HCMIU - Calculus I - Final-term
Semester 1 - Duration : 90 minutes
Answer Modified : Sunday, January 5^{th}, 2025

- The answers are only for reference because the scale can vary with respect to each lecturers. It is advisable to stick to your instructors for achieving absolute scores.
- The work is being done personally by 1^{st} year student.

SUGGESTED ANSWER

1. Evaluate the limit

$$\lim_{x\to 0}\frac{\int_0^{2x}[e^t+\sin t]\,dt}{\ln(1+10x)}$$

By replacing $x=0$, this equation has the type of $(\frac{0}{0})$. Therefore, we apply L'hospital rule:

$$I=\lim_{x\to 0}\frac{\dfrac{d}{dx}\int_0^{2x}[e^t+\sin t]\,dt}{\dfrac{d}{dx}\ln(1+10x)}$$

Substitution Rule \$ Fundamental Theorem Of Calculus I: $u=2x\Longrightarrow\dfrac{du}{dx}=2$

$$\Longrightarrow\quad\frac{d}{du}\cdot\frac{du}{dx}\int_0^u[e^t+\sin t]\,dt\quad\Longleftrightarrow\quad\frac{d}{du}\cdot 2\int_0^u[e^t+\sin t]\,dt$$

$$\Longleftrightarrow\quad 2\cdot\frac{d}{du}[F(u)-F(0)]\quad\Longleftrightarrow\quad 2\cdot[e^{2x}+sin(2x)]$$

$$\Longrightarrow\quad I=\lim_{x\to 0}\frac{2\cdot[e^{2x}+\sin 2x]\cdot(1+10x)}{10}=\frac{2\cdot[e^{2\cdot 0}+\sin(2\cdot 0)]\cdot(1+10\cdot 0)}{10}=\quad\frac{1}{5}$$

2. Find the linear approximation of the function $f(x)=\sqrt[3]{1+x}$ at $x_0=0$. Use this approximation to approximate the number $\sqrt[3]{0.97}$ and $\sqrt[3]{1.1}$.

Linear approximation of a function is identical to the tangent line:

$$y=f(x)=L(x)\approx f'(x_0)\cdot(x-x_0)+f(x_0)$$

Now, we need to find the prime and the value of the $f(x)$ at $x_0=0$.

$$f'(x)=\frac{1}{3}\cdot\frac{1}{\sqrt[3]{(1+x)^2}}\quad\Longrightarrow\quad f'(x_0)=\frac{1}{3}$$

$$f(x_0)=1$$

$\Longrightarrow\quad L(x)\approx\frac{1}{3}\cdot(x-0)+1\approx\frac{1}{3}x+1$

- $\sqrt[3]{1.1} = \sqrt[3]{1+0.1} \quad \Longrightarrow \quad x = 0.1 \quad \Longrightarrow \quad L(x) \approx \frac{1}{3} \cdot 0.1 + 1 \approx 1.0(3)$
- $\sqrt[3]{0.97} = \sqrt[3]{1-0.03} \quad \Longrightarrow \quad x = -0.03 \quad \Longrightarrow \quad L(x) \approx -\frac{1}{3} \cdot 0.03 + 1 \approx 0.99$

3. A spherical snowball is melting in such a way that its radius is decreasing at a rate of 1 cm/min. At what rate is the volume of the snowball decreasing when the radius is 9 cm?

   Formula of the volume of a sphere: $V = \frac{4}{3}\pi R^3$

$$\Longrightarrow \quad \frac{dV}{dt} = \frac{4}{3} \cdot \pi \cdot 3 \cdot R^2 \cdot \frac{dR}{dT} \quad \Longleftrightarrow \quad \frac{dV}{dt} = \frac{4}{3} \cdot \pi \cdot 3 \cdot 9^2 \cdot 1 = 324\pi \approx 1017,88(\text{cm/m})$$

4. (a) *Show that the equation $x^3 + x + 4 = 0$ has a unique real root on $\mathbb{R}$.*

   Let $y = f(x) = x^3 + x + 4 \Longrightarrow f'(x) = 3x^2 + 1 > 0, \forall x.$

   The function is continuous on $(-\infty, \infty)$. Therefore, it does not satisfy Rolle's Theorem.

   On the other hand, there exists a c unique real root on the interval $(-2, 0)$:

   Given $x_1 = -2, \quad x_2 = 0 \quad \longrightarrow \quad f(x_1) = -6,\ f(x_2) = 4.$

   $f(x_1) < 0 < f(x_2) \longrightarrow -2 < c < 0 \quad or \quad f(c) = 0.$ (Intermediate Value Theorem)

   (b) *Use Newton's method to approximate the root of $x^3 + x + 4 = 0$ correct to seven decimal places. Let x_0 = -1.5 be the initial approximation.*

   Let $f(x) = x^3 + x + 4 \Longrightarrow f'(x) = 3x^2 + 1 \quad \& \quad f(x_0) = -0.875, \quad f'(x_0) = 7.75$

   Now we are all set to apply Newton's method in identifying precisely approximate value:

$$\boxed{x_n + 1 = x_n - \frac{f(x_n)}{f'(x_n)}}$$

   - $x_1 = -1.5 - \frac{-0.875}{7.75} \approx -1.3870968$
   - $x_2 = -1.3870968 - \frac{-0.0559231}{6.7721125} \approx -1.3788389$
   - $x_3 = -1.3788389 - \frac{-2.828832 \cdot 10^{-4}}{6.7035901} \approx -1.3787967$
   - $x_4 \approx \boxed{-1.3787967}$

5. Find the derivative of the function

$$G(x) = \int_x^{3x^2} \ln(t^2 + 1)\, dt.$$

   For this type of function, we can use Fundamental Theorem of Calculus I to find its derivative. However, the integral bound is not ranging from a variable to a constant, which conflicts to the theorem.

Therefore, it is appropriate if we split into two distinctive bounds at $x = 0$

$$\implies \quad G(x) = \int_0^{3x^2} \ln(t^2+1)\,dt - \int_0^{x} \ln(t^2+1)\,dt$$

Substitution rule for each integral: $u = 3x^2 \iff du = 6x \cdot dx \iff \dfrac{du}{dx} = 6x$ & $v = x \longrightarrow \dfrac{dv}{dx} = 1$

$$\implies \quad \frac{d}{dx}[G(x)] = \frac{d}{dx}\int_0^{3x^2} [\ln(t^2+1)]\,dt - \frac{d}{dx}\int_0^{x} [\ln(t^2+1)]\,dt$$

$$= \frac{d}{du}\cdot\frac{du}{dx}\int_0^{u} [\ln(t^2+1)]\,dt - \frac{d}{dv}\cdot\frac{dv}{dx}\int_0^{v} [\ln(t^2+1)]\,dt$$

$$= 6x \cdot \frac{d}{du}\int_0^{u} [\ln(t^2+1)]\,dt - \frac{d}{dv}\int_0^{v} [\ln(t^2+1)]\,dt$$

$$= 6x \cdot \frac{d}{du}[F(u) - F(0)] - \frac{d}{dv}[F(v) - F(0)] = 6x \cdot \ln(9x^4+1) - \ln(x^2+1)$$

6. Evaluate the improper integral

$$\int_3^{\infty} \frac{1}{(x+6)\sqrt{x+6}}\,dx.$$

The integral above is improper type 1, thus we can also write this as (for convenience):

$$\int_3^{\infty} \frac{1}{(x+6)^{\frac{3}{2}}}\,dx \quad = \quad \lim_{t\to\infty}\int_3^{t} \frac{1}{(x+6)^{\frac{3}{2}}}\,dx \quad = \lim_{t\to\infty} \frac{-2}{(x+6)^{\frac{1}{2}}}\Big|_3^t = \lim_{t\to\infty} \frac{-2}{(t+6)^{\frac{1}{2}}} + \frac{2}{3} = \frac{2}{3} \quad (\mathit{Convergent})$$

7. *An oil storage tank cracks at time $t = 0$ and oil then leaks from the tank at a rate of $r(t) = t \cdot e^{-0.01\cdot t}$ liters per minute.*
   (a) *[5 points] Find the time at which the rate has its maximum value.*
   (b) *[5 points] How much oil leaks out during the first ten minutes?*

   Notes: Derivative of a function indicates the rate of change (velocity) of the oil leaking at t time

$$r(t) = t \cdot e^{-0.01\cdot t} \quad \implies \quad r'(t) = e^{-0.01\cdot t}(1 - 0.01\cdot t) = 0 \quad \iff \quad \boxed{t = 100}$$

Sketch the derivative test:

x	0		100		$+\infty$
$f'(x)$		$+$	0	$-$	
$f(x)$	0		$r(100)$		0

At $t = 100$, the rate of the oil leaking reaches maximum value $r(100) = 100 \cdot e^{-0.01 \cdot 100} \approx 36.79(\text{l/min})$

During the first 10 minutes, the proportion of oil leaked is:

$$\int_0^{10} r(t)\,dt = \int_0^{10} t \cdot e^{-0.01 \cdot t}\,dt = I \tag{1}$$

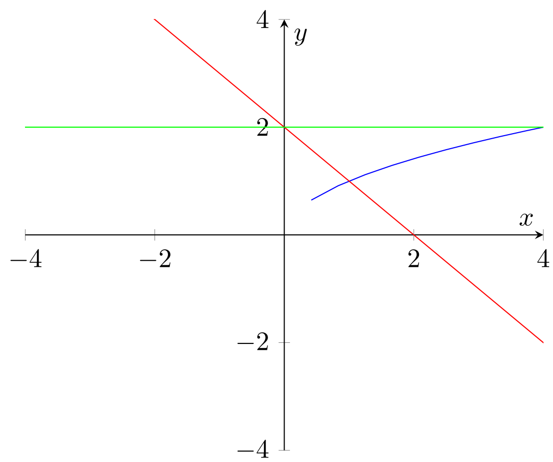
Integration by parts: $u = t \quad \& \quad dv = e^{-0.01 \cdot t} \cdot dt$

$$\iff \quad du = dt \quad \& \quad v = \frac{e^{-0.01 \cdot t}}{-0.01}$$

From (1)

$$\iff I = \left[-100 \cdot t \cdot e^{-0.01 \cdot t}\right]\Big|_0^{10} - \int_0^{10} \frac{e^{-0.01 \cdot t}}{-0.01}\,dt = \frac{-1000}{e^{0.1}} + 10000 \cdot \left[\frac{-1}{e^{0.1}} + 1\right] \approx 46.79 \quad (liters)$$

8. Sketch the region enclosed by the curves $y = 2$, $y = \sqrt{x}$ and $x + y = 2$. Then find the area of the region.



The equation of x-intercept: $y = \sqrt{x}$ and $y = 2 - x$ is $2 - x = \sqrt{x} \quad \Longrightarrow \quad x = 1 \quad \& \quad x = 4$

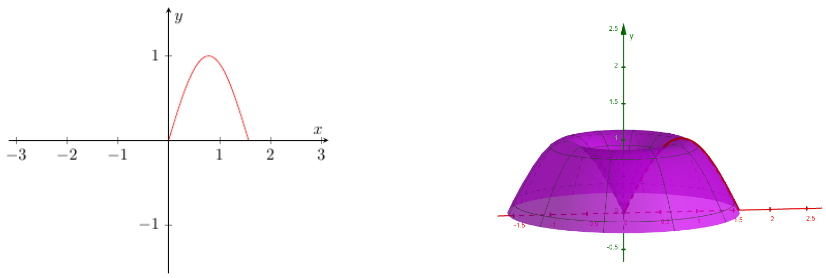
Area of the region bounded by the curves:

$$A = \int_0^1 2 - (2 - x)\,dx + \int_1^4 2 - \sqrt{x}\,dx = \left[\frac{x^2}{2}\right]\Big|_0^1 + \left[2x - \frac{2}{3} \cdot \sqrt[2]{x^3}\right]\Big|_1^4 = \frac{11}{6}$$

9. An object is moving along a straight line path with an initial velocity of $v(0) = 5\,\text{m/s}$. The table below presents the dependence of the acceleration of this object on time t (in m/s^2). Use the Trapezoidal Rule to approximate the velocity of the object at $t = 2$ seconds.

Time t (in seconds)	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
Acceleration (in m/s^2)	1	1.5	1.8	2	2.5	3	3.5	4	4.5	4.2	4

$$\int_0^2 a(t)\,dt \approx \frac{\Delta x}{2} \cdot [a(0) + 2 \cdot a(0,2) + 2 \cdot a(0.4) + ... + a(2)] \approx 5.9(m/s)$$



(a) Before rotating

(b) After rotating

Figure 1: Solid of revolution of the curve $y = sin(2 \cdot x)$

10. The region bounded by $y = sin(2x)$ with $0 \leqslant x \leqslant \frac{\pi}{2}$, and $y = 0$ is rotated about the y-axis. Find the volume of the solid of revolution.

    There are many ways to find the volume of the curve revolving y-axis including Washers method. However the problem would become complicated due to the variable needs to be as well aligned with the axis rotated $x = \frac{\arcsin y}{2}$ and thus the integration is much more difficult to solve.

    Hence we can think of using *cylindrical shell method*:

$$V(x) = \int_0^{\frac{\pi}{2}} 2 \cdot \pi \cdot x \cdot \sin(2x)\, dx$$

    Integration by parts for $u = x \quad \& \quad dv = \sin(2x) \cdot dx \Longrightarrow du = dx \quad \& \quad v = \frac{-cos(2x)}{2}$

$$V(x) = 2 \cdot \pi \left[\frac{-1}{2} \cdot x \cdot \cos(2x) \Big|_0^{\frac{\pi}{2}} + \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos(2x)\, dx \right] = \frac{\pi^2}{2}.$$