Homework4

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1 Problem 1

Consider the hyperelastic model generated by using the following strain energy density function Φ :

$$\Phi(E) = \lambda\frac{J^2-1}{4} - (\lambda/2+\mu)ln(J) + \mu E_{NN}$$

where λ, μ are the Lame parameters (see homework 1), E is the Green strain tensor discussed in class, and J = det F. The trace of E is indicated above by $E_{NN} = E_{11} + E_{22} + E_{33}$.

Solution:

Note: In order to complete this calculation we will need the derivation of $\frac{\partial J}{\partial E_{ij}}$. See appendix for calculation.

$$\begin{aligned}
\frac{\partial \Phi}{\partial E_{ij}} &= \frac{\partial}{\partial E_{ij}}\Big(\lambda\frac{J^2-1}{4} - (\lambda/2+\mu)ln(J) + \mu E_{rs}\delta_{rs}\Big) \\
&= \frac{\lambda}{4} * 2J\frac{\partial J}{\partial E_{ij}} - (\lambda/2+\mu)\frac{\frac{\partial J}{\partial E_{ij}}}{J} \\
&= \frac{\lambda}{2}JJF^{-1}_{Ii}F^{-1}_{Ji} - \frac{\lambda}{2}F^{-1}_{Ii}F^{-1}_{Ji} - \mu F^{-1}_{Ii}F^{-1}_{Ji} + \mu\delta_{IJ} \\
&= \mu[\delta_{IJ} - F^{-1}_{Ii}F^{-1}_{Ji}] + \frac{\lambda}{2}(J^2-1)F^{-1}_{Ii}F^{-1}_{Ji}
\end{aligned} \tag{1}$$

Now working within indices, show that the material stiffness modulus C_{IJKL} is given given by:

$$C_{IJKL} := \frac{\partial S_{IJ}}{\partial E_{KL}} = 2\mu\Big[1+\frac{\lambda}{2\mu}(1-J^2)\Big]F^{-1}_{Im}F^{-1}_{Jn}F^{-1}_{Km}F^{-1}_{Ln} + \lambda J^2F^{-1}_{Im}F^{-1}_{Jm}F^{-1}_{Kn}F^{-1}_{Ln}$$

Note the following relations:

1.
$$\frac{\partial(F^{-1}_{Ii}F^{-1}_{Ji})}{\partial E_{KL}} = \frac{\partial(F^{-1}F^{-T})_{IJ}}{\partial E_{KL}}$$

2.
$$C = F^TF$$

3.
$$C_{MI}C^{-1}_{IJ} = \delta_{MI}$$

4.
$$C_{MI} = 2E_{MI} - \delta^{MI}$$

See appendix for calculation of $\frac{\partial C^{-1}_{IJ}}{\partial E_{KL}}$

$$\begin{aligned}
C_{IJKL} &= \mu \frac{\partial \delta_{IJ}}{\partial E_{KL}} - \mu \frac{\partial}{\partial E_{KL}}(F^{-1}_{Ii}F^{-1}_{Ji}) + \frac{\lambda}{2}\frac{\partial}{\partial E_{KL}}\Big[(J^2-1)F^{-1}_{Ii}F^{-1}_{Ji}\Big] \\
&= 2\mu C^{-1}_{IJ}C^{-1}_{KL} + \frac{\lambda}{2}2J^2 F^{-1}_{Ki}F^{-1}_{Li}F^{-1}_{Im}F^{-1}_{Jm} + \lambda(J^2-1)C^{-1}_{IJ}C^{-1}_{KL} \\
&= 2\mu F^{-1}_{Ki}F^{-1}_{Li}F^{-1}_{Im}F^{-1}_{Jm} + \lambda J^2 F^{-1}_{Ki}F^{-1}_{Li}F^{-1}_{Im}F^{-1}_{Jm} + \lambda(J^2-1)F^{-1}_{Ki}F^{-1}_{Li}F^{-1}_{Im}F^{-1}_{Jm} \\
&= 2\mu\Big[1+\frac{\lambda}{2\mu}(1-J^2)\Big]F^{-1}_{Im}F^{-1}_{Jm}F^{-1}_{Ki}F^{-1}_{Li} + \lambda J^2 F^{-1}_{Im}F^{-1}_{Jm}F^{-1}_{Ki}F^{-1}_{Li}
\end{aligned} \tag{2}$$

Show now that

$$\sigma_{ij} = \frac{1}{J}F_{iI}S_{IJ}F_{jJ} = \frac{1}{J}\Big[\mu(F_{iJ}F_{jJ} - \delta_i j + \frac{\lambda}{2}(J^2-1)\delta_{ij})\Big]$$

$$\begin{aligned}
\sigma_{ij} &= \frac{1}{J}F_{iI}S_{IJ}F_{jJ} \\
&= \frac{1}{J}F_{iI}\Big(\mu[\delta_{IJ} - F^{-1}_{Ii}F^{-1}_{Ji}] + \frac{\lambda}{2}(J^2-1)F^{-1}_{Ii}F^{-1}_{Ji}\Big)F_{jJ} \\
&= \frac{1}{J}\Big(\mu[F_{iI}\delta_{IJ}F_{jJ} - F_{iI}F^{-1}_{Ii}F^{-1}_{Ji}F_{jJ}] + \frac{\lambda}{2}(J^2-1)F_{iI}F^{-1}_{Ii}F^{-1}_{Ji}F_{jJ}\Big) \\
&= \frac{1}{J}\Big(\mu[F_{iJ}F_{jJ} - \delta_{ij}] + \frac{\lambda}{2}(J^2-1)\delta)_{ij}\Big)
\end{aligned} \tag{3}$$

And finally show

$$c_{ijkl} = \frac{1}{J}F_{iI}F_{jJ}F_{kK}F_{lL}C_{IJKL} = \frac{1}{J}\Big\{\lambda J^2\delta_{ij}\delta_{kl} + 2\mu\Big[1+\frac{\lambda}{2\mu}(1-J^2)\Big]d_{ik}d_{kl}\Big\}$$

$$\begin{aligned}
c_{ijkl} &= \frac{1}{J}F_{iI}F_{jJ}F_{kK}F_{lL}C_{IJKL} \\
&= \frac{1}{J}F_{iI}F_{jJ}F_{kK}F_{lL}\Big\{2\mu\Big[1+\frac{\lambda}{2\mu}(1-J^2)\Big]F^{-1}_{Im}F^{-1}_{Jn}F^{-1}_{Km}F^{-1}_{Ln} + \lambda J^2 F^{-1}_{Im}F^{-1}_{Jm}F^{-1}_{Kn}F^{-1}_{Ln}\Big\} \\
&= \frac{1}{J}\Big\{2\mu\Big[1+\frac{\lambda}{2\mu}(1-J^2)\Big]F_{iI}F_{jJ}F^{-1}_{Im}F^{-1}_{Jn}F^{-1}_{Km}F^{-1}_{Ln}F_{kK}F_{lL} + \lambda J^2 F_{iI}F_{jJ}F^{-1}_{Im}F^{-1}_{Jm}F^{-1}_{Kn}F^{-1}_{Ln}JF_{kK}F_{lL}\Big\} \\
&= \frac{1}{J}\Big\{\lambda J^2\delta_{ij}\delta_{kl} + 2\mu\Big[1+\frac{\lambda}{2\mu}(1-J^2)\Big]d_{ik}d_{kl}\Big\}
\end{aligned} \tag{4}$$

If we consider small deformation then $J = 1$ then all the parts where we have $1 - J^2 = 0$

$$\sigma_{ij} = \frac{1}{J}F_{iI}S_{IJ}F_{jJ} = \frac{1}{J}\Big[\mu(F_{iJ}F_{jJ} - \delta_i j)\Big]$$

$$c_{ijkl} = \frac{1}{J}F_{iI}F_{jJ}F_{kK}F_{lL}C_{IJKL} = \frac{1}{J}\Big\{\lambda J^2\delta_{ij}\delta_{kl} + 2\mu d_{ik}d_{kl}\Big\}$$

2 Appendix

2.1 Derivations

2.1.1 $\frac{\partial J}{\partial E_{ij}}$

$$
\begin{aligned}
\frac{\partial J}{\partial E_{ij}} &= \frac{\partial}{\partial E_{ij}}\Big[(det(2E+I))^{1/2}\Big] \\
&= \frac{1}{2}(det(2E+I))^{1/2} * cof(2E+I)_{IJ} * 2 \\
&= \frac{1}{J} * det(F^TF)(F^TF)^{-1}_{IJ} \\
&= JF^{-1}_{Ii}F^{-1}_{Ji}
\end{aligned}
\tag{5}
$$

2.2 $\frac{\partial C^{-1}_{IJ}}{\partial E_{KL}}$

$$
\begin{aligned}
C_{MI}C^{-1}C_{IJ} &= \delta_{MI} \\
\frac{\partial C_{MI}}{\partial E_{KL}}C^{-1}_{IJ} - C_{MI}\frac{\partial C^{-1}_{IJ}}{\partial E_{KL}} &= 0 \\
\frac{\partial(2E_{MI} - \delta_{MI})}{\partial E_{KL}}C^{-1}_{IJ} &= C_{MI}\frac{\partial C^{-1}_{IJ}}{\partial E_{KL}} \\
C^{-1}_{MI}2\frac{\partial E_{MI}}{\partial E_{KL}} &= \frac{\partial C^{-1}_{IJ}}{E_{KL}} \\
2C^{-1}_{MI}\frac{\delta_{MK}\delta_{IL}\partial E_{KL}}{\partial E_{KL}} &= \frac{\partial C^{-1}_{IJ}}{E_{KL}} \\
\frac{\partial C^{-1}_{IJ}}{E_{KL}} &= 2C^{-1}_{ML}C^{-1}_{IJ}
\end{aligned}
\tag{6}
$$

3 Problem 2