Spectral analysis with the Wigner distribution

February 8, 2021

1 Spectrum of the autocorrelation function

A distribution's autocorrelation is essentially a moment with itself given some delay t,

$$\psi(t) = \int_{-\infty}^{\infty} f^*(t-\tau)f(\tau)d\tau. \tag{1}$$

The autocorrelation function is of great importance in stochastic processes, signal processing, etc. A particular instance of it is known as the Wigner distribution or Wigner-Ville transformation, which is the Fourier transformation of the *instantaneous* autocorrelation,

$$W(t,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f^*\Big(t-\frac{\tau}{2}\Big) f\Big(t+\frac{\tau}{2}\Big)e^{-i\omega\tau}d\tau. \tag{2}$$

The Wigner function is the distribution of the spectral intensity of a signal in its *phase space*. Although $W(t,\omega)$ may have negative values, it is the energy distribution because its marginals (or zeroth moments) recover the signal intensity in both space and frequency,

$$\int_{-\infty}^{\infty} W(t,\omega)d\omega = |f(t)|^2, \tag{3}$$

$$\int_{-\infty}^{\infty} W(t,\omega)dt = |\hat{f}(\omega)|^2. \tag{4}$$

1.1 Wigner distributions of periodic functions

Another remarkable property of the distribution is that it is straightforward (though computationally expensive) to calculate once a Fourier analysis of the function $f(t)$ has been determined. To see this, consider a periodic function expressed by its Fourier series,

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t}, \qquad \omega_n = \frac{2\pi}{L}n \tag{5}$$

where $f(t+L) = f(t)$. Note that the function $f(t)$ is continuous in t but discrete in its spectrum. The Wigner function inherits this property. Substituting into Eqn. 2, one has

$$W(t,\omega) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} (c_n e^{i\omega_n t})^*(c_m e^{i\omega_m t})\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i(\omega-(\omega_n+\omega_m)/2)\tau}d\tau \tag{6}$$

which then simplifies from the orthogonality of the Fourier basis into the expression

$$W(t, \omega) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (c_n e^{i\omega_n t})^* (c_m e^{i\omega_m t}) \delta\Big(\omega - \frac{1}{2}(\omega_n + \omega_m)\Big). \tag{7}$$

The summation is on the infinite lattice $(n, m) \in \mathbb{Z}$, yet the delta function restricts the frequency spectrum to be discrete. The comb picks out all frequencies ω which are integer multiples of *half* the fundamental frequency,

$$\omega = \frac{\pi}{L}\ell, \quad \ell \in \mathbb{Z}. \tag{8}$$

From the frequency-matching condition and summation on diagonals of the lattice one has

$$W\Big(t, \frac{\pi}{L}\ell\Big) = \sum_{n=-\infty}^{\infty} (c_n e^{i\omega_n t})^* (c_{\ell-n} e^{i\omega_{\ell-n} t}), \quad \ell \in \mathbb{Z} \tag{9}$$

as shown in Fig. 1. Note that if the function $f(t)$ is real, then the Wigner distribution $W(t, \omega)$ is also real. This calculation reveals the Wigner function as particularly simple in the Fourier basis, being basically a discrete convolution of the Fourier series with itself for each integer ℓ. For example, the zero-mode is the summation

$$W(t, 0) = \sum_{n=-\infty}^{\infty} |c_n|^2 e^{-i2\omega_n t} \tag{10}$$

which has zero time-average (contributing nothing to the spectrum $\hat{f}(\omega)$) but necessary to describe the *local* (in coordinate t) deviation from mean-zero which a function will naturally acquire in its spatial structure. As an energy it oscillates with twice the frequency of its corresponding mode. In order to avoid aliasing interference the distribution should be truncated at the Nyquist frequency of the series,

$$W\Big(t, \frac{\pi}{L}\ell\Big) = \sum_{n=-k}^{k} (c_n e^{i\omega_n t})^* (c_{\ell-n} e^{i\omega_{\ell-n} t}), \quad \ell \in \mathbb{Z}. \tag{11}$$

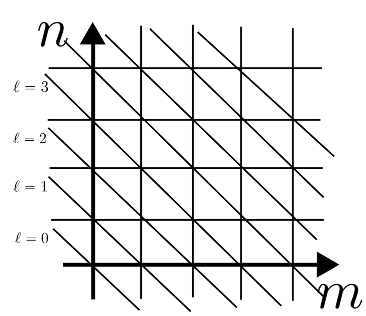

Figure 1: Summation on the lattice (n, m) resolves into a single summation on a diagonal satisfying $m = \ell - n$ for each frequency component ℓ of the Wigner distribution.

1.2 Wigner distribution of an analytic signal

The *analytic signal* $f_s(t)$ joins the Hilbert transform of a function $f(t)$ to itself,

$$f_s(t) = f(t) + i\mathcal{H}[f](t). \tag{12}$$

If a signal is real-valued then the spectrum of its analytic signal is the one-sided series

$$f_s = \sum_{n=0}^{\infty} 2c_n e^{i\omega_n t}, \quad \omega_n = \frac{2\pi}{L}. \tag{13}$$

In signal processing this is called the "real Fourier series". If desired, one can eliminate the interference effects at the low-frequencies present in the Wigner distribution of Eqn. 11. It should be noted that this process will discard the beat effects, and the marginal distribution will be altered to instead return the intensity spectrum of the analytic signal $|f_s(t)|^2$. This can be useful in data analysis however, and the formula for the Wigner distribution simplifies.

Repeating the calculation of Section 1.1, the distribution of $f_s(t)$ is given by

$$W\left(t, \frac{\pi}{L}\ell\right) = 2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(c_n e^{i\omega_n t})^*(c_m e^{i\omega_m t})\delta\left(\frac{\pi}{L}(\ell - (n+m))\right), \quad \ell \in \mathbb{Z}^+. \tag{14}$$

The summation is now in the first quadrant only of the lattice in Fig. 1, and the summation along the diagonal $m = \ell - n$ truncates. For a given mode ℓ, interference is possible only between frequencies up to and including ℓ, giving

$$W\left(t, \frac{\pi}{L}\ell\right) = 2\sum_{n=0}^{\ell}(c_n e^{i\omega_n t})^*(c_{\ell-n} e^{i\omega_{\ell-n} t}), \quad \ell \in \mathbb{Z}^+. \tag{15}$$

This reduces interferences, but discards important information regarding energetic interactions between *frequency differences*, instead describing only frequency-sum modulations. For example, beat frequencies $f_b = |f_2 - f_1|$ are important differences which are described differently by the Wigner distribution of an analytic signal.

1.3 Transform of the covariance function

If instead one considers the Fourier transform of the instantaneous *correlation* or covariance of two functions $x(t)$ and $y(t)$, a cross-term Wigner function can be defined

$$W_{\{x,y\}}(t,\omega) = \int_{-\infty}^{\infty} x^*\left(t - \frac{\tau}{2}\right) y\left(t + \frac{\tau}{2}\right) e^{-i\omega\tau} d\tau. \tag{16}$$

If $x(t)$ and $y(t)$ have the same period, then $W_{\{x,y\}}$ is the convolution of the two Fourier series,

$$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{i\omega_n t}, \qquad y(t) = \sum_{n=-\infty}^{\infty} b_n e^{i\omega_n t} \tag{17}$$

$$W_{\{x,y\}}(t, \frac{\pi}{L}\ell) = \sum_{n=-\infty}^{\infty} (a_n e^{i\omega_n t})^*(b_{\ell-n} e^{i\omega_{\ell-n} t}). \tag{18}$$

Its marginals are the correlation function in frequency and function $x(t) \times y(t)$ in space.

2 Examples of Wigner distributions

The following are simple examples of Wigner functions presented in order to build intuition. An objective of the following discussion is to point out that the interference patterns present in the distribution are physically meaningful, but are perhaps undesirable from a signal processing perspective, in particular if the signal is not truly periodic (e.g. a music recording).

2.1 Single sine function

The distribution of a basic sine function $f(t) = \sin(t) = -\frac{1}{2}ie^{it} + \frac{1}{2}ie^{-it}$ is given by

$$W\left(t, \frac{\pi}{L}\ell\right) = \frac{1}{4}\begin{cases} 1 & \ell = 2 \\ -2\cos(2t) & \ell = 0 \\ 1 & \ell = -2 \end{cases} \tag{19}$$

so that summing in ℓ, one finds the energy distribution $|f(t)|^2 = \frac{1}{2} - \frac{1}{2}\cos(2t) = \sin^2(t)$. The Wigner distribution is shown below. Evidently the interference pattern at $\ell = 0$ is due to combination of the positive and negative frequency components. In the signal processing context it is remarked that the zero-mode is “unexpected”, but this example demonstrates that it is necessary in order to describe the non-zero mean in energy of the sine wave. The zero-mode $\frac{1}{2}\cos(2t) = 0$ when $f(t) = |f(t)|^2$ and no more energy is needed than in $\ell = \pm 2$.

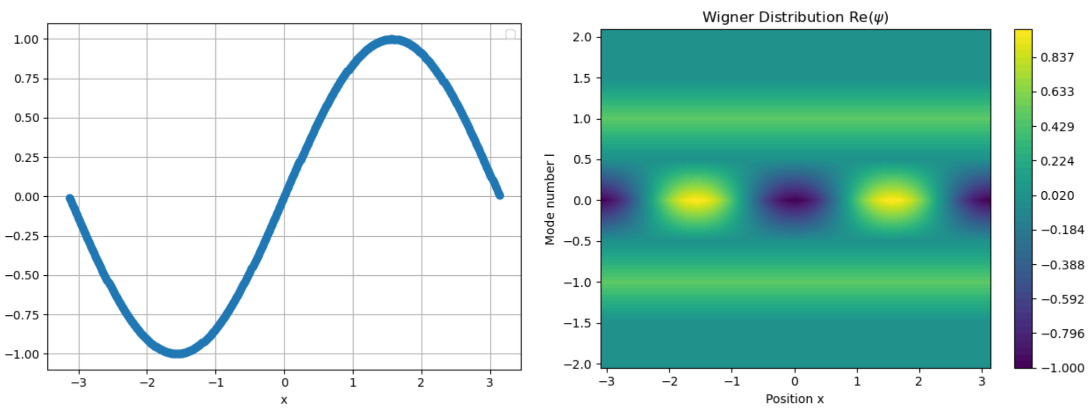

Figure 2: Wigner function of the sine wave (right), note interference between modes $n = \pm 1$.

2.2 Classic beat wave

Now consider a function with more spatially modulated structure than the sine wave,

$$f(t) = \cos(4t) + \cos(5t) = 2\cos\left(\frac{1}{2}t\right)\cos\left(\frac{9}{2}t\right). \tag{20}$$

This waveform has a classic beat structure, with a low-frequency of $\omega_B = 1/2$ as shown below and a modulation frequency of $\omega_M = 9/2$. The spectrum of the signal (its orthogonal components) is of course only present at $\omega = 4$ and 5, but the spatial structure of the waveform

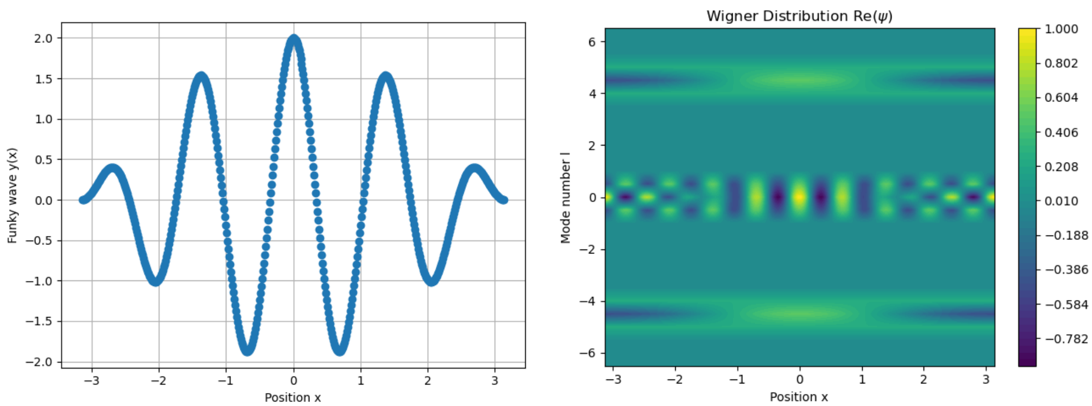

Figure 3: Spatial modulation in the superposition of $\cos(4t)$ and $\cos(5t)$ (left) and Wigner distribution (right). Low-frequency energy is present in modes $\ell = 0, \pm 1$ and high-frequency modulation in $\omega = 9/2$ which cancel out when considering the total frequency spectrum.

has a *low-frequency variation* which is present in the mode $\ell = 1$ and a *harmonic modulation frequency* at $\ell = 9$ (or $\omega = 9/2$). A generic spectrogram computed by a windowed Fourier transform will typically not see the energy present in the harmonic combinations resulting in spatial structure of the waveform. It appears instead as a bump in the spectrogram.

2.3 Elliptic cosine

Another good example of spatial structure is the elliptic cosine function with Fourier series

$$\mathrm{cn}(4K(m)x|m) = \frac{\pi}{K(m)\sqrt{m}} \sum_{n=0}^{\infty} \frac{\cos(2\pi(1+2n)x)}{\cosh(\frac{\pi}{2}(2n+1)\frac{K'(m)}{K(m)})}. \tag{21}$$

where $K(m)$ is a complete elliptic integral. For parameter m close to one the wave energy becomes very localized, shown below. Where the function is peaked there is a zero-mode

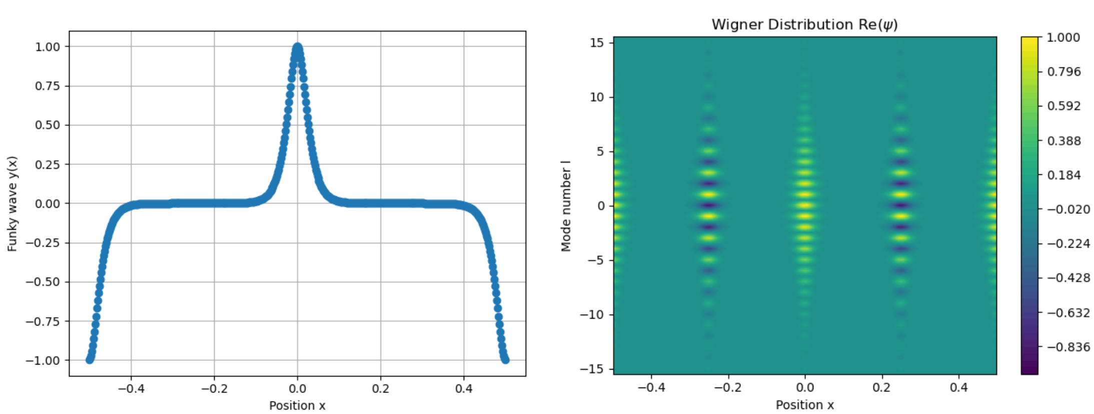

Figure 4: Elliptic cosine and its Wigner distribution for parameter $m = 1 - 1^{-10}$.

(DC) component to the energy! However in its totality there is no mean value to the function. In order to cancel out this zero-mode a *virtual frequency distribution* is present in coordinate x where the signal energy is entirely zero. This phenomenon can be understood as a "spatial beat" in constrast to the usual frequency beat. In the context of the energetic distribution of a truly periodic signal this interference phenomenon has physical significance.

2.4 D Major arpeggio on the guitar

Yet for analysis of a waveform *assumed* to be periodic the spatial beat effect is detrimental. An example of such is in analysis of a musical waveform. Below the Wigner function (of the analytic signal, see Section 1.2) is calculated of a 7.5 second recording of the four notes of a D Major chord played in succession. The four notes of the D chord are 144 Hz (D), 216 Hz (A), 288 Hz (D), and 364 Hz (F#). A windowed Fourier transform is also computed and shown for comparison, which has only four spectral lines (plus their integer harmonics).

The Wigner function results in more localized signals, but also encounters *time beats* when the recording is silent. It also predicts *interferences* between any pair of notes. One of the pleasant aspects of the D Major chord is its internal harmony, as shown in Table 1. The frequency of modulation between any pair of notes reinforce one another and fill out the succeeding intervals. The Wigner spectrogram picks up this interference pattern.

Chord Notes [Hz]	F#, 364		D, 288		A, 216		D, 144
First Neighbor Mean [Hz]		326		252		180	
Second Neighbor Mean [Hz]			290		216		
Center Frequency [Hz]				254			

Table 1: Frequencies of modulation in the D Major chord on the guitar.

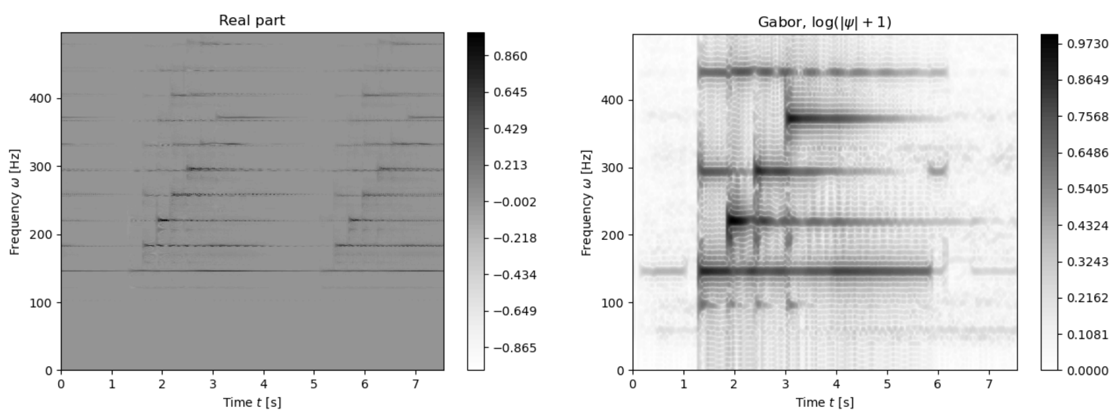

Figure 5: Wigner distribution (left) and typical spectrogram (right) of four notes played on the guitar, each calculated from the analytic signal (real fft) of the waveform. The Wigner function has an interference pattern where the signal is silent (0-1.1 s and the end). However, it also describes interferences between each pair of notes, in particular at the chord center frequency of 254 Hz, which plays a large part in the psychoacoustical perception of the chord.

2.4.1 Filtered Wigner distribution

The key point is that instantaneous spatial structure can actually be part of perception. Or on the other hand, in pure physics parlance a particle ensemble will undergo spatially-varying wave-particle interactions with beats and other frequencies of modulation in a turbulent wavefield, which can occupy modes which appear to have no energy when considering the total spectrum. The Wigner distribution provides an angle on this by describing the spatial distribution of wave energy in certain "virtual states" at the modulation frequencies.

However it's clear that autocorrelation is, at best, an incomplete model. For example, in the elliptic cosine problem a particle *will* experience a DC component of energy when passing through a wave peak. Yet it will experience no potential at all in the lacuna, where the Wigner distribution assigns a virtual frequency distribution to cancel out those modes.

In the context of signal processing the presence of the time beats, an artifact purely of the assumed periodicity of the signal, makes direct interpretation of the Wigner spectrogram difficult at first glance. The windowed Fourier transform, which for large windows does not see frequencies of modulation, helps to pick out where the signal is actually non-zero as shown in Fig. 5. An interesting compromise, equivalent to filtering the Wigner distribution, is to simply multiply the two spectrograms together. The result is shown below.

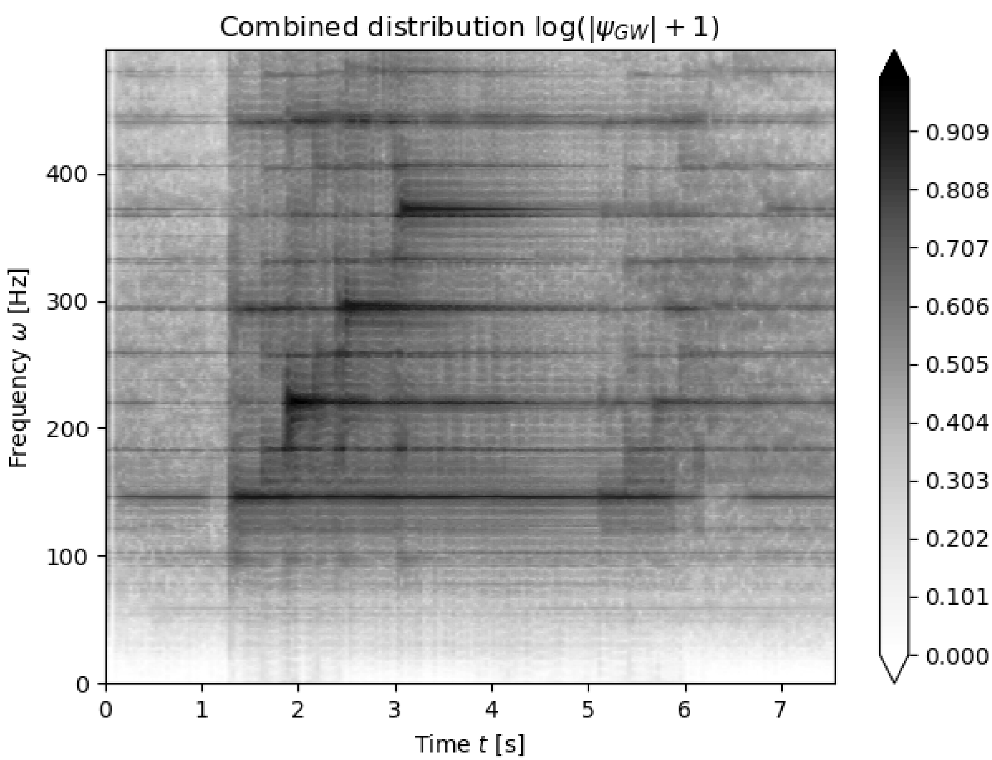

Figure 6: The combined spectrogram (shown here in log scale) multiplies togeter the windowed Fourier transform with the Wigner distribution. The resulting spectrogram separtes the true signal from its "time beat" phantom part and includes spectral lines at the frequencies of modulation between the component notes of the arpeggiated D chord.