Physics 760 PS 3

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1 Problem 1

Find an analytic function whose imaginary part is $(ycos(y)+xsin(y))e^x$

$$\frac{dv}{dy} = e^x(cos(y) - ysin(y) + xcos(y))$$

By CR equations we have

$$\frac{du}{dx} = e^x(cos(y) - ysin(y) + xcos(y))$$

Hence let's integrate wrt to x in order to get u...

$$u = \frac{\partial u}{\partial x}dc = \int e^x(cos(y)-ysin(y)+xcos(y))dx = e^x(cos(y)-ysin(y)+cos(y)x-cos(y))+C(y)$$

Integrating wrt to y we get...

$$\frac{\partial u}{\partial y} = e^x(sin(y) - ycos(y) - sin(y) - xsin(y) + sin(y)) + C'(y)$$

Again using CR relations we have

$$\frac{\partial v}{\partial x} = -e^x(sin(y) - ycos(y) - xsin(y)) + C'(y)$$

But if we differentiate the initial equation we have wrt x we get

$$e^x(cos(y)y + xsin(y) - sin(y))$$

Hence by comparison we quickly realize that $C'(y) = 0 \rightarrow C(y) =$ constant
Therefore

$$f(x,y) = e^x(cos(y) - ysin(y) + cos(y)x - cos(y)) + i * ((ycos(y) + xsin(y))e^x)$$

2 Problem 2

Determine the types of singularities at $z = 0$ and $z = \infty$

2.1 Part A

$$\frac{1}{z-2}$$
$$@z = 0 \;\; \text{Obviously analytic}$$
$$@z = \infty \text{ we have } \frac{1}{\frac{1}{\xi}-2} \to \frac{1}{\infty} = 0 \text{ So analytic}$$

2.2 Part B

$$\frac{1+z^3}{z^2} = \frac{1}{z^2} + z$$
$$@z = 0 \text{ Second order pole}$$
$$@z = \infty \;\; \to \xi^2 + \frac{1}{\xi} \text{ Simple Pole}$$

2.3 Part C

$$sinh(\frac{1}{z}) = \frac{1}{z} + \frac{!}{3!z^3} + \dots$$
$$@z = 0 \text{ Clearly Essential Singular Point}$$
$$@z = \infty \to \xi + \frac{!}{3!}\xi^3 + \dots \text{ hence analytic}$$

2.4 Part D

$$\frac{e^z}{z^3} = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2!z} + \frac{1}{3!} + \frac{z}{4!} + \dots$$
$$@z = 0 \text{ Singular pole of order 3}$$
$$@z = \infty \text{ Essential Singular since } \;\xi^3 + \xi^2 + \frac{1}{2}\xi + \frac{1}{6} + \frac{1}{4!\xi} + \dots$$

2.5 Part E

$$\frac{z^{1/2}}{(1+z^2)^{1/2}}$$
$$@z = 0 \text{ Branch point since it makes the value under the root zero}$$
$$@z = \infty \to \frac{\frac{1}{\xi}^{\frac{1}{2}}}{(1+\frac{1}{\xi^2})^{\frac{1}{2}}} \to 0 \text{ if } \xi \to 0 \text{ therefore its a Branch point}$$

3 Problem 3

Show that $exp(iaz^2)$ is analytic and then evaluate the following integral:

$$\int_0^\infty cos(at^2)dt$$

3.1 Part A

Show e^{iaz^2} is analytic

$$e^{iaz^2} = e^{ai(x^2-y^2)-2axy} = e^{-2axy}[cos(a(x^2-y^2)) + sin(a(x^2-y^2))]$$
$$\frac{du}{dx} = -2aye^{-2axy}cos(a(x^2-y^2)) - e^{-2axy}sin(a(x^2-y^2)) * 2ax$$
$$\frac{dv}{dy} = -2axe^{-2axy}sin(a(x^2-y^2)) - e^{-2axy}cos(a(x^2-y^2)) * 2ay$$

Since $\frac{du}{dx} == \frac{dv}{dy}$ our function is analytic!

3.2 Part B

Evaluation of integral (I have seen this before in a complex analysis course I took as an undergrad)...
First notice $e^{-iat^2} = cos(at^2) + isin(at^2)$ by the Euler Formula. Now lets just integrate e^{-iat^2} which is the well-known Gaussian Integral with a constant coefficient on t.

$$\int_{-\infty}^{\infty} e^{-iat^2}dt = \sqrt{\frac{\pi}{ia}}$$

Hence

$$\int_{0}^{\infty} e^{-iat^2}dt = \int_{0}^{\infty} cos(at^2) + isin(at^2)dt = \frac{1}{2}\sqrt{\frac{\pi}{ia}}$$

It is important to note that $\sqrt{\frac{1}{i}} = \sqrt{i}^{-1} = e^{-i\pi/4} = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$ by Euler's Formula. Hence we have

$$\int_{0}^{\infty} cos(at^2) + isin(at^2)dt = \sqrt{\frac{\pi}{8a}} - i\sqrt{\frac{\pi}{8a}}$$

Thus we can conclude the following:

$$\int_{0}^{\infty} cos(at^2)dt = \sqrt{\frac{\pi}{8a}}$$

4 Problem 4

4.1 Part A

4.2 Part B

Evaluate the following integral:

$$I = \int_{1}^{\infty} \frac{dx}{x(x^2-1)^{\frac{1}{2}}}$$

Clearly we have a branch point a $z = \pm i$. However due to the bounds we only need to worry about $z = i$.

$$2I = \oint_{\gamma} \frac{dz}{z(z^2-1)^{\frac{1}{2}}}$$

Where γ is a contour enclosing our branch point at $z = i$. In order to solve this integral we simply need ot find the residue at the Poles...

$$\lim_{z \to 0} z \frac{1}{z(z^2-1)^{\frac{1}{2}}} = \frac{1}{z(z^2-1)^{\frac{1}{2}}}$$

Now we can plug in our pole and go about our business:

$$\frac{1}{(i^2-1)^{\frac{1}{2}}} = \frac{1}{2i}$$

Thus our answer is

$$I = \frac{2\pi i}{2 * 2i} = \frac{\pi}{2}$$

Using $x = sec(t)$ and mathematica i was able to check my result (as you advised).

5 Problem 5

Solve the following:

$$\int_0^\infty \frac{dx}{1+x^n}$$

over a wedge of angle $2pi/n$

$$\oint \frac{dz}{1+z^n} = 2\pi i Res\Big(\frac{1}{1+z^n}, e^{i\pi/n}\Big) = 2\pi i \frac{1}{e^{-i\pi/n}(e^{i\pi/n} + e^{i\pi - i\pi/n})}$$

We can do that last step because we pull out $e^{i\pi/n}$ (which is the singular point) and then take the limit to find the residual!

$$2\pi i \frac{1}{e^{-i\pi/n}(e^{i\pi/n} + e^{i\pi - i\pi/n})} = 2\pi i \frac{1}{e^{i\pi/n} - e^{-i\pi/n}} = \frac{2\pi i csc(\frac{\pi}{n})}{2i} = \pi csc\Big(\frac{\pi}{n}\Big)$$

6 Problem 6

6.1 Part A

I will be using the residue theorem.

$$\begin{aligned} 2I = \int_{-\infty}^{\infty} \frac{ln(x)^2}{1+x^2} dx &= 2\pi i Res\Big(\frac{ln(x)^2}{1+x^2}, i\Big) \\ &= 2\pi i \lim_{z \to i} \frac{ln(z)^2}{(z-i)(z+i)}(z-i) \\ &= 2\pi i \frac{ln(i)^2}{2i} \\ &= \pi ln(i)^2 \\ &= -\frac{\pi^3}{4} \end{aligned}$$

Therefore $I = -\frac{\pi^3}{8}$.

6.2 Part B

Demonstrate the following:

$$\int_0^\infty \frac{ln(x)}{1+x^2}dx = 0$$

$$\begin{aligned}
\int_0^\infty \frac{ln(x)^2}{1+x^2}dx &= \int_0^\infty \frac{(ln(|z|)+i\pi)^2}{1+z^2}dz \\
&= \int_0^\infty \frac{ln(|z|)}{z^2+1}dz + 2\pi i \int_0^\infty \frac{ln(z)}{z^2+1}dz - \pi^2 \int_0^\infty \frac{1}{z^2+1}dz \\
&= -\frac{\pi^3}{4} \quad \text{By part A}
\end{aligned}$$

Hence we can easily see that

$$\int_0^\infty \frac{ln(x)}{x^2+1}dx = 0$$

by simply examining the real and imaginary parts of the previous expression!