Linear Algebra - HCMUS - April 22nd, 2025
Semester 1 : Year 2024 - 2025 - Time Estimation : 60 minutes

1. **(3.5pts = 1 + 1 + 0.5 + 1 pts)**
   - Solve the system of linear equations:

$$\begin{cases} -2t + x + 2y + 3z = 7 \\ -3t + 2x - y - 2z = -8 \\ 2t + 3x + 2y - z = -11 \\ t + 2x - 3y + 2z = -6 \end{cases}$$

   - Find the real value of m such that the determinant A = $\begin{pmatrix} 3 & 1 & m+1 \\ 0 & m+2 & 2 \\ 3 & m+3 & 2m+2 \end{pmatrix}$ has its rank is 2.

2. **(3.5pts = 1.5 + 2 pts)** Given the matrices $B = \frac{1}{2}\begin{pmatrix} -1 & 2 & 0 \\ 1 & 1 & -2 \\ 4 & 0 & -3 \end{pmatrix}$ and invertible A with $A^{-1} = \begin{pmatrix} 3 & 1 & 2 \\ 2 & 2 & 3 \\ 3 & 2 & 3 \end{pmatrix}$
   - Find the matrix A
   - Find all matrices X and Y that satisfies the following linear system of matrices $\begin{cases} A^T(X-Y) = I_3 \\ (2Y-3X)(2A)^{-1} = B \end{cases}$

3. **(2pts)**
   Given matrices $A = \begin{pmatrix} 1 & 1 & m+2 \\ -3 & 2 & 3m-3 \\ m-1 & -2 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} a & -ab & a(m+2) \\ -3 & -2b & 3m-3 \\ m-1 & 2b & -2 \end{pmatrix}$ with all real parametric variables m, a and b.

   Find the determinant of A, then derive the $det(B)$. In which case is the matrix A invertible?

4. **(1pts)** Given square matrices A and B at degree n ($n \geq 1$). Assuming that $\exists s \in \mathbb{N}^*$ such that $(AB)^s = I_n$. Show that A and B are all invertible and $(BA)^s = I_n$

END.

Question 1:

- Using either Gaussian or Gaussian - Jordan elimination on represented linear system as matrix form

$$\begin{bmatrix}1 & 2 & 3 & -2 & 7\\ 2 & -1 & -2 & -3 & -8\\ 3 & 2 & -1 & 2 & -11\\ 2 & -3 & 2 & 1 & -6\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 2 & 3 & -2 & 7\\ 0 & -5 & -8 & 1 & -22\\ 0 & -4 & -10 & 8 & -32\\ 0 & -7 & -4 & 5 & -20\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 2 & 3 & -2 & 7\\ 0 & -5 & -8 & 1 & -22\\ 0 & 0 & \frac{-18}{5} & \frac{36}{5} & \frac{-72}{5}\\ 0 & 0 & \frac{36}{5} & \frac{18}{5} & \frac{54}{5}\end{bmatrix}$$

$$\longrightarrow \begin{bmatrix}1 & 2 & 3 & -2 & 7\\ 0 & -5 & -8 & 1 & -22\\ 0 & 0 & -1 & 2 & -4\\ 0 & 0 & 2 & 1 & 3\end{bmatrix} \longrightarrow \begin{bmatrix}1 & 2 & 3 & -2 & 7\\ 0 & -5 & -8 & 1 & -22\\ 0 & 0 & -1 & 2 & -4\\ 0 & 0 & 0 & 5 & -5\end{bmatrix}$$

Back-substitue the variables, thus we obtain the particular solution: $x = \begin{bmatrix}-3\\ 1\\ 2\\ -1\end{bmatrix}$

- A matrix is linearly dependent if atleast one of the vectors (either columns or rows) can be determined as a linear combination of other vectors. By property, we can immediately find out solutions of parametric variable are $m = -2 \vee m = 1$.

  Alternatively, we might think of using Rouché - Capelli theorem as stated that if the rank of matrix is less than number of pivot columns, then $det(A) = 0$ or non-invertible.

$$det(A) = 3\begin{vmatrix}m+2 & 2\\ m+3 & 2m+2\end{vmatrix} - \begin{vmatrix}0 & 2\\ 3 & 2m+2\end{vmatrix} + (m+1)\begin{vmatrix}0 & m+2\\ 3 & m+3\end{vmatrix} = 3m^2 + 3m - 6 = 0 \Longrightarrow \begin{cases} m = 1\\ m = -2\end{cases}$$

Question 2:

(a) Observe that $AA^{-1} = I \Longrightarrow (A^{-1})^{-1} = A$. Let $X = A^{-1} \Longrightarrow X^{-1} = \frac{1}{\|X\|} \cdot adj(X)$

Remarks: Adjoint of a matrix is found by taking the transpose of the cofactor matrix of A.

$$|X| = 3\begin{vmatrix}2 & 3\\ 2 & 3\end{vmatrix} - \begin{vmatrix}2 & 3\\ 3 & 3\end{vmatrix} + 2\begin{vmatrix}2 & 2\\ 3 & 2\end{vmatrix} = -1\cdot(-3) + 2\cdot(-2) = -1$$

$$\mathbf{adj}(X) = \begin{bmatrix} +\begin{vmatrix}2 & 3\\ 2 & 3\end{vmatrix} & -\begin{vmatrix}2 & 3\\ 3 & 3\end{vmatrix} & +\begin{vmatrix}2 & 2\\ 3 & 2\end{vmatrix}\\ -\begin{vmatrix}1 & 2\\ 2 & 3\end{vmatrix} & +\begin{vmatrix}3 & 2\\ 3 & 3\end{vmatrix} & -\begin{vmatrix}3 & 1\\ 3 & 2\end{vmatrix}\\ +\begin{vmatrix}1 & 2\\ 2 & 3\end{vmatrix} & -\begin{vmatrix}3 & 2\\ 2 & 3\end{vmatrix} & +\begin{vmatrix}3 & 1\\ 2 & 2\end{vmatrix}\end{bmatrix}^T = \begin{bmatrix}0 & 3 & -2\\ 1 & 3 & -3\\ -1 & -5 & 4\end{bmatrix}^T = \begin{bmatrix}0 & 1 & -1\\ 3 & 3 & -5\\ -2 & -3 & 4\end{bmatrix}$$

$$\boxed{\Longrightarrow X^{-1} = \begin{bmatrix}0 & -1 & 1\\ -3 & -3 & 5\\ 2 & 3 & -4\end{bmatrix}}$$

(b)

$$\begin{cases} A^T(X-Y) = I_3\\ (2Y-3X)(2A)^{-1} = B\end{cases} \longrightarrow \begin{cases} (X-Y) = (A^T)^{-1}\\ (2Y-3X)(2A)^{-1} = B\end{cases} \longrightarrow \begin{cases} (X-Y) = (A^T)^{-1}\\ (2Y-3X)(2A)^{-1} = B\end{cases}$$

$$\longrightarrow \begin{cases} X = (A^{-1})^T + Y \\ (-Y - 3(A^{-1})^T)(2A)^{-1} = B \end{cases} \longrightarrow \begin{cases} X = (A^{-1})^T + Y \\ (-Y - 3(A^{-1})^T) = 2BA \end{cases} \longrightarrow \begin{cases} X = (A^{-1})^T + Y \\ Y = -2BA - 3(A^{-1})^T \end{cases}$$

Replacing with the given matrices A, B and A^{-1}, we get the solution of X and Y:

$$\boxed{\longrightarrow \begin{cases} X = \begin{bmatrix} 0 & 1 & -15 \\ 5 & 6 & -18 \\ 2 & 7 & -22 \end{bmatrix} \\ Y = \begin{bmatrix} -3 & -1 & -18 \\ 4 & 4 & -20 \\ 0 & 4 & -25 \end{bmatrix} \end{cases}}$$

Question 3:

$$det(A) = \begin{vmatrix} 2 & 3m-3 \\ -2 & -2 \end{vmatrix} - \begin{vmatrix} -3 & 3m-3 \\ m-1 & -2 \end{vmatrix} + (m+2) \begin{vmatrix} -3 & 2 \\ m-1 & -2 \end{vmatrix} = m^2 - 5 \neq 0 \Longrightarrow m \neq \pm 5$$

Notice that the determinant of a matrix is the cross product between components, which means their parametric variables a, b are scalar multiplications of that product. Hence, $\det(B) = -ab \cdot \det(A) = -ab(m^2 - 5)$

Question 4:

$\exists s \in \mathbb{N}^*, (AB)^s = I_n \Longrightarrow (AB)^{s-1}(AB) = I_n \Longleftrightarrow (AB)^{s-1} = (AB)^{-1}$. Hence, $(AB)^{-1}$ exists and AB is invertible.

AB invertible $\Longrightarrow \exists C$ such that $C(AB) = I$, but using associative property of matrix multiplication: $C(AB) = (CA)B = I \Longrightarrow$ B is invertible and $CA = B^{-1}$

Similarly we can prove this for matrix A with $(AB)D = I \Longrightarrow BD = A^{-1} \Longrightarrow$ A is also invertible.

To prove $(BA)^s = I$, elaborate the initial hypothesis $(AB)^s = I_n \Longrightarrow ABABABAB...AB = I_n \Longleftrightarrow A(BA)^{s-1}B = I_n \Longleftrightarrow A^{-1}A(BA)^{s-1}BB^{-1} = A^{-1}I_nB^{-1} \Longleftrightarrow (BA)^{s-1} = (BA)^{-1}$

$$(BA)^s = I_n \Longrightarrow (BA)^{s-1}(BA) = I \Longleftrightarrow \boxed{(BA)^{-1}(BA) = I_n}$$

Q.E.D