Betti diagrams of modules over short graded Gorenstein rings

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Motivation

In trying to understand modules over short Gorenstein rings, we balance two perspectives:

- **Modules are "built" from indecomposable modules.**
  This is a classical point of view; for rings to which the Krull–Remak–Schmidt Theorem applies, every module can be viewed as a direct sum of indecomposable modules.
- **Betti diagrams are "built" from indecomposable diagrams.**
  This more recent point of view, called *Boij–Söderberg theory*, replaces each module with coarser data (a numerical invariant called its *Betti diagram*) and aims to write this as a nonnegative rational linear combination of distinguished diagrams.

Short Gorenstein Rings

Let $\Bbbk$ be a field and let $e \geqslant 2$ be an integer. A short Gorenstein graded $\Bbbk$-algebra with multiplicity e is the quotient of a polynomial ring in e variables by a quadratic homogeneous ideal given the usual grading such that $\mathfrak{m}^3 = 0$ and $\dim_{\Bbbk} \mathfrak{m}^2 = 1$.

For example, $R = \Bbbk[x,y,z]/(xy, xz, yz, x^2 - y^2, x^2 - z^2)$ is a short Gorenstein graded $\Bbbk$-algebra.

Fix R, a short Gorenstein graded ring with multiplicity e. Let M be a finitely generated graded R-module with $M = \bigoplus_{j \in \mathbb{Z}} M_j$, where M_j is a $\Bbbk$-vector space generated in degree j. Finitely generated R-modules are finite dimensional $\Bbbk$-vector spaces, so there are integers

$$\inf M = \min\{j \mid M_j \neq 0\} \quad \text{and} \quad \sup M = \max\{j \mid M_j \neq 0\}.$$

The **Hilbert series of** M is the polynomial

$$\mathcal{H}_M(s) = \sum_{j=\inf M}^{\sup M} \dim_{\Bbbk}(M_j) s^j.$$

The **Betti diagram of** M is a matrix of invariants of M

$$\beta(M) = \begin{pmatrix} \vdots & \vdots & \vdots & \iddots \\ \beta_{0,-1}(M) & \beta_{1,0}(M) & \beta_{2,1}(M) & \cdots \\ \beta_{0,0}(M) & \beta_{1,1}(M) & \beta_{2,2}(M) & \cdots \\ \beta_{0,1}(M) & \beta_{1,2}(M) & \beta_{2,3}(M) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where the $\beta_{i,j}(M)$ are the **graded Betti numbers of** M.

Koszul and Nonlinear Modules

An R-module K is called **Koszul** provided

- K is generated in degree 0, and
- $\beta(K)$ has nonzero entries in only one row.

The notation $K^{(p,q)}$ denotes any Koszul module with Hilbert series $p + qs$. A **linear module** is one that can be written as $K(-j)$ for a Koszul module K and an integer j.

As an R-module, $\Bbbk$ is one of the $K^{(1,0)}$. Define the sequence $(b_n)_{n \geqslant 1}$ via $b_n = \beta_{n,n}(\Bbbk)$.

Nonlinear indecomposable modules are exactly the modules $C^{(n)} = (\operatorname{Syz}_{-n}(\Bbbk))(-n-1)$ and their twists [Sjö79, AIŞ08].

Applying Krull-Remak-Schmidt

Every R-module M has a finite direct sum decomposition

$$M = \bigoplus_{j=\inf M}^{\sup M} \left(R^{r_j} \oplus K^{(p_j,q_j)} \oplus \bigoplus_{n \geqslant 1} C^{(n)^{c_{n,j}}} \right)(-j).$$

Furthermore, the numbers r_j, p_j, q_j, and $c_{n,j}$ are uniquely determined.

Theorem 1 (G—, 2012)

Let R be a short Gorenstein graded $\Bbbk$-algebra with $e \geqslant 2$. Let M be a finitely generated R-module with no non-zero free summand, and set

$$u = \inf(M), \quad v = \sup(M), \quad \text{and} \quad \ell(M) = 1 + \max\left\{ n \;\middle|\; b_{n-1} \leqslant \max_j \{\dim_{\Bbbk} M_j\} \right\}.$$

Each one of the following sets of data determines each one of the others:

1. The Betti diagram $\beta(M)$.
2. The graded Betti numbers $\{\beta_{i,j}(M) \mid 0 \leqslant i \leqslant \ell,\ u \leqslant j - i \leqslant v\}$.
3. The numbers $c_{n,j}$, p_j, and q_j (as in the equation above) for all $j \in \mathbb{Z}$ and positive $n \in \mathbb{Z}$.

Some words about the proof.

Some key ingredients in this proof include:

- All modules have rational Poincaré series with a common denominator.
- The Poincaré series of linear modules and of indecomposable, nonlinear modules each have special forms.
- Partial fraction decompositions of rational functions are unique.

Example 1

For R with $e = 3$, there is no R-module M for which the nonzero entries in columns $0 \leqslant i \leqslant 4$ of $\beta(M)$ are given by

$$\begin{matrix} * \end{matrix}\begin{pmatrix} 3 & 1 & 1 & 2 & 5 \\ 2 & 3 & 6 & 15 & 39 \end{pmatrix},$$

where the asterisk marks row 0.
First we observe that $\ell(M) \leqslant 3$, so we have enough columns to apply the theorem. We find $p_1 = 2$ and $q_1 = 6$, but these values violate numerical conditions on Hilbert series of Koszul modules given in [AIŞ10]. They also violate the conditions in Theorem 2.

Example 2

For R with $e = 3$, is there an R-module M for which the nonzero entries in columns $0 \leqslant i \leqslant 4$ of $\beta(M)$ are given by

$$\begin{matrix} * \end{matrix}\begin{pmatrix} 3 & 1 & 1 & 2 & 5 \\ 1 & 3 & 8 & 21 & 55 \end{pmatrix}?$$

Again, the asterisk marks row 0.
Applying the theorem, we find that the question is equivalent to whether there exist Koszul modules with Hilbert series $2 + 5s$ and $1 + s$. After using a theorem in [AIŞ08], we need only determine if there is a Koszul module with Hilbert series $2 + 5s$.

Theorem 2 (G—, 2012)

Let R be a short Gorenstein ring with $e \geqslant 2$. There exists a Koszul R-module $K^{(p,q)}$ if and only if the integers p and q satisfy

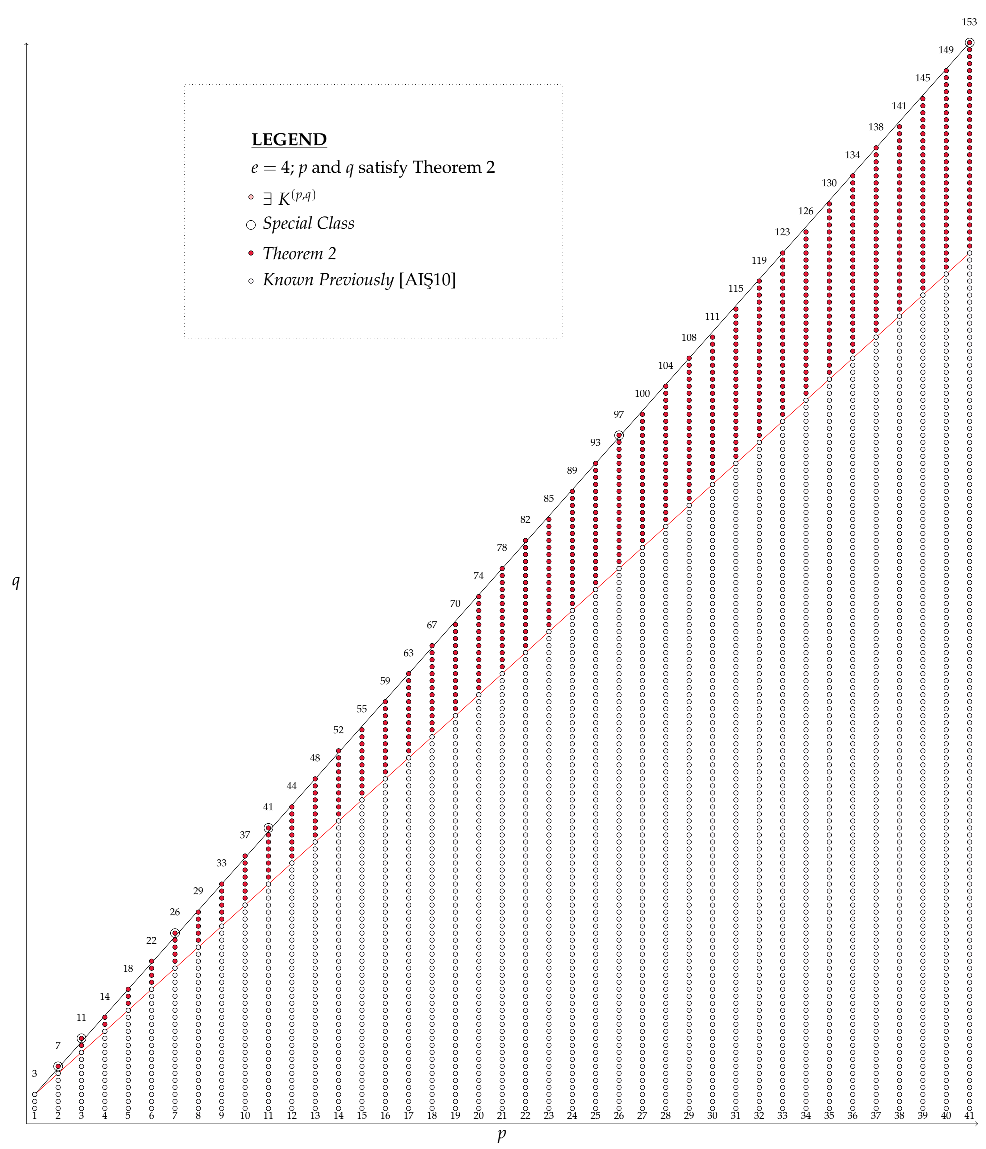
$$p \geqslant 1 \quad \text{and} \quad 0 \leqslant \frac{q}{p} \leqslant \left(\frac{e + \sqrt{e^2 - 4}}{2} \right).$$

Observations and Ongoing Work

Several questions remain:

- **Can indecomposable Koszul modules be detected by Betti diagram alone?**
  Unfortunately, not all of them can be detected this way. However, special types of indecomposable Koszul modules can be detected by their Betti diagrams (and even by their Hilbert series).
- **What diagrams form the collection of distinguished diagrams?**
  I.e., which diagrams (minimally) generate the cone of Betti diagrams over R? Ask me about my conjecture!
- **What are some descriptions of the cone of Betti diagrams?**

A Picture of Theorem 2



Acknowledgments & References

This research has been supported by GAANN, NebraskaMATH, NSF grant #DMS-1103176, and the University of Nebraska–Lincoln. Thanks to my advisors Luchezar Avramov and Roger Wiegand for their guidance.

References:

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