Incompressible MHD and Elsässer variables

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1 The scheme of magnetohydrodynamics

Magnetohydrodynamics consists of the fluid conservation laws of mass and momentum along with Laplace's $\boldsymbol{j} \times \boldsymbol{B}$ force and the magnetic Ampère and Faraday laws. The first states $\mu_0 \boldsymbol{j} = \nabla \times \boldsymbol{B}$ and is the $c \to \infty$ limit of the vector part of Maxwell's equations. In its basic form, Faraday's law

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = 0 \tag{1}$$

is the bivector part of Maxwell's equations describing the inductive evolution of $\boldsymbol{B}$. Further, under frame-change the local electric field is related to $\boldsymbol{B}$ by the Lorentz transformation

$$\boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = 0 \implies \boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B} \tag{2}$$

supposing an overall charge neutrality of the fluid. Eliminating $\boldsymbol{E}$, the MHD system consists of

$$\frac{d\rho}{dt} = -(\nabla \cdot \boldsymbol{v})\rho, \tag{3}$$

$$\rho \frac{d\boldsymbol{v}}{dt} = \frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} - \nabla p + \nu \nabla^2 \boldsymbol{v}, \tag{4}$$

$$\frac{d\boldsymbol{B}}{dt} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{v} - (\nabla \cdot \boldsymbol{v})\boldsymbol{B}, \tag{5}$$

having applied the product rule for $\nabla \times (\boldsymbol{v} \times \boldsymbol{B})$. Finally, one must include an equation of state for the pressure such as an isentropic relation.

2 Ideal incompressible MHD

The system is called ideal in the absence of dissipation and resistivity, $\nu = \eta = 0$. Further, for low velocities the velocity field is solenoidal $\nabla \cdot \boldsymbol{v} = 0$. These conditions reduce the system to

$$\rho \frac{d\boldsymbol{v}}{dt} = \frac{1}{\mu_0}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} - \nabla p, \tag{6}$$

$$\frac{d\boldsymbol{B}}{dt} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{v}. \tag{7}$$

Note that Eq. 7 is identical in form to the equation for vorticity $\omega = \nabla \times \boldsymbol{v}$. Thus the right-hand side of the induction equation in incompressible flow is a magnetic flux tube stretching term.

2.1 Divergence of the Laplace force

In an incompressible flow a pressure Poisson equation is obtained in involution from the constraint $\nabla \cdot \boldsymbol{v} = 0$ by considering the divergence of the advective momentum flux,

$$\nabla \cdot \frac{d\boldsymbol{v}}{dt} = \nabla \cdot ((\boldsymbol{v} \cdot \nabla)\boldsymbol{v}) = \partial_i v^j \partial_j v^i. \tag{8}$$

With the presence of additional body forces, their divergence must be considered as well. Since the magnetic field is solenoidal the Laplace force term expands into two components

$$\boldsymbol{j} \times \boldsymbol{B} = \frac{1}{\mu_0}(\boldsymbol{B} \cdot \nabla)\boldsymbol{B} - \nabla\big(\frac{1}{2\mu_0}B^2\big) \tag{9}$$

so that its divergence consists of the two terms

$$\nabla \cdot (\boldsymbol{j} \times \boldsymbol{B}) = \frac{1}{\mu_0}\nabla \cdot ((\boldsymbol{B} \cdot \nabla)\boldsymbol{B}) - \nabla^2\big(\frac{1}{2\mu_0}B^2\big). \tag{10}$$

The first term represents the divergence of magnetic tension, and is formally identical to the momentum flux divergence, while the second term is the Laplacian of magnetic pressure. Therefore, the pressure Poisson equation obtained in involution with $\nabla \cdot \boldsymbol{v} = 0$ is

$$\nabla^2(p + \frac{1}{2\mu_0}B^2) = \frac{1}{\mu_0}\partial_i B^j \partial_j B^i - \partial_i v^j \partial_j v^i. \tag{11}$$

In incompressible MHD, the total pressure $p_t \equiv p + \frac{1}{2\mu_0}B^2$ obeys a Poisson equation. The source term of the pressure field is given by two terms of opposite sign, the divergence of the magnetic tension and momentum flux respectively. The opposite signs of the effective force on a fluid element are responsible for the many kinds of MHD equilibria. Furthermore, Eq. 11 shows that it is possible for the total pressure to be a harmonic function.

2.2 Conservative forms of the equations

With the incompressible constraint considered as the pressure Poisson equation, the ideal MHD equations are given by

$$\partial_t \boldsymbol{v} + \nabla \cdot (\boldsymbol{v}\boldsymbol{v}) = \frac{1}{\mu_0 \rho}(\boldsymbol{B} \cdot \nabla)\boldsymbol{B} - \nabla p_t, \tag{12}$$

$$\partial_t \boldsymbol{B} + \nabla \cdot (\boldsymbol{v}\boldsymbol{B}) = (\boldsymbol{B} \cdot \nabla)\boldsymbol{v}, \tag{13}$$

$$\nabla^2 p_t = \frac{1}{\mu_0}\partial_i B^j \partial_j B^i - \partial_i v^j \partial_j v^j. \tag{14}$$

The Poisson source terms may also be understood as the double contraction of the gradient tensors $\nabla\boldsymbol{v}$, $\nabla\boldsymbol{B}$ with their transposes.

2.3 Symmetrization of the equations using Elsässer's variables

As written the roles played by the fluid velocity and magnetic field appear to be quite different. However, there is a great deal of symmetry between the two evolution equations. The velocity field $\boldsymbol{v}$ serves as the flux of both systems. Further, the magnetic stretching term $(\boldsymbol{B} \cdot \nabla)\boldsymbol{v}$ and the magnetic tension term $(\boldsymbol{B} \cdot \nabla)\boldsymbol{B}$ consist of the same linear operator acting on the two fields. This suggests that the two equations can be combined additively.

Letting $\boldsymbol{v}_A \equiv (\mu_0\rho)^{-1/2}\boldsymbol{B}$, define the Elsässer variables as $\boldsymbol{\mathcal{P}} \equiv \boldsymbol{v} + \boldsymbol{v}_A$ and $\boldsymbol{\mathcal{M}} \equiv \boldsymbol{v} - \boldsymbol{v}_A$. Adding and subtracting Eq. 13 from Eq. 12 produces the symmetrized form

$$\partial_t \boldsymbol{\mathcal{P}} + \nabla \cdot (\boldsymbol{\mathcal{M}}\boldsymbol{\mathcal{P}}) = -\nabla p_t \tag{15}$$

$$\partial_t \boldsymbol{\mathcal{M}} + \nabla \cdot (\boldsymbol{\mathcal{P}}\boldsymbol{\mathcal{M}}) = -\nabla p_t \tag{16}$$

$$\nabla^2 p_t = -\frac{1}{2}\Big(\partial_i \mathcal{M}^j \partial_j \mathcal{P}^i + \partial_i \mathcal{P}^j \partial_j \mathcal{M}^i\Big). \tag{17}$$

This form of the ideal MHD equations is physically signficant. Rescaling the magnetic flux to its Alfvénic form and combining with the velocity field reveals the essential activity of $\boldsymbol{B}$ in magnetohydrodynamics, namely an Alfvénic advection superimposed on the fluid velocity. Further, pressure disturbances have a global character due to the elliptical nature of the pressure Poisson equation. Alfvénic interactions due to the *advective nature* of the magnetic flux are present even in an incompressible state. This is a fully advective phenomenon, as no linearization is required.