HCMIU - Linear Algebra - Mid-term Test
Semester 3 - Year: 2023 ∽ 2024 - Duration : 75 minutes
Date Modified : Saturday, July $26^{\text{th}}$, 2025

**INSTRUCTIONS:** Each student is allowed a scientific calculator and a maximum of two double-sided sheets of reference material (size A4 or similar), stapled together and marked with their name and ID. All other documents and electronic devices are forbidden.

**Question 1.** *(20 marks)* Determine the matrices $\boldsymbol{A^T A}$ and $\boldsymbol{A A^T}$ if

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix}$$

**Question 2.** *(20 marks)* Solve the following system of equations using Gaussian elimination with back-substitution:

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 3 \\ 3x_1 - x_2 + 2x_3 &= 1 \\ x_1 - x_2 + x_3 &= -1 \\ 4x_1 + x_2 + x_3 &= 4 \end{aligned}$$

**Question 3.** *(20 marks)* Use Cramer's Rule to find the solution of the system

$$\begin{aligned} -2x_1 + 3x_2 - x_3 &= 1 \\ x_1 + 2x_2 - x_3 &= 4 \\ -2x_1 - x_2 + x_3 &= -3 \end{aligned}$$

**Question 4.** *(20 marks)* Evaluate the determinant of

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

**Question 5.** *(20 marks)*

(a) Let **A** and **B** be n x n matrices. Show that if **AB** is non-singular, then **A** and **B** must be non-singular.

(b) A square matrix is **skew-symmetric** if $\boldsymbol{A^T = -A}$. Show that if **B** is a square matrix of order n, then $\boldsymbol{A = \frac{1}{2}(B - B^T)}$ is skew-symmetric.

*END*

**SUGGESTED ANSWER**

**Question 1.** Key points to remember:

- For matrix multiplication, the number of columns in the first matrix must be equal to the number of rows in the second matrix and the operation is dot product between them.
- The transpose of a matrix is the process of flipping matrix over the diagonal.

$$A^TA = \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 10 & -2 & -11 \\ -2 & 4 & -8 \\ -11 & -8 & 41 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & -2 & -4 \\ 3 & 0 & -5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 21 & -17 \\ -17 & 34 \end{bmatrix}$$

**Question 2.** Perform Gaussian elimination by taking the first pivot as milestone to formulate upper triangular or trapezoidal shape

$$\left[\begin{array}{ccc:c} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 1 & -1 & 1 & -1 \\ 4 & 1 & 1 & 4 \end{array}\right] \longrightarrow \left[\begin{array}{ccc:c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -3 & 2 & -4 \\ 0 & -7 & 5 & -8 \end{array}\right] \longrightarrow \left[\begin{array}{ccc:c} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & -\frac{1}{7} & -\frac{4}{7} \\ 0 & 0 & 0 & 0 \end{array}\right] \Longrightarrow \boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 4 \end{bmatrix}}$$

**Question 3.** Cramer's rule for solving linear system of equations

$$x_i = \frac{det(A_i)}{det(A)} \quad i = 1, ..., n$$

Where $A_i$ is the matrix formed by replacing ith column with the vector b in $Ax = b$. Consider the system of equations under matrix form, we obtain the determinant of A and find the variable x, y, z based on the rule.

$$\left[\begin{array}{ccc:c} -2 & 3 & -1 & 1 \\ 1 & 2 & -1 & 4 \\ -2 & -1 & 1 & -3 \end{array}\right] \Longrightarrow \det(A) = \begin{vmatrix} -2 & 3 & -1 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{vmatrix} = -2 + 3 - 3 = -2$$

$$\left.\begin{aligned} x &= \frac{\det(A_1}{\det(A)} = -\frac{1}{2}\begin{vmatrix} 1 & 3 & -1 \\ 4 & 2 & -1 \\ -3 & -1 & 1 \end{vmatrix} = 2 \\ y &= \frac{\det(A_2)}{\det(A)} = -\frac{1}{2}\begin{vmatrix} -2 & 1 & -1 \\ 1 & 4 & -1 \\ -2 & -3 & 1 \end{vmatrix} = 3 \\ z &= \frac{\det(A_3)}{\det(A)} = -\frac{1}{2}\begin{vmatrix} -2 & 3 & 1 \\ 1 & 2 & 4 \\ -2 & -1 & -3 \end{vmatrix} = 4 \end{aligned}\right\} \Longrightarrow \boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}}$$

**Question 4.**

$$A = \left[\begin{array}{ccc:c} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{array}\right] \Longrightarrow \det(A) = 2 \cdot \begin{vmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & 3 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 3 \end{vmatrix} + 3 \cdot \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{vmatrix} - \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 6$$

**Question 5.**

(a) $A, B \in \mathbb{R}^{n\times n}$. Since AB is non-singular $\Longrightarrow \det(AB) = \det(A)\det(B) \neq 0 \Longleftrightarrow \begin{cases} \det(A) \neq 0 \\ \det(B) \neq 0 \end{cases}$ (Q.E.D)

(b) Skew-symmetric if $A^T = -A$, $\forall B \in \mathbb{R}^{n\times n}$

$$A = \frac{1}{2}(B - B^T) \quad \Longrightarrow \quad A^T = \frac{1}{2}(B - B^T)^T = \frac{1}{2}(B^T - B) = -\frac{1}{2}(B - B^T) = -A$$

Therefore, A is skew-symmetric (Q.E.D)

***END***