Notes on electromagnetism and charged particle motion

November 4, 2020

1 Origin of the electromagnetic field tensor

This introduction section establishes the field tensor and some of its properties.

1.1 The field tensor comes from least action and special relativity

In the scheme of electromagnetism, the field tensor comes about by considering the least action principle for a particle of mass m with coordinates $x^\mu = (ct, x, y, z)$ interacting with some spacetime vector field $A_\mu = (A_0, A_x, A_y, A_z)$[1]. This action is a balance of spacetime path length with the accumulated effect of that spacetime field,

$$S = -mc\int ds + q\int A_\mu dx^\mu \tag{1}$$

where the first term describes the arc-length of the particle's spacetime trajectory with

$$ds = \sqrt{-\eta_{\mu\nu}dx^\mu dx^\nu} = \sqrt{c^2dt^2 - dx^2 - dy^2 - dz^2}. \tag{2}$$

Here $\eta_{\mu\nu} = \mathrm{diag}(-c^2, 1, 1, 1)$ is the spacetime metric tensor. This says that spacetime is a 4D *hyperbolic manifold* where the shortest distance between two points occurs when $dx/dt = c$ (in which case $ds = 0$!) and the longest path is when one doesn't "move" at all.

Conformal mapping of 2D hyperbolic space into a unit disk (Escher: Circle Limit I).
"Straight lines" are circles joining the boundary disk at 90°.

The spacetime arc-length has a special parameterization defined by $ds = cd\tau \equiv \sqrt{c^2dt^2 + 0}$ when $d\boldsymbol{x} = 0$. This is the arc-length in the frame where the particle is not moving (its rest frame), and then the general ds can be written as

$$ds = d\tau\sqrt{c^2\frac{dt^2}{d\tau^2} - \frac{dx^2}{d\tau^2} - \frac{dy^2}{d\tau^2} - \frac{dz^2}{d\tau^2}} = d\tau\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} \tag{3}$$

so that $d\tau$ is called the particle's *proper time* (i.e. the arc-length in its rest frame). It is essentially just arc-length parameterization of a curve, so that $ds = cd\tau$.

1.1.1 Arc-length parametrization refresher

If someone had told me that arc-length parametrization was the key to understanding special relativity I probably would have paid more attention in Calculus class! Let's do a helix to refresh how the technique works. First, for some curve $r = r(\zeta)$, its line element is

$$ds = |r'(\zeta)|d\zeta = \int_a^b \sqrt{\left(\frac{dx}{d\zeta}\right)^2 + \left(\frac{dy}{d\zeta}\right)^2 + \left(\frac{dz}{d\zeta}\right)^2}d\zeta, \tag{4}$$

that's the Pythagorean theorem. Consider a helix with "angular velocity" parameterization:

$$r(t) = (a\cos(t), a\sin(t), bt). \tag{5}$$

The arc-length in this parameterization is

$$s \equiv \int_a^b |r'(t)|dt \tag{6}$$

so up to some t the arc-length is given by

$$s = \int_0^t \sqrt{a^2\sin^2(t') + a^2\cos^2(t') + b^2 dt'} = \int_0^t \sqrt{a^2 + b^2}dt' = \sqrt{a^2 + b^2}t. \tag{7}$$

Arc-length parameterization is when we use this arc-length as the parameter,

$$r(s) = \left(a\cos\left(\frac{s}{\sqrt{a^2 + b^2}}\right), a\sin\left(\frac{s}{\sqrt{a^2 + b^2}}\right), b\frac{s}{\sqrt{a^2 + b^2}}\right) \tag{8}$$

Now, the line element is given by $ds = |1|ds$. The arc-length is simply $\Delta s = \int_{s_0}^{s_1} ds$. This example makes the procedure look easy, but beware that in fact the square-root integral can't be done explicitly in many cases.

1.2 Back to the worldline action

When parameterized with some parameter ζ (basically, some time coordinate) the action is

$$S = -mc\int d\zeta\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\zeta}\frac{dx^\nu}{d\zeta}} + q\int d\zeta A_\mu\frac{dx^\mu}{d\zeta} \equiv \int d\zeta\mathcal{L}. \tag{9}$$

The action has two parts: the spacetime line element ds and the scalar $A_\mu\frac{dx^\mu}{d\zeta}$. Now the principle of least action is applied to yield the Euler-Lagrange equations,

$$\frac{\delta S}{\delta\tau} = 0 \quad \Longrightarrow \quad \frac{\partial\mathcal{L}}{\partial x^\mu} - \frac{d}{d\tau}\left(\frac{\partial\mathcal{L}}{\partial(dx^\mu/d\zeta)}\right) = 0. \tag{10}$$

1.2.1 Variation of the arc-length term

Variation of the first term (the length $\mathcal{L}_s = -mc\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\zeta}\frac{dx^\nu}{d\zeta}} = -mc(cd\tau)$) gives

$$-\frac{d}{d\zeta}\Big(\frac{\partial\mathcal{L}_s}{\partial(dx^\mu/d\zeta)}\Big) = mc\frac{d}{d\zeta}\Big(\frac{\eta_{\mu\nu}\frac{dx^\nu}{d\zeta}}{|\mathcal{L}_s|/mc}\Big). \tag{11}$$

Now if we do this in the arc-length parameterization in the rest frame (the proper-time $d\tau$!) then $|\mathcal{L}_s|/mc = c$. So variation of the first term *in the rest frame* just gives simple acceleration

$$-\frac{d}{d\tau}\Big(\frac{\partial\mathcal{L}_s}{\partial(dx^\mu/d\tau)}\Big) = \frac{d}{d\tau}(m\eta_{\mu\nu}\frac{dx^\nu}{d\tau}) = m\eta_{\mu\nu}\frac{d^2x^\nu}{d\tau^2}. \tag{12}$$

Note that the metric tensor $\eta_{\mu\nu}$ factors out of the derivative here. There is a complication here for $\eta_{\mu\nu} = \eta_{\mu\nu}(x)$. Further, if one had $\eta_{\mu\nu} = \eta_{\mu\nu}(x)$ then the derivative $\frac{\partial\mathcal{L}_s}{\partial x^\mu}$ would have come into play. This is basically how gravitational force works in the relativistic picture.

1.2.2 Variation of the spacetime field term

The second Lagrangian term is $\mathcal{L}_A = qA_\mu(x)\frac{dx^\mu}{d\tau}$. It has two parts to its variation,

$$\frac{\partial\mathcal{L}_A}{\partial x^\mu} = q\partial_\mu(A_\nu\frac{dx^\nu}{d\tau}) = q\partial_\mu A_\nu\frac{dx^\nu}{d\tau} \tag{13}$$

as the first derivative and as its second,

$$-\frac{d}{d\tau}\frac{\partial\mathcal{L}_A}{\partial(dx^\mu/d\tau)} = -q\frac{dA_\mu(x)}{d\tau} = -q\partial_\nu A_\mu\frac{dx^\mu}{d\tau}. \tag{14}$$

1.2.3 The electromagnetic tensor comes out of the Euler-Lagrange equation

Combining all of these terms into Eqn. 10 gives the equation of motion,

$$\frac{\delta S}{\delta\tau} = 0 \quad\Longrightarrow\quad m\eta_{\mu\nu}\frac{d^2x^\nu}{d\tau^2} = q(\partial_\mu A_\nu - \partial_\nu A_\mu)\frac{dx^\nu}{d\tau}. \tag{15}$$

Evidently the equation of motion involving the particle interacting with that spacetime vector field A_μ involves the antisymmetric second-order tensor[1]

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{16}$$

The equation of motion cleans up by changing an index, $F^\mu_\nu = \eta^{\mu\lambda}F_{\lambda\nu}$, to give

$$m\frac{d^2x^\mu}{d\tau^2} = qF^\mu_\nu\frac{dx^\nu}{d\tau}. \tag{17}$$

The above turns out to yield the Lorentz force by identifying those tensor components as

$$F_{\mu\nu} = \begin{bmatrix} 0 & E_1/c & E_2/c & E_3/c \\ -E_1/c & 0 & -B_3 & B_2 \\ -E_2/c & B_3 & 0 & -B_1 \\ -E_3/c & -B_2 & B_1 & 0 \end{bmatrix}. \tag{18}$$

[1]So it may be identified with a two-form! Note that it is a tensor because it is made of vector gradients (so that it transforms properly when those vector parts are rotated).

1.3 The electromagnetic energy-momentum tensor

So with just those simple assumptions, the Lorentz force comes out. Pretty remarkable! It is interesting to see how this relates to the field's energy-momentum tensor (which we have basically seen before, in MHD).

1.3.1 Maxwell's equations

Maxwell's equations come from two factors:

- From our study of differential forms, we know that the quantity $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the component matrix of a differential two-form $F = dA$ (with $x^0 = ct$),

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu. \tag{19}$$

  Therefore we'll have $dF = 0$. This will give two equations in one; this is sometimes called the *Bianchi identity* in differential geometry. Let's see how it works explicitly:

$$\begin{aligned}
F &= B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy && (20)\\
&\quad + E_x dx \wedge dt + E_y dy \wedge dt + E_z dz \wedge dt && (21)\\
\implies dF &= (\partial_x B_x dx + \partial_t B_x dt) \wedge dy \wedge dz && (22)\\
&\quad + (\partial_y B_y dy + \partial_t B_y dt) \wedge dz \wedge dx && (23)\\
&\quad + (\partial_z B_z dz + \partial_t B_z dt) \wedge dx \wedge dy && (24)\\
&\quad + (\partial_y E_x dy + \partial_z E_x dz) \wedge dx \wedge dt && (25)\\
&\quad + (\partial_x E_y dx + \partial_z E_y dz) \wedge dy \wedge dt && (26)\\
&\quad + (\partial_x E_z dx + \partial_y E_z dy) \wedge dz \wedge dt = 0 && (27)\\
\implies & (\partial_x B_x + \partial_y B_y + \partial_z B_z) dx \wedge dy \wedge dz && (28)\\
&\quad + (\partial_t B_x + \partial_y E_z - \partial_z E_y) dt \wedge dy \wedge dz && (29)\\
&\quad + (\partial_t B_y + \partial_z E_x - \partial_x E_z) dt \wedge dz \wedge dx && (30)\\
&\quad + (\partial_t B_z + \partial_x E_y - \partial_y E_x) dt \wedge dx \wedge dy = 0. && (31)
\end{aligned}$$

  The vanishing of these four volume forms gives the "auxiliary" Maxwell equations:

$$\nabla \cdot \boldsymbol{B} = 0, \tag{32}$$

$$\partial_t \boldsymbol{B} + \nabla \times \boldsymbol{E} = 0. \tag{33}$$

- The Maxwell equations with sources can be considered to come from two directions:

  1. Writing Maxwell's Lagrangian $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2}(\epsilon_0 |\boldsymbol{E}|^2 - \frac{1}{\mu_0} |\boldsymbol{B}|^2)$ and performing least action to find $\partial_\mu F^{\mu\nu} = -J^\nu$, containing Gauss's law as the time-part and Ampere's law as the space part;
  2. Or considering the *Hodge dual* $\star F$ of the field tensor's corresponding two-form and solving $d(\star F) = -J$ where J is the current density's corresponding 3-form:

$$J = -\rho_c dx \wedge dy \wedge dz + j_x dt \wedge dy \wedge dz + j_y dt \wedge dz \wedge dx + j_z dt \wedge dx \wedge dy. \tag{34}$$

     This introduces the idea of a *dual electromagnetic field tensor.*

These two approaches are equivalent!

1.4 The dual field tensor and the energy-momentum tensor

Textbooks will introduce the field's energy-momentum tensor as

$$T^{\mu\nu} = F^{\mu}_{\lambda} F^{\nu\lambda} - \frac{1}{4}\eta^{\mu\nu} F_{\sigma\rho} F^{\sigma\rho} = F^{\mu}_{\lambda} F^{\nu\lambda} + \frac{1}{2}\eta^{\mu\nu}(\epsilon_0 |\boldsymbol{E}|^2 - \frac{1}{\mu_0}|\boldsymbol{B}|^2). \tag{35}$$

It's important for a variety of reasons. One, it shows up in fluid models (like MHD) in the energy-momentum equations. Two, it is part of the source term for $g_{\mu\nu} = g(x)$ in Einstein's equations for the metric tensor. From this definition, one finds a matrix with familiar forms

$$T = \begin{bmatrix} \frac{1}{2}(\epsilon_0|\boldsymbol{E}|^2 + \frac{1}{\mu_0}|\boldsymbol{B}|^2) & \frac{1}{\mu_0}(\boldsymbol{E}\times\boldsymbol{B})^T/c \\ \frac{1}{\mu_0}(\boldsymbol{E}\times\boldsymbol{B})/c & -\sigma_{ij} \end{bmatrix} \tag{36}$$

$$\sigma_{ij} = \epsilon_0 E_i E_j + \frac{1}{\mu_0} B_i B_j - \frac{1}{2}(\epsilon_0|\boldsymbol{E}|^2 + \frac{1}{\mu_0}|\boldsymbol{B}|^2)\delta_{ij} \tag{37}$$

where σ is the Maxwell stress tensor. It combines the energy density, energy flux, and momentum fluxes all into one matrix. However its definition by Eqn. 35 is not very symmetric. One can find a more symmetric way of writing the energy-momentum tensor by considering the component matrix of the field two-form's Hodge dual[2]:

$$\star F = \frac{1}{2}\widetilde{F}_{\mu\nu} dx^{\mu} \wedge dx^{\nu} \tag{38}$$

so that $\widetilde{F}_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}F^{\lambda\sigma}$. This dual tensor turns out to have components

$$\widetilde{F}_{\mu\nu} = \begin{bmatrix} 0 & B_1 & B_2 & B_3 \\ -B_1 & 0 & -E_3 & E_2 \\ -B_2 & E_3 & 0 & -E_1 \\ -B_3 & -E_2 & E_1 & 0 \end{bmatrix} \tag{39}$$

which reverses the positions of B and E relative to $F_{\mu\nu}$! (And note the sign change on the spatial matrix part.) Now, one can write the energy-momentum tensor as the combination

$$T_{\mu\nu} = \frac{1}{2}\eta_{\lambda\sigma}(F^{\mu\lambda}F^{\nu\sigma} + \widetilde{F}^{\mu\lambda}\widetilde{F}^{\nu\sigma}) = \frac{1}{2}(F^{\mu}_{\lambda}F^{\nu\lambda} + \widetilde{F}^{\mu}_{\lambda}\widetilde{F}^{\nu\lambda}) \tag{40}$$

which is far more symmetric than Eqn. 35. This suggests that the dual tensor should play some role in the electrodynamics too, which will be seen in the coming section on charged particle motion in a field.

[2] In four dimensions, the Hodge dual of a two-form is another two-form (think Pascal's triangle). There is a sign change too because the space is hyperbolic.

2 Rotation groups

This section introduces the idea of generators of the Lie groups of rotations and Lorentz transformations. There is a connection between the Lorentz force tensor and the generator of Lorentz transformations which is explored in Section 3.

2.1 Two-dimensional rotations: the SO(2) group

Rotations in the plane through an angle θ are a transformation whose matrix is

$$R(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}. \tag{41}$$

Rotation matrices are orthogonal; $RR^T = I$. To unrotate, you just rotate in the other direction. The orthogonal 2x2 matrices also describe reflections; to describe rotations only requires $\det(R) = +1$, which is called "special". The plane rotation group is then called SO(2) (special, orthogonal, two-dimensional). It is a *Lie group* because it depends on a continuous parameter and has ∞ members, in contrast to the finite groups (like permutations).

2.1.1 Generator of SO(2)

The infinitesimal generator of a group is a "near identity" element in a special sense. To obtain the condition $RR^T = I$, one considers $R(\theta) = I + A(\theta) + \mathcal{O}(\theta^2)$ for A "small";

$$(I + A + \cdots)(I + A + \cdots)^T = I + A^T + A + \mathcal{O}(\theta^2) = I. \tag{42}$$

so that to first-order this generator A must satisfy

$$A^T + A = 0 \qquad \Longrightarrow \qquad A^T = -A. \tag{43}$$

But in 2D, there is really only one candidate, the "symplectic matrix"

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{44}$$

So solutions are, for any parameter θ (which we have considered small),

$$A = \theta J \qquad \Longrightarrow \qquad R \approx I + \theta J = \begin{bmatrix} 1 & \theta \\ -\theta & 1 \end{bmatrix}. \tag{45}$$

This is Lie's idea: to rotate through an angle θ, rotate n-times through θ/n and take the limit $n \to \infty$[2]. Lie's procedure yields the matrix exponential:

$$R(\theta) = \lim_{n\to\infty} (R(\theta/n))^n = \lim_{n\to\infty} (I + \frac{\theta}{n} J)^n = \exp(\theta J). \tag{46}$$

This works formally using power series:

$$\exp(\theta J) = \sum_{n=0}^{\infty} \frac{\theta^n}{n!} J^n = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} I + \sum_{n=0}^{\infty} \frac{(-1)^{2n+1} \theta^{2n+1}}{(2n+1)!} J \tag{47}$$

$$= \cos(\theta) I + \sin(\theta) J = R(\theta) \tag{48}$$

because the symplectic matrix acts like the "imaginary" unit: $J^0 = I$, $J^1 = J$, $J^2 = -I$, $J^3 = -J$, $J^4 = I$, etc. Therefore we call J the infinitesimal generator of SO(2). Any rotation in the plane through an angle θ can be specified symbolically by $\exp(\theta J)$.

2.2 Three-dimensional rotations

The very same idea holds in 3D. The rotation group in 3D has to satisfy the same special orthogonal properties, $RR^T = I$ and $\det(R) = 1$. We know from the study of Euler angles that any rotation in three dimensions can be decomposed into three component rotations about three independent planes, say (x,y), (y,z), (z, x) in a right-handed coordinate system. The order of these three component rotations is important!

The infinitesimal generator idea still holds. The generators about the three planes are

$$J_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad J_y = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad J_z = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{49}$$

and the general three-dimensional rotation can be written

$$R(\theta_x, \theta_y, \theta_z) = \exp(\theta_x J_x + \theta_y J_y + \theta_z J_z) \tag{50}$$

or in terms of one big generator

$$A(\theta_x, \theta_y, \theta_z) = \begin{bmatrix} 0 & \theta_z & -\theta_y \\ -\theta_z & 0 & \theta_x \\ \theta_y & -\theta_x & 0 \end{bmatrix}. \tag{51}$$

Note that A acts to take a cross product with the vector $\boldsymbol{\theta} = (\theta_x, \theta_y, \theta_z)$,

$$\begin{bmatrix} 0 & \theta_z & -\theta_y \\ -\theta_z & 0 & \theta_x \\ \theta_y & -\theta_x & 0 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} \theta_z u_y - \theta_y u_z \\ \theta_x u_z - \theta_z u_x \\ \theta_y u_x - \theta_x u_y \end{bmatrix} = \boldsymbol{u} \times \boldsymbol{\theta}. \tag{52}$$

Let's see how this works in the matrix exponential $R = \exp(A)$. First, the powers of A are,

$$A^2 = \begin{bmatrix} 0 & \theta_z & -\theta_y \\ -\theta_z & 0 & \theta_x \\ \theta_y & -\theta_x & 0 \end{bmatrix} \begin{bmatrix} 0 & \theta_z & -\theta_y \\ -\theta_z & 0 & \theta_x \\ \theta_y & -\theta_x & 0 \end{bmatrix} \tag{53}$$

$$= \begin{bmatrix} \theta_x^2 & \theta_x\theta_y & \theta_x\theta_z \\ \theta_x\theta_y & \theta_y^2 & \theta_y\theta_z \\ \theta_x\theta_z & \theta_y\theta_z & \theta_z^2 \end{bmatrix} - |\theta|^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boldsymbol{\theta\theta}^T - |\theta|^2 I \tag{54}$$

$$A^3 = A(\boldsymbol{\theta\theta}^T - |\theta|^2 I) = -|\theta|^2 A, \qquad (\text{as } \boldsymbol{\theta} \times \boldsymbol{\theta} = 0) \tag{55}$$

$$A^4 = -|\theta|^2 A^2 \tag{56}$$

$$A^5 = |\theta|^4 A \tag{57}$$

and so on, so that the matrix exponential sum can be reordered as follows (with $A^0 = I$),

$$R = \exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} |\theta|^{2k+1} \left[\frac{A}{|\theta|}\right] - \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!} |\theta|^{2n} \left[\frac{A^2}{|\theta|^2}\right] \tag{58}$$

$$= I + \sin(|\theta|)\frac{A}{|\theta|} + (1 - \cos(|\theta|))\frac{A^2}{|\theta|^2} \tag{59}$$

$$= \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}^T + \cos(|\theta|)(I - \hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}^T) + \sin(|\theta|)A/|\theta|. \tag{60}$$

This result is called the Euler-Rodrigues formula for a rotation matrix through an angle $|\theta| = \sqrt{\theta_x^2 + \theta_y^2 + \theta_z^2}$ about the axis $\hat{\boldsymbol{\theta}} = (\theta_x, \theta_y, \theta_z)/|\theta|$[3].

2.3 Four-dimensional Euclidean rotations

Up to four dimensional Euclidean space, there are now six independent planes of rotation. It turns out that the group SO(4) is identical to SO(3)⊗SO(3) near the identity. The six independent generators combine into one big generator

$$A(\theta_x, \theta_y, \theta_z, \alpha_x, \alpha_y, \alpha_z) = \begin{bmatrix} 0 & \theta_x & -\theta_y & \alpha_x \\ -\theta_x & 0 & \theta_z & \alpha_y \\ \theta_y & -\theta_z & 0 & \alpha_z \\ -\alpha_x & -\alpha_y & -\alpha_z & 0 \end{bmatrix} \tag{61}$$

and the general 4D Euclidean rotation may still be written in terms of an exponential map,

$$R(\theta_x, \theta_y, \theta_z, \alpha_x, \alpha_y, \alpha_z) = \exp(A). \tag{62}$$

This generator looks quite similar to the electromagnetic field tensor and its dual, with an overall minus sign in the spatial part to account for the space being hyperbolic.

2.4 Generator of the Lorentz group, SO(3,1)

The Lorentz transformation expresses the symmetries of an isotropic 3+1-dimensional hyperbolic manifold, like spacetime. It consists of ordinary rotations in the three spatial planes, and Lorentz boosts (like inverted rotations, sin → sinh) in the mixed space/time planes. Its generator is written similarly to the above, with time customarily taking the first coordinate, $x^\mu = (ct, x, y, z)$. In any spacetime plane the boost is generated by,

$$K = \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix} \quad \implies \quad \exp(K) = \sum_{n=0}^{\infty} \frac{K^n}{n!} = \begin{bmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{bmatrix} \tag{63}$$

which mirrors the "hyperbolic Euler identity", $\exp(\theta) = \cosh(\theta) + \sinh(\theta)$. With that, the SO(3,1) generator is typically written as

$$A(\theta_x, \theta_y, \theta_z, \alpha_x, \alpha_y, \alpha_z) = \begin{bmatrix} 0 & \alpha_x & \alpha_y & \alpha_z \\ \alpha_x & 0 & \theta_z & -\theta_y \\ \alpha_y & -\theta_z & 0 & \theta_x \\ \alpha_z & \theta_y & -\theta_x & 0 \end{bmatrix}. \tag{64}$$

Its exponential map $\exp(A)$ will be explored in the coming sections.

3 Lorentz force generates "Lorentz transformations"

This section illustrates how the Lorentz force tensor, as a (1,1)-rank object (i.e., a linear transform on the four-velocity) can produce solutions through exponentiation in the very same way that the generator of the Lorentz group would produce spacetime transformations.

3.1 Motion in perpendicular static fields

Let's consider the problem of $\boldsymbol{E} \times \boldsymbol{B}$ charged particle motion in perpendicular static fields. The equation of motion expressing the Lorentz force was found via least action to be

$$\frac{d}{d\tau}\frac{dx^\mu}{d\tau} = \frac{q}{m}F^\mu_\nu \frac{dx^\nu}{d\tau} \tag{65}$$

where the field tensor written with one lower and one upper index (thus acting as a linear transformation) was (with $E \equiv E/c$, with $c = 1$ until the end of the calculation),

$$F^\mu_\nu = \eta^{\mu\lambda}F_{\lambda\nu} = \begin{bmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix} \tag{66}$$

which we now see to be identical in form to the infinitesimal generator of spacetime Lorentz transformations. Let's consider the solution via Lie series for the special case of perpendicular fields, $\boldsymbol{E} = E\hat{y}$ and $\boldsymbol{B} = B\hat{z}$, giving as special case a tensor

$$A = \begin{bmatrix} 0 & 0 & E \\ 0 & 0 & B \\ E & -B & 0 \end{bmatrix} = E\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + B\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \tag{67}$$

where the z-axis has been suppressed for the calculation because there are no dynamics there. Computing powers of A for the matrix exponential, one finds

$$A^0 = I, \qquad A^1 = A \tag{68}$$

$$A^2 = \begin{bmatrix} E^2 & -EB & 0 \\ EB & -B^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + (E^2 - B^2)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{69}$$

$$A^3 = (E^2 - B^2)A \tag{70}$$

$$A^4 = (E^2 - B^2)\begin{bmatrix} E^2 & -EB & 0 \\ EB & -B^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + (E^2 - B^2)^2\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{71}$$

$$A^5 = (E^2 - B^2)^2 A \tag{72}$$

and so on. We recognize the factor of EB appearing in the space-time components as the Poynting flux $\boldsymbol{E} \times \boldsymbol{B}$. Name the component matrices appearing here as

$$J_z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad K_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} E^2 & -EB & 0 \\ EB & -B^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad I_y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{73}$$

Then the equation of motion is solved by Lie series,

$$\frac{d}{d\tau}\frac{dx^\mu}{d\tau} = \frac{q}{m}F^\mu_\nu\frac{dx^\nu}{d\tau} \qquad \Longrightarrow \qquad \frac{dx^\mu}{d\tau} = \exp\big(\frac{q}{m}\tau A\big)\Big(\frac{dx^\mu}{d\tau}\Big|_{\tau=0}\Big) \tag{74}$$

$$\exp\Big(\frac{q}{m}\tau A\Big) = \sum_{n=0}^{\infty}\frac{\tau^n}{n!}\Big(\frac{q}{m}\Big)^n A^n \tag{75}$$

$$= I + \sum_{k=0}^{\infty}\frac{\tau^{2k+1}}{(2k+1)!}\Big(\frac{q}{m}\Big)^{2k+1}(\sqrt{E^2-B^2})^{2k+1}\frac{EK_y+BJ_z}{\sqrt{E^2-B^2}} \tag{76}$$

$$+\sum_{k=1}^{\infty}\frac{\tau^{2k}}{(2k)!}\Big(\frac{q}{m}\Big)^{2k}(\sqrt{E^2-B^2})^{2k}\Big(\frac{P}{E^2-B^2}\Big) + \sum_{k=1}^{\infty}\frac{\tau^{2k}}{(2k!)}\Big(\frac{q}{m}\Big)^{2k}(\sqrt{E^2-B^2})^{2k}I_y \tag{77}$$

which are recognized as hyperbolic functions,

$$\exp\Big(\frac{q}{m}\tau A\Big) = I + \sinh\Big(\frac{q}{m}\sqrt{E^2-B^2}\tau\Big)\frac{EK_y+BJ_z}{\sqrt{E^2-B^2}} \tag{78}$$

$$+ (\cosh\Big(\frac{q}{m}\sqrt{E^2-B^2}\tau)\Big)-1)\Big(\frac{P}{E^2-B^2}+I_y\Big). \tag{79}$$

At this point, $1/c$ is reintroduced for $E \to E/c$. Observe that for $E/B < c$, the square-roots introduce a factor i. Using the identities $\sinh(ix) = i\sin(x)$, $\cosh(ix) = \cos(x)$, and introducing the factors:

$$v_E = \frac{E}{B}, \tag{80}$$

$$\gamma_E = \sqrt{1-\Big(\frac{v_E}{c}\Big)^2}, \tag{81}$$

$$\omega_c = \frac{qB}{m}, \tag{82}$$

$$\hat{P} = \frac{P}{B^2} = \begin{bmatrix} v_E^2/c^2 & v_E/c & 0 \\ -v_E/c & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{83}$$

one can reduce the solution by Lie series/exponential map to the form

$$\exp\Big(\frac{q}{m}\tau A\Big) = I + \sin(\omega_c\gamma_E\tau)\Big(\frac{v_E}{\gamma_E}K_y/c + J_z\Big) \tag{84}$$

$$+ (1-\cos(\omega_c\gamma_E\tau))\Big(\hat{P}/\gamma_E^2 + I_y\Big). \tag{85}$$

Observe the close similarity to the Euler-Rodrigues formula for rotations in three-dimensional Euclidean space!! In fact, this boosting/rotating matrix produces the $\boldsymbol{E}\times\boldsymbol{B}$ drift.

3.1.1 Initial value problem for a stationary particle: the $E \times B$ drift

So assuming a stationary particle with a four-velocity (with $dt/d\tau = 1$ initially),

$$\left.\frac{dx^\mu}{d\tau}\right|_{\tau=0} = \begin{bmatrix} c \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{86}$$

the solution using Eqn. 84-85 has components

$$\frac{dt}{d\tau} = 1 + \left(\frac{v_E/c}{\gamma_E}\right)^2 (1 - \cos(\omega_c \gamma_E \tau)) \tag{87}$$

$$\frac{dx}{d\tau} = \frac{v_E}{\gamma_E^2}(1 - \cos(\omega_c \gamma_E \tau)) \tag{88}$$

$$\frac{dy}{d\tau} = \frac{v_E}{\gamma_E}\sin(\omega_c \gamma_E \tau) \tag{89}$$

which recovers a relativistic solution for $\boldsymbol{E} \times \boldsymbol{B}$ drift for an initially stationary particle. Note the appearance of the time dilation factor γ_E which one can take to be slowing the effective oscillation frequency ω_c. Note also the asymmetry of the boost factor in the x and y directions. This is seen to appear because of the EM energy flux in the x direction.

The appearance of hyperbolic functions for $E > cB$ suggests that, as particles are unable to execute an $\boldsymbol{E} \times \boldsymbol{B}$ drift greater than the speed of light, the oscillatory functions become hyperbolic functions which produce an accelerating motion towards light speed, with an unbounded trajectory in both x and y. This is the case in a very weakly magnetized plasma, in which case the particles are mostly responding to the electric field (i.e., in the limit $v_E \gg c$ and $B \to 0$ the particles are not actually drifting much at all, unless E is very strong).

3.1.2 The case of $E = cB$

To treat the case of $E = cB$, note that the Lie series will truncate at the third term,

$$\exp\left(\frac{q}{m}\tau A\right) = I + \frac{q}{m}\tau A + \frac{1}{2}\frac{q^2}{m^2}\tau^2 A^2 \tag{90}$$

in which case the solutions are simply

$$\frac{dt}{d\tau} = 1 + \frac{1}{2}(\omega_c \tau)^2, \tag{91}$$

$$\frac{dx}{d\tau} = \frac{1}{2}c(\omega_c \tau)^2, \tag{92}$$

$$\frac{dy}{d\tau} = c(\omega_c \tau). \tag{93}$$

Although apparently weird, this is a consistent relativistic solution, satisfying:

$$c^2 dt^2 - dx^2 - dy^2 = c^2 d\tau^2. \tag{94}$$

To explore a little further, one may convert back to the laboratory frame by solving each equation for the coordinates in terms of the proper time parameter,

$$t(\tau) = \tau(1 + \frac{1}{6}(\omega_c \tau)^2), \tag{95}$$

$$x(\tau) = \frac{1}{6}(\omega_c \tau)^2(\tau c), \tag{96}$$

$$y(\tau) = \frac{1}{2}(\omega_c \tau)(\tau c). \tag{97}$$

Eliminating the parameter τ, the trajectory follows the curve

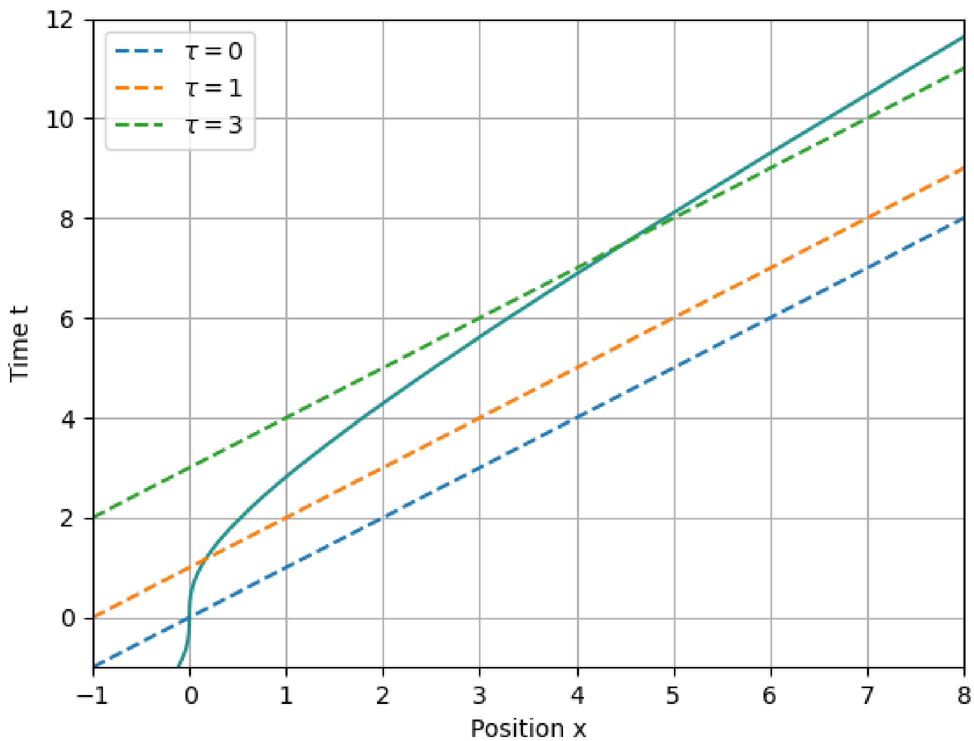
$$y^3 \sim x^2 \qquad \Longrightarrow \qquad y \sim x^{2/3}. \tag{98}$$

Also, $\frac{dx}{dt} \to 0$ in the weak field limit $\omega_c \to 0$. The exact solution for $x = x(t)$ is from a cubic polynomial and is not pretty, but the motion can be understood qualitatively as the intersection of the two curves

$$x = \frac{1}{6}(\omega_c t)^2 \Big(1 - \frac{x}{ct}\Big)^3 \tag{99}$$

$$c\tau = ct - x \tag{100}$$

where the proper time τ parameterizes the curve via the point of intersection. This is plotted below for $c = \omega_c = 1$. The particle can be seen to steadily accelerate in the $\boldsymbol{E} \times \boldsymbol{B}$ direction, limiting towards $\frac{dx}{dt} \to c$ after a few "periods" $1/\omega_c$. At this critical point of $E = cB$, there is no oscillation nor exponential growth. Instead there is steady *algebraic* growth of the particle velocity in the lab frame towards the speed of light.



3.1.3 The case of $E < cB$

Integrating Equations 87 to 89 gives the parameterized particle trajectories

$$t(\tau) = \tau\left(1 + \left(\frac{v_E}{c\gamma_E}\right)^2\right) - \left(\frac{v_E}{c\gamma_E}\right)^2 \frac{1}{\gamma_E\omega_c}\sin(\omega_c\gamma_E\tau), \tag{101}$$

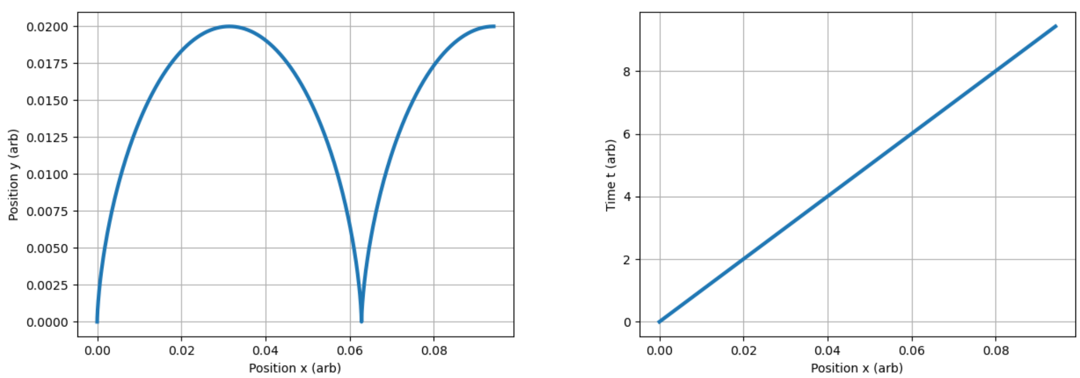
$$x(\tau) = \frac{v_E}{\gamma_E^2}\tau - \frac{v_E}{\gamma_E^3\omega_c}\sin(\omega_c\gamma_E\tau), \tag{102}$$

$$y(\tau) = \frac{v_E}{\gamma_E^2\omega_c}(1 - \cos(\omega_c\gamma_E\tau)). \tag{103}$$

Combining t and x yields the conversion between proper time and the lab frame,

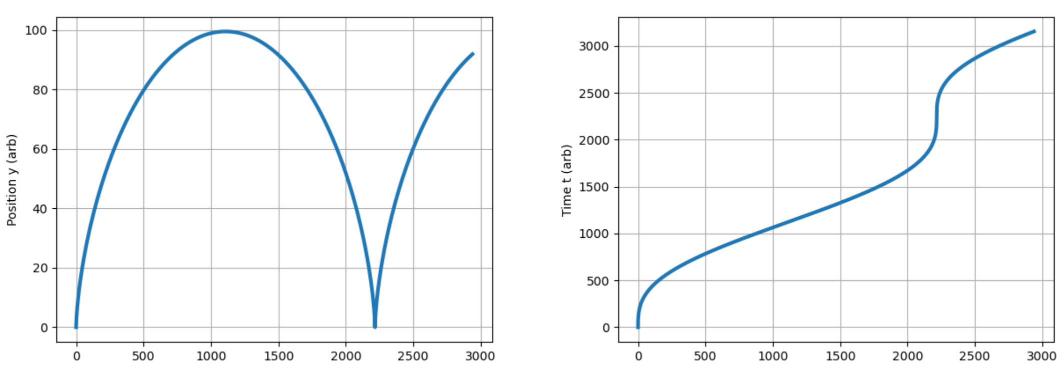
$$c\tau = ct - \left(\frac{v_E}{c}\right)x \tag{104}$$

as a more general condition than Eqn. 100. In the limit $v_E \ll c$ one finds $t = \tau$ and recovers the usual cycloidal trajectory. The below figure shows this for $\omega_c = c = 1$.



(a.) Trajectory and drift speed with $v_E/c = 0.01$. The $\boldsymbol{E} \times \boldsymbol{B}$ drift speed is constant.

On the other hand, as $v_E \to c$ the particle accelerates towards c. It eventually reaches zero velocity in the x direction and spends a long time at the cycloidal notches. In the limit $v_E \to c$ the behavior resembles the case $v_E = c$, of course.



(b.) Trajectory and drift speed for $v_e/c = 0.99$. The drift speed is no longer constant.

3.1.4 The case of $E > cB$

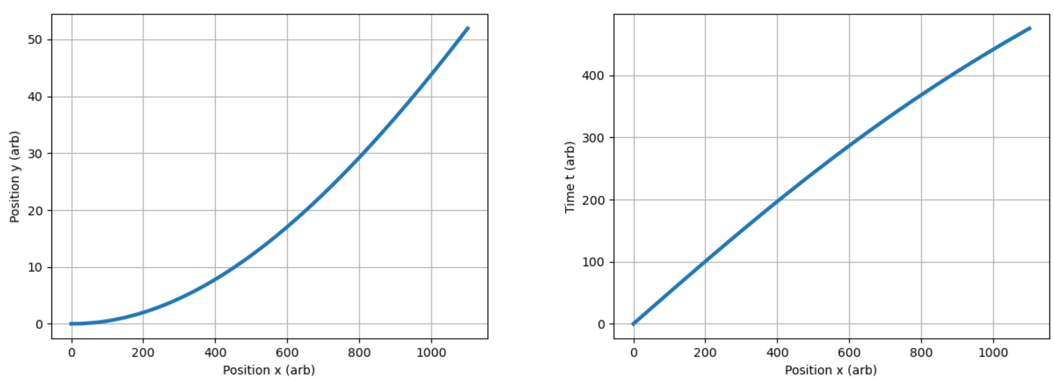
With a superluminal characteristic velocity $v_E > c$, the particle is unable to drift at the characteristic speed. This situation is relevant in weakly magnetized plasmas with a strong applied field. In the rotation/boost perspective on the Lorentz force, this case corresponds to a stronger boost than a rotation. The trajectories integrate to

$$t(\tau) = \tau\left(1 + \left(\frac{v_E}{c\gamma_E}\right)^2\right) + \left(\frac{v_E}{c\gamma_E}\right)^2 \frac{1}{\gamma_E \omega_c} \sinh(\omega_c \gamma_E \tau), \tag{105}$$

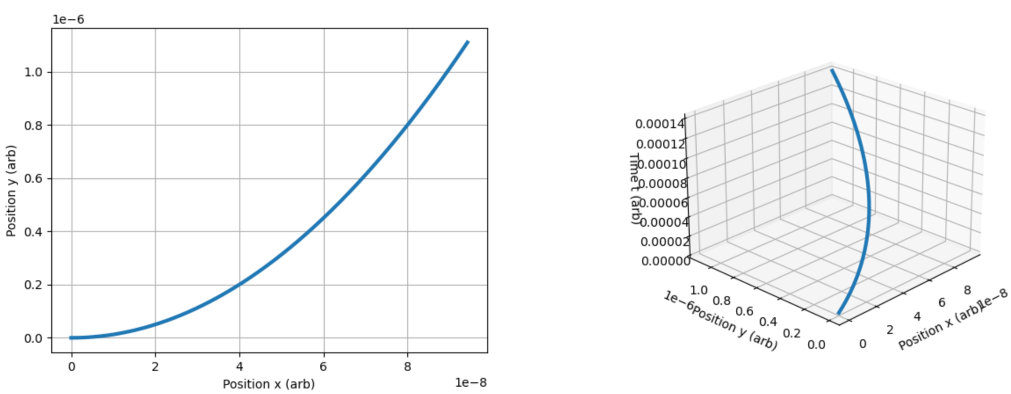
$$x(\tau) = \frac{v_E}{\gamma_E^2}\tau + \frac{v_E}{\gamma_E^3 \omega_c} \sinh(\omega_c \gamma_E \tau), \tag{106}$$

$$y(\tau) = \frac{v_E}{\gamma_E^2 \omega_c}(1 - \cosh(\omega_c \gamma_E \tau)). \tag{107}$$

The particle trajectory becomes unbounded in the y-direction, although the initial velocity is mostly x-directed. In the limit $\omega_c \tau \to \infty$, the ratio of y to x limits to $y/x \to \gamma_E$. Thus in the unmagnetized limit $E \gg cB$ the trajectory is a straight line mainly in the y-direction with a slope given by $\gamma_E = \sqrt{(E/cB)^2 - 1}$.



(c.) Trajectory and drift for $v_e/c = 1.01$. The trajectory is not bounded in the y-direction. Motion is primarily in the x-direction as the particle accelerates toward light speed.



(c.) Trajectory and drift for $v_e/c = 1000$. After a short time motion is primarily y-directed.

3.2 Motion in general uniform fields

In the case of $\boldsymbol{E} \cdot \boldsymbol{B} \neq 0$, the exponential map of the Lorentz force as a rotation generator is a little more complicated. This is because $\boldsymbol{E} \cdot \boldsymbol{B}$ is introduced as an additional parameter of the transformation besides $|E|^2 - |B|^2$. Since one may always rotate the coordinate frame to align with the magnetic axis, it suffices to study the generator

$$F^{\mu}_{\nu} \equiv A = \begin{bmatrix} 0 & 0 & E_y & E_z \\ 0 & 0 & B_z & 0 \\ E_y & -B_z & 0 & 0 \\ E_z & 0 & 0 & 0 \end{bmatrix} \tag{108}$$

corresponding to a choice of frame with $\boldsymbol{E} = (0, E_y, E_z)$ and $\boldsymbol{B} = (0, 0, B)$. Considering powers of this matrix, one finds

$$A^2 \equiv P = \begin{bmatrix} |E|^2 & -E_y B_z & 0 & 0 \\ E_y B_z & -B_z^2 & 0 & 0 \\ 0 & 0 & E_y^2 - B_z^2 & E_y E_z \\ 0 & 0 & E_y E_z & E_z^2 \end{bmatrix}. \tag{109}$$

The general case of A^2 can be inferred from this. Continuing, the third power is

$$A^3 = AP = \alpha A + \beta \widetilde{A} \tag{110}$$

after defining the scalars $\alpha = |E|^2 - |B|^2$ and $\beta = \boldsymbol{E} \cdot \boldsymbol{B}$ (note that these are the only two Lorentz-invariant scalars which are quadratic in the EM field) and the dual tensor

$$\widetilde{A} = \begin{bmatrix} 0 & 0 & 0 & B_z \\ 0 & 0 & -E_z & E_y \\ 0 & E_z & 0 & 0 \\ B_z & -E_y & 0 & 0 \end{bmatrix}. \tag{111}$$

Note that as defined, the product of $A\widetilde{A} = \beta I$. Then the fourth power of A is

$$A^4 = \alpha P + \beta^2 I. \tag{112}$$

Listing various powers,

$$A^0 = I \tag{113}$$
$$A^1 = A \tag{114}$$
$$A^2 = P \tag{115}$$
$$A^3 = \alpha A + \beta \widetilde{A} \tag{116}$$
$$A^4 = \alpha P + \beta^2 I \tag{117}$$
$$A^5 = (\alpha^2 + \beta^2) A + \alpha\beta \widetilde{A} \tag{118}$$
$$A^6 = (\alpha^2 + \beta^2) P + \alpha\beta^2 I \tag{119}$$
$$A^7 = (\alpha(\alpha^2 + \beta^2) + \alpha\beta^2)) A + \beta(\alpha^2 + \beta^2) \widetilde{A} \tag{120}$$
$$\cdots \quad \cdots \tag{121}$$

and so on, the coefficients of the even and odd powers of A,

$$A^{2n+1} = c_{n+1}A + d_{n+1}\widetilde{A}, \tag{122}$$
$$A^{2n} = c_n P + e_n I, \tag{123}$$

are observed to satisfy the linear recursion relations (excluding $n = 0$),

$$c_n = \alpha c_{n-1} + \beta^2 c_{n-2}, \tag{124}$$
$$d_n = \beta c_{n-1}, \tag{125}$$
$$e_n = \beta d_n, \tag{126}$$

with initial conditions $c_0 = 0$, $c_1 = 1$. Equation 124 is a linear second-order homogeneous recursion relation [4]. Its characteristic polynomial is

$$t^2 - \alpha t - \beta^2 = 0. \tag{127}$$

For convenience, redefine $\lambda = \alpha/2 = \frac{1}{2}(|E|^2 - |B|^2)$ (as the actual Maxwell Lagrangian). Provided that Eqn. 127 has distinct roots, then the solutions of the recursion relation are

$$c_n = \frac{r^n - s^n}{2\sqrt{\lambda^2 + \beta^2}}, \tag{128}$$
$$r = \lambda + \sqrt{\lambda^2 + \beta^2}, \tag{129}$$
$$s = \lambda - \sqrt{\lambda^2 + \beta^2}. \tag{130}$$

Therefore, the terms in the exponential map $\exp\left(\frac{q}{m}\tau A\right)$ may be grouped into separate series multiplying against four matrices. That is, the even and odd terms split like

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\left(\frac{q}{m}\tau\right)^{2n+1} A^{2n+1} = A\sum_{k=0}^{\infty} \frac{(\frac{q}{m}\tau)^{2k+1}}{(2k+1)!}c_{k+1} + \widetilde{A}\sum_{k=0}^{\infty} \frac{(\frac{q}{m}\tau)^{2k+1}}{(2k+1)!}d_{k+1}, \tag{131}$$
$$\sum_{n=0}^{\infty} \frac{1}{(2n)!}\left(\frac{q}{m}\tau\right)^{2n} A^{2n} = P\sum_{k=0}^{\infty} \frac{(\frac{q}{m}\tau)^{2k}}{(2k)!}c_k + I\sum_{k=1}^{\infty} \frac{(\frac{q}{m}\tau)^{2k}}{(2k)!}e_k. \tag{132}$$

For convenience, redefine $\sigma \equiv -s/B^2$ and $\rho \equiv r/B^2$ (which are both positive for $\lambda, \beta \neq 0$, as $s < 0$). Substituting the recursion solution, we have for example

$$\sum_{k=0}^{\infty} \frac{(\frac{q}{m}\tau)^{2k+1}}{(2k+1)!}c_{k+1} = \frac{1}{2\sqrt{\lambda^2+\beta^2}}\left(\sqrt{\rho}\sum_{k=0}^{\infty}\frac{(\omega_c\tau)^{2k+1}}{(2k+1)!}\sqrt{\rho}^{2k+1} + \sqrt{\sigma}\sum_{k=0}^{\infty}(-1)^k\frac{(\omega_c\tau)^{2k+1}}{(2k+1)!}\sqrt{\sigma}^{2k+1}\right) \tag{133}$$
$$= \frac{B}{2\sqrt{\lambda^2+\beta^2}}\left(\sqrt{\rho}\sinh(\omega_c\sqrt{\rho}\tau) + \sqrt{\sigma}\sin(\omega_c\sqrt{\sigma}\tau)\right) \tag{134}$$

where $\omega_c = qB/m$. The other sums are done entirely analogously. For proper normalization, define $\widehat{\lambda} \equiv \lambda/B^2$ and $\widehat{\beta} \equiv \beta/B^2$. Then, with a few more definitions,

$$\sqrt{\lambda^2+\beta^2} = B^2\sqrt{\widehat{\lambda}^2+\widehat{\beta}^2} \tag{135}$$
$$u \equiv |E|/(cB) \tag{136}$$
$$v \equiv (\boldsymbol{E}\cdot\boldsymbol{B})/(cB^2) \tag{137}$$

the resulting exponential map can be written

$$\exp\left(\frac{q}{m}\tau A\right) = I + \frac{1}{2}\Big(\sqrt{\rho}\sinh(\omega_c\sqrt{\rho}\tau) + \sqrt{\sigma}\sin(\omega_c\sqrt{\sigma}\tau)\Big)\frac{1}{\sqrt{\widehat{\lambda}^2+\widehat{\beta}^2}}\widehat{A} \tag{138}$$

$$+\frac{1}{2}\Big(\frac{\sinh(\omega_c\sqrt{\rho}\tau)}{\sqrt{\rho}} - \frac{\sin(\omega_c\sqrt{\sigma}\tau)}{\sqrt{\sigma}}\Big)\frac{\widehat{\beta}}{\sqrt{\widehat{\lambda}^2+\widehat{\beta}^2}}\widetilde{\widehat{A}} \tag{139}$$

$$+\frac{1}{2}\Big(\cosh(\omega_c\sqrt{\rho}\tau) - \cos(\omega_c\sqrt{\sigma}\tau)\Big)\frac{1}{\sqrt{\widehat{\lambda}^2+\widehat{\beta}^2}}\widehat{P} \tag{140}$$

$$+\frac{1}{2}\Big(\frac{\cosh(\omega_c\sqrt{\rho}\tau)-1}{\rho} + \frac{\cos(\omega_c\sqrt{\sigma}\tau)-1}{\sigma}\Big)\frac{\widehat{\beta}^2}{\sqrt{\widehat{\lambda}^2+\widehat{\beta}^2}}I \tag{141}$$

with normalized matrices $\widehat{A} = A/B$, $\widetilde{\widehat{A}} = \widetilde{A}/B$, and $\widehat{P} = P/B^2$. This matrix expresses a general Lorentz transformation involving the vectors $\boldsymbol{E} = (0, E_y, E_z)$, $\boldsymbol{B} = (0, 0, B_z)$. This reduces to the result for perpendicular fields by setting the parameter $\beta = \boldsymbol{E}\cdot\boldsymbol{B} = 0$.

3.2.1 Initial value problem for a static particle

Repeating the case of an initially static particle (with $dt/d\tau|_{\tau=0} = 1$),

$$\left.\frac{dx^\mu}{d\tau}\right|_{\tau=0} = \begin{bmatrix} c & 0 & 0 & 0\end{bmatrix}^T \tag{142}$$

the velocity at proper time τ is obtained through the exponential map $\exp(q/m\tau A)$,

$$\frac{dt}{d\tau} = 1 + \frac{1}{2\sqrt{\widehat{\lambda}^2+\widehat{\beta}^2}}\bigg(\Big(\frac{|E|}{cB_z}\Big)^2(\cosh(\omega_c\sqrt{\rho}\tau) - \cos(\omega_c\sqrt{\sigma}\tau)) \tag{143}$$

$$+\Big(\frac{E_z}{cB_z}\Big)^2\Big(\frac{\cosh(\omega_c\sqrt{\rho}\tau)-1}{\rho} + \frac{\cos(\omega_c\sqrt{\sigma}\tau)-1}{\sigma}\Big)\bigg), \tag{144}$$

$$\frac{dx}{d\tau} = \frac{E_y}{B_z}\frac{1}{2\sqrt{\widehat{\lambda}^2+\widehat{\beta}^2}}(\cosh(\omega_c\sqrt{\rho}\tau) - \cos(\omega_c\sqrt{\sigma}\tau)), \tag{145}$$

$$\frac{dy}{d\tau} = \frac{E_y}{B_z}\frac{1}{2\sqrt{\widehat{\lambda}^2+\widehat{\beta}^2}}(\sqrt{\rho}\sinh(\omega_c\sqrt{\rho}\tau) + \sqrt{\sigma}\sin(\omega_c\sqrt{\sigma}\tau)), \tag{146}$$

$$\frac{dz}{d\tau} = \frac{E_z}{B_z}\frac{1}{2\sqrt{\widehat{\lambda}^2+\widehat{\beta}^2}}\bigg(\Big(\sqrt{\rho}+\frac{1}{\sqrt{\rho}}\Big)\sinh(\omega_c\sqrt{\rho}\tau) + \Big(\sqrt{\sigma}-\frac{1}{\sqrt{\sigma}}\Big)\sin(\omega_c\sqrt{\sigma}\tau)\bigg), \tag{147}$$

for $v_E = E_y/B$ the velocity associated with the $\boldsymbol{E}\times\boldsymbol{B}$ Poynting flux. These equations are then directly integrated (with initial condition $(ct, x, y, z) = (0, 0, 0, 0)$) to yield parameterized trajectories describing the motion of a relativistic particle in general uniform fields,

$$t(\tau) = \tau + \frac{1}{2}\frac{\omega_c^{-1}}{\sqrt{\widehat{\lambda}^2 + \widehat{\beta}^2}}\Big(\Big(\frac{u}{c}\Big)^2\Big(\frac{\sinh(\omega_c\sqrt{\rho}\tau)}{\sqrt{\rho}} - \frac{\sin(\omega_c\sqrt{\sigma}\tau)}{\sqrt{\sigma}}\Big) \tag{148}$$

$$+ \Big(\frac{v_z}{c}\Big)^2\Big(\frac{\sinh(\omega_c\sqrt{\rho}\tau)/\sqrt{\rho} - \omega_c\tau}{\rho} + \frac{\sin(\omega_c\sqrt{\sigma}\tau)/\sqrt{\sigma} - \omega_c\tau}{\sigma}\Big)\Big), \tag{149}$$

$$x(\tau) = \frac{1}{2}v_y\frac{\omega_c^{-1}}{\sqrt{\widehat{\lambda}^2 + \widehat{\beta}^2}}\Big(\frac{\sinh(\omega_c\sqrt{\rho}\tau)}{\sqrt{\rho}} - \frac{\sin(\omega_c\sqrt{\sigma}\tau)}{\sqrt{\sigma}}\Big), \tag{150}$$

$$y(\tau) = \frac{1}{2}v_y\frac{\omega_c^{-1}}{\sqrt{\widehat{\lambda}^2 + \widehat{\beta}^2}}(\cosh(\omega_c\sqrt{\rho}\tau) - \cos(\omega_c\sqrt{\sigma}\tau)), \tag{151}$$

$$z(\tau) = \frac{1}{2}v_z\frac{\omega_c^{-1}}{\sqrt{\widehat{\lambda}^2 + \widehat{\beta}^2}}\Big(\Big(1 + \frac{1}{\rho}\Big)(\cosh(\omega_c\sqrt{\rho}\tau) - 1) + \Big(1 - \frac{1}{\sigma}\Big)(1 - \cos(\omega_c\sqrt{\sigma}\tau))\Big). \tag{152}$$

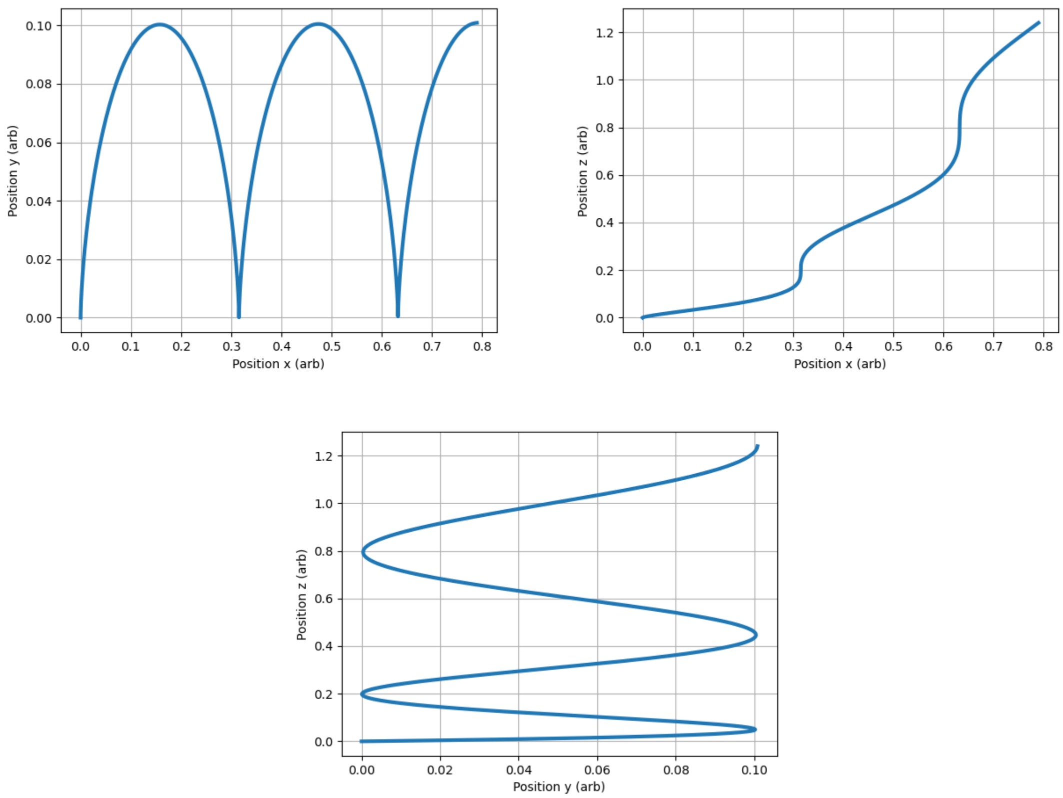
To recapitulate, the various parameters (normalized to B) defined so far are

$$v_y = \frac{E_y}{B}, \qquad v_z = \frac{E_z}{B}, \qquad u = \frac{|E|}{B} = \sqrt{v_y^2 + v_z^2} \tag{153}$$

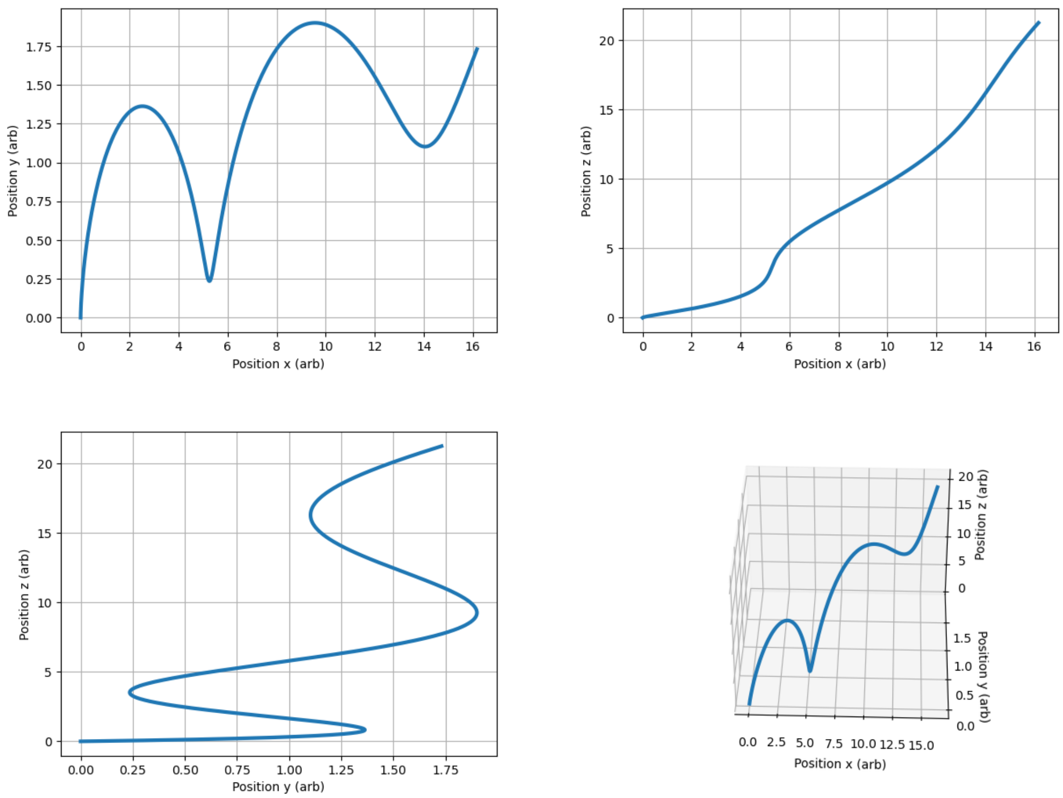
$$\widehat{\lambda} = \frac{1}{2}\Big(\Big(\frac{u}{c}\Big)^2 - 1\Big), \qquad \widehat{\beta} = \frac{v_z}{c} \tag{154}$$

$$\rho = \sqrt{\widehat{\lambda}^2 + \widehat{\beta}^2} + \widehat{\lambda}, \qquad \sigma = \sqrt{\widehat{\lambda}^2 + \widehat{\beta}^2} - \widehat{\lambda} \tag{155}$$

and of course $\omega_c = qB/m$ is the particle cyclotron frequency (with m the rest mass).



(a) Trajectory of a particle in the case $v_y/c = 0.05$, $v_z/c = 0.01$. This is "expected" drift behavior with a cycloidal $\boldsymbol{E} \times \boldsymbol{B}$ trajectory steadily accelerating in the direction of the field due to the component of E along B.



(b) With more "relativistic" fields (e.g. with weaker B), the drift behavior is modified. In the case $v_y/c = 0.5$, $v_z/c = 0.1$ a drift develops in the x-direction as well.

3.2.2 The golden ratio characterizes the special case of $\alpha = \beta \neq 0$

There is a special solution when the two characteristic values are equal, *i.e.* when $\alpha = |E|^2/c^2 - |B|^2$ and $\beta = \boldsymbol{E} \cdot \boldsymbol{B}/c$ satisy $\alpha = \beta$. The recurrence relation is then

$$c_n = \alpha c_{n-1} + \alpha^2 c_{n-2} \tag{156}$$

which has solution $c_n = F_n \alpha^{n-1}$ for F_n the nth Fibonacci number. Applying Binet's formula,

$$F_n = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}} \tag{157}$$

for $\phi = (1 + \sqrt{5})/2$ the golden ratio, the sums are consequently of the type,

$$\sum_{k=0}^{\infty} \frac{F_{k+1}}{(2k+1)!} \alpha^k = \frac{\sinh(\sqrt{\phi\alpha}) - \sin(\sqrt{\alpha/\phi})}{\sqrt{5\alpha}}. \tag{158}$$

Therefore the exponential map has the particular solution

$$\exp(A) = I + \frac{1}{2}\frac{\sqrt{\phi}\sinh(\sqrt{\phi\alpha}) - \sqrt{\phi^{-1}}\sin(\sqrt{\phi^{-1}\alpha})}{\sqrt{5}}\frac{A}{\sqrt{\alpha}} \tag{159}$$

$$+ \frac{1}{2}\frac{\sqrt{\phi^{-1}}\sinh(\sqrt{\phi\alpha}) - \sqrt{\phi}\sin(\sqrt{\phi^{-1}\alpha})}{\sqrt{5}}\frac{\widetilde{A}}{\sqrt{\alpha}} \tag{160}$$

$$+ \frac{1}{2}(\cosh(\sqrt{\phi\alpha}) - \cos(\sqrt{\phi^{-1}\alpha}))\Big(\frac{P}{\alpha} + I\Big) \tag{161}$$

giving characteristic boost/rotation parameters involving the golden ratio ϕ and the single characteristic parameter (one could say, with double multiplicity) $\alpha = |E|^2 - |B|^2 = \boldsymbol{E}\cdot\boldsymbol{B}$.

3.2.3 General matrices in the exponential map

For arbitrary fields $\boldsymbol{E} = (E_x, E_y, E_z)$ and $\boldsymbol{B} = (B_x, B_y, B_z)$, one may infer the general matrices appearing in the exponential map of the Lorentz force tensor from the structure of Eqn. 109. The structure of the map found previously still holds, with scalars $\alpha = |E|^2/c^2 - |B|^2$ and $\beta = \boldsymbol{E}\cdot\boldsymbol{B}/c$, in terms of matrices

$$A = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{bmatrix}, \quad \widetilde{A} = \begin{bmatrix} 0 & B_x & B_y & B_z \\ B_x & 0 & -E_z/c & E_y/c \\ B_y & E_z/c & 0 & -E_x/c \\ B_z & -E_y/c & E_x/c & 0 \end{bmatrix} \tag{162}$$

$$P = \begin{bmatrix} |E|^2/c^2 & -(\boldsymbol{E}\times\boldsymbol{B})_x/c & -(\boldsymbol{E}\times\boldsymbol{B})_y/c & -(\boldsymbol{E}\times\boldsymbol{B})_z/c \\ (\boldsymbol{E}\times\boldsymbol{B})_x/c & & & \\ (\boldsymbol{E}\times\boldsymbol{B})_y/c & & E_iE_j/c^2 + B_iB_j - |B|^2 I_3 & \\ (\boldsymbol{E}\times\boldsymbol{B})_z/c & & & \end{bmatrix} \tag{163}$$

The appearance of $\boldsymbol{E}\times\boldsymbol{B}$ terms in the matrix P is responsible for the $\boldsymbol{E}\times\boldsymbol{B}$ drift motion.

References

[1] A. Zee, *Einstein gravity in a nutshell.* Princeton University Press, 2013.

[2] A. Zee, *Group theory in a nutshell for physicists.* Princeton University Press, 2016.

[3] G. Gallego and A. Yezzi, “A compact formula for the derivative of a 3-d rotation in exponential coordinates,” *J Math Imaging Vis*, vol. 51, pp. 378–384, 2013.

[4] D. Roche, “Second-order linear homogeneous recurrence relations with constant coefficients.” Notes from course SM 242 at USNA, available here under 5.8.