Wintersemester 2020/21

Lineare Algebra - Übung 12

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Hausaufgabe 12.1: *Orthogonales Komplement*
Sei $(V, \langle | \rangle)$ ein Skalarproduktraum und $M \subseteq V$.

a) Beweis: $M^\perp = \{\vec{v} \in V \mid \forall \vec{u} \in M : \vec{v} \perp \vec{u}\} \subseteq V$:

1) $\vec{0} \in V \Rightarrow \vec{0} \in M^\perp$.

2) Sei $\lambda, \mu \in \mathbb{R}$ und $\vec{u}, \vec{v} \in M^\perp$.

Zz.: $\lambda\vec{u} + \mu\vec{v} \in M^\perp \Leftrightarrow \forall \vec{m} \in M : (\lambda\vec{u} + \mu\vec{v}) \perp \vec{m} \Leftrightarrow \forall \vec{m} \in M : \langle \lambda\vec{u} + \mu\vec{v} \mid \vec{m}\rangle = 0$.

$\langle \lambda\vec{u} + \mu\vec{v} \mid \vec{m}\rangle = \lambda \underbrace{\langle \vec{u} \mid \vec{m}\rangle}_{0} + \mu \underbrace{\langle \vec{v} \mid \vec{m}\rangle}_{0} = \lambda \cdot 0 + \mu \cdot 0 = 0.$ □

b) Beweis: $\forall \vec{m} \in M : \vec{m}$ ist zu allen $\vec{v} \in M^\perp$ orthogonal $\Rightarrow \vec{m} \in (M^\perp)^\perp \Rightarrow M \subseteq (M^\perp)^\perp$ □

c) Berechnung: $\{u\}^\perp = \{\vec{v} \in V \mid \vec{v} \perp \vec{u}\} = \{\vec{v} \in V \mid \langle \vec{v} \mid \vec{u}\rangle = 0\}$

$\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \rangle \overset{!}{=} 0 \Leftrightarrow x \cdot 1 + y \cdot (-2) + z \cdot 2 \overset{!}{=} 0 \Rightarrow x = 2y - 2z, y = y, z = z \Rightarrow \{\vec{u}\}^\perp = Span(\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix})$

Hausaufgabe 12.2: *Orthonormalisierung*
Sei $[\vec{u}_1, \ldots, \vec{u}_4] \subset \mathbb{R}^4, \vec{u}_1 := \begin{pmatrix} 2/3 \\ -2/3 \\ 0 \\ 1/3 \end{pmatrix}, \vec{u}_2 := \begin{pmatrix} -2/3 \\ -2/3 \\ 1/3 \\ 0 \end{pmatrix}, \vec{u}_3 := \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \vec{u}_4 := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.
Orthogonalisierung:

$$\vec{w}_1 = \vec{u}_1 \; und \; \langle \vec{w}_1 \mid \vec{w}_1 \rangle = (4/9) + (4/9) + (1/9) = 1$$

$$\vec{w}_2 = \vec{u}_2 - \frac{\langle \vec{u}_2 \mid \vec{w}_1\rangle}{\langle \vec{w}_1 \mid \vec{w}_1\rangle} \cdot \vec{w}_1 = \begin{pmatrix} -2/3 \\ -2/3 \\ 1/3 \\ 0 \end{pmatrix} - \frac{(-4/9) + (4/9)}{1} \cdot \begin{pmatrix} 2/3 \\ -2/3 \\ 0 \\ 1/3 \end{pmatrix} = \vec{u}_2$$

$$und \; \langle \vec{w}_2 \mid \vec{w}_2 \rangle = (4/9) + (4/9) + (1/9) = 1$$

$$\vec{w}_3 = \vec{u}_3 - \frac{\langle \vec{u}_3 \mid \vec{w}_1\rangle}{\langle \vec{w}_1 \mid \vec{w}_1\rangle} \cdot \vec{w}_1 - \frac{\langle \vec{u}_3 \mid \vec{w}_2\rangle}{\langle \vec{w}_2 \mid \vec{w}_2\rangle} \cdot \vec{w}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \frac{2/3}{1} \cdot \begin{pmatrix} 2/3 \\ -2/3 \\ 0 \\ 1/3 \end{pmatrix} + \frac{2/3}{1} \cdot \begin{pmatrix} -2/3 \\ -2/3 \\ 1/3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/9 \\ 2/9 \\ 2/9 \end{pmatrix}$$

$$und \; \langle \vec{w}_3 \mid \vec{w}_3 \rangle = \frac{1}{81} + \frac{4}{81} + \frac{4}{81} = \frac{9}{81} = \frac{1}{9}$$

$$\vec{w}_4 = \vec{u}_4 - \frac{\langle \vec{u}_4 \mid \vec{w}_1\rangle}{\langle \vec{w}_1 \mid \vec{w}_1\rangle} \cdot \vec{w}_1 - \frac{\langle \vec{u}_4 \mid \vec{w}_2\rangle}{\langle \vec{w}_2 \mid \vec{w}_2\rangle} \cdot \vec{w}_2 - \frac{\langle \vec{u}_4 \mid \vec{w}_3\rangle}{\langle \vec{w}_3 \mid \vec{w}_3\rangle} \cdot \vec{w}_3$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \frac{1/3}{1} \cdot \begin{pmatrix} 2/3 \\ -2/3 \\ 0 \\ 1/3 \end{pmatrix} - \frac{0}{1} \cdot \begin{pmatrix} -2/3 \\ -2/3 \\ 1/3 \\ 0 \end{pmatrix} - \frac{2/9}{1/9} \cdot \begin{pmatrix} 0 \\ 1/9 \\ 2/9 \\ 2/9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 2/9 \\ -2/9 \\ 0 \\ 1/9 \end{pmatrix} - \begin{pmatrix} 0 \\ 2/9 \\ 4/9 \\ 4/9 \end{pmatrix} = \begin{pmatrix} -2/9 \\ 0 \\ -4/9 \\ 4/9 \end{pmatrix}$$

Normalisierung:

$$\vec{w}_1 = \begin{pmatrix} 2/3 \\ -2/3 \\ 0 \\ 1/3 \end{pmatrix}, \vec{w}_2 = \begin{pmatrix} -2/3 \\ -2/3 \\ 1/3 \\ 0 \end{pmatrix}, \vec{w}_3 = \begin{pmatrix} 0 \\ 1/9 \\ 2/9 \\ 2/9 \end{pmatrix}, \vec{w}_4 = \begin{pmatrix} -2/9 \\ 0 \\ -4/9 \\ 4/9 \end{pmatrix} \; und \; \vec{v}_r = \frac{\vec{w}_r}{||\vec{w}_r||}$$

$$||\vec{w}_1|| = ||\vec{w}_2|| = 1$$

$$||\vec{w}_3|| = \sqrt{(1/81) + (4/81) + (4/81)} = \sqrt{1/9} = \frac{1}{3} \Rightarrow \vec{w}_3 = 3 \cdot \begin{pmatrix} 0 \\ 1/9 \\ 2/9 \\ 2/9 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}$$

$$||\vec{w}_4|| = \sqrt{(4/81) + (16/81) + (16/81)} = \sqrt{36/81} = \sqrt{4/9} = \frac{2}{3} \Rightarrow \vec{w}_4 = \frac{3}{2} \cdot \begin{pmatrix} -2/9 \\ 0 \\ -4/9 \\ 4/9 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 0 \\ -2/3 \\ 2/3 \end{pmatrix}$$

D.h. $\vec{w}_1 = \begin{pmatrix} 2/3 \\ -2/3 \\ 0 \\ 1/3 \end{pmatrix}, \vec{w}_2 = \begin{pmatrix} -2/3 \\ -2/3 \\ 1/3 \\ 0 \end{pmatrix}, \vec{w}_3 = \begin{pmatrix} 0 \\ 1/3 \\ 2/3 \\ 2/3 \end{pmatrix}, \vec{w}_4 = \begin{pmatrix} -1/3 \\ 0 \\ -2/3 \\ 2/3 \end{pmatrix}$

Hausaufgabe 12.3: *Hauptachsentransformation*
Sei $A := \begin{pmatrix} 1 & -1 & -2 \\ -1 & 1 & -2 \\ -2 & -2 & -2 \end{pmatrix} \in M_3(\mathbb{R})$.

Eigenwerte:

$$\begin{vmatrix} X-1 & 1 & 2 \\ 1 & X-1 & 2 \\ 2 & 2 & X+2 \end{vmatrix} \overset{(1)}{=} \begin{vmatrix} X-2 & 1 & 2 \\ 0 & X-1 & 2 \\ (-X/2)+1 & 2 & X+2 \end{vmatrix} \overset{(2)}{=} \begin{vmatrix} X-2 & 1 & 2 \\ 0 & X-1 & 2 \\ 0 & 5/2 & X+3 \end{vmatrix} \overset{(3)}{=} (X-2) \cdot \begin{vmatrix} X-1 & 2 \\ 5/2 & X+3 \end{vmatrix}$$

$$\overset{(4)}{=} (X-2)(X^2+2X-8) \Rightarrow X_1 = 2, X_2 = -1+\sqrt{9} = 2, X_3 = -1-\sqrt{9} = -4$$

(1) Spalte I minus $\frac{1}{2}$ Spalte III

(2) Zeile III plus $\frac{1}{2}$ Zeile I

(3) Laplace nach Spalte I

(4) Sarrus

Eigenvektoren:

$\lambda_{1,2}:$

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{pmatrix} \Rightarrow Span(\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix})$$

$\lambda_3:$

$$\begin{pmatrix} -5 & 1 & 2 \\ 1 & -5 & 2 \\ 2 & 2 & -2 \end{pmatrix} \Rightarrow Span(\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix})$$

Gram-Schmidt Verfahren:

$\lambda_{1,2}:$

$\vec{u}_1 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \vec{u}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$\vec{w}_1 = \vec{u}_1, \vec{w}_2 = \vec{u}_2 - \dfrac{\langle \vec{u}_2 \mid \vec{w}_1 \rangle}{\langle \vec{w}_1 \mid \vec{w}_1 \rangle} \cdot \vec{w}_1$

$\vec{w}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - \dfrac{2}{5} \cdot \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/5 \\ 1 \\ -2/5 \end{pmatrix}$

$||\vec{w}_1|| = \sqrt{5} \Rightarrow \vec{w}_1 = \dfrac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{pmatrix}$

$||\vec{w}_2|| = \sqrt{6/5} \Rightarrow \vec{w}_2 = \dfrac{1}{\sqrt{6/5}} \begin{pmatrix} -1/5 \\ 1 \\ -2/5 \end{pmatrix} = \begin{pmatrix} -\sqrt{30} \\ 1/\sqrt{6/5} \\ -\sqrt{30}/15 \end{pmatrix}$

$\lambda_3:$

$\vec{u}_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

$\vec{w}_3 = \vec{u}_3$

$||\vec{w}_3|| = \sqrt{6} \Rightarrow \vec{w}_3 = \dfrac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{pmatrix}$

Aus den Ergebnissen des Gram-Schmidt Verfahrens folgt $S = (\vec{w}_1, \vec{w}_2, \vec{w}_3)$:

$$S = \begin{pmatrix} -2/\sqrt{5} & -\sqrt{30}/30 & 1/\sqrt{6} \\ 0 & 1/\sqrt{6/5} & 1/\sqrt{6} \\ 1/\sqrt{5} & -\sqrt{30}/15 & 2/\sqrt{6} \end{pmatrix} \ \mathit{und} \ \begin{vmatrix} -2/\sqrt{5} & -\sqrt{30}/30 & 1/\sqrt{6} \\ 0 & 1/\sqrt{6/5} & 1/\sqrt{6} \\ 1/\sqrt{5} & -\sqrt{30}/15 & 2/\sqrt{6} \end{vmatrix} = -1$$

Da $det(S) = -1$ folgt $\tilde{S} = (-\vec{w}_1, \vec{w}_2, \vec{w}_3)$.

Anmerkung: Die Berechnung von $det(S)$ hab ich hier noch ergänzt:

$$det(S) = \begin{vmatrix} -2/\sqrt{5} & -\sqrt{30}/30 & 1/\sqrt{6} \\ 0 & 1/\sqrt{6/5} & 1/\sqrt{6} \\ 1/\sqrt{5} & -\sqrt{30}/15 & 2/\sqrt{6} \end{vmatrix} \overset{(1)}{=} \begin{vmatrix} -2/\sqrt{5} & -\sqrt{30}/30 & 1/\sqrt{6} \\ 0 & 1/\sqrt{6/5} & 1/\sqrt{6} \\ 0 & -\sqrt{30}/12 & 5/2\sqrt{6} \end{vmatrix} \overset{(2)}{=} -\frac{2}{\sqrt{5}} \begin{vmatrix} 1/\sqrt{6/5} & 1/\sqrt{6} \\ -\sqrt{30}/12 & 5/2\sqrt{6} \end{vmatrix}$$

$$\overset{Sarrus}{\Rightarrow} -\frac{2}{\sqrt{5}} \left[\left(\frac{1}{\sqrt{6/5}} \cdot \frac{5}{2\sqrt{6}} \right) + \left(\frac{\sqrt{30}}{12} \cdot \frac{1}{\sqrt{6}} \right) \right] = -\frac{2}{\sqrt{5}} \left(\frac{5\sqrt{5}}{12} + \frac{\sqrt{5}}{12} \right) = -\frac{12\sqrt{5}}{12\sqrt{5}} = -1$$

(1) III+$\frac{1}{2}$I

(2) Laplace nach Spalte I