Math 295: Homework 10

Carter Rhea

April 10, 2014

1. Chapter 5.4 # 2d Suppose that Y and Z are subsets of B. Will it always be true that $Y \subseteq Z \iff f^{-1}(Y) \subseteq f^{-1}(Z)$?
   $(\rightarrow)$ Assume $Y \subseteq Z$ and $Y \subseteq B$ and $Z \subseteq B$. Let $a \in f^{-1}(Y)$. So $f(a) = y$. Then, by assumption, $f(a) \in Z$.Thus $f^{-1}(f(a)) \subseteq f^{-1}(Z)$, which implies that $a \in f^{-1}(Z)$. Since a was arbitrary, $Y \subseteq Z \rightarrow f^{-1}(Y) \subseteq f^{-1}(Z)$.
   $(\leftarrow)$ Assume $f^{-1}(Y) \subseteq f^{-1}(Z)$ and also $Y \subseteq B$ and $Z \subseteq B$. Let $a \in f^{-1}(Y)$. So $f(a) = Y$. Since $f^{-1}(f(a)) \subseteq f^{-1}(Z)$, $f(a) \in Z$. Since a was arbitrary, $f^{-1}(Y) \subseteq f^{-1}(Z) \rightarrow Y \subseteq Z$.

2. Chapter 5.4 # 3 Suppose $X \subseteq A$. Will it always be true that $f^{-1}(f(X)) = X$?
   Take figure 1 from chapter 5.4 from the book as an example. The relation is not one-to-one. Take $X = \{1, 2\}$. Then $f(X) = f(\{1, 2\}) = \{4, 5\}$. Now take $f^{-1}(\{4, 5\}) = \{1, 2, 3\}$. Thus, it has been shown that $f^{-1}(f(X))$ does not always equal X. I hypothesize that if f is injective, then $f^{-1}(f(X)) = X$.

3. Chapter 6.1 # 2 Prove that for all $n \in \mathbb{N}, 0^2 + 1^2 + 2^2 + ... + n^2 = n(n + 1)(2n + 1)/6$
   Base Case: $0^2 = \frac{0(1)(1)}{6}$ which is equivalent to $0 = 0$ which is true. So $P(0)$ is true.
   Inductive Case: Let $n \in \mathbb{N}$. Assume $P(n)$ is true. Thus $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.
   Thus, adding $(n + 1)^2$ to both sides.
   $$\sum_{i=1}^{n} i^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$
   which implies
   $$\sum_{i=0}^{n+1} i^2 = \frac{n(n+1)(2n+1) + 6(n+1)^2}{6}$$
   which implies
   $$\sum_{i=0}^{n+1} i^2 = \frac{(n+1)(2n^2+7n+6)}{6}$$
   which implies
   $$\sum_{i=0}^{n+1} i^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$
   Which is $P(n + 1)$. Q.E.D.

4. Chapter 6.1 # 5 Prove that for all $n \in \mathbb{N}, 0 * 1 + 1 * 2 + 2 * 3 + ... + n(n + 1) = n(n + 1)(n + 2)/3$.
   Base Case: $P(0) = 0(1) = 0(1)(2)/3$ which is equivalent to $0 = 0$ which is true. Thus $P(0)$ is true.
   Inductive Case: Let $n \in \mathbb{N}$. Assume $P(n)$ is true. Thus, $\sum_{i=1}^{n} n(n+1) = \frac{n(n+1)(n+2)}{3}$
   Adding $(n + 1)(n + 2)$ to both sides,
   $$\sum_{i=1}^{n} n(n+1) + (n+1)(n+2) = \frac{n(n+1)(n+2)}{3} + (n+1)(n+2)$$
   which implies
   $$\sum_{i=1}^{n+1} n(n+1) = \frac{n(n+1)(n+2)}{3} + \frac{3(n+1)(n+2)}{3}$$
   which implies
   $$\sum_{i=1}^{n+1} n(n+1) = \frac{(n+1)(n^2+5n+6)}{3}$$

which implies

$$\sum_{i=1}^{n+1} n(n+1) = \frac{(n+1)(n+2)(n+3)}{3}$$

Which is $P(n+1)$. Q.E.D.

5. Chapter 6.1 # 12 Prove that for all integers a and b and all $n \in \mathbb{N}, (a-b) \mid (a^n + b^n)$.
   Base Case:$k(a-b) = 1 - 1 = 0, so(a-b) = 0$. Thus $a - b \mid a^n - b^n$
   Induction Case: Let n be an arbitrary natural number and suppose $a - b \mid a^n - b^n$. Then we can chose an integer k such that $(a-b)k = a^n - b^n$. Thus,

   $$a^{n+1} - b^{n+1} = a^n + a - b^n - b$$

   which is equal to

   $$(a^n - b^n) + (a - b)$$

   which is equal to

   $$k(a-b) + (a-b)$$

   which is equal to

   $$(k+1)(a-b)$$

   Let $m = k + 1$ such that $m \in \mathbb{Z}$, thus

   $$(k+1)(a-b) = m(a-b)$$

   Therefore $(a-b) \mid a^{n+1} - b^{n+1}$, as required. Q.E.D.