Physics 760 PS 2

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1 Problem 1

Find the sum of the first N terms and discuss convergence

1.1 Part A

$$\sum_{n=1}^{N} ln(\frac{n+1}{n}) = \sum_{n=1}^{N} ln(n+1) - ln(n)$$

Hence we can apply the summation in the textbook (difference method)

$$\lim_{n\to\infty} ln(N+1) - ln(0) = \infty + \infty = \infty$$

Therefore our series diverges!

1.2 Part B

$$\sum_{n=1}^{N} (-2)^n = \frac{1}{3}((-2)^{n+2} - (-2)^n)$$

Pulling the same trick we see

$$\lim_{n\to\infty} \frac{1}{3}((-2)^{N+3} - (-2)^0))) = \lim_{n\to\infty} \frac{1}{3}((-2)^{N+1} - 1)$$

This is oscillatory.

1.3 Part C

$$\sum_{n=1}^{N} (-1)^{n+1} \frac{n}{3^n} = \sum -n\Big(\frac{-1}{3}\Big)^n$$

This is an arithmogeometric series where $a = 0$, $n = n$, $d = -1$, $r = \sum_{n=1}^{\infty} \frac{2}{n^2} \frac{-1}{3}$. Hence we can simply use the summation formula for such a series which is:

$$S_N = \frac{a - [a + (N-1)d]r^n}{1-r} + \frac{rd(1 - r^{N-1})}{(1-r)^2}$$

Plugging in our values we get the following...

$$\frac{3}{16}[1 - (-3)^{-N}] + \frac{3}{4}N(-3)^{-N-1}$$

Which clearly converges to $\frac{3}{16}$

2 Problem 2

Prove that $cos(\theta) + \ldots cos(\theta + n\alpha) = \frac{sin(\frac{1}{2}(n+1)\alpha)}{sin(\frac{1}{2}\alpha)} cos(\theta + \frac{1}{2}n\alpha)$

$$\begin{aligned} cos(\theta) + \ldots cos(\theta + n\alpha) &= Re\{exp(i(\theta + n\alpha))\} \\ &= Re\{exp(i\theta)exp(i\alpha)^n\} \end{aligned}$$

We notice that this is a geometric series. Hence the n^{th} partial sum is the following:

$$\begin{aligned} Re\Big\{\frac{e^{(i\theta)}(1 - e^{(i\alpha)^{n+1}})}{1 - e^{(i\alpha)}}\Big\} &= Re\Big\{\frac{e^{(i\theta)}e^{(i\frac{(n+1)\alpha}{2})}(e^{(-i\frac{(n+1)\alpha}{2})} - e^{(i\frac{(n+1)\alpha}{2})}}{e^{(i\frac{\alpha}{2})}(e^{(-i\frac{\alpha}{2})} - e^{(i\frac{\alpha}{2})})}\Big\} \\ &= cos(\theta + \frac{n\alpha}{2}) * \frac{2isin(\frac{n+1}{2}\alpha)}{2isin(\frac{\alpha}{2})} \\ &= cos(\theta + \frac{n\alpha}{2}) * \frac{sin(\frac{n+1}{2}\alpha)}{sin(\frac{\alpha}{2})} \end{aligned}$$

Note that we used the beautiful formula:

$$sin(\theta) = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

3 Problem 3

Determine if the series converges

3.1 Part A

$$\sum_{n=1}^{\infty} \frac{2sin(n\theta)}{n(n+1)} \leq \sum \frac{2}{n(n+1)} < \sum \frac{2}{n^2}$$

Note that since $\sum \frac{2}{n^2}$ converges (see next question), $\sum_{n=1}^{\infty} \frac{2sin(n\theta)}{n(n+1)}$ converges as well by the comparison test!

3.2 Part B

$$\sum_{n=1}^{\infty} \frac{2}{n^2}$$

Let's use the beloved integral test...

$$\int_0^{\infty} \frac{2}{n^2} dn = -2n^{-1}\Big|_1^{\infty} = 2$$

Therefore $\sum_{n=1}^{\infty} \frac{2}{n^2}$ converges.

3.3 Part C

$$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}} > \sum_{n=1}^{\infty} \frac{1}{2n} \text{ Diverges by comparison test since } \sum_{n=1}^{\infty} \frac{1}{2n} \text{ diverges}$$

3.4 Part D

$$\sum_{n=1}^{\infty} \frac{(-1)^n (n^2+1)^{\frac{1}{2}}}{nln(n)}$$

Let's check for convergence using the alternating series test.

$$\lim_{x\to\infty} \frac{(n+1)^{\frac{1}{2}}}{nln(n)} = \frac{\infty}{\infty} \text{ By L'Hopital's Rule} = \lim_{x\to\infty} \frac{\frac{1}{2}(n+1)^{\frac{1}{2}}}{ln(n)+1} = 0$$

Hence the series converges by the convergence test.

3.5 Part E

$$\sum_{n=1}^{\infty} \frac{n^p}{n!}$$

Let's check for convergence using the Ratio Test...

$$\lim_{x\to\infty} \frac{(n+1)^p}{(n+1)!} \frac{n!}{n^p} = \lim_{x\to\infty} \frac{(n+1)^p}{(n+1)n^p} = 0$$

So we converge once again!!

4 Problem 4

Determine if the series converge

4.1 Part A

$$\sum \frac{x^n}{n+1}$$

We will use the ratio test...

$$\lim_{x\to\infty} \left| \frac{x^{n+1}(n+1)}{(n+1)x^n} \right| = |x| < 1$$

Hence it converges for $|x| < 1$.

4.2 Part B

$$\sum (sin(x))^n$$

Lets try the root test now...

$$\lim_{x\to\infty} \sqrt[n]{sin(x)^n} = sin(x)$$

Therefore $Convergent\ \forall x\ s.t.\ x \mod (\frac{(n+1)\pi}{2}) \neq 0$

4.3 Part C

$$\sum n^x$$

We can clearly see that if $x \geq -1$ then we have a series larger than the harmonic series and thus by comparison it diverges. Similarly, if $x < -1$,then we can once again note that we have a $p - series$ with $p > 1$ and thus we converge.
Hence we will converge as long as $x < -1$.

4.4 Part D

$$\sum e^{nx}$$

ROOT TEST!!!!

$$\lim_{x\to\infty} \sqrt[n]{e^{nx}} = e^x < 1 \text{ for convergence}$$

Therefore we have $x < 0$ for convergence!

4.5 Part E

$$\sum ln(n)^x$$

if $x \geq -1$ we can use the comparison test to compare our series to the divergent $p - series$.
Or let's just do the integral test:

$$\int_1^\infty ln(n)^x dn = x \int_1^\infty ln(n) = \infty \text{ used mathematica}$$

Hence our series diverges!

5 Problem 5

For what positive x values does this series converge

$$\sum \frac{x^{\frac{n}{2}} e^{-n}}{n}$$

Let's give the ratio test a go...

$$\lim_{x\to\infty} \frac{x^{\frac{n+1}{2}} e^{-(n+1)} n}{(n+1) x^{\frac{n}{2}} e^{-n}} = \lim_{x\to\infty} \frac{x^{\frac{1}{2}} e^{-1} n}{n+1} = x^{\frac{1}{2}} e^{-1}$$

Now by the definition of convergence through the ratio test is the following...

$$x^{\frac{1}{2}} e^{-1} < 1$$

$$x < e^2$$

6 Problem 6

Solve the following integral

$$\int_0^\infty \frac{\omega^3 d\omega}{e^{\omega/T} - 1}$$

Lets start by noting the very important definition:

$$\zeta(x) = \frac{1}{\Gamma(x)} \int_o^\infty \frac{1}{e^x - 1} x^{s-1} dx$$

$$\zeta(x)\Gamma(x) = \int_o^\infty \frac{1}{e^x - 1} x^{s-1} dx$$

For our integral we must first do the following substitution:
$u = \frac{w}{T} \to du = \frac{dW}{T}$. Thus we have the following:

$$\int_0^\infty \frac{u^3 T^3}{e^u - 1} T du = T^4 \int_0^\infty \frac{u^3}{e^u - 1} du = T^4 \Gamma(4)\zeta(4)$$