# Laplace equation on the circle

December 7, 2020

## 1 The Poisson kernel

Long ago, it was discovered that the solutions to the Dirichlet problem on the unit circle are elegantly expressed using complex analysis. Poisson discovered that the Laplace problem on the circle

$$\varphi_{xx} + \varphi_{yy} = 0, \tag{1}$$

with $\varphi|_{r=1} = \varphi_0(\theta)$ as boundary could be determined in terms of a certain integral kernel,

$$\varphi(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \varphi_0(\phi)\frac{1-r^2}{1-2r\cos(\phi-\theta)+r^2}d\phi \tag{2}$$

today called the Poisson kernel. Later, it was realized that the above form of Poisson's kernel manifests the real part of the analytic function

$$\Omega(z) = \varphi + i\psi = i\psi(0) + \frac{1}{2\pi}\int_0^{2\pi} \varphi_0(\theta)\frac{e^{i\theta}+z}{e^{i\theta}-z}d\theta \tag{3}$$

where $\psi = \psi(r,\theta)$, the conjugate part, gives the value of the corresponding problem with gradient boundary conditions. The value $i\psi(0)$ simply expresses the fact that the problem with a specified gradient is undetermined up to an added constant. Today, the function

$$\Phi(\zeta, z) = \frac{\zeta+z}{\zeta-z} \tag{4}$$

is called the Poisson kernel. It expresses (with $\zeta = e^{i\theta}$) harmonic functions on the circle in integral form in terms of their values on its boundary.

## 2 Solving Laplace's equation for harmonics

The integration $\int_0^{2\pi}$ should be understood as a contour integral about the unit circle $z = e^{i\theta}$ in the complex plane (Fig. 1). In that case, if $\varphi(\theta)$ is also function of $e^{i\theta}$ (on the circle), then the integral of Eqn. 3 can be resolved using Cauchy's residue theorem,

$$\oint_C f(z)dz = 2\pi i \sum \mathrm{Res}(f(z)). \tag{5}$$

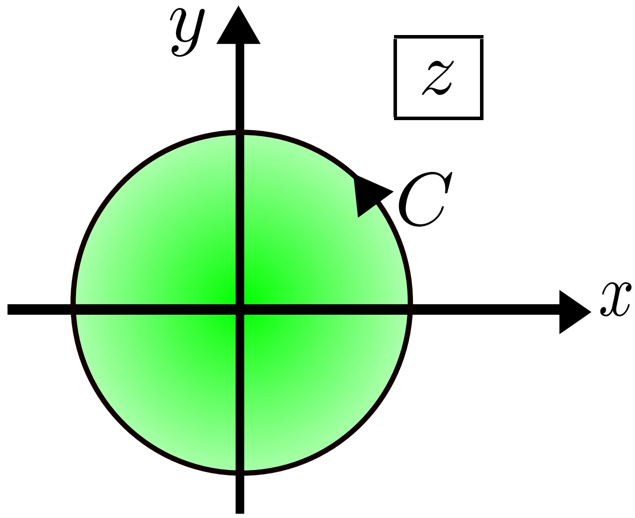


Figure 1: The integration $\int_0^{2\pi} d\theta$ is a closed path on the unit circle $z = e^{i\theta}$.

Suppose that the given boundary value is a fundamental harmonic, $\varphi(\theta) = \sin(n\theta)$. The solution of Laplace's equation is expressed in integral form using the Poisson kernel,

$$\Omega(z) = \frac{1}{2\pi}\int_0^{2\pi} \sin(n\theta)\frac{e^{i\theta}+z}{e^{i\theta}-z}d\theta \tag{6}$$

assuming that $\mathrm{Im}(\Omega)|_{r=0} = 0$ (to fix the arbitrary constant). Expanding the sine,

$$\Omega(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{in\theta}-e^{-in\theta}}{2i}\frac{e^{i\theta}+z}{e^{i\theta}-z}d\theta. \tag{7}$$

Now, one may change variables to $\zeta = e^{i\theta}$, with $d\theta = \frac{d\zeta}{i\zeta}$ to yield an integral on the circle

$$\Omega(z) = -\frac{1}{4\pi}\oint_C \frac{\zeta^{2n}-1}{\zeta^{n+1}}\frac{\zeta+z}{\zeta-z} \equiv -\frac{1}{4\pi}\oint_C f(\zeta,z)d\zeta. \tag{8}$$

The integrand has a simple pole at $\zeta = z$ and a pole of order $n+1$ at the origin, so

$$\oint_C f(\zeta)d\zeta = 2\pi i\Big(\mathrm{Res}(f(\zeta=0), n+1) + \mathrm{Res}(f(\zeta=z),0)\Big). \tag{9}$$

The residue at the simple pole is evaluated simply, while the other requires a limit,

$$\mathrm{Res}(f(\zeta=z),0) = \frac{(z^{2n}-1)2z}{z^{n+1}} = 2(z^n - z^{-n}), \tag{10}$$

$$\mathrm{Res}(f(\zeta=0),n+1) = \frac{1}{n!}\lim_{\zeta\to 0}\frac{\partial^n}{\partial\zeta^n}\Big[(\zeta^{2n}-1)\Big(\frac{\zeta+z}{\zeta-z}\Big)\Big]. \tag{11}$$

For the latter, split the derivatives into two terms and evaluate each with Leibniz's rule,

$$\frac{\partial^n}{\partial\zeta^n}\Big[\zeta^{2n}\Big(\frac{\zeta+z}{\zeta-z}\Big)\Big] = \sum_{m=0}^{n}\binom{n}{m}\partial^m(\zeta^{2n})\partial^{(n-m)}\Big(\frac{\zeta+z}{\zeta-z}\Big), \tag{12}$$

$$-\frac{\partial^n}{\partial\zeta^n}\Big[\Big(\frac{\zeta+z}{\zeta-z}\Big)\Big] = 2\frac{n!}{(\zeta-z)^{n+1}}. \tag{13}$$

When taking the limit $\zeta \to 0$, every term of the first set of derivatives, Eqn. 12, must vanish as $m < 2n$ and every term has at least a linear coefficient of $\zeta$. Only the second set of derivatives, of Eqn. 13, contributes to the residue, so that it is simply

$$\implies \text{Res}(f(\zeta = 0), n + 1) = 2z^{-n}. \tag{14}$$

The residue of the pole at the origin cancels the divergent term of the residue at $\zeta = z$. Therefore, the integration evaluates to simply

$$\Omega(z) = \frac{2\pi i}{-4\pi}(2z^n) = \frac{1}{i}z^n. \tag{15}$$

## 2.1 Harmonic extension of the circle functions

Equation 15 may be understood in polar form $z = re^{i\theta}$,

$$\Omega = \frac{1}{i}r^n e^{in\theta} = \frac{r^n}{i}(\cos(n\theta) + i\sin(n\theta)) \equiv \varphi(r, \theta) + i\psi(r, \theta) \tag{16}$$

revealing the solutions of the Dirichlet and Neumann problems within the circle

$$\varphi(r, \theta) = r^n \sin(n\theta) \tag{17}$$

$$\psi(r, \theta) = -r^n \cos(n\theta) \tag{18}$$

as the harmonic extensions of the Fourier modes on the disk. The factor of $-i$ has appeared by choosing the boundary value $\varphi = \sin(n\theta)$. Instead choosing $\varphi(\theta) = \cos(n\theta)$ gives

$$\Omega(z) = z^n = r^n \cos(n\theta) + ir^n \sin(n\theta) \tag{19}$$

and reveals the complex monomials to be simply the analytic continuation of the Fourier modes of the circle into the entire complex plane (though truly, onto the Riemann sphere!).