Note on superpotentials in theories of collisional relaxation (Rosenbluth) and gravitational equilibria (Chandrasekhar)

Daniel W. Crews

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1 Introduction

This note reviews the history and basic theory of fourth-order equations for scalar potentials, in both the problem of Newtonian gravitation and also collisional relaxation of collectively-interacting systems as described by the Rosenbluth potentials. The intention of the note is to demonstrate that, due to the parallels with abstract potential theory, the Rosenbluth potentials are dimension-dependent. Due to this, the usual formulation as a sequence of Poisson equations does not lead to an equilibrium normal distribution in a strictly one-dimensional velocity space. This suggests issues with physicality of Rosenbluth potentials used in one- or two-dimensional velocity spaces.

2 The superpotential in Newtonian gravitational theory

Classic potential theory focuses on the scalar potential Φ and vector potential $\boldsymbol{A}$. In Newtonian gravity, or in the electromagnetic Coulomb gauge, the scalar potential satisfies Poisson's second-order equation $\nabla^2\Phi = 4\pi\rho$. The scalar potential Φ, well known to generate vector force fields in static theories by $\boldsymbol{F} = -\nabla\Phi$, is related to its source distribution $\rho(\boldsymbol{x})$ by

$$\Phi(\boldsymbol{x}) = -\int \rho(\boldsymbol{x}')\frac{d\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|}. \tag{1}$$

The scalar potential is also understood as the weighting of the distribution's potential energy by

$$W = \frac{1}{2}\int \rho(\boldsymbol{x})\Phi(\boldsymbol{x})d\boldsymbol{x} = -\frac{1}{2}\int\int \rho(\boldsymbol{x})\rho(\boldsymbol{x}')\frac{d\boldsymbol{x}d\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|}. \tag{2}$$

It's less well known that the scalar potential is itself the Laplacian of a scalar $\nabla^2\chi = -2\Phi$, with χ termed a superpotential by Chandrasekhar [1]. This superpotential χ arises in a variety of situations where the potential energy must be considered as a tensor rather than a scalar quantity, namely

$$W_{ij} = -\frac{1}{2}\int\int \rho(\boldsymbol{x})\rho(\boldsymbol{x}')\frac{(x_i - x_i')(x_j - x_j')}{|\boldsymbol{x}-\boldsymbol{x}'|^3}d\boldsymbol{x}d\boldsymbol{x}'. \tag{3}$$

This situation comes about, for example as considered by Chandrasekhar, in the virial theorem of a gravitating, rotating, magnetized system in general Cartesian coordinates, and the dispersion relation of small oscillations about equilibria of such systems [2]. The particular system considered in Ref. [1] introducing the superpotential was the stability of a rotating ellipsoidal mass distribution.

2.1 The tensor potential and the vector potential of torque

Equation 3 suggests the potential energy to have an associated tensor potential

$$\Phi_{ij}(\boldsymbol{x}) = -\int \rho(\boldsymbol{x}') \frac{(x_i - x_i')(x_j - x_j')}{|\boldsymbol{x} - \boldsymbol{x}'|^3} d\boldsymbol{x}'. \tag{4}$$

The scalar potential still satisfies Poisson's equation, so the tensor potential must be related to the scalar potential in some way. Following Chandrasekhar and Leibovitz, define a vector

$$\boldsymbol{D}(\boldsymbol{x}) \equiv -\int \rho(\boldsymbol{x}') \frac{\boldsymbol{x}'}{|\boldsymbol{x} - \boldsymbol{x}'|} d\boldsymbol{x}' \tag{5}$$

to investigate this relationship. To identify the meaning of $\boldsymbol{D}$, consider the two quantities

$$\frac{\partial D_i}{\partial x_j} = \int \rho(\boldsymbol{x}') \frac{x_i'(x_j - x_j')}{|\boldsymbol{x} - \boldsymbol{x}'|^3} d\boldsymbol{x}' \tag{6}$$

$$-x_i \frac{\partial \Phi}{\partial x_j} = \int \rho(\boldsymbol{x}') \frac{-x_i(x_j - x_j')}{|\boldsymbol{x} - \boldsymbol{x}'|^3} d\boldsymbol{x}' \tag{7}$$

Addition of Eqs. 6 and 7 results in an identity for the tensor potential Φ_{ij}

$$\Phi_{ij} = \frac{\partial D_i}{\partial x_j} - x_i \frac{\partial \Phi}{\partial x_j} = \frac{\partial D_j}{\partial x_i} - x_j \frac{\partial \Phi}{\partial x_i} \tag{8}$$

with the identity symmetric in (i, j) because Φ_{ij} is symmetric by definition. Rearranging Eq. 8 leads to an identity relating $\boldsymbol{D}$ and $\boldsymbol{x}\Phi$ and therefore to an interpretation of $\boldsymbol{D}$,

$$\frac{\partial D_i}{\partial x_j} - \frac{\partial D_j}{\partial x_i} = x_i \frac{\partial \Phi}{\partial x_j} - x_j \frac{\partial \Phi}{\partial x_i} \tag{9}$$

$$\implies \nabla \times \boldsymbol{D} = (\nabla \Phi) \times \boldsymbol{x} \tag{10}$$

$$= \nabla \times (\boldsymbol{x}\Phi). \tag{11}$$

Equation 10 reveals $\boldsymbol{D}$ to be the vector potential of the force field $-\nabla\Phi$'s torque $\boldsymbol{T} = -\boldsymbol{x} \times \nabla\Phi$ in a particular frame of reference. This is the physical interpretation of the field $\boldsymbol{D}$, and unlike Φ is frame-dependent. Equation 11 means that $\boldsymbol{D}$ differs from $\boldsymbol{x}\Phi$ by the gradient of a scalar function,

$$\boldsymbol{D} = \boldsymbol{x}\Phi + \nabla\chi. \tag{12}$$

It is easy to imagine a field with torque; consider the lines of force of a dipole field.

2.2 The Poisson equation for the superpotential χ

The torque $\boldsymbol{T}$ is not independent from the force $-\nabla\Phi$, so there is no gauge freedom in χ. Rather, substitution of Eq. 12 into Eq. 8 leads to an equation for the Hessian matrix of the scalar χ,

$$\frac{\partial^2 \chi}{\partial x_i \partial x_j} = \Phi_{ij} - \Phi\delta_{ij} \tag{13}$$

in terms of the scalar potential Φ and the tensor potential Φ_{ij}. Contraction of Eq. 13 shows that the scalar χ satisfies a Poisson equation with a multiple of the potential Φ as its source density,

$$\nabla^2 \chi = (1 - \nabla \cdot \boldsymbol{x})\Phi. \tag{14}$$

In the typical case of three-dimensional space $\mathrm{div}(\boldsymbol{x}) = 3$, producing the system of Poisson equations

$$\nabla^2 \Phi = 4\pi\rho(\boldsymbol{x}), \tag{15}$$

$$\nabla^2 \chi = -2\Phi. \tag{16}$$

These two equations combine to form a single fourth-order equation for the superpotential,

$$\nabla^4 \chi = -8\pi\rho(\boldsymbol{x}). \tag{17}$$

Given a density $\rho(\boldsymbol{x})$, having solved Eq. 17 the tensor potential Φ_{ij} is obtained by differentiation,

$$\Phi_{ij} = \frac{\partial^2 \chi}{\partial x_i \partial x_j} - \frac{1}{2}(\nabla^2 \chi)\delta_{ij} \tag{18}$$

Because the tensor potential can't be found by the scalar Φ alone, and instead depends on solution of a higher-order differential equation whose source is the mass density, Chandrasekhar considered the scalar χ to be a more fundamental quantity than Φ and termed it the superpotential. Critically, Eq. 14 depends on the dimensionality of the space. In one-dimensional space $\mathrm{div}(\boldsymbol{x}) = 1$ and the superpotential is not necessary as it is superfluous; the tensor potential contains only one quantity, namely the scalar potential Φ. One can verify that the superpotential has the integral form

$$\chi(\boldsymbol{x}) = -\int \rho(\boldsymbol{x}')|\boldsymbol{x} - \boldsymbol{x}'|d\boldsymbol{x}' \tag{19}$$

or the first relative moment of the distribution.

3 Landau's collision operator and its Rosenbluth potentials

Representations of the collision operator, or interaction term, in the kinetic equation

$$\frac{df}{dt} = \Big(\frac{\partial f}{\partial t}\Big)_c \tag{20}$$

from particle-particle scattering are of considerable interest for plasma physicists. Landau developed a form for the collision operator due to accumulated two-body Coulomb scattering [3] and (in hindsight) neglecting dressed-particle effects from collective interactions [4]. By integrating over all solid angles and introducing a physically-motivated ad-hoc cut-off at the Debye length, Landau's operator is of the form of a velocity-space divergence

$$\Big(\frac{\partial f}{\partial t}\Big)_c = \nabla_v \cdot \boldsymbol{J} \tag{21}$$

$$J_i = \Lambda \int a^{ij}(\boldsymbol{v} - \boldsymbol{v}')\Big(f(\boldsymbol{v}')\frac{\partial f}{\partial v_j} - f(\boldsymbol{v})\frac{\partial f}{\partial v'_j}\Big)d\boldsymbol{v}' \tag{22}$$

with Λ the plasma parameter, and where the matrix $a^{ij}(\boldsymbol{z})$ has elements $a^{ij}(\boldsymbol{z}) = \frac{1}{|z|}\Big(\delta_{ij} - \frac{z_i z_j}{|z|^2}\Big)$ for a 3D Coulomb force [5]. Rosenbluth observed that rather than casting the probability source as the divergence of a certain phase-space current, Landau's operator can equivalently be written in a Fokker-Planck form [6]

$$\frac{1}{\Lambda}\Big(\frac{\partial f}{\partial t}\Big)_c = -\frac{\partial}{\partial v_i}\Big(f\langle \Delta v^i\rangle\Big) + \frac{1}{2}\frac{\partial^2}{\partial v^i \partial v^j}\Big(f\langle \Delta v^i \Delta v^j\rangle\Big) \tag{23}$$

in terms of dispersion coefficients $\langle \Delta v^i \rangle$ and $\langle \Delta v^i \Delta v^j \rangle$. In Rosenbluth's consideration, the dispersion coefficients take the form (considering like-particle scattering for example)

$$\langle \Delta v^i \rangle = \frac{\partial h}{\partial v^i}, \tag{24}$$

$$\langle \Delta v^i \Delta v^j \rangle = \frac{\partial^2 g}{\partial v^i \partial v^j}; \tag{25}$$

in other words as derivatives of functions $h(\boldsymbol{v})$ and $g(\boldsymbol{v})$, moment-like integrals of the distribution,

$$H(\boldsymbol{v}) = \int 2 f(\boldsymbol{v}') \frac{1}{|\boldsymbol{v} - \boldsymbol{v}'|} d\boldsymbol{v}', \tag{26}$$

$$G(\boldsymbol{v}) = \int f(\boldsymbol{v}') |\boldsymbol{v} - \boldsymbol{v}'| d\boldsymbol{v}'. \tag{27}$$

The factor of 2 arises from the mass ratio $\frac{m_a + m_b}{m_b}$ in unlike-mass collisions. Chandrasekhar had done early work in the 1950's on both pinch stability [7] and the Fokker-Planck equation for gravitating systems where he identified dynamical friction as the necessary balance to diffusive scattering in equilibrium [8]. Rosenbluth was familiar with Chandrasekhar's work and noted (without reference, as Chandrasekhar 1962 was not yet published) the formal similarity of Eqs. 26 and 27 to potential theory [6]. By analogy, the integral forms of $H(\boldsymbol{v})$ and $G(\boldsymbol{v})$ are inverted to a series of Poisson equations,

$$\nabla_v^2 H = -4\pi (2 f(\boldsymbol{v})), \tag{28}$$

$$\nabla_v^2 G = 2H \tag{29}$$

by use of the identities $\nabla(r^{-1}) = -\boldsymbol{r}/r^3$ and $\nabla \cdot (\boldsymbol{r}/r^3) = 4\pi\delta(\boldsymbol{r})$ (the latter holding only in three-dimensional space). The quantities $H(\boldsymbol{v})$ and $G(\boldsymbol{v})$ are now referred to as Rosenbluth potentials. Therefore one need only solve one fourth-order differential equation for the superpotential $g(\boldsymbol{v})$ to obtain the coefficients of dynamical friction and diffusion for the Fokker-Planck form of the Landau collision operator for a homogeneous, unmagnetized plasma.

3.1 The three-dimensional equilibrium potentials

The functions $H(\boldsymbol{v})$ and $G(\boldsymbol{v})$ represent the potentials of dynamical friction and diffusion respectively. Given a spherically-symmetric Maxwellian distribution

$$f_0(v) = \frac{1}{\pi^{3/2}} e^{-v^2} \tag{30}$$

the potentials of thermal equilibrium can be found by integration [9]. Solution of the first Poisson is an integrable second-order ODE,

$$\nabla^2 H = -8\pi f(v) \quad \Longrightarrow \quad H(v) = 2\frac{\mathrm{erf}(v)}{v} \tag{31}$$

and the solution of the second using Eq. 31 as its source is

$$\nabla^2 G = 2H(v) \quad \Longrightarrow \quad G(v) = -2\Big(\frac{e^{-v^2}}{\sqrt{\pi}} + \mathrm{erf}(v)\Big(v + \frac{1}{2v}\Big)\Big) \tag{32}$$

The dynamical friction force is

$$F_D = -(\nabla_v H) \cdot \hat{v} = \frac{2}{v^2}\Big(\mathrm{erf(v)} + \frac{1}{\sqrt{\pi}}\frac{d}{dv}e^{-v^2}\Big) \tag{33}$$

and the coefficient of diffusion is

$$D_{rr} = (\nabla_v \nabla_v G) \cdot \hat{v}\hat{v} = \frac{1}{v^3}\Big(\mathrm{erf(v)} + \frac{1}{\sqrt{\pi}}\frac{d}{dv}e^{-v^2}\Big) \tag{34}$$

so that the tendencies balance out to a net-zero probability current

$$\boldsymbol{J}_v \cdot \hat{v} = -F_D f + \frac{1}{2}D_{rr}\frac{df}{dv} = 0. \tag{35}$$

3.2 Discussion on the Rosenbluth potentials in one-dimensional velocity space

Based on the discussion in Section 2, the potential theory underlying the inversion of the integral potentials of Eqs. 26 and 27 into differential equations is dimension-dependent. It was observed that the superpotential is meaningless in one-dimensional space. Similarly, in a one-dimensional velocity space the scattering theory breaks down. This appears mathematically as the identity for $\nabla^2(1/r)$ not holding. In a sense, the scattering is blocked by the imposed dimensionality of velocity. This phenomenon is noted and discussed in great detail in the context of the proper Lenard-Balescu collision operator in [10]. Since the Landau operator can be considered as an excellent approximation to the dressed-particle operator, much of the intuition of Ref. [10] continues to hold for the results of the basic two-body inverse-square law scattering approach.

The best approximation for a one-dimensional scattering is discussed in the second-half of Rosenbluth 1957 [6]. The additional two components of velocity can be represented by an expansion of the distribution function in spherical harmonics, assuming that the disturbance is azimuthally-symmetric in velocity space. From a point of view of computational tractability, however, one cannot avoid discretization of at least a 2D velocity-space, accounting for the axial and transverse coordinates of the azimuthally-symmetric velocity space. The issue of dimensionality strongly motivates the use of reduced, linear operators such as the Lenard-Bernstein collision term.

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