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1. **Optimal Consumption with Diversifiable Income**

Consider the particular case of the income fluctuation problem discussed in class where labor income is diversifiable and the only option for saving is a risky asset. More precisely, at time $t = 0$ the individual receives in cash an amount equal to the present-discounted value of her lifetime income, which is part of initial financial assets A_0.

The sequence formulation of the problem at time $t = 0$ is the following:

$$\begin{aligned} \max_{c_0, c_1, \ldots} \quad & \mathrm{E}_0 \sum_{t \geq 0} \gamma^t \log c_t, \\ \text{s.t.} \quad & A_{t+1} = R_t(A_t - c_t), \\ & A_0,\ R_0 \text{ given,} \end{aligned}$$

where

- γ: subjective discount factor, the corresponding subjective discount rate δ is defined via $\gamma = 1/(1+\delta)$,
- E_0: expectation conditional on period 0 information (which includes R_0 and A_0),
- A_t: beginning of period t financial assets,
- R_t: period t gross-return on savings, which we assume stochastic (for simplicity there is no riskless asset),
- c_t: period t consumption.

Assume that the (log of the) gross return on savings, $\log R_t$, follows a first-order autoregression:

$$\log R_t = (1 - \phi) \log \overline{R} + \phi \log R_{t-1} + e_t,$$

with $\overline{R}$ constant, $\phi \in [0, 1)$ and e_t i.i.d. with zero mean. ,

(a) Write the Bellman Equation for this problem. Indicate the state and decision variables. Provide the economic intuition for whether you can or cannot write the problem with only one state variable.

(b) Assuming that existence and uniqueness of a solution to the Bellman equation has been established, show that the solution is of the form:

$$v = k_0 + k_1 \log A_t + k_2 \log R_t. \tag{1}$$

Find closed form expressions for k_1 and k_2. You do not need to solve for k_0.

(c) Find an explicit expression for current consumption, c_t.

(d) The expression you found in (c) should depend on γ and A_t but not on R_t. Does this mean that the path of optimal consumption does not depend on the path of interest rates? Explain.

(e) Show that $\log A_t$ and $\log c_t$ follow processes that belongs to the ARIMA family. Find the impulse response function of consumption growth $\Delta \log c_t$ to innovations in the gross return of earnings, e_t.

(f) The expression you obtain for v in part (c) is of the form:

$$v(A_t, R_t) = c_0 + c_1 \log A_t + c_2[\log R_t - \log \bar{R}],$$

where c_0, c_1 and c_2 are constants and only c_2 depends on ϕ. Would you expect c_2 to be increasing or decreasing in ϕ?[1] Justify your answer using economic intuition.

2. **Ecuación para tiempos tormentosos**

La prensa frecuentemente afirma que una caída en el ahorro corriente presagia menor crecimiento futuro. En este problema veremos que no necesariamente es así.

(a) Considere el modelo de equivalencia cierta (utilidad cuadrática, no hay activo riesgoso, $r = \delta$). Entonces el consumo óptimo viene dado por:

$$C_t = \frac{r}{1+r}\left\{\sum_{k\geq 0} \beta^k \mathrm{E}_t[Y_{L,t+k}] + A_t\right\},$$

donde $\beta = 1/(1+r)$, $Y_{L,t}$ denota el ingreso laboral en t, A_t activos financieros al comienzo del período t y suponemos que el timing es tal que el ingreso financiero durante el período t, $Y_{K,t}$, es igual a $r(A_{t-1} + Y_{L,t-1} - C_{t-1})$.

Recordando que, por definición, ahorro durante t, S_t, es igual a la diferencia entre ingreso total y consumo, muestre que:

$$S_t = Y_{L,t} - r\sum_{k\geq 0} \beta^{k+1} \mathrm{E}_t[Y_{L,t+k}].$$

A continuación muestre, a partir de la expresión anterior, que:

$$S_t = -\sum_{k\geq 1} \beta^k \mathrm{E}_t[\Delta Y_{L,t+k}], \tag{2}$$

donde $\Delta Y_{L,t} \equiv Y_{L,t} - Y_{L,t-1}$. Explique por qué este resultado muestra que una reducción en el ahorro no necesariamente presagia menor crecimmiento en el futuro. También explique por qué esta ecuación se conoce como la "ecuación de días tormentosos".

(b) A continuación usamos el resultado anterior para predecir cambios futuros en el ingreso en base al ahorro corriente. Suponemos que el ingreso sigue un proceso ARIMA(0,1,1):

$$\Delta Y_t = g + \varepsilon_t + \phi\varepsilon_{t-1},$$

con ε_t i.i.d. con media nula y varianza σ^2. Use la ecuación de días tormentosos (2) para mostrar cómo se puede utilizar los ahorros del período t para predecir el cambio de ingreso entre t y $t+1$.

[1] No need to do the math.

3. **Certainty Equivalence and a Simple Fiscal Rule**

The assumptions ensuring certainty equivalence hold (the interest rate r is equal to the subjective discount rate δ, quadratic utility), so that consumption is given by:

$$C_t = \frac{r}{1+r}\left\{A_t + \sum_{s\geq 0}\beta^s \mathrm{E}_t[Y_{t+s}]\right\}, \tag{3}$$

with $\beta \equiv 1/(1+r)$. Also, timing conventions are such that beginning of period assets satisfy:

$$A_{t+1} = (1+r)(A_t + Y_t - C_t). \tag{4}$$

(a) Use (3) and (4) to show that:

$$\Delta A_{t+1} = Y_t - r\sum_{s\geq 1}\beta^s \mathrm{E}_t[Y_{t+s}],$$

Assume now that income follows an AR(1) process:

$$Y_t - \mu = \phi(Y_{t-1} - \mu) + \epsilon_t, \tag{5}$$

with $0 \leq \phi < 1$ and ϵ_t an innovation process (i.i.d. with zero mean and variance σ^2).

(b) Use (3) to find an expression for C_t as a function of A_t and Y_t.

(c) Use the expression you obtained in part (a) to prove that A_t is an integrated process (i.e., it is not stationary but its first difference is).

The price of oil (and other natural resources) has skyrocketed in recent years, leading to major windfalls in government revenues for oil exporting countries. The usual recommendation from the World Bank and IMF is to create an Oil Fund, that saves part of the windfall for the future. The savings/spending rules for these funds typically take the form:

$$G_t = rF_t + \mu + r(Y_t - \mu), \tag{6}$$

where G_t denotes government expenditures out of oil resources, F_t beginning-of-period resources in the Oil Fund and Y_t net oil revenues. Huge fluctuations in assets in the Oil Fund have been observed in countries that have followed this prescription, often forcing governments to abandon the rule or the fund altogether.

(d) Assume that the government maximizes the expected present discounted quadratic utility of consumption out of oil income with a discount rate equal to the interest rate (which we assume constant and exogenous). Also assume that the oil income process has no persistence. Show that under these assumptions rule (6) is (approximately) optimal.

In practice, oil revenues are highly persistent: the price of oil follows a process close to a random walk, so that ϕ is close to one. It follows that the rule (6) is not optimal, even under the stringent assumptions considered in part (d).

(e) Assume the true value of ϕ is strictly positive. Find the ratio of the standard deviation of ΔF_t when the government uses (6) and the corresponding standard deviation when the government uses the rule corresponding to the true value of ϕ derived in part (c). Can the large values of ΔF_t observed in practice be due to the fact that (6) ignores the persistence of oil revenues?

(f) There are many first-order effects that were ignored when showing that rule (6) is (approximately) optimal in part (d). One is that the price of oil is highly persistent. Mention two additional effects and briefly explain (no formal derivations needed, but state the intuition underlying your statements as clearly as possible) how incorporating each one of them would affect the magnitude of fluctuations of assets in the Oil Fund and the responsiveness of current government expenditures to a positive oil shock.

4. **Precautionary Saving with CARA Utility**

Consider the setup for the general consumption problem covered in class, with no risky assets and CARA felicity function

$$u(c) = -\frac{1}{\theta}e^{-\theta c},$$

where $\theta > 0$ denotes the household's absolute risk aversion coefficient. The subjective discount rate is δ, the interest rate is r and both are constant. Labor income, y_t, is i.i.d., so that

$$y_t = \bar{y} + \varepsilon_t,$$

with the ε_t i.i.d. $\mathcal{N}(0, \sigma^2)$.

The household's beginning-of-period assets evolve according to:

$$A_{t+1} = (1 + r)[A_t + y_t - c_t]. \tag{7}$$

It can be shown that there exists a unique solution to the Bellman equation and that this is in a one-to-one correspondence with the solution to the Euler equation. You do not need to do prove this. It follows that there exists a unique solution for the Euler equation, which is what we focus on in this problem.

We assume that the solution to the Euler equation is of the form:

$$c_t = \frac{r}{1+r}\left\{A_t + y_t + \frac{1}{r}\bar{y}\right\} - P(r, \theta, \delta, \sigma), \tag{8}$$

where P is a constant (that depends on r, θ, δ and σ). We find a P such that (8) solves the Euler equation.

Hint: Note that many parts of the problem can be solved if you did not answer the preceding parts.

(a) Assuming (8) holds, use (7) and some algebra to show that:

$$\Delta c_t = \frac{r}{1+r}\varepsilon_t + rP. \tag{9}$$

(b) Use (9) and the problem's Euler equation to find an explicit expression for P.

(c) Show that P is increasing in σ and decreasing in δ. Interpret both results.

(d) Next assume $r = \delta$. Precautionary saving is defined as the difference between actual saving and saving prescribed by certainty-equivalence. Show that precautionary saving is equal to P and that P is positive.