Chapter 1

Review: Formulation of Linear Static Problems

1.1 Preliminaries

Let $n_{sd}(= 2 \text{ or } 3)$ denote the *number of space dimensions* of the problem under consideration. Let $\Omega \subset \mathbb{R}^{n_{sd}}$ be an open set with piecewise smooth boundary Γ. A general point in $\mathbb{R}^{n_{sd}}$ is denoted by x. We will identify the point x with its position vector emanating from the origin of $\mathbb{R}^{n_{sd}}$. The *unit outward normal vector to* Γ is denoted by n.

We shall employ the following alternative representations for x and n:

$$(n_{sd} = 2): \qquad x = \{x_i\} = \left\{ \begin{array}{c} x_1 \\ x_2 \end{array} \right\} = \left\{ \begin{array}{c} x \\ y \end{array} \right\} \qquad n = \{n_i\} = \left\{ \begin{array}{c} n_1 \\ n_2 \end{array} \right\} = \left\{ \begin{array}{c} n_x \\ n_y \end{array} \right\} \tag{1.1}$$

where x_i and n_i, $1 \leq i \leq n_{sd}$, are the Cartesian components of x and n, respectively, see Fig. 1.1. Unless otherwise specified we shall work in terms of Cartesian components of vectors and tensors.

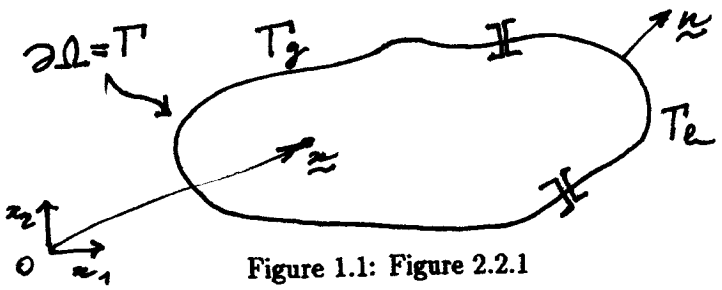

Figure 1.1: Figure 2.2.1

We assume that Γ admits the decomposition

$$\Gamma = \overline{\Gamma_g \bigcup \Gamma_h} \tag{1.2}$$

where

$$\Gamma_g \bigcap \Gamma_h = \emptyset \tag{1.3}$$

and Γ_g and Γ_h are open sets in Γ. The notations are defined as follows: $\bigcup$ is the *set union* symbol. Thus $\Gamma_g \bigcup \Gamma_h$ means the set of all points x contained in either Γ_g or Γ_h. Also, $\bigcap$ is the *set intersection* symbol. Thus $\Gamma_g \bigcap \Gamma_h$ means the set of all points contained in both Γ_g and Γ_h. The *empty set* is denoted by $\emptyset$. Thus (1.3) means that there is no point x contained in both Γ_g and Γ_h (i.e., Γ_g and Γ_h do not intersect or overlap). A bar above a set means *set closure*, i.e., the union of the set with its boundary. Thus

$$\overline{\Omega} = \Omega \bigcup \Gamma \tag{1.4}$$

For further discussion, see [1].

We shall assume throughout that $\Gamma_g \neq \emptyset$ but allow for the case $\Gamma_h = \emptyset$.

Let the indices i, j, k, l, run over the values $1, \ldots, n_{sd}$. Differentiation is denoted by a comma (e.g., $u_{,i} = u_{,x_i} = \partial u / \partial x_i$) and repeated indices imply summation (e.g., in $\mathbb{R}^3$, $u_{,ii} = u_{,11} + u_{,22} + u_{,33} = \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2$). The summation convention only applies to the indices i, j, k and l, and only to two repeated indices. If there are three, or more, repeated indices in an

expression, then the summation convention is *not* in effect. (This is in keeping with the usual convention.)

Divergence theorem. Let $f:\overline{\Omega} \to \mathbf{R}$ be C^1. Then

$$\int_\Omega f_{,i}\, d\Omega = \int_\Gamma f n_i\, d\Gamma \tag{1.5}$$

Integration by parts. Let f be as above and also let $g:\overline{\Omega} \to \mathbf{R}$ be C^1. Then

$$\int_\Omega f_{,i} g\, d\Omega = -\int_\Omega f g_{,i}\, d\Gamma + \int_\Gamma f g n_i\, d\Gamma \tag{1.6}$$

1.2 Classical Linear Heat Conduction: Strong and Weak Forms; Equivalence

Let q_i denote (Cartesian components of) the *heat flux vector*, let u be the *temperature* and let f be the *heat supply per unit volume*. Assume the heat flux vector is defined in terms of the temperature gradient by the *generalized Fourier law*:

$$q_i = \kappa_{ij} u_{,j}, \qquad \kappa_{ij} = \kappa_{ji} \quad \text{(symmetry)} \tag{1.7}$$

where the *conductivities*, κ_{ij}'s, are given functions of x. (If the κ_{ij}'s are constant throughout Ω, the body is said to be *homogeneous*.) The *conductivity matrix* $\kappa = [\kappa_{ij}]$, is assumed positive definite (namely, $p^T \kappa p \geq 0$ for all n_{sd}-vectors p, and $p^T \kappa p = 0$ implies $p = 0$). The most common situation Def. pos.-def. in practice is the *isotropic case* in which $\kappa_{ij}(x) = \kappa(x)\delta_{ij}$, where δ_{ij} is the Kronecker delta.

A formal statement of the strong form of the boundary-value problem is as follows:

(S) Given $f: \Omega \to \mathbf{R}$, $g: \Gamma_g \to \mathbf{R}$ and $h: \Gamma_h \to \mathbf{R}$, find $u: \overline{\Omega} \to \mathbf{R}$ such that

$$q_{i,i} = f \quad \text{in } \Omega \quad (\textit{heat equation}) \tag{1.8}$$

$$u = g \quad \text{on } \Gamma_g \tag{1.9}$$

$$-q_i n_i = h \quad \text{on } \Gamma_h \tag{1.10}$$

where q_i is defined by (1.7).

The functions g and h are the *prescribed boundary temperature and heat flux*, respectively. This problem possesses a unique solution for appropriate restrictions on the given data.

In a more mathematical terminology, (1.8) is a generalized Poisson equation, (1.9) is a Dirichlet boundary condition and (1.10) is a Neumann boundary condition.

We shall now recall the weak formulation of the boundary-value problem. (1.9) and (1.10) will be treated as essential and natural boundary conditions, respectively. Let $\mathcal{S}$ denote the trial solution space and $\mathcal{V}$ the variation space. $\mathcal{S}$ and $\mathcal{V}$ consist of real-valued functions defined on $\overline{\Omega}$ satisfying certain smoothness requirements, such that all members of $\mathcal{S}$ satisfy (1.9), whereas if $w \in \mathcal{V}$, then

$$w = 0 \quad \text{on } \Gamma_g \tag{1.11}$$

The weak formulation of the problem goes as follows:

(W) Given $f:\Omega \to \mathbf{R}$, $g:\Gamma_g \to \mathbf{R}$ and $h:\Gamma_h \to \mathbf{R}$, find $u \in \mathcal{S}$ such that for all $w \in \mathcal{V}$

$$-\int_\Omega w_{,i} q_i \, d\Omega = \int_\Omega w f \, d\Omega + \int_{\Gamma_h} w h \, d\Gamma \tag{1.12}$$

where q_i is defined by (1.7).

Theorem. If all functions involved are sufficiently smooth then a solution of (S) is a solution of (W) and vice versa.

It is convenient to introduce an abstract version of (1.12). Let

$$a(w,u) = \int_\Omega w_{,i} \kappa_{ij} u_{,j} \, d\Omega \tag{1.13}$$

$$(w,f) = \int_\Gamma w f \, d\Omega \tag{1.14}$$

$$(w,h)_\Gamma = \int_{\Gamma_h} w h \, d\Gamma \tag{1.15}$$

Then (1.12) may be written as

$$a(w,u) = (w,f) + (w,h)_\Gamma \tag{1.16}$$

It is useful to also adopt an *index-free* notation for (1.12). Aside from stemming the proliferation of indices, this formulation is conductive to the computer implementation of the element arrays, especially in more complicated situations such as elasticity.

In introducing our index-free notation we shall assume for definiteness that $n_{sd} = 2$. Let ∇

denote the gradient operator; thus

$$\nabla u = \{u_{,i}\} = \begin{Bmatrix} u_{,1} \\ u_{,2} \end{Bmatrix} \tag{1.17}$$

$$\nabla w = \{w_{,i}\} = \begin{Bmatrix} w_{,1} \\ w_{,2} \end{Bmatrix} \tag{1.18}$$

In the case of two space dimensions, the conductivity matrix may be written as

$$\kappa = [\kappa_{ij}] = \begin{bmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{bmatrix} \quad \text{(symmetric)} \tag{1.19}$$

In the isotropic case, (1.19) simplifies to

$$\kappa = \kappa[\delta_{ij}] = \kappa \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{1.20}$$

In terms of the above expressions, the integrand of (1.13) may be written in index-free fashion:

$$w_{,i}\kappa_{ij}u_{,j} = (\nabla w)^T \kappa (\nabla u) \tag{1.21}$$

Thus in place of (1.13) we may write

$$a(w,u) = \int_\Omega (\nabla w)^T \kappa (\nabla u)\, d\Omega \tag{1.22}$$

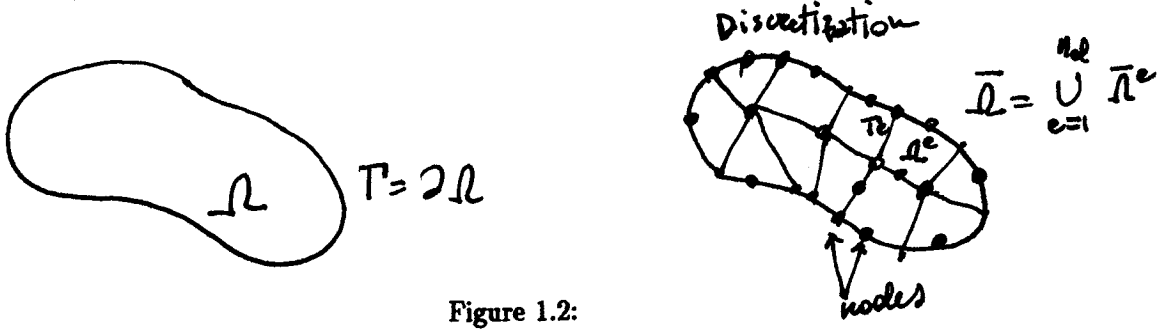

Figure 1.2:

1.3 Heat Conduction: Galerkin Formulation; Symmetry and Positive-definiteness of K

Let $\mathcal{S}^h$ and $\mathcal{V}^h$ be finite-dimensional approximations to $\mathcal{S}$ and $\mathcal{V}$, respectively. We assume all members of $\mathcal{V}^h$ vanish, or vanish approximately, on Γ_g and that each member of $\mathcal{S}^h$ admits the representation

$$u^h = v^h + g^h \tag{1.23}$$

where $v^h \in \mathcal{V}^h$ and g^h results in satisfaction, or at least approximate satisfaction, of the boundary condition $u = g$ on Γ_g.

The Galerkin formulation is given as follows: Def. cf.

$$(G)\left\{\begin{array}{l} \text{Given } f, g \text{ and } h \text{ (as in } (W)\text{), find } u^h = v^h + g^h \in \mathcal{S}^h \text{ such that for all } w^h \in \mathcal{V}^h \\ \\ a(w^h, v^h) = (w^h, f) + (w^h, h)_\Gamma - a(w^h, g^h) \end{array}\right. \tag{1.24}$$

We now view our domain as "discretized" into element domains Ω^e, $1 \leq e \leq n_{el}$. In two dimensions the element domains might be simply triangles and quadrilaterals; see Fig. 1.2. Nodal points may exist anywhere on the domain but most frequently appear at the element vertices and interelement boundaries and less often in the interiors.

In simple one-dimensional models the global nodal ordering and ordering of equations in the matrix system may coincide. In multidimensional applications this would prove to be an inconvenient restriction with regard to data preparation.

In what follows, a more flexible scheme is described. Let $\eta = \{1, 2, \ldots n_{np}\}$, the *set of global node numbers* where n_{np} is the *number of nodal points*. By the terminology g-node we shall mean a node, A, at which it is prescribed that $u^h = g$. Let $\eta_g \subset \eta$ be the set of "g-nodes." The *complement of* η_g in η, denoted by $\eta - \eta_g$, is the set of nodes at which u^h is to be determined. The number of nodes in $\eta - \eta_g$ equals n_{eq}, the *number of equations*.

A typical member of $\mathcal{V}^h$ is assumed to have the form

$$w^h(x) = \sum_{A \in \eta - \eta_g} N_A(x) c_A \tag{1.25}$$

where N_A is the shape function associated with node number A and c_A is a constant. We assume throughout that $w^h = 0$ if and only if $c_A = 0$ for each $A \in \eta - \eta_g$.

$$v^h(x) = \sum_{A \in \eta - \eta_g} N_A(x) d_A \tag{1.26}$$

where d_A is the unknown at node A (i.e., temperature) and

$$g^h(x) = \sum_{A \in \eta_g} N_A(x) g_A, \qquad g_A = g(x_A) \tag{1.27}$$

From (1.27), we see that g^h has been chosen to be the nodal interpolate of g by way of the shape functions. Consequently, g^h will be, generally, only an approximation of g. See Fig. 1.3. Additional sources of error are (1) the use of approximations f^h and h^h in place of f and h, respectively; and (2) domain approximations in which the element boundaries do not exactly coincide with Γ. Analyses

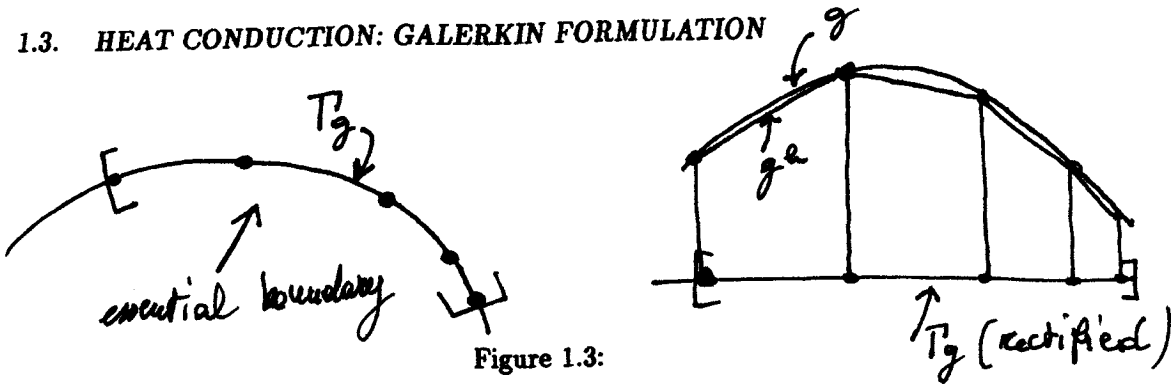

Figure 1.3:

of these approximations are presented in Strang and Fix [2].

Substituting (1.25)–(1.27) into (1.24) and making use of the fact that the c_A's are arbitrary (see [1] for more details), results in

$$\sum_{B\in\eta-\eta_g} a(N_A,N_B)d_B = (N_A,f)+(N_A,h)_\Gamma - \sum_{B\in\eta_g} a(N_A,N_B)g_B, \qquad A\in\eta-\eta_g \tag{1.28}$$

To define the global stiffness matrix and force vector, we need to first specify the global ordering of equations. For this purpose we introduce the ID array, sometimes called the *destination array*, which assigns to node A the corresponding global equation number, viz.,

$$\mathrm{ID}(A) \;=\; \begin{cases} \overbrace{P}^{\text{Global equation number}} & \text{if } A\in\eta-\eta_g \\ 0 & \text{if } A\in\eta_g \end{cases} \tag{1.29}$$

where $1\le P\le n_{eq}$. The dimension of ID is n_{np}. As may be seen from (1.29), nodes at which g is prescribed are assigned "equation number" zero. An example of the setup of ID and other important data-processing arrays is presented in Sec. 1.5.

The matrix equivalent of (1.28) is given as follows:

$$\boldsymbol{K}\boldsymbol{d} \;=\; \boldsymbol{F} \tag{1.30}$$

$$K = [K_{PQ}], \quad d = \{d_Q\}, \quad F = \{F_P\}, \quad 1 \le P,Q \le n_{eq} \tag{1.31}$$

$$K_{PQ} = a(N_A, N_B), \quad P = \mathrm{ID}(A), \quad Q = \mathrm{ID}(B) \tag{1.32}$$

$$F_P = (N_A, f) - (N_A, h)_\Gamma - \sum_{B \in \eta_g} a(N_A, N_B) g_B \tag{1.33}$$

The main properties of K are symmetry and positive-definiteness.

1.4 Heat Conduction: Element Stiffness Matrix and Force Vector

We can break up the global arrays into sums of elemental contributions

$$K = \sum_{e=1}^{n_{el}} K^e, \qquad K^e = [K^e_{PQ}] \tag{1.34}$$

$$F = \sum_{e=1}^{n_{el}} F^e, \qquad F^e = \{F^e_P\} \tag{1.35}$$

where

$$K^e_{PQ} = a(N_A, N_B)^e = \int_{\Omega^e} (\nabla N_A)^T \kappa (\nabla N_B)\, d\Omega \tag{1.36}$$

$$\begin{aligned} F^e_P &= (N_A, f)^e + (N_A, h)^e_\Gamma - \sum_{B \in \eta_g} a(N_A, N_B)^e g_B \\ &= \int_{\Omega^e} N_A f\, d\Omega + \int_{\Gamma^e_h} N_A h\, d\Gamma - \sum_{B \in \eta_g} a(N_A, N_B)^e g_B \end{aligned} \tag{1.37}$$

$$\Gamma^e_h = \Gamma_h \bigcap \Gamma^e, \qquad P = \mathrm{ID}(A), \qquad Q = \mathrm{ID}(B) \tag{1.38}$$

See Fig. 1.4 for an illustration of Γ^e_h

The element stiffness, k^e, and element force vector, f^e, may be deduced from these equations:

$$k^e = [k^e_{ab}], \quad f^e = \{f^e_a\}, \quad 1 \le a,b \le n_{en} \tag{1.39}$$

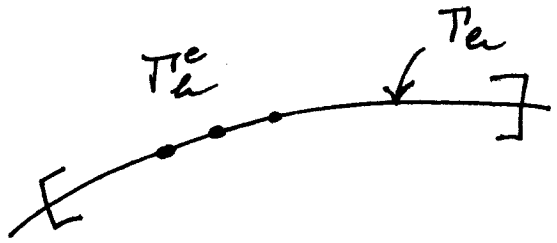

Figure 1.4: Figure 2.5.1

$$k^e_{ab} = a(N_a, N_b)^e = \int_{\Omega^e} (\nabla N_a)^T \kappa (\nabla N_b)\, d\Omega \tag{1.40}$$

$$f^e_a = \int_{\Omega^e} N_a f\, d\Omega + \int_{\Gamma^e_h} N_a h\, d\Gamma - \sum_{b=1}^{n_{en}} k^e_{ab} g^e_b \tag{1.41}$$

where n_{en} is the number of element nodes, and $g^e_b = g(x^e_b)$ of g is prescribed at node number b Del. recall and *equals zero otherwise*. An implicit assumption in localizing the g-term is that if x_A is not a node attached to element e then $N_A(x) = 0$ for all $x \in \overline{\Omega^e}$. Otherwise, the last term in (1.37) may involve g-data of nodes not attached to element e, which is not accounted for in (1.41).

The global arrays, $\boldsymbol{K}$ and $\boldsymbol{F}$ may be formed from the element arrays $\boldsymbol{k}^e$ and $\mathbf{f}^e$, respectively, by way of an assembly algorithm as described in [1].

The element stiffness matrix can be written in a standard form convenient for programming:

$$\boldsymbol{k}^e = \int_{\Omega^e} \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B}\, d\Omega \tag{1.42}$$

where, in the present case,

$$\underbrace{\boldsymbol{D}}_{n_{sd} \times n_{sd}} = \kappa \tag{1.43}$$

$$\underbrace{\boldsymbol{B}}_{n_{sd} \times n_{en}} = [\boldsymbol{B}_1, \ldots, \boldsymbol{B}_{n_{en}}] \tag{1.44}$$

$$\underbrace{\boldsymbol{B}_a}_{n_{sd} \times 1} = \nabla N_a \tag{1.45}$$

The component version of (1.42) is

$$k^e_{ab} = \int_{\Omega^e} B^T_a D B_b \, d\Omega \tag{1.46}$$

Let

$$\underbrace{d^e}_{n_{en} \times 1} = \{d^e_a\} = \begin{Bmatrix} d^e_1 \\ d^e_2 \\ \vdots \\ d^e_{n_{en}} \end{Bmatrix} \tag{1.47}$$

where

$$d^e_a = u^h(x^e_a) \tag{1.48}$$

d^e is called the *element temperature vector*. It can be shown that the heat flux at point $x \in \Omega^e$ can be calculated from the formula

$$q(x) = -D(x)B(x)d^e = -D(x)\sum_{a=1}^{n_{en}} B_a d^e_a \tag{1.49}$$

1.5 Heat Conduction: Data Processing Arrays; ID, IEN and LM

The element nodal data is stored in the array IEN, the *element nodes array*, which relates local node numbers to global node numbers, viz.,

$$\text{IEN}(\underbrace{a}_{\substack{\text{Local}\\ \text{node}\\ \text{number}}}, \underbrace{e}_{\substack{\text{Element}\\ \text{number}}}) = \underbrace{A}_{\substack{\text{Global}\\ \text{node}\\ \text{number}}} \tag{1.50}$$

The relationship between global node numbers and global equation numbers as well as nodal boundary condition information is stored in the ID array, (see (1.29)). In practice, the IEN and ID arrays are set up from input data. The LM array is defined from the relation

$$\text{LM}(a,e) = \text{ID}(\text{IEN}(a,e)) \tag{1.51}$$

Because of the previous relationship, we often think of LM as the element "localization" of ID. Strictly speaking the LM array is redundant. However, it is generally convenient in computing to set up LM once and for all, rather than make use of (1.51) repeatedly. We postpone further description of these arrays until we consider elasticity which is more general than the present case.

In terms of the IEN and LM arrays, a precise definition of the g_a^e's may be given (see (1.41)):

$$g_a^e = \begin{cases} 0 & \text{if } \text{LM}(a,e) \neq 0 \\ g_A & \text{if } \text{LM}(a,e) = 0, \text{ where } A = \text{IEN}(a,e) \end{cases} \tag{1.52}$$

This definition may be easily programmed.

1.6 Classical Linear Elastostatics: Strong and Weak Forms

Classical elastostatics is a rich subject in its own right. See [3]–[7] for background and references to the literature. These references range from the very physical to the very mathematical. The most physical book is the one by Timoshenko and Goodier. In ascending order of mathematical content are Sokolnikoff, Gurtin, Duvant-Lions, Fichera.

The reader is reminded that indices i, j, k and l take on values $1,\ldots,n_{sd}$, where n_{sd} is the number of spatial dimensions, and the summation convention applies to repeated indices i, j, k and l *only*.

Let σ_{ij} denote (Cartesian components of) the (Cauchy) *stress tensor*, let u_i denote the *displacement vector*, and let f_i be the *prescribed body force per unit volume*. The *(infinitesimal) strain tensor* is defined to be the symmetric part of the displacement gradients, viz.,

$$u_{(i,j)} \stackrel{\text{def.}}{=} \frac{u_{i,j} + u_{j,i}}{2} \qquad \textit{(strain-displacement equations)} \tag{1.53}$$

The stress tensor is defined in terms of the strain tensor by the *generalized Hooke's law:*

$$\sigma_{ij} = c_{ijkl} u_{(k,l)} \tag{1.54}$$

where the c_{ijkl}'s, the *elastic coefficients*, are given functions of x. (If the c_{ijkl}'s are constants throughout, the body is called *homogeneous*.) The elastic coefficients are assumed to satisfy the following properties:

Symmetry

$$c_{ijkl} = c_{klij} \qquad \text{(major symmetry)} \tag{1.55}$$

$$\left.\begin{aligned} c_{ijkl} &= c_{jikl} \\ c_{ijkl} &= c_{ijlk} \end{aligned}\right\} \qquad \text{(minor symmetries)} \tag{1.56}$$

Positive-definiteness

$$c_{ijkl}(x)\psi_{ij}\psi_{kl} \geq 0 \tag{1.57}$$

$$c_{ijkl}(x)\psi_{ij}\psi_{kl} = 0 \quad \text{implies} \quad \psi_{ij} = 0 \tag{1.58}$$

for all $x \in \overline{\Omega}$ and all $\psi_{ij} = \psi_{ji}$.

Note. The positive-definiteness condition is in terms of symmetric arrays, ψ_{ij}.

A consequence of the major symmetry (1.55) is that $\boldsymbol{K}$ is symmetric. The first minor symmetry Change ref. implies the symmetry of the stress tensor (i.e., $\sigma_{ij} = \sigma_{ji}$). The positive-definiteness condition, when combined with appropriate boundary conditions on the displacement, leads to the positive-definiteness of $\boldsymbol{K}$.

In the present theory, the unknown is a vector (i.e., the displacement vector).

A formal statement of the strong form of the boundary-value problem goes as follows:

(S) Given $f_i: \Omega \to \mathbf{R}$, $g_i: \Gamma_g \to \mathbf{R}$ and $h_i: \Gamma_h \to \mathbf{R}$, find $u: \overline{\Omega} \to \mathbf{R}$ such that

$$\sigma_{ij,i} + f_i = 0 \quad \text{in } \Omega \quad (\textit{equilibium equations}) \tag{1.59}$$

$$u_i = g_i \quad \text{on } \Gamma_g \tag{1.60}$$

$$\sigma_{ij} n_j = h_i \quad \text{on } \Gamma_h \tag{1.61}$$

where σ_{ij} is defined in terms of u_i by (1.53) and (1.54).

Remark

The functions g_i and h_i are called the *prescribed boundary displacements* and *tractions*, respectively. (S) is sometimes referred to as the *mixed boundary-value problem of linear elastostatics*. Under appropriate hypotheses on the data, (S) possesses a unique solution (see Fichera [4]). The additional complexity of the present theory, when compared with heat conduction, is that the unknown (i.e., $\boldsymbol{u} = \{u_i\}$) is vector-valued rather than scalar-valued. In practice, it is important to be able to deal with somewhat more complicated boundary-condition specification. See [1] for further discussion.

Let $\mathcal{S}_i$ denote the trial solution space and $\mathcal{V}_i$ the variation space. Each member $u_i \in \mathcal{S}_i$ satisfies

$u_i = g_i$ on Γ_g, whereas each $w_i \in \mathcal{V}_i$ satisfies $w_i = 0$ on Γ_g. Equation (1.61) will be a natural boundary condition.

The weak formulation goes as follows

(W) Given $f_i: \Omega \to \mathbf{R}$, $g_i: \Gamma_g \to \mathbf{R}$ and $h_i: \Gamma_h \to \mathbf{R}$, find $u_i \in \mathcal{S}_i$ such that for all $w_i \in \mathcal{V}_i$

$$\int_\Omega w_{i,j}\sigma_{ij}\,d\Omega = \int_\Omega w_i f_i\,d\Omega + \int_{\Gamma_h} w_i h_i\,d\Gamma \tag{1.62}$$

where σ_{ij} is defined in terms of u_i by (1.53) and (1.54).

In the solid mechanics literature, (W) is sometimes referred to as the *principle of virtual work*, or *principle of virtual displacements*, w_i being the *virtual displacements*.

It can be shown that a solution of (S) is a solution of (W), and vice-versa. See [1].

The abstract notation for the present case is

$$a(w,u) = \int_\Omega w_{(i,j)} c_{ijkl} u_{(k,l)}\,d\Omega \tag{1.63}$$

$$(w,f) = \int_\Omega w_i f_i\,d\Omega \tag{1.64}$$

$$(w,h)_\Gamma = \int_{\Gamma_h} w_i h_i\,d\Gamma \tag{1.65}$$

Let $\mathcal{S} = \{u \mid u_i \in \mathcal{S}_i\}$ and let $\mathcal{V} = \{w \mid w_i \in \mathcal{V}_i\}$. Then the weak form can be concisely written in terms of (1.63)–(1.65) as follows:

$$(W)\begin{cases} \text{Given } \boldsymbol{f}, g \text{ and } h \text{ (in which the components are defined in } (W)\text{), find } u \in \mathcal{S} \text{ such that} \\ \text{for all } w \in \mathcal{V} \\ \qquad a(w,u) = (w,f) + (w,h)_\Gamma \end{cases} \tag{1.66}$$

As discussed in Sec. 1.2, it is desirable to construct index-free counterparts of the expression on the right-hand sides of (1.63)–(1.65). For concreteness we shall assume that $n_{sd} = 2$; thus $1 \le i,j,k,l \le 2$.

According to conventional notation $\epsilon = [u_{(i,j)}]$ is the strain tensor. However, we will have no Modify need for this matrix, and consequently we reserve ϵ for the "vector of strain components"

$$\epsilon(u) = \{\epsilon_I(u)\} = \begin{Bmatrix} u_{1,1} \\ u_{2,2} \\ u_{1,2} + u_{2,1} \end{Bmatrix} \tag{1.67}$$

$$\epsilon(w) = \{\epsilon_I(w)\} = \begin{Bmatrix} w_{1,1} \\ w_{2,2} \\ w_{1,2} + w_{2,1} \end{Bmatrix} \tag{1.68}$$

This will considerably simplify subsequent writing. Note that factors of one-half have been eliminated from the *shearing components* (i.e., last components) of (1.67) and (1.68). Compare (1.67) and (1.68) with (1.53). A similar notational redefinition occurs with respect to the "vector of stress components"

$$\sigma = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} \tag{1.69}$$

Table 1.1: Table 2.7.1

I J	i k	j l
1	1	1
3	1	2
3	2	1
2	2	2

Let

$$\boldsymbol{D} = [D_{IJ}] = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ & D_{22} & D_{23} \\ \text{symmetric} & & D_{33} \end{bmatrix} \tag{1.70}$$

where

$$D_{ij} = c_{ijkl} \tag{1.71}$$

in which the indices are related by Table 1.1.

As should be clear, we have "collapsed" pairs of indices (i, j, and k, l) into single indices (I and J, respectively) taking account of the symmetries of c_{ijkl}, $u_{(k,l)}$ and $w_{(i,j)}$. Observe that the indices I and J take on values 1, 2 and 3. In n_{sd} dimensions, the I and J indices will take on values $1, 2, \ldots n_{sd}(n_{sd}+1)/2$.

It can be shown that

$$w_{(i,j)} c_{ijkl} u_{(k,l)} = \epsilon(w)^T \boldsymbol{D} \epsilon(u) \tag{1.72}$$

and so

$$a(w, u) = \int_\Omega \epsilon(w)^T \boldsymbol{D} \epsilon(u)\, d\Omega \tag{1.73}$$

1.7 Elastostatics: Galerkin Formulation; Symmetry, and Positive-definiteness of K

Let $\mathcal{S}^h$ and $\mathcal{V}^h$ be finite-dimensional approximations to $\mathcal{S}$ and $\mathcal{V}$, respectively. We assume members $w^h \in \mathcal{V}^h$ result in satisfaction, or approximate satisfaction of the boundary condition $w_i = 0$ on Γ_g and members of $\mathcal{S}^h$ admit the decomposition

$$u^h = v^h + g^h \tag{1.74}$$

where $v^h \in \mathcal{V}^h$ and g^h results in satisfaction, or approximate satisfaction, of the boundary condition $u_i = g_i$ on Γ_g.

The Galerkin formulation of our problem is given as follows:

$$(G)\left\{\begin{array}{l} \text{Given } f, g \text{ and } h \text{ (as in } (W)\text{), find } u^h = v^h + g^h \in \mathcal{S}^h \text{ such that for all } w^h \in \mathcal{V}^h \\ \\ a(w^h, v^h) = (w^h, f) + (w^h, h)_\Gamma - a(w^h, g^h) \end{array}\right. \tag{1.75}$$

To define the global stiffness matrix and force vector for elasticity, it is necessary to introduce the ID array. This entails a generalization of the definition given in Sec. 1.5, since in the present case there will be more than 1 degree of freedom per node. For elasticity there are n_{sd} degrees of freedom per node, but in order to include in our definition cases such as heat conduction, we shall take the fully general situation in which it is assumed that there are n_{dof} *degrees of freedom*. In

this case

$$\mathrm{ID}(\underbrace{i}_{\substack{\text{Degree of}\\ \text{freedom}\\ \text{number}}}, \underbrace{A}_{\substack{\text{Global}\\ \text{node}\\ \text{number}}}) = \begin{cases} \overbrace{P}^{\substack{\text{Global}\\ \text{equation}\\ \text{number}}} & \text{if } A \in \eta - \eta_g \\ 0 & \text{if } A \in \eta_g \end{cases} \tag{1.76}$$

where $1 \leq i \leq n_{dof}$. Thus ID has dimensions $n_{dof} \times n_{dof}$ If $n_{dof} = 1$, we reduce to the case considered previously in Sec. 1.5 (i.e., $\mathrm{ID}(i, A) = \mathrm{ID}(A)$).

Recall that $\eta = \{1, 2, \ldots, n_{np}\}$ denotes the set of global node numbers. Let $\eta_g \subset \eta$ be the set of nodes at which $u_i^h = g_i$ and let $\eta - \eta_g$ be the complement of η_g. For each node in $\eta - \eta_g$ the nodal value of u_i^h is to be determined.

The explicit representations of v_i^h and g_i^h, in terms of the shape functions and nodal values are

$$v_i^h = \sum_{A \in \eta - \eta_g} N_A \; d \underbrace{_{iA}}_{\substack{\text{Degree of}\\ \text{freedom}\\ \text{number}}} \underbrace{}_{\substack{\text{Global}\\ \text{node}\\ \text{number}}} \tag{1.77}$$

$$g_i^h = \sum_{A \in \eta_g} N_A g_{iA} \tag{1.78}$$

Let e_i denote the ith Euclidean basis vector for $\mathbb{R}^{n_{sd}}$, e_i has a 1 in slot i and zeros elsewhere. For example

$$(n_{sd} = 2) \qquad e_1 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}, \quad e_2 = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \tag{1.79}$$

The vector versions of (1.77) and (1.78) may be defined with the aid of e_i, viz.,

$$v^h = v_i^h e_i \tag{1.80}$$

$$g^h = g_i^h e_i \tag{1.81}$$

Likewise, a typical member $w^h \in \mathcal{V}^h$ has the representation

$$w^h = w_i^h e^i, \quad w_i^h = \sum_{A \in \eta - \eta_g} N_A c_{iA} \tag{1.82}$$

Substituting (1.77), (1.78), and (1.80)–(1.82) into (1.75) and making use of the fact that the c_{iA}'s are arbitrary, results in (verify!)

$$\begin{aligned} \sum_{j=1}^{n_{dof}} \sum_{B \in \eta - \eta_g} a(N_A e_i, N_B e_j) d_{jB} &= (N_A e_i, f) + (N_A e_i, h)_\Gamma \\ &- \sum_{j=1}^{n_{dof}} \sum_{B \in \eta_g} a(N_A e_j, N_B e_j) d_{jB}, \qquad A \in \eta - \eta_g, \qquad 1 \le i \le n_{dof} \end{aligned} \tag{1.83}$$

This is equivalent to the matrix equation

$$\boldsymbol{K}\boldsymbol{d} = \boldsymbol{F} \tag{1.84}$$

where

$$\boldsymbol{K} = [K_{PQ}] \tag{1.85}$$

$$\boldsymbol{d} = \{d_Q\} \tag{1.86}$$

$$\boldsymbol{F} = \{F_P\} \tag{1.87}$$

$$K_{PQ} = a(N_A e_i, N_B e_j) \tag{1.88}$$

$$F_P = (N_A e_i, f) + (N_A e_i, h)_\Gamma - \sum_{j=1}^{n_{dof}} \sum_{B \in \eta_g} a(N_A e_i, N_B e_j) g_{jB} \tag{1.89}$$

in which

$$P = \mathrm{ID}(i, A), \qquad Q = \mathrm{ID}(j, B) \tag{1.90}$$

Equation (1.88) may be written in more explicit form by using (1.73) and noting that (see (1.67) and (1.68)):

$$\boldsymbol{\epsilon}(N_A \mathbf{e}_i) = \boldsymbol{B}_A \mathbf{e}_i \tag{1.91}$$

where, for example,

$$(n_{sd} = 2) \qquad \boldsymbol{B}_A = \begin{bmatrix} N_{A,1} & 0 \\ 0 & N_{A,2} \\ N_{A,2} & N_{A,1} \end{bmatrix} \tag{1.92}$$

With these, (1.88) becomes

$$\underbrace{K_{PQ}}_{\substack{\text{Global}\\ \text{equation}\\ \text{numbers}}} = \mathbf{e}_i^T \overbrace{\int_\Omega \underbrace{\boldsymbol{B}^T{}_A \boldsymbol{D} \boldsymbol{B}_B}_{\text{Global node numbers}} d\Omega}^{\text{Degree of freedom numbers}} \mathbf{e}_j \tag{1.93}$$

and the indices are related by (1.90).

Equation (1.89) is also amenable to explication. Note that, by (1.64)

$$(N_A \mathbf{e}_i, \boldsymbol{f}) = \int_\Omega N_A f_i \, d\Omega \tag{1.94}$$

and likewise by (1.65)

$$(N_A \mathbf{e}_i, \boldsymbol{h})_\Gamma = \int_{\Gamma_h} N_A h_i \, d\Gamma \tag{1.95}$$

Thus (1.89) may be written as

$$F_P = \int_\Omega N_A f_i \, d\Omega + \int_{\Gamma_h} N_A h_i \, d\Gamma - \sum_{j=1}^{n_{dof}} \sum_{B \in \eta_g} a(N_A e_i, N_B e_j) g_{jB} \tag{1.96}$$

Now that we have defined $\boldsymbol{K}$, we can establish its fundamental properties.

$\boldsymbol{K}$ is symmetric and if appropriate displacement boundary conditions are prescribed then $\boldsymbol{K}$ is also positive definite.

Positive-definiteness of $\boldsymbol{K}$ is based upon two requirements: a positive-definiteness condition on the constitutive coefficients and suitable boundary conditions being incorporated into $\mathcal{V}^h$. See [1] for details.

1.8 Elastostatics: Element Stiffness Matrix and Force Vector

As usual $\boldsymbol{K}$ and $\boldsymbol{F}$ may be decomposed into sums of elemental contributions. These results will be omitted here as the reader should now be familiar with the ideas involved (cf. Sec. 1.4). We will proceed directly to the definitions of $\boldsymbol{k}^e$ and $\mathbf{f}^e$:

$$\boldsymbol{k}^e = [k^e_{pq}], \quad \mathbf{f}^e = \{f^e_p\}, \quad 1 \le p, q \le n_{ee} = n_{ed} n_{en} \tag{1.97}$$

$$k^e_{pq} = e_i^T \int_{\Omega e} \boldsymbol{B}_a^T \boldsymbol{D} \boldsymbol{B}_b \, d\Omega \, e_j, \quad p = n_{ed}(a-1) + i, \quad q = n_{ed}(b-1) + j \tag{1.98}$$

$$\boldsymbol{B}_a = \begin{bmatrix} N_{a,1} & 0 \\ 0 & N_{a,2} \\ N_{a,2} & N_{a,1} \end{bmatrix} \qquad (n_{sd} = 2) \tag{1.99}$$

and

$$f_p^e = \int_{\Omega^e} N_a f_i \, d\Omega + \int_{\Gamma_h^e} N_a h_i \, d\Gamma - \sum_{q=1}^{n_{ee}} k_{pq}^e g_q^e, \quad \Gamma_h^e = \Gamma_h \bigcap \Gamma^e \tag{1.100}$$

where $g_q^e = g_{jb}^e = g_j(x_b^e)$ if g_j is prescribed at node b, and equals zero otherwise.

It is useful for programming purposes to define the *nodal submatrix*

$$\underbrace{k_{ab}^e}_{n_{ed} \times n_{ed}} = \int_{\Omega^e} \boldsymbol{B}_a^T \boldsymbol{D} \boldsymbol{B}_b \, d\Omega \tag{1.101}$$

From (1.98) we see that

$$k_{pq}^e = e_i^T k_{ab}^e e_j \tag{1.102}$$

This means, "the pq-component of $\boldsymbol{k}^e$ is the ij-component of the submatrix $\boldsymbol{k}_{ab}^e$."

By (1.97) through (1.99), we see that $\boldsymbol{k}^e$ may be written as

$$\boldsymbol{k}^e = \int_{\Omega^e} \boldsymbol{B}^T \boldsymbol{D} \boldsymbol{B} \, d\Omega \tag{1.103}$$

where

$$\boldsymbol{B} = [\boldsymbol{B}_1, \boldsymbol{B}_2, \ldots, \boldsymbol{B}_{n_{en}}] \tag{1.104}$$

For example, in the case of a two-dimensional (i.e., $n_{sd} = n_{ed} = 2$), four-noded element, $\boldsymbol{k}^e$ takes the form

$$\underbrace{\boldsymbol{k}^e}_{8 \times 8} = \left[\begin{array}{cccc} \boldsymbol{k}_{11}^e & \boldsymbol{k}_{12}^e & \boldsymbol{k}_{13}^e & \boldsymbol{k}_{14}^e \\ \boldsymbol{k}_{21}^e & \boldsymbol{k}_{22}^e & \boldsymbol{k}_{23}^e & \boldsymbol{k}_{24}^e \\ \boldsymbol{k}_{31}^e & \boldsymbol{k}_{32}^e & \boldsymbol{k}_{33}^e & \boldsymbol{k}_{34}^e \\ \boldsymbol{k}_{41}^e & \boldsymbol{k}_{42}^e & \boldsymbol{k}_{43}^e & \boldsymbol{k}_{44}^e \end{array}\right] \tag{1.105}$$

In practice, the submatrices above the dashed lines are computed and those below, if required, are determined by symmetry.

The global arrays $\boldsymbol{K}$ and $\boldsymbol{F}$ may be formed from the element arrays $\boldsymbol{k}^e$ and $\mathbf{f}^e$, respectively, by way of an assembly algorithm as outlined in [1].

Let

$$\underbrace{\boldsymbol{d}^e}_{n_{ee}\times 1} = \{d_a^e\} = \begin{Bmatrix} d_1^e \\ d_2^e \\ \vdots \\ d_{n_{en}}^e \end{Bmatrix} \tag{1.106}$$

$$(n_{ed}=2) \qquad d_a^e = \begin{Bmatrix} d_{1a}^e \\ d_{2a}^e \end{Bmatrix} \tag{1.107}$$

where

$$d_{ia}^e = u_i^h(x_a^e) \tag{1.108}$$

$\boldsymbol{d}^e$ is called the *element displacement vector*. The vector of stress components (1.69) at point $x \in \Omega^e$ can be calculated from the formula

$$\sigma(x) = \boldsymbol{D}(x)\boldsymbol{B}(x)\boldsymbol{d}^e = \boldsymbol{D}(x)\sum_{a=1}^{n_{en}} \boldsymbol{B}_a(x)\boldsymbol{d}_a^e \tag{1.109}$$

1.9 Elastostatics: Data Processing Arrays ID, IEN, and LM

We have already noted that the definition of the ID array must be generalized for the present case as indicated in Sec. 1.7. We must also generalize our definition of the LM array. However, the IEN array remains the same as before.

In the present and fully general cases, the LM array is three-dimensional, with dimensions

$n_{ed} \times n_{en} \times n_{el}$, and is defined by

$$\underset{\substack{\text{Degree of}\\ \text{freedom}\\ \text{number}}}{\text{LM}(\underbrace{i}, \underbrace{a}_{\substack{\text{Local}\\ \text{node}\\ \text{number}}}, \underbrace{e}_{\substack{\text{Element}\\ \text{number}}})} = \text{ID}(i, \text{IEN}(a, e)) \tag{1.110}$$

Alternatively, it is sometimes convenient to think of LM as two-dimensional, with dimensions $n_{ee} \times n_{el}$, viz.,

$$\underset{\substack{\text{Local}\\ \text{equation}\\ \text{number}}}{\text{LM}(\underbrace{p}, \underbrace{e}_{\substack{\text{Element}\\ \text{number}}})} = \text{LM}(i, a, e), \qquad p = n_{ed}(a-1) + i \tag{1.111}$$

To see how everything works in practice, it is helpful to run through a simple example.

Example 1

Consider the mesh of four-node, rectangular elements illustrated in Fig. 1.5. We assume that the local node numbering begins in the lower left-hand corner for each element and proceeds in counterclockwise fashion. This is shown for element 4, which is typical. Four displacement (i.e., "g-type") boundary conditions are specified; namely, the horizontal displacement is specified at nodes 1 and 10, and the vertical displacement is specified as nodes 1 and 3. Since $n_{np} = 12$, $n_{dof} = n_{ed} = 2$, and 4 displacement degrees of freedom are specified, we have $n_{eq} = 20$. We adopt the convention that the global equation numbers run in ascending order with respect to the ascending order of global node numbers. However, in practice, equation numbers are often renumbered internally to minimize the bandwidth of K and thus decrease storage and solution effort. The ID, EN, and LM arrays are given in Fig. 1.6. The reader is urged to verify the results.

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