Mathematics 445/545 2016, Assignment 1, corrected

Due Thursday January 28

Notes

- Assignments are intended primarily as learning activities, and I expect questions and discussions in class and/or in office hours or online.

  So start early: I expect to hear your questions by next Tuesday at the latest, and better yet by this Thursday!

- The final exercise is only required of students enrolled in Math 545; however Math 445 students are welcome to do it to, for some extra credit.

1. Consider the task of computing the positive n-th root $r = a^{1/n}$ of a positive number a by fixed-point iteration with

$$g(x) = \frac{(n-1)x + a/x^{n-1}}{n}. \tag{1}$$

   (a) Show that the sequence of iterations given by $x_{k+1} = g(x_k)$ does indeed converge to $r = a^{1/n}$ for any positive initial value x_0.

   (b) Show that this convergence is *super-linear*; that is:

$$\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k|} = 0, \text{ where } e_k = r - x_k. \tag{2}$$

   Do this using either Taylor's Theorem or results from class about fixed-point iteration.

2. For the case of a *double root*, $f(r) = 0$, $f'(r) = 0$, $f''(r) \neq 0$, show that the convergence of Newton's method is only linear, with contraction constant $1/2$; that is:

$$\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k|} = 1/2. \tag{3}$$

   (Aside: thus it is no faster than the bisection method.)

3. **For Math 545 students:** Iterative methods can often be accelerated by increasing the size of the step taken by a carefully chosen factor $\omega > 1$. (This will be particularly useful with linear algebra, where it is called *over-relaxation.*) For Newton's method, an accelerated version is

$$x_{k+1} = x_k - \omega \frac{f(x_k)}{f'(x_k)}. \tag{4}$$

   For the above case of a double root, find a value of the acceleration factor ω that restores super-linear convergence.