HCMIU - Calculus - Final-term Test

Semester 2 (Group 3 & 4) - Year: 2021 - 2022 - Duration : 120 minutes

Date Modified : Saturday, July 26$^{\text{th}}$, 2025

INSTRUCTIONS:

- Each student is allowed one doubled-sized sheet of reference material (size A4 of similar). All other documents and electronic devices are forbidden, except scientific calculators.
- There is a total of 5 (five) questions. Each one carries 20 points.

Question 1. Let $f(x, y) = xe^{-3y}$

(a) Find the tangent plane to the graph of f at (1, 0, 1)

(b) Find the directional derivative $D_{\mathbf{u}}f(1, 0)$, where $\mathbf{u} = \left(\frac{1}{\sqrt{2}}\right)\mathbf{\imath} + \left(\frac{1}{\sqrt{2}}\right)\mathbf{\jmath}$

Question 2. Find the local maximum - minimum values and saddle point(s) of the function

$$f(x, y) = e^x x^2 - e^x (y - 1)^2$$

Question 3. Evaluate the multiple integrals

(a) $\iint_D e^x y \, dA$ where D is bounded by the curve $y = 2\sqrt{x}$ and the lines $y = 0, x = 1$.

(b) $\iiint_E 2x \, dV$, where E is bounded by the parabolic cylinders $y = x^2$ and $x = y^2$ and the planes $z = 0$ and $z = 2x + y$

Question 4. Let $\mathbb{F}(x, y) = (x + y^2)\imath + (2xy + 1)\jmath$.

(a) Show that $\mathbb{F}(x, y)$ is a conservative vector field.

(b) Find a potential function f such that $\mathbf{F} = \nabla f$.

(c) Use the part b) to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the curve C, where C is the arc of the curve $y = 2x \sin\left(\frac{\pi x^3}{6}\right)$ from (0, 0) to (1, 1).

Question 5. Do the following requests

(a) Find curl $\mathbb{F}$ and div $\mathbb{F}$ if $\mathbb{F}(x, y, z) = z^2 x\imath + x^2 y\jmath + y^2 z k$

(b) Evaluate the surface integral of the vector field $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x\mathbf{\imath} + y\mathbf{\jmath} + z^2\mathbf{k}$, and S is the part of the cone $z = \sqrt{x^2 + y^2}$ between the planes z = 1 and z = 2 with upward orientation.

END

SUGGESTED ANSWER

Question 1. Directional derivative is displayed by the following formula. Note that for every vectors v must be normalized as unit vector $\vec{u} = \frac{\vec{v}}{\|\vec{v}\|}$

$$D_u f(x,y) = \nabla f(x,y) \cdot \mathbf{u} = \left\langle \frac{\partial f}{\partial x}; \frac{\partial f}{\partial y} \right\rangle \cdot \mathbf{u} = \left\langle e^{-3y}, -3xe^{-3y} \right\rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \Longrightarrow \boxed{D_u f(1,0) = \langle 1, -3 \rangle}$$

The tangent plane to the function at point P(1, 0, 1):

$$\text{(P)}: \quad z - z_0 = \nabla_{f_x}(x - x_0) + \nabla_{f_y}(y - y_0) \Longleftrightarrow \quad \boxed{\text{(P)}: x - 3y - z = 0}$$

Question 2. Find all possible critical points of the multivariable function

$$\begin{cases} f_x = e^x x^2 - e^x (y-1)^2 = 0 \\ f_y = -2e^x (y-1) = 0 \end{cases} \Longrightarrow \begin{cases} y = 1 \vee x = 0 \\ y = 1 \vee x = -2 \end{cases}$$

Second derivative test:

$$D(x,y) = f_{xx} \cdot f_{yy} - (f_{xy})^2$$

Find each respective components:

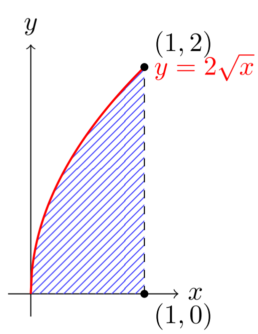
$$f_{xx} = e^x x^2 + 2e^x x + 2e^x x + 2e^x - e^x (y-1)^2 = e^x \left[x^2 + 4x + 2 - (y-1)^2\right]; \quad f_{yy} = -2e^x; \quad f_{xy} = e^x(2-2y)$$

$$\Longrightarrow D(x,y) = -2e^{2x} \left[x^2 + 4x + 2 - (y-1)^2\right]$$

Considering existing critical points and check the second derivative with respect to x if necessary.

1. Critical Point (0, 1): $D(0,1) = -4 < 0 \Longrightarrow$ Saddle point
2. Critical Point (-2, 1): $D(-2,1) \approx 0.073 > 0$ & $f_{xx} \approx -0.27 < 0$ with a value of $f(-2,1) \approx 0.54 \Longrightarrow$ Relative Maximum

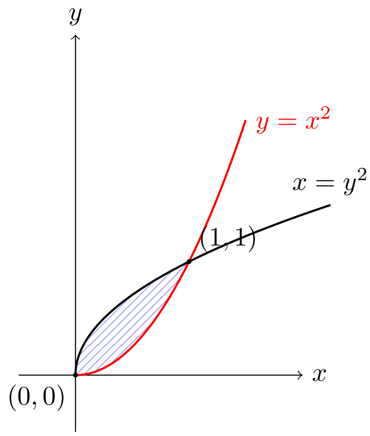
Question 3. (a) Find the constraints by sketching the curves $y = 2\sqrt{x}$, $y = 0$ & $x = 1$



Therefore, the integral domain is : $(x,y) : (0,0) \to (1, 2\sqrt{x})$. Solve the integral sequentially from inside to outside, specifically with respect to y, then w.r.t to x and apply integration by parts if required following by LIATE rule.

$$\iint_D e^x y\, dA = \int_0^1 \int_0^{2\sqrt{x}} e^x y\, dy\, dx = \int_0^1 2e^x x\, dx = 2$$

(b) General approach for triple integral solution: z-simple, xy-plane. This becomes double integral, thus do the similar process as previously being made.



$$\iiint_E 2x\, dV = \int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^{2x+y} 2x\, dz\, dy\, dx = \int_0^1 \int_{x^2}^{\sqrt{x}} 2x(2x+y)\, dy\, dx = \int_0^1 -x^5-4x^4+4x^2\sqrt{x}+x^2\, dx = \frac{107}{210}$$

Question 4. $P = x + y^2, \quad Q = 2xy + 1$

(a) F is conservative field if and only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 2y$.

(b) There are two primary methods of detecting potential function: Integrate the components with respect to their axes and find their common terms or integrate the x-component (leaving constant that can contain variables) and derivative with respect to y - compare with the y-component in the conservative vector to find that constant.

$$\int P\, dx = \int x + y^2\, dx = \frac{x^2}{2} + y^2 x + C(y)$$

$$\frac{\partial}{\partial y}\left[\frac{x^2}{2} + xy^2 + C(y)\right] = 2xy + C'(y) = 2xy + 1 \Longrightarrow \quad C'(y) = 1 \Longrightarrow \quad C^y = y + C$$

Hence, the potential function of f(x, y) is:

$$\boxed{f(x,y) = \frac{x^2}{2} + xy^2 + y + C}$$

(c) Fundamental Theorem Of Calculus II (Line Integrals) For Conservative Vector Field

$$\int_C F \cdot dr = f(\vec{r}(b)) - f(\vec{r}(a)) = f(\vec{r}(1,1)) - f(r(0,0)) = \frac{5}{2} + C - C = \frac{5}{2}$$

Question 5. $F(x,y,z) = z^2x\mathbf{i} + x^2y\mathbf{j} + y^2z\mathbf{k}$

(a) $$\mathbf{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = (y^4z - x^2y^3)\vec{\mathbf{i}} - (x^2y^2z - z^2y^2x)\vec{\mathbf{j}} + z^2(x^2y - y^2x)\vec{\mathbf{k}}$$

$$\mathbf{div}(\vec{F}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = z^2 + x^2 + y^2$$

(b) $\iint_S \mathbf{F} \cdot dS;\quad F(x,y,z) = x\mathbf{i} + y\mathbf{j} + z^2\mathbf{k};\quad S: z = \sqrt{x^2+y^2} \in [1;2];\quad x, y > 0$

Surface integral of vector field where F describes the velocity of flow / fluid at arbitrary point across the surface is called flux

$$\iint_S \mathbf{F} \cdot dS = \iint_S \vec{\mathbf{F}}\vec{n}\, dS = \iint_D (-Pg_x - Qg_y + R)\, dA = \iint_D \frac{-(x^2+y^2)}{\sqrt{x^2+y^2}} + x^2 + y^2\, dA$$

Transform the D region into polar coordinates, respectively for each z values form a circle on xy-plane with radius of 1 and 2: $1 \le r \le 2$ and the angle from $0 \le \theta \le 2\pi$ since there is no constraints.

$$\Longrightarrow \boxed{\int_0^{2\pi}\int_1^2 (r^3 - r^2)\, dr\, d\theta = \frac{11}{2}\pi}$$

END