HCMIU - Calculus - Final-term Test

Semester 2 (Group 3 & 4) - Year: 2023 - 2024 - Duration : 120 minutes

Date Modified : Monday, Sep. 1st, 2025

INSTRUCTIONS:

- Each student is allowed one doubled-sized sheet of reference material (size A4 of similar). All other documents and electronic devices are forbidden, except scientific calculators.
- There is a total of 10 (ten) questions. Each one carries 10 points.

Question 1. The flow of heat along a thin conducting rod is governed by the one-dimensional *heat equation*

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

where $u(x,t)$ is the temperature at a location x on the rod at time t. The positive constant k is related to the conductivity of the material. Find the constant k such that the function

$$u(x,t) = 2e^{-4t}\cos(2x)$$

satisfies the heat equation.

Question 2. Let $f(x,y,z) = x^3\sqrt{y^2+z^2}$. Find the linear approximation of the function $f(x,y,z)$ at the point $(2,3,4)$ and use it to approximate $f(2.01, 3.08, 3.95)$.

Question 3. Find the critical points of

$$f(x,y) = x^4 + 4x^2(y-2) + 8(y-1)^2$$

Determine for each critical point whether it is a local maximum, local minimum or a saddle point.

Question 4. Use Lagrange multipliers to find absolute minimum and absolute maximum values of the function

$$f(x,y) = x - y, \quad \text{subject to} \quad x^2 + y^2 - 3xy = 20$$

Question 5. Let D be the planar domain bounded by the parabolas $y = (x-1)^2$ and $x = 1 - y^2$. Find the area of D.

Question 6. Evaluate the line integral $\int_C xy^2\,ds$ where C is the right half of the unit circle centered at the origin.

Question 7. Find the volume of the region S that lies between the paraboloid $z = 24 - x^2 - y^2$ and the cone $z = 2\sqrt{x^2+y^2}$.

Question 8. Evaluate the triple integral $\iiint_E (x-y)\,dV$, where E is the solid bounded by the three coordinates planes and the plane $x+y+z=2$.

Question 9. Find a function $f(x,y)$ so that $\nabla f(x,y) = \langle y\cos(xy), x\cos(xy) + 2y\rangle$.

Question 10. Use Green's Theorem to evaluate

$$\oint_C \sqrt{1+x^3}\,dx + 2xy\,dy$$

where C is the triangle from (0; 0) to (1; 0) to (1; 3) to (0; 0).

END

SUGGESTED ANSWER

Question 1.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \iff -8e^{-4t}\cos(2x) = k\left[-8e^{-4t}\cos(2x)\right] \iff k = 1$$

Question 2. Linear approximation at specific point (x, y, z)

$$L(x,y,z) \approx \frac{\partial f}{\partial x}(x-x_0) + \frac{\partial f}{\partial y}(y-y_0) + \frac{\partial f}{\partial z}(z-z_0) \Longrightarrow L(x,y,z) \approx 60x + \frac{24}{5}y + \frac{32}{5}z - 160$$

Find each desirable components:

- $$\frac{\partial f}{\partial x} = 3x^2\sqrt{y^2+z^2} \Longrightarrow \frac{\partial f}{\partial x}|_{(2,3,4)} = 60$$
- $$\frac{\partial f}{\partial y} = \frac{x^3 y}{\sqrt{y^2+z^2}} \Longrightarrow \frac{\partial f}{\partial y}|_{(2,3,4)} = \frac{24}{5}$$
- $$\frac{\partial f}{\partial z} = \frac{x^3}{\sqrt{y^2+z^2}} = \frac{32}{5}$$

$$\boxed{L(2.01, 3.08, 3.95) \approx 60\cdot 2.01 + \frac{24}{5}\cdot 3.08 + \frac{32}{5}\cdot 3.95 - 160 \approx 90.664}$$

Question 3.

$$\begin{cases} f_x = 4x^3 + 8x(x-2) = 0 \\ f_y = 4x^2 + 16(x-1) = 0 \end{cases} \longrightarrow \begin{cases} y = 1 - \frac{1}{4}x^2 \\ 4x^3 + 8x(-\frac{1}{4}x^2) = 0 \end{cases} \Longrightarrow \begin{cases} x = 0 \wedge y = 1 \\ x = 2 \wedge y = 0 \\ x = -2 \wedge y = 0 \end{cases}$$

Critical Points: $(0,1)$, $(2,0)$, $(-2,0)$

Second Derivative Test

$$D(x,y) = f_{xx}\cdot f_{yy} - (f_{xy})^2 = 128x^2 + 86y - 192; \quad \begin{cases} f_{xx} = 12x^2 + 8(y-2) \\ f_{yy} = 16 \\ f_{xy} = 8x \end{cases}$$

- $(0,1) \Longrightarrow$ $D(0,1) = -96 < 0 \Longrightarrow$ Saddle Point.
- $(2,0) \Longrightarrow$ $D(2,0) = 320 > 0$ & $f_{xx} = 32 > 0 \Longrightarrow$ Relative Minimum.
- $(-2,0) \Longrightarrow$ $D(-2,0) = 320 > 0$ & $f_{xx} = 32 > 0 \Longrightarrow$ Relative Minimum.

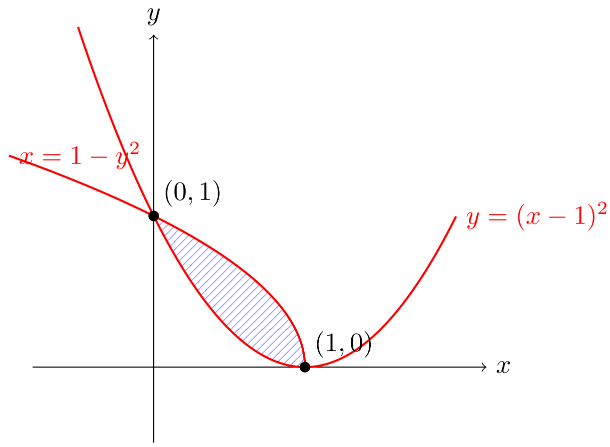
Question 4. LaGrange Multipliers

$$\nabla f(x,y) = \lambda \nabla g(x,y) \Longrightarrow \begin{cases} 1 = \lambda(2x - 3y) \\ -1 = \lambda(2y - 3x) \end{cases} \Longrightarrow x = -y$$

- Critical Points: $(2,-2)$, $(-2,2)$
- Absolute Maximum Value: 4.

- Absolute Minimum Value: -4.

Question 5. The constraints for the planar domain is the graph bounded by two curves, which ranges from $x = 0 \to x = 1$ & $y = (x-1)^2 \to y = \sqrt{1-x}$

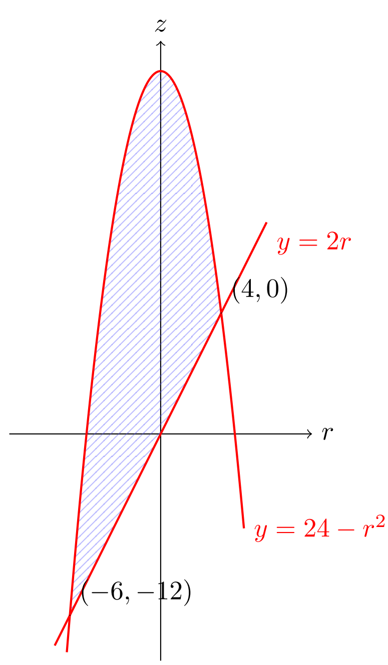


$$D = A = \int_0^1 \sqrt{1-x} - (x-1)^2 = \frac{1}{3}$$

Question 6. Right of the unit circle centered at the origin means the angle of θ ranges from $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

$$\int_C xy^2 \, dS = 2 \cdot \int_0^{\frac{\pi}{2}} \cos(\theta)\sin(\theta)^2 \, d\theta = \frac{2}{3}$$

Question 7. The volume of the region S is the triple integral whose constant is 1 with following constraints: $2r \leq z \leq 24 - r^2, \;\; 0 \leq r \leq 4, \;\; 0 \leq \theta \leq 2\pi$



$$\boxed{V = \int_0^{2\pi} \int_0^4 \int_{2r}^{24-r^2} dz \, dr \, d\theta = \int_0^{2\pi} \int_0^4 \left(-r^2 - 2r + 24\right) \, dr \, d\theta = \frac{352}{3}\pi}$$

Question 8. $x = 0, y = 0, z = 0; \quad x + y + z = 2$

1. z-simple: $0 \le z \le 2 - (x + y)$
2. xy-plane: $0 \le x \le 2, \quad 2 - x \le y \le 0$

$$V = \iiint_E = (x - y)\,dV = \int_0^2 \int_0^{x-2} \int_0^{2-(x+y)} x - y\,dz\,dy\,dx = \frac{80}{3}$$

Question 9. F is a conservative field

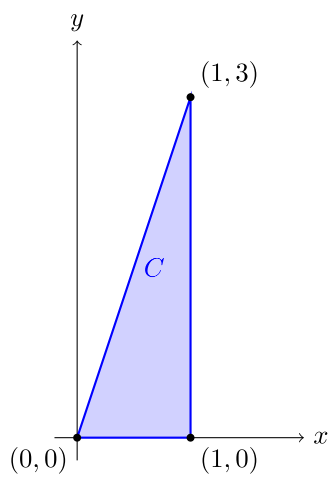
$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = \cos(xy) - xy\sin(xy)$$

$$\int y\cos(xy)\,dx = \sin(xy) + C(y) \Longrightarrow \frac{\partial P}{\partial y} = x\cos(xy) + C'(y) = x\cos(xy) + 2y$$

$$\Longrightarrow C(y) = y^2 + C$$

Hence, the potential function is $\boxed{f(x, y) = \sin(xy) + y^2 + C}$

Question 10. Sketch the graph to set up the constraints from $0 \le x \le 1, \quad y = 3x$



$$\boxed{\oint_C \sqrt{1 + x^3}\,dx + 2xy\,dy = \int_0^1 \int_0^{3x} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dy\,dx = \int_0^1 \int_0^{3x} 2y\,dy\,dx = 3}$$

END