**CEE 530/ME 524. Introduction to the FEM**
**Duke University**
**Homework 2, Due Fri 9/16, 2016 by 4:00 PM**

1. Consider the one dimensional rod of unit area, length $L$, and modulus $E$ shown in the figure below. The bar is subjected to a sinusoidal body force $b = b_0 \sin(\pi x/L)$ and boundary conditions $u(0) = 0$ and $Eu'(L) = 0$.

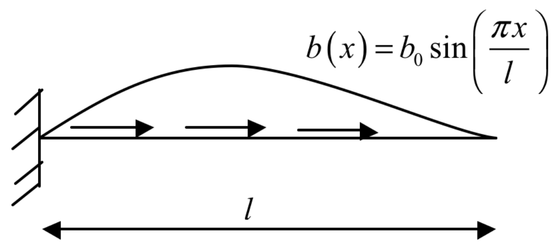


Figure 1: Rod under body load

The exact solution for this boundary value problem is given as

$$u = \frac{b_0 L^2}{\pi^2 E}\left(\sin\frac{\pi x}{L} + \frac{\pi x}{L}\right)$$

$$\sigma = \frac{b_0 L}{\pi}\left(\cos\frac{\pi x}{L} + 1\right)$$

(a) Verify that the above solution satisfies the governing differential equation

$$\sigma' + b = 0, \quad \sigma = Eu'$$

(b) Use the weak-form Galerkin method to find an approximate solution $u_h$ to this problem. Use the following functions to construct trial and test functions.

$$\psi_1 = \sin\left(\frac{\pi x}{4L}\right), \quad \psi_2 = \sin\left(\frac{\pi x}{2L}\right)$$

Show that these functions are appropriate for the weak-form Galerkin method. Compare the approximate displacement function, $u_h$ , and the approximate stress function, $\sigma_h$ , with the exact solutions. For this, plot both the exact and approximate solutions in the same graph. Use $b_0 = 10$, $E = L = 1$.

2. Consider the following boundary value problems. Determine if the given solution approximations are acceptable for the weak-form Galerkin method. If an approximation is not acceptable, specify how it can be fixed.

   (a)

$$
\begin{aligned}
(Eu')' + b &= 0, x \in (0, L) \\
Eu'(0) &= p, \\
u(L) &= 0 \\
u_h &= a_1 \frac{x}{L} + a_2 \frac{x^2}{L^2}
\end{aligned}
$$

   (b)

$$
\begin{aligned}
(Eu')' + b &= 0, x \in (0, L) \\
u(0) &= 1, \\
u(L) &= 2 \\
u_h &= a_1 \frac{x}{L} + a_2 \frac{x^2}{L^2} + a_3 \left( \frac{x}{L} - \frac{x^2}{L^2} \right)
\end{aligned}
$$

3. Derive the weak form of the following advection-diffusion problem with a convective boundary condition at $x = 1$.

$$
\begin{aligned}
(\kappa T')' - cT' + s(x) = 0, \quad x \in (0, 1) \\
T(0) = 1, \\
h\left(T(1) - 10\right) + \kappa T'(1) = 0
\end{aligned}
$$

   In the above equations, $\kappa$, $c$, and $h$ are given constants.