Probability - HCMUS - April 25th, 2025

Semester 2 : Year 2024 - 2025 - Time Duration: 60 minutes

Remarks:

- Standard distribution table (Gaussian table) is located on the second page of the test.
- Every calculations must be rounded to 4 decimal places; do not write results under fractional form

Question 1. (3.0 points) During a football match, there are 3 footballers labeled A, B and C reserving. The probabilities of substitution to be made are respectively $40\%, 35\%, 25\%$. For each footballers, the probabilities of scoring goals in the next 15 minutes are accordingly $15\%, 10\%$ and 8%.

(a) Find the probability of one of them scores goals in the next 15 minutes.

(b) Given that one of the footballers scored goals in the next 15 minutes, of which highest rate is belonged to?

(c) Given that there is not any goals being scored in 15 minutes after substitution, what is the probability of the footballer "C" entering the lawn?

Question 2. (3.0 points) The life expectancy of a light bulb (in hours) is random variable $Y = 100 \times X$, where X is the random variable consisting of PDF (probability density function) is given by:

$$f(x) = \begin{cases} \frac{C}{x^4}, & \text{if} \quad x \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

where C is an unknown constant

(a) Find the constant C.

(b) Calculate $\mathbb{E}(Y)$, $Var(Y)$.

(c) Consider two following events:

- A: Life expectancy of light bulb is atleast 100 hours.
- B: Life expectancy of light bulb is not more than 200 hours.

Are these events independent? How come?

Question 3. (4.0 points) The weight of a food packet for cattle has a normal distribution with a mean of 1.5kg and standard deviation of 250g. A farm buys 500 packets for its own cattle.

(a) Calculate the probability of a packet weighing more than 2kg.

(b) Every week, the cattle needs to be fed about 175kg food proportion. Estimate the probability that the farm still has atleast 55kg left in the end of month. Assuming a month comprises 4 weeks, and the food proprotion at each week is equally likely.

(c) Before in use, the farmer firstly measures the food packet so as to ensure the sufficient amount for cattle. If its weight is less than 1kg, the farmer will have to exchange with the appropriate one. Estimate the probability of farmer request of minimum 10 food packets.

END

Standard Distribution Table $Z \approx N(0,1) : \Theta(z) = \mathbb{P}(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{\frac{-x^2}{2}}\, dx$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

SUGGESTED ANSWER

Question 1. Let S be the event that one of the footballers scores the goals in the next 15 minutes

We already know the probability of each football substitution are accordingly:

$$P(A) = 40\%, \quad P(B) = 35\%, \quad P(C) = 25\%$$

Notes: In order to make the event happen, each footballers must be sent to the lawn beforehand. Hence, the following rates align with Bayes' rule.

$$P(S|A) = 15\%, \quad P(S|B) = 10\%, \quad P(S|C) = 8\%$$

(a) According to law of total probability, the likelihood of the goal is scored in the next 15 minutes (i.e any footballers stated could be the one) is:

$$\begin{aligned} P(S) &= P(A) \cdot P(S|A) + P(B) \cdot P(S|B) + P(C) \cdot P(S|C) \\ &= 40\% \cdot 15\% + 35\% \cdot 10\% + 25\% \cdot 8\% \\ &= \frac{23}{200} \Longrightarrow \boxed{P(S) = 0.115} \end{aligned}$$

(b) From the overall rate of scoring arbitrary goals, apply conditional probability to find distinctive ones (A, B, C) then decide the highest rate.

$$\begin{aligned} P(A|S) &= \frac{P(A \cap S)}{P(S)} = \frac{P(A) \cdot P(S|A)}{P(S)} = \frac{0.4 \cdot 0.15}{0.115} \approx \boxed{0.5217} \\ P(B|S) &= \frac{P(B \cap S)}{P(S)} = \frac{P(B) \cdot P(S|B)}{P(S)} = \frac{0.35 \cdot 0.1}{0.115} \approx 0.3043 \\ P(C|S) &= \frac{P(C \cap S)}{P(S)} = \frac{P(C) \cdot P(S|C)}{P(S)} = \frac{0.25 \cdot 0.08}{0.115} \approx 0.1739 \end{aligned}$$

Implications: Footballer A has the higher chance of scoring goals than footballers B and C. So A rate is accepted.

(c) Any goals is not scored in the next 15 minutes, so the chance is negated. Do similarly to the (a) question and find $P(C|S^C)$

$$\begin{aligned} P(S^C) &= P(A) \cdot P(S^C|A) + P(B) \cdot P(S^C|B) + P(C) \cdot P(S^C|C) \\ &= 0.4 \cdot 0.85 + 0.35 \cdot 0.9 + 0.25 \cdot 0.92 \\ &= 0.885 \\ P(C|S^C) &= \frac{P(C \cap S^C)}{P(S^C)} = \frac{P(C) \cdot P(S^C|C)}{P(S^C)} \\ &= \frac{0.25 \cdot 0.92}{0.885} = 0.2599 \Longrightarrow \boxed{P(C|S^C) = 0.2599} \end{aligned}$$

Question 2. We are given the probability density function with a constant C. It is important to know that the summation of all probabilities within the event is added to one $\sum_{n=1}^{\infty} P(A_n) = 1$

(a) Apply the previous remark and continuous random variable properties into calculating C

$$P\left(-\infty < x < \infty\right) = \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx = 0 + \int_{1}^{\infty} \frac{C}{x^4}\,dx = \lim_{t\to\infty}\int_{1}^{t} \frac{C}{x^4}\,dx = 1$$

$$\Longrightarrow -\frac{C}{3}\cdot\lim_{t\to\infty}\left[\frac{1}{t^5}-1\right] = \frac{C}{3} = 1 \Longleftrightarrow \boxed{C=3}$$

(b) The revised PDF after C replacement

$$f(x) = \begin{cases} \frac{3}{x^4}, & \text{if} \quad x \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

Observe that $Y = 100 \times X$ and $\mathbb{E}\left(aX+b\right) = a\mathbb{E}(X) + b$. Consequently, we can also derive the expectation of squared linear equation to find the variance. In most scenarios, the general idea will be solving the integrals.

$$\mathbb{E}(Y) = 100\cdot\mathbb{E}(X) = 100\cdot\int_{1}^{\infty} x\cdot\frac{3}{x^4}\,dx = 100\cdot\lim_{t\to\infty}\int_{1}^{t}\frac{3}{x^3}\,dx = -100\cdot\frac{3}{2}\left(\lim_{t\to\infty}\left[\frac{1}{t^2}-1\right]\right) = 150 \text{ (hours)}$$

$$\mathbb{E}(Y^2) = 100^2\cdot\mathbb{E}(X^2) = 100^2\cdot\int_{1}^{\infty} x^2\cdot\frac{3}{x^4}\,dx = 100^2\cdot\left(\lim_{t\to\infty}\left[-\frac{3}{t}+3\right]\right) = 30000 \text{ (hours)}$$

$$Var(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = 30000 - 150^2 = 7500 \text{ (hours)}$$

$$\Longrightarrow \quad \boxed{\mathbb{E}(Y) = 150} \quad \& \quad \boxed{Var(Y) = 7500} \quad \text{(hours)}$$

(c) Check for the independence of two following events A and B by the formula $P(A\cap B) = P(A)\cdot P(B)$. In other words, find the probability of life expectancy of the light bulb in accordance to the different constraints.

$$P(Y\geq 100) \Longleftrightarrow P(X\geq 1) = \int_{1}^{\infty} f(x)\,dx = \lim_{t\to\infty}\left[-\frac{1}{t^3}+1\right] = 1$$

$$P(Y\leq 200) \Longleftrightarrow P(X\leq 2) = \int_{2}^{-\infty} = \int_{-\infty}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx = \int_{1}^{2}\frac{3}{x^4} = \frac{7}{8}$$

Let A and B be the constraints of random variables such that $A = \{X \mid X\geq 1\}$, $B = \{X \mid X \leq 2\}$. Then the intersection of A and B will be $A\cap B = \{X \mid 1\leq X\leq 2\}$

$$P(A\cap B) = \int_{1}^{2} f_X(x)\,dx = \int_{1}^{2}\frac{3}{x^4}\,dx = \frac{7}{8} = P(X\geq 1)\cdot P(X\leq 2)$$

Hence, A and B are independent events.

Question 3.

Summary: A farm buy 500 food packets for the cattle, one of which has $\mu = 1.5$ (kg), $\sigma = 0.25$ (kg)

(a) To identify the occurrence of the food packet weighing more than 2kg, use standard distribution probability and normalize the values based on the given table.

$$X \sim \mathbb{N}(1.5, 0.25^2) \Longrightarrow P(X>2) = 1 - P(X\leq 2) \approx 1 - P\left(\mathbb{Z}\leq\frac{2-1.5}{0.25}\right) \approx 1 - P(\mathbb{Z}\leq 2) \approx 0.028$$

(b) Given the fact that the cattle need food proportion of 175kg per week, i.e 700kg of total 500 food packets he (or) she previously purchased, additionally maintaining atleast 55kg of food packet at the end of month, the farmer have to store at most $700 + 55 = 755$ (kg).

Since this model has now been increased with $n = 500$ (initial assumption), we apply Central Limit Theorem to converge to a normalized sample distribution

$$X \sim \mathbb{N}(n\mu, \sqrt{n}\sigma) \Longrightarrow \mathbb{Z}_n = \frac{\overline{X} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{X_1 + X_2 + ... + X_n - n\mu}{\sqrt{n}\sigma} \Longrightarrow \begin{cases} n\mu = 750(kg) \\ \sqrt{n}\sigma \approx 5.5902 \end{cases}$$

The probability that the total food available from 500 packets must be 755(kg) is:

$$P(X \geq 755) = 1 - P(X < 755) \approx 1 - P\left(\mathbb{Z} < \frac{755 - 750}{0.25\sqrt{500}}\right) \approx 1 - P(\mathbb{Z} < 0.8944) \approx 0.1867$$

(c) Now we are told that extra step is added by inspecting the food packets under standards (less than 1kg will be replaced with current one), which is roughly rephrased into: If $X < 1$, that packet will be replaced, i.e the probability for minimum 10 food packets are rejected.

We are interested to find out the chance for a single packet to be rejected in such amount:

$$P(X < 1) = P(\mathbb{Z} < \frac{1 - 1.5}{0.25}) \approx 0.0228$$

Evidently, it is impossible to find the occurrence of atleast 10 rejected packets in the sample size $n = 500$, or given the time is allowed, the calculation will be complicated. Therefore, normal approximation comes in handy (Binomial).

For n = 500 (packets), we want to test out how likely is the number of rejected ones. Let Y be the new Binomial approximation.

$$Y \sim \mathbb{N}(np, np(1-p)) \Longrightarrow \begin{cases} \mathbb{E}(Y) = np \approx 11.4 \\ Var(Y) = np(1-p) \approx 11.14 \end{cases}$$

Apply the continuity correction in approximating discrete probability distribution

$$P(Y \leq 10) = P(Y \leq 10 + 0.5) \approx P\left(\mathbb{Z} \leq \frac{10 + 0.5 - 11.4}{\sqrt{11.4(1 - 0.0228)}}\right) \approx P(\mathbb{Z} \leq -0.27) \approx 0.3936$$

Hence, the probability of farmer request of minimum 10 food packets is approximately 0.3936.

END