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Title: Population Growth of Cyanobacteria

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Analysis of the Problem:

When an initial amount of cyanobacteria are placed on a Petri dish, the population will grow because of the division of cells at a certain rate. However, because cyanobacteria depend on sunlight to make energy, the rate of division will depend on the time of day. Additionally, the Petri dish itself limits the population because there are not enough resources such as food and space to support an infinite number of cyanobacteria. As a result, the growth rate should also depend on the population so that as the population increases the growth rate decreases. At a certain value called the carrying capacity, there will be zero growth and the population will stabilize. Consequently, a model of the population growth of cyanobacteria with a certain initial value and carrying capacity would include a growth rate that depends on the time of day and the amount of cyanobacteria in the Petri dish. The total change in the population will then be proportional to the growth rate and the current number of cyanobacteria. A graph of the population versus time could then be expected to have an overall s shape because of the initial lag in cell division and the influence of the carrying capacity. The local maximums on the graph would representing the time of day when cell division is the highest.

Model Design:

Computational models are simplified representations of phenomena, systems, or processes. They are used to help us study and understand different aspects of life and can lead to predictions involving responses in un-tested scenarios. Often because of certain assumptions and simplifications, they are much easier to work with than the systems they represent and can point to new insight and comprehension. To translate a problem into a computational model there are five main steps: analysis of a problem, model development and formulation, model implementation, model verification, and model interpretation. The first step, analysis of a problem, involves determining what the question and unknowns are and what the answer would look like. The second step, model development and formulation, includes gathering relevant data, making assumptions, determining variable relationships and units, and writing down specific equations or rules. Model implementation involves writing a program in a computer language to solve the problem while model verification involves testing special cases whose answers are known to confirm the solution is accurate and the program does what it is supposed to do. Finally, model interpretation includes answering the main question sometimes with figures or tables and discussing more about the solution: if it was expected, what it means and how it could change if the assumptions are relaxed.

In the model for the population growth of cyanobacteria, several assumptions were made before the model was implemented. The model was assumed to be dynamic since it depends on time and deterministic since there is no element of chance and the solution is expected to be the same under the same conditions. It is also assumed to be a point model since the solution does not depend on the location in space. Because the individual cyanobacteria will be dividing at different times and taking up all available space, time and space are assumed to be continuous in the model. Furthermore, the number of bacteria is assumed to be continuous because the amount of cyanobacteria can include all points on the number line. But, the initial number of bacteria was chosen to be 1,000 so that even if the solution involved a fraction, the fraction would still represent a whole bacterium. Additionally, the initial number of bacteria is assumed to be large enough so that new bacteria will be created even if the period of time is small. From reference(1), the initial growth rate was chosen to be .99 day^{-1} and the carrying capacity was chosen to be 10,000 x10^3 cells. The value of dt was chosen to be .2/24 day so every half hour a data point would be produced and the simulation time was chosen to be 30 days so the simulation would run long enough for the trends on the graphs to be determined.

In the model development, several variables and equations were used to help find the solution. The initial number of cyanobacteria, n_o cells, is a constant as is the initial growth rate, r_o day^{-1}, the change in time, Δt days and the carrying capacity, c cells. The number of cyanobacteria, n cells, is a container variable and the growth rate, r day^{-1}, is a dependent variable. The differential equation $dn/dt = r(n,t)n(t)$ can be rewritten as the finite difference equation

$$n(t+\Delta t) = n(t) + r(n,t)*n(t)*\Delta t \qquad (1)$$

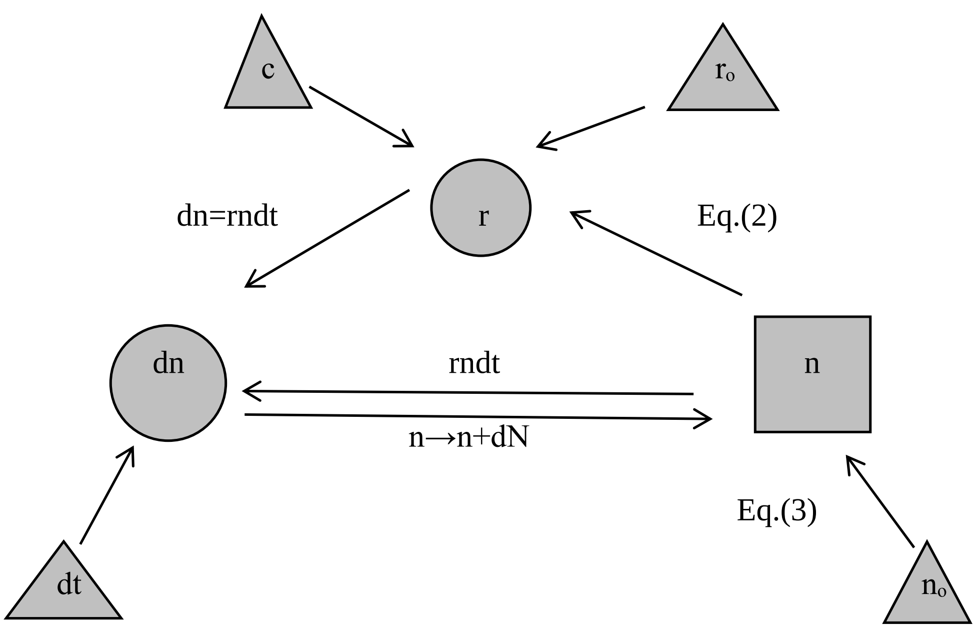
which can be used to develop the model and find a numerical solution if Δt is small enough. In Eq.(1), r(n,t) is assumed to decrease linearly and is given by the equation

$$r(n,t) = (r_o - k(n - n_o))\sin^2(\tau\pi t) \qquad (2)$$

where $k = r_o/(c-n_o)$ τ is a unit correction factor equal to 1 day^{-1}. The $\sin^2(\tau\pi t)$ is used to ensure the growth rate is never negative and to make it dependent on time where the period of π represents 24 hours. So when it is midnight, the time is zero or some multiple of π and as a result the growth rate is zero. When it is noon, the time is some multiple of $\pi/2$ and the growth rate is a maximum. By plugging Eq.(2) into Eq.(1), n(t) can be solved for to get

$$n(t) = \frac{r_o + kn_o}{\frac{r_o}{n_o}\exp\left(\frac{r_o+kn_o}{2}*\left(\frac{\sin(2\tau\pi t)}{2\pi\tau} - t\right)\right)+k} \qquad (3)$$

which gives the exact solution to the model. Overall, the variables and equations are related by the following diagram.



Model Solution:

To find the solution, the code below was run in Python. First initial values and constants were defined. Then a method was written that would return values of time, and the corresponding the numerical and exact solutions, and the growth rate. Specifically, inside the previous method, a method was written that used Eq.(2) defined growth rate as a function of population and time. Additionally, arrays to be filled with values of population and time were created. Eq.(3) was used to find the exact the exact values of the population and Eq.(1) was used to find the numerical values of the population. Finally, plots were created that show the growth rate over time and the cyanobacteria population over time. To verify the solution, plots were created with different values of initial population and growth rate and compared with the solutions that were known. Additionally, every plot contained both the numerical and exact solution which could be compared to each other to determine validity.

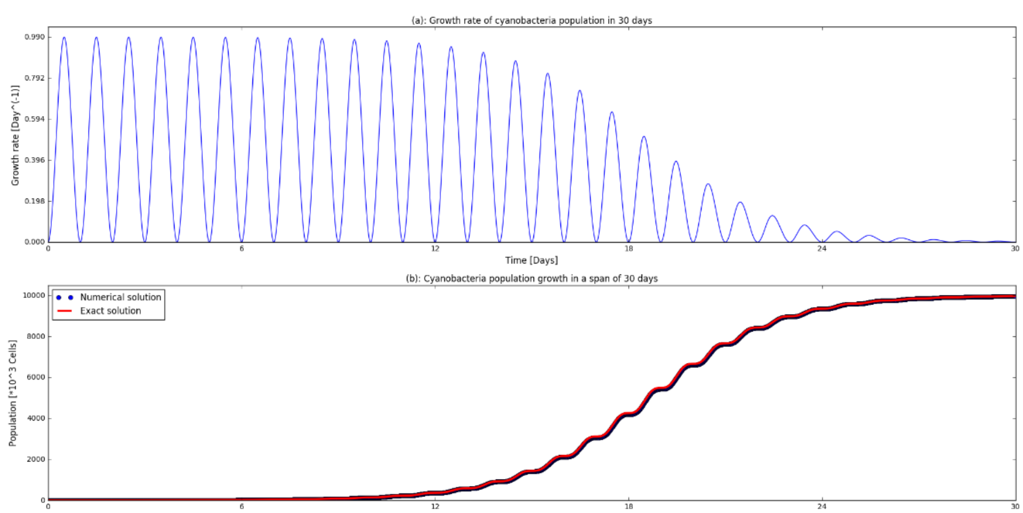
```
#Module1: Cyanobacteria growth under sunlight
import numpy as np
import matplotlib.pyplot as plt
# default parameters: final
large, small = 12, 10  #font sizes
simulationTime = 30.0 #days
dt = 0.2/24 # day
pinitial = 1.0 #*10**3 cells, the initinal population
pmax = 10000.0 #*10**3 cells, the carrying capacity
rmax = .99 #day^-1, the initial growth rate
#%%
def populationDisCont_rate(dt, pinitial, pmax, rmax):
#returns the values of time, and the corresponding exact and numerical
#solution, and the growth rate

    def growthRate(n,t):#defines the growth rate
        return (rmax - k * (n-pinitial)) * np.sin(np.pi * t)**2

    k = rmax / (pmax - pinitial) #(day*cells)^-1

    time = np.arange(0, simulationTime+dt, dt)
    #creates an array for the values of time when data is collected
    population = np.zeros(time.size)
    #creates an array to be filled with the population at a certain time
    population[0] = pinitial

    populationCont = (rmax+k*pinitial) / (k + rmax/pinitial* \
    np.exp((rmax+k*pinitial)/2* (np.sin(2*np.pi*time)/(2*np.pi) - time)))
    #gives the exact solution
    rate = growthRate(populationCont, time) # via *exact solution* and time
    for i in np.arange(1, time.size):
        population[i] = population[i-1] + dt * growthRate(population[i-1], \
        time[i-1]) * population[i-1]
        #creates an array filled with the values of the numerical solution

    return [time, populationCont,population, rate]
#%% default plot
time, populationCont,population, rate = populationDisCont_rate(dt, pinitial,
\
                                                              pmax, rmax)

plt.subplot(2,1,1)
plt.plot(time, rate)
plt.title('(a): Growth rate of cyanobacteria population in 30 days', \
          fontsize = large)
```

```
plt.xlim(0,simulationTime)
plt.ylim(0,1.05*np.max(rate))
plt.xlabel("Time [Days]", size = large)
plt.ylabel("Growth rate [Day^(-1)]", size = large)
plt.xticks(np.linspace(0,simulationTime,6), fontsize = small)
plt.yticks(np.linspace(0, np.max(rate), 6), fontsize = small)

plt.subplot(2,1,2)
plt.plot(time, population, 'bo', label='Numerical solution')
plt.plot(time, populationCont,'r-', linewidth = 3, label='Exact solution')
plt.legend(fontsize = large, loc = 2)
plt.title('(b): Cyanobacteria population growth in a span of 30 days', \
          fontsize = large)
plt.xlim(0,simulationTime)
plt.ylim(0,1.05*pmax)
plt.xlabel("Time [Days]", size = large)
plt.ylabel("Population [*10^3 Cells]", size = large)
plt.xticks(np.linspace(0,simulationTime,6), fontsize = small)
plt.yticks(np.linspace(0, pmax, 6), fontsize = small)
#%% More plots to verify the solution
#rmax --> 0.002
time, populationCont1,population1, bla = populationDisCont_rate(dt, pinitial,
pmax, 0.002)

plt.subplot(2,2,1)
plt.plot(time, population, 'bo', label='Numerical solution')
plt.plot(time, populationCont,'r-', linewidth = 3, label='Exact solution')
plt.legend(fontsize = large, loc = 2)
plt.title('(a): Population growth with r_0 = 0.02 day^(-1)', fontsize =
large)
plt.xlim(0,simulationTime)
plt.ylim(0,1.05*pmax)
plt.xlabel("Time [Days]", size = large)
plt.ylabel("Population [*10^3 Cells]", size = large)
plt.xticks(np.linspace(0,simulationTime,6), fontsize = small)
plt.yticks(np.linspace(0, pmax, 6), fontsize = small)

#rmax --> 200
time, populationCont2,population2, bla = populationDisCont_rate(dt, pinitial,
pmax, 200)

plt.subplot(2,2,2)
plt.plot(time, population2, 'bo', label='Numerical solution')
plt.plot(time, populationCont2,'r-', linewidth = 3, label='Exact solution')
plt.legend(fontsize = large, loc = 4)
plt.title('(b): Population growth with r_0 = 200 day^(-1)', fontsize = large)
plt.xlim(0,simulationTime)
plt.ylim(0,1.05*pmax)
plt.xlabel("Time [Days]", size = large)
plt.ylabel("Population [*10^3 Cells]", size = large)
plt.xticks(np.linspace(0,simulationTime,6), fontsize = small)
plt.yticks(np.linspace(0, pmax, 6), fontsize = small)

#pinitial --> 0
time, populationCont2,population2, bla = populationDisCont_rate(dt,
0.0000001, pmax, rmax)
```

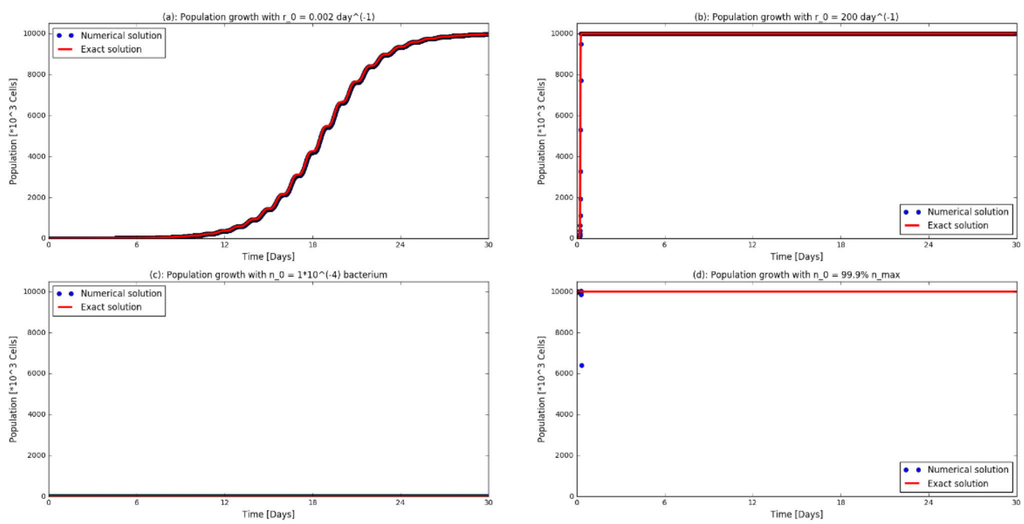
```
plt.subplot(2,2,3)
plt.plot(time, population2, 'bo', label='Numerical solution')
plt.plot(time, populationCont2,'r-', linewidth = 3, label='Exact solution')
plt.legend(fontsize = large, loc = 2)
plt.title('(c): Population growth with n_0 = 1*10^(-4) bacterium', fontsize =
large)
plt.xlim(0,simulationTime)
plt.ylim(0,1.05*pmax)
plt.xlabel("Time [Days]", size = large)
plt.ylabel("Population [*10^3 Cells]", size = large)
plt.xticks(np.linspace(0,simulationTime,6), fontsize = small)
plt.yticks(np.linspace(0, pmax, 6), fontsize = small)

#pinitial --> pmax
time, populationCont2,population2, bla = populationDisCont_rate(dt,
0.999*pmax, pmax, rmax)

plt.subplot(2,2,4)
plt.plot(time, population2, 'bo', label='Numerical solution')
plt.plot(time, populationCont2,'r-', linewidth = 3, label='Exact solution')
plt.legend(fontsize = large, loc = 4)
plt.title('(d): Population growth with n_0 = 99.9% n_max', fontsize = large)
plt.xlim(0,simulationTime)
plt.ylim(0,1.05*pmax)
plt.xlabel("Time [Days]", size = large)
plt.ylabel("Population [*10^3 Cells]", size = large)
plt.xticks(np.linspace(0,simulationTime,6), fontsize = small)
plt.yticks(np.linspace(0, pmax, 6), fontsize = small)
```

Results, Verification, and Conclusion:

The following figure contains plots that show the growth rate of cyanobacteria over time and the cyanobacteria population over time. The data produced results that were expected because growth rate is dependent on the time of day and decreases with time. Additionally, the graph of the population of cyanobacteria has an overall s shape because of the initial lag in cell division and the influence of the carrying capacity. As was predicted, it also and contains local maximums during the time of day where the cell division is a maximum.



To verify the solution, both the numerical an exact solutions are included on the graph. Both solutions are very close to each other, indicating the results are valid. However, the exact solution is slightly larger than the numerical solution, because the numerical solution has a linear relationship between points and cannot change as quickly as the exact solution. In the future, an even smaller value of dt could be used so the difference between the numerical and exact solutions decreases. Furthermore, to verify the solution special cases were plotted whose trends were already known. For instance, if the initial growth rate were changed to .002 day^{-1} it would be expected that the new graph would have the same shape as graph with an initial growth rate of .99 day^{-1} except it would not be as steep. If the initial growth rate were changed to 200 day^{-1} it would be expected that initially the population would experience a large growth but would quickly level out at the value of the carrying capacity. Also if the initial population were changed to be extremely close to zero, it would be expected that the population would not experience any growth. Additionally, if the initial population were changed to be extremely close to the value of the carrying capacity, it would be expected that the population would not experience any growth. As shown by the figure below, all plots of the special cases match what was expected which verifies the solution.



Overall, growth of cyanobacteria in a Petri dish was modeled. The solution shows, that the population of cyanobacteria is not only influenced by the growth rate and the carrying capacity, but also by the initial value of the population. The solution was expected because the graphs of the population over time and the growth rate over time had the predicted shapes and trends. If some assumptions were relaxed, such as setting the growth rate as a constant, there would be no local maximums and the population would increase exponentially. Next time, in addition to decreasing the value of dt the model could be improved by including a variable that would account for cyanobacteria death since in reality not all bacteria would survive.

References:

1) https://www.derwentestuary.org.au/assets/WSUD_Guidelines_-_Appendix_D_-_Cyanobacterial_growth.pdf