HCMIU - Calculus I - Final Test Sample

Semester 1 - Year: 2024 $\backsim$ 2025 - Duration : 120 minutes

Date Modified : Wednesday, January 15^{th}, 2025

INSTRUCTIONS:

- Use of calculator is allowed. Each student is allowed one doubled-sized sheet of reference material (size A4 of similar). All other documents and electronic devices are forbidden
- You must explain your answers in detail; no points will be given for the answer alone.
- There are a total of 10 (ten) questions. Each one carries 10 points

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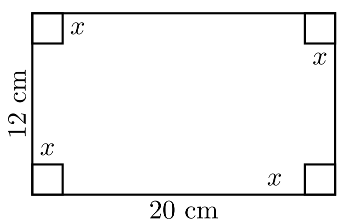
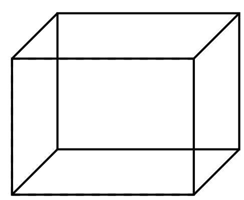
1. Find the limit

$$\lim_{x\to\infty} x \ln\left(1+\frac{2}{x}\right)$$

2. You are constructing a cardboard box with the dimension 12cm by 20cm. You then cut equal-size squares from each corner so you may fold the edges. What are the dimensions of the box with the largest volume? Refer to the figure below.





3. Use the Newton's method to find an approximate solution of the equation with the initial $x_0 = -3$, correct to 7 decimal places.

$$\frac{x^3}{3}+\frac{x^2}{2}+3=0$$

4. Let $F(x) = \int_x^{x^2} f(t)\,dt$. Assume that $F'(1) = 3$. Find $f(1)$.

5. Use the L'Hospital rule to find the limit

$$\lim_{x\to 0} \frac{\int_0^{3x^2} \ln\left(t^2+2\right)dt}{x^2}.$$

6. Evaluate the integral

$$\int_0^1 \frac{x^2}{(x^3+1)^4}\,dt$$

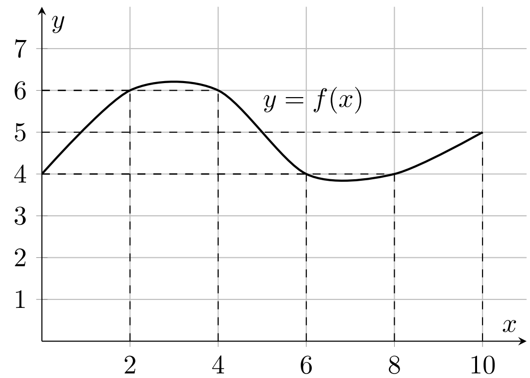
7. Evaluate the improper integral

$$\int_0^{\infty} x^2 e^{-2x}\,dx.$$

8. Find the volume of the solid generated by revolving the region under the curve $y = \frac{5}{x}$ about the x-axis, for x ranging from $x = 3$ to $x = \infty$.

9. Let R be the region in the first quadrant bounded by the curves $y = x^3$ and $y = 2x - x^2$. Calculate the area of R and the volume obtained by rotating R about the y-axis.

10. Approximate the area under the curve $f(x)$ between $x = 0$ and $x = 10$ (as shown in the figure below), using the Trapezoidal Rule with n = 5 sub-intervals.



END

SUGGESTED ANSWER

Question 1. Apply L'hospital Rule for the following outputs that have forms: $\frac{0}{0}$ & $\frac{\infty}{\infty}$, other than those such as ∞^{∞} will be effective with natural logarithm method.

$$\lim_{x\to\infty} x\ln\left(1+\frac{2}{x}\right) = \lim_{x\to\infty}\frac{\ln(1+\frac{2}{x})}{\frac{1}{x}} \quad \left(\frac{0}{0}\right) \overset{\text{L'H}}{=} \lim_{x\to\infty}\frac{2x^2}{x(x+2)} = \lim_{x\to\infty}\frac{2x}{x+2} = 2$$

Question 2. Given initial length of each sides and after cutting equal-size squares, the current length is subtracted by 2x, exceptionally the height is flipped with x. In order to find the possible maximum volume, sketch the derivative test with the following revised components with width: $12-2x$ (cm), height: x (cm) and length $20-2x$ (cm) $(x \le 6)$

$$\Longrightarrow V(x) = x(12-2x)(20-2x) \Longleftrightarrow V'(x) = 240-120x+12x^2 = 0 \Longrightarrow \begin{cases} x_1 = \frac{16+2\sqrt{19}}{3} & \text{(Excluded since } x_1 > 0) \\ x_2 = \frac{16-2\sqrt{19}}{3} & \text{(Accepted)} \end{cases}$$

x	$-\infty$		0		$\frac{16-2\sqrt{19}}{3}$		6		$+\infty$
$f'(x)$				$-$	0	$+$			
$f(x)$			$V(0)$	$\nearrow$	$V\left(\frac{16-2\sqrt{19}}{3}\right)$	$\searrow$	$V(0)$		

$$\Longrightarrow V_{\max}(x) = V\left(\frac{16-2\sqrt{19}}{3}\right) = 263.68(cm^3)$$

Question 3. Let $f(x) = \frac{x^3}{3} + \frac{x^2}{2} + 3 \Longrightarrow f'(x) = x^2 + x \quad \& \quad f(x_0) = -1.5, \quad f(x_0) = 6$

Now we are all set to apply Newton's method in identifying precisely approximate value:

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

- $x_1 = -3 - \frac{-1.5}{6} \approx -\frac{11}{4}$
- $x_2 = -\frac{11}{4} - \frac{f(x_1)}{f'(x_1)} \approx -\frac{628}{231}$
- $x_3 = -\frac{628}{231} - \frac{f(x_2)}{f'(x_2)} \approx -2.718142564$
- $x_4 = -2.718142564 - \frac{f(x_3)}{f'(x_3)} \approx -2.718142458$
- $x_5 = -2.718142458 - \frac{f(x_4)}{f'(x_4)} \approx \boxed{-2.718142458}$

Question 4. Fundamental Theorem Of Calculus I

$$\frac{d}{dx}\left[F(x)\right]=\frac{d}{dx}\left[\int_x^{x^2} f(t)\,dt\right]\Longrightarrow F'(x)=\frac{d}{dx}\left[\int_0^{x^2} f(t)\,dt\right]+\frac{d}{dx}\left[\int_x^{0} f(t)\,dt\right]$$

$$=\frac{d}{du}\cdot\frac{du}{dx}\left[\int_0^{u} f(t)\,dt\right]+\frac{d}{dv}\cdot\frac{dv}{dx}\left[\int_v^{0} f(t)\,dt\right]$$

$$=2x\cdot\frac{d}{du}\left[\int_0^{u} f(t)\,dt\right]+\frac{d}{dv}\left[\int_v^{0} f(t)\,dt\right]$$

$$=2x\cdot\left[F(u)-F(0)\right]+\frac{d}{dv}\left[F(0)-F(v)\right]$$

$$=2x\cdot f(u)-f(v)=2x\cdot f(x^2)-f(x)$$

$$\boxed{F'(1)=3\Longrightarrow F'(1)=2\cdot 1\cdot f(1)-f(1)\Longrightarrow f(1)=3}$$

Question 5. Do similarly to Question 4 + L'hospital rule

$$\lim_{x\to 0}\frac{\int_0^{3x^2}\ln(t^2+2)\,dt}{x^2}\quad\left(\frac{0}{0}\right)\overset{\text{L'h}}{=}\lim_{x\to 0}\frac{\frac{d}{dx}\left[\int_0^{3x^2}\ln\left(t^2+2\right)\,dt\right]}{2x}$$

$$=\lim_{x\to 0}\frac{\frac{d}{du}\cdot\frac{du}{dx}\left[\int_0^{u}\ln\left(t^2+2\right)\,dt\right]}{2x}$$

$$=\lim_{x\to 0}\frac{6x\cdot\frac{d}{du}\left[\int_0^{u}\ln\left(t^2+2\right)\right]}{2x}$$

$$=\lim_{x\to 0}3\cdot\ln\left(9x^4+2\right)=3\ln(2)$$

Question 6. Apply Substitution rule with $u=x^3+1 \implies du=3x^2\,dx \iff \frac{du}{3}=x^2\,dx$

$$\int_0^1\frac{x^2}{\left(x^3+1\right)^4}\,dx=\int_1^2\frac{1}{3}\cdot\frac{du}{u^4}=\frac{1}{3}\cdot\frac{1}{u^4}\,du=\frac{1}{3}\cdot-\frac{u^{-3}}{3}\Big|_1^2=\frac{1}{3}\cdot\left[\frac{2^{-3}}{-3}+\frac{1^{-3}}{3}\right]=\frac{7}{72}$$

Question 7. Improper Integral Type 1 & Integration By Parts

$$\int_0^{\infty}x^2e^{-2x}\,dx\Longrightarrow\lim_{t\to\infty}\int_0^{t}x^2e^{-2x}\,dx.\quad I=\int_0^{t}x^2e^{-2x}\,dx$$

Proceed the partial integration based on LIATE tricks (Logs - Inverse - Algebraric - Trig - Exponential)

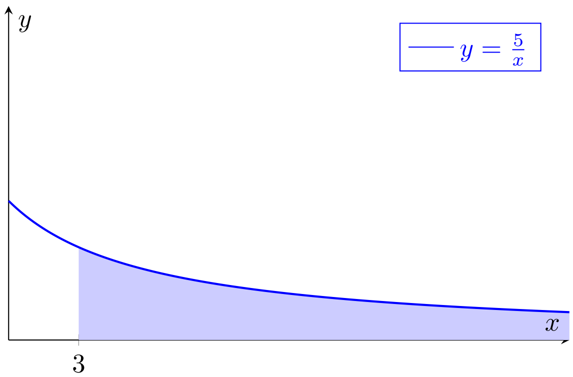
$$\begin{cases}u=x^2\\ dv=e^{-2x}\,dx\end{cases}\Longrightarrow\begin{cases}du=2x\,dx\\ v=-\frac{e^{-2x}}{2}\end{cases}\Longrightarrow I=\left[-\frac{x^2e^{-2x}}{2}\right]_0^t+\int_0^t xe^{-2x}\,dx$$

$$=\frac{-1}{2}\left\{e^{-2t}t^2\right\}+\underbrace{\int_0^t xe^{-2x}\,dx}_{\begin{cases}u=x\\ dv=e^{-2x}\,dx\end{cases}}$$

$$=-\frac{e^{-2t}t^2}{2}-\frac{1}{2}e^{-2t}-\frac{1}{4}e^{-2t}+\frac{1}{4}$$

$$\Longrightarrow\lim_{t\to\infty}\left[\frac{-e^{-2t}t^2}{2}-\frac{3}{4}e^{-2t}+\frac{1}{4}\right]=\lim_{t\to\infty}\left[\frac{-t^2e^{-2t}}{2}\right]\ \left(\frac{\infty}{\infty}\right)+\frac{1}{4}\overset{\text{L'h}}{=}\lim_{t\to\infty}\left[\frac{-2t}{4e^{2t}}\right]+\frac{1}{4}\ \left(\frac{\infty}{\infty}\right)+\frac{1}{4}\overset{\text{L'h}}{=}\left[\frac{-t}{4e^{2t}}\right]+\frac{1}{4}=\frac{1}{4}$$

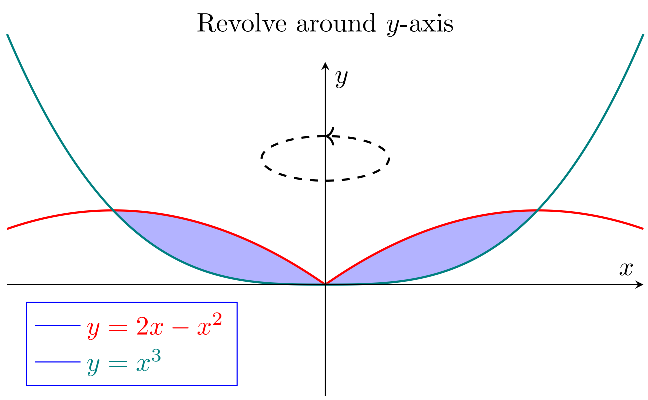
Question 8. Calculating the volume rotated by the function $y=\frac{5}{x}$ around x-axis, which also derives the formation of improper integral type 1 when x approaches ∞



Using Disk Method for calculating the volume of the function revolving around x-axis:

$$V = \int_3^{\infty} \pi \left[\frac{5}{x}\right]^2 dx = 25\pi \cdot \lim_{t\to\infty} \int_3^t \frac{1}{x^2}\, dx = 25\pi \cdot \lim_{t\to\infty} \left[-\frac{1}{t} + \frac{1}{3}\right] = \frac{25\pi}{3}$$

Question 9.



Find the intercepts between two curves by considering horizontal equation:

$$2x - x^2 = x^3 \Longrightarrow \begin{cases} x = 0 \longrightarrow y = 0 \\ x = 1 \longrightarrow y = 1 \\ x = -2 \longrightarrow y = -8 \end{cases}$$

It goes without saying that the region bounded by two curves, revolves from $x = 0 \to x = 2$ (x-axis) and $y = 0 \to y = 1$ (y-axis).

The area of R in the first quadrant can be found by the following formula:

$$V = \int_a^b [f(x) - g(x)]\, dx = \int_0^1 \left[2x - x^2 - x^3\right] dx = \left[x^2 - \frac{x^3}{3} - \frac{x^4}{4}\right]\Big|_0^1 = \frac{5}{12}$$

To find the solid of revolution around y-axis, use the Cylindrical shells. However, the functions must depend on y variables and be converted by quadratic formula

$$y = x^3 \to x = \sqrt[3]{y}, s \;\; y = 2x - x^2 \Longrightarrow x^2 - 2x + y = 0 \Longrightarrow \begin{cases} x_1 = \frac{-b+\sqrt{\Delta}}{2a} \\ x_2 = \frac{-b-\sqrt{\Delta}}{2a} \end{cases} \Longleftrightarrow \begin{cases} x_1 = 1 + \sqrt{1-y} \\ x_2 = 1 - \sqrt{1-y} \;\; \text{(Excluded)} \end{cases}$$

$$
\begin{aligned}
V = \int_a^b 2\pi y g(y)\, dy = \int_0^1 2\pi y \left[1 + \sqrt{1-y} - \sqrt[3]{y}\right] dy &= 2\pi \cdot \int_0^1 \left[y + y\sqrt{1-y} - y\sqrt[3]{y}\right] dy \\
&= 2\pi \cdot \left[\frac{y^2}{2}\Big|_0^1 + \left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right)\Big|_1^0 - \frac{3}{7}y^{\frac{7}{3}}\right] \\
&= 2\pi \cdot \left[\frac{1}{2} - \frac{2}{5} + \frac{2}{3} - \frac{3}{7}\right] \\
&= \frac{71}{105}\pi
\end{aligned}
$$

Question 10. Area approximation under the curve f(x) with 5 sub-intervals using Trapezoidal Rule

$$
\boxed{T_n = \frac{\Delta x}{2} \cdot \left[f(x_0) + 2f(x_1) + ... + 2f(x_{n-1}) + f(x_n)\right], \;\; \Delta x = \frac{b-a}{n}}
$$

$$
\Delta x = 2 \Longrightarrow T_5 = f(0) + 2f(2) + 2f(4) + 2f(6) + 2f(8) + f(10) = 4 + 12 + 12 + 8 + 8 + 5 = 49
$$