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Guías de ejercicios No. 4

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1 Incertidumbre del modelo lleva a tomar cautela[1]

La NKPC viene dada por

$$\pi_t = \beta \mathrm{E}\pi_{t+1} + \kappa_t x_t + e_t. \tag{1}$$

Donde la pendiente de la curva de Phillips, $\kappa_t = \bar{\kappa} + v_t$, varía en el tiempo y estas variaciones no son observadas por la autoridad monetaria que sólo conoce la media correspondiente, $\bar{\kappa}$.
También tenemos un shock de costos, e_t, el cual es observado por la autoridad monetaria.[2] Las variables v_t y e_t son ruidos blancos no correlacionados de media nula y varianza σ_v^2 y σ_e^2.
En este problema consideramos equilibrios perfectos de Markov, es decir, discrecionales. La función de pérdida social viene dada por

$$L_t = \frac{1}{2}\mathrm{E}_t[\pi_t^2 + \lambda x_t^2]. \tag{2}$$

1. Tomando $\mathrm{E}_t\pi_{t+1}$ como dado, determine el valor de x_t que minimiza la función objetivo. Antes de hacer ningún cálculo, explique por qué x_t dependerá de e_t y *no* dependerá de v_t.

   Respuesta: x_t no dependerá de v_t porque esta varible no es observada por la autoridad. En cambio, puede depender de e_t porque esta variable se observa.

   Denotando $\pi^e = \mathrm{E}_t\pi_{t+1}$ en (1), y reemplazando esta expresión en (2), se obtiene:

   $$L_t = \frac{1}{2}\left\{(\lambda + \bar{\kappa}^2 + \sigma_v^2)x_t^2 + 2(\beta\pi^e + e_t)\bar{\kappa}x_t + \text{t.i.p.}\right\}.$$

   De la CPO correspondiente se obtiene

   $$x_t = -\frac{(\beta\pi^e + e_t)\bar{\kappa}}{\lambda + \bar{\kappa}^2 + \sigma_v^2}.$$

2. Utilice su resultado de la parte (a) para mostrar que $\mathrm{E}_t\pi_{t+1} = 0$.

   Respuesta Sustituyendo x_t en (1) por la expresión anterior y tomando E_{t-1} a ambos lados se concluye que $\pi^e = 0$. El motivo es que estamos suponiendo $x^* = 0$.

[1] Basado en Brainard (AER, 1967).
[2] Estos pueden deberse, por ejemplo, a variaciones en las tasas de impuestos.

3. Obtenga expresiones para x_t y π_t en función de e_t y los parámetros del modelo.

   Respuesta Con $\pi^e = 0$, la expresión que obtuvimos para x_t pasa a:

$$x_t = -\frac{\bar{\kappa}}{\lambda + \bar{\kappa}^2 + \sigma_v^2} e_t.$$

   Y luego, de (1), también con $\pi^e = 0$:

$$\pi_t = \frac{\lambda + \sigma_v^2 - \bar{\kappa} v_t}{\lambda + \sigma_v^2 + \bar{\kappa}^2} e_t$$

4. Obtenga expresiones para la media y desviación estándar (o varianza) de x_t y π_t.

   Respuesta De las expresiones de la parte anterior, las medias de x_t y π_t son zero. Las desviaciones estándar correspondientes son:

$$\begin{aligned}
\sigma(x_t) &= \frac{\bar{\kappa}}{\lambda + \bar{\kappa}^2 + \sigma_v^2} \sigma_e, \\
\sigma^2(\pi_t) &= \left(\frac{\lambda + \sigma_v^2}{\lambda + \sigma_v^2 + \bar{\kappa}^2} \right)^2 \sigma_e^2 + \left(\frac{\bar{\kappa}^2}{\lambda + \sigma_v^2 + \bar{\kappa}^2} \right)^2 \sigma_v^2 \sigma_e^2.
\end{aligned}$$

5. Concluya que cuando hay incertidumbre respecto de la pendiente de la NKPC, la respuesta óptima (bajo discreción) será tener una brecha, x_t, menos volátil y una inflación más volátil. ¿Puede dar la intuición para este resultado?

   Respuesta Es fácil ver que cuando hay incertidumbre respecto de κ, x_t es menos volátil y π_t más volatil. Cuando hay incertidumbre respecto de κ la autoridad elige menos variabilidad en x_t, que es afectado directamente por κ (error multiplicativo) a cambio de más volatilidad en π.

2 The Cagan Model and the Lucas Critique

Consider a rational-expectations version of the Cagan model, in which the demand for money is of the form

$$\log M_t - \log P_t = -\gamma - \alpha \mathrm{E}_t \pi_{t+1}, \tag{3}$$

for some coefficients γ and $\alpha > 0$. Suppose that the money supply, M_t^s, evolves according to a stochastic process

$$\mu_t = \mu_{t-1} + \varepsilon_t - \theta \varepsilon_{t-1}, \tag{4}$$

where $\mu_t \equiv \log M_t^s - \log M_{t-1}^s$, is the rate of growth of the money supply, ε_t is an i.i.d. random variable with mean zero and bounded support, and $0 < \theta < 1$. (Specification (4) implies that when an unexpected increase in the rate of growth of the money supply occurs, a fraction θ of the increase is "taken back" the next period. The rest —a fraction $1 - \theta$ of the shock— is a permanent increase in the rate of money growth, on average.)

1. Show that (4) implies that $\Delta\mu_t \equiv \mu_t - \mu_{t-1}$ is a bounded stochastic process. Show furthermore that it implies that one can solve for ε_t uniquely as a function of the past history of the observations of the rate of money growth, μ_{t-j}, for all $j \geq 0$. [This assumes that one knows that money growth evolves

according to (4), and that one knows the distribution from which ε_{t-j} has been drawn at all past dates (at least that it is bounded). The result is important for the discussion below, as it implies that if the history of the money supply up through period t is public information at date t, then the entire history of the shocks ε_{t-j} for $j \geq 0$ is also public information at date t.]

Respuesta:

$$\begin{aligned}
\Delta\mu_t &= \mu_t - \mu_{t-1} \\
&= \varepsilon_t - \theta\varepsilon_{t-1},
\end{aligned}$$

which is bounded since ε_t is bounded. Rearranging this equation, we have

$$\varepsilon_t = \Delta\mu_t + \theta\varepsilon_{t-1}.$$

Lagging this equation one period and substituting back in, we have

$$\begin{aligned}
\varepsilon_t &= \Delta\mu_t + \theta\left[\Delta\mu_{t-1} + \theta\varepsilon_{t-2}\right] \\
&= \Delta\mu_t + \theta\Delta\mu_{t-1} + \theta^2\varepsilon_{t-2} \\
&= \ldots \\
&= \Delta\mu_t + \theta\Delta\mu_{t-1} + \theta^2\Delta\mu_{t-2} + \ldots + \lim_{j\to\infty}\theta^j\varepsilon_{t-j} \\
&= \sum_{j=0}^{\infty}\theta^j\Delta\mu_{t-j} \\
&= \sum_{j=0}^{\infty}\theta^j\left(\mu_{t-j} - \mu_{t-1-j}\right).
\end{aligned}$$

2. Assume that the money supply evolves according to (4), money demand evolves according to (3) and that the entire past history of the money supply is public information. Define what would constitute a rational expectations equilibrium. Show that there is a unique rational expectations equilibrium with the property that the rate of acceleration of inflation $\Delta\pi_t$ is bounded, where $\pi_t \equiv \log P_t - \log P_{t-1}$ is the rate of inflation. Solve for this inflation process, expressing inflation as a function of the history of the money growth rate,

$$\pi_t = \sum_{j\geq 0}\phi_j\mu_{t-j}. \tag{5}$$

(Hint: Use the money demand equation to express inflation as a function of the current money growth rate and the change in inflation expectations [imposing money demand equals money supply in equilibrium]. Iterate this function forward j periods, take time t expectations and show that you can write $E_t\pi_{t+j}$ as a function of expected money growth at time $t+j$ and inflation expectations at time $t+j+1$. Use this to write $E_t\pi_{t+1}$ as a function of all future money growth and a limit term in inflation. Taking this expression one period back, we get a similar expression for $E_{t-1}\pi_t$. Compute the difference between $E_t\pi_{t+1}$ and $E_{t-1}\pi_t$, imposing bounded inflation acceleration, and substitute this into the original expression for π_t to get current inflation as a function of current and expected future money growth. Now use (4) to express expected money growth in terms of current money growth and current shocks.

This implies that π_t only depends on current money growth and current shocks. Use the result from part (a) to get an equation for π_t of the form in (5)).

Respuesta:

$$\log M_t = \log P_t - \gamma - \alpha E_t \pi_{t+1}$$

so

$$\mu_t = \log P_t - \log P_{t-1} - \alpha\left[E_t \pi_{t+1} - E_{t-1}\pi_t\right]$$

or

$$\pi_t = \mu_t + \alpha\left[E_t \pi_{t+1} - E_{t-1}\pi_t\right]$$

Updating one period, we have

$$\pi_{t+1} = \mu_{t+1} + \alpha\left[E_{t+1} \pi_{t+2} - E_t\pi_{t+1}\right]$$

Taking expectations, we have

$$E_t\pi_{t+1} = E_t\mu_{t+1} + \alpha\left[E_t \pi_{t+2} - E_t\pi_{t+1}\right]$$

or

$$E_t\pi_{t+1} = \frac{E_t\mu_{t+1}}{1+\alpha} + \frac{\alpha}{1+\alpha}E_t\pi_{t+2}.$$

We can do the same thing for any j to get

$$E_t\pi_{t+j} = \frac{E_t\mu_{t+j}}{1+\alpha} + \frac{\alpha}{1+\alpha}E_t\pi_{t+j+1}.$$

Substituting this repeatedly, we have

$$\begin{aligned}
E_t\pi_{t+1} &= \frac{E_t\mu_{t+1}}{1+\alpha} + \frac{\alpha}{1+\alpha}\left[\frac{E_t\mu_{t+2}}{1+\alpha} + \frac{\alpha}{1+\alpha}E_t\pi_{t+3}\right] \\
&= \frac{E_t\mu_{t+1}}{1+\alpha} + \frac{\alpha E_t\mu_{t+2}}{(1+\alpha)^2} + \frac{\alpha^2}{(1+\alpha)^2}E_t\pi_{t+3} \\
&= \frac{E_t\mu_{t+1}}{1+\alpha} + \frac{\alpha E_t\mu_{t+2}}{(1+\alpha)^2} + \frac{\alpha^2}{(1+\alpha)^2}\left[\frac{E_t\mu_{t+3}}{1+\alpha} + \frac{\alpha}{1+\alpha}E_t\pi_{t+4}\right] \\
&= \frac{E_t\mu_{t+1}}{1+\alpha} + \frac{\alpha E_t\mu_{t+2}}{(1+\alpha)^2} + \frac{\alpha^2 E_t\mu_{t+3}}{(1+\alpha)^3} + \frac{\alpha^3}{(1+\alpha)^3}E_t\pi_{t+4} \\
&= \ldots \\
&= \sum_{j=1}^{\infty}\frac{\alpha^{j-1}}{(1+\alpha)^j}E_t\mu_{t+j} + \lim_{j\to\infty}\frac{\alpha^{j-1}}{(1+\alpha)^{j-1}}E_t\pi_{t+j}
\end{aligned}$$

Substituting into

$$\pi_t = \mu_t + \alpha\left[E_t \pi_{t+1} - E_{t-1}\pi_t\right]$$

then gives

$$\pi_t = \mu_t + \sum_{j=1}^{\infty} \frac{\alpha^j}{(1+\alpha)^j} E_t \mu_{t+j} - \sum_{j=1}^{\infty} \frac{\alpha^j}{(1+\alpha)^j} E_{t-1} \mu_{t-1+j} + \lim_{j\to\infty} \left(\frac{\alpha^j}{(1+\alpha)^j} \left(E_t \pi_{t+j} - E_{t-1} \pi_{t-1+j} \right) \right).$$

The limit is equal to zero if inflation acceleration is bounded and thus

$$\begin{aligned}
\pi_t &= \mu_t + \sum_{j=1}^{\infty} \frac{\alpha^j}{(1+\alpha)^j} E_t \mu_{t+j} - \sum_{j=1}^{\infty} \frac{\alpha^j}{(1+\alpha)^j} E_{t-1} \mu_{t-1+j} \\
&= \mu_t + \sum_{j=1}^{\infty} \frac{\alpha^j}{(1+\alpha)^j} \left(E_t \mu_{t+j} - E_{t-1} \mu_{t-1+j} \right)
\end{aligned}$$

Now note that

$$\begin{aligned}
E_t \mu_{t+2} &= E_t \left[\mu_{t+2-1} + \varepsilon_{t+2} - \theta \varepsilon_{t+2-1} \right] \\
&= E_t \left[\mu_{t+1} + \varepsilon_{t+2} - \theta \varepsilon_{t+1} \right] \\
&= E_t \mu_{t+1} \\
&= E_t \left[\mu_t + \varepsilon_{t+1} - \theta \varepsilon_t \right] \\
&= \mu_t - \theta \varepsilon_t
\end{aligned}$$

Applying the same argument gives that

$$E_t \mu_{t+j} = \mu_t - \theta \varepsilon_t \ \forall j > 0.$$

Similarly,

$$E_{t-1} \mu_{t-1+j} = \mu_{t-1} - \theta \varepsilon_{t-1} \ \forall j > 0.$$

Thus

$$\begin{aligned}
\pi_t &= \mu_t + \sum_{j=1}^{\infty} \frac{\alpha^j}{(1+\alpha)^j} \left(\mu_t - \theta \varepsilon_t - [\mu_{t-1} - \theta \varepsilon_{t-1}] \right) \\
&= \mu_t + \sum_{j=1}^{\infty} \frac{\alpha^j}{(1+\alpha)^j} \left(\mu_t - \theta \varepsilon_t - [\mu_t - \varepsilon_t] \right) \\
&= \mu_t + \sum_{j=1}^{\infty} \frac{\alpha^j}{(1+\alpha)^j} (1-\theta) \varepsilon_t \\
&= \mu_t + \sum_{j=0}^{\infty} \frac{\alpha^{j+1}}{(1+\alpha)^{j+1}} (1-\theta) \varepsilon_t \\
&= \mu_t + \frac{\alpha}{(1+\alpha)} \sum_{j=0}^{\infty} \frac{\alpha^j}{(1+\alpha)^j} (1-\theta) \varepsilon_t \\
&= \mu_t + \alpha (1-\theta) \varepsilon_t
\end{aligned}$$

From part 1, we then have

$$\pi_t = \mu_t + \alpha(1-\theta)\sum_{j=0}^{\infty}\theta^j(\mu_{t-j}-\mu_{t-1-j})$$

which is of the desired form.

3. Given the solution in part (b), compute the *impulse response function* for inflation, in response to an unexpected increase in the money growth rate of one additional percentage point per month. That is, compute the sequence of coefficients

$$\gamma_j = \frac{\partial}{\partial\varepsilon_t}[\mathrm{E}_t\pi_{t+j} - \mathrm{E}_{t-1}\pi_{t+j}].$$

(Hint: This might be easier if you use the expression derived in (4) just before you substitute in for ε_t from (3).)

Respuesta:

$$\begin{aligned}
\gamma_j &= \frac{\partial}{\partial\varepsilon_t}[E_t\pi_{t+j} - E_{t-1}\pi_{t+j}] \\
&= \frac{\partial}{\partial\varepsilon_t}[E_t[\mu_{t+j} + \alpha(1-\theta)\varepsilon_{t+j}] - E_{t-1}[\mu_{t+j} + \alpha(1-\theta)\varepsilon_{t+j}]]
\end{aligned}$$

For $j=0$ this is equal to

$$\frac{\partial}{\partial\varepsilon_t}[\mu_t + \alpha(1-\theta)\varepsilon_t - E_{t-1}\mu_t]$$

and for $j>0$ it is equal to

$$\frac{\partial}{\partial\varepsilon_t}[E_t\mu_{t+j} - E_{t-1}\mu_{t+j}].$$

Again, use

$$\begin{aligned}
E_t\mu_{t+j} &= \mu_t - \theta\varepsilon_t \\
&= \mu_{t-1} + (1-\theta)\varepsilon_t - \theta\varepsilon_{t-1}\forall j>0
\end{aligned}$$

and

$$\begin{aligned}
E_t\mu_t &= E_t\mu_{t-1} + E_t\varepsilon_t - E_t\theta\varepsilon_{t-1} \\
&= \mu_{t-1} + \varepsilon_t - \theta\varepsilon_{t-1}
\end{aligned}$$

and

$$E_{t-1}\mu_{t+j} = \mu_{t-1} - \theta\varepsilon_{t-1}\forall j > -1$$

to get

$$
\begin{aligned}
\gamma_0 &= \frac{\partial}{\partial \varepsilon_t}\left[\mu_t + \alpha(1-\theta)\varepsilon_t - E_{t-1}\mu_t\right] \\
&= \frac{\partial}{\partial \varepsilon_t}\left[\mu_t + \alpha(1-\theta)\varepsilon_t - \mu_{t-1} + \theta\varepsilon_{t-1}\right] \\
&= \frac{\partial}{\partial \varepsilon_t}\left[\mu_{t-1} + \varepsilon_t + \alpha(1-\theta)\varepsilon_t - \mu_{t-1}\right] \\
&= 1 + \alpha(1-\theta).
\end{aligned}
$$

$$
\begin{aligned}
\gamma_j &= \frac{\partial}{\partial \varepsilon_t}\left[E_t\mu_{t+j} - E_{t-1}\mu_{t+j}\right] \\
&= \frac{\partial}{\partial \varepsilon_t}\left[\mu_{t-1} + (1-\theta)\varepsilon_t - \theta\varepsilon_{t-1} - \mu_{t-1} + \theta\varepsilon_{t-1}\right] \\
&= (1-\theta) \quad \forall j > 0.
\end{aligned}
$$

4. Given the solution in part (b), compute how inflation expectations $\pi_t^e \equiv \mathrm{E}_t\pi_{t+1}$t should evolve in the rational expectations equilibrium. Show that inflation expectations will evolve in accordance with a Cagan-type error-correction (or adaptive expectations) formula,

$$\pi_t^e = \lambda\pi_t + (1-\lambda)\pi_{t-1}^e, \tag{6}$$

for some value of the adjustment coefficient $0 < \lambda < 1$. Compute the value of λ as a function of model parameters.

(Hint: Compute π_t^e, π_t and π_{t-1}^e using formulas from part (b) (again this might be easier before substituting in for ε_t). Then using these values solve (6) for λ.)

Respuesta:

If

$$\pi_t = \mu_t + \alpha(1-\theta)\varepsilon_t$$

then we have

$$
\begin{aligned}
E_t\pi_{t+1} &= \mu_t - \theta\varepsilon_t \\
\pi_t &= \mu_t + \alpha(1-\theta)\varepsilon_t \\
E_{t-1}\pi_t &= \mu_{t-1} - \theta\varepsilon_{t-1} = \mu_t - \varepsilon_t
\end{aligned}
$$

We then want to solve for λ so that

$$E_t\pi_{t+1} = \lambda\pi_t + (1-\lambda)E_{t-1}\pi_t$$

or

$$
\begin{aligned}
\mu_t - \theta\varepsilon_t &= \lambda\left[\mu_t + \alpha(1-\theta)\varepsilon_t\right] \\
&\quad +(1-\lambda)\left[\mu_t - \varepsilon_t\right] \\
&= \mu_t + \lambda\alpha(1-\theta)\varepsilon_t - (1-\lambda)\varepsilon_t
\end{aligned}
$$

this implies that

$$
\begin{aligned}
\lambda\alpha(1-\theta) - (1-\lambda) + \theta &= 0 \\
\lambda\left[\alpha(1-\theta)+1\right] &= 1-\theta \\
\lambda &= \frac{1-\theta}{\left[\alpha(1-\theta)+1\right]}
\end{aligned}
$$

5. Suppose that this model describes the evolution of money growth and inflation in a certain economy, and that an economist who collects data on these two series seeks to interpret them using the Cagan model (the original model, with adaptive expectations!), described by equations (3) and (6). (That is, inflation expectations are described by (6) and inflation is described by first-differenced (3)). Would the economist find that the Cagan equations fit these time series, or would it be possible to reject the model once long enough data series were available?

   Suppose the economist estimates the parameters γ, α and λ of the Cagan model using the time series for money growth and inflation, and then uses this estimated econometric model to compute an implied".policy multiplier" $\psi_j \equiv \partial\pi_{t+j}/\partial\mu_t$. What values will be computed? Will this value be consistent with that computed accounting for the true, rational expectations?

   (Note: The impulse response calculated in part (c) is not necessarily the same as the impulse response calculated in this part. Why?)

   Respuesta: As long as parameters remain unchanged, (6) is the evolution of the rational expectations model as well as the adaptive expectations model, so the two models are identical, so it wouldn't be rejected with arbitrarily long data.

   From the money demand equation,

$$
\pi_t = \mu_t + \alpha\left[E_t\pi_{t+1} - E_{t-1}\pi_t\right],
$$

   then substituting the estimated relationship for inflation expectations,

$$
\begin{aligned}
\pi_t &= \mu_t + \alpha\left[E_t\pi_{t+1} - E_{t-1}\pi_t\right] \\
&= \mu_t + \alpha\left[\lambda\pi_t + (1-\lambda)E_{t-1}\pi_t - E_{t-1}\pi_t\right] \\
&= \mu_t + \alpha\left[\lambda\pi_t - \lambda E_{t-1}\pi_t\right]
\end{aligned}
$$

   so

$$
\pi_t = \frac{\mu_t - \alpha\lambda E_{t-1}\pi_t}{(1-\alpha\lambda)}
$$

so

$$\begin{aligned}\psi &= \frac{\partial \pi_t}{\partial \mu_t} \\ &= \frac{1}{1-\alpha\lambda} \\ &= 1+\alpha(1-\theta),\end{aligned}$$

which is the same as we would get from part b.

6. What would happen to the path of inflation if the central bank were to move from a monetary policy summarized by (4) to a monetary policy where the money growth rate follows a random walk:

$$\mu_t = \mu_{t-1} + \varepsilon_t.$$

(Assume the policy change is public and is understood by the private sector.) Would the policy multiplier computed in part (e) give the right answer to this question? If not, what is the mistake involved in that calculation, and what is the actual multiplier? Relate this result to the Lucas Critique.

Respuesta: The policy multiplier should give the wrong result, essentially because λ has changed since it's a function of θ but this won't be picked up for a long time.

$$\begin{aligned}\frac{\frac{1-\theta}{[\alpha(1-\theta)+1]}}{\frac{1}{[\alpha+1]}} &= \frac{[\alpha+1](1-\theta)}{[\alpha(1-\theta)+1]} \\ &= \frac{1+\alpha-\alpha\theta-\theta}{[1+\alpha-\alpha\theta]} \\ &< 1\end{aligned}$$

so λ is increased after the change so now inflation responds more to the monetary shock. The Lucas Critique essentially says that if individuals reoptimize in response to changes in government policy then the equilibrium response can be different than that estimated under old policy parameters. This is exactly what happens here since the money supply equation has changed, individuals immediately take this into account and change their behavior. Under adaptive expectations this change will take a long time to show up through λ.