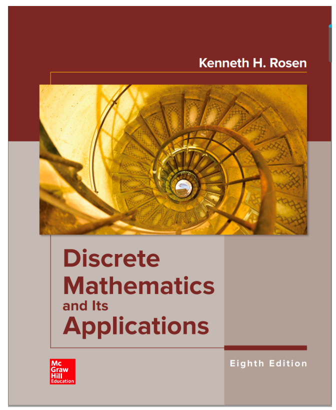
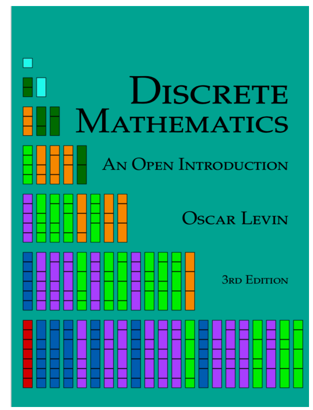
Vietnam National University of HCMC
International University
School of Computer Science and Engineering





Boolean Algebras

Assoc.Prof.Dr.Nguyen Van Sinh
nvsinh@hcmiu.edu.vn

Outline

- Boolean functions
- Representing Boolean functions
- Logic Gates
- Minimization of circuits

Refer: chapter 12

Basic Law of Boolean Algebra

x	y	x · y	x + y
1	1	1	1
1	0	0	1
0	1	0	1
0	0	0	0

'·' mean AND
'+' mean OR

- $1 + 1 = 1$, $1 + 0 = 1$, $0 + 1 = 1$, $0 + 0 = 0$
- $1 \cdot 1 = 1$, $1 \cdot 0 = 0$, $0 \cdot 1 = 0$, $0 \cdot 0 = 0$

Example 1

- $F(x, y) = x \cdot \overline{y}$ ($\bar{y}$: not y)

x	y	$\overline{y}$	$x \cdot \overline{y}$
1	1	0	0
1	0	1	1
0	1	0	0
0	0	1	0

Example 2

- $F(x, y, z) = xy + \overline{z}$

x	y	z	xy	$\overline{z}$	$xy + \overline{z}$
1	1	1	1	0	1
1	1	0	1	1	1
1	0	1	0	0	0
1	0	0	0	1	1
0	1	1	0	0	0
0	1	0	0	1	1
0	0	1	0	0	0
0	0	0	0	1	1

Law of Boolean Algebra

(1) Law of the double negation

- $\overline{\overline{x}} = x$

(2) Idempotent laws

- $x + x = x$
- $x \cdot x = x$

Law of Boolean Algebra

(3) Identity laws

- $x + 0 = x$
- $x \cdot 1 = x$

(4) Domination laws

- $x + 1 = 1$
- $x \cdot 0 = 0$

Law of Boolean Algebra

(5) Commutative laws

- $x + y \quad = \quad y + x$
- $x \cdot y \quad = \quad y \cdot x$

(6) Associative laws

- $x + (y + z) \quad = \quad (x + y) + z$
- $x \cdot (y \cdot z) \quad = \quad (x \cdot y) \cdot z$

Law of Boolean Algebra

(7) Distributive laws

- $x + (y \cdot z) = (x + y) \cdot (x + z)$
- $x \cdot (y + z) = x \cdot y + x \cdot z$

(8) De Morgan's laws

- $\overline{x \cdot y} = \overline{x} + \overline{y}$
- $\overline{x + y} = \overline{x} \cdot \overline{y}$

Law of Boolean Algebra

(9) Absorption laws

- $x + x \cdot y \quad = \quad x$
- $x \cdot (x + y) \quad = \quad x$

(10) Unit property

- $x + \overline{x} \quad = \quad 1$

(11) Zero property

- $x \cdot \overline{x} \quad = \quad 0$

Representing Boolean functions

- There are two problem:
  - Given the values of Boolean function, How can a Boolean expression that represents this function be found?
  - Is there a smaller set of operators that can be used to represent all Boolean functions?

- We check the following examples to illustrate:

Example: find Boolean expression

- Find Boolean expression that represent the functions F(x,y,z) and G(x,y,z) which are given in below table:

x	y	z	F	G
1	1	1	0	0
1	1	0	0	1
1	0	1	1	0
1	0	0	0	0
0	1	1	0	0
0	1	0	0	1
0	0	1	0	0
0	0	0	0	0

- $F(x,y,z) = x\,\bar{y}\,z$
- $G(x,y,z) = x\,y\,\bar{z} + \bar{x}\,y\,\bar{z}$

Example: find function expansion

- Find function expansion for the function $F(x,y,z) = (x + y)\ \bar{z}$ and determine the function **sum-of-products**

$F(x,y,z)$	$=$	$(x + y)\ \bar{z}$
Distributive law	$=$	$x\ \bar{z} + y\ \bar{z}$
Identity law	$=$	$x\ 1\ \bar{z} + 1\ y\ \bar{z}$
Unit property	$=$	$x\ (y + \bar{y})\ \bar{z} + (x + \bar{x})\ y\ \bar{z}$
Distributive law	$=$	$xy\ \bar{z} + x\ \bar{y}\ \bar{z} + xy\ \bar{z} + \bar{x}y\ \bar{z}$
Idempotent law	$=$	$xy\ \bar{z} + x\ \bar{y}\ \bar{z} + \bar{x}y\ \bar{z}$

Example: find function expansion

- Check in below table where the value of F is 1

x	y	z	$x+y$	$\bar{z}$	$(x+y)\bar{z}$
1	1	1	1	0	0
1	1	0	1	1	1
1	0	1	1	0	0
1	0	0	1	1	1
0	1	1	1	0	0
0	1	0	1	1	1
0	0	1	0	0	0
0	0	0	0	1	0

$$xy\bar{z} + x\bar{y}\bar{z} + \bar{x}y\bar{z}$$

Disjunctive normal form (DNF)

- A DNF is a standardization (or normalization) of a logical formula which is a disjunction of conjunctive clauses (minterm); it is an OR of ANDs also known as a sum of products.

- A DNF formula is in full disjunctive normal form (FDNF) if each of its variables appears exactly once in every minterm. Each minterm (contains n variables) is a conjunction (AND) of one or more literals.

Example: finding a DNF

$$E = \overline{\overline{xy}z}\ \overline{(\overline{x}+y)(x+z)}$$

$$E = (\overline{\overline{xy}} + \overline{z})((\overline{\overline{x}+y}) + (\overline{x+z}))$$

$$E = (xy + \overline{z})(\overline{\overline{x}}\,\overline{y} + \overline{x}\,\overline{z})$$

$$E = (xy + \overline{z})(x\overline{y} + \overline{x}\,\overline{z})$$

$$E = xyx\overline{y} + xy\overline{x}\,\overline{z} + x\overline{y}\,\overline{z} + \overline{x}\,\overline{z}\,\overline{z}$$

$$E = 0 + 0 + x\overline{y}\,\overline{z} + \overline{x}\,\overline{z} = x\overline{y}\,\overline{z} + \overline{x}\,\overline{z}$$

Example: finding an FDNF

$$E(x,y,z) = yz + x\bar{z}$$

$$E = (x+\bar{x})yz + x\bar{z}(y+\bar{y})$$

$$E = xyz + \bar{x}yz + xy\bar{z} + x\bar{y}\bar{z}$$

Example: finding an FDNF

$E(x,y,z) = xy+z$

$E = xy(z+\bar{z}) + (x+\bar{x})z$

$E = xyz + xy\bar{z} + xz + \bar{x}z$

$E = xyz+xy\bar{z}+xz(y+\bar{y})+\bar{x}z(y+\bar{y})$

$E = xyz+xy\bar{z}+x\bar{y}z+\bar{x}yz+\bar{x}\bar{y}z$

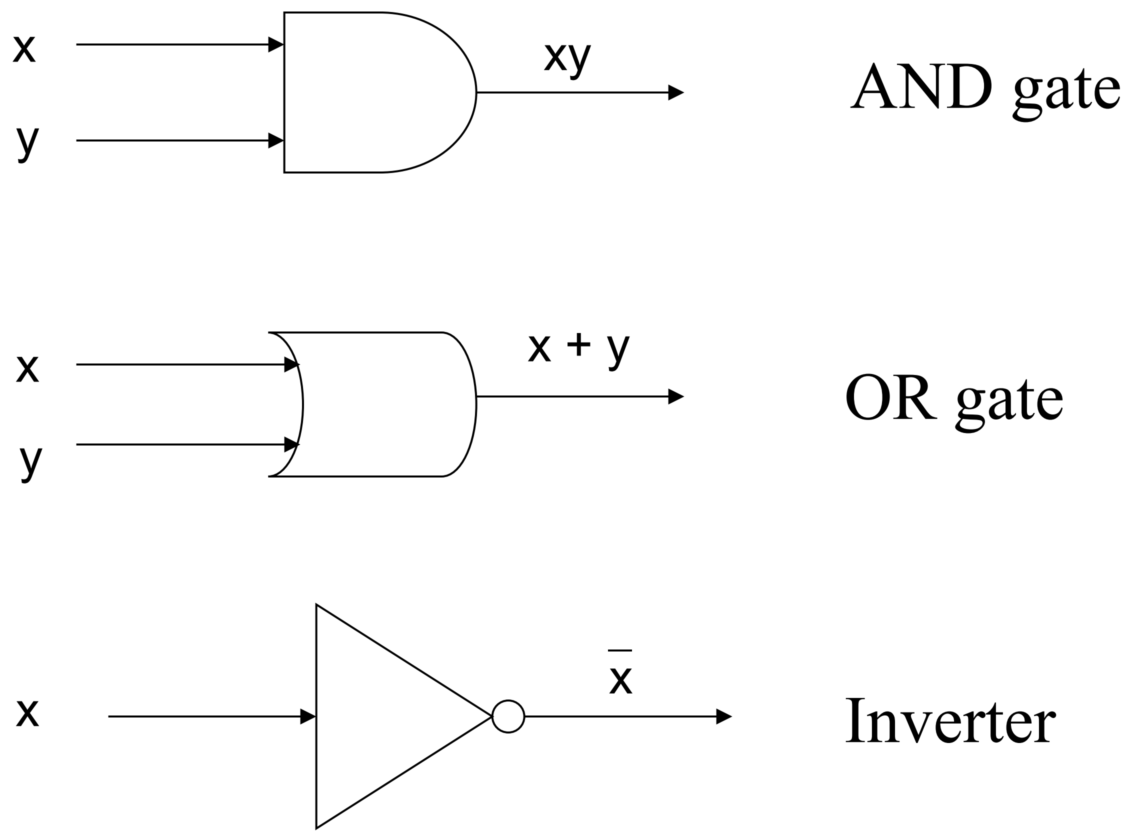
Example: finding an FDNF

$$E = xyz + xy\bar{z} + x\bar{y}z + \bar{x}yz + \bar{x}\,\bar{y}z$$

The true table:

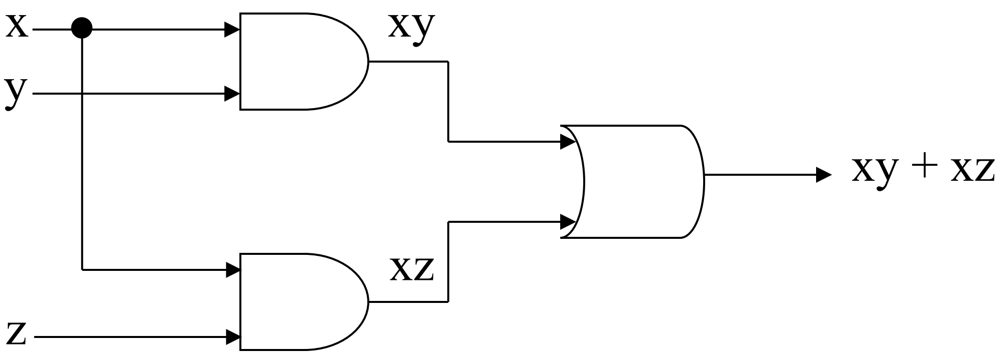
x	y	z	E(x,y,z)
1	1	1	1
1	1	0	1
1	0	1	1
1	0	0	0
0	1	1	1
0	1	0	0
0	0	1	1
0	0	0	0

Logic gates

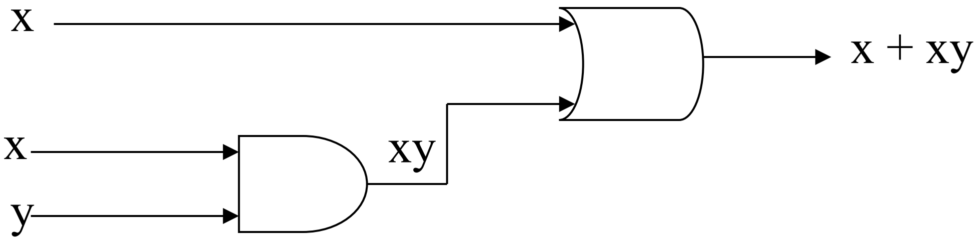


Combination of gate (1)

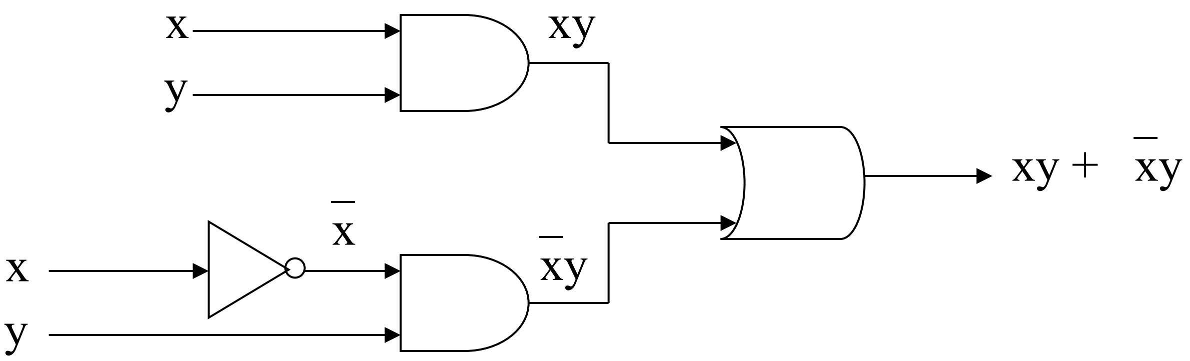
- xy + xz

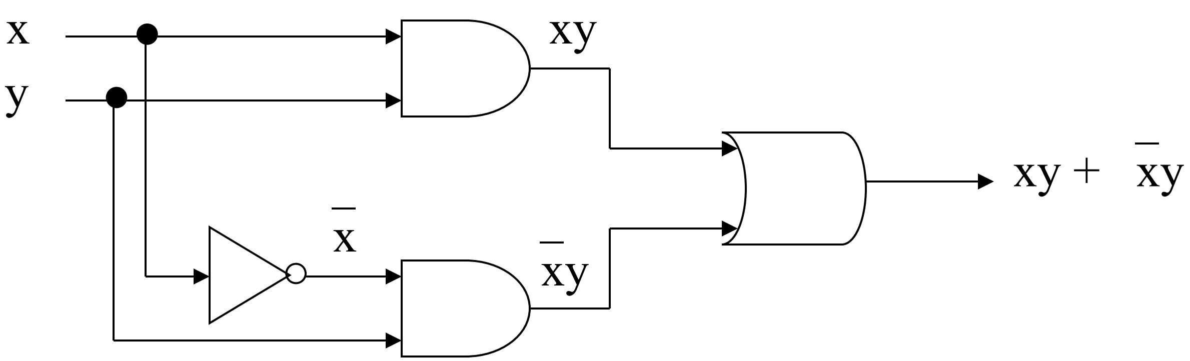


- x + xy



Combination of gate (2)





Example: combination of gate

Construct circuits that produce the following outputs:

1. $(x + y + z)\ \bar{x}\ \bar{y}\ \bar{z}$
2. $(x + y)\ \bar{x}$
3. $\bar{x}\ (y + \bar{z})$
4. $xy + xz + yz$
5. $xy + \bar{x}\ \bar{y}$
6. $xyz + x\ \bar{y}\ \bar{z} + \bar{x}\ y\ \bar{z} + \bar{x}\ \bar{y}\ z$

Minimization of circuits using laws (1)

- $xyz + x\,\bar{y}\,z \;=\; (y + \bar{y})(xz)$
  $= 1 \cdot xz$
  $= xz$

- $x + x \;=\; (x + x) \cdot 1$
  $= (x + x) \cdot (x + \bar{x})$
  $= x + (x \cdot \bar{x})$
  $= x + 0$
  $= x$

Minimization of circuits using laws (2)

- $x + xy = x \cdot 1 + xy$ Identity laws
  $= x(1 + y)$ Distributive laws
  $= x(y + 1)$ Commutative laws
  $= x \cdot 1$ Domination laws
  $= x$ Identity laws
- $x + 1 = (x + 1) \cdot 1$ Identity laws
  $= (x + 1) \cdot (x + \overline{x})$ Unit property
  $= x + 1 \cdot \overline{x}$ Distributive laws
  $= x + \overline{x}$ Identity laws
  $= 1$ Unit property

Minimization of circuits using Karnaugh Maps (k-maps)

Karnaugh Maps of Boolean expression with degree n contains 2^n cells, each cell equivalent to one minterm. Cells are said to be adjacent if the minterms that they represent differ in exactly one literal. Each minterm is assigned number 1 respectively.

Minimization of circuits using Karnaugh Maps (k-maps)

K-maps with 2 variables

	y	$\bar{y}$
x	xy	$x\bar{y}$
$\bar{x}$	$\bar{x}y$	$\bar{x}\bar{y}$

	y	$\bar{y}$
x	xy	$x\bar{y}$
$\bar{x}$	$\bar{x}y$	$\bar{x}\bar{y}$

	y	$\bar{y}$
x	xy	$x\bar{y}$
$\bar{x}$	$\bar{x}y$	$\bar{x}\bar{y}$

K-maps with 3 variables

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	xyz	$xy\bar{z}$	$x\bar{y}\bar{z}$	$x\bar{y}z$
$\bar{x}$	$\bar{x}yz$	$\bar{x}y\bar{z}$	$\bar{x}\bar{y}\bar{z}$	$\bar{x}\bar{y}z$

Example: minimization of circuits using k-maps

$$xy$$

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	xyz	$xy\bar{z}$	$x\bar{y}\bar{z}$	$x\bar{y}z$
$\bar{x}$	$\bar{x}yz$	$\bar{x}y\bar{z}$	$\bar{x}\bar{y}\bar{z}$	$\bar{x}\bar{y}z$

Example: minimization of circuits using k-maps

$$y\bar{z}$$

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	xyz	$xy\bar{z}$	$x\bar{y}\bar{z}$	$x\bar{y}z$
$\bar{x}$	$\bar{x}yz$	$\bar{x}y\bar{z}$	$\bar{x}\bar{y}\bar{z}$	$\bar{x}\bar{y}z$

Example: minimization of circuits using k-maps

$$xz$$

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	xyz	$xy\bar{z}$	$x\bar{y}\bar{z}$	$x\bar{y}z$
$\bar{x}$	$\bar{x}yz$	$\bar{x}y\bar{z}$	$\bar{x}\bar{y}\bar{z}$	$\bar{x}\bar{y}z$

Example: minimization of circuits using k-maps

$\overline{x}$

	yz	$y\overline{z}$	$\overline{y}\,\overline{z}$	$\overline{y}z$
x	xyz	$xy\overline{z}$	$x\overline{y}\,\overline{z}$	$x\overline{y}z$
$\overline{x}$	$\overline{x}yz$	$\overline{x}y\overline{z}$	$\overline{x}\,\overline{y}\,\overline{z}$	$\overline{x}\,\overline{y}z$

Example: minimization of circuits using k-maps

Z

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	xyz	$xy\bar{z}$	$x\bar{y}\bar{z}$	$x\bar{y}z$
$\bar{x}$	$\bar{x}yz$	$\bar{x}y\bar{z}$	$\bar{x}\bar{y}\bar{z}$	$\bar{x}\bar{y}z$

Example: minimization of circuits using k-maps

- $xy + \bar{x}y$

	y	$\bar{y}$
x	1	
$\bar{x}$	1	

$= y$

- $x\,\bar{y} + \bar{x}\,y$

	y	$\bar{y}$
x		1
$\bar{x}$	1	

$= x\,\bar{y} + \bar{x}\,y$

- $x\,\bar{y} + \bar{x}\,y + \bar{x}\,\bar{y}$

	y	$\bar{y}$
x		1
$\bar{x}$	1	1

$= \bar{x} + \bar{y}$

Example: minimization of circuits using k-maps

- $x\,\overline{y}\,\overline{z} + \overline{x}\,\overline{y}\,\overline{z}$

	yz	$y\overline{z}$	$\overline{y}\,\overline{z}$	$\overline{y}z$
x			1	
$\overline{x}$			1	

$= \overline{y}\,\overline{z}$

- $xy\overline{z} + x\,\overline{y}\,\overline{z} + \overline{x}y\overline{z} + \overline{x}\,\overline{y}\,\overline{z}$

	yz	$y\overline{z}$	$\overline{y}\,\overline{z}$	$\overline{y}z$
x		1	1	
$\overline{x}$		1	1	

$= \overline{z}$

- $\overline{x}yz + \overline{x}y\overline{z} + \overline{x}\,\overline{y}\,\overline{z} + \overline{x}\,\overline{y}z$

	yz	$y\overline{z}$	$\overline{y}\,\overline{z}$	$\overline{y}z$
x				
$\overline{x}$	1	1	1	1

$= \overline{x}$

Example: minimization of circuits using k-maps

Expression: $E = xy + \bar{x}y + \bar{x}\bar{y}$

	y	$\bar{y}$
x	1	
$\bar{x}$	1	1

Minimal Expression: $E = \bar{x} + y$

$\bar{x}$

Example: minimization of circuits using k-maps

Expression: $E = xyz + xy\bar{z} + \bar{x}y\bar{z} + \bar{x}\bar{y}z$

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	1	1		
$\bar{x}$		1		1

Minimal Expression: $E = xy + y\bar{z} + \bar{x}\bar{y}z$

$\bar{x}$

Example: minimization of circuits using k-maps

Expression: $E = xyz + xy\bar{z} + x\bar{y}z + \bar{x}yz + \bar{x}\bar{y}z$

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	1	1		1
$\bar{x}$	1			1

Minimal Expression: $E = xy + z$

$\overline{x}$

Example: minimization of circuits using k-maps

Expression: $E = xyz + xy\overline{z} + \overline{x}y\overline{z} + \overline{x}\,\overline{y}z + \overline{x}\,\overline{y}\,\overline{z}$

	yz	$y\overline{z}$	$\overline{y}\,\overline{z}$	$\overline{y}z$
x	1	1		
$\overline{x}$		1	1	1

Minimal Expression: $E = xy + \overline{x}\,\overline{z} + \overline{x}\,\overline{y}$

$\bar{x}$

Example: minimization of circuits using k-maps

Expression: $E = xyz + xy\bar{z} + \bar{x}y\bar{z} + \bar{x}\bar{y}z + \bar{x}\bar{y}\bar{z}$

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	1	1		
$\bar{x}$		1	1	1

Minimal Expression: $E = xy + y\bar{z} + \bar{x}\bar{y}$

Duality

- $1 \longrightarrow 0$
- $0 \longrightarrow 1$
- $+ \longrightarrow \cdot$
- $\cdot \longrightarrow +$

Example: Duality

- $a + b = 0$
  $a \cdot b = 1$

- $(a + 0) + (1 \cdot \overline{a}) = 1$
  $(a \cdot 1) \cdot (0 + \overline{a}) = 0$

- $a \cdot (\overline{a} + b) = a \cdot b$
  $a + (\overline{a} \cdot b) = a + b$

Homework

- Finish all examples in slide 23
- Exercises: 6, 10, 12 page 854.
- Exercises: 4, 6, 10, 12 page 858
- Exercises: 8, 14, 16 page 864.
- Exercises: 8, 12, 14 pages 878

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