1 Stationarity of AR(1)

Let's define a initial value $x_0 = h$ for our serie $\{x_j\}_0^{T-1}$. We also know that this process follows an AR(1). Then the next value x_1 will be:

$$x_1 = \phi h + \sigma_1 + c \tag{1}$$

Consequently, we can describe the whole process as follows:

$$\begin{aligned} x_1 &= \phi h + \sigma_1 + c && (2)\\ x_2 &= \phi(\phi h + \sigma_1 + c) + \sigma_2 + c && (3)\\ \vdots &\ \vdots\ \vdots && (4)\\ x_T &= \phi^T h + \sum_{j=0}^{T-1} \phi^j \sigma_{T-j} + c \sum_{j=0}^{T-1} \phi^j && (5) \end{aligned}$$

By definition $\mathbb{E}[\sigma] = 0 \Rightarrow \mathbb{E}[\sum_{j=0}^{T-1} \phi^j \sigma_{T-j}] = 0$. Then, the expectation of x_t should satisfies:

$$\mathbb{E}[x_t] = \phi^T h + c \sum_{j=0}^{T-1} \phi^j \tag{6}$$

For the process to be stationary the mean should be constant and therefore, the time should not affect the final result.

1. if $|\phi| < 1$, then the second term $c \sum_{j=0}^{T-1} \phi^j$ converge to $\frac{c}{1-\phi}$ constant
2. and the first term converge to zero