{ \begin{bmatrix} -3 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}}$$

The image of a linear transformation spans the pivots (3 and 2) of column spaces Col(T):

$$\boxed{B_{\mathrm{Im}(T)} = \left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \right\}}$$

Question 3:

(a)

$$\det(A - \lambda I) = \begin{vmatrix} 13-\lambda & 1 & 4 \\ 1 & 13-\lambda & 4 \\ 4 & 4 & 10-\lambda \end{vmatrix} = 0 \Longrightarrow -\lambda^3 + 36\lambda^2 - 396\lambda + 1296 = 0 \Longrightarrow \begin{cases} \lambda = 6 \\ \lambda = 18 \\ \lambda = 12 \end{cases}$$

1. $\lambda = 6$:

$$(A - 6I)\vec{x} = \vec{0} \Longrightarrow \begin{bmatrix} 7 & 1 & 4 \\ 1 & 7 & 4 \\ 4 & 4 & 4 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \vec{x} = \begin{bmatrix} 1 & 1 & -2 \end{bmatrix}$$

2. $\lambda = 18$:

$$(A - 18I)\vec{x} = \vec{0} \Longrightarrow \begin{bmatrix} -5 & 1 & 4 \\ 1 & -5 & 4 \\ 4 & 4 & 8 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \vec{x} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

3. $\lambda = 12$:

$$(A - 12I)\vec{x} = \vec{0} \Longrightarrow \begin{bmatrix} 1 & 1 & 4 \\ 1 & 1 & 4 \\ 4 & 4 & -2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \vec{x} = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}$$

Hence the eigenmatrix of A is:

$$\vec{E}_A(\lambda) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ -2 & 1 & 0 \end{bmatrix}$$

(b) Since all eigenvectors are unique and $v_1^T v_2 = v_1^T v_3 = v_2^T v_3 = 0$, A is diagonalizable with orthogonal vectors as evident by $A = PDP^T$

$$A = \frac{1}{6} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 12 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} = Q\Lambda Q^T$$

For A^{2023}, orthogonal matrices remain unchanged when being raised to the power of 2023, while the pivots in diagonal matrix get powered.

$$A^{2023} = Q\Lambda^{2023}Q^T = \frac{1}{6}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ -2 & 1 & 0 \end{bmatrix}\begin{bmatrix} 6^{2023} & 0 & 0 \\ 0 & 18^{2023} & 0 \\ 0 & 0 & 12^{2023} \end{bmatrix}\begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

(c) The general solution of the linear system of differential equations

$$y = c_1 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} e^{-6t} + c_2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{-18t} + c_3 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{-12}$$

Find the respective constants given the initial condition

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ -2 & 1 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \Longrightarrow \begin{cases} c_1 = -\frac{1}{6} \\ c_2 = \frac{1}{2} \\ c_3 = -\frac{1}{3} \end{cases}$$

$$\boxed{y = -\frac{1}{6}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} e^{-6t} + \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{-18t} + -\frac{1}{3}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{-12}}$$

END