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A correct, reversible Trotter splitting for the evolution operator in molecular dynamics simulations of molecular systems with constraints

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ABSTRACT

For generating an isobaric–isothermal ensemble in molecular dynamics simulations of atomic systems a correct ensemble distribution can be obtained by the approach of Martyna, Tobias and Klein [J. Chem. Phys. **101**, 4177–4189 (1994)]. The constituting equations of the latter approach have been also generalised to molecular systems [M. E. Tuckerman, *Statistical Mechanics: Theory and Molecular Simulation*, Oxford Graduate Texts (2010)] using the molecular virial instead of the atomic one. An isothermal–isobaric method for systems with holonomic molecular constraints has been also introduced in the past [G. Kalibaeva, M. Ferrario and G. Ciccotti, Mol. Phys. **101** 765–778 (2003)] but the factorisation was not worked out completely, resulting in a non-reversible first-order algorithm. Here, we propose the correct factorisation and implement a reversible integrator based on Martyna, Tobias and Klein equations for a system of molecules subjected to holonomic constraints. We add to the algorithm the Suzuki–Yoshida treatment that improves the energy conservation or permits to go to larger time steps. Finally, we test our algorithm by applying it to the dynamics of a Ortotherphenyl model.
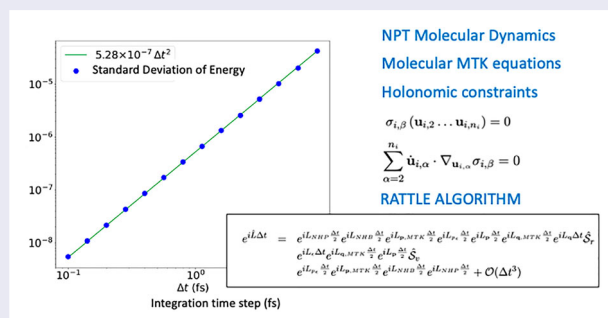
1. Introduction

The need in molecular dynamics (MD) simulations to go beyond the original microcanonical simulations in which the state variables are E, V and N has been understood in the 80s and the program has been essentially achieved with the general formulation provided by Tuckerman *et al.* in the late 90s. Of particular interest is the description of a system subjected to the action of a piston and in contact with a heat bath characterised thermodynamically by pressure, temperature and number of particles. Let's indicate this ensemble, the isothermal–isobaric ensemble, by NPT. The reason of the particular interest is that the NPT ensemble often represents the conditions under which experiments are carried out. Moreover,

under these conditions the density and energy of the system have no difficulty in adjusting automatically to the internal pressure and temperature of the corresponding external values avoiding artificial metastabilities. In the past years, a number of variances of the original methods, introduced by [1–3], have been proposed to simulate the NPT ensemble [4–6]. Finally, a general formulation and justification has been provided by [7], also using the atomic expression for the virial [8]. There is another ingredient, important in MD to simulate realistically and efficiently molecular systems. Indeed, it is often necessary to impose a set of holonomic molecular constraints on the system [9]. This is done by freezing some internal degrees of freedom, such as bond lengths or bending

angles, to eliminate high frequency modes which can limit the integration time step (the generally most useful multiple time-step approach [10] is not always the best solution). It is, therefore, important to be able to simulate a constrained system in the NPT ensemble. This has been achieved in the past in various ways, using either molecular or atomic expressions for the virial [8,11–13]. MD simulations, in the isothermal–isobaric ensembles, require to use equations of motion not derived from a Hamiltonian, where the phase space is extended to include a few new dynamical variables in addition to the usual $\{r, p\}$ variables. The new variables act as controls which equilibrate internal and external temperature ('thermostat' variables) and/or pressure ('barostat' variables) of the system. The thermostat variables act on the momenta of the particles, while the barostat variables act on both positions and momenta. In non-Hamiltonian dynamics, the simultaneous use of the NPT dynamics and constraints is less straightforward than one could hope and full formulations of the model can be found, e.g. in [11,12]. The NPT equations of motion can be written in two different ways, using the atomic or the molecular expression for the virial. Although both approaches have been correctly formulated, here we will discuss only the molecular virial case since, especially in presence of constraints, the atomic virial case becomes cumbersome [8]. While the formulation of the non-Hamiltonian equations of motions represents a solved problem [7], the question of adequate algorithms to solve those equations has been only worked out completely for the case in which the atomic virial is used [13]. Instead, the algorithms presented in [11,12] are unfortunately only precise to the first order in the time step and no more reversible. The solution given in Refs. [13,14] satisfies the required second-order precision and reversibility but making use of the atomic virial it does not present the problems on which we focus in the present paper. Further on that, we have recalled and used the Suzuki–Yoshida decomposition to improve energy conservation or to allow larger time steps.

Section 2 defines the model used to describe the coupling with the molecular centres of mass and writes down the corresponding equations of motion. Section 3 gives a full description of the algorithm to be used for the integrations of these non-trivial non-Hamiltonian equations, which is precise to the second order in the time step. In Section 4, we introduce and apply the Suzuki–Yoshida decomposition. Finally in Section 5 we present a mechanical test of the algorithm for a very simple model, confirming the usefulness of the developed approach. Section 6 presents some concluding remarks.

2. Methods

2.1. Molecular equations for the isobaric ensemble

Martyna, Tobias and Klein (MTK) equations based on molecular pressure read [6,10,15]:

$$\dot{\mathbf{r}}_{i,\alpha} = \frac{\mathbf{p}_{i,\alpha}}{m_{i,\alpha}} + \frac{p_\epsilon}{W}\mathbf{R}_i$$

$$\dot{\mathbf{p}}_{i,\alpha} = \mathbf{f}_{i,\alpha} - \left(1 + \frac{3}{N_{fm}}\right)\frac{p_\epsilon}{W}\frac{m_{i,\alpha}}{M_i}\mathbf{P}_i - \frac{p_{\eta_1}}{Q_{\eta,1}}\mathbf{p}_{i,\alpha}$$

$$\dot{V} = \frac{3Vp_\epsilon}{W}$$

$$\dot{p}_\epsilon = 3V(\mathcal{P}_{mol} - P) + \frac{3}{N_{fm}}\sum_i \frac{\mathbf{P}_i^2}{M_i} - \frac{p_{\xi_1}}{Q_{\xi,1}}p_\epsilon$$

$$\dot{\eta}_j = \frac{p_{\eta_j}}{Q_{\eta,j}} \quad j = 1, \ldots, M_\eta$$

$$\dot{p}_{\eta_1} = \left[\sum_{i,\alpha} \frac{\mathbf{p}_{i,\alpha}^2}{m_{i,\alpha}} - N_f k_B T\right] - \frac{p_{\eta_2}}{Q_{\eta,2}}p_{\eta_1}$$

$$\dot{p}_{\eta_j} = \left[\frac{p_{\eta_{j-1}}^2}{Q_{\eta,j-1}} - k_B T\right] - \frac{p_{\eta_{j+1}}}{Q_{\eta,j+1}}p_{\eta_j}$$

$$j = 2, \ldots, M_\eta - 1$$

$$\dot{p}_{\eta_{M_\eta}} = \left[\frac{p_{\eta_{M_\eta-1}}^2}{Q_{\eta,M_\eta-1}} - k_B T\right]$$

$$\dot{\xi}_j = \frac{p_{\xi_j}}{Q_{\xi,j}} \quad j = 1, \ldots, M_\xi$$

$$\dot{p}_{\xi_1} = \left[\frac{p_\epsilon^2}{W} - k_B T\right] - \frac{p_{\xi_2}}{Q_{\xi,2}}p_{\xi_1}$$

$$\dot{p}_{\xi_j} = \left[\frac{p_{\xi_{j-1}}^2}{Q_{\xi,j-1}} - k_B T\right] - \frac{p_{\xi_{j+1}}}{Q_{\xi,j+1}}p_{\xi_j}$$

$$j = 2, \ldots, M_\xi - 1$$

$$\dot{p}_{\xi_{M_\xi}} = \left[\frac{p_{\eta_{M_\xi-1}}^2}{Q_{\xi,M_\xi-1}} - k_B T\right] \tag{1}$$

where $i = 1 \ldots N$ runs over all N molecules and α over all n_i atoms within a molecule, $\mathcal{P}_{mol}$ is the molecular pressure [16], $\mathbf{P}_i = \sum_\alpha \mathbf{p}_{i,\alpha}$,

$$\mathbf{R}_i = \sum_\alpha \frac{m_{i,\alpha}\mathbf{r}_{i,\alpha}}{M_i} \tag{2}$$

$\mathbf{f}_{i,\alpha}$ is the total force acting on atom α of molecule i and we also coupled two independent Nosè–Hoover chains to particles (whose associated variables are η_j and p_{η_j} with $j = 1 \ldots M_\eta$) and to barostat (whose associated

variables are ξ_j and p_{ξ_j} with $j = 1 \ldots M_\xi$). In Equation (1) $N_f = \sum_i^N 3n_i$ or $N_f' = N_f - 3$ if the initial total momentum is zero, N_{fm} is the number of degrees of freedom associated to the molecules, $Q_{\eta,j}$ and $Q_{\xi,j}$ are thermostat 'masses' and W is the barostat 'mass'. To numerically integrate these equations, we implement a reversible integrator algorithm. If we define the state of the system as:

$$\Gamma = (\mathbf{r}^{N_{at}}, \mathbf{p}^{N_{at}}, V, p_\epsilon, \eta_1 \ldots \eta_{M_\eta}, p_{\eta_1} \cdots p_{\eta_{M_\eta}},$$

$$\xi_1 \ldots \xi_{M_\xi}, p_{\xi_1} \cdots p_{\xi_{M_\xi}}) \tag{3}$$

where $N_{at} = \sum_i n_i$ is the total number of atoms in the system and Γ encompasses all the $6N_{at} + 2M_\eta + 2M_\xi + 2$ variables of the extended phase space, the equations of motion of a phase function explicitly time-independent can be written as follows:

$$\dot{f}(\Gamma) = \sum_\lambda \dot{\Gamma}_\lambda \frac{\partial}{\partial \Gamma_\lambda} f(\Gamma) = i\hat{L}f \tag{4}$$

with $\hat{L}$, the Liouville operator.

2.2. Solution method without constraints

In preparation of the computable Liouvillian, we start suggesting a preliminary splitting of the Liouvillian that permits to arrive at the end to a fully computable evolution operator. To integrate the equations of motion, we need a decomposition of the operator $e^{i\hat{L}\Delta t}$, reduced to terms of the type $e^{f(\{\Gamma'\})} \frac{\partial}{\partial \Gamma_\lambda}$ with Γ' not containing Γ_λ or such that it can be reformulated in similar shape (e.g. terms of the type $e^{q \frac{\partial}{\partial q}} = e^{\frac{\partial}{\partial \ln q}}$),

In the present case, $\widehat{iL} = \sum_\lambda \dot{\Gamma}_\lambda \frac{\partial}{\partial \Gamma_\lambda}$, can be explicitly written as e.g.

$$i\hat{L} = iL_{NHB} + iL_{NHP} + iL_{p_\epsilon} + iL_{\mathbf{p},MTK0} + iL_{\mathbf{p},MTK1}$$

$$+ iL_{\mathbf{p}} + iL_{\mathbf{q},MTK0} + iL_{\mathbf{q},MTK1} + iL_{\mathbf{q}} + iL_\epsilon \tag{5}$$

where iL_{NHB} and iL_{NHP} are the time evolution operators of the two Nosè–Hoover chains [15] coupled to the barostat (NHB) and particles (NHP) respectively, i.e.

$$iL_{NHP} = -\sum_{i,\alpha} \frac{p_{\eta_1}}{Q_{\eta,1}} \mathbf{p}_{i,\alpha} \frac{\partial}{\partial \mathbf{p}_{i,\alpha}} + \sum_{j=1}^{M_\eta} \frac{p_{\eta_j}}{Q_{\eta,j}} \frac{\partial}{\partial \eta_j}$$

$$+ \sum_{j=1}^{M_\eta - 1} \left(G_{\eta,j} - p_{\eta_j} \frac{p_{\eta_{j+1}}}{Q_{\eta,j+1}} \right) \frac{\partial}{\partial p_{\eta_j}}$$

$$+ G_{\eta,M_\eta} \frac{\partial}{\partial p_{\eta_{M_\eta}}} \tag{6}$$

$$iL_{NHB} = -\frac{p_{\xi_1}}{Q_{\xi,1}} p_\epsilon \frac{\partial}{\partial p_\epsilon} + \sum_{j=1}^{M_\xi} \frac{p_{\xi_j}}{Q_{\xi,j}} \frac{\partial}{\partial \xi_j}$$

$$+ \sum_{j=1}^{M_\xi - 1} \left(G_{\xi,j} - p_{\xi_j} \frac{p_{\xi_{j+1}}}{Q_{\xi,j+1}} \right) \frac{\partial}{\partial p_{\xi_j}}$$

$$+ G_{\xi,M_\xi} \frac{\partial}{\partial p_{\xi_{M_\xi}}} \tag{7}$$

where

$$G_{\eta,1} = \sum_{i,\alpha} \frac{\mathbf{p}_{i,\alpha}^2}{m_{i,\alpha}} - N_f k_B T \tag{8}$$

$$G_{\eta,j} = \frac{p_{\eta_{j-1}}^2}{Q_{\eta,j-1}} - k_B T \quad j = 2 \ldots M_\eta \tag{9}$$

$$G_{\xi,1} = \frac{p_\epsilon^2}{W} - k_B T \tag{10}$$

$$G_{\xi,j} = \frac{p_{\xi_{j-1}}^2}{Q_{\xi,j-1}} - k_B T \quad j = 2 \ldots M_\xi \tag{11}$$

and

$$iL_{p_\epsilon} = \left[3V(\mathcal{P}_{mol} - P) + \frac{3}{N_{fm}} \sum_i \frac{\mathbf{P}_i^2}{M_i} \right] \frac{\partial}{\partial p_\epsilon} \tag{12}$$

$$iL_{\mathbf{p},MTK0} = -\left(1 + \frac{3}{N_{fm}}\right) \sum_i \sum_\alpha \frac{p_\epsilon}{W} \frac{m_{i,\alpha}}{M_i} \mathbf{p}_{i,\alpha} \frac{\partial}{\partial \mathbf{p}_{i,\alpha}} \tag{13}$$

$$iL_{\mathbf{p},MTK1} = -\left(1 + \frac{3}{N_{fm}}\right) \sum_i \frac{p_\epsilon}{W} \sum_{\substack{\alpha,\alpha' \\ \alpha' \neq \alpha}} \frac{m_{i,\alpha} \mathbf{p}_{i,\alpha'}}{M_i} \frac{\partial}{\partial \mathbf{p}_{i,\alpha}} \tag{14}$$

$$iL_{\mathbf{q},MTK0} = \sum_i \sum_\alpha \frac{p_\epsilon}{W} \frac{m_{i,\alpha}}{M_i} \mathbf{r}_{i,\alpha} \frac{\partial}{\partial \mathbf{r}_{i,\alpha}} \tag{15}$$

$$iL_{\mathbf{q},MTK1} = \sum_i \frac{p_\epsilon}{W} \sum_{\substack{\alpha,\alpha' \\ \alpha' \neq \alpha}} \frac{m_{i,\alpha'} \mathbf{r}_{i,\alpha'}}{M_i} \frac{\partial}{\partial \mathbf{r}_{i,\alpha}} \tag{16}$$

$$iL_{\mathbf{p}} = \sum_i \sum_\alpha \mathbf{f}_{i,\alpha} \frac{\partial}{\partial \mathbf{p}_{i,\alpha}} \tag{17}$$

$$iL_{\mathbf{q}} = \sum_i \sum_\alpha \frac{\mathbf{p}_{i,\alpha}}{m_{i,\alpha}} \frac{\partial}{\partial \mathbf{r}_{i,\alpha}} \tag{18}$$

$$iL_\epsilon = \frac{p_\epsilon}{W} 3V \frac{\partial}{\partial V} \tag{19}$$

Note that the splitting of the operators

$$iL_{\mathbf{q},MTK} = iL_{\mathbf{q},MTK0} + iL_{\mathbf{q},MTK1} \tag{20}$$

and

$$iL_{\mathbf{p},MTK} = iL_{\mathbf{p},MTK0} + iL_{\mathbf{p},MTK1} \tag{21}$$

is necessary to have an algorithm precise to the order Δt^2 instead of Δt as done in Ref. [11]. First we consider the case of atoms which are not subjected to forces due to holonomic constraints. In this case to integrate the equations of motion, we present an approach which permits to build an explicit workable decomposition of $e^{i\hat{L}\Delta t}$ in terms of computable expressions. Let iL_g be a generic exponentiated Liouvillian that can be decomposed as follows:

$$iL_g = iL_1 + iL_2 + iL_3, \qquad (22)$$

where it is assumed that $e^{iL_1\Delta t}$, $e^{iL_2\Delta t}$ and $e^{iL_3\Delta t}$ give computable expressions. By applying a first Trotter decomposition of the time evolution operator $e^{iL_g\Delta t}$, one has

$$e^{iL_g\Delta t} = e^{iL_3\frac{\Delta t}{2}}e^{(iL_1+iL_2)\Delta t}e^{iL_3\frac{\Delta t}{2}} + \mathcal{O}(\Delta t^3) \qquad (23)$$

and, then, by further Trotter decomposing the operator $e^{(iL_1+iL_2)\Delta t}$

$$e^{(iL_1+iL_2)\Delta t} = e^{iL_2\frac{\Delta t}{2}}e^{iL_1\Delta t}e^{iL_2\frac{\Delta t}{2}} + \mathcal{O}(\Delta t^3) \qquad (24)$$

one gets for the operator $e^{iL_g\Delta t}$:

$$e^{iL_g\Delta t} = e^{iL_3\frac{\Delta t}{2}}e^{iL_2\frac{\Delta t}{2}}e^{iL_1\Delta t}e^{iL_2\frac{\Delta t}{2}}e^{iL_3\frac{\Delta t}{2}} + \mathcal{O}(\Delta t^3). \qquad (25)$$

Using the decomposition in Equation (25) and in reference to Equation (5), we can write the operator $e^{i\hat{L}\Delta t}$ as follows:

$$e^{i\hat{L}\Delta t} = e^{iL_{NHP}\frac{\Delta t}{2}}e^{iL_{NHB}\frac{\Delta t}{2}}e^{iL_{\mathbf{p},MTK}\frac{\Delta t}{2}}e^{iL_{p\epsilon}\frac{\Delta t}{2}}e^{iL_{\mathbf{p}}\frac{\Delta t}{2}}e^{iL_{\mathbf{q},MTK}\frac{\Delta t}{2}}$$
$$\times e^{iL_{\mathbf{q}}\Delta t}e^{iL_{\epsilon}\Delta t}e^{iL_{\mathbf{q},MTK}\frac{\Delta t}{2}}e^{iL_{\mathbf{p}}\frac{\Delta t}{2}}e^{iL_{p\epsilon}\frac{\Delta t}{2}}e^{iL_{\mathbf{p},MTK}\frac{\Delta t}{2}}$$
$$\times e^{iL_{NHB}\frac{\Delta t}{2}}e^{iL_{NHP}\frac{\Delta t}{2}} + \mathcal{O}(\Delta t^3), \qquad (26)$$

where, we note, L_{ϵ} and $L_{\mathbf{q}}$ operators commute and all the operators are applied from left to right, to guarantee recursivity [11], i.e. the first operator from the left $e^{iL_{NHP}\Delta t/2}$ updates all the other operators and the function to which it is applied. For example, if we consider two operators $e^{a(y)\frac{\partial}{\partial x}\Delta t}$ and $e^{b(x)\frac{\partial}{\partial y}\Delta t}$, which can be applied to functions which depend on x and y, and a generic function $f(x,y)$, one has

$$e^{a(y)\frac{\partial}{\partial x}\Delta t}e^{b(x)\frac{\partial}{\partial y}\Delta t}f(x,y) = e^{b(x')\frac{\partial}{\partial y}\Delta t}f(x',y')$$
$$= f(x'',y''), \qquad (27)$$

where

$$x' = x + a(y), \quad y' = y$$
$$x'' = x', \quad y'' = y' + b(x'). \qquad (28)$$

We remind that we assumed that particles are not subjected to constraint forces, i.e. only explicit interaction forces enter in the operators $e^{iL_{\mathbf{p}}\frac{\Delta t}{2}}$.

The operators $e^{iL_{NHP}\Delta t/2}$ and $e^{iL_{NHB}\Delta t/2}$ in Equation (26) have to be further factorised, since the terms into which they are split (see Equations (6) and (7)) are not simple translation operators. We now discuss the approach, which can be found in Ref. [15], to achieve a factorisation such that it can be straightforwardly translated into a computer algorithm. If one has a generic operator iL_g, such that

$$iL_g = iL_1' + iL_2' + iL_3' + iL_4' \qquad (29)$$

resorting to the same Trotter factorisation used to arrive in Equation (25) with $iL_1 = iL_1'$, $iL_2 = iL_2' + iL_3'$ and $iL_3 = iL_4'$, one obtains

$$e^{iL_g\Delta t} = e^{iL_4\frac{\Delta t}{2}}e^{i(L_2+L_3)\frac{\Delta t}{2}}e^{iL_1\Delta t}e^{i(L_2+L_3)\frac{\Delta t}{2}}e^{iL_4\frac{\Delta t}{2}}$$
$$+ \mathcal{O}(\Delta t^3) \qquad (30)$$

and by factorising again the operators $e^{i(L_2+L_3)\frac{\Delta t}{2}}$, i.e.

$$e^{i(L_2+L_3)\frac{\Delta t}{2}} = e^{iL_3\frac{\Delta t}{4}}e^{iL_2\frac{\Delta t}{2}}e^{iL_3\frac{\Delta t}{4}} + \mathcal{O}(\Delta t^3) \qquad (31)$$

one finally has

$$e^{iL_g\Delta t} = e^{iL_4\frac{\Delta t}{2}}e^{iL_3\frac{\Delta t}{4}}e^{iL_2\frac{\Delta t}{2}}e^{iL_3\frac{\Delta t}{4}}e^{iL_1\Delta t}e^{iL_3\frac{\Delta t}{4}}$$
$$\times e^{iL_2\frac{\Delta t}{2}}e^{iL_3\frac{\Delta t}{4}}e^{iL_4\frac{\Delta t}{2}} + \mathcal{O}(\Delta t^3). \qquad (32)$$

We note that the factorisation used here is not unique and that the overall integrator will still be $\mathcal{O}(\Delta t^3)$. In principle, any factorisation based on Trotter decomposition is eligible to be used here.

Recalling Equations (6), by leveraging several times Trotter decomposition, similarly to what done to obtain Equation (32), one has:

$$e^{iL_{NHP}\Delta t/2} = e^{\frac{\Delta t}{4}G_{\eta,M_\eta}\frac{\partial}{\partial p_{\eta M_\eta}}}\prod_{j=M_\eta-1}^{1}$$
$$\times \left(e^{-\frac{\Delta t}{8}\frac{p_{\eta j+1}}{Q_{\eta,j+1}}\frac{\partial}{\partial p_{\eta j}}}e^{\frac{\Delta t}{4}G_{\eta,j}\frac{\partial}{\partial p_{\eta j}}}e^{-\frac{\Delta t}{8}\frac{p_{\eta j+1}}{Q_{\eta,j+1}}\frac{\partial}{\partial p_{\eta j}}}\right)$$
$$\times \prod_{\alpha=1}^{N_{at}}e^{-\frac{\Delta t}{2}\frac{p_{\eta 1}}{Q_{\eta,1}}\mathbf{P}_{i,\alpha}\cdot\frac{\partial}{\partial \mathbf{P}_{i,\alpha}}}\prod_{j=1}^{M_\eta}e^{-\frac{\Delta t}{2}\frac{p_{\eta j}}{Q_{\eta,j}}\frac{\partial}{\partial \eta_j}}$$
$$\times \prod_{j=1}^{M_\eta-1}\left(e^{-\frac{\Delta t}{8}\frac{p_{\eta j+1}}{Q_{\eta,j+1}}\frac{\partial}{\partial p_{\eta j}}}e^{\frac{\Delta t}{4}G_{\eta,j}\frac{\partial}{\partial p_{\eta j}}}e^{-\frac{\Delta t}{8}\frac{p_{\eta j+1}}{Q_{\eta,j+1}}\frac{\partial}{\partial p_{\eta j}}}\right)$$
$$\times e^{\frac{\Delta t}{4}G_{\eta,M_\eta}\frac{\partial}{\partial p_{\eta M_\eta}}} + \mathcal{O}(\Delta t^3), \qquad (33)$$

where the indices in the product symbols are written so that the order of the factors, starting from the left, ensure the correct writing of the partial evolution operator in the lhs. With a similar treatment applied to (7), one obtains

$$e^{iL_{NHB}\Delta t/2} = e^{\frac{\Delta t}{4}G_{\xi,M_\xi}\frac{\partial}{\partial p_{M_\xi}}} \prod_{j=M_\xi-1}^{1}$$

$$\times \left(e^{-\frac{\Delta t}{8}\frac{p_{\xi,j+1}}{Q_{\xi,j+1}}\frac{\partial}{\partial p_{\xi,j}}} e^{\frac{\Delta t}{4}G_{\xi,j}\frac{\partial}{\partial p_{\xi,j}}} e^{-\frac{\Delta t}{8}\frac{p_{\xi,j+1}}{Q_{\xi,j+1}}\frac{\partial}{\partial p_{\xi,j}}} \right)$$

$$\times e^{-\frac{\Delta t}{2}\frac{p_{\xi,1}}{Q_{\xi,1}}p_\epsilon\frac{\partial}{\partial p_\epsilon}} \prod_{j=1}^{M_\xi} e^{-\frac{\Delta t}{2}\frac{p_{\xi,j}}{Q_{\xi,j}}\frac{\partial}{\partial\xi_j}}$$

$$\times \prod_{j=1}^{M_\xi-1} \left(e^{-\frac{\Delta t}{8}\frac{p_{\xi,j+1}}{Q_{\xi,j+1}}\frac{\partial}{\partial p_{\xi,j}}} e^{\frac{\Delta t}{4}G_{\xi,j}\frac{\partial}{\partial p_{\xi,j}}} e^{-\frac{\Delta t}{8}\frac{p_{\xi,j+1}}{Q_{\xi,j+1}}\frac{\partial}{\partial p_{\xi,j}}} \right)$$

$$\times e^{\frac{\Delta t}{4}G_{\xi,M_\xi}\frac{\partial}{\partial p_{\xi M_\xi}}} + \mathcal{O}(\Delta t^3). \tag{34}$$

Note that these factorisations are identical to the ones proposed in Ref. [10,15] and that they are constituted of operators which can be readily translated into a computer code.

As an alternative to Nose–Hoover chains, one can employ a stochastic reservoirs which have some advantages in terms of stability, thus possibly allowing for larger integration time steps [15]. For example the algorithm BAOAB proposed in Refs. [17,18] can be readily implemented by eliminating in Equation (26) the NH propagator coupled to particles (i.e. $e^{iL_{NHP}}\Delta t$) and replacing the evolution operator $e^{iL_\mathbf{q}\Delta t}$ with the following operator:

$$e^{iL_\mathbf{q}\frac{\Delta t}{2}} e^{iL_O\Delta t} e^{iL_\mathbf{q}\frac{\Delta t}{2}}, \tag{35}$$

where iL_O is the Liouvillian operator of Langevin dynamics, as discussed in Refs. [17,19]. To do the same for the barostat, one has to eliminate in Equation (26) the NH propagator coupled to the barostat (i.e. $e^{iL_{NHB}\Delta t}$) introducing a stochastic barostat as discussed in Ref. [20]. Many other stochastic methods have been proposed in the literature [21–24], but they do not need an individual treatment for the specific problem we are addressing here. In any case one can find a thorough comparison and discussion of them in Ref. [18].

We can treat the operators $iL_{\mathbf{q},MTK}$ and $iL_{\mathbf{p},MTK}$, similarly to operators iL_{NHP} and iL_{NHB}, i.e. we can employ the Trotter factorisation to have

$$e^{iL_{\mathbf{q},MTK}\frac{\Delta t}{2}} = \prod_i \prod_{\alpha=1}^{n_i} e^{\sum_{\alpha'\neq\alpha}\frac{\Delta t}{4}\frac{p_\epsilon}{W}\frac{m_{i,\alpha'}\mathbf{r}_{i,\alpha'}}{M_i}\frac{\partial}{\partial\mathbf{r}_{i,\alpha}}}$$

$$\times e^{\sum_\alpha\frac{\Delta t}{2}\frac{p_\epsilon}{W}\frac{m_{i,\alpha}}{M_i}\mathbf{r}_{i,\alpha}\frac{\partial}{\partial\mathbf{r}_{i,\alpha}}}$$

$$\times \prod_{\alpha=n_i}^{1} e^{\sum_{\alpha'\neq\alpha}\frac{\Delta t}{4}\frac{p_\epsilon}{W}\frac{m_{i,\alpha'}\mathbf{r}_{i,\alpha'}}{M_i}\frac{\partial}{\partial\mathbf{r}_{i,\alpha}}} + \mathcal{O}(\Delta t^3) \tag{36}$$

$$e^{iL_{\mathbf{p},MTK}\frac{\Delta t}{2}} = \prod_i \prod_{\alpha=1}^{n_i} e^{-\sum_{\alpha'\neq\alpha}\frac{\Delta t}{4}\left(1+\frac{3}{N_{fm}}\right)\frac{p_\epsilon}{W}\frac{m_{i,\alpha'}\mathbf{p}_{i,\alpha'}}{M_i}\frac{\partial}{\partial\mathbf{p}_{i,\alpha}}}$$

$$\times e^{-\sum_\alpha\frac{\Delta t}{2}\left(1+\frac{3}{N_{fm}}\right)\frac{p_\epsilon}{W}\frac{m_{i,\alpha}}{M_i}\mathbf{p}_{i,\alpha}\frac{\partial}{\partial\mathbf{p}_{i,\alpha}}}$$

$$\times \prod_{\alpha=n_i}^{1} e^{-\sum_{\alpha'\neq\alpha}\frac{\Delta t}{4}\left(1+\frac{3}{N_{fm}}\right)\frac{p_\epsilon}{W}\frac{m_{i,\alpha'}\mathbf{p}_{i,\alpha'}}{M_i}\frac{\partial}{\partial\mathbf{p}_{i,\alpha}}} \tag{1}$$

$$+ \mathcal{O}(\Delta t^3). \tag{37}$$

So far we discussed how to develop a code for implementing a measure-preserving (with respect to the invariant measure defined in Ref. [7]) isobaric–isothermal ensemble of particles without holonomic constraints. For the Nosè–Hoover thermostats, we adopted a well-established strategy already discussed [15], while we derived a suitable reversible scheme for MTK equations (see Equation 1).

2.3. Solution method with constraints

We now consider the case where the atoms within each molecule are constrained through a translation-invariant function, i.e. we assume that the atoms of the N molecules are subjected to the following holonomic constraints:

$$\sigma_{i,\beta}\left(\mathbf{u}_{i,2}\ldots\mathbf{u}_{i,n_i}\right) = 0, \tag{38}$$

where $\mathbf{u}_{i,\alpha} = \mathbf{r}_{i,\alpha} - \mathbf{r}_{i,1}$ and $\beta = 1\ldots l_i$ runs over all l_i constraints of ith molecule. We note that the relative velocity $\dot{\mathbf{u}}_{i,\alpha} = \mathbf{v}_{i,\alpha} - \mathbf{v}_{i,1}$ have also to fulfil the holonomic constraints in their differential form, i.e.

$$\sum_{\alpha=2}^{n_i} \dot{\mathbf{u}}_{i,\alpha} \cdot \nabla_{\mathbf{u}_{i,\alpha}}\sigma_{i,\beta} = 0, \tag{39}$$

which states that the collective relative velocity $\dot{\mathbf{u}}_{i,\alpha}$ with $\alpha = 2,\ldots,n_i$ has to be orthogonal to the normal of the hypersurface σ.

Before considering the isobaric ensemble let us consider the velocity Verlet algorithm in the micro-canonical ensemble for a system composed of N molecules where the constituting atoms are subjected to the holonomic constraints given in Equation (38). The full propagator $e^{i\hat{L}\Delta t}$, by which positions and momenta are updated over the time step Δt, in the case of velocity Verlet algorithm reads:

$$e^{i\hat{L}\Delta t} = e^{iL_\mathbf{p}^A\frac{\Delta t}{2}} e^{iL_\mathbf{q}\Delta t} e^{iL_\mathbf{p}^B\frac{\Delta t}{2}} + \mathcal{O}(\Delta t^3), \tag{40}$$

where the two operators $e^{iL_{\mathbf{p}}^{A}\Delta/2}$ and $e^{iL_{\mathbf{p}}^{B}\Delta/2}$ evolve momenta according to forces evaluated at time t and $t + \Delta t$ respectively and they include the constraint forces acting on each atom (i, α) at time t and $t + \Delta t$. These constraint forces can be expressed in terms of two different sets of lagrangian multipliers $\lambda_{i,\beta}$ and $\lambda'_{i,\beta}$, so that the two operators $e^{iL_{\mathbf{p}}^{A}\Delta/2}$ and $e^{iL_{\mathbf{p}}^{B}\Delta/2}$ in Equation (40) can be written as follows:

$$e^{iL_{\mathbf{p}}^{A}\Delta t/2} = e^{\left[\sum_{i,\alpha} \mathbf{f}_{i,\alpha}(t) + \mathbf{c}_{i,\alpha}\right]\frac{\partial}{\partial \mathbf{p}_{i,\alpha}}}$$

$$e^{iL_{\mathbf{p}}^{B}\Delta t/2} = e^{\left[\sum_{i,\alpha} \mathbf{f}_{i,\alpha}(t+\Delta t) + \mathbf{c}'_{i,\alpha}\right]\frac{\partial}{\partial \mathbf{p}_{i,\alpha}}}, \quad (41)$$

where $\mathbf{c}_{i,\alpha}$ and $\mathbf{c}'_{i,\alpha}$ are the constraint forces associated to the lagrangian multipliers $\lambda_{i,\beta}$ and $\lambda'_{i,\beta}$ respectively, i.e.

$$\mathbf{c}_{i,\alpha} = \sum_{\beta} \lambda_{i,\beta} \nabla_{\mathbf{r}_{i,\alpha}} \sigma_{i,\beta}(t)$$

$$\mathbf{c}'_{i,\alpha} = \sum_{\beta} \lambda'_{i,\beta} \nabla_{\mathbf{r}_{i,\alpha}} \sigma_{i,\beta}(t + \Delta t), \quad (42)$$

where $\alpha = 1 \ldots n_i$. According to RATTLE algorithm [25], the positions are first updated neglecting the constraint forces through the action of the operator $e^{iL_{\mathbf{p}}\frac{\Delta t}{2}} e^{iL_{\mathbf{q}}\Delta t}$ and then they are adjusted, together with the contribution to velocities, to fulfil the constraints in Equation (38). Afterwards, the momenta are updated through the operator $e^{iL_{\mathbf{p}}\frac{\Delta t}{2}}$ and then the velocities of the atoms are adjusted to fulfil the constraints in Equation (39). Hence, RATTLE algorithm amounts to make the following replacements in Equation (40):

$$e^{iL_{\mathbf{p}}^{A}\frac{\Delta t}{2}} e^{iL_{\mathbf{q}}\Delta t} \rightarrow e^{iL_{\mathbf{p}}\frac{\Delta t}{2}} e^{iL_{\mathbf{q}}\Delta t} \hat{\mathcal{S}}_r \quad (43)$$

$$e^{iL_{\mathbf{p}}^{B}\frac{\Delta t}{2}} \rightarrow e^{iL_{\mathbf{p}}\frac{\Delta t}{2}} \hat{\mathcal{S}}_v, \quad (44)$$

where the operator $\hat{\mathcal{S}}_r$ adjusts positions and its contribution to velocities of atoms to take into account the action of the constraint forces $\mathbf{c}_{i,\alpha}$, while the operator $\hat{\mathcal{S}}_v$ adjusts the velocities according to constraint forces $\mathbf{c}'_{i,\alpha}$ and the operators $e^{iL_{\mathbf{p}}\frac{\Delta t}{2}}$ evolve momenta according to non-constraint forces $\mathbf{f}_{i,\alpha}$, i.e.

$$e^{iL_{\mathbf{p}}\frac{\Delta t}{2}} = e^{\sum_{i,\alpha} \mathbf{f}_{i,\alpha}\frac{\Delta t}{2}\frac{\partial}{\partial \mathbf{p}_{i,\alpha}}}. \quad (45)$$

In view of Equations (43) and (44), the full propagator of the RATTLE algorithm can be written as follows:

$$e^{i\hat{L}\Delta t} = e^{iL_{\mathbf{p}}\frac{\Delta t}{2}} e^{iL_{\mathbf{q}}\Delta t} \hat{\mathcal{S}}_r e^{iL_{\mathbf{p}}\frac{\Delta t}{2}} \hat{\mathcal{S}}_v + \mathcal{O}(\Delta t^3). \quad (46)$$

Concerning the operator $\hat{\mathcal{S}}_r$, it adjusts first the positions of particles to fulfil the constraints in Equation (38), which is equivalent to say that

$$\hat{\mathcal{S}}_r \mathbf{r}_{i,\alpha} = \mathbf{r}_{i,\alpha} + \frac{\Delta t^2}{2 m_{i,\alpha}} \mathbf{c}_{i,\alpha}, \quad (47)$$

where we remind that $\mathbf{r}_{i,\alpha}$ is the position of atom α belonging to molecule i. Then it also updates (by using the constraints forces already calculated for adjusting the positions), consistently with Equation (47), the velocity of the atom, i.e. $\hat{\mathcal{S}}_r$ acts on atom velocity as follows:

$$\hat{\mathcal{S}}_r \mathbf{v}_{i,\alpha} = \mathbf{v}_{i,\alpha} + \frac{\Delta t}{2 m_{i,\alpha}} \mathbf{c}_{i,\alpha}. \quad (48)$$

The evaluation of the lagrangian multipliers is achieved iteratively as in the standard SHAKE approach [26], where the constraints are adjusted one by one until convergence is reached. The operator $\hat{\mathcal{S}}_v$ updates only the velocities to fulfil the constraints in Equation (39), i.e.

$$\hat{\mathcal{S}}_v \mathbf{v}'_{i,\alpha} = \mathbf{v}'_{i,\alpha} + \frac{\Delta t}{2 m_{i,\alpha}} \mathbf{c}'_{i,\alpha}, \quad (49)$$

where velocities $\mathbf{v}'_{i,\alpha}$ are those obtained after the application of operator $\hat{\mathcal{S}}_r$ (see above) and again these adjustments are done iteratively over each constraint until convergence is reached for all constraints.

We now have what we need to treat the case of isobaric MD equations. The full propagator, if atoms are subjected to holonomic constraints as in Equation (38), is

$$e^{i\hat{L}\Delta t} = e^{iL_{NHP}\frac{\Delta t}{2}} e^{iL_{NHB}\frac{\Delta t}{2}} e^{iL_{\mathbf{p},MTK}\frac{\Delta t}{2}} e^{iL_{p_{\epsilon}}\frac{\Delta t}{2}}$$

$$\times \overbrace{e^{iL_{\mathbf{p}}^{A}\frac{\Delta t}{2}} e^{iL_{\mathbf{q},MTK}\frac{\Delta t}{2}} e^{iL_{\mathbf{q}}\Delta t}} e^{iL_{\epsilon}\Delta t}$$

$$\times e^{iL_{\mathbf{q},MTK}\frac{\Delta t}{2}} \overbrace{e^{iL_{\mathbf{p}}^{B}\frac{\Delta t}{2}}} e^{iL_{p_{\epsilon}}\frac{\Delta t}{2}} e^{iL_{\mathbf{p},MTK}\frac{\Delta t}{2}} e^{iL_{NHB}\frac{\Delta t}{2}}$$

$$\times e^{iL_{NHP}\frac{\Delta t}{2}} + \mathcal{O}(\Delta t^3), \quad (50)$$

where the operators $e^{iL_{\mathbf{p}}^{A}\Delta t/2}$ and $e^{iL_{\mathbf{p}}^{B}\Delta t/2}$ include the constraint forces. In the spirit of the RATTLE approach which has been used for the velocity Verlet algorithm in the micro-canonical ensemble, we make the following replacements of the terms highlighted in Equation (50):

$$e^{iL_{\mathbf{p}}^{A}\frac{\Delta t}{2}} e^{iL_{\mathbf{q},MTK}\frac{\Delta t}{2}} e^{iL_{\mathbf{q}}\Delta t} \rightarrow e^{iL_{\mathbf{p}}\frac{\Delta t}{2}} e^{iL_{\mathbf{q},MTK}\frac{\Delta t}{2}} e^{iL_{\mathbf{q}}\Delta t} \hat{\mathcal{S}}_r \quad (51)$$

$$e^{iL_{\mathbf{p}}^{B}\frac{\Delta t}{2}} \rightarrow e^{iL_{\mathbf{p}}\frac{\Delta t}{2}} \hat{\mathcal{S}}_v. \quad (52)$$

In the present case the operator $\hat{\mathcal{S}}_r$ properly corrects positions and impart velocities only according to the constraint forces related to the lagrangian multipliers $\lambda_{i,\beta}$ (and not $\lambda'_{i,\beta}$) In view of the replacements in Equations (51) and (52), the full propagator $e^{i\hat{L}\Delta t}$ becomes

$$e^{i\hat{L}\Delta t} = e^{iL_{NHP}\frac{\Delta t}{2}} e^{iL_{NHB}\frac{\Delta t}{2}} e^{iL_{\mathbf{p},MTK}\frac{\Delta t}{2}} e^{iL_{p_{\epsilon}}\frac{\Delta t}{2}} e^{iL_{\mathbf{p}}\frac{\Delta t}{2}} e^{iL_{\mathbf{q},MTK}\frac{\Delta t}{2}}$$

$$\times e^{iL_{\mathbf{q}}\Delta t} \hat{\mathcal{S}}_r e^{iL_{\epsilon}\Delta t} e^{iL_{\mathbf{q},MTK}\frac{\Delta t}{2}} e^{iL_{\mathbf{p}}\frac{\Delta t}{2}} \hat{\mathcal{S}}_v$$

$$\times e^{iL_{p_{\epsilon}}\frac{\Delta t}{2}} e^{iL_{\mathbf{p},MTK}\frac{\Delta t}{2}} e^{iL_{NHB}\frac{\Delta t}{2}} e^{iL_{NHP}\frac{\Delta t}{2}} + \mathcal{O}(\Delta t^3). \quad (53)$$

In previous equation, the operators $e^{iL_{\mathbf{q},MTK}\frac{\Delta t}{2}}$ and $e^{iL_{\mathbf{p},MTK}\frac{\Delta t}{2}}$ are also applied after the operators $\hat{S}_r$ and $\hat{S}_v$, respectively. Moreover, while the replacement in Equation (52) is exactly what one does for the micro-canonical velocity Verlet algorithm (cf. Equation 44), the replacement in Equation (51) differs from the one in Equation (43), in that, to calculate the lagrangian multipliers $\lambda_{i,\alpha}$, one has to take into account two times the action of the operator $e^{iL_{\mathbf{q},MTK}\frac{\Delta t}{2}}$. However, since from Equation (20) it follows that

$$iL_{\mathbf{q},MTK}(\mathbf{r}_{i,\alpha} - \mathbf{r}_{i,\alpha'}) = \frac{p_\epsilon}{W} \sum_{i',\alpha''} \mathbf{R}_{i'} \frac{\partial}{\partial \mathbf{r}_{i',\alpha''}} (\mathbf{r}_{i,\alpha} - \mathbf{r}_{i,\alpha'}) = 0,$$

(54)

where $\mathbf{r}_{i,\alpha}$ and $\mathbf{r}_{i,\alpha'}$ are positions of two atoms belonging to molecule i, one has that

$$e^{iL_{\mathbf{q},MTK}\frac{\Delta t}{2}} (\mathbf{r}_{i,\alpha} - \mathbf{r}_{i,\alpha'}) = \mathbf{r}_{i,\alpha} - \mathbf{r}_{i,\alpha'} \qquad (55)$$

thus proving that the operator $e^{iL_{\mathbf{q},MTK}\frac{\Delta t}{2}}$ preserves the relative position of atom pairs, i.e. that the full propagator in Equation (53) guarantees the correct conservation of translational invariant constraints.

It is left to prove that the relative velocity of atoms along the constraint direction is preserved by the operator $e^{iL_{\mathbf{p},MTK}\frac{\Delta t}{2}}$, i.e. by the full propagator $e^{i\hat{L}\Delta t}$, however, considering that from Equation (21) it follows that

$$iL_{\mathbf{p},MTK}(\mathbf{v}_{i,\alpha} - \mathbf{v}_{i,\alpha'}) = -\left(1 + \frac{3}{N_{fm}}\right) \frac{p_\epsilon}{W} \sum_{i',\alpha''} \frac{\mathbf{P}_{i'}}{M_{i'}}$$
$$\times \frac{\partial}{\partial \mathbf{v}_{i',\alpha''}} (\mathbf{v}_{i,\alpha} - \mathbf{v}_{i,\alpha'}) = 0,$$

(56)

where $\mathbf{v}_{i,\alpha}$ and $\mathbf{v}_{i,\alpha'}$ are velocities of two atoms belonging to molecule i, one has that

$$e^{iL_{\mathbf{p},MTK}\frac{\Delta t}{2}} (\mathbf{v}_{i,\alpha} - \mathbf{v}_{i,\alpha'}) = \mathbf{v}_{i,\alpha} - \mathbf{v}_{i,\alpha'}, \qquad (57)$$

which proves that also the operator $e^{iL_{\mathbf{p},MTK}\frac{\Delta t}{2}}$ preserves the relative velocity of atoms.

In spite of the analytical property of the above operators, the propagation of the numerical error produced by the operators in Equations (36) and (37) is not forbidden. The proper writing in the presence of constraints is summarised in Equation (53). There it is seen that the last application of SHAKE is not at the end of the step, so that a numerical discrepancy can appear. However, the re-application of SHAKE in the next step with the original values of the constraints, resets automatically coordinates and velocities, forbidding the propagation of the error. Therefore, the overall trajectory is stable, preserves the constraints and remains precise to third order.

3. Suzuki–Yoshida treatment

The factorisation of the operators $e^{iL_{NHP}\Delta t/2}$ and $e^{iL_{NHB}\Delta t/2}$ in Equations (33) and (34) and of the operators $iL_{\mathbf{q},MTK}$ and $iL_{\mathbf{p},MTK}$ in Equations (20) and (21) can be further improved by using the Suzuki–Yoshida (SY) treatment discussed in Refs. [15,27,28]. The idea behind the treatment is to start from an approximate computable representation of the exact operator to use in the integration and by suitably combining the computable operator to reconstruct a better approximation of the exact operator. Below we sketch the idea of the procedure for the improvement to lowest order and then in the application we will use a higher order approximation derived by Suzuki and Yoshida [15]. Without any reference to the specific case, let us consider a Liouvillian composed by two non commuting operators L_1 and L_2. We have

$$S_{ex}(\Delta t) = e^{(iL_1 + iL_2)\Delta t}. \qquad (58)$$

This operator can be approximated through, for example, the second-order symmetric Trotter factorisation of S_{ex}, i.e.

$$S_2(\Delta t) = e^{iL_1 \frac{\Delta t}{2}} e^{iL_2 \Delta t} e^{iL_1 \frac{\Delta t}{2}} \qquad (59)$$

so that $S_{ex}(\Delta t) = S_2(\Delta t) + \mathcal{O}(\Delta t^3)$.

To use the given decomposition but getting an higher approximation, Suzuki and Yoshida proceed formally in the way described below, using the Baker–Campbell–Hausdorff (BCH) formula.

By applying the BCH formula to the product of two computable operators, we can write

$$e^Y e^X = e^{Y+X+\frac{1}{2}[Y,X]+\frac{1}{12}([Y,[Y,X]]+[X,[X,Y]])+\cdots} \qquad (60)$$

where X and Y are two non-commutative operators and $[X,Y] = XY - YX$.

For the symmetric product $e^X e^Y e^X$ the BCH formula applied to $e^X e^Y$ gives

$$e^X e^Y e^X = e^X e^{Y+X+\frac{1}{2}[Y,X]+\frac{1}{12}([Y,[Y,X]]+[X,[X,Y]])+\cdots} \qquad (61)$$

Let us call

$$Z = Y + X + \frac{1}{2}[Y,X] + \frac{1}{12}([Y,[Y,X]] + [X,[X,Y]]) + \ldots \qquad (62)$$

and apply again BCH formula to $e^X e^Z$ to find:

$$e^X e^Z = e^{X+Z+\frac{1}{2}[X,Z]+\frac{1}{12}([X,[X,Z]]+[Z,[Z,X]])+\cdots} \qquad (63)$$

We now observe that if we substitute to operator Z in the right-hand side of the previous equation following

its explicit definition given in Equation (62), we can rearrange the exponent and find

$$e^X e^Z = e^X e^Y e^X = e^{2X+Y+\frac{1}{6}[Y,[Y,X]]-\frac{1}{6}[X,[X,Y]]+\dots} \quad (64)$$

where $2X + Y$ is a linear contribution and the next term is a cubic one. This fundamental relation we will be now used at two different levels of abstraction to get the improved computer operator we are seeking for.

As for the first step, it is convenient to go back to the meaning of operators X and Y as $2X = 2(iL_1\Delta t/2)$ and $Y = iL_2\Delta t$. That gives us for S_2 the computable expression given in Equation (59). At the same time, the BCH provides us with a formula for $\ln(S_2)$ not directly computable but useful for what follows. After some algebra, based on Equation (64) for the operators X and Y identified above, the exponent of S_2 can be written as

$$\ln[S_2(\Delta t)] = \Delta t\gamma_1 + \Delta t^3\gamma_3 + \Delta t^5\gamma_5 + \dots$$
$$= \Delta t\gamma_1 + \Delta t^3\gamma_3 + \mathcal{O}(\Delta t^5), \quad (65)$$

where $\gamma_1 = iL_1 + iL_2$, $\gamma_3 = \frac{1}{12}[iL_2,[iL_2,iL_1]] - \frac{1}{24}[iL_1,[iL_1,iL_2]]$ and γ_5 is an operator which does not depend on Δt and whose specific form is not important for the following derivations. Note that the first-order operator γ_1 reconstructs the full Liouvillian.

The next step is to build a computable symmetric combination of operators S_2 to create an algorithm of higher order. Let us define the computable expression

$$S_4(\Delta t) = S_2(w_1\Delta t)S_2(w_2\Delta t)S_2(w_1\Delta t)$$
$$= e^{\ln[S_2(w_1\Delta t)]}e^{\ln[S_2(w_2\Delta t)]}e^{\ln[S_2(w_1\Delta t)]} \quad (66)$$

and apply to it the BCH expression given in Equation (64). We now require that, thanks to a suitable choice of the coefficients w_1 and w_2 and to the BCH, the operator $S_4(\Delta t)$ be a 4th order approximation of S_{ex}, i.e.

$$S_{ex}(\Delta t) = S_4(\Delta t) + \mathcal{O}(\Delta t^5). \quad (67)$$

Indeed, if we plug Equation (65) into Equation (66) and, then, we use the BCH result, we have

$$S_4(\Delta t) = e^{\ln[S_2(w_1\Delta t)]}e^{\ln[S_2(w_2\Delta t)]}e^{\ln[S_2(w_1\Delta t)]}$$
$$= e^{2\ln[S_2(w_1\Delta t)]+\ln[S_2(w_2\Delta t)]}$$
$$= e^{\Delta t(2w_1+w_2)\gamma_1+\Delta t^3(2w_1^3+w_2^3)\gamma_3+\mathcal{O}(\Delta t^5)}$$
$$= e^{\Delta t(2w_1+w_2)\gamma_1+\Delta t^3(2w_1^3+w_2^3)\gamma_3} + \mathcal{O}(\Delta t^5), \quad (68)$$

where we have taken advantage of the fact that operators $S_2(w_1\Delta t)$ and $S_2(w_2\Delta t)$ commute. To ensure that operator $S_4(\Delta t)$ is identical to $S_{ex}(\Delta t)$ up to $\mathcal{O}(\Delta t^5)$ terms, in Equation (68) the coefficient of the term proportional

to Δt must be equal to 1, while the term proportional to Δt^3 must be equal to 0, i.e.

$$2w_1 + w_2 = 1$$
$$2w_1^3 + w_2^3 = 0. \quad (69)$$

By solving for w_2 in the first equation and plugging the result into the second one, one obtains

$$6w_1^3 - 12w_1^2 + 6w_1 - 1 = 0. \quad (70)$$

This third-order equation has an explicit solution formula [29], which gives for the only real solution $w_1 = \frac{1}{2-2^{1/3}}$, so that $w_2 = 1 - \frac{2}{2-2^{1/3}}$.

With that, we have completed the demonstration that the operator S_4, computable with its expression given by the right-hand side of Equation (66), provides an algorithm precise to the 4th order in Δt.

In our numerical simulations discussed in the next section, for testing the method we have used the generalisation to sixth order given in Ref. [15]. The operators to which we apply this approach are $S^I = e^{iL_{NHP}\Delta t/2}$ $S^{II} = e^{iL_{NHB}\Delta t/2}$, $S^{III} = e^{iL_{q,MTK}\Delta t/2}$ and $S^{IV} = e^{iL_{p,MTK}\Delta t/2}$. For the operators S^I, S^{II}, S^{III} and S^{IV} the second-order computable expressions S_2^α, with $\alpha = I, II, III, IV$, are the right-hand side of Equations (33), (34), (36) and (37), respectively. To get the approximation to sixth order by the SY treatment we need a symmetric combination of 7 operators S_2^α, i.e.

$$S^\alpha(\Delta t) \approx \prod_{k=1}^{3} S_2^\alpha(w_k\Delta t)S_2^\alpha(w_4\Delta t)\prod_{k=3}^{1} S_2^\alpha(w_k\Delta t), \quad (71)$$

where

$$w_1 = 0.78451361047755726381949763386644$$
$$w_2 = 0.23557321335935813368479318297853$$
$$w_3 = -1.17767998417887100694641568096434$$
$$w_4 = 1.31518632068391121888424972822389. \quad (72)$$

Finally, in our numerical simulations, just for the two contributions S^α, $\alpha = I, II$ r-RESPA procedure, applied to the total Liouvillian, is also combined with SY decomposition by applying the S^α operator in Equation (71) n_r times with a time step $w_k\Delta t/n_r$ [10], thus having

$$S^\alpha(\Delta t) = \left[S^\alpha(\Delta t/n_r)\right]^{n_r}$$
$$\approx \left[\prod_{k=1}^{3} S_2^\alpha(w_k\Delta t/n_r)S_2^\alpha(w_{n_4}\Delta t/n_r)\right.$$
$$\left.\prod_{k=3}^{1} S_2^\alpha(w_k\Delta t/n_r)\right]^{n_r}. \quad (73)$$

4. Test of the algorithm

In this section, we test the integrator previously introduced in a simple model problem, i.e. a realistic Orthotherphenyl (OTP) model.

4.1. Model

We employed the OTP model first proposed in Ref. [30] and afterwards widely used in the literature [31–34]. It is summarised in the following. The OTP molecule is represented by three atoms of mass $m = 78$ amu placed onto a rigid isosceles triangle, where the length of two edges of equal length is $l = 0.483$ nm and the angle between them is $\theta = 75°$ Atoms belonging to different molecules and at a distance r interact via the following Lennard–Jones (LJ) potential:

$$v_{LJ}(r) = 4\epsilon \left[\left(\frac{\sigma}{r} \right)^{12} - \left(\frac{\sigma}{r} \right)^{6} \right] \tag{74}$$

with $\epsilon = 5.276$ kJ/mol and $\sigma = 0.483$ nm. As typical in MD simulations, the interaction potential is truncated so that the interaction potential between atoms becomes

$$u(r) = \begin{cases} v_{LJ}(r) & r < r_c \\ 0 & \text{otherwise} \end{cases} \tag{75}$$

with $r_c = 2.5\sigma$. Hence, a suitable smoothing of the potential is necessary. A convenient approach, proposed in Ref. [35], to avoid the discontinuity at $r = r_c$, consists in smoothing the interaction potential in such a way that at least the second derivative of $u(r)$ is continuous. The LJ potential is smoothed with the following replacement:

$$u(r) = \begin{cases} [v_{LJ}(r) - v_{LJ}(r_c)] \dfrac{(r_c - r)^n}{(r_c - r)^n + \alpha^n} & r < r_c \\ 0 & \text{otherwise,} \end{cases} \tag{76}$$

where $\alpha > 0$ sets the range of smoothing and r_c is the cutoff radius. In our simulations, we set $n = 4$, so as to have an interaction potential continuous up to the fourth order at $r = r_c$, $\alpha = 0.1$ as in Ref. [35] and $r_c = 1.2616$ nm, which is the same value used in Ref. [31].

The masses of mechanical and thermal pistons W, $Q_{\eta,j}$ and $Q_{\xi,j}$ have been chosen according to the prescription suggested in Ref. [10]:

$$\begin{aligned} Q_{1,\eta} &= N_f k_B T \tau_\eta^2 \\ Q_{j,\eta} &= k_B T \tau_\eta^2 \quad j = 2 \dots M_\eta \\ Q_{j,\xi} &= k_B T \tau_\xi^2 \quad j = 1 \dots M_\xi \\ W &= (N_f + 3) k_B T \tau_b^2, \end{aligned} \tag{77}$$

where $N_f = 3N_{at} - 3 - \sum_{i=1}^{N} l_i$, τ_η, τ_ξ and τ_b are characteristic times associated to the pistons. We set these characteristic times as $\tau_\eta = 1$ ps, $\tau_\xi = 1$ ps and $\tau_b = 3$ ps. In our simulations, we used NH chains constituted of four elements, i.e. we set $M_\eta = 4$ and $M_\xi = 4$. The calculation of the constraint forces, needed to apply the operators $\hat{S}_r$ and $\hat{S}_v$, is implemented iteratively as discussed in Refs. [19,25,26], with a tolerance set to 10^{-10} for adjusting both positions and velocities.

4.2. Results and discussion

Using the model discussed in the previous section, we tested the NPT algorithm illustrated in Section 2. Equation (1) ensures the conservation of the following quantity:

$$\begin{aligned} E(t) = {}& \mathcal{H}(\mathbf{r}^{N_{at}}, \mathbf{p}^{N_{at}}) + \frac{p_\epsilon^2}{2W} + PV + \sum_{k=1}^{M_\eta} \frac{p_{\eta_k}^2}{2Q_k} \\ & + \sum_{k=1}^{M_\xi} \frac{p_{\xi_k}^2}{2Q_k} + N_f k_B T \eta_1 \\ & + k_B T \sum_{k=2}^{M_\eta} \eta_k + k_B T \sum_{k=1}^{M_\xi} \xi_k. \end{aligned} \tag{78}$$

In all simulations, we will use $N = 784$ OTP molecules and take our production runs only after having properly equilibrated the systems by inspecting the reached stationarity of $E(t)$ on time. All runs were performed at $P = 1.66$ MPa and $T = 345.8$ K. We carried out simulations both using SY plus r-RESPA for operators $e^{iL_{NHP}\Delta t/2}$, $e^{iL_{NHB}\Delta t/2}$ and just SY for operators $iL_{\mathbf{q},MTK}$ and $iL_{\mathbf{p},MTK}$ and simulations without using these techniques. When both SY decomposition and r-RESPA are used as in Equation (73), we set $n_r = 4$. First, we calculated the standard deviation $\sigma_E(\Delta t)$ of the conserved quantity $E(t)$ as a function of the integration time step and the results are reported in Figure 1. As expected the $\sigma_E(\Delta t)$ exhibits a quadratic dependence on Δt.

To test the stability of the algorithm, we calculated for several integration time steps Δt the instantaneous quantity
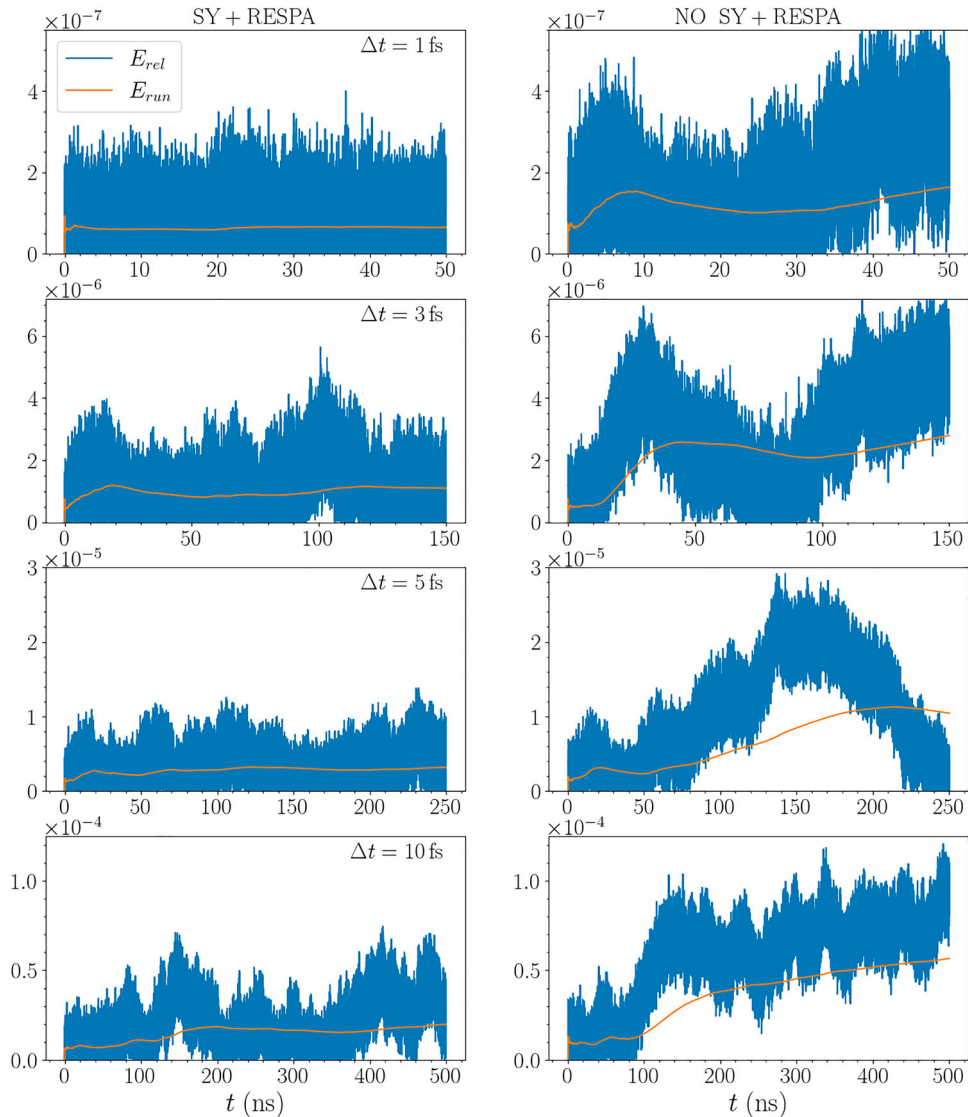
$$E_{rel}(t) = \left| \frac{E(t) - E(0)}{E(0)} \right| \tag{79}$$

and the running average

$$E_{run}(t) = \frac{1}{N_c} \sum_{i=1}^{N_c} E_{rel}(t_i), \tag{80}$$
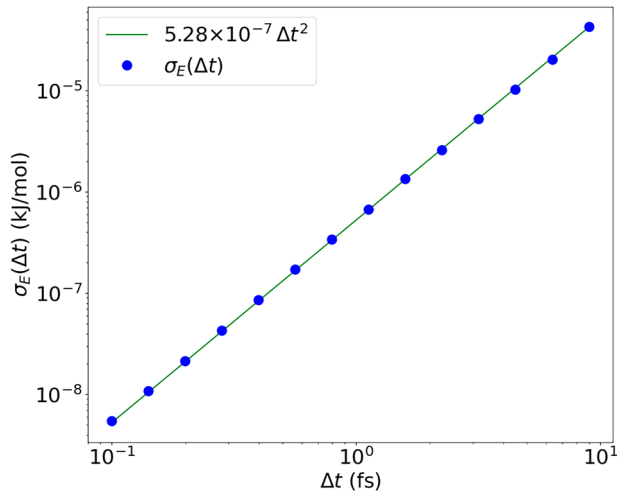
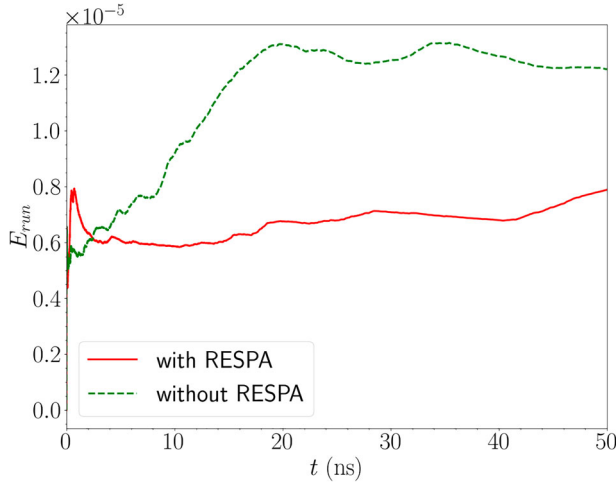**Figure 1.** Standard deviation of energy $\sigma_E(\Delta t)$ together with a quadratic fit of the data.

where the average is calculated over a set of N_c configurations taken at times t_i, with $i = 1 \ldots N_c$, where $t_1 = 0$ and $t_{N_c} = t$.

Figure 2 shows E_{rel} and E_{run} for different integration time steps Δt. In this figure, plots on left show results from simulations employing both SY decomposition and r-RESPA, while plots on right show results from simulations where SY and r-RESPA are not used. It can be seen that energy in all cases is well conserved, i.e. there is no significant drift, although the use of SY and r-RESPA improves energy conservation [10,14].

In previous simulations, we used both r-RESPA and SY decomposition for the operators $e^{iL_{NHP}\Delta t/2}$ and $e^{iL_{NHB}\Delta t/2}$. To disentangle the role played by these two techniques in improving energy conservation, we carried out a simulation with $\Delta t = 10fs$ with and without using r-RESPA. The results are shown in Figure 3. We see that

Figure 2. Plot of E_{rel} and E_{run} for $\Delta t = 1$ fs, 3 fs, 5 fs and 10 fs (from top to bottom). Left: both SY decomposition and r-RESPA are used; Right: no SY or r-RESPA are used.

Figure 3. Plot of E_{run} with and without using r-RESPA.

r-RESPA does contribute to improve energy conservation, even though this is rather minor.

As discussed in Section 2.3, in our algorithms, the operators $e^{iL_{\mathbf{q},MTK}\frac{\Delta t}{2}}$ and $e^{iL_{\mathbf{p},MTK}\frac{\Delta t}{2}}$ are applied after having adjusted positions and velocities respectively.

Following Ref. [19], when positions and velocities are adjusted iteratively by RATTLE algorithm, the termination criteria ensure that for all constraints one has

$$\mu_r = \frac{|d^2 - \Delta\mathbf{r} \cdot \Delta\mathbf{r}|}{2\,d^2} < \xi_r \tag{81}$$

and

$$\mu_v = \frac{m|\Delta\mathbf{v} \cdot \Delta\mathbf{r}|}{2\,d^2} < \xi_v, \tag{82}$$

where $\Delta\mathbf{r}$ is the distance between two constrained atoms, $\Delta\mathbf{v}$ is their relative velocity and μ_v has dimensions of a mass over time . In Figure 4, μ_r and μ_v are shown as a function of time during a simulation with $\Delta t = 10$ fs,
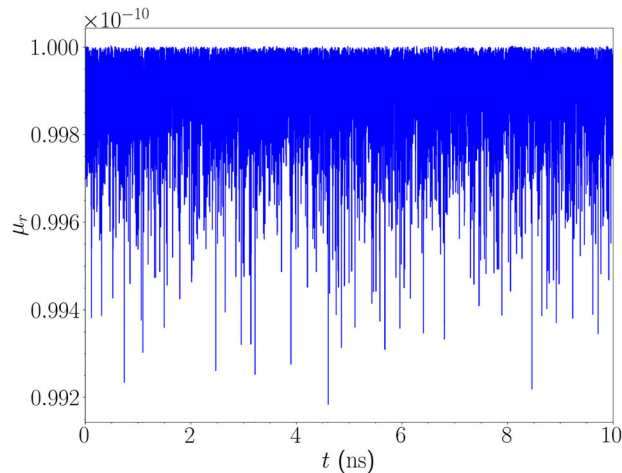
where we set $\xi_r = \xi_v = 10^{-10}$. Note that these quantities are calculated after the full propagator $e^{i\hat{L}\Delta t}$ is applied. It can be seen that our algorithm preserve constraints very well.

5. Conclusion

We have proposed a numerical algorithm to solve MTK equations of motion for a system with holonomic molecular constraints which is accurate up to second order in the integration time step.[1] Our algorithm is measure-preserving and corrects the one proposed in Ref. [11], which remains precise only to first order, due to the overlooking of the terms not sufficiently factorised. For a popular model of a molecular glass former, the orthotherphenyl, we also provided a careful test of our approach, by proving that the standard deviation of energy depends quadratically on the integration time step and that no drift of energy is observed even on very long simulations times (hundreds of nanoseconds). Finally, we showed how to SY decomposition and r-RESPA methods help keeping a more stable trajectory and also improving energy conservation. This result can be used alternatively to have larger integration time steps.
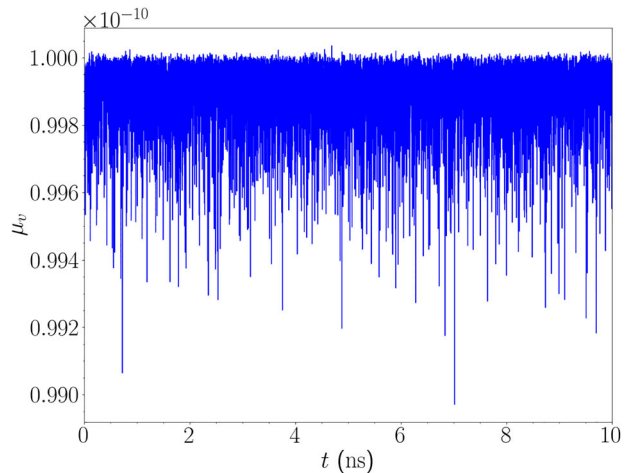
Note

1. Source code written in C++ is available on request and it can be integrated into main MD simulation packages.

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Figure 4. Plot of μ_r (left) and μ_v (right) over time.

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