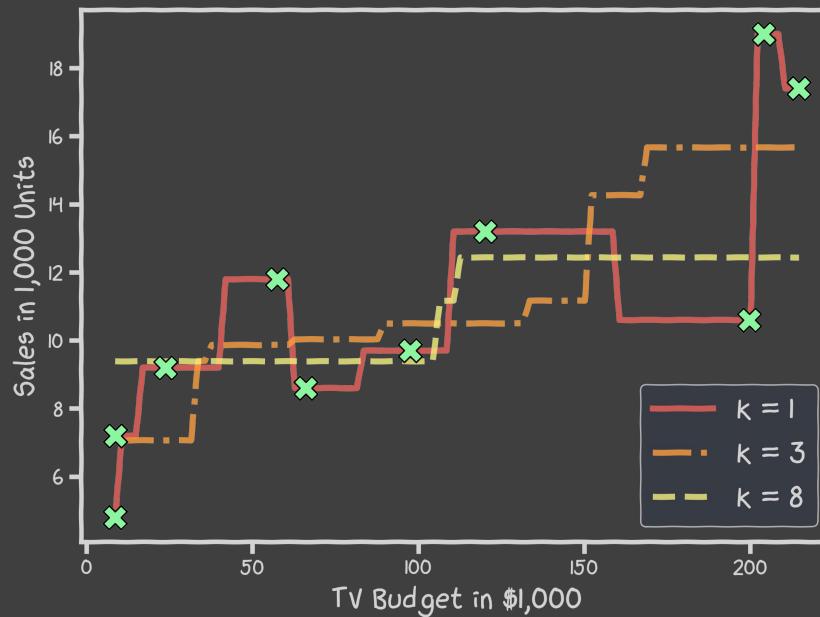


# Linear Models



Note that in building our kNN model for prediction (non-parametric), we did not compute a closed form for  $\hat{f}$ .

What if we ask the question:

“how much more sales do we expect if we double the TV advertising budget?”

# Linear Models

We can build a model by first assuming a simple form of  $f$ :

$$f(x) = \beta_0 + \beta_1 X$$

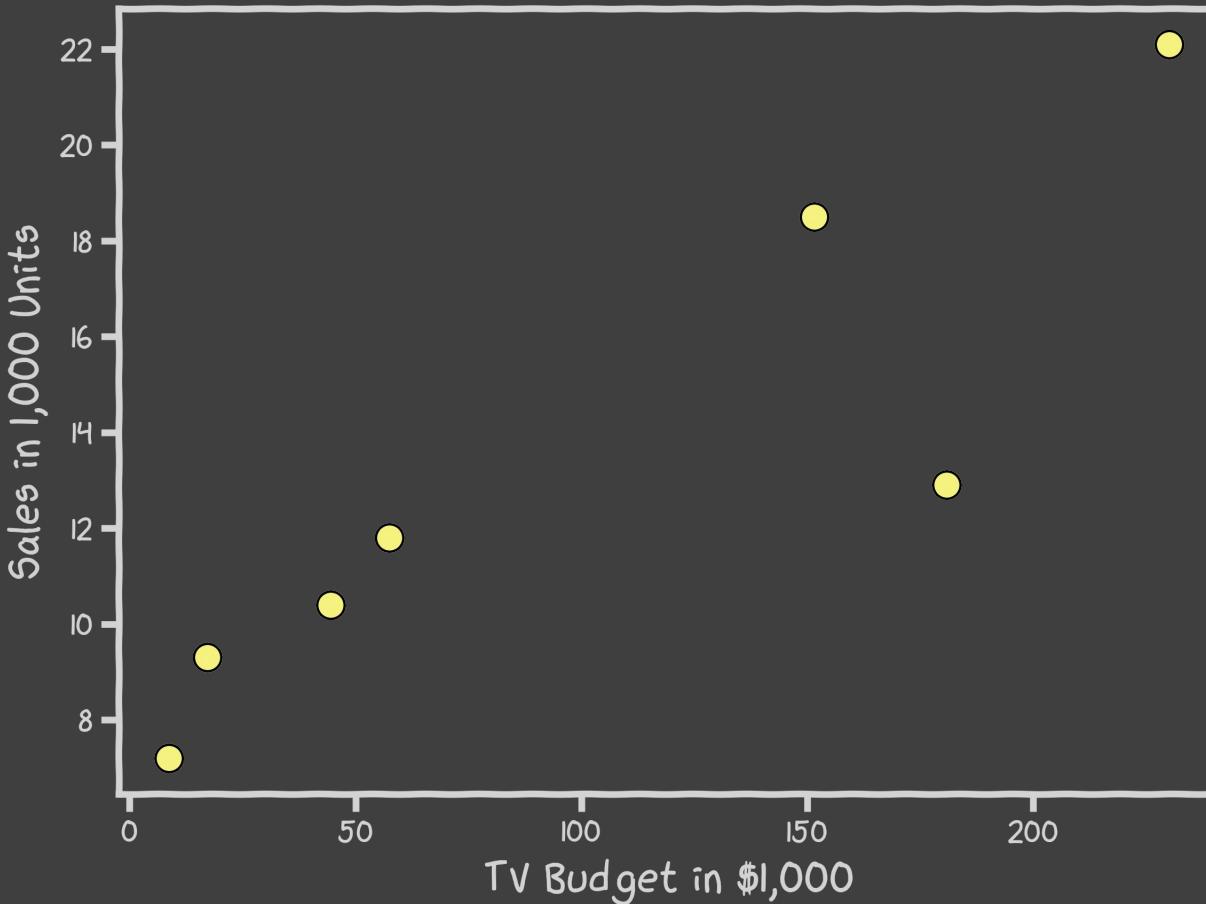
... then it follows that our estimate is:

$$\hat{Y} = \hat{f}(x) = \hat{\beta}_0 + \hat{\beta}_1 X$$

where  $\hat{\beta}_1$  and  $\hat{\beta}_0$  are **estimates** of  $\beta_1$  and  $\beta_0$  respectively, that we compute using observations.

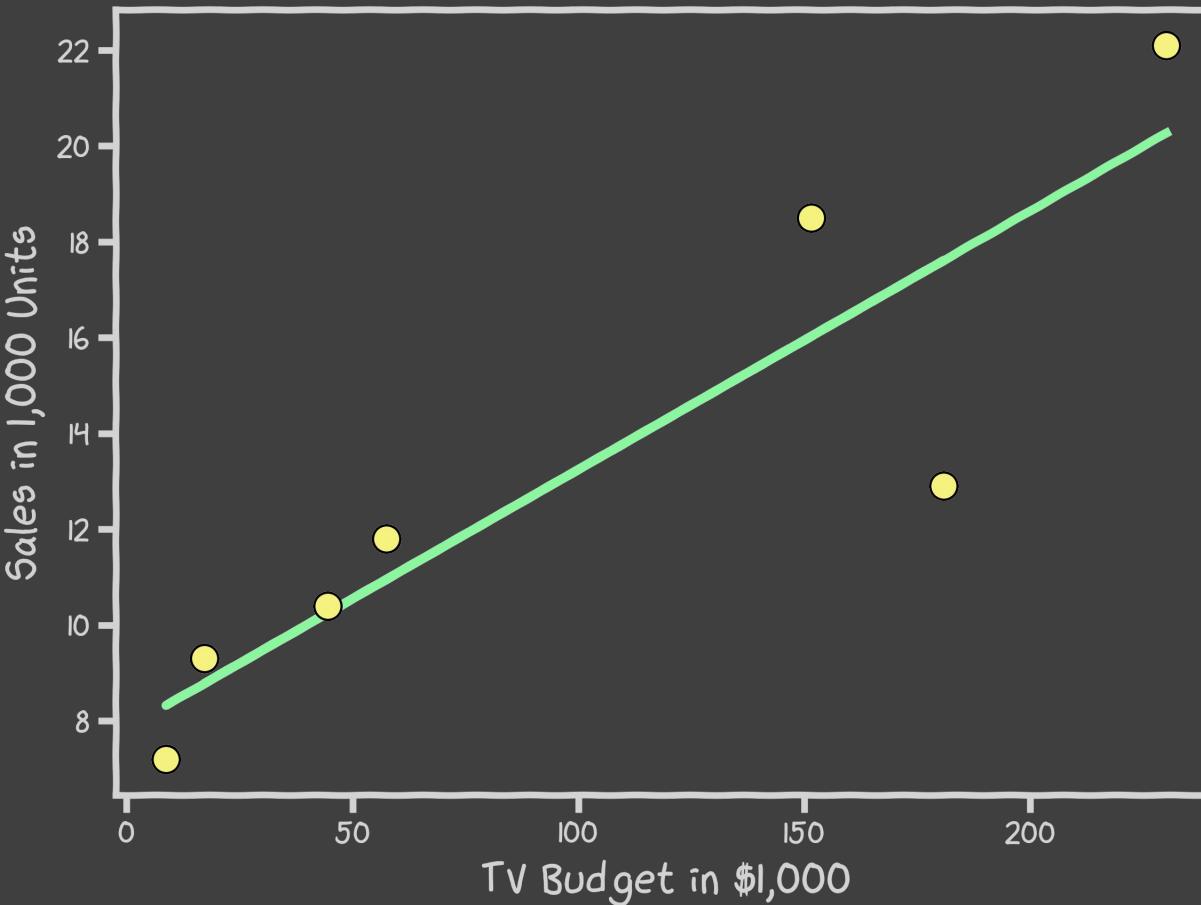
# Estimate of the regression coefficients

For a given data set



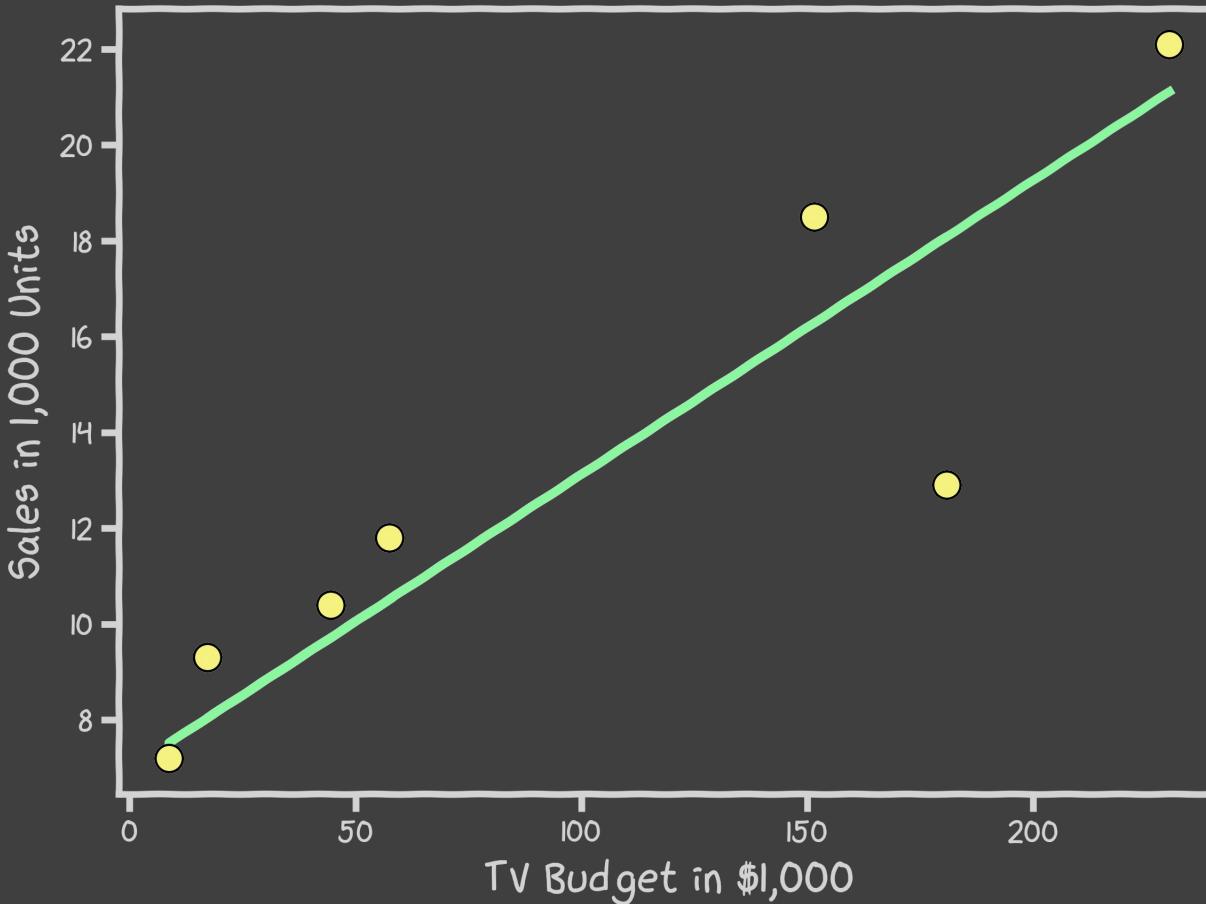
# Estimate of the regression coefficients (cont)

Is this line good?



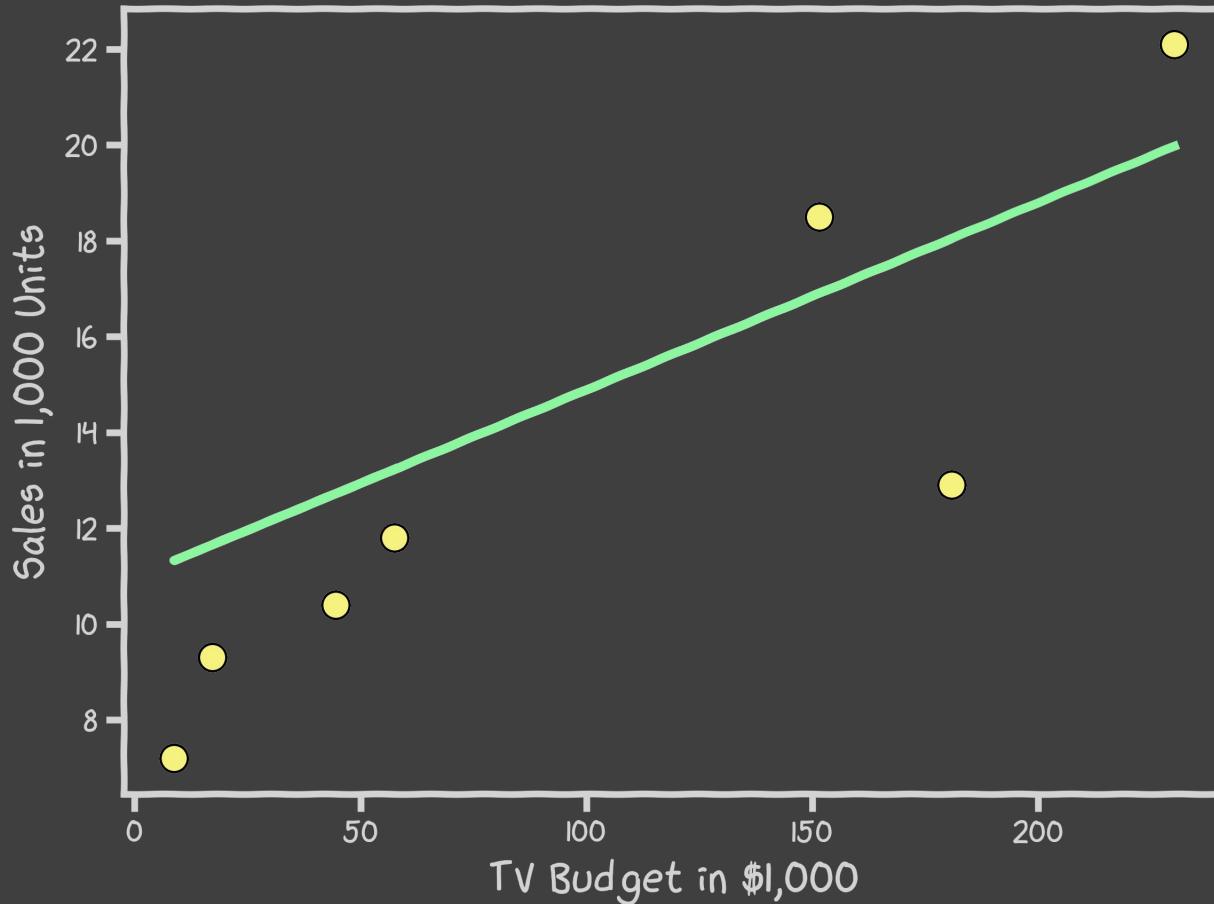
# Estimate of the regression coefficients (cont)

Maybe this one?



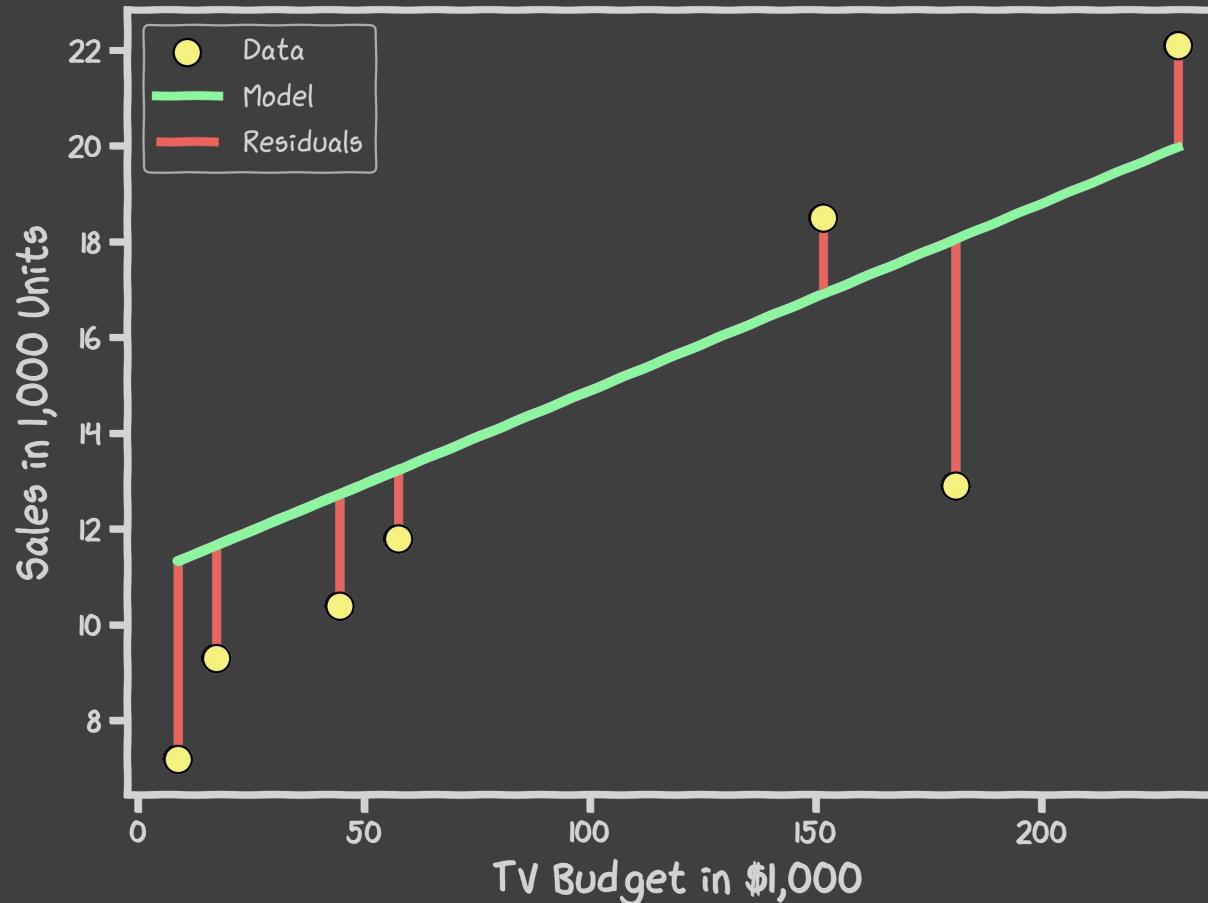
# Estimate of the regression coefficients (cont)

Or this one?



# Estimate of the regression coefficients (cont.)

Question: Which line is the best?



As before, for each observation  $(x_n, y_n)$ , the absolute residuals,  $r_i = |y_i - \hat{y}_i|$  quantify the error at each observation.

# Estimate of the regression coefficients (cont.)

AGAIN, we use the **MSE** as our loss function,

$$L(\beta_0, \beta_1) = \frac{1}{n} \sum_{i=1}^n (y_i - \hat{y}_i)^2$$

We choose  $\beta_1$  and  $\beta_0$  that minimizes the predictive errors made by our model, i.e., minimize our loss function.

Then the optimal values,  $\hat{\beta}_0$  and  $\hat{\beta}_1$ , should be:

$$\hat{\beta}_0, \hat{\beta}_1 = \operatorname{argmin}_{\beta_0, \beta_1} L(\beta_0, \beta_1).$$

WE CALL THIS **FITTING**  
OR **TRAINING** THE  
MODEL

Introducing...



# sklearn.linear\_model.LinearRegression

## Methods

**fit(X, y[, sample\_weight])** Fit linear model.

**get\_params([deep])** Get parameters for this estimator.

**predict(X)** Predict using the linear model.

**score(X, y[, sample\_weight])** Return the coefficient of determination  $R^2$  of the prediction.

```
>>> from sklearn.linear_model import LinearRegression
>>> reg = LinearRegression()
>>> reg.fit(X, y)
>>> reg.coef_
array([1.2])
>>> reg.intercept_
3.2
>>> reg.predict(np.array([[3]]))
array([16.])
```

## RECAP: Exercise

```
>>> from sklearn.linear_model import LinearRegression  
>>> df = pd.read_csv('Advertising.csv')  
>>> X= df[['TV']].values  
>>> y = df['Sales'].values
```

## RECAP: Exercise

```
>>> from sklearn.linear_model import LinearRegression  
>>> df = pd.read_csv('Advertising.csv')  
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>>> y = df['Sales'].values  
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```

## RECAP: Exercise

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>>> from sklearn.linear_model import LinearRegression  
>>> df = pd.read_csv('Advertising.csv')  
>>> X= df[['TV']].values  
>>> y = df['Sales'].values  
>>> reg = LinearRegression()  
>>> reg.fit(X, y)  
>>> reg.coef_  
array([[0.04665056]])  
>>> reg.intercept_  
array([7.08543108])  
>>> reg.predict(np.array([[100]]))  
array([[11.75048733]])
```

```
>>> reg.fit(X, y)
```



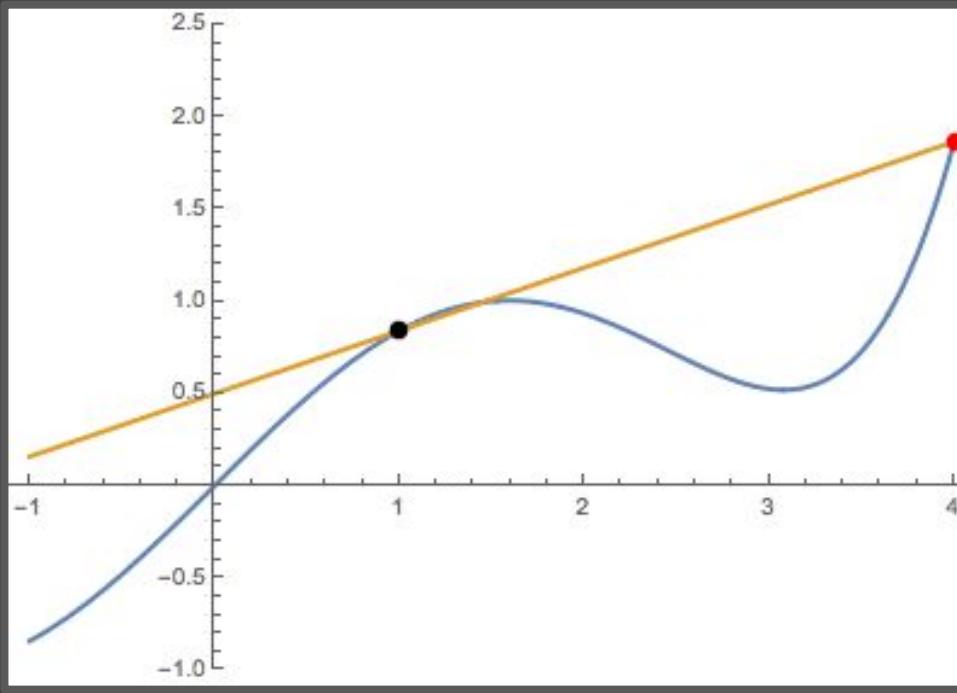
What happened here!



# Derivative definition

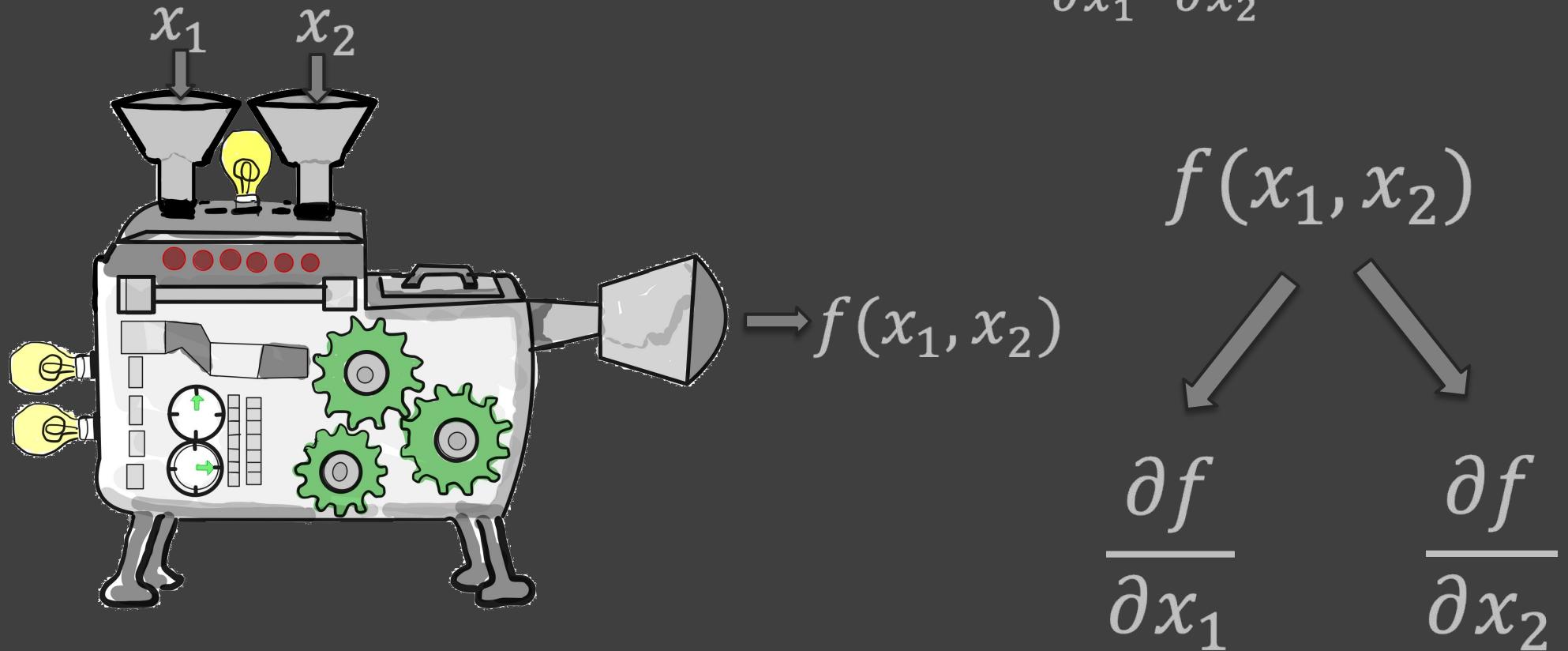
A derivative is the instantaneous rate of change of a single valued function. Given a function  $f(x)$  the derivative can be defined as:

$$f'(x) = \frac{df}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$$



# Partial derivatives

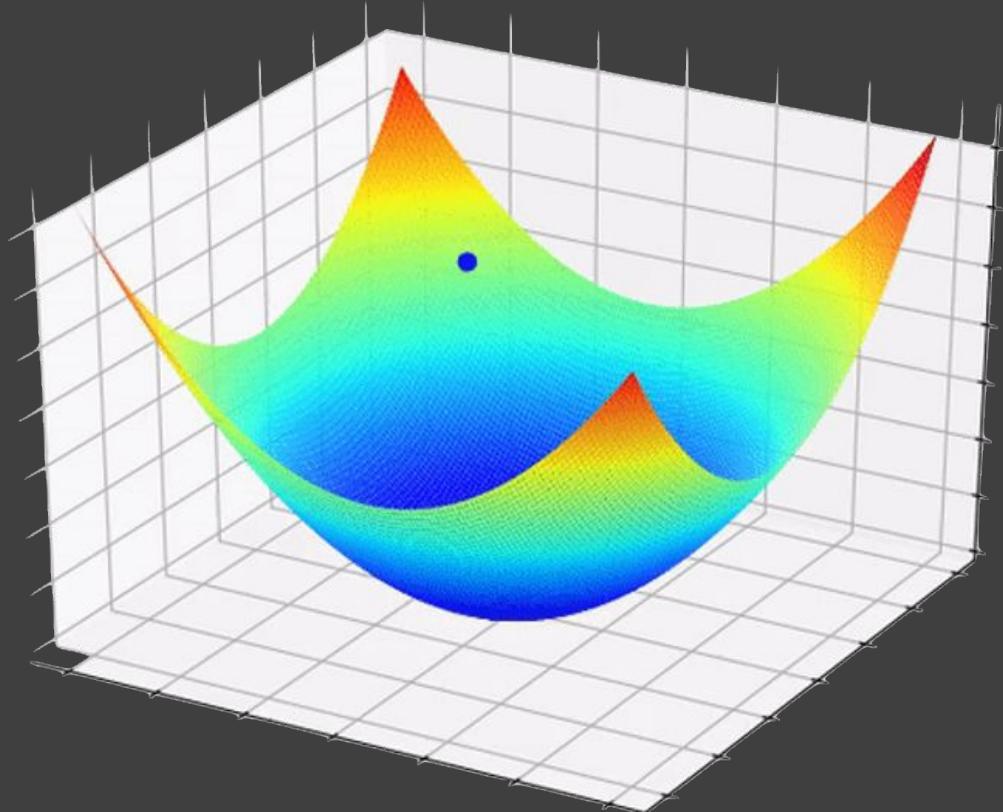
For a function  $f$ , the partial derivative is written as:  $\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots$



What is the rate of change of the function with respect to one variable with the others held fixed?

# Optimization

How does one minimize a loss function?



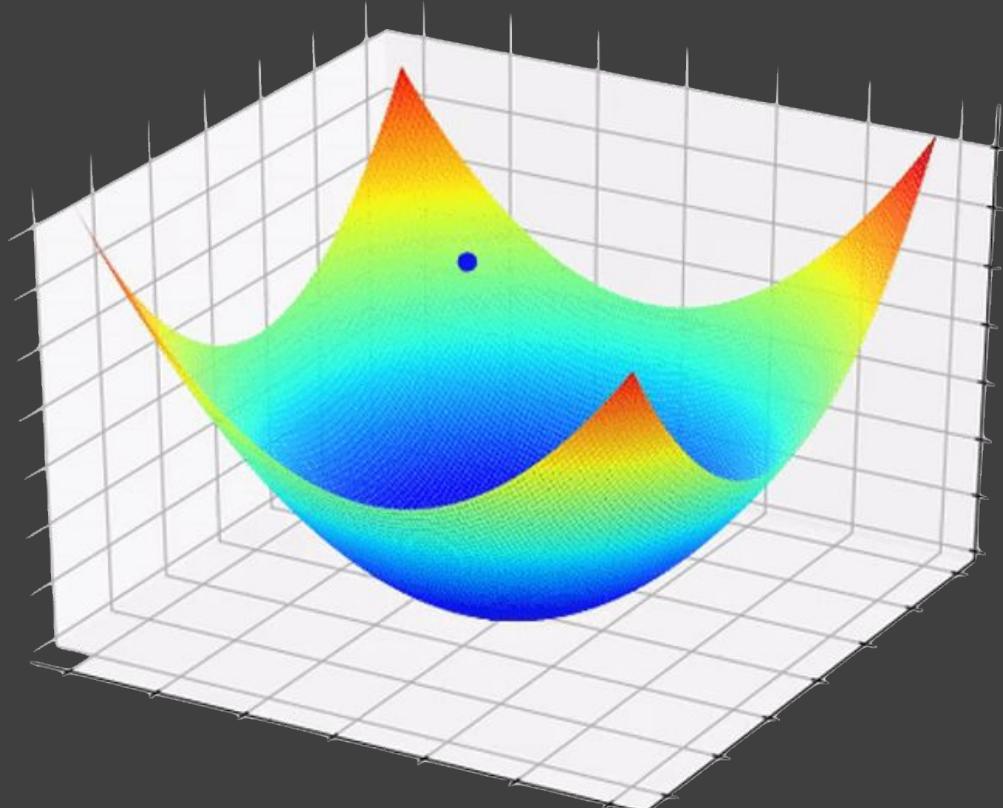
The global minima or maxima of  $L(\beta_0, \beta_1)$  must occur at a point where the gradient (slope)

$$\nabla L = \left[ \frac{\partial L}{\partial \beta_0}, \frac{\partial L}{\partial \beta_1} \right] = 0$$

- Brute Force: Try every combination
- Exact: Solve the above equation
- Greedy Algorithm: Gradient Descent

# Optimization

How does one minimize a loss function?



The global minima or maxima of  $L(\beta_0, \beta_1)$  must occur at a point where the gradient (slope)

$$\nabla L = \left[ \frac{\partial L}{\partial \beta_0}, \frac{\partial L}{\partial \beta_1} \right] = 0$$

- Brute Force: Try every combination
- **Exact: Solve the above equation**
- Greedy Algorithm: Gradient Descent

# Partial derivative example

If  $L(\beta_0, \beta_1) = (y - (\beta_1 x + \beta_0))^2$  then what is.  $\frac{\partial l}{\partial \beta}$  ?

Looks like we're going  
to need the chain rule,  
but what is it? I forget



# Partial derivative example

If  $L(\beta_0, \beta_1) = (y - (\beta_1 x + \beta_0))^2$  then what is  $\frac{\partial l}{\partial \beta}$  ?

$$\frac{\partial L(f(\beta_0))}{\partial \beta_0} = \frac{\partial L}{\partial f} \frac{\partial f}{\partial \beta_0}$$



# Partial derivative $\frac{\partial L}{\partial \beta_0}$

If  $L(\beta_0, \beta_1) = (y - (\beta_1 x + \beta_0))^2$  then what is  $\frac{\partial l}{\partial \beta}$  ?

$$L = \underbrace{(y - \beta_1 x - \beta_0)^2}_{f}$$

$$\frac{\partial L}{\partial \beta_0} = \frac{\partial L}{\partial f} \frac{\partial f}{\partial \beta_0} \quad L = f^2 \quad \frac{\partial L}{\partial f} = 2f \quad f = y - \beta_1 x - \beta_0 \quad \frac{\partial f}{\partial \beta_0} = -1$$

$$\frac{\partial L}{\partial \beta_0} = \frac{\partial L}{\partial f} \frac{\partial f}{\partial \beta_0} = -2f = -2(y - \beta_1 x - \beta_0)$$

# Partial derivative $\frac{\partial L}{\partial \beta_1}$

If  $L(\beta_0, \beta_1) = (y - (\beta_1 x + \beta_0))^2$  then what is  $\frac{\partial l}{\partial \beta}$  ?

$$L = (y - \beta_1 x - \beta_0)^2$$

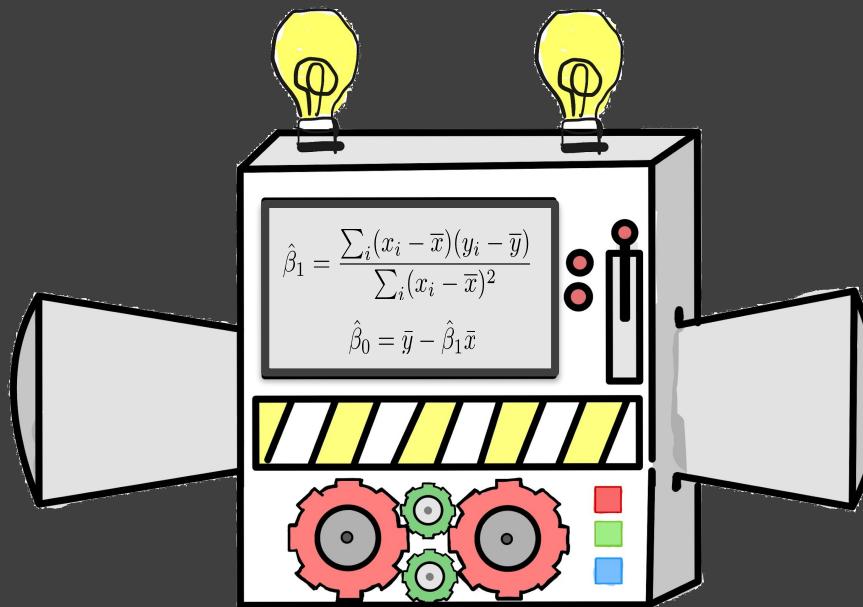
$$\frac{\partial L}{\partial \beta_1} = \frac{\partial L}{\partial f} \frac{\partial f}{\partial \beta_1} \quad L = f^2 \quad \frac{\partial L}{\partial f} = 2f \quad f = y - \beta_1 x - \beta_0 \quad \frac{\partial f}{\partial \beta_1} = -x$$

$$\frac{\partial L}{\partial \beta_1} = \frac{\partial L}{\partial f} \frac{\partial f}{\partial \beta_1} = -2x f = -2x(y - \beta_1 x - \beta_0)$$

# Optimization

$$\nabla L = \left[ \frac{\partial L}{\partial \beta_0}, \frac{\partial L}{\partial \beta_1} \right] = 0$$

$$\frac{\partial L}{\partial \beta_0} = -2(y - \beta_1 x - \beta_0) = 0$$



$$\frac{\partial L}{\partial \beta_1} = -2x(y - \beta_1 x - \beta_0) = 0$$

# Optimization

$$\hat{\beta}_1 = \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sum_i (x_i - \bar{x})^2}$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

# Summary: Estimate of the regression coefficients

We use MSE as our loss function,

$$L(\beta_0, \beta_1) = \frac{1}{n} \sum_{i=1}^n (y_i - \hat{y}_i)^2 = \frac{1}{n} \sum_{i=1}^n [y_i - (\beta_1 x_i + \beta_0)]^2$$

We choose  $\beta_1$  and  $\beta_0$  in order to minimize the predictive errors made by our model, i.e. minimize our loss function.

Then the optimal values for  $\hat{\beta}_0$  and  $\hat{\beta}_1$  should be:

$$\hat{\beta}_0, \hat{\beta}_1 = \operatorname{argmin}_{\beta_0, \beta_1} L(\beta_0, \beta_1).$$

WE CALL THIS **FITTING**  
OR **TRAINING** THE  
MODEL

# Estimate of the regression coefficients: analytical solution

Take the gradient of the loss function and find the gradient is zero:  $\nabla L = \left[ \frac{\partial L}{\partial \beta_0}, \frac{\partial L}{\partial \beta_1} \right] = 0$

Finding the exact solution only works for rare cases. Linear regression is one of such rare cases.

$$\hat{\beta}_1 = \frac{\sum_i (x_i - \bar{x})(y_i - \bar{y})}{\sum_i (x_i - \bar{x})^2}$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

where  $\bar{y}$  and  $\bar{x}$  are sample means.

The line:

is called the **regression line**.

$$\hat{Y} = \hat{\beta}_1 X + \hat{\beta}_0$$