Math 295: Test 2 Take-home

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1. Prove the following statement: Let $x, y \in R$. If x and y are nonnegative, then $\frac{x+y}{2} \geq \sqrt{xy}$.
   Proof by contradiction: Assume x and y are negative ($x < 0$ and $y < 0$) and $\frac{x+y}{2} \geq \sqrt{xy}$. If x and y are both negative, then $\frac{x+y}{2} < 0$ because two negative numbers added together and subsequently divided by two is still a negative number. Also, if x and y are both negative, then $\sqrt{xy} > 0$ because the product of two negative numbers is positive. Therefore $\frac{x+y}{2} < \sqrt{xy}$ because a negative number is always smaller than a positive number. However, this contradicts the fact that $\frac{x+y}{2} \geq \sqrt{xy}$. Therefore, x and y need to be nonnegative.
   Note: The cases involving either x or y (exclusively) being negative was shown because if either was negative, then $\sqrt{xy}$ would equal an imaginary number.

2. Suppose A, B, and C are sets. Prove that $A \cup C \subseteq B \cup C$ if and only if $A \setminus C \subseteq B \setminus C$.
   Proof: $(\rightarrow)$ Assume $A \setminus B \subseteq B \setminus C$ and also assume $A \cup C$.
   Case 1: Assume $x \in A$ and $x \notin C$. If $x \in A$ and $x \notin C$, then $x \in B$ by the given $(A \setminus B \subseteq B \setminus C)$. Thus $x \in (B \cup C)$, as required.
   Case 2: Assume $x \in C$ and $x \notin A$. If $x \in C$, then $x \in (B \cup C)$, as required.
   Case 3: Assume $x \in A$ and $x \in C$. If $x \in C$, then $x \in (B \cup C)$, a required.
   Therefore, if $x \in (A \cup C)$,then $x \in (B \cup C)$. Which means $(A \cup C \subseteq B \cup C)$, as required.
   $(\leftarrow)$ Assume $(A \cup C) \subseteq (B \cup C)$, and also assume $A \setminus C$. $x \in A$ and $x \notin C$. According to the assumed statement, $((A \cup C) \subseteq (B \cup C))$, since $x \in A$, then $x \in (B \cup C)$. However, since $x \notin C$ (this is from the original assumption $x \in A \setminus C$), $x \in B$. Thus, if $x \in A \setminus C$, then $x \in B \setminus C$.
   Therefore, $A \cup C \subseteq B \cup C$ if and only if $A \setminus C \subseteq B \setminus C$.

3. Prove that $Dom(S \circ R) \subseteq Dom(R)$.
   Assume $x \in Dom(S \circ R)$. Thus, by the definition of the domain of S composed with R, $Dom(S \circ R) = \{a \in A | \exists c \in C((a, c) \in S \circ R)\}, x \in a \in A$, which means $x \in A$. By the definition of a composition, $R \in A \times B$ and $S \in B \times C$. Therefore, by the definition of the domain of R, $\{a \in A | \exists b \in B((a, b) \in R)\}$, $x \in Dom(R)$ since $x \in A$. Thus, if $x \in Dom(S \circ R)$, then $x \in Dom(R)$, as required $(Dom(S \circ R) \subseteq Dom(R)$

4. Show by example that $Dom(S \circ R) = Dom(R)$ may be false.
   Proof: Let $R = \{(1, 3)(1, 4)(2, 5)(3, 6)\}$
   Let $S = \{(3, 2)(4, 3)\}$
   Thus, $S \circ R = \{(1, 2)(1, 3)\}$
   Then, $Dom(S \circ R) = \{1\}$ and $Dom(R) = \{1, 2, 3\}$.
   Therefore $Dom(S \circ R) \neq Dom(R)$.