Math 245 Test 2 2015

All calculations should be done by hand except basic calculator tasks like arithmetic and evaluating the polynomial and exponential functions that arise.

Do not discuss the problems with anyone except me; if you do have questions, such as clarifications, contact me. I will in general respond by sending a clarification by email to all students.

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(1) (a) Find the polynomial of lowest degree that agrees with the function $f(x) = \cos x$ at the nodes $x_0 = 0$, $x_1 = \pi/4$, and $x_2 = 3\pi/4$.

(b) Without using the actual value of the cosine at any argument except the above three, use this polynomial to approximate $\cos(1)$, and compute an upper bound on the error in this approximation.

(c) Compute the actual error in this approximation of $\cos(1)$, and compare to your result in part (b).

(2) Two quantities x and y are known to be related by the form $y = ae^x + be^{-x}$.

(**Aside:** this is true for example if they come from a solution of the differential equation $d^2y/dx^2 = y$).

For the following six x values (known exactly), y values have been measured, with most of the inaccuracy in the y values just due to rounding, but with a mistake in recording one of the y values; this gives the following table of values:

x	0	.2	.4	.6	.8	1
y	4.3	4.1245	4.1146	4.1382	4.5964	5.107

(a) Set up the least squares procedure for this pair of basis functions, and determine the best estimates of a and b in the least squares sense.

(b) Check the accuracy by computing the error at each point, and using this, try to identify the point where the mistaken y value is.

(c) Propose a plausible replacement for that erroneous value.

(3) (a) Derive the approximation of $f'(x)$ of the form

$$f'(a) \approx Af(a) + Bf(a+2h) + Cf(a+3h),$$

that is best the sense of having the highest possible **degree of precision**, p.

(Note that the value at $a + h$ is missing!)

(b) Using Taylor polynomials of degree p with error term, verify that this approximate has error $O(h^p)$.

(4) The composite Simpson's rule (which is fourth order accurate) has been used to approximate a definite integral $I = \int_a^b f(x)\,dx$ with 10 and 20 intervals, giving the values

$$S_{10} = 7.38084362852, \quad S_{20} = 7.35531055134.$$

(a) Use Richardson extrapolation to get a better approximation to the value I of this integral.

(b) Give a practical estimate of the error in this approximation of the integral.

(c) Given the extra information that $S_{60} = 7.35439923178$, use Richardson extrapolation with S_{20} and S_{60} to get another, hopefully even more accurate approximation of the integral.