CEE 630 / ME 525
Nonlinear Finite Element Analysis

Spring 2017
Professor Guglielmo Scovazzi

Homework # 1
Due in class (paper format), on Tuesday January 31, 2017

If you need help, refer to Hughes [1987] (mentioned in the syllabus) or the review notes posted on Sakai, or any other introductory finite element book of your choice. Show all work.

1. Beginning with the following strong form of the linear elasticity boundary value problem:

$$\left\{ \begin{array}{c} \sigma_{ij,j} + f_i = 0 \quad \text{in } \Omega \\ u_i = g_i \quad \text{on } \Gamma_g \\ \sigma_{ij} n_j = h_i \quad \text{on } \Gamma_h \end{array} \right\} \quad (\mathcal{S})$$

where

$$\sigma_{ij} = c_{ijkl}\epsilon_{kl}$$
$$\epsilon_{kl} = u_{(k,l)} = \tfrac{1}{2}(u_{k,l} + u_{l,k})$$

use an argument analogous to that given in class for heat conduction to show that

$$\int\limits_{\Omega} w_{(i,j)}\sigma_{ij}\, d\Omega = \int\limits_{\Omega} w_i f_i\, d\Omega + \int\limits_{\Gamma_h} w_i h_i\, d\Gamma \quad (\mathcal{W})$$

If we define solution and weighting spaces $\mathcal{S}_i$ and $\mathcal{V}_i$ such that

$$\mathcal{S}_i = \{u_i : u_i = g_i \quad \text{on } \Gamma_g \text{ and } u_i \text{ is smooth}\}$$

$$\mathcal{V}_i = \{w_i : w_i = 0 \quad \text{on } \Gamma_g \text{ and } w_i \text{ is smooth}\}$$

the so-called *weak form* of $(\mathcal{S})$ amounts to enforcing $(\mathcal{W})$ for all $w_i \in \mathcal{V}_i, i = 1, \ldots, n_{comp}$.

2. For isotropic elasticity, the fourth order elasticity tensor $\mathbf{c}$ has the following form:

$$c_{ijkl} = \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda\delta_{ij}\delta_{kl}$$

where λ and μ, the Lame parameters, are given in terms of Young's modulus E and Poisson's ratio ν via

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

$$\mu = \frac{E}{2(1+\nu)}$$

a. Find a matrix $\mathbf{D} = [D_{IJ}]$, where I, J are the reduced indices discussed in class, such that

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\epsilon}(\mathbf{u})$$

For simplicity, you may work in two dimensions.

b. Starting from

$$a(\mathbf{w}, \mathbf{u}) = \int_{\Omega} w_{(i,j)} c_{ijkl} u_{(k,l)} \, d\Omega,$$

show that $a(\mathbf{w}, \mathbf{u})$ can also be written as

$$\begin{aligned} a(\mathbf{w}, \mathbf{u}) &= \int_{\Omega} \epsilon(\mathbf{w})^T \mathbf{D} \epsilon(\mathbf{u}) \, d\Omega \\ &= \int_{\Omega} \epsilon(\mathbf{w})^T \boldsymbol{\sigma} \, d\Omega \end{aligned}$$

(again, feel free to work in two dimensions)

3. Consider the Galerkin approximation to the weak form

$$a(\mathbf{w}^h, \mathbf{u}^h) = (\mathbf{w}^h, \mathbf{f}) + (\mathbf{w}^h, \mathbf{h})_{\Gamma_h}$$

where $\mathbf{w}^h, \mathbf{u}^h$ have the representations given in class.

a. Use similar reasoning to that applied for heat conduction to argue that

$$\begin{aligned} \sum_{j=1}^{n_{comp}} \left(\sum_{B \in \eta - \eta_g} a(N_A \mathbf{e}_i, N_B \mathbf{e}_j) d_{jB} \right) &= (N_A \mathbf{e}_i, \mathbf{f}) + (N_A \mathbf{e}_i, \mathbf{h})_{\Gamma_h} \\ &\quad - \sum_{j=1}^{n_{comp}} \left(\sum_{B \in \eta_g} a(N_A \mathbf{e}_i, N_B \mathbf{e}_j) g_{jB} \right) \end{aligned}$$

which must hold for all $A \in \eta - \eta_g, 1 \leq i \leq n_{comp}$.

b. Convert the expression in (a) to the desired matrix equation

$$\mathbf{K}\mathbf{d} = \mathbf{F}$$

Use the concept of the ID array, and specify $\mathbf{K} = [K_{PQ}], \mathbf{d} = \{d_Q\}$, and $\mathbf{F} = \{F_P\}$ fully.

c. We usually define a matrix $\mathbf{B}_A$, associated with node A, such that

$$\boldsymbol{\epsilon}(N_A \mathbf{e}_i) = \mathbf{B}_A \mathbf{e}_i.$$

Give the expression for $\mathbf{B}_A$ in two dimensions (hint: it is a three by two matrix). What is the new expression for K_{PQ} if you incorporate $\mathbf{B}_A$?

4. In this problem we develop expressions for the local element stiffness matrix $\mathbf{k}^e$ (the procedure for the element force vector $\mathbf{f}^e$ is similar).
a. Suppose we have n_{en} element nodes and n_{comp} components per node. Let p, q be local equation numbers:

$$p = n_{comp}(a-1) + i$$

$$q = n_{comp}(b-1) + j$$

Write an expression for k^e_{pq} in terms of $\mathbf{B}_A$, $\mathbf{B}_B$, $\mathbf{D}$, $\mathbf{e}_i$ and $\mathbf{e}_j$.
b. Define an element matrix $\mathbf{B}$ via

$$\mathbf{B} = [\mathbf{B}_1, \mathbf{B}_2, \ldots, \mathbf{B}_{n_{en}}]$$

Show that

$$\mathbf{k}^e = \int_{\Omega^e} \mathbf{B}^T \mathbf{D} \mathbf{B} \, d\Omega.$$

c. Consider a vector of displacements, i.e., $\mathbf{d}^e = \{d^e_{ia}\}, i = 1, \ldots, n_{comp}, \quad a = 1, \ldots, n_{en}$. Write expressions for the vector stress and strain using only $\mathbf{d}^e$, $\mathbf{D}$, and the matrix operator $\mathbf{B}$ from (b).
d. Following our discussion of heat conduction, outline how the calculation of $\mathbf{k}^e$ may be done in an element subroutine. Include (in outline form only) information about quadrature, the required change of variables, and the concept of a shape function subroutine.
e. Thinking ahead to the nonlinear case, suppose we want to write the global equations in the form

$$\begin{aligned} \mathbf{F}^{int} &= \mathbf{F}^{ext}, \quad \text{where} \\ \mathbf{F}^{int} &= \boldsymbol{A}_{e=1}^{n_{el}} \mathbf{f}^{int^e} \end{aligned}$$

In the case of linear elasticity, how would you define $\mathbf{f}^{int^e}$ in this approach?