Math 295: Homework 9

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1. Chapter 5.2 # 3 Which of the functions in exercise 3 of § 5.1 are one-to-one and which are onto?

   (a) Let $A = \{a, b, c, \}$ and $B = \{a, b\}$, and $\{(a, b), (b, b), (c, a)\}$.
   f is not one-to-one but it is onto.

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by the formula $f(x) = x^2 - 2x$.
   f is neither one-to-one or onto.

   (c) Let $f = \{(x, n) \in \mathbb{R} \times \mathbb{Z} | n \leq x < n + 1\}$.
   f is not one-to-one, but it is onto.

2. Chapter 5.2 #6 Let $A = \mathcal{P}(\mathbb{R})$. Define $f : \mathbb{R} \to A$ by the formula $f(x) = \{y \in \mathbb{R} | y^2 < x\}$.

   (a) Find $f(2)$.
   $f(2) = (-\sqrt{2}, \sqrt{2})$

   (b) Is f one-to-one? Is it onto?
   f is one-to-one, and f is not onto.

3. Chapter 5.2 # 8 Suppose $f : A \to B$ and $g : B \to C$.

   (a) Prove that if $g \circ f$ is onto then g is onto.
   Proof: Assume $(g \circ f)(x)$ is onto. Thus, by the definition of compisition $(g \circ f)(x) = g(f(x))$. Since, $g(f(x))$ is onto, for every $c \in C$, there exists and $a \in A$ such that $g(f(a)) = c$. Since $f : A \to B, f(a) = b$. Thus $g(f(a)) = g(b)$. And therefore for all $c \in C$ there exists at least one $b \in B$ such that $g(b) = c$. Thus $g(x)$ is onto.

   (b) Prove that if $g \circ f$ is one-to-one then f is one-to-one.
   Proof: Assume $(g \circ f)(x)$ is one-to-one. Thus, by the definition of composition, $g(f(x))$ is one-to-one. Given $a_1, a_2 \in A$, by the definition of one-to-one $(g \circ f)(a_1) = (g \circ f)(a_2)$, which implies $a_1 = a_2$. Also $g(f(a_1)) = g(f(a_2))$ which now implies $f(a_1) = f(a_2)$. Thus, by the definition of one-to-one, f is one-to-one.

4. Chapter 5.3 # 2 Let F be the funtion defined in exercise 4(b) of § 5.1. If $X \in B$, what is $F^{-1}(X)$? $F^{-1}(X) = A \setminus X$

5. Chapter 5.3 # 6 Let $A = \mathbb{R} \setminus \{2\}$, and let f be the function with domain A defined by the formula

$$f(x) = \frac{3x}{x - 2}$$

   (a) Show that f is a one-to-one, onto function from A to B for some set $B \subseteq \mathbb{R}$. What is the set B.
   One-to-one: According to the definition of f, we have

$$f(a_1) = f(a_2)$$

   iff

$$\frac{3a_1}{a_1 - 2} = \frac{3a_2}{a_2 - 2}$$

   iff

$$3a_1(a_2 - 2) = 3a_2(a_1 - 2)$$

   iff

$$3a_1a_2 - 6a_1 = 3a_1a_22 - 6a_2$$

   iff

$$-6a_1 = -6a_2$$

iff

$$a_1 = a_2$$

Thus there can be no real numbers a_1 and a_2 for which $f(a_1) = f(a_2)$ and $a_1 \neq a_2$. Thus f is one-to-one.
Onto: Let y be an arbitrary real number. Let $x = \frac{-2x}{3-y}$. Then $g(x) = \frac{3x}{x-2} = \frac{3\frac{-2y}{3-y}}{\frac{-2y}{3-y}-2} = \frac{\frac{-6y}{3-y}}{\frac{-2y-6+2y}{3-y}} = \frac{-6y}{-6} = y$. Thus $\forall y \in \mathbb{R} \exists x \in \mathbb{R}(g(x) = y)$.Therefore, f is an arbitrary function.
The set $B = \mathbb{R} \setminus \{3\}$.
It has been shown that f is one-to-one and unto from A to B for some set $B \subseteq \mathbb{R}$.

(b) Find a formula for $f^{-1}(x)$.
$f^{-1} = \frac{2x}{x-3}$

6. Chapter 5.3 # 13 Suppose $f : A \to B$ and f is onto. Let $R = \{(x, y) \in A \times A | f(x) = f(y)\}$. By exercise 17(a) of § 5.1, R is an equivalence relation on A.

(a) Prove that there is a function $h : A \setminus R \to B$ such that for all $x \in A$, $h([x]_R) = f(x)$.
Let $h = \{(X, y) \in A \setminus R \times B | \exists x \in X(f(x) = y)\}$. Pick $[x] \in A \setminus R$. Suppose $[a] = [x]$ which implies aRx, which by the definition of a relation, $f(a) = f(x)$.
Then,

$$h([a]) = f(a)$$

and

$$h([x]) = f(x)$$

So, $h([a]) = h([x])$ Thus, h is a function.

(b) One-to-one: Suppose $h([a]) = h([b])$. Then $f(a) = f(b)$. So by the definition of R, aRb implies $a \in [b]$. Furthermore, bRa implies $b \in [a]$. Thus based on Lemma (4.6.5), $[a] = [b]$. Thus h is one-to-one.
Onto: Since f is onto, $\forall b \in B \exists a \in A(f(a) = b)$. Since, $A \setminus R = \{[x] \mid x \in A\}, a \in [a] \in A \setminus R$. Thus $h([a]) = f(a) = b$. Therefore h is onto because we found for any arbitrary $b \in B \exists a \in [a](h([a]) = b)$.