FINITE ELEMENT ANALYSIS FOR ADVECTION DIFFUSION EQUATION

CARTER RHEA

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Contents

1. Abstract

In order to better understand the Finite Element Method from a programmers point of view, this project is a tutorial in programming a robust finite element solver in python from scratch. A large component of the project is consumed with the development of Object Oriented Programming techniques allowing for a generalized solver. Moreover, this paper explores the complexities of the Advection Diffusion Equation and how it leads to the use of stability techniques in Finite Element Analysis. The primary stability technique examined in this paper is the Galerkin Lease Squares method.

2. Introduction to the Advection Diffusion Equation

$$-k * \nabla^2 u + q \cdot \nabla u - f = 0 \quad u \in \Omega$$
$$u = 0 \quad u \in \Gamma$$
$$k > 0 \; ; \; q \; is \; a \; constant \; vector \; field$$

2.1. Weak Formulation.

We will be following the common steps in putting the normal (strong) advection diffusion equation into its weak, or variational, form.
Note:

$$\int_\Omega w \nabla^2 u d\Omega = \int_{\delta\Gamma} w \nabla u \cdot \hat{n} d\Gamma - \int_\Omega \nabla w \nabla u d\Omega$$

Derivation:

$$\int w(-k * \nabla^2 u + q \cdot \nabla u - f) d\Omega$$

$$-\int_{\delta\Gamma} wk\nabla u \cdot \hat{n} d\Gamma + \int_\Omega \nabla wk \nabla u d\Omega + \int_\Omega wq \cdot \nabla u d\Omega - \int_\Omega wf d\Omega = 0$$

$$\int_\Omega \nabla wk \nabla u d\Omega + \int_\Omega wq \cdot \nabla u d\Omega = \int_\Omega wf d\Omega$$

For the sake of compactness, we shall use the following two defines:

$$a(u, w) = \int_\Omega \nabla wk \nabla u d\Omega + \int_\Omega wq \cdot \nabla u d\Omega$$
$$< l, w >= \int_\Omega wf d\Omega$$

2.2. Error Analysis.

Lets define some constants to be used in the analysis of stability:

$$\alpha \; - \; level \; of \; "ellipticity" \; of \; PDE$$
$$\alpha = \frac{k}{1 + C}$$
$$C \; - \; Poincaré - Friedrich \; constant$$

M is the Lipschitz continuity constant which is essentially a bounding value for the derivative of a function.

$$M = K + \sqrt{C} max|q|$$

Thus we can write the error bound as the following:

$$||u - u_h||_{1,\Omega} \leq C\frac{M}{\alpha}H^k|u|_{1,\Omega}$$

Hence, the as the ratio $\frac{max|q|}{k}$ grows, the error grows! Thus the error is determined by the constants q and k. We need to find a way to deal with this...

2.3. **Stability.** In order to reduce the error we need to uncouple it from constants. To do this we modify the weak form of the PDE such that we subvert the undesired error dependency on the constants whilst retaining the consistency of the solution. We will have to add terms to the variational form such that it is still a solution to the initial PDE.

Let $R(u_h)$ be the residual. Hence

$$R(u_h) = -\kappa\nabla^2 u + q \cdot \nabla u_h - f$$

Hence the general structure of our stabilized form is

$$a(u, w) + \sum \int P(w_h)\tau R(u_h) d\Omega - < l, w_h > = 0$$

where P is a chosen operator applied to the test function and τ is the stability parameter.

The many different stability methods arise from the choice of P.

2.4. **Galerkin Least Squares.** We will choose the Galerkin Least Squares method to drive our choice in P. The main concept of this stability method is to minimize the L^2 norm of the residual.

(NEED TO ADD STABILITY PART)

How does this translate to the advection-diffusion equation?

$$a(u_h, w_h) + \sum_e \int_{\Omega_e} \tau(\kappa\nabla^2 w_h - q \cdot \nabla w_h) \cdot (\kappa\nabla^2 u_h - q \cdot \nabla u_h - f) d\Omega_e = < l, w_h >$$

Now we are able to go about formulating our finite element method for the stabilized weak form. As always,

$$u_h = [N]\{d\} \textit{ and } w_h = [N]\{d\}$$

$$\nabla u_h = [B]\{d\} \textit{ and } \nabla w_h = [B]\{d\}$$

$$\nabla^2 u_h = [H]\{d\} \textit{ and } \nabla^2 w_h = [H]\{d\}$$

So we have the following,

$$[K] = \sum_e \int_{\Omega_e} k[B]^T[B] + [N]^T q^T [B] d\Omega_e$$

$$[Q] = \sum_e \int_{\Omega_e} \kappa^2 \tau [H]^T [H] d\Omega$$

$$[P] = \sum_e \int_{\Omega_e} \kappa \tau [H]^T q^T [B] d\Omega$$

$$[S] = \sum_e \int_{\Omega_e} \tau [B]^T q q^T [B] d\Omega$$

$$\{F\} = \sum_e \int_{\Omega_e} [N]^T f d\Omega$$

$$\{L\} = \sum_e \int_{\Omega_e} \tau(\kappa [H]^T - [B]^T q) f d\Omega$$

Substituting these in to the weak form we have,

$$([K] + [Q] + [P] + [P^T] + [S])\{u\} = \{F\} + \{L\}$$

Finally, we have to deal with the τ that I have up until now included in my formulas, but neglected to formally define. τ is the stability coefficient, which for our use can be considered a constant value. The following relationships demonstrate the calculation of τ.

$$\tau(x, P_e) = \frac{h_e}{2 * |q|} \xi(P_e)$$

$$P_e = \frac{M_i |q| h_e}{2k}$$

$$\xi(P_e) = \begin{cases} P_e & 0 \leq P_e \geq 1 \\ 1 & P_e > 1 \end{cases}$$

$$m_i = min\{\frac{1}{3}, 2C_i\}$$

$$h_e = \sqrt{2}\frac{Area}{Diagonal}$$

And for this case, we know that $m_i = \frac{1}{12}$ [1]. Hence we can calculate τ.

3. Coding

In order to implement the GLS method for Advection-Diffusion, I first needed to have a FE solver; I opted to use the FE solver employed in my lab: MOOSE.
MOOSE is a Finite element software that relies heavily on object oriented programming and is written in C++. In order to construct physical models in MOOSE, the user is required to create a class which contains a piece of the weak form. For instance, in order to properly formulate the stabilized advection-diffusion problem in MOOSE, I created a number of kernels which dealt with each piece of the weak form (I.E. I had a kernel for [K]). However, in MOOSE one is required to input the non-finite formulation (i.e. weak form). So my kernels were inputted as follows:

$$K = a(u, w)$$

$$Q = \kappa^2 \tau \nabla^2 w \nabla^2 u$$

$$P = \kappa \tau \nabla^2 w q \nabla u$$

$$N = \tau q \kappa \nabla w \nabla^2 u = P^T$$

$$S = \tau \nabla w q q^T \nabla u$$

$$L = -\tau f(\kappa \nabla^2 w - q \nabla u)$$

$$F = -wf$$

4. Appendices

4.1. **Appendix I.** The following is a a kernel in the Moose program designed to incorporate Q into the formulation.

References

[1] Franca et al. in their paper ?Stabilized FEM: I. Application to the advective- diffuse model? (CMAME 95, 1992).