1 Convergence to an unknown solution — April 8, 2021

Oftentimes one needs to determine the rate of convergence of a numerical method to an unknown solution, for example in nonlinear problems without an explicit solution. Generically, the error ϵ between the numerical solution $x(h)$ and the true solution x^* using a method with step-size h will have a leading-order power law relationship,

$$\epsilon \equiv x^* - x(h) = ch^\alpha + \mathscr{O}(h^\beta) \tag{1}$$

where c is some constant, α is the rate of convergence, and $\mathscr{O}(h^\beta)$ is the next highest-order error term. If x^*, c, and α are all unknown parameters, then by considering the higher-order terms to be the same order of magnitude, one can write three equations for the unknowns with three numerical solutions of distinct step-sizes and solve a nonlinear equation for the convergence rate α. However, this is not really an exact method because the higher-order errors $\mathscr{O}(h^\beta)$ do not cancel exactly. On the other hand, one can cast the problem of determining the convergence rate α in terms of a sequence of numerical solutions of varying step-size.

2 Estimating the rate of convergence

Consider a sequence of numerical solutions $\{x_0, x_1, x_2, \cdots\}$ each using step-sizes $\{h_0, h_1, h_2, \cdots\}$ and having errors $\epsilon_n \equiv x^* - x_n$. If the method is convergent then as the step-size goes to zero, i.e. $\lim_{n\to\infty} h_n \to 0$, the sequence $\{x_n\}_{n=0}^{\infty}$ will converge to the true solution x^*. For example, it's convenient to use a multiplicative relationship between step-sizes, $h_n = a^{-n}h_0$, such as successively halving the step size using $a = 2$. To find the rate of convergence from this sequence, consider the following theorem.

Theorem 1. *If the ratio of successive errors $\{\epsilon_n\}_{n=0}^{\infty}$ converges to an asymptotic constant,*

$$\lim_{n\to\infty} \frac{|x_{n+1} - x^*|}{|x_n - x^*|} \to \mu \tag{2}$$

then the difference of elements of the sequence limits to a factor of the error, $\lim_{n\to\infty} |x_{n+1} - x_n| \le (1+\mu)\epsilon_n$.

Proof: *Consider the limit*

$$\lim_{n\to\infty} \frac{|x_{n+1} - x_n|}{|x_n - x^*|} = \lim_{n\to\infty} \frac{|(x_{n+1} - x^*) - (x_n - x^*)|}{|x_n - x^*|} \le \lim_{n\to\infty} \frac{|x_{n+1} - x^*|}{|x_n - x^*|} + 1 = \mu + 1 \tag{3}$$

so that in the limit $n \to \infty$, one has $|x_{n+1} - x_n| \le (1+\mu)\epsilon_n$.

Typical factors are $\mu = 0$ for "superlinear" convergence of the sequence, and if linearly then $0 < \mu < 1$ (see [1]). Since $\epsilon_n \to 0$ as $n \to \infty$ (as the method is convergent), then the successive differences can be used to estimate the error to within a factor, $\epsilon_n \sim \nu|x_{n+1} - x_n|$. It is often stated that the error can be approximated by the successive difference of terms, and the above explains why. Now the rate of convergence of the numerical method can be approximated by considering the successive errors,

$$\epsilon_{n+1} \sim \nu|x_{n+1} - x_n| \sim c(h/a^{n+1})^\alpha, \tag{4}$$

$$\epsilon_n \sim \nu|x_n - x_{n-1}| \sim c(h/a^n)^\alpha. \tag{5}$$

Dividing the two and solving for the rate α gives the estimate as

$$\alpha_n \equiv \log_a\left(\frac{|x_{n+1} - x_n|}{|x_n - x_{n-1}|}\right), \quad \text{and} \quad \lim_{n\to\infty} \alpha_n \to \alpha. \tag{6}$$

If a non-integer a is chosen, use the change of base formula to obtain an appropriate logarithm. Otherwise, if one successively halves the step size ($a = 2$) then the logarithm is base two. Because Thm. 1 holds in the limit $n \to \infty$, the estimate of Eqn. 6 converges to the actual rate of the numerical method as the refinement continues. In practice, this may take a few doublings to achieve. For instance, a test case with an RK method on a nonlinear problem observed $\alpha_1 \sim 0.885$, $\alpha_2 \sim 0.933$, $\alpha_3 \sim 0.974$, and $\alpha_4 = 1.000$ for a method with linear convergence.

References

[1] R. Wang, "Order and rate of convergence," [Online] available here.