Phys 760: PS 1 Solutions

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1 Problem 1

Explain why a positive definite inner product is non-degenerate
Proof: Assume we have a positive definite inner product $(\cdot,\cdot)$. Then given some function g we have $(g, g) = 0 \Leftrightarrow g = 0$.
Claim: Given any function f such that $(f, g) = 0$, then $g = 0$.
By Contradiction: Assume $g \neq 0$. Then there exists some f such that $f = g$ where $(f, g) = 0$ and $g \neq 0$ which contradicts the definition of a positive definite inner product. Therefore $g = 0$, as required.

2 Problem 2

Show that a non-negative inner product with the property that $(f, f) = 0$ implies that $\forall g(g, f) = 0$
Proof: Assume we have a non-negative inner product $(\forall f(f, f) \geq 0)$.
Claim: $(f, f) = 0 \rightarrow \forall g(g, f) = 0$
By Contradiction: Assume $\exists g s.t.(g, f) \neq 0$. However, if we allow $f = g$ then by the definition of a non-negative inner product $(g, f) = (g, g) = 0$ which is a contradiction to the initial assumption. Therefore $\forall g(g, f) = 0$.

3 Problem 3

Show that if a linear operator is represented by a hermitian matrix in an orthonormal basis, then $(g, Af) = (Ag, f)$.
Proof: Assume we have a linear operator A such that A is a hermitian matrix $(A = A^\dagger)$ in an orthonormal bases $(A * A^T = 0)$. Then we have the following:

$$(g, Af) = (g, A^\dagger f) = (Ag, AA^\dagger f) = (Ag, f)$$

4 Problem 4

Show The Epsilon-Delta Correspondence
Claim:

$$\sum_{i=1}^{3} \epsilon_{ijk}\epsilon_{inm} = \delta_{jn}\delta_{km} - \delta_{jm}\delta_{kn}$$

$$\sum_{i=1}^{3} \epsilon_{ijk}\epsilon_{inm} = \epsilon_{1jk}\epsilon_{1nm} + \epsilon_{2jk}\epsilon_{2nm} + \epsilon_{3jk}\epsilon_{3nm} = \begin{cases} 1 & j = n \;\&\; k = m \\ -1 & j = m \;\&\; k = n \\ 0 & else \end{cases}$$

$$\delta_{jn}\delta_{km} - \delta_{jm}\delta_{kn} = \begin{cases} 1 & j = n \;\&\; k = m \\ -1 & j = m \;\&\; k = n \\ 0 & else \end{cases}$$

5 Problem 5

5.1 Part A

Non-singular NxN matrices form a vector space of dimension N^2
This is false because the 0 matrix is singular and thus not a part of the set created from non-singular matrices.

5.2 Part B

Singular NxN matrices form a vector space of dimension N^2
False. The addition of two singular matrices is not always singular!

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{1}$$

5.3 Part C

Complex numbers form a vector space of dimension 2
True!! It is very easily shown that the addition of two complex numbers is a complex number, the scalar multiplication of a complex number is still complex, and the number 0 is a complex number (albeit a very boring one!).

5.4 Part D

Polynomial functions of x form an infinite-dimensional vector space
True

5.5 Part e

Series $\{a_0, a_1, \ldots, a_N\}$ for which $\sum_{n=0}^{N} |a_n|^2 = 1$ form an $N - dimensional$ vector space.
False. It is simply not possible to have the zero vector (or zero series) satisfying the summation.

6 Problem 6

6.1 Part A

If A is Hermitian and U is unitary then $U^{-1}AU$ is Hermitian
Proof: Assume A is Hermitian and U is unitary. Then we have the following properties:

$$A = A^\dagger$$

$$UU^\dagger = I \Rightarrow U = U^{-\dagger} \Rightarrow U^{-1} = U^\dagger$$

Hence we can do the following...

$$\begin{aligned}
U^{-1}AU &= \sum_{jk} u_{ij}^{-1} a_{jk} ukl \\
&= u_{ji}^\star a_{kl}^\star u_{kl} \quad \text{(by properties above)} \\
&= u_{ji}^\star a_{kj}^\star u_{lk}^{-\star} \quad \text{(by properties above)} \\
&= (UAU^{-1})^\star \\
&= (U^{-1}AU)^\dagger
\end{aligned} \tag{2}$$

Therefore

$$U^{-1}AU = (U^{-1}AU)^\dagger$$

6.2 Part B

If A is antiHermitian then iA is Hermitian

Note: Definition of antiHermitian $:= a_{ij} = -a_{ji}^\star$.

Let $a_{ij} = c + di$. By definition of skew-hermitian we have, $a_{ij} = -aji^\star$.

$$\begin{aligned}
a_{ji} = c + di \rightarrow -a_{ji}^\star &= c + di \\
\rightarrow a_{ji}^\star &= -c - di \\
\rightarrow a_{ji} &= -c + di
\end{aligned}$$

Let now $B = iA \leftarrow b_{ij} = ci - d$ Hence we have :

$$\begin{aligned}
b_{ij} = ci - d \rightarrow b_{ji} &= ci - d \quad \text{(from definition of } a_{ji}) \\
\rightarrow b_{ji}^\star &= ci - d
\end{aligned} \tag{3}$$

Therefore $B = iA$ is Hermitian!

6.3 Part C

The product of two Hermitian matrices A and B is Hermitian iff A and B commute

($\Rightarrow$) Assume the product of two matrices Hermitian matrices is Hermitian.

By Contradiction: Assume $AB \neq BA$. By definition of Hermitian we have:

$$\sum_{k=1}^{N} a_{ik}b_{kj} = AB = (AB)^\dagger = \Big(\sum_k a_{ik}b_{kj}\Big)^\dagger = \Big(\sum_k a_{ik}^\star b_{kj}^\star\Big)^T = \Big(\sum_k b_{jk}^\star a_{ki}^\star\Big)$$

However we must note that

$$\begin{aligned}
a_{ik}b_{kj} &\neq b_{kj}a_{ik} \quad \text{(by definition of non-commuting)} \\
&= b_{jk}^\star a_{ik}^\star \quad \text{(by definition of Hermitian)}
\end{aligned}$$

Hence we have a contradiction

($\Leftarrow$) Assume $AB = BA$ and A, B are Hermitian matrices.

By Contradiction: Assume $AB \neq (AB)^\dagger$. Hence,

$$\sum_k a_{ik}b_{kj} \neq b_{jk}^\star a_{ki}^\star$$

But since A and B are Hermitian we know,

$$a_{ij} = a_{ji}^\star \ \& \ b_{ij} = b_{ji}^\star$$

Furthermore, since $AB = BA$ we can pick an arbitrary element such that we have,

$$\begin{aligned}\sum_k a_{ij}b_{jk} &= b_{kj}a_{ji} \quad \text{by definition of commutative} \\ &= b_{jk}^\star a_i j^\star \quad \text{since } A \text{ and } B \text{ are Hermitian}\end{aligned} \tag{4}$$

However this is a contradiction. Therefore AB must be Hermitian!

6.4 Part D

If S is a real antisymmetric matrix then $A = (! - S)(1 + S)^{-1}$ is orthogonal

Note that $S^T = -S$ by definition of antisymmetric.

$$\begin{aligned}A^T A &= (1+S)^{-T}(1-S)^T(1-S)(1+S)^{-1} \\ &= (1+S)^{-T}(1-S^T)(1-S)(1+S)^{-1} \\ &= (1+S)^{-T}(1+S)(1-S)(1+S)^{-1} \\ &= (1-S)^{-1}(1-S^2)(1+S)^{-1} \\ &= (1-S)^{-1}(1-S)(1+S)(1+S)^{-1} \\ &= I * I \\ &= I\end{aligned}$$

6.5 Part E

If K is antihermitian, then $V = (1+K)(1-K)^{-1}$ is unitary

Note $K^\dagger = -K$. Claim: $V^\dagger V = 1$

$$\begin{aligned}VV^\dagger &= (1+K)(1-K)^{-1}(1-K^\dagger)^{-1}(1+K^\dagger) \\ &= (1-K^\dagger)(1+K^\dagger)^{-1}(1-K^\dagger)^{-1}(1+K^\dagger) \\ &= I * I \\ &= I\end{aligned}$$

7 Problem 7

Two anticommuting matrices A and B satisfy $AB + BA = 0$. If $A^@ = B^2 = 1$ and $[A, B] = 2iC$

7.1 Prove $C^2 = 1$ and that $[B, C] = 2iA$

$$\begin{aligned}(AB - BA)^2 &= (2iC)^2 \\ ABAB - ABBA - BAAB - BABA &= -4C^2 \\ -1 - ABBA - BAAB - 1 &= -4C^2 \mid \text{ABAB = BABA = -1 through the anticommuting property} \\ -1 - 1 - 1 - 1 &= -4C^2 \\ 1 &= C^2\end{aligned}$$

$$\begin{aligned}
[B,C] &= BC - CB \\
&= B(\frac{AB-BA}{2i}) - \frac{AB-BA}{2i}B \\
&= \frac{BAB-BBA}{2i} - \frac{ABB-BAB}{2i} \\
&= \frac{1}{2i}(BAB-BBA-ABB+BAB) \\
&= \frac{1}{2i}(-BBA-BBA-BBA-BBA) \\
&= \frac{-4BBA}{2i} \\
&= 2iA
\end{aligned}$$

7.2 Evaluate $[[[A,B],[B,C]],[A,B]]$

$$\begin{aligned}
[[[A,B],[B,C]],[A,B]] &= [[2iC,2iA],[2iC]] \\
&= [-4CA-4AC,2ic] \\
&= -8iCAC-8iACC-8iCCA-8iCAC \\
&= -16iA-16iCAC \\
&= -32iA
\end{aligned}$$

8 Problem 8

Show $x = (A^\dagger A)^{-1}A^\dagger y$ is the minimization for our error norm E
Let $E' = M * E^2$ for our sanity. Note that now we can define E' in the following manner:

$$E' = (y-Ax)^\dagger(y-Ax) = (y^\dagger - A^\dagger x^\dagger)(y-Ax) = (y^\dagger y - A^\dagger x^\dagger y - y^\dagger Ax + A^\dagger x^\dagger Ax)$$

Now our main goal will be the common minimization technique of taking the derivative and setting it equal to zero...

$$\frac{dE'}{dx} = \frac{d}{dx}\left\{y^\dagger y - A^\dagger x^\dagger y - y^\dagger Ax + A^\dagger x^\dagger Ax\right\} = -y^\dagger A - y^\dagger A + 2x^\dagger A^\dagger A$$

$$\begin{gathered}
y^\dagger A = x^\dagger A^\dagger A \\
x^\dagger = y^\dagger A(A^\dagger A)^{-1} \\
x = (y^\dagger A(A^{-1}A^{-1}))^\dagger = (A^\dagger A)^{-1}A^\dagger y
\end{gathered}$$

9 Problem 9

Find the SVD for our matrix... Using python this is trivial...
$numpy.linalg.svd(np.matrix([[0,-1],[1,1],[-1,0]]))$ returns the following:

```
U is
 [[  4.08248290e-01   7.07106781e-01   5.77350269e-01]
 [ -8.16496581e-01  -2.22044605e-16   5.77350269e-01]
 [  4.08248290e-01  -7.07106781e-01   5.77350269e-01]]
s is
 [ 1.73205081  1.        ]
V is
 [[-0.70710678 -0.70710678]
 [ 0.70710678 -0.70710678]]
```

10 Problem 10

I found all of the problems were quite reasonable and a good portion of my time was spend typing everything up!