# Jacobi theta functions from a Gaussian-cosecant convolution

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### 0.1 Background: The Faddeeva, or plasma dispersion, function

A function of some interest is the convolution of a Gaussian with a simple pole,

$$Z(\zeta) \equiv \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-z^2}}{z-\zeta} dz, \quad \text{Im}(\zeta) > 0 \tag{1}$$

defined initially in the upper half $\zeta$-plane. The function is defined and rescaled in various ways, and goes by names such as the Faddeeva function, Kramp function, or plasma dispersion function. Typically $Z(\zeta)$ is defined in the upper half-plane and continued into the lower half. Continued in this way, a convenient form valid throughout the complex $\zeta$-plane is

$$Z(\zeta) = i\sqrt{\pi} e^{-\zeta^2}(1 + \text{erf}(i\zeta)). \tag{2}$$

### 0.2 A convolution with resonance at $z = n \in \mathbb{Z}$

The cosecant function $\pi \csc(\pi z)$ is locally a simple pole at each integer, due to its Laurent series

$$\pi \csc(\pi z) = \sum_{n=-\infty}^{\infty} \frac{z(-1)^n}{z^2 - n^2} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{z-n}, \tag{3}$$

$$\implies \lim_{z \to m} \pi \csc(\pi z) \to \frac{(-1)^m}{z-m}. \tag{4}$$

Now consider the convolution of a Gaussian with the cosecant function,

$$\mathcal{J}(\zeta) \equiv \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \pi \csc(\pi(z-\zeta)) e^{-z^2} dz, \quad \text{Im}(\zeta) > 0. \tag{5}$$

#### 0.2.1 Representation of the convolution as a series of Faddeeva functions

**Theorem 1** *One representation of the integral is*

$$\mathcal{J}(\zeta) = \sum_{n=-\infty}^{\infty} (-1)^n Z(\zeta - n) \tag{6}$$

*as a series in terms of, essentially, Faddeeva functions.*

***Proof***: *Apply the expansion in Eq. 3 and integrate term-by-term.*

An issue with this series representation is that in order to resolve the behavior at large real argument, terms are needed up to $|n| \geq |\text{Re}(\zeta)|$.

### 0.2.2 Representation of the convolution as a Fourier-like series

**Theorem 2** *An alternative representation of the integral is the Fourier-like series*

$$\mathcal{J}(\zeta) = 2\pi i \sum_{n=0}^{\infty} e^{2\pi i(n+\frac{1}{2})\zeta} e^{-(\pi(n+\frac{1}{2}))^2} \tag{7}$$

***Proof**: To find an alternative representation, expand the cosecant in geometric series*

$$\pi \csc(\pi(z-\zeta)) = 2\pi i \frac{e^{-\pi i(z-\zeta)}}{1-e^{-2\pi i(z-\zeta)}} \tag{8}$$

$$= 2\pi i \sum_{n=0}^{\infty} e^{-2\pi i(z-\zeta)(n+\frac{1}{2})}, \tag{9}$$

*also called q-series in terms of the nome $q = \exp(-2\pi i(z-\zeta))$, convergent for $Im(\zeta) > 0$. Then $\mathcal{J}(\zeta)$ may be integrated term-by-term as a series of Gaussian integrals*

$$\mathcal{J} = 2\pi i \sum_{n=0}^{\infty} e^{2\pi i(n+\frac{1}{2})\zeta} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2-2\pi iz(n+\frac{1}{2})} dz \tag{10}$$

$$= 2\pi i \sum_{n=0}^{\infty} e^{2\pi i(n+\frac{1}{2})\zeta} e^{-(\pi(n+\frac{1}{2}))^2} \tag{11}$$

$$= 2\pi i e^{-\zeta^2} \sum_{n=0}^{\infty} e^{-(i\zeta-\pi(n+\frac{1}{2}))^2} \tag{12}$$

The last equality above is another representation similar in form to that for $Z(\zeta)$ as the product of a Gaussian and an error function. However, the form as written in the theorem is more obviously convergent for $\mathrm{Im}(\zeta) > 0$. It is interesting to observe that $\mathcal{J}(\zeta)$ is "half" of a $\vartheta$-function.

### 0.2.3 Continuation of $\mathcal{J}(\zeta)$ into the lower half-plane

Now consider the extension of $\mathcal{J}(\zeta)$ defined in Eq. 5 into the lower half-plane. The series of Eq. 7 in fact gives the desired function.

**Theorem 3** *The series given by Eq. 7 is valid throughout the entire complex $\zeta$-plane as the continuation given by adding the sum of residues when $\zeta$ crosses the real axis from the upper half-plane.*

***Proof**: Denote the function of Eq. 5 as $\mathcal{J}^+(\zeta)$. Using the geometric series for $Im(\zeta) < 0$ gives*

$$\mathcal{J}^-(\zeta) = -2\pi i \sum_{n=0}^{\infty} e^{-2\pi i(n+\frac{1}{2})\zeta} e^{-(\pi(n+\frac{1}{2}))^2}. \tag{13}$$

*Then the jump across the real axis is given by*

$$\Delta\mathcal{J} = \mathcal{J}^+ - \mathcal{J}^- = 2\pi i \sum_{n=0}^{\infty} e^{-(\pi(n+\frac{1}{2}))^2} (e^{2\pi i(n+\frac{1}{2})\zeta} + e^{-2\pi i(n+\frac{1}{2})\zeta}) \tag{14}$$

$$= 4\pi i \sum_{n=0}^{\infty} \cos\Big(\pi(1+2n)\zeta\Big) e^{-(\pi(n+\frac{1}{2}))^2} \tag{15}$$

$$= 2\pi i \vartheta_2(\pi\zeta, e^{-\pi^2}) \tag{16}$$

*in terms of the Jacobi theta function* $\vartheta_2(z,q) = 2\sum_{n=0}^{\infty}\cos((1+2n)z)q^{(n+\frac{1}{2})^2}$.

*On the other hand, the residue of the integrand at any simple pole* $z = \zeta + n$ *is*

$$Res\Big(\pi\csc(\pi(z-\zeta))e^{-\zeta^2}\Big)\Big|_{z=\zeta+n} = (-1)^n e^{-(\zeta+n)^2} \tag{17}$$

$$\implies \sum Res\Big(\pi\csc(\pi(z-\zeta))e^{-\zeta^2}\Big) = \sum_{n=-\infty}^{\infty}(-1)^n\frac{e^{-(\zeta+n)^2}}{\sqrt{\pi}} \tag{18}$$

$$= e^{i\pi\zeta}\sum_{n=-\infty}^{\infty}\frac{1}{\sqrt{\pi}}e^{-(\zeta+n)^2}e^{-i\pi(\zeta+n)} \tag{19}$$

*Recall Poisson's summation formula relating a periodic summation to one of its Fourier transform,*

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty}\hat{f}(n)e^{i2\pi nx}, \tag{20}$$

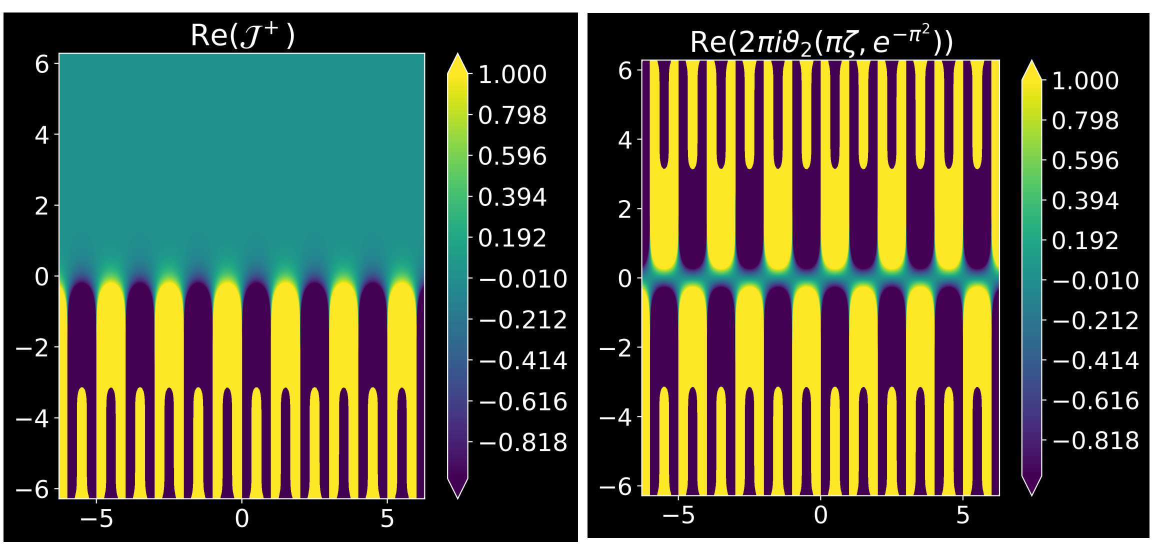
*and observe that the Fourier transform of the summand* $f(x+n)$ *in Eq. 19 is* $\hat{f} = e^{-(\pi(m+\frac{1}{2}))^2}$, *so*

$$\sum_{n=-\infty}^{\infty}(-1)^n\frac{e^{-(\zeta+n)^2}}{\sqrt{\pi}} = \sum_{n=-\infty}^{\infty} e^{2\pi i(n+\frac{1}{2})\zeta}e^{-(\pi(n+\frac{1}{2}))^2} = \vartheta_2(\pi\zeta, e^{-\pi^2}). \tag{21}$$

*Therefore,* $\Delta\mathcal{J} = 2\pi i\sum(Res(\mathcal{J}^+))$. *Adding the sum of residues back to* $\mathcal{J}^-$ *in the lower half-plane results in the desired continued function,*

$$\mathcal{J}(\zeta) = \mathcal{J}^- + 2\pi i\sum(Res(\mathcal{J}^+)) = \mathcal{J}^- + \mathcal{J}^+ - \mathcal{J}^- = \mathcal{J}^+, \quad Im(\zeta) < 0. \tag{22}$$

*This proves the theorem.*




(a) The continued function $\mathcal{J}(\zeta)$ (b) The theta function $2\pi i\vartheta_2(\pi\zeta, e^{-\pi^2})$

Figure 1: Comparison of the function considered ("half" a $\vartheta$-function) and a full theta function, given by the sum of residues of the integrand. Plots are on the complex $\zeta$-plane.