ENECO630 (Macroeconomía I)
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Guia 4: Soluciones

1. Optimal Consumption with Diversifiable Income

(a) State variables: assets at the beginning of the period and current interest rate (A and R in what follows). Decision variable: either consumption, c, or next period's assets, A'. Since $A' = R(A-c)$, the BE then is:

$$V(A,R) = \max_c \{\log(c) + \gamma \mathrm{E}[V(R(A-c),R')|A,R]\}.$$

(b) With the suggested functional form for V, the r.h.s. of the BE becomes:

$$\max_c \{\log(c) + \gamma k_0 + \gamma k_1(\log R + \log(A-c)) + \gamma k_2 \mathrm{E}[\log R'|R]\},$$

with

$$\mathrm{E}[\log R'|R] = (1-\phi)\log \bar{R} + \phi \log R$$

The corresponding FOC and some straightforward algebra lead to:

$$c = \frac{A}{1+\gamma k_1},$$

Substituting back in the BE we obtain:

$$k_0 + k_1 \log A + k_2 \log R =$$

$$= \gamma k_0 + \gamma k_1 \log(\gamma k_1) - (1+\gamma k_1)\log(1+\gamma k_1) + \gamma k_2(1-\phi)\log \bar{R} + [1+\gamma k_1]\log A + \gamma(k_1 + k_2\phi)\log R.$$

First we equate the coefficients of $\log A$ leading to:

$$k_1 = \frac{1}{1-\gamma}.$$

Next we equate coefficients for $\log R$,

$$k_2 = \frac{\gamma}{(1-\gamma)(1-\gamma\phi)}.$$

(c) Substituting $k_1 = 1/(1-\gamma)$ in the solution for consumption,

$$c_t = (1-\gamma)A_t.$$

(d) The path of interest rates influences the path of A_t and therefore the path for consumption.

(e) We have that $A_{t+1} = R_t(A_t - c_t)$ and substituting with the previous expressions we have $A_{t+1} = \gamma\, R_t\, A_t$. Therefore,

$$\log A_{t+1} = \log \gamma + \log R_t + \log A_t$$

So that $\Delta \log A_t$ is AR(1) and therefore $\log A_t$ is ARIMA(1,1,0).

Since $c_t = (1-\gamma)A_t$, we have that

$$\log \Delta c_{t+1} = \Delta \log A_{t+1} = \log \gamma + \log R_t$$

which is also ARIMA(1,1,0).

The impulse-response of $\log \Delta c_{t+k}$ to e_t is equal to ϕ^{k-1} for $k = 1, 2, 3, \ldots$ and zero otherwise.

(f) The expression $(\log R_t - \log \bar{R})$ captures the extent to which current returns on the risky asset are above their average value. Since ϕ measures the extent to which these returns are likely to persist into the future, large values of ϕ are associated with higher than average expected discounted utility if $\log R_t > \log \bar{R}$ and with lower than average expected utility if $\log R_t < \log \bar{R}$. It follows that c_2 should be increasing in ϕ.

2. Ecuación para tiempos tormentosos (20 puntos)

(a) Tenemos que:

$$S_t = Y_t - C_t = Y_{K,t} + Y_{L,t} - \frac{r}{1+r}\{\sum_{k\geq 0} \beta^k E_t\left[Y_{L,t+k}\right] + A_t\}$$

Comparando la expresión anterior para A_t e $Y_{K,t}$ tenemos que:

$$Y_{K,t} = \frac{r}{1+r} A_t,$$

que combinadas con la expresión anterior para S_t y usando que bajo equivalencia cierta $\beta(1+r) = 1$, llegamos a:

$$\begin{aligned} S_t &= \frac{r}{1+r} A_t + Y_{L,t} - \frac{r}{1+r}\{\sum_{k\geq 0} \beta^k E_t\left[Y_{L,t+k}\right] + A_t\} \\ &= Y_{L,t} - r\sum_{k\geq 0} \beta^{k+1} E_t\left[Y_{L,t+k}\right] \end{aligned}$$

Para derivar la segunda expresión, se comienza con la expresión a la que se quiere llegar:

$$S_t = -\sum_{k\geq 1} \beta^k E_t\left[\Delta Y_{L,t+k}\right] = -\sum_{k\geq 1} (\beta^k E_t Y_{L,t+k} - \beta^k E_t Y_{L,t+k-1}).$$

Si se abre la suma, el coeficiente de $E_t\left[Y_{L,t+k}\right]$ es $-\beta^k + \beta^{k+1}$ si $k \geq 1$ y β si $k = 0$. Y por lo tanto $-\beta^k + \beta^{k+1} = -r\beta^{k+1}$, tenemos que:

$$\begin{aligned} -\sum_{k\geq 1} \beta^k E_t\left[\Delta Y_{L,t+k}\right] &= -r\sum_{k\geq 1} \beta^{k+1} E_t\left[Y_{L,t+k}\right] + \beta Y_{L,t} = \\ &= -r\sum_{k\geq 0} \beta^{k+1} E_t\left[Y_{L,t+k}\right] + (\beta + r\beta) Y_{L,t} \end{aligned}$$

Recordando que $\beta + r\beta = 1$ completa la demostración.

Esta expresión para S_t dice que el ahorro es igual el valor presente esperado de futuras caídas del ingreso laboral. Por eso una caída en el ahorro no presagia necesariamente menor crecimiento en el futuro. Por esta misma razón se conoce como la "ecuación de días tormentosos", porque se ahorra para cubrir futuras y esperadas caídas en el ingreso laboral.

(b) Una derivación directa muestra que:

$$\mathrm{E}_t[\Delta Y_{L,t+k}] = \begin{cases} g, & \text{for } k \geq 2, \\ g + \theta\varepsilon_t & \text{for } k = 1 \end{cases}$$

Usando la segunda expresión para S_t que se obtuvo antes, tenemos:

$$\begin{aligned}
S_t &= -\sum_{k\geq 1}\beta^k E_t\left[\Delta Y_{L,t+k}\right] \\
&= -\beta E_t\left[\Delta Y_{L,t+1}\right] - \sum_{k\geq 2}\beta^k E_t\left[\Delta Y_{L,t+k}\right] \\
&= -\beta E_t\left[\Delta Y_{L,t+1}\right] - \sum_{k\geq 2}\beta^k g \\
&= -\beta E_t\left[\Delta Y_{L,t+1}\right] - g\beta^2 \sum_{k\geq 0}\beta^k \\
&= -\beta E_t\left[\Delta Y_{L,t+1}\right] - \frac{g\beta^2}{1-\beta}
\end{aligned}$$

Y por lo tanto:

$$E_t\left[\Delta Y_{L,t+1}\right] = -\frac{g\beta}{1-\beta} - \frac{S_t}{\beta} = -\frac{g}{r} - (1+r)\,S_t.$$

Se concluye por lo tanto, que una caída en el ahorro presente genera un aumento en el ingreso futuro esperado.

3. Certainty Equivalence and a Simple Fiscal Rule

(a) Under certainty equivalence, we know that

$$C_t = \frac{r}{1+r}\left\{A_t + \sum_{s\geq 0}\beta^s E_t[Y_{t+s}]\right\} \tag{1}$$

Substituting our expression for C_t in (1) into the asset accumulation equation gives

$$\begin{aligned}
A_{t+1} &= R(A_t + Y_t - C_t) \\
&= (1+r)\left[A_t + Y_t - \frac{r}{1+r}\left\{A_t + \sum_{s\geq 0}\beta^s E_t[Y_{t+s}]\right\}\right]
\end{aligned}$$

Rearranging gives

$$\begin{aligned}
\Delta A_{t+1} &= \left[rA_t - rA_t + (1+r)Y_t - rY_t - r\sum_{s\geq 1}\beta^s E_t[Y_{t+s}]\right] \\
&= Y_t - r\sum_{s\geq 1}\beta^s E_t[Y_{t+s}]
\end{aligned}$$

(b) For an AR(1) we have that

$$E_t[Y_{t+s}] = \mu + \phi^s(Y_t - \mu)$$

plugging that into (1) gives

$$\begin{aligned}
C_t &= \frac{r}{1+r}\left\{A_t + \sum_{s\geq 0}\beta^s(\mu + \phi^s(Y_t-\mu))\right\} \\
&= \frac{r}{1+r}\left\{A_t + \frac{\mu}{1-\beta} + \frac{Y_t-\mu}{1-\beta\phi}\right\}
\end{aligned}$$

And using that $\beta = \frac{1}{1+r}$ gives,

$$C_t = \frac{r}{R}A_t + \mu + \frac{r}{R-\phi}(Y_t - \mu)$$

(c) Substitute our expression for $E_t[Y_{t+s}]$ into what we derived in part (a)

$$\begin{aligned}
\Delta A_{t+1} &= Y_t - r\sum_{s\geq 1}\beta^s E_t[Y_{t+s}] \\
&= Y_t - r\sum_{s\geq 1}\beta^s\left(\mu + \phi^s(Y_t-\mu)\right) \\
&= Y_t - \left(\frac{r\beta\mu}{1-\beta}\right) - \left(\frac{r\beta\phi(Y_t-\mu)}{1-\beta\phi}\right) \\
&= \left(1-\frac{r\beta\phi}{1-\beta\phi}\right)Y_t - \left(\frac{r\beta}{1-\beta} - \frac{r\beta\phi}{1-\beta\phi}\right)\mu \\
&= \left(1-\frac{r\beta\phi}{1-\beta\phi}\right)Y_t - \left(1-\frac{r\beta\phi}{1-\beta\phi}\right)\mu \\
\Delta A_{t+1} &= \frac{(1-\phi)(1+r)}{1+r-\phi}(Y_t-\mu)
\end{aligned}$$

Thus ΔA_{t+1} is an integrated process because $(Y_t-\mu)$ is a stationary AR(1). Note that $A_t = RA_{t-1}+\cdots$ with $|R|>1$ so in levels A_t was not stationary.

(d) Equation (6) in problem set corresponds to the consumption rule we derived in part (b) when $\phi = 0$ and under the approximation $\frac{r}{R} \simeq r$ and the conditions that we usually have in certainty equivalence problems (quadratic utility, no risky asset, $r = \delta$, no insurance)

(e) If the oil fund follows equation (5) in the problem set, we have that ΔA_{t+1} is what we derived in (c) when $\phi = 0$. This gives

$$\Delta A_{t+1} = Y_t - \mu$$

However, if the correct value of ϕ is used for the expenditure rule then

$$\Delta A_{t+1} = \frac{(1-\phi)(1+r)}{1+r-\phi}(Y_t-\mu)$$

It follows that the ratio of the standard deviations is equal to

$$\frac{1+r-\phi}{(1-\phi)(1+r)} > 1$$

This ratio is increasing in ϕ and tends to infinity as ϕ tends to 1, so the large swings in practice can be a result of assuming the wrong level of persistence of natural resource.

(f) -Precautionary savings: F_t will be larger on average, but there is no clear implication on whether ΔF_t or the response of G_t to a positive oil shock will be larger or smaller. In fact, based on the explicit expressions we derived in class for the CARA case, there would be no effect in both cases.

-Costs in adjusting government expenditures (both economic and political) are missing from the utility function. Once these are included, government expenditures will respond less to the current shock (and asses in the fund will fluctuate more)

- Ignoring consumption of goods that are not financed from oil (e.g. private production) is a major shortcoming. The problems with considering a utility function with a publicly and private provided good is that, if the economy is expected to grow in future decades, then consumption smoothing will lead to spending today the incomes from future generations, which highlights that more careful thought needs to be given to the policy instruments available to the government.

- the government could use financial derivatives to reduce the risk exposure. This would reduce the size of the fluctuation in F_t and the size of the response of G_t to oil shocks.

-The framework ignores investment, both by the government and the private sector.

-the framework assumes the nature resource is not exhaustible. For an exhaustible resource, the government will save more and spend less, so as to have the resources to spend when the natural resource is exhausted.

4. Precautionary Saving with CARA Utility

Consider the setup for the general consumption problem covered in class, with no risky assets and CARA felicity function

$$u(c) = -\frac{1}{\theta}e^{-\theta c},$$

where $\theta > 0$ denotes the consumer's absolute risk aversion coefficient. The subjective discount rate is δ, the interest rate is r and both are constant. Labor income, y_t, is i.i.d., so that

$$y_t = \bar{y} + \varepsilon_t,$$

with the ε_t i.i.d. $\mathcal{N}(0, \sigma^2)$.

The consumer's beginning-of-period assets evolve according to:

$$A_{t+1} = (1+r)[A_t + y_t - c_t]. \tag{2}$$

In this problem we will show that the solution to the consumer's problem is of the form:

$$c_t = \frac{r}{1+r}\left\{A_t + y_t + \frac{1}{r}\bar{y}\right\} - P(r, \theta, \delta, \sigma), \tag{3}$$

where P is a constant (that depends on r, A, δ and σ). We will also find an explicit expression for P.

(a) Assuming (3) holds, use (2) and some algebra to show that:

$$\Delta c_t = \frac{r}{1+r}\varepsilon_t + rP. \tag{4}$$

Solution: using (3) for c_t and c_{t-1} we get

$$\Delta c_t = c_t - c_{t-1} = \frac{r}{1+r}[A_t + y_t - A_{t-1} - y_{t-1}]$$

using (2)

$$\begin{aligned}
\Delta c_t &= \frac{r}{1+r}[(1+r)(A_{t-1} + y_{t-1} - c_{t-1}) + y_t - A_{t-1} - y_{t-1}] \\
&= \frac{r}{1+r}[rA_{t-1} + ry_{t-1} - (1+r)c_{t-1} + y_t]
\end{aligned}$$

Finally, using (3) for c_{t-1} again

$$\begin{aligned}
\Delta c_t &= \frac{r}{1+r}[y_t - \bar{y} + (1+r)P] \\
&= \frac{r}{1+r}\varepsilon_t + rP
\end{aligned}$$

(b) Use (4) and the problem's Euler equation to find an explicit expression for P.
Solution: define $\beta \equiv 1/(1+\delta)$, then the Euler equation is

$$u'(c_t) = \beta(1+r)E_t[u'(c_{t+1})]$$

and using the given utility function we get

$$1 = \beta(1+r)E_t[e^{-\theta \Delta c_{t+1}}]$$

Using (3) we get

$$1 = \beta(1+r)e^{-r\theta P}E_t[e^{-\frac{r\theta}{1+r}\varepsilon_{t+1}}]$$

Next we use the following result: if X is a normal random variable with mean μ and variance σ^2 then for any real number t,

$$E[e^{tX}] = e^{\mu t + \frac{1}{2}t^2\sigma^2}$$

Applying this to our last result we get

$$1 = \beta(1+r)e^{-r\theta P}e^{\frac{(\sigma r\theta)^2}{2(1+r)^2}}$$

taking logs we get

$$0 = \log(\beta(1+r)) - r\theta P + \frac{(\sigma r\theta)^2}{2(1+r)^2}$$

and finally, solving for P gives

$$P = \frac{\log(\beta(1+r))}{\theta r} + \frac{\sigma^2 r\theta}{2(1+r)^2}$$

(c) Show that P is increasing in σ and decreasing in δ. Interpret both results.
Solution: from last part, we have that $\frac{\partial P}{\partial \beta} > 0$, and hence P depends negatively on δ. As people get more impatient, they want to consume more today, and so they subtract less from consumption. Also from last part, $\frac{\partial P}{\partial \sigma} > 0$, thus as uncertainty increases, people want to save more, because of precautionary saving.

(d) Next assume $r = \delta$. Precautionary saving is defined as the difference between actual saving and saving prescribed by certainty-equivalence. Show that precautionary saving is equal to P and that P is positive.
Solution: in this case,

$$P = \frac{\sigma^2 r\theta}{2(1+r)^2} > 0$$

From class we know that under certainty-equivalence (and using that $r = \delta$),

$$c_t = \frac{r}{1+r}\left\{A_t + \sum_{s\geq 0}\frac{E_t[y_{t+s}]}{(1+r)^s}\right\}$$

let's work on this formula

$$\begin{aligned}
c_t &= \frac{r}{1+r}\left\{A_t + y_t + E_t\sum_{s\geq 1}\frac{y_{t+s}}{(1+r)^s}\right\} \\
&= \frac{r}{1+r}\left\{A_t + y_t + \bar{y}\sum_{s\geq 1}\frac{1}{(1+r)^s}\right\} \\
&= \frac{r}{1+r}\left\{A_t + y_t + \frac{\bar{y}}{r}\right\}
\end{aligned}$$

therefore, the precautionary savings are given by

$$S_t - S_t^{certain} = C_t^{certain} - C_t = P$$

5. Liquidity Constraints

(a) The idea is that the maximum you can consume in period 1 is your income Y_1 plus the maximum you can borrow on the account of Y_2, which is $Y_2/(1+r_D)$, implying a consumption of $Y_1 + Y_2/(1+r_D)$. On the other hand, if you do not consume in period 1, and thus save all your income Y_1, then in period 2 your consumption will be equal to $Y_2 + Y_1(1+r_S)$. Therefore, the budget constraint (in the (C_1, C_2) space) will be made up of two lines: one that goes from $(Y_1 + Y_2/(1+r_D), 0)$ until (Y_1, Y_2), and thus has a slope of $-(1+r_D)$. The other line goes from (Y_1, Y_2) until $Y_2 + Y_1(1+r_S)$, and thus has a slope of $-(1+r_S)$.

(b) Let $S \geq 0$ represent savings and $B \geq 0$ borrowings, then $C_1 = Y_1 - S + B$ and $C_2 = Y_2 + (1 + r_S)S - (1 + r_D)B$. Then the problem we have to solve is

$$\max_{S,B} U(Y_1 - S + B, Y_2 + (1 + r_S)S - (1 + r_D)B) \quad \text{s.t.} \quad S, B \geq 0$$

The FOC of this problem are

$$B: \quad \frac{\partial U}{\partial C_1}(C_1, C_2) - \frac{\partial U}{\partial C_2}(C_1, C_2)[1 + r_D] \leq 0$$

$$S: \quad -\frac{\partial U}{\partial C_1}(C_1, C_2) + \frac{\partial U}{\partial C_2}(C_1, C_2)[1 + r_S] \leq 0$$

If we want consumption in period 1 and 2 to be Y_1 and Y_2, we need $S = B = 0$ to satisfy these inequalities, and thus what we need is that

$$\frac{\partial U}{\partial C_1}(Y_1, Y_2) \leq \frac{\partial U}{\partial C_2}(Y_1, Y_2)[1 + r_D] \quad \text{and} \quad \frac{\partial U}{\partial C_1}(Y_1, Y_2) \geq \frac{\partial U}{\partial C_2}(Y_1, Y_2)[1 + r_S]$$

Thus, given that U is increasing and concave, necessary and sufficient conditions for consumption in periods 1 and 2 to be Y_1 and Y_2 respectively, are that

$$\left(\frac{\left.\frac{\partial U}{\partial C_1}\right|_{(Y_1, Y_2)}}{\left.\frac{\partial U}{\partial C_2}\right|_{(Y_1, Y_2)}} \right) \in [1 + r_S, 1 + r_D]$$

(c) In this part we assume that $U(C_1, C_2) = u_1(C_1) + u_2(C_2)$, and the inequalities are now:

$$u_1'(Y_1) \leq u_2'(Y_2)[1 + r_D] \quad \text{and} \quad u_1'(Y_1) \geq u_2'(Y_2)[1 + r_S]$$

(d) You all did the graph correctly: you have to shift the BC to the right in Δy units. The idea here is that as the derivatives are continuous and the conditions hold with strict inequality, if Δy is small enough the conditions will still hold for the new period 1 income $Y_1 + \Delta y$, and thus the individual will eat his entire income each period. Therefore he will consume $Y_1 + \Delta y$ in the first period implying $\Delta C_1 = \Delta y$.

(e) As we can not borrow, but we can still save, what changes in the BC is that the line that went from $(Y_1 + Y_2/(1 + r_D), 0)$ until (Y_1, Y_2), now is replaced by the straight line that goes from $(Y_1, 0)$ until (Y_1, Y_2), and thus the slope of this line is infinite. The other line is the same.

The condition for $C_1 = Y_1$ and $C_2 = Y_2$ in this case is that

$$\frac{\partial U}{\partial C_1}(Y_1, Y_2) \geq \frac{\partial U}{\partial C_2}(Y_1, Y_2)[1 + r_S]$$

and in the case of separable utility

$$u_1'(Y_1) \geq u_2'(Y_2)[1 + r_S]$$

The same arguments as before are used when analysing a small increase in period 1 income.