CEE 630 / ME 525
Nonlinear Finite Element Analysis
Spring 2017
Professor Guglielmo Scovazzi
Homework # 4
Due Tuesday, March 21 2017.

1. Consider the hyperelastic model generated by using the following strain energy density function Φ:

$$\Phi(\mathbf{E}) := \lambda \frac{J^2 - 1}{4} - (\lambda/2 + \mu) \ln J + \mu E_{NN}$$

where λ, μ are the Lame parameters (see homework 1), $\mathbf{E}$ is the Green strain tensor discussed in class, and $J = \det \mathbf{F}$. The *trace* of $\mathbf{E}$ is indicated above by $E_{NN} = E_{11} + E_{22} + E_{33}$.

a. Working in indices, show that

$$S_{IJ} := \frac{\partial \Phi}{\partial E_{IJ}} = \mu[\delta_{IJ} - F_{Ii}^{-1} F_{Ji}^{-1}] + \frac{\lambda}{2}(J^2 - 1) F_{Ii}^{-1} F_{Ji}^{-1}$$

by direct differentiation of Φ.

Important hint: In your derivation, you will need to calculate $\frac{\partial J}{\partial E_{IJ}}$. This can be done by recalling that:

$$\begin{aligned} 2\mathbf{E} + \mathbf{I} &= \mathbf{F}^T \mathbf{F}, \text{so that} \\ \det(2\mathbf{E} + \mathbf{I}) &= \det(\mathbf{F}^T \mathbf{F}) = J^2, \text{giving} \\ J &= [\det(2\mathbf{E} + \mathbf{I})]^{1/2} \end{aligned}$$

We next note two important identities from tensor calculus, which apply to any nonsingular square matrix $\mathbf{A}$:

$$\begin{aligned} \frac{\partial \det \mathbf{A}}{\partial A_{IJ}} &= (\text{cof}\mathbf{A})_{IJ}, \text{and} \\ (\text{cof}\mathbf{A})_{IJ} &= (\det \mathbf{A}) A_{JI}^{-1} \end{aligned}$$

where cof$\mathbf{A}$ is the matrix of cofactors of $\mathbf{A}$ (i.e., $(\text{cof}\mathbf{A})_{IJ}$ is the determinant of the matrix left over when you delete the I^{th} row and J^{th}

column of $\mathbf{A}$, with a minus sign if $(-1)^{I+J} < 0$). We can use these relationships (you should do this!) to conclude that

$$\frac{\partial J}{\partial E_{IJ}} = J F^{-1}_{Ii} F^{-1}_{Ji}$$

from which the desired result easily follows.

b. Still working in indices, show that the material stiffness moduli C_{IJKL} are given by

$$C_{IJKL} := \frac{\partial S_{IJ}}{\partial E_{KL}} = 2\mu \left[1 + \frac{\lambda}{2\mu}(1 - J^2)\right] F^{-1}_{Im} F^{-1}_{Jn} F^{-1}_{Km} F^{-1}_{Ln} + \lambda J^2 F^{-1}_{Im} F^{-1}_{Jm} F^{-1}_{Kn} F^{-1}_{Ln}$$

Another important hint: In this calculation, you need to calculate

$$\frac{\partial (F^{-1}_{Ii} F^{-1}_{Ji})}{\partial E_{KL}} = \frac{\partial (\mathbf{F}^{-1} \mathbf{F}^{-T})_{IJ}}{\partial E_{KL}}$$

There are several ways to do this; one possibility is to define the right Cauchy-green tensor via $\mathbf{C} = \mathbf{F}^T \mathbf{F}$; then the calculation amounts to getting

$$\frac{\partial C^{-1}_{IJ}}{\partial E_{KL}}$$

I would recommend doing this by differentiating the identity

$$C_{MI} C^{-1}_{IJ} = \delta_{MJ}$$

and noting that

$$C_{MI} = 2E_{MI} - \delta_{MI}$$

c. Finally, verify the following results for the spatial moduli c_{ijkl} and the Cauchy stress σ_{ij}:

$$\sigma_{ij} = \frac{1}{J} F_{iI} S_{IJ} F_{jJ} = \frac{1}{J}\left[\mu(F_{iJ} F_{jJ} - \delta_{ij}) + \frac{\lambda}{2}(J^2 - 1)\delta_{ij}\right]$$

and

$$\begin{aligned} c_{ijkl} &= \frac{1}{J} F_{iI} F_{jJ} F_{kK} F_{lL} C_{IJKL} \\ &= \frac{1}{J}\left\{\lambda J^2 \delta_{ij}\delta_{kl} + 2\mu \left[1 + \frac{\lambda}{2\mu}(1 - J^2)\right] \delta_{ik}\delta_{jl}\right\} \end{aligned}$$

Since we are dealing with stresses and strains that possess symmetries between i, j and k, l, we usually write c_{ijkl} in the following form, which is equivalent in this case:

$$c_{ijkl} = \frac{1}{J}\left\{\lambda J^2 \delta_{ij}\delta_{kl} + \mu\left[1 + \frac{\lambda}{2\mu}(1 - J^2)\right][\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}]\right\}$$

In the limit of small deformations ($\mathbf{F} \approx \mathbf{I}$) how do the expressions for σ_{ij} and c_{ijkl} simplify? Are they what we should expect?

2. Modify your one-dimensional elastic rod Matlab code (with incremental loading) to model elasto-plasticity. Use the elastic-plastic constitutive law detailed in class with $E = 10$, $\sigma_Y = 5$, and $H = 4$. Modify the incremental loading scheme to first load, and then unload. Use boundary conditions of $g = 0.0$ and $h = 10$, and set the loading to occur over 10 load steps. Add an additional figure which plots the stress in the second element as a function of strain over the entire loading history. Submit a gzipped folder of all routines via email.