

HCMIU - Calculus I - Final-term  
 Semester 1 - Duration : 90 minutes  
 Answer Modified : Sunday, January 5<sup>th</sup>, 2025

- The answers are only for reference because the scale can vary with respect to each lecturers. It is advisable to stick to your instructors for achieving absolute scores.
- The work is being done personally by 1<sup>st</sup> year student.

### SUGGESTED ANSWER

1. Evaluate the limit

$$\lim_{x \rightarrow 0} \frac{\int_0^{2x} [e^t + \sin t] dt}{\ln(1 + 10x)}$$

By replacing  $x = 0$ , this equation has the type of  $\left(\frac{0}{0}\right)$ . Therefore, we apply L'hospital rule:

$$I = \lim_{x \rightarrow 0} \frac{\frac{d}{dx} \int_0^{2x} [e^t + \sin t] dt}{\frac{d}{dx} \ln(1 + 10x)}$$

Substitution Rule § Fundamental Theorem Of Calculus I:  $u = 2x \implies \frac{du}{dx} = 2$

$$\implies \frac{d}{du} \cdot \frac{du}{dx} \int_0^u [e^t + \sin t] dt \iff \frac{d}{du} \cdot 2 \int_0^u [e^t + \sin t] dt$$

$$\iff 2 \cdot \frac{d}{du} [F(u) - F(0)] \iff 2 \cdot [e^{2x} + \sin(2x)]$$

$$\implies I = \lim_{x \rightarrow 0} \frac{2 \cdot [e^{2x} + \sin 2x] \cdot (1 + 10x)}{10} = \frac{2 \cdot [e^{2 \cdot 0} + \sin(2 \cdot 0)] \cdot (1 + 10 \cdot 0)}{10} = \frac{1}{5}$$

2. Find the linear approximation of the function  $f(x) = \sqrt[3]{1+x}$  at  $x_0 = 0$ . Use this approximation to approximate the number  $\sqrt[3]{0.97}$  and  $\sqrt[3]{1.1}$ .

Linear approximation of a function is identical to the tangent line:

$$y = f(x) = L(x) \approx f'(x_0) \cdot (x - x_0) + f(x_0)$$

Now, we need to find the prime and the value of the  $f(x)$  at  $x_0 = 0$ .

$$f'(x) = \frac{1}{3} \cdot \frac{1}{\sqrt[3]{(1+x)^2}} \implies f'(x_0) = \frac{1}{3}$$

$$f(x_0) = 1$$

$$\implies L(x) \approx \frac{1}{3} \cdot (x - 0) + 1 \approx \frac{1}{3}x + 1$$

- $\sqrt[3]{1.1} = \sqrt[3]{1+0.1} \implies x = 0.1 \implies L(x) \approx \frac{1}{3} \cdot 0.1 + 1 \approx 1.0(3)$
- $\sqrt[3]{0.97} = \sqrt[3]{1-0.03} \implies x = -0.03 \implies L(x) \approx -\frac{1}{3} \cdot 0.03 + 1 \approx 0.99$

3. A spherical snowball is melting in such a way that its radius is decreasing at a rate of 1 cm/min. At what rate is the volume of the snowball decreasing when the radius is 9 cm?

Formula of the volume of a sphere:  $V = \frac{4}{3}\pi R^3$

$$\implies \frac{dV}{dt} = \frac{4}{3} \cdot \pi \cdot 3 \cdot R^2 \cdot \frac{dR}{dT} \iff \frac{dV}{dt} = \frac{4}{3} \cdot \pi \cdot 3 \cdot 9^2 \cdot 1 = 324\pi \approx 1017,88(\text{cm}^3/\text{m})$$

4. (a) Show that the equation  $x^3 + x + 4 = 0$  has a unique real root on  $\mathbb{R}$ .

$$\text{Let } y = f(x) = x^3 + x + 4 \implies f'(x) = 3x^2 + 1 > 0, \forall x.$$

The function is continuous on  $(-\infty, \infty)$ . Therefore, it does not satisfy Rolle's Theorem.

On the other hand, there exists a c unique real root on the interval  $(-2, 0)$ :

$$\text{Given } x_1 = -2, \quad x_2 = 0 \quad \longrightarrow \quad f(x_1) = -6, \quad f(x_2) = 4.$$

$$f(x_1) < 0 < f(x_2) \longrightarrow -2 < c < 0 \quad \text{or} \quad f(c) = 0. \quad (\text{Intermediate Value Theorem})$$

- (b) Use Newton's method to approximate the root of  $x^3 + x + 4 = 0$  correct to seven decimal places. Let  $x_0 = -1.5$  be the initial approximation.

$$\text{Let } f(x) = x^3 + x + 4 \implies f'(x) = 3x^2 + 1 \quad \& \quad f(x_0) = -0.875, \quad f'(x_0) = 7.75$$

Now we are all set to apply Newton's method in identifying precisely approximate value:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- $x_1 = -1.5 - \frac{-0.875}{7.75} \approx -1.3870968$
- $x_2 = -1.3870968 - \frac{-0.0559231}{6.7721125} \approx -1.3788389$
- $x_3 = -1.3788389 - \frac{-2.828832 \cdot 10^{-4}}{6.7035901} \approx -1.3787967$
- $x_4 \approx \boxed{-1.3787967}$

5. Find the derivative of the function

$$G(x) = \int_x^{3x^2} \ln(t^2 + 1) dt.$$

For this type of function, we can use Fundamental Theorem of Calculus I to find its derivative. However, the integral bound is not ranging from a variable to a constant, which conflicts to the theorem.

Therefore, it is appropriate if we split into two distinctive bounds at  $x = 0$

$$\Rightarrow G(x) = \int_0^{3x^2} \ln(t^2 + 1) dt - \int_0^x \ln(t^2 + 1) dt$$

Substitution rule for each integral:  $u = 3x^2 \iff du = 6x \cdot dx \iff \frac{du}{dx} = 6x$  &  $v = x \longrightarrow \frac{dv}{dx} = 1$

$$\begin{aligned} \Rightarrow \frac{d}{dx}[G(x)] &= \frac{d}{dx} \int_0^{3x^2} [\ln(t^2 + 1)] dt - \frac{d}{dx} \int_0^x [\ln(t^2 + 1)] dt \\ &= \frac{d}{du} \cdot \frac{du}{dx} \int_0^u [\ln(t^2 + 1)] dt - \frac{d}{dv} \cdot \frac{dv}{dx} \int_0^v [\ln(t^2 + 1)] dt \\ &= 6x \cdot \frac{d}{du} \int_0^u [\ln(t^2 + 1)] dt - \frac{d}{dv} \int_0^v [\ln(t^2 + 1)] dt \\ &= 6x \cdot \frac{d}{du} [F(u) - F(0)] - \frac{d}{dv} [F(v) - F(0)] = 6x \cdot \ln(9x^4 + 1) - \ln(x^2 + 1) \end{aligned}$$

6. Evaluate the improper integral

$$\int_3^\infty \frac{1}{(x+6)\sqrt{x+6}} dx.$$

The integral above is improper type 1, thus we can also write this as (for convenience):

$$\int_3^\infty \frac{1}{(x+6)^{\frac{3}{2}}} dx = \lim_{t \rightarrow \infty} \int_3^t \frac{1}{(x+6)^{\frac{3}{2}}} dx = \lim_{t \rightarrow \infty} \left. \frac{-2}{(x+6)^{\frac{1}{2}}} \right|_3^t = \lim_{t \rightarrow \infty} \frac{-2}{(t+6)^{\frac{1}{2}}} + \frac{2}{3} = \frac{2}{3} \quad (\text{Convergent})$$

7. An oil storage tank cracks at time  $t = 0$  and oil then leaks from the tank at a rate of  $r(t) = t \cdot e^{-0.01 \cdot t}$  liters per minute.

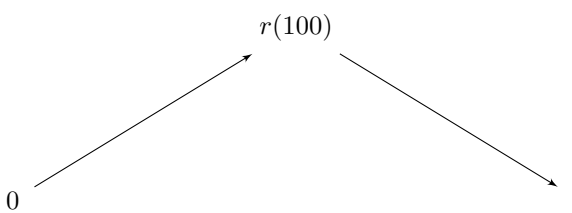
(a) [5 points] Find the time at which the rate has its maximum value.

(b) [5 points] How much oil leaks out during the first ten minutes?

**Notes:** Derivative of a function indicates the rate of change (velocity) of the oil leaking at  $t$  time

$$r(t) = t \cdot e^{-0.01 \cdot t} \implies r'(t) = e^{-0.01 \cdot t} (1 - 0.01 \cdot t) = 0 \iff \boxed{t = 100}$$

Sketch the derivative test:

$x$	0	100	$+\infty$
$f'(x)$	+	0	-
$f(x)$			

At  $t = 100$ , the rate of the oil leaking reaches maximum value  $r(100) = 100 \cdot e^{-0.01 \cdot 100} \approx 36.79(\text{l/min})$

During the first 10 minutes, the proportion of oil leaked is:

$$\int_0^{10} r(t) dt = \int_0^{10} t \cdot e^{-0.01 \cdot t} dt = I \quad (1)$$

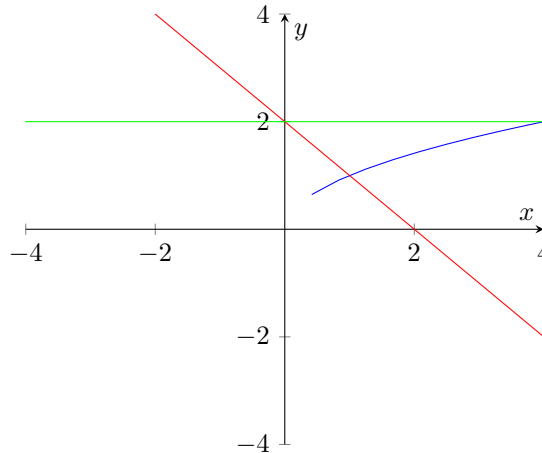
Integration by parts:  $u = t$  &  $dv = e^{-0.01 \cdot t} \cdot dt$

$$\Longleftrightarrow du = dt \quad \& \quad v = \frac{e^{-0.01 \cdot t}}{-0.01}$$

From (1)

$$\Longleftrightarrow I = \left[ -100 \cdot t \cdot e^{-0.01 \cdot t} \right]_0^{10} - \int_0^{10} \frac{e^{-0.01 \cdot t}}{-0.01} dt = \frac{-1000}{e^{0.1}} + 10000 \cdot \left[ \frac{-1}{e^{0.1}} + 1 \right] \approx 46.79 \quad (\text{liters})$$

8. Sketch the region enclosed by the curves  $y = 2$ ,  $y = \sqrt{x}$  and  $x + y = 2$ . Then find the area of the region.



The equation of x-intercept:  $y = \sqrt{x}$  and  $y = 2 - x$  is  $2 - x = \sqrt{x} \implies x = 1 \quad \& \quad x = 4$

Area of the region bounded by the curves:

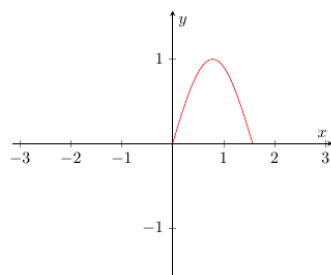
$$A = \int_0^1 2 - (2 - x) dx + \int_1^4 2 - \sqrt{x} dx = \left[ \frac{x^2}{2} \right]_0^1 + \left[ 2x - \frac{2}{3} \cdot \sqrt[3]{x^3} \right]_1^4 = \frac{11}{6}$$

9. An object is moving along a straight line path with an initial velocity of  $v(0) = 5 \text{ m/s}$ . The table below presents the dependence of the acceleration of this object on time  $t$  (in  $\text{m/s}^2$ ). Use the Trapezoidal Rule to approximate the velocity of the object at  $t = 2$  seconds.

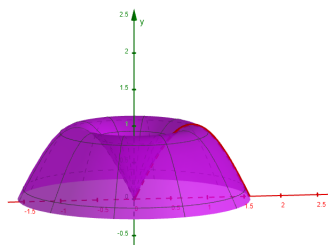
Time $t$ (in seconds)	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
Acceleration (in $\text{m/s}^2$ )	1	1.5	1.8	2	2.5	3	3.5	4	4.5	4.2	4

The velocity of the object at  $t = 2(s)$

$$v(t) = v(0) + \int_0^2 a(t) dt = 5 + \frac{\Delta x}{2} \cdot [a(0) + 2 \cdot a(0.2) + 2 \cdot a(0.4) + \dots + a(2)] \approx 10.9(\text{m/s})$$



(a) Before rotating



(b) After rotating

Figure 1: Solid of revolution of the curve  $y = \sin(2 \cdot x)$

10. The region bounded by  $y = \sin(2x)$  with  $0 \leq x \leq \frac{\pi}{2}$ , and  $y = 0$  is rotated about the y-axis. Find the volume of the solid of revolution.

There are many ways to find the volume of the curve revolving y-axis including Washers method. However the problem would become complicated due to the variable needs to be as well aligned with the axis rotated  $x = \frac{\arcsin y}{2}$  and thus the integration is much more difficult to solve.

Hence we can think of using *cylindrical shell method*:

$$V(x) = \int_0^{\frac{\pi}{2}} 2 \cdot \pi \cdot x \cdot \sin(2x) dx$$

Integration by parts for  $u = x$  &  $dv = \sin(2x) \cdot dx \implies du = dx$  &  $v = \frac{-\cos(2x)}{2}$

$$V(x) = 2 \cdot \pi \left[ \frac{-1}{2} \cdot x \cdot \cos(2x) \Big|_0^{\frac{\pi}{2}} + \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos(2x) dx \right] = \frac{\pi^2}{2}.$$