Linear Algebra - HCMUS - April 25th, 2024

Semester 2 : Year 2023 - 2024 - Time Duration: 90 minutes

Question 1. (3 points) Given two matrices $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix}$

(a) Calculate $B^T(A - 3I_2^{2024})$.

(b) Solve the equation $AX = B$.

(c) Does there exist such positive integer n that $A^n = \left(\sqrt{2}\right)^n I_2$?

Question 2. (3 points) For each pairs $(x, y) \in \mathbb{R}^2$, denote $A_{x,y} = \begin{pmatrix} -2 & 3 & 5 & 1 \\ 2 & 4 & -3 & x \\ 1 & -5 & 3 & 1 \\ 7 & -21 & -1 & y \end{pmatrix}$

(a) Prove that $\det A_{x,y}$ is independent of x variable.

(b) Find x, y to formulate trapezoidal form of $A_{x,y}$ $\begin{pmatrix} 1 & 0 & 0 & \square \\ 0 & 1 & 0 & \triangle \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Make out appropriate coefficients to fill in $\square, \triangle$.

(c) Does there exist or not a lower triangular matrix B such that for each pairs $(x, y) \in \mathbb{R}^2, B^{-1}A_{x,y}$ forms an upper triangular matrix?

Question 3. (2 points) Solve and analyze the linear system $\begin{cases} \lambda x + y + z = 1 \\ x + \lambda y + z = 1 - \lambda \\ x + y + \lambda z = 2\lambda \end{cases}$ where λ is a real parameter. Find λ so that the linear system of equations has only unique solution (x_0, y_0, z_0), additionally one of the variables must be equal to $-\frac{2}{3}$

Question 4. (2 points) For each $k \in \mathbb{N}$, let $f_k(x) = 1 + 2x + 3x^2 + ... + (k+1)x^k$

(a) Given $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $n \in \mathbb{N}$. Show that $f_n(A) = I_2 + n2^n A$.

(b) Prove the following determinant:

$$\begin{vmatrix} f_0(1) & f_1(1) & f_2(1) & \cdots & f_{2023}(1) \\ f_0(2) & f_1(2) & f_2(2) & \cdots & f_{2023}(2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_0(2024) & f_1(2024) & f_2(2024) & \cdots & f_{2023}(2024) \end{vmatrix} = \prod_{k=1}^{2024} k!.$$

END

SUGGESTED ANSWER

Question 1. Notice the power to 2024th term did not affect to the identical identity as it always remains the same. The (b) question has nothing to do, while brief words were given to the last question - the eigenvalues come into the play to investigate the potential for nth-term square root.

(a)

$$B^T(A - 3I_2^{2024}) = B^T(A - 3I_2) = \begin{pmatrix} -1 & 3 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 5 & -5 \\ 1 & -2 \\ -3 & -4 \end{pmatrix}$$

(b)

$$AX = B \Longrightarrow X = A^{-1}B = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 1 & 3 \\ 4 & 1 & -1 \end{pmatrix}$$

(c) Let $A - \lambda I$ be the eigenvalue equation, thus we forms the determinant for it.

$$\det\begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1 = 0 \Longrightarrow \lambda = 1 \pm i$$

Convert the eigenvalues to the polar form:

$$z = re^{i\theta} = \sqrt{2}e^{\pm i\frac{\pi}{4}} \Longrightarrow z^n = (\sqrt{2})^n e^{\frac{n\pi i}{4}}$$

Using Euler's method to approximate the exponential (explicitly derive the condition)

$$e^{\frac{i\pi}{4}} = \left(\cos\frac{n\pi}{4} + i\cdot\sin\frac{n\pi}{4}\right) \Longrightarrow \left(\cos\frac{n\pi}{4} + i\cdot\sin\frac{n\pi}{4}\right) = 1 \text{ iff} \frac{n\pi}{4} = k2\pi \Longleftrightarrow \boxed{n = 8k}$$

Hence, for smallest positive n = 8. The multiplication satisfies $A^8 = (\sqrt{2})^8 I_2 = \begin{pmatrix} 16 & 0 \\ 0 & 16 \end{pmatrix}$

Question 2. Since we want to eliminate the x coefficient in the determinant, we perform column-wise operation on the 2nd one. Consequently, the determinant must be 0 to show that the row-reduced echelon form (RREF) reflects the singularity - finding y, additionally we could also derive x variable. The short question for (c) is doesn't exist, but validating the explanation is tedious and out of the author's knowledge.

(a)

$$\det(A) = \sum_{j=1}^{4}(-1)^{2+j}a_{2j}M_{2j} = -2C_{12} + 4C_{22} + 3C_{23} + xC_{42}$$

$$= -2\begin{vmatrix} 3 & 5 & 1 \\ -5 & 3 & 1 \\ -21 & -1 & y \end{vmatrix} + 4\begin{pmatrix} -2 & 5 & 1 \\ 1 & 3 & 1 \\ 7 & -1 & y \end{pmatrix} + 3\begin{vmatrix} -2 & 3 & 1 \\ 1 & -5 & 1 \\ 7 & -21 & y \end{vmatrix} + x\begin{vmatrix} -2 & 3 & 5 \\ 1 & -5 & 3 \\ 7 & -21 & -1 \end{vmatrix} = 91(1-y)$$

(b) The matrix is linearly dependent if $\det(A) = 0 \Longleftrightarrow y = 1$

$$\Longrightarrow \begin{pmatrix} -2 & 3 & 5 & 1 \\ 2 & 4 & -3 & x \\ 1 & -5 & 3 & 1 \\ 7 & -21 & -1 & 1 \end{pmatrix} \Longleftrightarrow \begin{pmatrix} 1 & -5 & 3 & 1 \\ 0 & 14 & -9 & x-2 \\ 0 & -7 & 11 & 3 \\ 0 & 14 & -26 & -6 \end{pmatrix} \Longleftrightarrow \begin{pmatrix} 1 & -5 & 3 & 1 \\ 0 & 14 & -9 & x-2 \\ 0 & -7 & 11 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Longleftrightarrow \begin{pmatrix} 1 & -5 & 3 & 1 \\ 0 & 14 & -9 & x-2 \\ 0 & 0 & \frac{13}{2} & \frac{1}{2}x+2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Deriving the third equation,

$$\begin{pmatrix} 0 & 0 & 1 & \frac{1}{13}x + \frac{4}{13} \end{pmatrix} \iff \frac{1}{13}x + \frac{4}{13} = 6 \iff \boxed{x = 74}$$

Continue with the elimination by replacing x value

$$\begin{pmatrix} 1 & -5 & 3 & 1 \\ 0 & 14 & -9 & 72 \\ 0 & 0 & \frac{13}{2} & 39 \end{pmatrix} \iff \begin{pmatrix} 1 & 0 & 0 & 28 \\ 0 & 1 & 0 & 9 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \iff \boxed{\square = 28, \triangle = 9}$$

Question 3. Perform the Gaussian elimination with matrix form:

$$\begin{pmatrix} \lambda & 1 & 1 & 1 \\ 1 & \lambda & 1 & 1-\lambda \\ 1 & 1 & \lambda & 2\lambda \end{pmatrix} \implies \begin{pmatrix} 1 & \lambda & 1 & 1-\lambda \\ \lambda & 1 & 1 & 1 \\ 1 & -1 & \lambda & 2\lambda \end{pmatrix} \implies \begin{pmatrix} 1 & \lambda & 1 & 1-\lambda \\ 0 & 1-\lambda^2 & 1-\lambda & 1-\lambda+\lambda^2 \\ 0 & -1-\lambda & \lambda-1 & 3\lambda-1 \end{pmatrix}$$

$$\implies \begin{pmatrix} 1 & \lambda & 1 & 1-\lambda \\ 0 & 1-\lambda^2 & 1-\lambda & 1-\lambda+\lambda^2 \\ 0 & 0 & \frac{\lambda^2-\lambda}{\lambda+1} & \frac{4\lambda^2+\lambda}{1+\lambda} \end{pmatrix} \iff \begin{cases} z = \frac{4\lambda+1}{\lambda-1} \\ y = \frac{\lambda+2}{1-\lambda} \end{cases} \quad x = \frac{-2\lambda+2}{\lambda-1}$$

Atleast one of the values in (x_0, y_0, z_0) is equal to $-\frac{2}{3}$, try with each respective values we find that y-value make λ possible to solve.

$$y = \frac{\lambda+2}{1-\lambda} = -\frac{2}{3} \iff \boxed{\lambda = -8}$$

Question 4. Hint: Watch closely the power pattern with singular matrix A.

(a)

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \implies A^2 = 2\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} .A^n = 2^{n-1}A (n \geq 1)$$

Formally rewrite the polynomial for each $k \in \mathbb{N}$

$$\begin{aligned} f_n(x) = \sum_{k=0}^{n}(k+1)x^k \implies \quad f_n(A) &= \sum_{m=0}^{n}(m+1)A^m \\ &= 1 \cdot A^0 + \sum_{m=1}^{n}(m+1)A^m \\ &= I_2 + \sum_{m=1}^{n}(m+1)\cdot A^m \\ &= I_2 + \underbrace{\sum_{m=1}^{n}(m+1)\cdot 2^{m-1}}\, A \quad (\text{since } A^m = 2^{m-1}A) \end{aligned}$$

Since the summation is not in definitive form, such methods like generating functions and induction are acquired to calculate $\sum_{m=1}^{n}(m+1)\cdot 2^{m-1}$.

$$S(n) = \sum_{m=1}^{n}(m+1)\cdot 2^{m-1} = \frac{1}{2}\left(\sum_{m=1}^{n} m \cdot 2^m + 2^m\right)$$

Calculate $\sum_{m=1}^{n} m \cdot 2^m$. Let $G = \sum_{m=0}^{n} r^m$ be general generating function.

$$\Longrightarrow G = \sum_{m=1}^{n} m \cdot 2^m = \frac{1 - r^{n+1}}{1 - r^n} \Longrightarrow G' = \sum_{m=0}^{n} m r^{m-1}$$

Multiply both sides with r to formulate the isomorphic summation with the target

$$rG' = \sum_{m=0}^{n} m \cdot r^m \Longleftrightarrow \sum_{m=0}^{n} m \cdot r^m = r\frac{1 - (n-1)r^n + nr^{n+1}}{1 - r^2}$$

$$\sum_{m=0}^{n} m \cdot r^m = 0 \cdot r^0 + \sum_{m=1}^{n} m \cdot r^m \Longrightarrow \sum_{m=1}^{n} m \cdot r^m = r\frac{1 - (n-1)r^n + nr^{n+1}}{1 - r^2}$$

Place in the current geometric value r = 2

$$\sum_{m=1}^{n} m2^m = 2\left[1 - (n+1)2^n + n2^{n+1}\right] = 2 + (n-1)2^{n+1}$$

$$\Longrightarrow S(n) = \frac{1}{2}\left[2 + (n-1)2^{n+1} + 2^{n+1} - 2\right] = n2^n \Longrightarrow \boxed{f_n(A) = I_2 + n2^n A} \quad (n \geq 1) \quad \text{(Q.E.D)}$$

(b) Hint: Vandermonde Matrix

$$\delta = \begin{vmatrix} 1 & x_0 & x_0^2 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & \dots & x_1^n \\ \dots & \dots & \dots & \dots & \dots \\ 1 & x_n & x_n^2 & \dots & x_n^n \end{vmatrix} = \prod_{0 \leq i < j \leq n} (x_j - x_i)$$

The determinant above is the composition of Vandermonde matrix and coefficient matrix of polynomial, thus

$$\begin{aligned} M = V \cdot C \longrightarrow \det(M) = \det(V) \cdot \det(C) &= \prod_{1 \leq i < j \leq n} (x_j - x_i) \cdot \prod_{k=0}^{n-1} (a_k) \\ &= \prod_{k=2}^{n} \prod_{i=1}^{k-1} (k - i) \cdot \prod_{k=0}^{n-1} (k+1) \\ &= \prod_{k=2}^{n} (k-1)! \cdot n! \end{aligned}$$

$$\boxed{\det(M) = n! \cdot \prod_{k=1}^{n-1} k! = \prod_{k=1}^{n} k!}$$

This determinant is valid for 2024th term (Q.E.D).

END