HCMIU - Calculus - Final-term Test

Semester 2 (Group 3 & 4) - Year: 2021 - 2022 - Duration : 120 minutes

Date Modified : Saturday, July 26^{th}, 2025

INSTRUCTIONS:

- Each student is allowed one doubled-sized sheet of reference material (size A4 of similar). All other documents and electronic devices are forbidden, except scientific calculators.
- There is a total of 10 (ten) questions. Each one carries 10 points.

Question 1. Find the following limit if it exists, or show that it does not

$$\lim_{(x,y)\to(0,0)} \sin\left(\frac{x^2-y^2}{x^2+y^2}\right)$$

Question 2. Let $u(x,y,z) = x^2y + xyz - z^2x$. Find

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z \partial y}$$

Question 3. Find the absolute maximum and minimum values of

$$f(x,y) = x^2 - xy + y^2 - 3x + 3y$$

on the triangle bounded by the lines $x = 0, y = 0$ and $x = y + 3$.

Question 4. Evaluate $\iint_T y \sin x \, dA$, where T is the region bounded by the parabola $x = y^2$ and the lines $x = \frac{\pi}{2}, y = 0$.

Question 5. Let $\mathbf{F} = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}, z \right\rangle$. Find curl $\mathbf{F}$.

Question 6. Find the work done by the force

$$\mathbf{F} = \left\langle x^2, xy, z^2 \right\rangle$$

on a particle that travels along the curve $\langle 2\sin t, 2\cos t, t\rangle, 0 \le t \le 3$.

Question 7. Let S be the surface parameterized by $\mathbf{r}(u,v) = \left\langle u - v, u^2 - v^2, v + 1 \right\rangle$. Find the tangent plane to S at the point (1, 3, 2).

Question 8. Let $\mathbf{F} = \left\langle ze^{xz}, 2yz, xe^{xz} + y^2 - 2z \right\rangle$. Find a function $f(x,y,z)$ such that $\nabla f = \mathbf{F}$ in $\mathbb{R}^3$.

Question 9. Evaluate $\iint_S z^2 \, dS$ where S is the part of the surface $z = \sqrt{4 - x^2 - y^2}$ lying between $z = 0$ and $z = \sqrt{3}$.

Question 10. Let D be the region bounded by the parabolas $y = (x-1)^2$ and $y = 5 - (x-2)^2$ and let C be its boundary, oriented positively. Evaluate $\int_C F \cdot d\mathbf{r}$ if $\mathbf{F} = \left\langle \cos\left(x^2\right) + 3y, x^2 + e^y \right\rangle$

END

SUGGESTED ANSWER

Question 1. Consider the limit of multivariable function along $x = 0, y = 0$, for certainty we could consider along $y = mx$ if previous ones are already identical.

1. Along $x = 0$:
$$\lim_{y\to 0} \sin\left(-\frac{y^2}{y^2}\right) = -\frac{\pi}{2}$$
2. Along $y = 0$:
$$\lim_{x\to 0} \sin\left(\frac{x^2}{x^2}\right) = \frac{\pi}{2}$$
3. Along $y = mx$:
$$\lim_{x\to 0} \sin\left(\frac{1-m^2}{1+m^2}\right)$$

Hence, the limit does not exist (D.N.E).

Question 2. $u(x, y, z) = x^2y + xyz - z^2x$

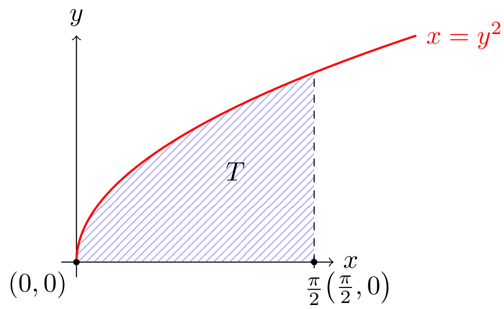
- $$\frac{\partial}{\partial z}\left[\frac{\partial u}{\partial y}\right] = \frac{\partial}{\partial z}\left[x^2 + xz\right] = x$$
- $$\frac{\partial^2 u}{\partial x^2} = 2y$$
- $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z \partial y} = 2y + x$$

Question 3. $f(x, y) = x^2 - xy + y^2 - 3x + 3y$

- Along $y = 0$:
$$f(x) = x^2 - 3x \Longrightarrow f'(x) = 2x - 3 = 0 \Longrightarrow x = \frac{3}{2}$$
- Along $x = 0$:
$$f(y) = y^2 + 3y \Longrightarrow f'(y) = 2y + 3 = 0 \Longrightarrow y = -\frac{3}{2}$$
- Along $y = x - 3$
$$f(3 + y, y) = y^2 + 3y \Longrightarrow 2y + 3 = 0 \Longrightarrow y = -\frac{3}{2}$$

1. End points: $(3, 0), (0, 0), (0, -3) \Longrightarrow 0$
2. Critical points: $\left(\frac{3}{2}, 0\right), \left(0, -\frac{3}{2}\right), \left(\frac{3}{2}, -\frac{3}{2}\right) \Longrightarrow -\frac{9}{4}$
3. Absolute maximum: 0 Absolute Minimum: $-\frac{9}{4}$

Question 4. Rewrite the constraints by sketching the graph



$$\boxed{\int_0^{\frac{\pi}{2}} \int_0^{\sqrt{x}} y \cdot \sin(x)\, dy\, dx = \int_0^{\frac{\pi}{2}} \frac{x}{2} \sin(x)\, dx = \frac{1}{2}}$$

Question 5.

$$\operatorname{curl}(\vec{\mathbf{F}}) = \nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{-2xy}{(x^2+y^2)^2} - \frac{x}{x^2+y^2} \right) \mathbf{i} - \left(\frac{2xyz}{(x^2+y^2)^2} + \frac{y}{x^2+y^2} \right) \mathbf{j} + \left(\frac{2x^2y + 2xy^2}{(x^2+y^2)^3} \right) \mathbf{k}$$

Question 6. Optional preference: Check if the work is conservative, then apply Fundamental Theorem Of Calculus II. Otherwise, find the parameterized line integral as usual.

$$\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}}$$

Apparently, the condition was not satisfied ($0 = 0 \neq y$), so the work depends on the path.

$$\int_C F \cdot dr = \int_0^3 F\left(r(t)\right) \cdot r'(t)\, dt = \int_0^3 t^2\, dt = 9$$

Question 7. General approach: Solve parametric equations given at a point to obtain u and v, find the normal vector of the surface and set up the tangent plane to S. Likewise the procedure to the plane (P).

$$\begin{cases} u - v = 1 \\ u^2 - v^2 = 3 \\ v + 1 = 2 \end{cases} \Longrightarrow P(1,3,2) \to r(2,1)$$

$$\frac{\partial r}{\partial u} = \langle 1; 2u; 0 \rangle = \langle 1; 4; 0 \rangle, \quad \frac{\partial r}{\partial v} = \langle -1; -2v; 1 \rangle = \langle -1, -2, 1 \rangle \Longrightarrow \vec{n} = \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 4 & 0 \\ -1 & -2 & 1 \end{vmatrix} = \langle 4; -1; 2 \rangle$$

$$\Longrightarrow \boxed{\text{(P)}: 4x - y + 2z - 5 = 0}$$

Question 8. F is a conservative field

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 0, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y} = 2y, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} = e^{xz}(1 + xz)$$

For the vector field that consists of three or more variables. It's advisable to integrate each components and select from most-common to least-common variables, which formulates the potential function.

$$P = f_x = ze^{xz}. \quad R = f_z = xe^{xz} + y^2 - 2z, \quad Q = f_y = 2yz$$

- $$\int ze^{xz}\,dx = e^{xz} + C(y,z)$$
- $$\int 2yz\,dy = y^2z + C(x,z)$$
- $$\int xe^{xz} + y^2 - 2z\,dz = xze^{xz} + y^2z - z^2 + C(x,y)$$

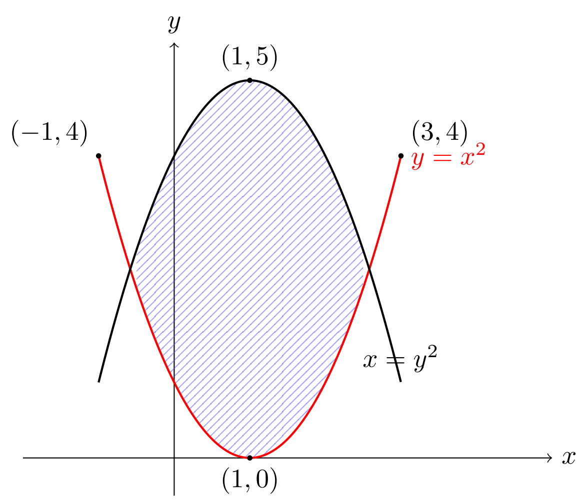
$$\boxed{f(x,y,z) = e^{xz} + y^2z - z^2 + C}$$

Question 9. Hint: z-simple to transform into the appropriate forms: xy-variables, spherical or cylindrical.

$$z = \sqrt{4 - 4\sin(\phi)^2}; \quad 0 \le \phi \le \frac{\sqrt{3}}{2} \Longleftrightarrow \frac{\pi}{6} \le \phi \le \frac{\pi}{2}; \quad 0 \le \theta \le 2\pi$$

$$\iint_S z^2 \cdot dS = \int_0^{2\pi} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4 - 4\sin(\phi)^2\,dS = \int_0^{2\pi} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4\sin(\phi)\cos(\phi)\left[4 - 4\sin(\phi)^2\right]\,d\phi\,d\theta = \frac{9}{2}\pi$$

Question 10. Apply Green Theorem onto positively oriented counterclockwise curve F:



$$\boxed{\int_C \mathbf{F} \cdot dr = \oint_C P\,dx + Q\,dy = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \int_0^3 \int_{(x-1)^2}^{5-(x-2)^2} 2x - 3\,dy\,dx = 0}$$

END