Assignment 3

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Problem 1

One way to get an intuitive assessment of the accuracy and stability of a numerical method for solving ODE initial value problems in the form

$$\frac{dx}{dt} = f(t, x), x(a) = xa$$

is to apply it to a single time-step in each of two basic test cases:

(i) The integration $\frac{dx}{dt} = f(t)$, exact solution $x(t + h) = x(t) + \int_t^{t+h} f(s)ds$.

(ii) The exponential equation $\frac{dx}{dt} = \lambda x$, exact solution $x(t + h) = x(t)e^{h\lambda}$.

Apply this idea to a single step of the (classical) four-stage Runge-Kutta method:

a)

Verify that for the integration case (i), the result is the same as with Simpson’s rule, so that the Runge-Kutta method is fourth-order accurate in this case.

SOLUTION

Using the classical Runge-Kutta 4^{th} order we have,

$$\int_t^{t+h} f(s)ds = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

Based on the differential equation we have $f(t, x) = f(t)$ Hence plugging this into the RK we have

$$K_1 = hf(t)$$

$$K_2 = hf(t + 1/2h)$$

$$K_3 = hf(t + 1/2h)$$

$$K_4 = hf(t + h)$$

Thus

$$x(t + h) = x(t) + \frac{1}{6}\Big(h(f(t) + 4hf(t + \frac{1}{2}h) + hf(t + h)\Big)$$

which is Simpson's rule. Therefore we have RK as a 4^{th} order method for this basic case!

b)

Verify that for case (ii), the time-stepping formula is

$$x(t+h) \approx x(t)p(h\lambda)$$

where $p(\xi)$ is a polynomial in $\xi = h\lambda$.

SOLUTION

Based on the differential equation we have $f(t,x) = \lambda x$. Hence plugging this into the RK we have

$$K_1 = h\lambda x$$

$$K_2 = h\lambda\big(x + 1/2(h\lambda x)\big) = h\lambda x + \frac{1}{2}h^2\lambda^2 x$$

$$K_3 = h\lambda\left(x + \frac{1}{2}\left(h\lambda x + \frac{1}{2}h^2\lambda^2 x\right)\right) = h\lambda x + \frac{1}{2}h^2\lambda^2 x + \frac{1}{4}h^3\lambda^3 x$$

$$K_4 = h\lambda(x + h\lambda x + \frac{1}{2}h^2\lambda^2 x + \frac{1}{4}h^3\lambda^3 x) = h\lambda x + h^2\lambda^2 x + \frac{1}{2}h^3\lambda^3 x + \frac{1}{4}h^4\lambda^4 x$$

Hence,

$$\begin{aligned} x(t+h) = x\frac{1}{6}\Big(h\lambda x + 2h\lambda x + h^2\lambda^2 x + 2h\lambda x + h^2\lambda^2 x + \frac{1}{2}h^3\lambda^3 x h\lambda x + h^2\lambda^2 x + \frac{1}{2}h^3\lambda^3 x + \frac{1}{4}h^4\lambda^4\Big) \\ = x + \frac{1}{6}\Big(6h\lambda x + 3h^2\lambda^2 x + h^3\lambda^3 x + \frac{1}{4}h^4\lambda^4 x\Big) \\ = x + h\lambda x + \frac{1}{2}h^2\lambda^2 x + \frac{1}{6}h^3\lambda^3 x + \frac{1}{24}h^4\lambda^4 x \\ = x + x\Big(h\lambda + \frac{1}{2}h^2\lambda^2 + \frac{1}{6}h^3\lambda^3 + \frac{1}{24}h^4\lambda^4\Big) \\ = x\Big(1 + h\lambda + \frac{1}{2}h^2\lambda^2 + \frac{1}{6}h^3\lambda^3 + \frac{1}{24}h^4\lambda^4\Big) \end{aligned}$$

Thus we have what we were trying to demonstrate since if we allow $h\lambda = \xi$ we then have a polynomial in ξ, as required.

c)

By comparing $p(h\lambda)$ to the factor $e^{h\lambda}$ in the exact solution, show that the error in this single step is $O(h^5)$, so that the method is fourth-order accurate for this case too.

SOLUTION

Note that our polynomial is the 4^{th} order Taylor polynomial for $e^{h\lambda}$. Therefore the error in a single step is $O(h^5)$!

Problem 2

The initial value problem $\frac{dx}{dt} = x^{1/3}, x(0) = 0$ has two solutions for $t \geq 0 : x_1(t) = 0$, and $x_2(t) = 2t$. If the Runge-Kutta method is applied, what happens? Can you see what happens with other ones-step methods seen so far? What is the moral here?

Here we have $f(x,t) = x^{1/3}$

SOLUTION

$$K_1 = hx^{1/3}$$

$$K_2 = h\left(x + \frac{1}{2}hx^{1/3}\right)^{1/3}$$

$$K_3 = h\left(x + h\left(x + \frac{1}{2}hx^{1/3}\right)^{1/3}\right)^{1/3}$$

$$K_4 = h\left(x + h\left(x + h\left(x + \frac{1}{2}hx^{1/3}\right)^{1/3}\right)^{1/3}\right)^{1/3}$$

Thus starting with our initial condition $x(0) = 0$ we get

$$K_1 = 0, K_2 = 0, K_3 = 0, K_4 = 0$$

Hence x(t+h) = 0 for any time. It is readily seen how this would continue for further t values. But in case you dont believe this I will show using RK4 code from last semester (or project 2)...

In [1]:

```
from ODE_IVP_module import *
```

In [2]:

```
def f(t,x): return x**(1/3)
init = 0
a = 0
b = 10
n = 1000
y4 = Runge_Kutta(f,init,a,b,n)
print("The best estimate according to the Runge-Kutta method for the differential e
```

```
The best estimate according to the Runge-Kutta method for the differen
tial equation dx/dt = x^(1/3) is 0
```

Hence we can see it runs into the solution of $x_1(t) = 0$. The same will happen for other one-step methods since they are so dependent on the initial condition (which should be no surprise). The moral of this is that the Initial Value Matters!

Problem 3

Suppose that with a certain machine arithmetic model (such as IEEE 64-bit), the Runge-Kutta method applied to a certain ODE IVP on the interval $[a, b]$ using fixed step size $h = \frac{b-a}{n}$ has truncation error $9nh^5$ and round-off error $36n2^{-50}$.

What is the optimal value of h, in the sense of minimizing total error, how many time steps are involved, and what is that minimal possible attainable error?

SOLUTION

To find an optimal value of h we should simply add our truncation error and rounding error as our error function and then optimize it...

Note that $n = \frac{b-a}{h}$

Hence,

$$E(h) = 9nh^5 + 36n2^{-50} = 9\left(\frac{b-a}{h}\right)h^5 + 36\left(\frac{b-a}{h}\right)2^{-50} = 9(b-a)h^4 + 36\left(\frac{b-a}{h}\right)2^{-50}$$

$$E'(h) = 63(b-a)h^3 - \frac{36(b-a)2^{-50}}{h^2} = 0$$

Hence we get,

$$h^5 = 2^{-50}$$

$$h = \sqrt[5]{2^{-50}} = \frac{1}{1024}$$

Hence our minimal amount of time steps is

$$n = \frac{b-a}{\frac{1}{1024}} = 1024(b-a)$$

And plugging these values into $E(h)$ we get,

$$E(h) = 9(b-a)\left(\frac{1}{1024}\right)^4 + 36(1024(b-a))2^{-50} = \frac{45(b-a)}{1099511627776}$$