Math 295: Homework 8

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1. Chapter 4.6 # 4 Which of the following relations $\mathbb{R}$ are equivalence relations? For those that are equivalence relations, what are the equivalence classes?

   (a) $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} | x - y \in \mathbb{N}\}$
   R is not an equivalance relation because it fails to satify the requirement regarding symmetry in order to be an equivalence relation. $x, y \in R$ and xRy. Then $x - y \in \mathbb{N}$. Thus $y - x = -(x - y) \notin \mathbb{N}$ since the negative of a natural number is not a natural number because by definition $\mathbb{N} = \{0, 1, 2, 3...\}$.
   Example: Allow $x = 2$ and $y = 1$. Then $x - y = 2 - 1 = 1 \in \mathbb{N}$, but $y - x = 1 - 2 = -1 \notin \mathbb{N}$.

   (b) $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} | x - y \in \mathbb{Q}\}$
   Reflexive: Suppose $x \in \mathbb{R}$. Then $x - x = o \in \mathbb{Q}$, so $(x, x) \in S$, and therefore S is reflexive.
   Symmetric: Suppose $(x, y) \in S$. By the definition of S, this means that $x - y \in \mathbb{Q}$. Then $y - x = -(x - y) \in \mathbb{Q}$ since the negative of a rational number is also a rational number, thus $(y, x) \in S$. Because (x, y) was an arbitrary element of S, S is symmetric.
   Transitive: Suppose $(x, y) \in S$ and $(y, z) \in S$.Then $x - y \in \mathbb{Q}$ and $y - z \in \mathbb{Q}$. It follows that the sum $x - y + y - z = x - z \in \mathbb{Q}$, so $(x, z) \in S$, as required.
   Thus S is an equivalence relation on $\mathbb{R}$.
   Equivalence Class: $\mathbb{R}/S = \{[0]\} \cup \{[q] \mid q \in \mathbb{R} \setminus \mathbb{Q}\}$, where $[0] = \mathbb{Q}$, and $\{[q] \mid q \in \mathbb{R} \setminus \mathbb{Q}\}$ such that each irrational number added with a rational is the equivalence class of the irrational number alone.

   (c) $T = \{(x, y) \in \mathbb{R} \times \mathbb{R} | \exists n \in \mathbb{Z}(y = x10^n\}$
   Reflexive: Suppose $x \in \mathbb{R}$. Then $\exists n \in \mathbb{Z}(x = x10^n)$, $n = 0$ satisfies the equations, so $(x, x) \in T$, and therefore S is reflexive.
   Symmetric: Suppose $(x, y) \in T$. By the definition of S, this means $\exists n \in \mathbb{Z}(y = x10^n)$. Then $\exists n \in \mathbb{Z}(x = y10^n)$, thus $(y, x) \in T$. Since (x, y) was an arbitrary element of T, T is symmetric.
   Transitive: Suppose $(x, y) \in T$ and $(y, z) \in T$. Then $\exists n \in \mathbb{Z}(y = x10^n)$ and $\exists m \in \mathbb{Z}(z = y10^m)$. Thus $\exists n \in \mathbb{Z}\ \exists m \in \mathbb{Z}\ (z10^{-m} = x10^n)$. Then $\exists n \in \mathbb{Z}\ \exists m \in \mathbb{Z}(z = x10^{n+m})$. By substuting $n + m = s$, $\exists s \in \mathbb{Z}(z = x10^s)$, so $(x, z) \in T$,as required.
   Thus T is an equivalence relation on $\mathbb{R}$.
   Equivalence Class: $\mathbb{R}/T = \{[x] \mid x \in \mathbb{R}\}$ where $[b] = \{a \in \mathbb{R} \mid \exists n \in \mathbb{Z} a = b10^n\}$

2. Chapter 4.6 #10 Let C_m be the congruence mod m relation defined in the text, for a positive integer m.

   (a) Complete the proof that C_m is an equivalence relation on $\mathbb{Z}$ by showing that it is reflexive and symmetric.
   $C_m = \exists k \in \mathbb{Z}(x - y = km)$
   Reflexive: Suppose $x \in \mathbb{R}$. Then $x - x = 0 = km$ for some $k \in \mathbb{Z}$, namely, $z = 0$. So $(x, x) \in C_m$, and therefore C_m is reflexive.
   Symmetric: Suppose $(x, y) \in C_m$. By the definition of C_m, $\exists k \in \mathbb{Z}(x - y = km)$. Then $\exists k \in \mathbb{Z}(y - x = -(x - y) = km)$.Thus, since k can be negative, $(y, x) \in C_m$. Since (x, y) was an arbitrary element of C_m, C_m is symmetric.

   (b) Find all of the equivalence classes for C_2 and C_3. How many equivalence classes are there in each case? In general, how many equivalence classes do you think there are for C_m?
   Equivalence class for C_2: $\{[x] | x \in \mathbb{Z}\} = \{[0], [1]\}$, where $[0] = \{n \in \mathbb{Z} | 2n\}$ and $[1] = \{m \in \mathbb{Z} | 2m + 1\}$
   .
   Equivalence class for C_3: $\{[x] | x \in \mathbb{Z}\} = \{[0], [1], [2]\}$, where $[0] = \{n \in \mathbb{Z} | 3n\}$, $[1] = \{m \in \mathbb{Z} | 3m + 1\}$, and $[2] = \{s \in \mathbb{Z} | 3s + 2\}$.
   In general, C_m will have m equivalence classes.

3. Chapter 4.6 # 11 Prove that for every integer n, either $n^2 \equiv 0(mod4)$ or $n^2 \equiv 1(mod4)$.
Rewritten in symbols: $\forall n(\exists k \in \mathbb{Z}(n^2 = 4k)$ or $\exists w \in \mathbb{Z}(n^2 = 4w + 1))$.
Proof: Let $n \in \mathbb{Z}$, thus n is either even or odd.
Case 1: Assume n is even. If n is even, then $n = 2r$ such that $r \in \mathbb{Z}$. Thus $n^2 = (2r)^2 = 4r^2$. Thus $n^2 = 4k$ with $k \in \mathbb{Z}$.
Case 2: Assume n is odd. If n is odd, then $n = 2s+1$ such that $s \in \mathbb{Z}$. Thus $n^2 = (2s+1)^2 = 4s^2+4s+1 = 4(s^2+s)+1$. Thus $n^2 = 4w+1$ with $4w \in \mathbb{Z}$.
Thus for every integer n, either $n^2 \equiv 0(mod4)$ or $n^2 \equiv 1(mod4)$.

4. Chapter 5.1 # 1

   (a) Let $A = \{1, 2, 3\}, B = \{4\}$, and $f = \{(1, 4), (2, 4), (3, 4)\}$. Is f a function from A to B?
   Yes, because there exists only one y for each x, and each x is used.

   (b) Let $A = \{1\}, B = \{2, 3, 4\}$, and $f = \{(1, 2), (1, 3), (1, 4)\}$. Is f a function from A to B?
   No, because there exists multiple y-values for each x.

   (c) Let C be the set of all cars registered in your state, and let S be the set of all finite sequences of letters and digits. Let $L = \{(c, s) \in C \times S|$ the liscnce plate number of the car c is $s\}$. Is L a funciton from C to S?
   Yes, because there exists only one liscence plate ,s, for each car, c, registered in the state. Thus, each c has one and only one corresponding s, so L qualifies as a function from C to S.

5. Chapter 5.1 # 4

   (a) Let N be the set of all countries and C the set of all cities. Let $H : N \to C$ be the function defined by the rule that for every country n, $H(n)$ =the capital of the country n. What is $H(Italy)$?
   Rome!

   (b) Let $A = \{1, 2, 3\}$ and $B = \mathcal{P}(A)$. Let $F : B \to B$ be the function defined by the formula $F(X) = A \setminus X$. What is $F(\{1, 3\})$?
   $F(\{1, 3\}) = A \setminus \{1, 3\} = \{2\}$.

   (c) Let $f = \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ be the function defined by the formula $f(x) = (x + 1, x - 1)$. What is $f(2)$?
   $f(2) = (2 + 1, 2 - 1) = (3, 1)$

6. Chapter 5.1 # 6 Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$ defined by the following formulas:

$$f(x) = \frac{1}{x^2 + 2} \qquad g(x) = 2x - 1$$

Find formulas for $(f \circ g)(x)$ and $(g \circ f)(x)$.

$$(f \circ g)(x) = \frac{1}{(2x-1)^2 + 2} = \frac{1}{4x^2 - 4x + 1 + 2} = \frac{1}{4x^2 - 4x + 3}$$

$$(g \circ f)(x) = 2\frac{1}{x^2 + 2} - 1 = \frac{2}{x^2 + 2} - 1 = \frac{2}{x^2 + 2} - \frac{x^2 + 2}{x^2 + 2} = \frac{-x^2}{x^2 + 2}$$