Assignment 4

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Problem 1

Consider the problem of solving the following system of second order ODEs with either your code for the system for the Runge_kutta method or with one of the methods provided in scipy.integrate.ode:

$$\frac{d^2y}{dt^2} = sin(yz) + cos(t)z$$
$$\frac{d^2x}{dt^2} = cos(yz) + e^t y$$

a)

Rewrite as a first order system of four equations

First we can rewrite the variables as such:

$$y = x_1$$
$$y' = x_2$$
$$z = x_3$$
$$z' = x_4$$

Giving us the differential equations:

$$x_1' = x_2$$

$$x_2' = sin(x_1 x_2) + cos(t)x_3$$
$$x_3' = x_4$$
$$x_4' = cos(x_1 x_3) + e^t x_1$$

b)

Now make it autonomous!

$$
\begin{aligned}
t &= x_0 \\
\\
y &= x_1 \\
y' &= x_2 \\
z &= x_3 \\
z' &= x_4
\end{aligned}
$$

With the differential equations

$$
\begin{aligned}
x_0' &= 1 \\
\\
x_1' &= x_2 \\
x_2' &= sin(x_1 x_2) + cos(x_0)x_3 \\
x_3' &= x_4 \\
x_4' &= cos(x_1 x_3) + e^{x_0} x_1
\end{aligned}
$$

Problem 2

The simplest implicit method, sometimes useful for handling stiff equations, is the Backward Euler, which is based on the backward difference approximation of the derivative,

$$\frac{du}{dt} = \frac{u(t) - u(t-h)}{h}$$

leading to

$$u(t+h) \approx u(t) + hf(t+h, u(t+h))$$

Thus we are solving

$$u_{i+1} = u_i + hf(t_{i+1}, u_{i+1})$$

a)

Solve this for the test equation $du/dt = -Ku$, and show that it never has the catastrophic geometric growth.

SOLUTION

$$
\begin{aligned}
u_{i+1} &= u_i + hku_{i+1} \\
u_{i+1}(1 - hk) &= u_i \\
u_{i+1} &= \frac{u_i}{1 - hk} \\
u_n &= \frac{u_0}{(1 - hk)^n}
\end{aligned}
$$

Thus if we consider the case where $K < 0$ it is clearly seen that $1 - hk > 0$ for all values of k and h since $h > 0$ by default. Thus we never have geometric growth and only decay!

b)

Determine its order of accuracy.

$$u(t+h) = u(t) + hf(t+h, u(t+h))$$

So we need to check out the taylor expansion for $f(t+h, u(t+h))$ For simplicity's sake lets call $K = u(t+h) - u(t) \approx u' h + u'' h^2$

$$f(t+h, u+k) \approx f(t,u) + \frac{df}{dt}h + \frac{df}{du}k + O(h^2 + k^2)$$

Note that

$$\frac{d}{dt}[f(t, u(t))] = \frac{d^2u}{dt^2} = \frac{df}{dt} + \frac{df}{du}\frac{du}{dt}$$

Thus with some substitution we get,

$$f(t+h, u+k) \approx f(t,u) + \frac{df}{dt}h + \frac{df}{du}\left(\frac{du}{dt}h + \frac{1}{2}\frac{d^2u}{dt^2}h^2\right) + O(h^2 + k^2)$$

$$f(t+h, u+k) \approx f(t,u) + \left(\frac{df}{dt}h + \frac{df}{du}\frac{du}{dt}h\right) + \frac{1}{2}\frac{df}{du}\frac{d^2u}{dt^2}h^2 + O(h^2 + k^2)$$

$$f(t+h, u+k) \approx f(t,u) + \frac{d^2u}{dt^2}h + \frac{1}{2}\frac{df}{du}\frac{d^2u}{dt^2}h^2 + O(h^2 + k^2)$$

Problem 4

Consider an implicit multistep method for the format

$$u_i = a_1 u_{i-1} + a_2 u_{i-2} + b_0 hf(t_i, u_i)$$

a)

Verify that this is second order accurate with the coefficient choices

$$a_1 = 4/3, a_2 = -1/3, b_0 = 2/3$$

SOLUTION

$$u(t) \approx a_1 u(t-h) + a_2 u(t-2h) + b_0 hf(t, u(t))$$

So lets approximate $u(t-h)$ and $u(t-2h)$ around center t and in power of h.

$$u(t-h) \approx u(t) - hf(t, u(t)) + O(h^2)$$
$$u(t-2h) \approx u(t) - 2hf(t, u(t)) + O(h^2)$$

$$u(t) \approx \frac{4}{3}\left(u(t) - hf(t,u) + O(h^2)\right) - \frac{1}{3}\left(u(t) - 2hf(t,u) + O(h^2)\right) + \frac{2}{3}hf(t, u(t))$$
$$u(t) \approx \frac{4}{3}u(t) - \frac{4}{3}hf(t,u) + O(h^2) - \frac{1}{3}u(t) + \frac{2}{3}hf(t,u) + O(h^2) + \frac{2}{3}hf(t, u(t))$$
$$u(t) \approx u(t) + 0(h^2)$$

Problem 5

If A is any $6x6$ matrix with the eigenvalues $-8, -3, 1, 2, 5, 7,$ determine which eigenvalues each of the following methods will find:

(a) The Power Method

(b) The inverse power method

(c) The shifted inverse power method with shift $s = 4$

Solution

(a) The power method finds the eigenvalue with the largest absolute value so it would find $\lambda = -8$

(b) The inverse power method finds the eigenvalue with the smallest absolute value so it would find $\lambda = 1$

(c) The shifted inverse power method with shift $s = 4$ would go four from the smallest (1) and thus pick up $\lambda = 5$.

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