CSE291
Topics in Computer Graphics
Mesh Animation

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Today

- Review introduction to FEM
- Elastic bodies in 3D

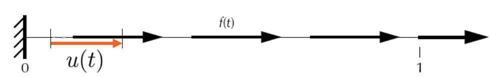
Finite Element Analysis (FEA)

- Class of techniques to solve PDEs
- FEM recipe

0. Strong formulation
1. Weak formulation
2. Trial solutions and test functions
3. Galerkin approximation
4. Matrix formulation

1D example

- One dimensional bar, unit length
- Subject to distributed load (forces) $f(t)$
- Boundary conditions
- Find longitudinal displacement $u(t)$



Strong (differential) formulation

- Constitutive equation (1D Poisson problem)

$$\frac{d^2u(t)}{dt^2} + f(t) = 0$$

Strong (differential) formulation

- Constitutive equation (1D Poisson problem)

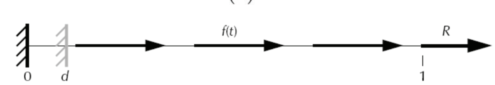
$$\frac{d^2u(t)}{dt^2} + f(t) = 0$$

- Boundary conditions
  - Prescribed displacement at the beginning (*essential* or *geometric* boundary condition)

$$u(0) = d$$

  - Concentrated force at the end (*natural* or *force* boundary condition)

$$u'(1) = R$$



Weak formulation

- Constitutive equation holds "in average"

$$-\int_0^1 v(t)\left(\frac{d^2u(t)}{dt^2}+f(t)\right)dt=0 \quad (\#)$$

  where $v(t)$ is a *test function*
- Above needs to be true for arbitrary test function

Boundary conditions

- Restrict *trial solutions* $u(t)$ to satisfy *geometric* boundary condition

$$u(t)\in\mathcal{U}=\{u(t)|u(0)=d\}$$

- Restrict test function $v(t)$ to satisfy *homogeneous* boundary conditions

$$v(t)\in\mathcal{V}=\{v(t)|v(0)=0\}$$

Weak formulation

- Integrate (#) by parts
- Principle of *virtual displacements* or principle *of virtual work*

$$\underbrace{\int_0^1 \frac{du}{dt}\frac{dv}{dt}dt}_{\text{Internal work}}=\underbrace{\int_0^1 vf\,dt+Rv(1)}_{\text{External work}}$$

- Body is in equilibrium

Solving the weak formulation

- Find $u(t)\in\mathcal{U}$ such that for all $v(t)\in\mathcal{V}$

$$\int_0^1 \frac{du}{dt}\frac{dv}{dt}dt=\int_0^1 vf\,dt+Rv(1)$$

Galerkin approximation

- Restrict test functions and trial solutions to finite dimensional function spaces

$$v^h(t)=\sum_{i=1}^{n}\hat{v}_iN_i(t)\in\mathcal{V}^h\subset\mathcal{V}$$

$$u^h(t)=\sum_{i=1}^{n}\hat{w}_iN_i(t)+dN_0(t)\in\mathcal{U}^h\subset\mathcal{U}$$

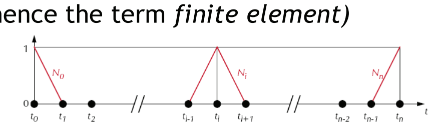
$$N_i(0)=0, i=1,\ldots,n \quad \text{and} \quad N_0(0)=1$$

$N_i(t)$ basis (shape) functions

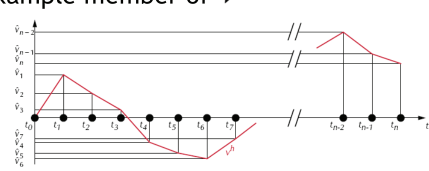
$\hat{v}_i, \hat{w}_i$ scalar weights

Galerkin approximation

- Shape functions usually have local support (hence the term *finite element*)



- Example member of $\mathcal{V}^h$



Galerkin approximation

- Plug test, trial functions into weak form

$$\sum_{j=1}^{n} \hat{w}_j a(N_i, N_j) = (f, N_i) + N_i(1)R - da(N_i, N_0)$$

$$\text{for } i = 1, \ldots, n$$

where

$$a(N_i, N_j) = \int_0^1 \frac{dN_i}{dt}\frac{dN_j}{dt} dt$$

$$(f, N_i) = \int_0^1 N_i f \, dt$$

Matrix formulation

- Define

$$K_{ij} = a(N_i, N_j)$$

$$F_i = (f, N_i) + N_i(1)R - da(N_i, N_0)$$

- Matrix form

$$\boxed{\mathbf{Kw} = \mathbf{F}}$$

Stiffness matrix $\mathbf{K}$
Force vector $\mathbf{F}$

Matrix formulation

Problem statement

- Given stiffness matrix and force vector, find parameters of displacement function
- I.e., solve $\mathbf{Kw} = \mathbf{F}$ for $\mathbf{w}$
- Stiffness matrix is positive definite, sparse
- Use standard techniques, e.g., conjugate gradient solvers

Summary

Basic "FEA recipe" (one of many)

- Derive weak formulation of PDE
  - Multiply PDE with test function
  - Integrate, apply integration by parts
  - Specify constraints on test and trial functions to fulfill boundary conditions
- Galerkin approximation
  - Discretize test and trial functions
  - Plug into weak formulation
  - Derive matrix equation

Questions?

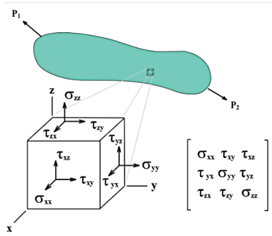
Linear elastic bodies

Physically-based model

- Stress, body forces
- Equilibrium conditions
- Deformations and displacements
- Strain
- Stress-strain relationship, constitutive equations

Stress

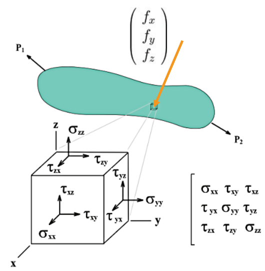
- Forces exerted by a piece of material onto its environment
- Measured as force per unit area
- Normal, shear stress
- Stress tensor
- Symmetric!



Stress tensor

Body forces

- External force applied to a piece of material
- Force per volume



Equilibrium conditions

- Independent of material properties

$$\begin{pmatrix} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + f_y \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + f_z \end{pmatrix} = 0$$

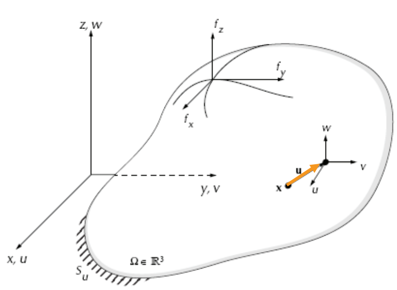
where f_x, f_y, f_z are body forces (force per volume)

- Holds for fluids too

Questions?

Deformations and displacments

- Deformation represented by displacement field $\mathbf{u}(\mathbf{x})$



Strain

- Geometric deformation
- Green tensor, or Cauchy's infinitesimal strain tensor

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

where u is the diplacement field

- "Symmetric part" of Jacobian of u
- Also called "kinematic equation"

Strain

- Matrix form

$$\varepsilon = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Stress-strain relationship

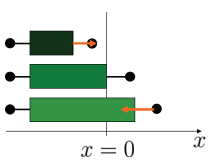
- Given geometric deformation of a piece of a body, what are strains (forces)?
- Stress-strain relationship captures physical material properties

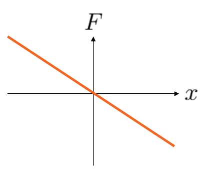
Linear-elastic bodies

- Hooke's law in 1D (springs)

$$F = -kx$$

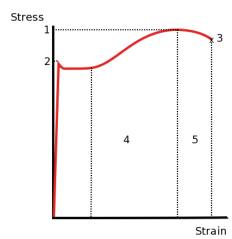
force F, spring constant k, displacement from rest length x





Stress-strain curve

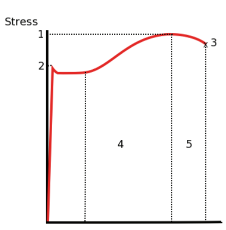
Stress-strain curves



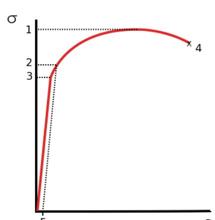
Steel

1. Ultimate strength
2. Yield strength
3. Rupture
4. Strain hardening region
5. Necking region

Stress-strain curves



Steel



Aluminum

Questions?

Linear elastic bodies

- Generalization of Hooke's law to 3D bodies
- Isotropic materials

$$\sigma = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \frac{E}{d} \begin{bmatrix} (1-\nu) & \nu & \nu & 0 & 0 & 0 \\ \nu & (1-\nu) & \nu & 0 & 0 & 0 \\ \nu & \nu & (1-\nu) & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-2\nu)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (1-2\nu)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (1-2\nu)/2 \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{bmatrix}$$

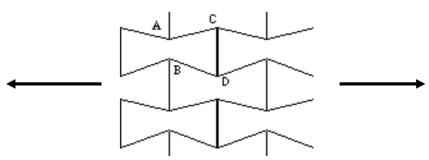
$$d = (1+\nu)(1-2\nu)$$

Linear elastic bodies

- E Young's modulus of elasiticity, slope of stress-strain curve
- ν Poisson's ratio
- Measure of how much thinner the material gets in one direction as you pull in the other
- Indicates change in volume
- 0.5 corresponds to no change in volume, i.e., incompressible

Negative Poisson's ratio

- http://silver.neep.wisc.edu/~lakes/Poisson.html



Constitutive equations

Strong formulation

- Equilibrium conditions

$$\mathcal{A}(\mathbf{u}) = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + f_y \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + f_z \end{pmatrix}$$

and linear stress-strain relationship

- Linear elastic bodies: linear displacement-strain, linear strain-stress, linear constitutive equations

Questions?

FEA recipe

- Derive weak formulation of PDE
  - Multiply PDE with test function
  - Integrate, apply integration by parts
  - Specify constraints on test and trial functions to fulfill boundary conditions
- Galerkin approximation
  - Discretize test and trial functions
  - Plug into weak formulation
  - Derive matrix equation

Derivation of weak form

- Multiplication of constitutive equations by test function $\mathbf{v} \equiv \delta\mathbf{u} = [\delta u, \delta v, \delta w]^T$ (also called *virtual displacements*)

$$\int_\Omega \mathbf{v}^T \mathcal{A}(\mathbf{u}) d\Omega =$$
$$-\int_\Omega \delta u \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x \right) + \delta v(A_2) + \delta w(A_3) d\Omega$$

- Integration by parts (Green's formula)
- No details here

Weak form

- Weak form, or principle of virtual work

$$\int_\Omega \delta\varepsilon^T \sigma d\Omega - \int_\Omega \delta\mathbf{u}^T \mathbf{f} d\Omega = 0$$

- Virtual strain $\delta\varepsilon \in \mathbf{R}^{6\times 1}$
- Strain vector $\sigma \in \mathbf{R}^{6\times 1}$

Weak form

- Virtual strain

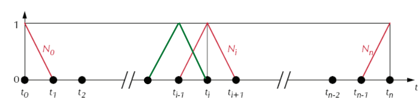
$$\delta\varepsilon = \left(\frac{\partial \delta u}{\partial x}, \frac{\partial \delta v}{\partial y}, \frac{\partial \delta w}{\partial z}, \frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x}, \ldots \right)^T = \mathcal{S}\delta\mathbf{u}$$

- Matrix form

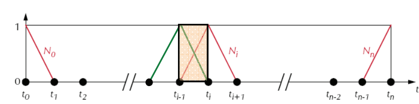
$$\mathcal{S}\delta\mathbf{u} = \begin{pmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} \delta u \\ \delta v \\ \delta w \end{pmatrix}$$

Discretization

Elements vs. shape functions



Shape functions



Elements

Discretization

- Galerkin approximation
- Element point of view
- Displacement in each element is

$$\mathbf{u}^{(m)}(x,y,z) = \sum_{i=0}^{k-1} \hat{\mathbf{u}}_i^{(m)} N_i^{(m)}(x,y,z)$$

where $\hat{\mathbf{u}}_i^{(m)} \in \mathbf{R}^3$
number of shape functions in element k
and (x,y,z) lies within element m

Discretization

- Maintain map relating element indices and global indices

$$(m,i) \rightarrow j, \quad j = 0, \ldots n-1$$

where total number of shape functions is

- Note this mapping is many-to-one
- Correspondence

$$\hat{\mathbf{u}}_i^{(m)} \equiv \hat{\mathbf{u}}_j$$
$$N_i^{(m)} \equiv N_j$$

Questions?

Discretization

- Matrix form

$$\mathbf{u}^{(m)}(x,y,z) = \mathbf{H}^{(m)}(x,y,z)\hat{\mathbf{u}}^{(m)}$$

where

$$\mathbf{H}^{(m)}(x,y,z) = \begin{pmatrix} N_0^{(m)},\dots,N_{k-1}^{(m)} & 0 & 0 \\ 0 & N_0^{(m)},\dots,N_{k-1}^{(m)} & 0 \\ 0 & 0 & N_0^{(m)},\dots,N_{k-1}^{(m)} \end{pmatrix} \in \mathbf{R}^{3\times 3k}$$

$$\hat{\mathbf{u}}^{(m)} = \left(\hat{u}_{x_0}^{(m)},\dots,\hat{u}_{x_{k-1}}^{(m)},\hat{u}_{y_0}^{(m)},\dots,\hat{u}_{y_{k-1}}^{(m)},\hat{u}_{z_0}^{(m)},\dots,\hat{u}_{z_{k-1}}^{(m)}\right)^T \in \mathbf{R}^{3k}$$

Discretization

- Strain

$$\varepsilon^{(m)}(x,y,z) = \mathcal{S}\mathbf{u}^{(m)}(x,y,z) = \mathbf{B}^{(m)}(x,y,z)\hat{\mathbf{u}}^{(m)}$$

- Where

$$\mathbf{B}^{(m)}(x,y,z) = \begin{pmatrix} \frac{\partial}{\partial x}[N_0^{(m)},\dots,N_{k-1}^{(m)}] & 0 & 0 \\ 0 & \frac{\partial}{\partial y}[N_0^{(m)},\dots,N_{k-1}^{(m)}] & 0 \\ 0 & 0 & \frac{\partial}{\partial z}[N_0^{(m)},\dots,N_{k-1}^{(m)}] \\ \frac{\partial}{\partial y}[N_0^{(m)},\dots,N_{k-1}^{(m)}] & \frac{\partial}{\partial x}[N_0^{(m)},\dots,N_{k-1}^{(m)}] & 0 \\ 0 & \frac{\partial}{\partial z}[N_0^{(m)},\dots,N_{k-1}^{(m)}] & \frac{\partial}{\partial y}[N_0^{(m)},\dots,N_{k-1}^{(m)}] \\ \frac{\partial}{\partial z}[N_0^{(m)},\dots,N_{k-1}^{(m)}] & 0 & \frac{\partial}{\partial x}[N_0^{(m)},\dots,N_{k-1}^{(m)}] \end{pmatrix} \in \mathbf{R}^{6\times 3k}$$

- Note: stress is

$$\sigma^{(m)}(x,y,z) = \mathbf{C}^{(m)}\varepsilon^{(m)}(x,y,z) = \mathbf{C}^{(m)}\mathbf{B}^{(m)}(x,y,z)\hat{\mathbf{u}}^{(m)}$$

Discretization

- Remember: weak form

$$\int_\Omega \delta\varepsilon^T \sigma d\Omega = \int_\Omega \delta\mathbf{u}^T \mathbf{f} d\Omega$$

- Use same per-element interpolation for displacement/strain and virtual displacement/strain
- Per-element perspective

$$\sum_m \int_{V^m} \delta\varepsilon^T \sigma d\Omega = \sum_m \int_{V^{(m)}} \delta\mathbf{u}^T \mathbf{f} d\Omega$$

Discretization

- Galerkin approximation of weak form

$$\delta\hat{\mathbf{U}}^T\left[\sum_m \int_{V^{(m)}} \mathbf{B}_*^{(m)^T}\mathbf{C}_*^{(m)}\mathbf{B}_*^{(m)} dV^{(m)}\right]\hat{\mathbf{U}} = \delta\hat{\mathbf{U}}^T\left[\sum_m \int_{V^{(m)}} \mathbf{H}_*^{(m)^T}\mathbf{f}^{(m)} dV^{(m)}\right]$$

$$\Rightarrow \left[\sum_m \int_{V^{(m)}} \mathbf{B}_*^{(m)^T}\mathbf{C}_*^{(m)}\mathbf{B}_*^{(m)} dV^{(m)}\right]\hat{\mathbf{U}} = \left[\sum_m \int_{V^{(m)}} \mathbf{H}_*^{(m)^T}\mathbf{f}^{(m)} dV^{(m)}\right]$$

where $\hat{\mathbf{U}} = \{\hat{\mathbf{u}}_j\}, j = 1,\dots,n$

- Assembly (map element indices to global indices)

$$\mathbf{B}_*^{(m)}[u,v] \equiv \mathbf{B}^{(m)}[s,t], \text{ where } (m,s)\rightarrow u, (m,t)\rightarrow v$$

Matrix formulation

- Global stiffness matrix

$$\mathbf{K} = \sum_m \int_{V^{(m)}} \mathbf{B}_*^{(m)^T}\mathbf{C}_*^{(m)}\mathbf{B}_*^{(m)} dV^{(m)}$$

- Global force vector

$$\mathbf{R} = \sum_m \int_{V^{(m)}} \mathbf{H}_*^{(m)^T}\mathbf{f}^{(m)} dV^{(m)}$$

- Solve for nodal weights

$$\mathbf{K}\hat{\mathbf{U}} = \mathbf{R}$$

Questions?

Dynamic equation

$$\partial_j \sigma_{ij} + f_i = \rho \partial_{tt} u_i$$

- Note equilibrium equation is the same, just with zero right hand side

Viscoelastic and plastic materials

- Viscoelastic: material that has damping, some energy during deformation converted to heat
- Plastic: when stress exceeds threshold, material changes shape permanently

Next time

- “Pose space deformation”, Lewis et al.
- Presentation by Iman Mostafavi