Modules over short graded Gorenstein rings

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Background

Let $\Bbbk$ be an algebraically closed field. Every short standard graded Gorenstein $\Bbbk$-algebra R with multiplicity e is isomorphic to

$$\Bbbk[x_1, \ldots, x_e] \Big/ \left(x_1^2 - x_\ell^2, x_i x_j \;\middle|\; \substack{1 \leq i \lneq j \leq e \\ 1 < \ell \leq e} \right)$$

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$$\begin{array}{lc} \text{Degree 0:} & \Bbbk \\ & \oplus \\ \text{Degree 1:} & \Bbbk x_1 \oplus \cdots \oplus \Bbbk x_e \\ & \oplus \\ \text{Degree 2:} & \Bbbk x_e^2 \end{array}$$

Modules over R

We consider only finitely generated, graded R-modules $M = \oplus_{j \in \mathbb{Z}} M_j$.

The Hilbert series of M is defined to be

$$\mathcal{H}_M(s) = \sum \dim_{\Bbbk} M_j s^j.$$

Example

For example, $\mathcal{H}_R(s) = 1 + es + s^2$.

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Koszul R-modules

An R-module K is called Koszul provided there exist free modules F_i and matrices ∂_i of linear forms (and zeroes) such that the following sequence is exact:

$$0 \leftarrow K \leftarrow F_0 \xleftarrow{\partial_1} F_1 \xleftarrow{\partial_2} F_2 \xleftarrow{\partial_3} \cdots$$

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Example

The R-module $\Bbbk \cong R/(x_1, \dots, x_e)$ is Koszul:

$$0 \leftarrow \Bbbk \leftarrow R \xleftarrow{(x_1,\dots,x_e)} R(-1)^e \xleftarrow{\begin{pmatrix} 0 & 0 & z & 0 & 0 & y & 0 & x \\ 0 & -z & -z & -z & y & -z & x-z & 0 \\ z & y & 0 & x & 0 & 0 & 0 & 0 \end{pmatrix}} \cdots$$

What Hilbert series are possible for Koszul R-modules?

Theorem (G—, 2012)

Given a short graded Gorenstein ring R with multiplicity e, there exists a Koszul module with Hilbert series $p + qs$ if and only if the integers p and q satisfy

$$1 \leq p \quad \text{and} \quad 0 \leq \frac{q}{p} \leq \frac{e + \sqrt{e^2 - 4}}{2}.$$

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Proof idea.

We use previously known results due to L. Avramov, S. Iyengar, and L. Şega for $0 \leq \frac{q}{p} \leq e - 1$.

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Special Koszul Modules

Definition

Given a free resolution of $\mathbb{k}$,

$$0 \leftarrow \mathbb{k} \leftarrow R(0)^1 \leftarrow R(-1)^e \leftarrow R(-2)^{e^2-1} \leftarrow \cdots]$$

Define the sequence $b_n = (b_0 = 1, b_1 = e, b_2 = e^2 - 1, \cdots)$.

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Special Koszul Modules

Theorem (G—, 2012)

Let M be an indecomposable module. Fix a positive integer $m \leq e - 1$. The following statements are equivalent:

1. $M \cong \mathrm{Syz}_{-n-1}(I)$ where I is an ideal minimally generated by m linear forms.
2. $\mathcal{H}_M(s) = (b_n - mb_{n-1}) + (b_{n+1} - mb_n)s$.

If one of the above holds, then M is Koszul.

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Acknowledgments & References

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