Linear Algebra - HCMUS - June 13rd, 2024
Semester 2 : Year 2023 - 2024 - Time Duration: 90 minutes

Question 1. (2.5 points) Do the following requests

(a) Is the set χ comprising all matrices whose traces are 0 in $M_n(\mathbb{R})$ a subspace of $M_n(\mathbb{R})$? If valid, then find $\dim\chi$.

(b) In $\mathbb{R}^4$ space, given $V = \langle(-7,-6,-1,4),(-2,3,1,-1),(4,5,1,-3)\rangle$ and U is a solution space of linear system $\begin{cases} x_1+2x_2+x_3+5x_4 &= 0 \\ 2x_1-2x_2+2x_3+x_4 &= 0 \\ -4x_1+10x_2-4x_3+7x_4 &= 0 \\ x_1+x_3+2x_4 &= 0 \end{cases}$. Find a basis for $U+V$ and $U\cap V$.

Question 2. (2 points) In $\mathbb{R}_3[x]$, given the following polynomials

$$f_1 = (1+x)^2, f_2 = 1-x^2+x^3 \text{ and } f_3 = 2x+3x^2$$

(a) Show that linear set $\mathcal{B} = \{f_1, f_2, f_3\}$ is linearly independent. Find a polynomial $f_4 \in \mathbb{R}^3[x]$ to make $\mathcal{B}\cup\{f_4\}$ a basis for $\mathbb{R}_3[x]$.

(b) Given $U = \langle f_1, f_2, f_3\rangle$ and $f = a_0 + a_1x + a_2x^2 + a_3x^3 \in \mathbb{R}_3^x$. Find the condition of coefficients a_0, a_1, a_2, a_3 such that $f \in U$. With that constraint, calculate $[f]_{\mathcal{B}}$.

(c) Does there exist a basis $\mathcal{B}'$ for U such that $(\mathcal{B}\to\mathcal{B}')^{2024} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ -3 & 0 & -1 \end{pmatrix}$

Question 3. (2.5 points) Let $\mathcal{B}_1 = \{u_1 = (1,1,2), u_2 = (2,1,0), u_3 = (-1,-3,1)\}$ be a basis for $\mathbb{R}^3$ and $\mathcal{B}_2 = \{v_1 = (5,2), v_2 = (4,1)\}$ be a basis for $\mathbb{R}^2$. Knowing that $f \in \mathcal{L}(\mathbb{R}^3,\mathbb{R}^2)$ satisfies $f(u_1) = v_1, f(u_2) = v_2, f(u_3) = 6v_1 - 9v_2$. Consequently, find $\dim \operatorname{Im}(f)$, $\dim \operatorname{Ker}(f)$, and $[f]_{\mathcal{B}_1,\mathcal{B}_2}$.

Question 4. (2 points) Let $A = \begin{pmatrix} 5 & 4 & x \\ 0 & 2 & 0 \\ -3 & -4 & y \end{pmatrix}$ be isomorphic to $B = \begin{pmatrix} 17 & 5 & -80 \\ 6 & 4 & -32 \\ 3 & 1 & -14 \end{pmatrix}$. Find x, y and A^{2024}.

Question 5. (1 point) Given $A = (a_{ij}) \in M_2(\mathbb{R}), B = (b_{ij}) \in M_3(\mathbb{R})$ and $C = (c_{ij}) \in M_4(\mathbb{R})$. Set

$$X = \begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 & a_{12} & 0 \\ 0 & b_{11} & 0 & 0 & b_{12} & b_{13} & 0 & 0 \\ 0 & 0 & c_{11} & c_{12} & 0 & c_{13} & 0 & c_{14} \\ 0 & 0 & c_{21} & c_{22} & 0 & c_{23} & 0 & c_{24} \\ 0 & b_{21} & 0 & 0 & b_{22} & b_{23} & 0 & 0 \\ 0 & b_{31} & 0 & 0 & b_{32} & b_{33} & 0 & 0 \\ 0 & 0 & c_{31} & c_{32} & 0 & c_{33} & 0 & c_{34} \\ a_{21} & 0 & 0 & 0 & 0 & 0 & a_{22} & 0 \\ 0 & 0 & c_{41} & c_{42} & 0 & c_{43} & 0 & c_{44} \end{pmatrix}.$$

Prove that $P_X(\lambda) = P_A(\lambda)P_B(\lambda)P_C(\lambda)$ and if A, B, C are diagonalizable, then X is also diagonalizable.

END

SUGGESTED ANSWER

Question 1. Given a set $\chi \in M_n(\mathbb{R}), \operatorname{tr}(\chi) = 0$

(a) A valid subspace must satisfy two conditions: Closure under addition (zero vector) and multiplication

- $v + w = \operatorname{tr}(v) + \operatorname{tr}(w) = 0$
- $c \cdot v = c \cdot \operatorname{tr}(v) = 0$

Hence, χ is a subspace for $M_n(\mathbb{R})$. Therefore, the set of matrices are dependent, resulting from zero summation on the trace, and the domain for the set are off-diagonals $(n^2 - n)$. For the dimension of χ, by Rank - Nullity theorem:

$$\dim(\mathrm{D}) = \dim(\operatorname{Ker}(tr)) + \dim(\operatorname{Im}(tr)) \iff \dim(\mathrm{D}) = n^2 - n + n - 1$$

$$\implies \boxed{\dim(\mathrm{D}) = \dim(\chi) = n^2 - 1}$$

(b) Find the basis for separate sets, then find their general basis. Since the solution is only 0, thus we find no intersection and the dimension for $U \cup W$ is 0. Apply Rank - Nullity theorem once again to find their union space.

$$V = \begin{pmatrix} -7 & -6 & -1 & 4 \\ -2 & 3 & 1 & -1 \\ 4 & 5 & 1 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} -7 & -6 & -1 & 4 \\ 0 & \frac{33}{7} & \frac{9}{7} & -\frac{15}{7} \\ 0 & \frac{11}{7} & \frac{3}{7} & -\frac{5}{7} \end{pmatrix} \longrightarrow \begin{pmatrix} -7 & -6 & -1 & 4 \\ 0 & \frac{33}{7} & \frac{9}{7} & -\frac{15}{7} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & -\frac{1}{11} & -\frac{2}{11} \\ 0 & 1 & \frac{3}{1} & -\frac{5}{11} \end{pmatrix} \implies \begin{cases} x_1 = \frac{1}{3}t_2 - \frac{1}{3}t_1 \\ x_2 = t_1 \\ x_3 = \frac{5}{2}t_2 - \frac{11}{3}t_1 \end{cases}$$

The vector set for V is:

$$\boxed{x \in \left\{ \begin{bmatrix} -\frac{1}{3} \\ 1 \\ -\frac{11}{3} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{3} \\ 0 \\ \frac{5}{3} \\ 1 \end{bmatrix} \right\}}, \quad \dim(V) = 2$$

Check for linear independence

$$\alpha_1 \cdot c_1 + \alpha_2 \cdot c_2 = 0 \implies \begin{cases} -\frac{1}{3}c_1 + \frac{1}{3}c_2 = 0 \\ -\frac{1}{3}c_1 + \frac{5}{3}c_2 = 0 \\ c_1 = 0 \\ c_2 = 0 \end{cases} \implies c_1 = c_2 = 0$$

Hence, the vector set is a basis for V. Perform the similar process for subspace U

$$\begin{pmatrix} 1 & 2 & 1 & 5 \\ 2 & -2 & 2 & 1 \\ -4 & 10 & -4 & 7 \\ 1 & 0 & 1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & 5 \\ 0 & -6 & 0 & -9 \\ 0 & 18 & 0 & 27 \\ 0 & -2 & 0 & -3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 1 & 5 \\ 0 & -6 & 0 & -9 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \implies \begin{cases} x_1 = -\frac{7}{2}t_2 - t_1 \\ x_2 = -\frac{3}{2}t_2 \\ x_3 = t_1 \\ x_4 = t_2 \end{cases}$$

The vector set for U is:

$$\boxed{x \in \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{7}{2} \\ -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix} \right\}}$$

Check for linear independence

$$\alpha_1 \cdot c_1 + \alpha_2 \cdot c_2 = 0 \Longrightarrow \begin{cases} -t_1 - \frac{7}{2}t_2 = 0 \\ -\frac{3}{2}c_2 = 0 \\ c_1 = 0 \\ c_2 = 0 \end{cases} \Longrightarrow c_1 = c_2 = 0$$

Find the solution for the intersection $U \cap V$

$$\begin{bmatrix} -\frac{1}{3} & 1 & -\frac{11}{3} & 0 \\ \frac{1}{3} & 0 & \frac{5}{3} & 1 \\ -1 & 0 & 1 & 0 \\ -\frac{7}{2} & -\frac{3}{2} & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} -\frac{1}{3} & 1 & -\frac{11}{3} & 0 \\ 0 & 1 & 2 & 1 \\ 0 & -3 & 12 & 0 \\ 0 & -12 & \frac{77}{2} & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} -\frac{1}{3} & 1 & -\frac{11}{3} & 0 \\ 0 & 1 & 2 & 1 \\ 0 & -3 & 12 & 0 \\ 0 & 0 & -\frac{19}{2} & 0 \end{bmatrix} \Longrightarrow x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The unique solution for the intersection is zero vector, thus the dimension for $\dim(U \cap V) = 0$ and for the union of U, V: $\dim(U \cup V) = \dim(U) + \dim(V) - \dim(U \cap V) = 4$

Question 2. Observe that the degree for 3rd-order polynomial is 4, which is also the rank. We could rewrite the set $\mathcal{B}$ under the vector form and likewise the procedures. Additionally the problem asks us to add another polynomial such that the nullspace is only zero vector (linearly independent), thereby forms a basis for $\mathbb{R}_3[x]$

(a)

$$f = \alpha_1 \cdot c_1 + \alpha_2 \cdot c_2 + \alpha_3 \cdot c_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \cdot c_1 + \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} c_2 + \begin{bmatrix} 0 \\ 2 \\ 3 \\ 0 \end{bmatrix} \cdot c_3 \Longrightarrow \begin{cases} c_1 + c_2 = 0 \\ 2c_1 + 2c_3 = 0 \\ c_1 - c_2 + 3c_3 = 0 \\ c_2 = 0 \end{cases} \Longrightarrow c_1 = c_2 = 0$$

Hence, $\mathcal{B} = \{f_1, f_2, f_3\}$ is linearly independent.

(b) If we try to add $f_4 = x^4$ or $f_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. Then, $\begin{cases} c_1 + c_2 = 0 \\ 2c_1 + 2c_3 = 0 \\ c_1 - c_2 + 3c_3 = 0 \\ c_2 + c_4 = 0 \end{cases}$

This indicates that vector added did not increase the span from initial linear combination (non-zero component on 4th coordinate). Therefore, $B \cup \{f_4\}$ is a basis for $\mathbb{R}_3[x]$

(c) Solve the combination with respect to a vector a

$$f = c_1 \cdot f_1 + c_2 \cdot f_2 + c_3 \cdot f_3 = a \Longrightarrow \begin{pmatrix} 1 & 1 & 0 & a_1 \\ 2 & 0 & 2 & a_2 \\ 0 & -2 & 3 & a_3 \\ 0 & 1 & 0 & a_3 \end{pmatrix} \longrightarrow \boxed{a_1 = 2a_3}$$

From the condition of coefficients, place in the linear systems to uncover the sets

$$\Longrightarrow \begin{cases} x_1 = a_0 - \frac{1}{2}a_1 \\ x_2 = \frac{1}{2}a_1 \\ x_3 = a_0 \end{cases} \Longrightarrow \boxed{x \in \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} \right\}}$$

Check for linear independence, likewise the process which would give us $c_1 = c_2 = 0$. Hence, we could derive the basis of f with respect to $\mathcal{B}$

$$[f]_{\mathcal{B}} = \left([f_1]_{\mathcal{B}}\,[f_2]_{\mathcal{B}}\right) = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$$

(d) The linear basis transformation from $\mathcal{B}$ to $\mathcal{B}'$ would not be possible as the matrix is singular with $\det = 0$, so the basis $\mathcal{B}'$ does not exist.

Question 3. The key detail is in the transformation of $f_{(\mathcal{B}_1)}$ regarding $\mathcal{B}_2$, i.e the feasibility of forming linear combination of the latter over the former.

$$[f]_{\mathcal{B}_1,\mathcal{B}_2} = \left(f(u_1)_{\mathcal{B}_2} f(u_2)_{\mathcal{B}_2} f(u_3)_{\mathcal{B}_2}\right) = \begin{pmatrix} 1 & 0 & 6 \\ 0 & 1 & -9 \end{pmatrix}$$

Where,

- $$[f(u_1)]_{\mathcal{B}_2} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} = \alpha_1 \cdot \begin{pmatrix} 5 \\ 2 \end{pmatrix} + \alpha_2 \cdot \begin{pmatrix} 4 \\ 1 \end{pmatrix} \Longrightarrow \vec{\alpha} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$
- $$[f(u_2)]_{\mathcal{B}_2} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} = \alpha_1 \cdot \begin{pmatrix} 5 \\ 2 \end{pmatrix} + \alpha_2 \cdot \begin{pmatrix} 4 \\ 1 \end{pmatrix} \Longrightarrow \vec{\alpha} = \begin{pmatrix} 0 & 1 \end{pmatrix}$$
- $$[f(u_3)]_{\mathcal{B}_2} = \begin{pmatrix} -6 \\ 3 \end{pmatrix} = \alpha_1 \cdot \begin{pmatrix} 5 \\ 2 \end{pmatrix} + \alpha_2 \cdot \begin{pmatrix} 4 \\ 1 \end{pmatrix} \Longrightarrow \vec{\alpha} = \begin{pmatrix} 6 & -9 \end{pmatrix}$$

Remark: The image of the function spans the pivot of columns and the kernel spans the nullspace.

$$\mathrm{Ker}(f) \approxeq N(f) \Longrightarrow \dim(\mathrm{Ker}) = 1,\ \ \mathrm{Im}(f) \approxeq C(f) \Longrightarrow \dim(\mathrm{Im}(f)) = 2$$

Question 4. A is isomorphic to B if they contains the same trace, determinant and eigenvalues. A factorization to the power 2024th can be either the diagonals or Jordan's blocks (explicitly to say, happens when exists repeated eigenvalues).

- Find x, y

$$\mathrm{tr}(A) = \mathrm{tr}(B) \Longleftrightarrow 7 + y = 7 \Longrightarrow \boxed{y = 0},\ \ \det(A) = \det(B) \Longleftrightarrow 6x = 12 \Longrightarrow \boxed{x = 2}$$

- Factorization of A

$$\det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 4 & 2 \\ 0 & 2-\lambda & 0 \\ -3 & -4 & -\lambda \end{vmatrix} = \lambda^3 + 7\lambda^2 - 16\lambda + 12 = 0 \Longleftrightarrow \lambda_1 = 3, \lambda_2 = 2\ \ (2)$$

- Obviously there exist repeated eigenvalues with respective eigenvectors $\vec{x} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$, which forms Jordan factorization J:

$$J_3 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \Longrightarrow J_3 = \lambda I_3 + N = \begin{pmatrix} J_2 & \\ & J_1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- The Jordan matrix to the power of n is expressed as binomial coefficient of $(\lambda I_3 + N)$

$$\boxed{J_3^n = (\lambda I_3 + N)^n = \binom{n}{0}\lambda^n I_3^n N^0 + \binom{n}{1}\lambda^{n-1} I_3^{n-1} N + ... + \binom{n-1}{n}\lambda I_3 N^{n-1} + N^n}$$

$$\implies J_2^{2024} = \sum_{n=0}^{2024}\binom{2024}{n}(\lambda I_2 + N)^n = 2^{2024} \cdot I_2^{2024} \cdot N^0 + 2024 \cdot 2^{2023} I_2^{2023} \cdot N = \begin{pmatrix} 2^{2024} & 2024 \cdot 2^{2023} & 0 \\ 0 & 2^{2024} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- J_1 is simply the power to the other eigenvalue: 3^{2024}. Therefore, we obtain the 2024th power Jordan matrix and matrix with corresponding eigenvectors

$$\boxed{A^{2024} = XJX^{-1} = \begin{pmatrix} 0 & -2 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2^{2024} & 2024 \cdot 2^{2023} & 0 \\ 0 & 2^{2024} & 0 \\ 0 & 0 & 3^{2024} \end{pmatrix}\begin{pmatrix} 0 & -2 & -2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1}}$$

Question 5. Rearrange the X matrix into ordered diagonal of sub matrices A, B and C.

$$X = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{11} & b_{12} & b_{13} & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{21} & b_{22} & b_{23} & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{31} & b_{32} & b_{33} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{11} & c_{12} & c_{13} & c_{14} \\ 0 & 0 & 0 & 0 & 0 & c_{21} & c_{22} & c_{23} & c_{24} \\ 0 & 0 & 0 & 0 & 0 & c_{31} & c_{32} & c_{33} & c_{34} \\ 0 & 0 & 0 & 0 & 0 & c_{41} & c_{42} & c_{43} & c_{44} \end{pmatrix} \longrightarrow \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}$$

The general determinant for eigenvalues, and characteristic equation is

$$\det(A - \lambda I) = \det(A - \lambda I)\det(B - \lambda I)\det(C - \lambda I) \iff P_X(\lambda) = P_A(\lambda)P_B(\lambda)P_C(\lambda)$$

Since A, B, C are diagonalizable:

$$A = S\Lambda_1 S^{-1},\ B = S\Lambda_2 S^{-1},\ C = S\Lambda_3 S^{-1} \implies \Lambda_1 = S^{-1}AS, \quad \Lambda_2 = S^{-1}BS, \quad \Lambda_3 = S^{-1}CS$$

Thereby, the factorization of X matrix:

$$X = S\Lambda S^{-1} \implies \Lambda = S^{-1}XS = \begin{pmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 0 & \Lambda_3 \end{pmatrix} = \begin{pmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & D_3 \end{pmatrix}$$

As a result, X is diagonalizable. (Q.E.D)

END