Quantum Field Theory

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Contents

0 Introduction **9**
0.1 Notation . . . 9

I non-Relativistic world **11**

1 Quantum Mechanics **13**
1.1 Space Basis and Momentum Basis . . . 13
1.2 Different Pictures . . . 14
1.2.1 Schrodigner's Picture . . . 14
1.2.2 Heisenberg's Picture . . . 15
1.2.3 Interaction Picture . . . 15
1.3 Harmonic Oscillator . . . 17
1.4 Coherent State . . . 18
1.5 Fermionic Harmonic Oscillator . . . 20
1.6 Fermionic Coherent State . . . 21
1.7 Spin Coherent State . . . 22
1.8 Perturbation theory . . . 24

2 Path Integral **27**
2.1 The Green's Function . . . 27
2.2 Path Integral Method . . . 28
2.3 Real Space Path Integral . . . 29
2.4 Propogator of Harmonic Oscillator . . . 30
2.5 Stationary Phase Approximation . . . 31
2.6 Wick Rotation . . . 32
2.7 Instanton Gas . . . 33
2.8 Fate of the False Vacuum . . . 37
2.9 Is K Real or Imaginary? . . . 39
2.10 Path Integral for Spin . . . 40

3 Second Quantization **41**
3.1 Identical Particles . . . 41
3.2 Creation & Annihilation Operators . . . 44

3.3 Second Quantization of Operators . . . 46

4 Green's Function for Many-body System 49
4.1 Green's Function . . . 49
4.2 Propagator of Free System . . . 49
4.2.1 Free Fermions . . . 49
4.2.2 Free Bosons . . . 51
4.3 Different Green's Functions . . . 52
4.3.1 Feynman Propogator . . . 53
4.3.2 Retarded Green's Function . . . 53
4.3.3 Advanced Green's Function . . . 54
4.3.4 Relation between Different Green's Functions . . . 55
4.3.5 Spectral Function . . . 55
4.3.6 Correlation Function . . . 56
4.4 Green's Function of Interacting System, the Gell-Mann-Low Theorem . . . 56
4.5 Perturbative Expansion and Wick's Theorem . . . 57
4.5.1 Perturbative Expansion . . . 57
4.5.2 Wick's Theorem . . . 58

5 Functional Field Integral 63
5.1 Path Integral of the Scalar Field . . . 63
5.2 Path Integral of the Spinor Field . . . 64
5.3 Path Integral Representation of the Green's Function . . . 65
5.4 Generating Functional Method . . . 66
5.4.1 The Free Bosonic System . . . 66
5.4.2 The Free Fermionic System . . . 67
5.4.3 Interacting System . . . 67
5.5 The Wick's Theorem Rediscovered . . . 68

6 Green's Function at Finite Temperature 71
6.1 Matsubara Green's Function . . . 71
6.1.1 Periodicity of Green's Functions . . . 71
6.1.2 Matsubara Green's Functions of Free System . . . 72
6.2 The Contour Integral Method . . . 73
6.3 Different Green's Functions . . . 74
6.3.1 Matsubara Green's Function . . . 74
6.3.2 Finite Temperature Feynman Propogator . . . 74
6.3.3 Finite Temperature Retarded Green's Function . . . 75
6.3.4 Finite Temperature Advanced Green's Function . . . 75
6.3.5 Relation between Different Green's Functions . . . 76
6.3.6 Spectral Function . . . 76
6.3.7 Correlation Function . . . 77
6.4 Matsubara Green's Function for Interacting System . . . 77
6.4.1 The Wick's theorem . . . 78

6.5 Path Integral at Finite Temperature . . . 79

7 Feynmann Diagram 81
7.1 Basic Construction . . . 81
7.2 Feynman Rule . . . 84
7.2.1 Overall Phase Factor of Dirac Fermions . . . 85
7.2.2 Feynman Rule, an Example . . . 86
7.3 Cancelation of Vacuum Diagrams . . . 88
7.4 Linked-cluster Theorem . . . 92

8 Linear Response Theory and Dissipation-Fluctuation Theorem 93
8.1 Linear Response Theory . . . 93
8.2 Linear Response at Finite Temperature . . . 94
8.3 Dissipation-Fluctuation Theorem . . . 95

9 Boson System 97

10 Fermion System 99
10.1 Hartee-Fock Approximation . . . 99
10.1.1 Interaction Energy . . . 99
10.1.2 Exchange Correlation . . . 100
10.2 Potential Scattering . . . 101
10.3 Self Energy . . . 102
10.4 Magnetic Susceptibility of Free Electron System . . . 104
10.5 Jellium Model . . . 104

II Relativistic World 105

11 Classical Field Theory 107
11.1 Lagrangian Formalism . . . 107
11.2 Hamiltonian Formalism . . . 108
11.2.1 Generalized Hamilton's Principle . . . 109
11.2.2 Dynamical Quantity . . . 110
11.2.3 Poisson Bracket . . . 111
11.3 Symmetry . . . 112
11.3.1 Noether's Theorem . . . 113
11.3.2 Change of Action by an Arbitrary Spatial Difference . . . 115
11.3.3 Poincaré Symmetry . . . 116
11.3.4 Conserved Current of Poincaré Symmetry . . . 117
11.3.5 Scale Invariance and It's Currents . . . 119
11.3.6 Symmetry in Hamiltonian Formalism . . . 120
11.4 Klein-Gordon Scalar Field . . . 121
11.5 Electromagnetic Field . . . 123
11.6 Gauge Field . . . 129

11.6.1 Non-abelian instanton . . . 130
11.7 Spinor . . . 133
11.7.1 Connection between $SL(2,\mathbb{C})$ and $L_+^\uparrow$ . . . 133
11.7.2 Weyl Spinor . . . 134
11.7.3 Dirac Spinor and Majorana Spinor . . . 136
11.8 Spinor from Clifford Algebra* . . . 137
11.8.1 Clifford Algebra . . . 137
11.8.2 Classification of Clifford Algebra . . . 139
11.8.3 Pin Group and Spin Group . . . 142
11.8.4 Gamma Matrices and Spinor . . . 144
11.9 Spinor Field . . . 147
11.10 Dirac Field . . . 147
11.10.1 Gamma Matrices . . . 147
11.10.2 Dirac Equation . . . 148
11.10.3 Solution to Dirac equation . . . 148
11.10.4 Orthonormal relation and Spin sums . . . 149
11.10.5 Lorentz Symmetry . . . 150
11.10.6 CPT . . . 150

12 Second Quantization aka. Canonical Quantization 151

13 Path Integral Approach 153

III Conformal Field Theory 155

14 Conformal Field theory 157
14.1 Conformal Transformation . . . 157
14.1.1 Conformal Transformation when $d \geq 3$ . . . 157
14.1.2 Conformal Transformation when $d = 2$ . . . 161
14.1.3 Virasoro Algebra . . . 164
14.2 Classical Conformal Symmetry . . . 165
14.2.1 Traceless Energy-Momentum Tensor . . . 165
14.3 Correlation Function . . . 166
14.3.1 2 Point Correlation Function . . . 167
14.3.2 3 Point Correlation Function . . . 168
14.4 Ward Identity . . . 169
14.4.1 Ward Identity for Translation . . . 171
14.4.2 Ward Identity for Rotation . . . 171
14.4.3 Ward Identity for Scale Transformation . . . 172
14.4.4 Ward Identity in Harmonic Coordinates . . . 173
14.4.5 Ward Identity for Arbitrary Transformation . . . 173
14.5 Operator Product Expansion . . . 174
14.5.1 Free Bosonic System . . . 175

14.5.2 Transformation of the Energy-Momentum Tensor 176
14.6 Radial Quantization 176
14.6.1 Operator Commutator 177
14.6.2 Radial Ordering 177

A Functional derivative **179**

B Poisson Bracket of Conserved Charge **181**

C Bogliubov Transformation **185**

D Some differential geometry **187**

Chapter 0

Introduction

This is the lecture notes of condensed matter field theory for internal use in LX He's group in the Key Lab of Quantum Information, CAS.

The following is the convention of notations.

0.1 Notation

The metric of the 3+1-D Minkowski space is

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{1}$$

Latin indices i, j, k run over the three spatial coordinate labels, while Greek indices μ, ν... run over four space-time coordinate labels. Einstein's sum rule is always assumed. $x^\mu = (x^0, x^1, x^2, x^3)$ corresponds to (t, x, y, z). For 4-vector x^μ and k^μ, $k \cdot x$ means $k_\mu x^\mu$ and x^2 means $x_\mu x^\mu$.

Total anti-symmetric tensor is defined such that $\epsilon_{0123} = 1$.

For a classical field ϕ with n components, we may write ϕ_i to emphasize its ith component, or ϕ to treat the field as a whole.

The Fourier transformation is defined as

The standard spin-j representation is

$$(J_j^1)_{mn} = \frac{1}{2}(\sqrt{(j+n+1)(j-n)}\delta_{m,n+1} + \sqrt{(j+m+1)(j-m)}\delta_{m,n-1}) \tag{2}$$

$$(J_j^2)_{mn} = \frac{i}{2}(-\sqrt{(j+n+1)(j-n)}\delta_{m,n+1} + \sqrt{(j+m+1)(j-m)}\delta_{m,n-1}) \tag{3}$$

$$(J_j^3)_{mn} = m\delta_{m,n} \tag{4}$$

where $m, n = -j \cdots j$.

For example, for spin-0, $J_0^i = 0$; for spin-1/2, $J_{1/2}^i = \sigma^i/2$ and for spin-1

$$J_1^1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_1^2 = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_1^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{5}$$

where m/n runs from j to $-j$ as the row/column number increases.

When summing the nearest neighbors, $\sum_{\langle i,j \rangle}$ means that (i,j) and (j,i) are summed twice, and $\sum_{\langle i,j \rangle'}$ means that (i,j) and (j,i) are just summed once.

When treating bosons and fermions in a uniformed manner, we use a parameter η. $\eta = 1$ for boson and $\eta = -1$ for fermion.

Part I

non-Relativistic world

Chapter 1

Quantum Mechanics

The following equations are heavily used in this chapter.

Firstly

$$e^Y X e^{-Y} = \sum_{n=0} \frac{1}{n!}[Y^n X] \tag{1.1}$$

where

$$Y^n = \underbrace{[Y, [Y \cdots [Y,}_{n} \tag{1.2}$$

Secondly, we have the Baker-Campbell-Hausdorff formula

$$\log(e^X e^Y) = \sum_{n=0} \frac{(-1)^{(n-1)}}{n!} \sum_{r_i+s_i>0} \frac{[X^{r_1} Y^{s_1} \cdots X^{r_n} Y^{s_n}]}{r_i! s_1! \cdots r_n! s_n! \sum_i (r_i + s_i)} \tag{1.3}$$

where

$$[X^{r_1} Y^{s_1} \cdots X^{r_n} Y^{s_n}] = \underbrace{[X, [X \cdots [X,}_{r_1} \underbrace{[Y, [Y \cdots [Y,}_{s_1} \cdots \underbrace{[X, [X \cdots [X,}_{r_n} \underbrace{[Y, [Y \cdots Y]}_{s_n} \cdots] \tag{1.4}$$

Especially, when $[X, [X, Y]] = [Y, [X, Y]] = 0$, we have

$$\log(e^X e^Y) = X + Y + [X, Y]/2 \tag{1.5}$$

or

$$e^X e^Y = e^{X+Y+[X,Y]/2} = e^{X+Y} e^{[X,Y]/2} \tag{1.6}$$

1.1 Space Basis and Momentum Basis

We start from the complete ortho-normal basis $|x\rangle (x \in \mathbb{R})$ where $\hat{x}|x\rangle = x|x\rangle$. With the canonical commuting relation $[\hat{x}, \hat{p}] = i\hbar$, we can uniquely determine $\hat{p}$.

Use the formula Eqn. 1.1, we have

$$e^{\frac{i}{\hbar} a \cdot \hat{p}} \hat{x} e^{-\frac{i}{\hbar} a \cdot \hat{p}} = \hat{x} + [\frac{i}{\hbar} a \cdot \hat{p}, \hat{x}] = \hat{x} + a \tag{1.7}$$

So

$$\hat{x}e^{-\frac{i}{\hbar}a\cdot\hat{p}}|x\rangle = e^{-\frac{i}{\hbar}a\cdot\hat{p}}(\hat{x}+a)|x\rangle = (x+a)e^{-\frac{i}{\hbar}a\cdot\hat{p}}|x\rangle \tag{1.8}$$

So

$$e^{-\frac{i}{\hbar}a\cdot\hat{p}}|x\rangle = |x+a\rangle \tag{1.9}$$

where $\hat{p}$ is assumed to be Hermitian.

Then

$$\langle x|e^{-\frac{i}{\hbar}a\cdot\hat{p}}|x'\rangle = \delta(x-(x'+a)) \tag{1.10}$$

So

$$\langle x|p^n|x'\rangle = (i\hbar\partial_a)^n\langle x|e^{-\frac{i}{\hbar}a\cdot\hat{p}}|x'\rangle|_{a=0} = (i\hbar\partial_{x'})^n\delta(x-x') = (-i\hbar\partial_x)^n\delta(x-x') \tag{1.11}$$

Next we find the complete ortho-normal eigen-basis for $\hat{p}$. Suppose we have $\hat{p}|p\rangle = p|p\rangle$. Then

$$\langle x|\hat{p}|p\rangle = \langle x|\hat{p}|x'\rangle\langle x'|p\rangle = -i\hbar\partial_x\langle x|p\rangle = p\langle x|p\rangle \tag{1.12}$$

The solution to this equation is

$$\langle x|p\rangle = Ce^{ip\cdot x/\hbar} \tag{1.13}$$

So

$$|p\rangle = Ce^{ip\cdot x/\hbar}|x\rangle \tag{1.14}$$

Let $C = \frac{1}{\sqrt{2\pi\hbar}}$, we have $\langle p|p'\rangle = \delta(p-p')$.

So a complete ortho-normal eigen-basis for $\hat{p}$ is

$$|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{ip\cdot x/\hbar}|x\rangle \tag{1.15}$$

One should not be confused by the normality of $|x\rangle$ and $|p\rangle$. Since $\langle x|x\rangle = \infty \neq 1$ and $\langle p|p\rangle = \infty \neq 1$. They alone are not physical states, though one can construct a state very localized at some x or p.

1.2 Different Pictures

1.2.1 Schrodigner's Picture

Let $O_S(t)$ and $|\phi_S(t)\rangle$ be time-dependent operator and state in Schrodigner's picture.

If $|\phi_S(t)\rangle$ is on-shell, we have

$$|\phi_S(t)\rangle = U_H(0\to t)|\phi_S(0)\rangle, \tag{1.16}$$

where $U_H(t\to 0)$ is the time-evolve operator under H from 0 to t, which satisfies

$$\begin{cases} \frac{d}{dt_2}U_H(t_1\to t_2) = -iH_S(t_2)U_H(t_1\to t_2), \\ U_H(t\to t) = I. \end{cases} \tag{1.17}$$

In integral form, we have

$$U_H(t_1\to t_2) \quad = \quad I + \int_{t_1}^{t_2} d\tau\frac{d}{d\tau}U_H(t_1\to\tau) \tag{1.18}$$

$$
\begin{aligned}
&= I + (-i)\int_{t_1}^{t_2} d\tau H_S(\tau)U_H(t_1 \to \tau) && (1.19)\\
&= I + (-i)\int_{t_1}^{t_2} d\tau_1 H_S(\tau_1) + \cdots + (-i)^n \int_{t_1}^{t_2} d\tau_1 \int_{t_1}^{\tau_1} d\tau_2 \cdots \int_{t_1}^{\tau_{n-1}} d\tau_n \\
&\quad H_S(\tau_1)\cdots H_S(\tau_n) + \cdots && (1.20)\\
&= I + (-i)\int_{t_1}^{t_2} d\tau_1 H_S(\tau_1) + \cdots + \frac{(-i)^n}{n!}\int_{t_1}^{t_2} d\tau_1 \cdots \int_{t_1}^{t_2} d\tau_n \\
&\quad \mathcal{T}[H_S(\tau_1)\cdots H_S(\tau_n)] + \cdots && (1.21)\\
&= \mathcal{T}[exp(-i\int_{t_1}^{t_2} d\tau H_S(\tau))] && (1.22)
\end{aligned}
$$

A static operator should satisfy

$$\dot{O}_S(t) = 0 \tag{1.23}$$

At finite temperature, the evolvement of the density matrix is

$$\rho_S(t) = U_H(0 \to t)\rho_S(t)U_H(t \to 0) \tag{1.24}$$

The equation of motion is

$$i\dot{\rho}_S(t) = [H_S(t), \rho_S(t)] \tag{1.25}$$

1.2.2 Heisenberg's Picture

With reference to Hamiltonian $H(t)$, the operator and state in Heisenberg's picture would be

$$
\begin{aligned}
O_H(t) &= U_H(t \to 0)O_S(t)U_H(0 \to t) && (1.26)\\
\phi_H(t) &= U_H(t \to 0)\phi_S(t) && (1.27)
\end{aligned}
$$

We have

$$\langle \xi_H(t)|O_H(t)|\phi_H(t)\rangle = \langle \xi_S(t)|O_S(t)|\phi_S(t)\rangle \tag{1.28}$$

If $|\phi_H(t)\rangle$ is on-shell, we have

$$|\phi_H(t)\rangle = |\phi_H(0)\rangle. \tag{1.29}$$

A static operator should satisfy

$$\dot{O}_H(t) = -i[O_H(t), H_H(t)]. \tag{1.30}$$

At finite temperature, density matrix in Heisenberg's picture is static.

1.2.3 Interaction Picture

Let $H = H_0 + H_{int}$ where H_0 is viewed as the free part and H_{int} is viewed as the interaction part. We define the operator and state in Heisenberg's picture as

$$O_I(t) = U_{H_0}(t \to 0)O_S(t)U_{H_0}(0 \to t) \tag{1.31}$$

$$\phi_I(t) \quad = \quad U_{H_0}(t \to 0)\phi_S(t) \tag{1.32}$$

We have

$$\langle \xi_I(t)|O_I(t)|\phi_I(t)\rangle = \langle \xi_S(t)|O_S(t)|\phi_S(t)\rangle \tag{1.33}$$

We may define the time-evolve operator in interaction picture as:

$$S_H(0 \to t) = U_{H_0}(t \to 0)U_H(0 \to t) \tag{1.34}$$

We may also define

$$\begin{aligned} S_H(t_1 \to t_2) &= S_H(0 \to t_2)S_H^\dagger(0 \to t_1) && (1.35)\\ &= U_{H_0}(t_2 \to 0)U_H(t_1 \to t_2)U_{H_0}(0 \to t_1) && (1.36) \end{aligned}$$

Obviously

$$S_H(t_2 \to t_3)S_H(t_1 \to t_2) = S_H(t_1 \to t_3) \tag{1.37}$$

It's easy to see that $S_H(t_1 \to t_2)$ satisfies

$$\begin{cases} \frac{d}{dt_2}S_H(t_1 \to t_2) = -iH_{int,I}(t_2)S_H(t_1 \to t_2), \\ S_H(t \to t) = I. \end{cases} \tag{1.38}$$

In integral form, we have similarly

$$S_H(t_1 \to t_2) = \mathcal{T}[exp(-i\int_{t_1}^{t_2} d\tau H_{int,I}(\tau))] \tag{1.39}$$

It's also useful to note

$$O_H(t) = S_H(t \to 0)O_I(t)S_H(0 \to t) \tag{1.40}$$

If $|\phi_I(t)\rangle$ is on-shell, we have

$$\begin{aligned} |\phi_I(t)\rangle &= U_{H_0}(t \to 0)U_H(0 \to t)|\phi_I(0)\rangle && (1.41)\\ &= S_H(0 \to t)|\phi_I(0)\rangle. && (1.42) \end{aligned}$$

A static operator should satisfy

$$\dot{O}_I(t) = -i[O_I(t), H_{0,I}(t)]. \tag{1.43}$$

At finite temperature, the evolvement of the density matrix is

$$\rho_I(t) = U_{H_0}(0 \to t)\rho_S(t)U_{H_0}(t \to 0) \tag{1.44}$$

The equation of motion is

$$i\dot{\rho}_I(t) = [H_{int,I}(t), \rho_I(t)] \tag{1.45}$$

1.3 Harmonic Oscillator

The Hamiltonian reads

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 \tag{1.46}$$

$$= \omega(\sqrt{\frac{m\omega}{2}}x - i\frac{p}{\sqrt{2m\omega}})(\sqrt{\frac{m\omega}{2}}x + i\frac{p}{\sqrt{2m\omega}}) + \frac{\omega}{2} \tag{1.47}$$

let $a = \sqrt{\frac{m\omega}{2}}x + i\frac{p}{\sqrt{2m\omega}}$, we have

$$[a, a^\dagger] = -i[x, p] = 1 \tag{1.48}$$

So $H = \omega(a^\dagger a + \frac{1}{2})$.

Define $N = a^\dagger a$. Then the eigenvectors of H are also eigenvectors of N. We label them by $|n, \lambda\rangle$, where

$$N|n, \lambda\rangle = n|n, \lambda\rangle \tag{1.49}$$

and λ labels the degeneracy.

Since

$$||a|n, \lambda\rangle|| = \langle n, \lambda|a^\dagger a|n, \lambda\rangle = n \tag{1.50}$$

we have $n \geq 0$ and $a|0, \lambda\rangle = 0$.

It's easy to check that

$$[N, a] = -a \tag{1.51}$$

$$[N, a^\dagger] = a^\dagger \tag{1.52}$$

So one can derive that

$$a^{\dagger n}a^n = \prod_{i=0}^{n-1}(N - i) \tag{1.53}$$

So if $n \geq 0$ is not an integer, one have m such that

$$||a^m|n, \lambda\rangle|| = \langle n, \lambda|a^{\dagger m}a^m|n, \lambda\rangle = \prod_{i=0}^{m-1}(n - i) < 0 \tag{1.54}$$

So for the eigenvectors $|n, \lambda\rangle = 0$, we have that $n \geq 0$ and n are integers.

When $n > 0$, we have

$$Na|n, \lambda\rangle = (n-1)a|n, \lambda\rangle \tag{1.55}$$

$$||a|n, \lambda\rangle|| = n \tag{1.56}$$

So $\frac{1}{\sqrt{n}}a|n, \lambda\rangle$ is in the space of $n - 1$.

Similarly when $n \geq 0$ $\frac{1}{\sqrt{n+1}}a^\dagger|n, \lambda\rangle$ is in the space of $n + 1$.

Besides, when $n > 0$, $a^\dagger a|n, \lambda\rangle = n|n, \lambda\rangle$. And when $n \geq 0$, $aa^\dagger|n, \lambda\rangle = (n+1)|n, \lambda\rangle$.

From the facts above, we can deduce that the spaces of different n have the same dimension.

Let's analysis the space of $n = 0$. We have

$$\begin{aligned}
ax + ibp|0, \lambda\rangle &= 0 && (1.57)\\
\langle x|ax + ibp|0, \lambda\rangle &= 0 && (1.58)\\
(ax + b\partial_x)\langle x|0, \lambda\rangle &= 0 && (1.59)\\
\partial_x(e^{\frac{ax^2}{2b}}\langle x|0, \lambda\rangle) &= 0 && (1.60)\\
e^{\frac{ax^2}{2b}}\langle x|0, \lambda\rangle &= C && (1.61)\\
\langle x|0, \lambda\rangle &= Ce^{-\frac{ax^2}{2b}} && (1.62)
\end{aligned}$$

where $a = \sqrt{\frac{m\omega}{2}}$ and $b = \frac{p}{\sqrt{2m\omega}}$.

After normalization, we have

$$\langle x|0, \lambda\rangle = \left(\frac{m\omega}{\pi}\right)^{1/4} e^{-\frac{m\omega}{2}x^2} \tag{1.63}$$

So the dimension of the space with $n = 0$ is 1. So the dimension of the space with each n is 1. So we may drop the λ label.

So $\frac{1}{\sqrt{n+1}}a^\dagger|n\rangle$ and $|n + 1\rangle$ differ by a phase factor. We may redefine $|n\rangle$ by

$$|n\rangle = \frac{a^{\dagger n}}{\sqrt{n!}}|0\rangle \tag{1.64}$$

Then

$$\begin{aligned}
a^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle && (1.65)\\
a|n\rangle &= \sqrt{n}|n-1\rangle && (1.66)
\end{aligned}$$

In Heisenberg's picture

$$\begin{aligned}
a(t) &= e^{iHt}ae^{-iHt} && (1.67)\\
&= \sum_n \frac{1}{n!}[(iHt)^n, a] && (1.68)\\
&= \sum_n \frac{1}{n!}(-i\omega t)^n a && (1.69)\\
&= e^{-i\omega t}a && (1.70)\\
a^\dagger(t) &= e^{i\omega t}a^\dagger && (1.71)
\end{aligned}$$

1.4 Coherent State

Let a be an annihilation operator. We call $|\alpha\rangle(\alpha \in \mathbb{C})$ a coherent state if

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{1.72}$$

Clearly $|0\rangle$ is a coherent state. Too find more coherent states, we define the displacement operator $D(\alpha)$ such that $D(\alpha)$ is unitary and

$$D(\alpha)^\dagger a D(\alpha) = a + \alpha \tag{1.73}$$

Since $[a, a^\dagger] = 1$, we have

$$e^{-\alpha a^\dagger + \beta a} a e^{\alpha a^\dagger - \beta a} = a + \alpha \tag{1.74}$$

For unitarity, we let $\beta = \alpha^*$. So a possible choice of $D(\alpha)$ is

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a} \tag{1.75}$$

It's easy to see that

$$|\alpha\rangle = D(\alpha)|0\rangle \tag{1.76}$$

Clearly

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a} = e^{-|a|^2/2} e^{\alpha a^\dagger} e^{-\alpha^* a} = e^{|a|^2/2} e^{-\alpha^* a} e^{\alpha a^\dagger} \tag{1.77}$$

So

$$\begin{aligned} |\alpha\rangle &= e^{-|a|^2/2} e^{\alpha a^\dagger} |0\rangle & (1.78) \\ &= e^{-|a|^2/2} \sum_n \frac{(\alpha a^\dagger)^n}{n!} |0\rangle & (1.79) \\ &= e^{-|a|^2/2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle & (1.80) \end{aligned}$$

It's easy to see that $D(\alpha)D(\beta) = e^{(\alpha\beta^* - \alpha^*\beta)/2} D(\alpha + \beta)$ and $D(\alpha)^\dagger = D(-\alpha)$.
Here are some properties of the coherent states:

1. $\langle\alpha|a|\alpha\rangle = \alpha$, $\langle\alpha|a^\dagger|\alpha\rangle = \alpha^*$, $\langle\alpha|n|\alpha\rangle = |\alpha|^2$.

2. $\langle\alpha|x|\alpha\rangle = \frac{\alpha+\alpha^*}{\sqrt{2m\omega}}$, $\langle\alpha|p|\alpha\rangle = \frac{\alpha-\alpha^*}{2i}\sqrt{2m\omega}$

3. $\Delta x = \frac{1}{\sqrt{2m\omega}}$, $\Delta p = \sqrt{\frac{m\omega}{2}}$, $\Delta x \Delta p = \frac{1}{2}$.

4.
$$\begin{aligned} \langle\alpha'|\alpha\rangle &= \langle 0|D(-\alpha')D(\alpha)|0\rangle & (1.81) \\ &= e^{(-\alpha'\alpha^* + \alpha'^*\alpha)/2} \langle 0|D(\alpha - \alpha')|0\rangle & (1.82) \\ &= e^{(-\alpha'\alpha^* + \alpha'^*\alpha)/2} e^{-|\alpha-\alpha'|^2/2} & (1.83) \\ &= e^{-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\alpha'|^2 + \alpha'^*\alpha} & (1.84) \end{aligned}$$

5. $\frac{1}{\pi}\int d\alpha^2 |\alpha\rangle\langle\alpha| = 1$. The integration is carried out over the entire complex plane.

6.

$$\begin{aligned}
\langle x|\alpha\rangle &= \langle x|D(\alpha)|0\rangle && (1.85)\\
&= \langle x|e^{\sqrt{\frac{m\omega}{2}}(\alpha-\alpha^*)x-i\frac{\alpha+\alpha^*}{\sqrt{2m\omega}}p}|0\rangle && (1.86)\\
&= e^{\frac{1}{4}(\alpha^2-\alpha^{*2})}\langle x|e^{-i\frac{\alpha+\alpha^*}{\sqrt{2m\omega}}p}e^{\sqrt{\frac{m\omega}{2}}(\alpha-\alpha^*)x}|0\rangle && (1.87)\\
&= e^{\frac{1}{4}(\alpha^2-\alpha^{*2})}\left\langle x-\frac{\alpha+\alpha^*}{\sqrt{2m\omega}}\right|e^{\sqrt{\frac{m\omega}{2}}(\alpha-\alpha^*)x}\left|0\right\rangle && (1.88)\\
&= e^{\frac{1}{4}(\alpha^2-\alpha^{*2})}e^{\sqrt{\frac{m\omega}{2}}(\alpha-\alpha^*)(x-\frac{\alpha+\alpha^*}{\sqrt{2m\omega}})}\left\langle x-\frac{\alpha+\alpha^*}{\sqrt{2m\omega}}\right|\left.0\right\rangle && (1.89)\\
&= e^{\frac{1}{4}(\alpha^2-\alpha^{*2})}e^{\sqrt{\frac{m\omega}{2}}(\alpha-\alpha^*)(x-\frac{\alpha+\alpha^*}{\sqrt{2m\omega}})}\left(\frac{m\omega}{\pi}\right)^{1/4}e^{-\frac{m\omega}{2}(x-\frac{\alpha+\alpha^*}{\sqrt{2m\omega}})^2} && (1.90)\\
&= \left(\frac{m\omega}{\pi}\right)^{1/4}e^{-\frac{m\omega}{2}(x-\frac{2\alpha}{\sqrt{2m\omega}})^2+\frac{\alpha^2-|\alpha|^2}{2}} && (1.91)
\end{aligned}$$

Now let's study how coherent states evolve with time. It's easy to check that

$$e^{-iHt}D(\alpha)e^{iHt}=e^{\alpha e^{-iHt}a^\dagger e^{iHt}-\alpha^*e^{-iHt}ae^{iHt}}=e^{\alpha e^{-i\omega t}a^\dagger-\alpha^*e^{i\omega t}a}=D(e^{-i\omega t}\alpha) \tag{1.92}$$

So

$$e^{-iHt}|\alpha\rangle=e^{-iHt}D(\alpha)e^{iHt}e^{-iHt}|0\rangle=D(e^{-i\omega t}\alpha)e^{-i\omega t/2}|0\rangle=e^{-i\omega t/2}|e^{-i\omega t}\alpha\rangle \tag{1.93}$$

1.5 Fermionic Harmonic Oscillator

Let

$$H=Ec^\dagger c \tag{1.94}$$

where

$$\begin{aligned}
\{c,c^\dagger\} &= 1 && (1.95)\\
\{c,c\}=\{c^\dagger,c^\dagger\} &= 0 && (1.96)
\end{aligned}$$

Define $N=a^\dagger a$. As in the bosonic case, we label eigenvectors of H them by $|n,\lambda\rangle$, where

$$N|n,\lambda\rangle=n|n,\lambda\rangle \tag{1.97}$$

and λ labels the degeneracy.

We have

$$\begin{aligned}
N^2|n,\lambda\rangle &= n^2|n,\lambda\rangle && (1.98)\\
&= c^\dagger cc^\dagger c|n,\lambda\rangle && (1.99)\\
&= c^\dagger(1-c^\dagger c)c|n,\lambda\rangle && (1.100)\\
&= (N-c^\dagger c^\dagger cc)|n,\lambda\rangle && (1.101)\\
&= N|n,\lambda\rangle && (1.102)
\end{aligned}$$

$$= \quad n|n,\lambda\rangle \tag{1.103}$$

So $n = 0$ or 1.

As in the bosonic case, we ha ve

$$\begin{aligned} [N,c] &= -c && (1.104)\\ [N,c^\dagger] &= c^\dagger && (1.105) \end{aligned}$$

Then we can deduce that the spaces of different n have the same dimension.

Let's assume the space that $n = 0$ is non-degenerate, with basis $|0\rangle$.

Then define $|1\rangle = c^\dagger|0\rangle$. Clearly

$$\begin{aligned} c^\dagger|1\rangle &= 0 && (1.106)\\ c|1\rangle &= |0\rangle && (1.107)\\ c|0\rangle &= 0 && (1.108) \end{aligned}$$

In Heisenberg's picture we have

$$\begin{aligned} c(t) &= e^{-iEt}c && (1.109)\\ c^\dagger(t) &= e^{iEt}c^\dagger && (1.110) \end{aligned}$$

1.6 Fermionic Coherent State

Let η and $\bar{\eta}$ be two independent Grassman numbers. We should be very cautious here since the Grassman algebra is not an integral domian.

Let's define

$$\begin{aligned} |\eta\rangle &= e^{-\eta c^\dagger}|0\rangle && (1.111)\\ \langle\bar{\eta}| &= \langle 0|e^{\bar{\eta}c} && (1.112) \end{aligned}$$

where the exponential is defined by power series.

It's easy to see that

$$\begin{aligned} |\eta\rangle &= (1-\eta c^\dagger)|0\rangle = |0\rangle - \eta|1\rangle && (1.113)\\ \langle\bar{\eta}| &= \langle 0|(1+\bar{\eta}c) = \langle 0| + \bar{\eta}\langle 1| && (1.114) \end{aligned}$$

and

$$\begin{aligned} c|\eta\rangle &= \eta|\eta\rangle && (1.115)\\ \langle\bar{\eta}|c^\dagger &= \bar{\eta}\langle\bar{\eta}| && (1.116) \end{aligned}$$

Then

$$\begin{aligned} \langle\bar{\eta}|\eta\rangle &= \langle 0|0\rangle - \bar{\eta}\langle 1|\eta|1\rangle && (1.117)\\ &= 1+\bar{\eta}\eta && (1.118) \end{aligned}$$

$$= e^{\bar{\eta}\eta} \tag{1.119}$$

and

$$\begin{aligned}
\int d\bar{\eta}d\eta e^{-\bar{\eta}\eta}|\eta\rangle\langle\bar{\eta}| &= \int d\bar{\eta}d\eta(1-\bar{\eta}\eta)(|0\rangle - \eta|1\rangle)(\langle 0| + \bar{\eta}\langle 1|) && (1.120)\\
&= \int d\bar{\eta}d\eta(1-\bar{\eta}\eta)(|0\rangle\langle 0| - \eta|1\rangle\langle 0| + \bar{\eta}|0\rangle\langle 1| + \eta\bar{\eta}|1\rangle\langle 1|) && (1.121)\\
&= \int d\bar{\eta}d\eta((1-\bar{\eta}\eta)|0\rangle\langle 0| - \eta|1\rangle\langle 0| + \bar{\eta}|0\rangle\langle 1| + \eta\bar{\eta}|1\rangle\langle 1|) && (1.122)\\
&= |0\rangle\langle 0| + |1\rangle\langle 1| && (1.123)\\
&= I && (1.124)
\end{aligned}$$

For a system with n fermion modes, since $\eta c^\dagger$ and $\bar{\eta}c$ are bosonic operators, we can define the fermionic coherent state by

$$\begin{aligned}
|\eta_1 \ldots \eta_n\rangle &= \prod_i e^{-\eta_i c_i^\dagger}|0\rangle && (1.125)\\
\langle\bar{\eta}_1 \ldots \bar{\eta}_n| &= \langle 0|\prod_i e^{\bar{\eta}_i c_i} && (1.126)
\end{aligned}$$

Then similarly

$$\begin{aligned}
\langle\bar{\eta}_1 \ldots \bar{\eta}_n|\eta_1 \ldots \eta_n\rangle &= e^{\sum_i \bar{\eta}_i\eta_i} && (1.127)\\
\int\prod_i(d\bar{\eta}_i d\eta_i)e^{-\sum_i \bar{\eta}_i\eta_i}|\eta_1 \ldots \eta_n\rangle\langle\bar{\eta}_1 \ldots \bar{\eta}_n| &= I && (1.128)
\end{aligned}$$

1.7 Spin Coherent State

Let $|sn\rangle$ be the state of the maximal weight of S_z. For each $g \in SU(2)$, we define

$$|g\rangle = g|s\rangle \tag{1.129}$$

Under Haar measure, we have

$$\int dg|g\rangle\langle g| = CI \tag{1.130}$$

Taking trace on both sides, we have $C = \frac{D}{V}$, where D is the dimension of the group representation and $V = \int dg1$.

Recall that $Ad : SU(2) \mapsto SO(3)$ is a 2 to 1 covering map and a group homomorphism. Note that $Ad_g\hat{S}_i = g\hat{S}_i g^{-1} = (Ad_g)_{ji}\hat{S}_j$.

For each g, define

$$\vec{n}_g = Ad_g \cdot \vec{z} \tag{1.131}$$

Then

$$\langle g|\hat{S}_i|g\rangle = \langle s|g^{-1}\hat{S}_i g|s\rangle \tag{1.132}$$

$$\begin{aligned}
&= \langle s|Ad_{g^{-1}}\hat{S}_i|s\rangle && (1.133)\\
&= \langle s|(Ad_{g^{-1}})_{ji}\hat{S}_j|s\rangle && (1.134)\\
&= (Ad_{g^{-1}})_{3i} && (1.135)\\
&= (Ad_g)_{i3} && (1.136)\\
&= Ad_g \cdot \vec{z} && (1.137)\\
&= \vec{n}_g && (1.138)
\end{aligned}$$

and

$$\begin{aligned}
\vec{n}_g \cdot \hat{S}|g\rangle &= \vec{n}_g \cdot \hat{S}g|s\rangle && (1.139)\\
&= gg^{-1}\vec{n}_g \cdot \hat{S}g|s\rangle && (1.140)\\
&= g\vec{n}_g \cdot Ad_{g^{-1}}\hat{S}|s\rangle && (1.141)\\
&= g(\vec{n}_g)_i(Ad_{g^{-1}})_{ji}\hat{S}_j|s\rangle && (1.142)\\
&= g(Ad_{g^{-1}}\vec{n}_g) \cdot \hat{S}|s\rangle && (1.143)\\
&= g\vec{z} \cdot \hat{S}|s\rangle && (1.144)\\
&= g\hat{S}_z|s\rangle && (1.145)\\
&= S|g\rangle && (1.146)
\end{aligned}$$

SU(2) Lie algebra can be represented by Schwinger boson

$$\begin{aligned}
S_+ &= a_1^\dagger a_2 && (1.147)\\
S_- &= a_2^\dagger a_1 && (1.148)\\
S_z &= \frac{1}{2}(a_1^\dagger a_1 - b_2^\dagger b_2) && (1.149)
\end{aligned}$$

It's easy to see that

$$|S,m\rangle = \frac{(a_1^\dagger)^{S+m}}{\sqrt{(S+m)!}}\frac{(a_2^\dagger)^{S-m}}{\sqrt{(S-m)!}}|0\rangle \qquad (1.150)$$

$(a_1^\dagger, a_2^\dagger)$ is closed under the adjoint operation of $SU(2)$, and

$$Ad_g a_i^\dagger = g a_i^\dagger g^{-1} = T(g)_{ji} a_j^\dagger \qquad (1.151)$$

where T is the standard 2D representation.

So

$$\begin{aligned}
|g\rangle &= g|s\rangle && (1.152)\\
&= g\frac{(a_1^\dagger)^{2S}}{\sqrt{(2S)!}}|0\rangle && (1.153)\\
&= \frac{(T(g)_{j1}a_j^\dagger)^{2S}}{\sqrt{(2S)!}}|0\rangle && (1.154)
\end{aligned}$$

$$= \sqrt{(2S)!}\sum_m \frac{T(g)_{11}^{S+m}T(g)_{21}^{S-m}}{\sqrt{(S+m)!}\sqrt{(S-m)!}}|S,m\rangle \tag{1.155}$$

So

$$\langle g'|g\rangle = (T(g')_{11}^* T(g)_{11} + T(g')_{21}^* T(g)_{21})^{2S} = (T(g')^\dagger T(g))_{11} \tag{1.156}$$

If we parameterize $SU(2)$ by

$$g(\phi,\theta,\chi) = e^{-i\phi\hat{S}_3}e^{-i\theta\hat{S}_2}e^{-i(\chi-\phi)\hat{S}_3} \tag{1.157}$$

$$= \begin{pmatrix} e^{-i\phi/2} & \\ & e^{i\phi/2}\end{pmatrix}\begin{pmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}\begin{pmatrix} e^{-i(\chi-\phi)/2} & \\ & e^{i(\chi-\phi)/2}\end{pmatrix} \tag{1.158}$$

$$= \begin{pmatrix} e^{-i\chi/2}\cos\frac{\theta}{2} & -e^{-i\phi}e^{i\chi/2}\sin\frac{\theta}{2} \\ e^{i\phi}e^{-i\chi/2}\sin\frac{\theta}{2} & e^{i\chi/2}\cos\frac{\theta}{2}\end{pmatrix} \tag{1.159}$$

Then

$$n_g = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \tag{1.160}$$

$$T(g)_{i1} = e^{-i\chi/2}(\cos\frac{\theta}{2}, e^{i\phi}\sin\frac{\theta}{2}) \tag{1.161}$$

So

$$\langle g'|g\rangle = (T(g')^\dagger T(g))_{11} \tag{1.162}$$

$$= \left(\frac{1+n_{g'}\cdot n_g}{2}\right)^S e^{iS\left((\chi'-\chi)+2\arctan\frac{\sin\theta\sin\theta'\sin(\phi-\phi')}{(1+\cos\theta)(1+\cos\theta')+\sin\theta\sin\theta'\cos(\phi-\phi')}\right)} \tag{1.163}$$

$$= \left(\frac{1+n_{g'}\cdot n_g}{2}\right)^S e^{iS((\chi'-\chi)+\Phi(\vec{z},n_g,n_{g'}))} \tag{1.164}$$

where $\Phi(\vec{z},n_g,n_{g'})$ is the area of the spherical triangle with vertices $\vec{z},n_g,n_{g'}$.

A Haar measure is $\sin\theta d\theta d\phi d\chi$.

1.8 Perturbation theory

The Schrödinger equation reads

$$i\frac{\partial}{\partial t}|\Psi\rangle = H|\Psi\rangle \tag{1.165}$$

Suppose $H = H_0 + H_1$ and doesn't t explicitly.

We can first diagonalize H:

$$H|\psi_n\rangle = E_n|\psi_n\rangle \tag{1.166}$$

Then the general solution of $|\Psi\rangle$ is

$$|\Psi\rangle = C_n e^{-iE_n t}|\psi_n\rangle \tag{1.167}$$

The equation Eqn.1.166 can be reformulated as

$$(E_n - H_0)|\psi_n\rangle = H_1|\psi_n\rangle \tag{1.168}$$

Let's define the free Green's operator

$$G_{0E}^{\pm} = \frac{1}{E - H_0 \pm i\epsilon} \tag{1.169}$$

where we have used ϵ to avoid singularity.

Suppose both H and H_0 has a continuous spectrum, and we label the eigen-state of H_0 as

$$H_0|\phi_n\rangle = E_n|\phi_n\rangle \tag{1.170}$$

We have

$$(E_n - H_0)(|\psi_n\rangle - |\phi_n\rangle) = H_1|\phi_n\rangle \tag{1.171}$$

$$|\psi_n\rangle - |\phi_n\rangle = G_{0E_n}^{\pm} H_1|\phi_n\rangle \tag{1.172}$$

$$|\psi_n\rangle = |\phi_n\rangle + G_{0E_n}^{\pm} H_1|\psi_n\rangle \tag{1.173}$$

$$= |\phi_n\rangle + G_{0E_n}^{\pm} H_1|\phi_n\rangle + G_{0E_n}^{\pm} H_1 V G_{0E_n}^{\pm} H_1|\phi_n\rangle + \cdots \tag{1.174}$$

If we assume V to be a small value of the 1st order, then this is a series expansion.

Let's define

$$G_E^{\pm} = G_{0E}^{\pm} + G_{0E}^{\pm} H_1 G_{0E}^{\pm} + \cdots \tag{1.175}$$

It's easy to see that

$$G_E^{\pm} = G_{0E}^{\pm} + G_{0E}^{\pm} H_1 G_E^{\pm} \tag{1.176}$$

$$G_E^{\pm} = \frac{1}{E - H \pm i\epsilon} \tag{1.177}$$

$$|\psi_n\rangle = |\phi_n\rangle + G_{E_n}^{\pm} H_1|\phi_n\rangle \tag{1.178}$$

Chapter 2

Path Integral

2.1 The Green's Function

Suppose a quantum system has a complete ortho-normal basis $|i\rangle$ labeled by i, which takes value in some (at least interpreted as) classical space that we can take path in.

As we know, the dynamics of a system is encoded in the time evolving operator $U(t_i \to t_j)$. Sometimes we prefer to express the time evolving operator in some representation

$$\langle j|U(t_i \to t_j)|i\rangle \tag{2.1}$$

The Green's function is defined by

$$G(j, t_j; i, t_i) = (-i)\langle j|U(t_i \to t_j)|i\rangle\theta(t_j - t_i) \tag{2.2}$$

The Green's function satisfies the equation

$$(i\partial_{t_j} - \hat{H})G(j, t_j; i, t_i) = \delta(t_j - t_i) \tag{2.3}$$

which can be used as the definition.

Let $K(j, t_j; i, t_i) = \langle j|U(t_i \to t_j)|i\rangle$. Then $G(j, t_j; i, t_i) = -iK(j, t_j; i, t_i)\theta(t_j - t_i)$

When the Hamiltonian is time independent, G and K only depend on $t_j - t_i$. So we may write $G(j, t_j; i, t_i) = G(j, i, t_j - t_i)$ and $K(j, t_j; i, t_i) = K(j, i, t_j - t_i)$.

Suppose the Hamiltonian H is time-independent and can be diagonalized

$$H|n\rangle = E_n|n\rangle \tag{2.4}$$

Then

$$U(t_i \to t_j) = \sum_n e^{-iE_n(t_j - t_i)}|n\rangle\langle n| \tag{2.5}$$

So

$$G(j, t_j; i, t_i) = (-i)\sum_n e^{-iE_n(t_j - t_i)}\langle j|n\rangle\langle n|i\rangle\theta(t_j - t_i) \tag{2.6}$$

For example, let $H = \frac{p^2}{2m}$ be the Hamiltonian of free particle (in 3D). We can diagonalize the Hamiltonian as

$$H|p\rangle = \frac{p^2}{2m}|p\rangle \tag{2.7}$$

and

$$\langle p|x\rangle = \frac{1}{(2\pi)^{3/2}}e^{ip\cdot x} \tag{2.8}$$

So

$$\begin{aligned} G(x_2,t_2;x_1,t_1) &= (-i)\int dp\frac{1}{(2\pi)^3}e^{ip\cdot(x_1-x_2)-i\frac{p^2}{2m}(t_2-t_1)}\theta(t_j-t_i) && (2.9) \\ &= (-i)\Big(\frac{m}{i\pi(t_2-t_1)}\Big)^{3/2}e^{\frac{im(x_2-x_1)^2}{t_2-t_1}}\theta(t_j-t_i) && (2.10) \end{aligned}$$

2.2 Path Integral Method

If we have

$$\langle j|U(t\to t+\Delta t)|i\rangle = C[e^{iL(i,\partial_0 i,t)\Delta t}+\mathcal{O}(\Delta t^2)] \tag{2.11}$$

for a short time period Δt, where

$$\partial_0 i = \frac{j-i}{\Delta t} \tag{2.12}$$

to the 1st order of Δt, and C is a constant.

Thus

$$\begin{aligned} \langle j|U(t_i\to t_j)|i\rangle &= C^N\langle j|U(t_N\to t_j)|x_N\rangle\cdots\langle x_2|U(t_1\to t_2)|x_1\rangle\langle x_1|U(t_i\to t_1)|i\rangle && (2.13) \\ &= C^N\int\prod dx_i e^{\sum_n iL(x_n,\partial_0 x_n,t_n)\Delta t} && (2.14) \\ &= C^N\int\mathcal{D}x e^{i\int_{t_i}^{t_j}dtL(x,\partial_0 x,t)} && (2.15) \end{aligned}$$

That is, the Green's function from $|i\rangle$ at t_i to $|j\rangle$ at t_j is the sum of phases of every possible path from i at t_i to j at t_j.

Sometimes we want to calculate

$$\langle j|U(t_j,t_m)OU(t_m,t_i)|i\rangle\ (t_j>t_m>t_i) \tag{2.16}$$

Suppose

$$O|m\rangle\langle m| \text{ or } |m\rangle\langle m|O = O(m)|m\rangle\langle m| \tag{2.17}$$

then

$$\begin{aligned} \langle j|U(t_j,t_m)OU(t_m,t_i)|i\rangle &= \langle j|U(t_j,t_m)|m\rangle O(m)\langle m|U(t_m,t_i)|i\rangle && (2.18) \\ &= C\int dm\int\mathcal{D}x e^{i\int_{t_m}^{t_j}dtL(x,\partial_0 x,t)}O(m)\int\mathcal{D}x e^{i\int_{t_i}^{t_m}dtL(x,\partial_0 x,t)} && (2.19) \end{aligned}$$

$$= \quad C \int \mathcal{D}x O(x(t_m)) e^{i \int_{t_i}^{t_j} dt L(x, \partial_0 x, t)} \tag{2.20}$$

We may construct a path integral

$$Z[J] = C \int \mathcal{D}x e^{i \int_{t_i}^{t_j} dt L(x, \partial_0 x, t) + O(x) J(t)} \tag{2.21}$$

This is just the path integral for the Hamiltonian $H + OJ(t)$. Then obviously

$$C \int \mathcal{D}x O(x(t_m)) e^{i \int_{t_i}^{t_j} L(x, \partial_0 x, t)} = \left. \frac{\delta Z[J]}{\delta i J(t_m)} \right|_{J=0} \tag{2.22}$$

Similarly

$$\langle j | U(t_j, 0) T[\prod_n O_H(t_n)] U(0, t_i) | i \rangle \quad = \quad C \int \mathcal{D}x \prod_n O(x(t_n)) e^{i \int_{t_i}^{t_j} dt L(x, \partial_0 x, t)} \tag{2.23}$$

$$= \quad \prod_n \frac{\delta}{\delta i J(t_n)} Z[J] |_{J=0} \tag{2.24}$$

where $t_i < t_n < t_j$

2.3 Real Space Path Integral

For a single particle Hilbert space, the decomposition of identity is

$$1 = \int dx |x\rangle\langle x| = \int \frac{dp}{2\pi} |p\rangle\langle p| \tag{2.25}$$

For the Hamiltonian

$$H = \frac{p^2}{2m} + U(x) \tag{2.26}$$

Then

$$\langle x_j | U(t_i \to t_j) | x_i \rangle \quad = \quad \langle x_j | e^{-iH\Delta t} | x_i \rangle \tag{2.27}$$

$$= \quad \int \frac{dp}{2\pi} \langle x_j | p \rangle \langle p | e^{-iH\Delta t} | x_i \rangle \tag{2.28}$$

$$= \quad \int \frac{dp}{2\pi} e^{ip \cdot x_j} e^{-i(\frac{p^2}{2m} + U(x_i))\Delta t} e^{-ip \cdot x_i} \tag{2.29}$$

$$= \quad \int \frac{dp}{2\pi} e^{i(p \cdot \frac{\Delta x}{\Delta t} - \frac{p^2}{2m} - U(x_i))\Delta t} \tag{2.30}$$

$$= \quad \sqrt{\frac{m}{2\pi i \Delta t}} e^{i(\frac{1}{2} m (\frac{\Delta x}{\Delta t})^2 - U(x_i))\Delta t} \tag{2.31}$$

$$= \quad \sqrt{\frac{m}{2\pi i \Delta t}} e^{iL(x_i, \partial_0 x_i)\Delta t} \tag{2.32}$$

where

$$L(x, \partial_0 x) = \frac{1}{2} m (\partial_0 x)^2 - U(x) \tag{2.33}$$

So

$$\begin{aligned}
K(x_j, x_i, t_j - t_i) &= \langle x_j | U(t_i \to t_j) | x_i \rangle && (2.34)\\
&= \langle x_j | U(t_N \to t_j) | x_N \rangle \cdots \langle x_2 | U(t_1 \to t_2) | x_1 \rangle \langle x_1 | U(t_i \to t_1) | x_i \rangle && (2.35)\\
&= \int \prod dx_i \left(\sqrt{\frac{m}{2\pi i \Delta t}} \right)^{N+1} e^{\sum_n i L(x_n, \partial_0 x_n, t_n) \Delta t} && (2.36)\\
&= \sqrt{\frac{m}{2\pi i \Delta t}} \int \mathcal{D} \left(\sqrt{\frac{m}{2\pi i \Delta t}} x \right) e^{i \int_{t_i}^{t_j} dt L(x, \partial_0 x, t)} && (2.37)
\end{aligned}$$

2.4 Propogator of Harmonic Oscillator

The potential of harmonic oscillator is

$$U(x) = \frac{1}{2} m \omega^2 x^2 \tag{2.38}$$

Let $A_n(\lambda, a) = \begin{pmatrix} \lambda & a & & \\ a & \lambda & \ddots & \\ & \ddots & \ddots & a \\ & & a & \lambda \end{pmatrix}_{n \times n}$

Then

$$\begin{aligned}
\det(A_n(\lambda, a)) &= \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2} && (2.39)\\
A_n^{-1}(\lambda, a)_{11} = A_n^{-1}(\lambda, a)_{nn} &= \frac{\lambda_1^n - \lambda_2^n}{\lambda_1^{n+1} - \lambda_2^{n+1}} && (2.40)\\
A_n^{-1}(\lambda, a)_{1n} = A_n^{-1}(\lambda, a)_{n1} &= \frac{(-a)^{n-1}(\lambda_1 - \lambda_2)}{\lambda_1^{n+1} - \lambda_2^{n+1}} && (2.41)
\end{aligned}$$

where λ_1 and λ_2 are two roots of the equation $x^2 - \lambda x + a^2 = 0$.

$$\begin{aligned}
K(x_f, x_i, t_f - t_i) &= \int \prod dx_i \left(\sqrt{\frac{m}{2\pi i \Delta t}} \right)^{N+1} e^{\sum_n i (\frac{1}{2} m (\frac{\Delta x}{\Delta t})^2 - \frac{1}{2} m \omega^2 x_i^2) \Delta t} && (2.42)\\
&= \int \prod dx_i \left(\sqrt{\frac{m}{2\pi i \Delta t}} \right)^{N+1} e^{-\frac{1}{2} x^T A x + y^T x - i \Delta t/(2m) y^T y + o(1)} && (2.43)
\end{aligned}$$

where $A = A_N(i[-2m/\Delta t + m\omega^2 \Delta t], im/\Delta t)$ and $y = -im/\Delta t (x_i, 0, \ldots, 0, x_f)$.

We have $\lambda_1 = -im/\Delta t + m\omega$ and $\lambda_2 = -im/\Delta t - m\omega$ to the 0th order of Δt.

Then

$$
\begin{aligned}
A^{-1}_{1N} = A^{-1}_{N1} &= \frac{(-im/\Delta t)^{N-1}2m\omega}{(-im/\Delta t + m\omega)^{N+1} - (-im/\Delta t - m\omega)^{N+1}} && (2.44)\\
&= \frac{(\frac{\Delta t}{im})^2 2m\omega}{2i\sin(\omega(t_f - t_i))} + o(\Delta t^2) && (2.45)\\
A^{-1}_{11} = A^{-1}_{NN} &= \frac{(-im/\Delta t + m\omega)^{N} - (-im/\Delta t - m\omega)^{N}}{(-im/\Delta t + m\omega)^{N+1} - (-im/\Delta t - m\omega)^{N+1}} && (2.46)\\
&= \frac{\Delta t}{-im}(1 - \omega\Delta t\cot(\omega(t_f - t_i))) + o(\Delta t^2) && (2.47)\\
\det(A) &= \frac{(-im/\Delta t + m\omega)^{N+1} - (-im/\Delta t - m\omega)^{N+1}}{2m\omega} && (2.48)\\
&= \left(\frac{-im}{\Delta t}\right)^{N+1}\frac{2i\sin(\omega(t_f - t_i))}{2m\omega} && (2.49)
\end{aligned}
$$

So

$$
\begin{aligned}
K(x_f, x_i, t_f - t_i) &= \int\prod dx_i\left(\sqrt{\frac{m}{2\pi i\Delta t}}\right)^{N+1} e^{-\frac{1}{2}x^TAx + y^Tx - i\Delta t/(2m)y^Ty + o(1)} && (2.50)\\
&= \left(\sqrt{\frac{m}{2\pi i\Delta t}}\right)^{N+1}(2\pi)^{N/2}\det(A)^{-1/2}e^{\frac{1}{2}y^TA^{-1}y - i\Delta t/(2m)y^Ty + o(1)} && (2.51)\\
&= \sqrt{\frac{m\omega}{2\pi i\sin(\omega T)}}e^{\frac{im\omega}{2\sin(\omega T)}(\cos(\omega T)(x_i^2 + x_f^2) - 2x_ix_f)} && (2.52)
\end{aligned}
$$

where $T = t_f - t_i$.

2.5 Stationary Phase Approximation

Let's consider the path integral

$$K(x_f, x_i, t_f - t_i) = C\int\mathcal{D}xe^{iS[x]} = C\int\mathcal{D}xe^{i\int_{t_i}^{t_j}dtL(x,\partial_0 x,t)} \tag{2.53}$$

A stationary path is a path $x_{cl}(t)$ such that $\frac{\delta S}{\delta x}\Big|_{x_{cl}(t)} = 0$. Around the stationary path we have expansion

$$S[x] = S[x_{cl}] + \frac{1}{2}\int dtdt'\frac{\delta^2 S}{\delta x\delta x'}\Delta x(t)\Delta x(t') + o(\Delta t^2) \tag{2.54}$$

We assume that $S[x]$ far away from x_{cl} are fast oscillating and doesn't contribute to the path integral. Near x_{cl}, we expand $S[x]$ to second order of Δt. So

$$C\int\mathcal{D}xe^{iS[x]} \simeq Ce^{iS[x_{cl}]}\int\mathcal{D}xe^{i\frac{1}{2}\int dtdt'\frac{\delta^2 S}{\delta x\delta x'}\Delta x(t)\Delta x(t')} = C(2\pi)^{N/2}\det(-i\frac{\delta^2 S}{\delta x\delta x'})^{-1/2}e^{iS[x_{cl}]} \tag{2.55}$$

For harmonic oscillator, we have

$$S[x]=S[x_{cl}]+S[\Delta x] \tag{2.56}$$

Note that is is an exact expansion.

It's easy to calculate that

$$x_{cl}=\frac{x_f\sin\omega(t-t_i)-x_i\sin\omega(t-t_f)}{\sin\omega T} \tag{2.57}$$

and

$$S[x_{cl}]=\frac{m\omega}{2\sin(\omega T)}(\cos(\omega T)(x_i^2+x_f^2)-2x_ix_f) \tag{2.58}$$

where $T=t_f-t_i$.

So

$$\begin{aligned}
K(x_f,x_i,t_f-t_i) &= \int\prod dx_i\left(\sqrt{\frac{m}{2\pi i\Delta t}}\right)^{N+1}e^{\sum_n iL(x_n,\partial_0 x_n,t_n)\Delta t} && (2.59)\\
&= e^{iS[x_{cl}]}\int\prod dx_i\left(\sqrt{\frac{m}{2\pi i\Delta t}}\right)^{N+1}e^{\sum_n iL(\Delta x_n,\partial_0\Delta x_n,t_n)\Delta t} && (2.60)\\
&= e^{iS[x_{cl}]}\left(\sqrt{\frac{m}{2\pi i\Delta t}}\right)^{N+1}(2\pi)^{N/2}\det(A)^{-1/2} && (2.61)\\
&= \sqrt{\frac{m\omega}{2\pi i\sin(\omega T)}}e^{\frac{im\omega}{2\sin(\omega T)}(\cos(\omega T)(x_i^2+x_f^2)-2x_ix_f)} && (2.62)
\end{aligned}$$

where A is the same as that in the last section.

2.6 Wick Rotation

Let's do analytic continuation on $K(j,i,t_j-t_i)$ to complex time. We claim that

$$K(j,i,e^{i\theta}(t_j-t_i))=C\int\mathcal{D}xe^{i\int_{e^{i\theta}t_i}^{e^{i\theta}t_j}dtL(x,\partial_t x,t)} \tag{2.63}$$

where t_i,t_f are real, and t is integrated on the line segment from $e^{i\theta}t_i$ to $e^{i\theta}t_f$.

The process $\theta:0\to\theta_0$ is called the Wick's rotation, as shown in Fig. 2.1.

The case $\theta=-\frac{\pi}{2}$ is called the Euclidean path integral:

$$K_E(j,i,t_j-t_i)\doteq K(j,i,-i(t_j-t_i))=C\int\mathcal{D}xe^{i\int_{-it_i}^{-it_j}dtL(x,\partial_t x,t)}=C\int\mathcal{D}xe^{\int_{t_i}^{t_j}d\tau L(x,i\partial_\tau x,-i\tau)} \tag{2.64}$$

where $t=i\tau$ and τ is real.

Then we can calculate $K_E(j,i,t_j-t_i)$ and do analytic continuation to complex time. Then

$$K(j,i,t_j-t_i)=K_E(j,i,i(t_j-t_i)) \tag{2.65}$$

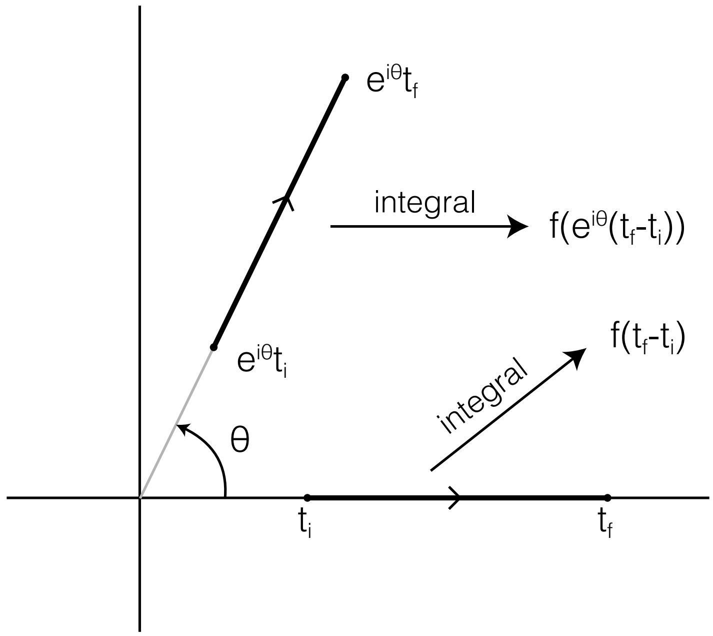


Figure 2.1: An illustration of Wick's rotation.

Sometimes the calculation of $K_E(j,i,t_j-t_i)$ is simpler that the calculation of $K(j,i,t_j-t_i)$. In this case we can get $K(j,i,t_j-t_i)$ by analytic continuation of $K_E(j,i,t_j-t_i)$.

Suppose that

$$L(x,\partial_t x,t) = \frac{1}{2}m(\partial_t x)^t - U(x) \tag{2.66}$$

Let's define

$$L_E(x,\partial_t x,\tau) = -L(x,i\partial_t x,-i\tau) = \frac{1}{2}m(\partial_t x)^2 + U(x) \tag{2.67}$$

Then

$$K_E(j,i,t_j-t_i) = C\int \mathcal{D}x e^{-\int_{t_i}^{t_j} d\tau L_E(x,\partial_\tau x,\tau)} \tag{2.68}$$

Note that L_E is the lagragian with potential $-U(x)$.

2.7 Instanton Gas

The simplest model which exhibit the instanton gas is a particle in double well potential. An example of double well potential is $V(x) = (x^2-a^2)^2$, whose shape is shown in Fig. 2.2. Classically, an unmoved particle start at $x=-a$ or $x=a$ would stay at $x=-a$ or $x=a$ forever. In quantum mechanics, the particle oscillates between $x=-a$ and $x=a$.

We would like to calculate

$$K(a,\pm a,t) = \langle \pm a|U(0\to t)|a\rangle \tag{2.69}$$

which is very difficult.

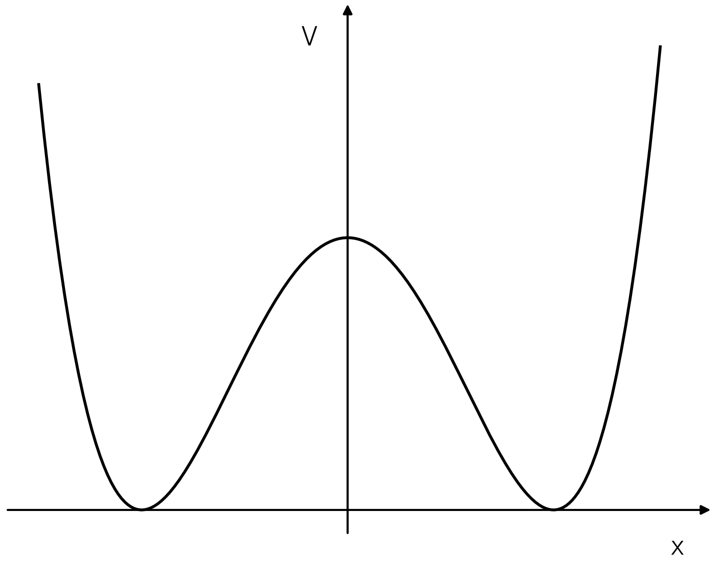


Figure 2.2: A double well potential $V(x) = (x^2 - a^2)^2$.

However, we can calculate

$$K_E(a, \pm a, t) = C \int \mathcal{D}x e^{-\int_0^t d\tau L_E(x, \partial_\tau x, \tau)} \tag{2.70}$$

instead, and get $K(a, \pm a, t)$ by analytic continuation.

For L_E, the potential is $-V(x)$, as shown in Fig. 2.3. This is a single well around $x = 0$.

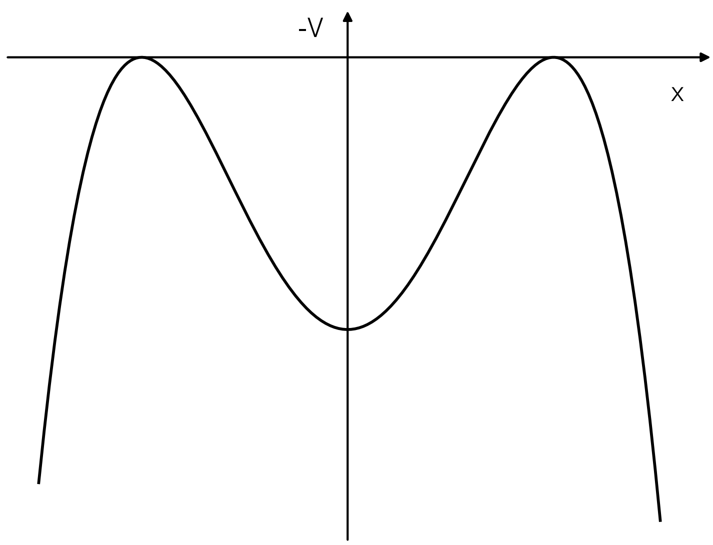


Figure 2.3: The potential $-V(x) = -(x^2 - a^2)^2$.

As for the imaginary path integral, we can do stationary phase approximation on Euclidian phase integral

$$C \int \mathcal{D}x e^{-S_E[x]} \simeq C(2\pi)^{N/2} \det(\frac{\delta^2 S}{\delta x \delta x'})^{-1/2} e^{-S_E[x_{cl}]} \tag{2.71}$$

Let's consider the Euclidian phase integral relative to 2 different classical path

1. As τ goes from $0 \to T$, x stays at $\pm a$. The Euclidian phase integral is

$$\sqrt{\frac{m\omega}{2\pi \sinh(\omega T)}} \sim e^{-\omega T/2} \tag{2.72}$$

   Note that T here is arbitrary, where ω is the frequency of the harmonic oscillator near $x = \pm a$.

2. As τ goes from $0 \to T$, x goes from $\pm a$ to $\mp a$, with energy $E = 0$. The Euclidian phase integral is

$$Ke^{-S_E[x_{cl}]} \tag{2.73}$$

   The energy conservation equation is

$$\frac{1}{2}m(\partial_\tau x)^2 - V(x) = 0 \tag{2.74}$$

   Then

$$S_{inst} = \int_0^T d\tau(\frac{1}{2}m(\partial_\tau x)^2 + V(x)) = \int_0^T d\tau m(\partial_\tau x)^2 = \int_{-a}^a dx m\partial_\tau x = \int_{-a}^a dx\sqrt{2mV(x)} \tag{2.75}$$

   Note that $T \sim \omega_2^{-1}$ here is fixed, where ω_2 is the frequency of the harmonic oscillator near $x = 0$.

s

Then a stationary path from $(0, -a)$ to (T, a) is:

1. Stay at $-a$ for time t_1.
2. Tunnel from $-a$ to a.
3. Stay at a for time t_2.
4. Tunnel from a to $-a$.
5. Stay at $-a$ for time t_3.
6. ……

This is shown in Fig. 2.4

Assume that $\omega_2^{-1} << T$, then the path integral contribution of the unmoving part is

$$\sim e^{-\omega T/2} \tag{2.76}$$

The path integral contribution of the tunneling part is

$$Ke^{-S_{inst}} \tag{2.77}$$

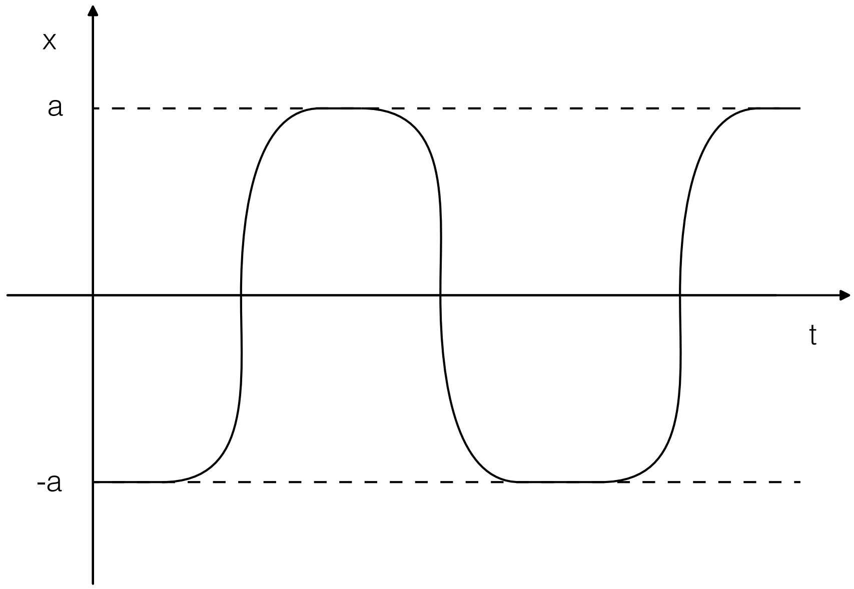


Figure 2.4: A stationary path for double-well potential

So the path integral is

$$K_E(a,-a,T) \sim e^{-\omega T/2}\Big(Ke^{-S_{inst}}\Big)^n \tag{2.78}$$

where n is the number of the tunneling process, and n is odd.

However, we should sum up over all possible stationary path: over all odd n and all possible time of the tunneling processes. So

$$\begin{aligned} K_E(a,-a,T) &\sim \sum_{n,odd} e^{-\omega T/2}\frac{1}{n!}\Big(\prod_i \int_0^T d\tau_i K e^{-S_{inst}}\Big) && (2.79)\\ &= e^{-\omega T/2}\sum_{n,odd}\frac{1}{n!}\Big(TKe^{-S_{inst}}\Big)^n && (2.80)\\ &= e^{-\omega T/2}\sinh\big(TKe^{-S_{inst}}\big) && (2.81) \end{aligned}$$

Similarly

$$K_E(a,a,T) \sim e^{-\omega T/2}\cosh\big(TKe^{-S_{inst}}\big) \tag{2.82}$$

Then

$$\begin{aligned} K(a,-a,t) &\sim e^{-i\omega t/2}\sin\big(Ke^{-S_{inst}}t\big) && (2.83)\\ K(a,a,t) &\sim e^{-i\omega t/2}\cos\big(Ke^{-S_{inst}}t\big) && (2.84) \end{aligned}$$

Later we'll show that K is real.

2.8 Fate of the False Vacuum

A model that exhibits the fate of the false vacuum is a particle in the potential whose shape is shown in Fig. 2.5. Classically, an unmoved particle start at $x = -a$ would stay at $x = -a$ forever. In quantum mechanics, the particle decay slowly from $x = -a$ to $x = +\infty$. The position $x = -a$ is called the false vacuum.

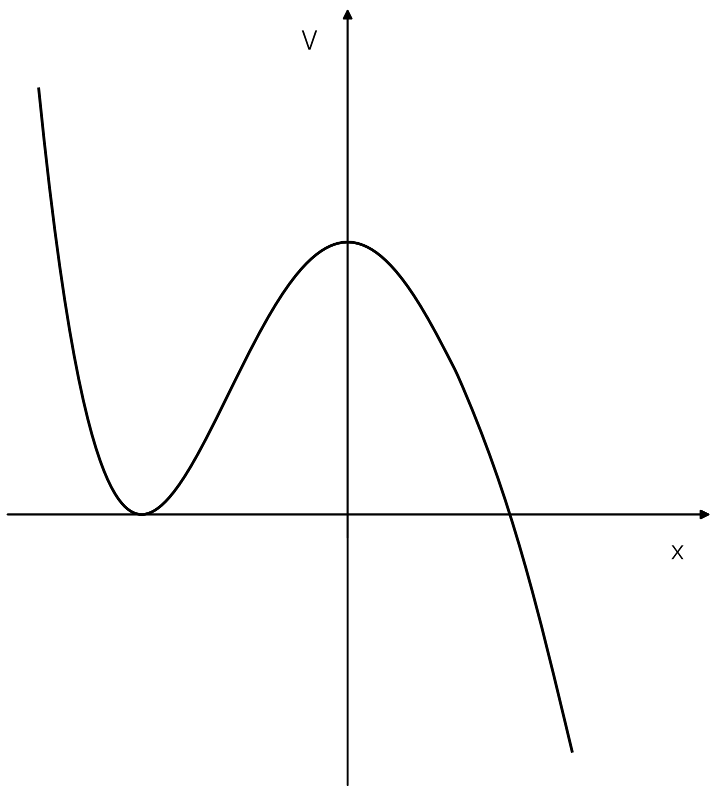


Figure 2.5: A potential with a meta-stable state.

We would like to calculate

$$K(-a, -a, t) = \langle \pm a|U(0 \to t)|a\rangle \tag{2.85}$$

Similarly to the discussion in the last section, we calculate the Euclidean path integral

$$K_E(-a, -a, t) = C \int \mathcal{D}x e^{-\int_0^t d\tau L_E(x, \partial_\tau x, \tau)} \tag{2.86}$$

whose potential is shown in Fig. 2.6.

Similarly, there're two types of typical classical path

1. As τ goes from $0 \to T$, x stays at $-a$. The Euclidian phase integral is

$$\sqrt{\frac{m\omega}{2\pi \sinh(\omega T)}} \sim e^{-\omega T/2} \tag{2.87}$$

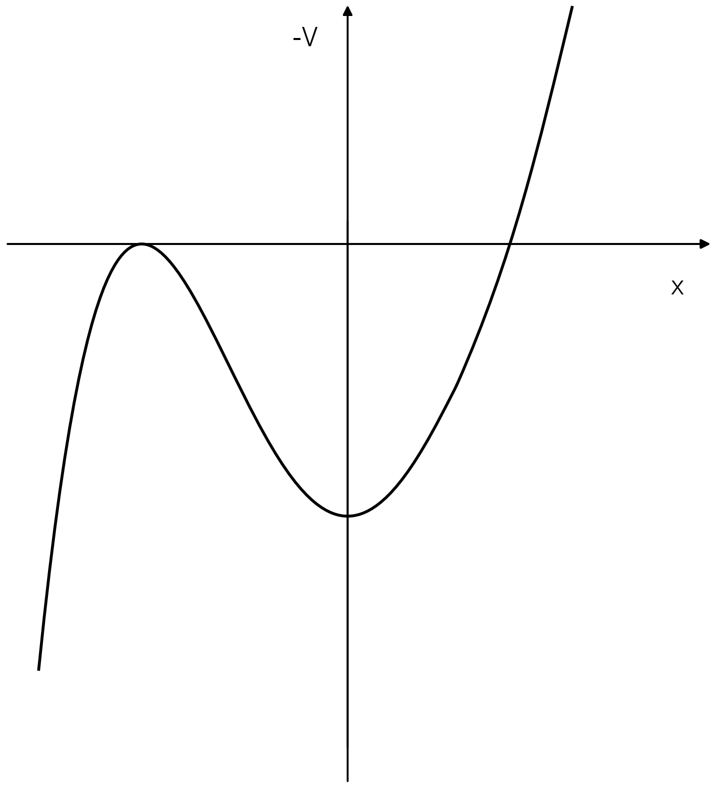


Figure 2.6: The inverse of the potential with a meta-stable state.

2. As τ goes from $0 \to T$, x goes from $-a$ to b ($V(b) = 0$) then back to $-a$, with energy $E = 0$. The Euclidian phase integral is
$$Ke^{-S_{bounce}} \tag{2.88}$$

Then a stationary path from $(0, -a)$ to $(T, -a)$ is:

1. Stay at $-a$ for time t_1.
2. Bounce from $-a$ to b then back to $-a$.
3. Stay at $-a$ for time t_2.
4. Bounce from $-a$ to b then back to $-a$.
5. Stay at $-a$ for time t_3.
6. ……

This is shown in Fig. 2.7

Similar to the double-well potential case, the Euclidean path integral is

$$K_E(-a, -a, T) \sim \sum_n e^{-\omega T/2} \frac{1}{n!} \Big(\prod_i \int_0^T d\tau_i K e^{-S_{bounce}} \Big) = e^{-\omega T/2} e^{TKe^{-S_{bounce}}} \tag{2.89}$$

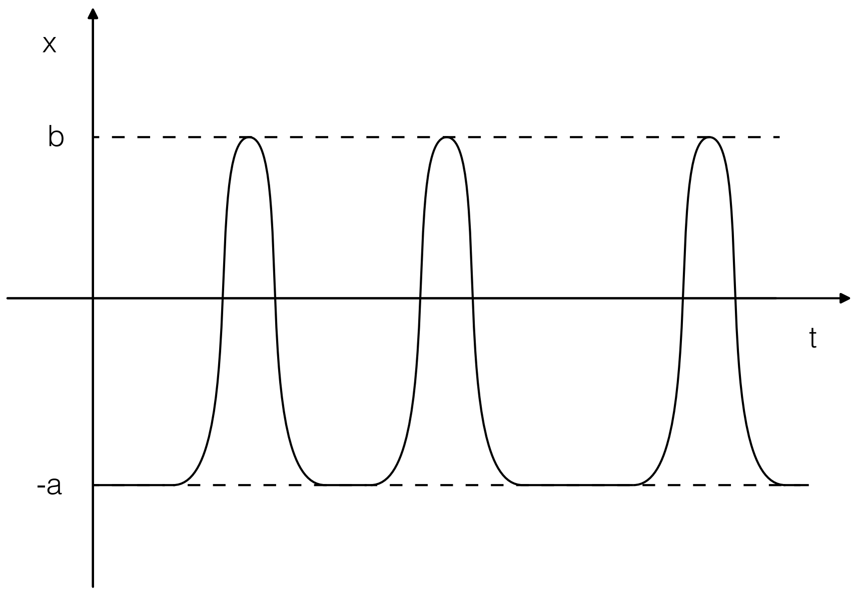


Figure 2.7: A stationary path for meta-stable potential

Then

$$K(-a,-a,t) \sim e^{-i\omega t/2} e^{-|K| e^{-S_{bounce}} t} \tag{2.90}$$

Later we'll show that K is imaginary.

2.9 Is K Real or Imaginary?

$$S_E[x] \simeq S_E[x_{cl,\tau}] + \frac{1}{2}\int_{t_i}^{t_j} d\tau \delta x(-m\partial_\tau^2 + V(x_{cl,\tau})'')\delta x \tag{2.91}$$

where $x_{cl,\tau}$ is the instaton or the bounce process at τ.

Then

$$K \sim det(-m\partial_\tau^2 + V(x_{cl,\tau})'')^{-1/2} \tag{2.92}$$

However, the classical contribution is integrated over τ, so in $det(-m\partial_\tau^2 + V(x_{cl,\tau})'')$, one should only consider the δx perpendicular to $\partial_\tau x_{cl,\tau}$.

It can be shown that

$$(-m\partial_\tau^2 + V(x_{cl,\tau})'')\partial_\tau x_{cl,\tau} = 0 \tag{2.93}$$

and the operator $-m\partial_\tau^2 + V(x_{cl,\tau})''$ is non-degenerate.

So

$$K \sim det'(-m\partial_\tau^2 + V(x_{cl,\tau})'')^{-1/2} \tag{2.94}$$

where the zero eigenvalue is omitted.

If the potential is double-well, $\partial_\tau x_{cl,\tau}$ has no node, so $-m\partial_\tau^2 + V(x_{cl,\tau})''$ has no negative eigenvalue. So $det(-m\partial_\tau^2 + V(x_{cl,\tau})'') > 0$ and K is real.

If the potential is of the shape in Fig. 2.5, $\partial_\tau x_{cl,\tau}$ has a node, so $-m\partial_\tau^2 + V(x_{cl,\tau})''$ has a negative eigenvalue. So $det(-m\partial_\tau^2 + V(x_{cl,\tau})'') < 0$ and K is imaginary.

2.10 Path Integral for Spin

Let

$$H = B \cdot \hat{S} \tag{2.95}$$

We have

$$\begin{aligned}
\langle g'|U(t \to t+\Delta t)|g\rangle &= \langle g'|e^{-i\Delta t H}|g\rangle && (2.96)\\
&= \langle g'|g\rangle - i\Delta t\langle g'|H|g\rangle + \mathcal{O}(\Delta t^2) && (2.97)\\
&= e^{\ln\langle g'|g\rangle - i\Delta t\frac{\langle g'|H|g\rangle}{\langle g'|g\rangle}} && (2.98)\\
&= e^{S\ln[(1+n_{g'}\cdot n_g)/2] + iS(\Delta\chi + \Phi(\vec{z}, n_g, n_{g'})) - i\Delta t\frac{\langle g'|H|g\rangle}{\langle g'|g\rangle}} && (2.99)
\end{aligned}$$

Let's consider mild oscillating trajectories, that is, $\Delta g \sim \Delta t$. Then

$$\langle g'|U(t \to t+\Delta t)|g\rangle = e^{-\frac{1}{4}S(\Delta n_g)^2 + iS(\Delta\chi + \Phi(\vec{z}, n_g, n_{g'})) - i\Delta t B\cdot n_g} \tag{2.100}$$

So

$$\begin{aligned}
\langle g_f|U(t_i \to t_f)|g_i\rangle &= \int \mathcal{D}g e^{iS_{WZ} + i\int_{t_i}^{t_f} dt(\frac{i}{4}S\Delta t(\dot{n}_g)^2 + S\dot{\chi} - B\cdot n_g} && (2.101)\\
&= e^{iS(\chi_f - \chi_i)}\int \mathcal{D}n_g e^{iS_{WZ} + i\int_{t_i}^{t_f} dt(i\epsilon(\dot{n}_g)^2 - B\cdot n_g)} && (2.102)
\end{aligned}$$

where $\mathcal{D}g$ is integrated over $d\theta d\phi d\chi$ and $\mathcal{D}n_g$ is integrated over $d\theta d\phi$, and S_{WZ} is S times the area enclosed by the path $\gamma : \vec{z} \to n_{g_i} \xrightarrow{n_{g(t)}} n_{g_f} \to \vec{z}$.

Let $A = \frac{1-\cos\theta}{\sin\theta}\hat{e}_\phi$, then $B' = \nabla \times A = \hat{e}_r$. So

$$S_{WZ} = S\int dS \cdot \hat{e}_r = S\int dS \cdot (\nabla \times A) = S\int_\gamma dn \cdot A \tag{2.103}$$

Then

$$\langle g_f|U(t_i \to t_f)|g_i\rangle = e^{iS(\chi_f - \chi_i)}\int \mathcal{D}n_g e^{i\int_{t_i}^{t_f} dt(S\dot{n}_g\cdot A + i\epsilon(\dot{n}_g)^2 - B\cdot n_g)} \tag{2.104}$$

since $dn \cdot A = 0$ along a meridian.

Chapter 3

Second Quantization

In this chapter, we discuss the quantum mechanics many-particle situation. We'll show how the states and operators are expressed by creation/annihilation operators.

3.1 Identical Particles

In a single particle Hilbert space $\mathbb{H}$, we have orthogonal-normalized basis: $|0\rangle$, $|1\rangle$, $|2\rangle$ ... We can construct a Hilbert space for N particles by tensor product N single particle Hilbert spaces.

We have a basis $|\{\lambda_i\}\rangle = |\lambda_1\lambda_2\dots\lambda_N\rangle$ where $\lambda_i = 0, 1, \cdots$ denotes the state of ith particle. We define the inner product in $\mathbb{H}^{\otimes N}$ by requiring that $|\{\lambda_i\}\rangle$ form a orthogonal-normalized basis. This definition of inner product of $\mathbb{H}^{\otimes N}$ is independent of the choice of orthogonal-normalized basis in $\mathbb{H}$.

We define the linear action P of the permutation group S_N on $\mathbb{H}^{\otimes N}$ by defining that its act on the basis of $\mathbb{H}^{\otimes N}$ as

$$P(g)|\lambda_1\lambda_2\dots\lambda_N\rangle = |\lambda_{g(1)}\lambda_{g(2)}\dots\lambda_{g(N)}\rangle \quad g \in S_N \tag{3.1}$$

This definition of action of S_N is also independent of the choice of basis in $\mathbb{H}$.

The principle of quantum mechanics states that particles of the same type are identical and indistinguishable. So if a many-body state lies in $\mathbb{H}^{\otimes N}$, it must be left invariant under permutation, since nothing has been actually done. It means if $|\Phi\rangle$ is a many-body state,

$$P(g)|\Phi\rangle = e^{i\theta}|\Phi\rangle \tag{3.2}$$

So $|\Phi\rangle$ forms the space for 1-D representation of S_N. The only two 1-D representations of S_N are $P(g) = 1$ and $P(g) = \mathrm{sgn}(g)$. We define bosons to be particles with wave-function such that $P(g)|\Phi\rangle = |\Phi\rangle$. Such states form a subspace of $\mathbb{H}^{\otimes N}$ called $\mathrm{sym}(\mathbb{H}^{\otimes N})$. And we define fermions to be particles with wave-function such that $P(g)|\Phi\rangle = \mathrm{sgn}(g)|\Phi\rangle$. Such states form a subspace of $\mathbb{H}^{\otimes N}$ called $\mathrm{asm}(\mathbb{H}^{\otimes N})$.

Define

$$S_+ = \frac{1}{N!}\sum_{g\in S_N} P(g) \tag{3.3}$$

$$S_- = \frac{1}{N!}\sum_{g\in S_N} P(g)\mathrm{sgn}(g) \tag{3.4}$$

$$(3.5)$$

It's easy to see that S_+ and S_- are orthogonal projective operators:

$$S_+S_+ = S_+ \tag{3.6}$$
$$S_-S_- = S_- \tag{3.7}$$
$$S_+S_- = S_-S_+ = 0 \tag{3.8}$$

Theorem 3.1.1.

$$\mathrm{sym}(\mathbb{H}^{\otimes N}) = S_+\mathbb{H}^{\otimes N} \tag{3.9}$$
$$\mathrm{asm}(\mathbb{H}^{\otimes N}) = S_-\mathbb{H}^{\otimes N} \tag{3.10}$$

Proof. For $|\Phi\rangle \in \mathrm{sym}(\mathbb{H}^{\otimes N})$,

$$\sum_{g\in S_N} P(g)|\Phi\rangle = N!|\Phi\rangle \tag{3.11}$$
$$|\Phi\rangle = \frac{1}{N!}\sum_{g\in S_N} P(g)|\Phi\rangle \tag{3.12}$$
$$= S_+|\Phi\rangle \tag{3.13}$$

For $|\Phi\rangle \in \mathrm{asm}(\mathbb{H}^{\otimes N})$,

$$\sum_{g\in S_N} \mathrm{sgn}(g)P(g)|\Phi\rangle = N!|\Phi\rangle \tag{3.14}$$
$$|\Phi\rangle = \frac{1}{N!}\sum_{g\in S_N} \mathrm{sgn}(g)P(g)|\Phi\rangle \tag{3.15}$$
$$= S_-|\Phi\rangle \tag{3.16}$$

Thus

$$\mathrm{sym}(\mathbb{H}^{\otimes N}) = S_+\mathrm{sym}(\mathbb{H}^{\otimes N}) \subset S_+\mathbb{H}^{\otimes N} \tag{3.17}$$
$$\mathrm{asm}(\mathbb{H}^{\otimes N}) = S_-\mathrm{asm}(\mathbb{H}^{\otimes N}) \subset S_-\mathbb{H}^{\otimes N} \tag{3.18}$$

It's easy to check that

$$P(g)S_+ = S_+P(g) = S_+ \tag{3.19}$$
$$P(g)S_- = S_-P(g) = \mathrm{sgn}(g)S_- \tag{3.20}$$

Thus for any $|\Psi\rangle$, $S_\pm|\Psi\rangle \in \mathrm{sym}(\mathbb{H}^{\otimes N})/\mathrm{asm}(\mathbb{H}^{\otimes N})$.
Thus

$$S_+(\mathbb{H}^{\otimes N}) \subset \mathrm{sym}(\mathbb{H}^{\otimes N}) \tag{3.21}$$
$$S_-(\mathbb{H}^{\otimes N}) \subset \mathrm{asm}(\mathbb{H}^{\otimes N}) \tag{3.22}$$

$\square$

Thus the basis for the space $\text{sym}(\mathbb{H}^{\otimes N})/\text{asm}(\mathbb{H}^{\otimes N})$ is $S_\pm|\{\lambda_i\}\rangle$.

For boson, it can be easily shown that

$$\begin{aligned} S_+|\{\lambda_i\}\rangle &\neq 0 && (3.23)\\ \langle\{\lambda_i\}|S_+S_+|\{\lambda_i\}\rangle &= \langle\{\lambda_i\}|S_+|\{\lambda_i\}\rangle = \frac{\prod_i n_i!}{N!} && (3.24)\\ S_+|\{\lambda_i\}\rangle &= S_+|\{\lambda'_i\}\rangle \text{ iff } \exists g : |\{\lambda_i\}\rangle = P(g)|\{\lambda'_i\}\rangle && (3.25)\\ S_+|\{\lambda_i\}\rangle &\perp S_+|\{\lambda'_i\}\rangle \text{ if } S_+|\{\lambda_i\}\rangle \neq S_+|\{\lambda'_i\}\rangle && (3.26) \end{aligned}$$

Thus $\sqrt{\frac{N!}{\prod_i n_i!}}S_+|\{\lambda_i\}\rangle$ is a orthogonal basis for $\text{sym}(\mathbb{H}^{\otimes N})$, where n_i is the number of particles at ith state. And each set of physically equivalent basis vectors in $\mathbb{H}^{\otimes N}$ give only one basis vector in $\text{sym}(\mathbb{H}^{\otimes N})$.

For fermion, first consider $|\{\lambda_i\}\rangle$ with multiple occupation, that is, $\lambda_i = \lambda_j$ for some $i \neq j$.

$$\begin{aligned} |\{\lambda_i\}\rangle &= P((i,j))|\{\lambda_i\}\rangle && (3.27)\\ S_-|\{\lambda_i\}\rangle &= S_-P((i,j))|\{\lambda_i\}\rangle && (3.28)\\ S_-|\{\lambda_i\}\rangle &= -S_-|\{\lambda_i\}\rangle && (3.29)\\ S_-|\{\lambda_i\}\rangle &= 0 && (3.30) \end{aligned}$$

For $|\{\lambda_i\}\rangle$ without multiple occupation, it can be easily shown that

$$\begin{aligned} S_-|\{\lambda_i\}\rangle &\neq 0 && (3.31)\\ \langle\{\lambda_i\}|S_-S_-|\{\lambda_i\}\rangle &= \langle\{\lambda_i\}|S_-|\{\lambda_i\}\rangle = \frac{1}{N!} && (3.32)\\ S_-|\{\lambda_i\}\rangle &= \pm S_-|\{\lambda'_i\}\rangle \text{ iff } \exists g : |\{\lambda_i\}\rangle = P(g)|\{\lambda'_i\}\rangle && (3.33)\\ S_-|\{\lambda_i\}\rangle &\perp S_-|\{\lambda'_i\}\rangle \text{ if } S_-|\{\lambda_i\}\rangle \neq \pm S_-|\{\lambda'_i\}\rangle && (3.34) \end{aligned}$$

Thus $\sqrt{N!}S_-|\{\lambda_i\}\rangle$ is a normalized-orthogonal basis for $\text{asm}(\mathbb{H}^{\otimes N})$, and we can still express it as $\sqrt{\frac{N!}{\prod_i n_i!}}S_+|\{\lambda_i\}\rangle$ as in the boson case.

To describe a system with variable particle numbers, we define the **Fock space** for boson as

$$\bigoplus_{N=0}^{\infty} \text{sym}(\mathbb{H}^{\otimes N}) \tag{3.35}$$

and for fermion as

$$\bigoplus_{N=0}^{\infty} \text{asm}(\mathbb{H}^{\otimes N}) \tag{3.36}$$

where $\text{sym}(\mathbb{H}^{\otimes 0}) = \text{asm}(\mathbb{H}^{\otimes 0}) = (|vac\rangle)$ is a 1-D space spanned by the vacuum state, which stands for the state with no particles.

It's convenient to express the basis of Fock space in **particle number representation**. That is, define

$$|n_0 n_1 ...\rangle = \sqrt{\frac{N!}{\prod_i n_i!}}S_\pm|\{\lambda_i\}\rangle \tag{3.37}$$

where n_i is the number of particles with ith state in $|\{\lambda_i\}\rangle$. It's easy to see, this definition has an ambiguity of ± 1 for fermion. Thus to eliminate this ambiguity, we require that the $|\{\lambda_i\}\rangle$ in the definition to be of increasing order, that is, $\lambda_1 \leq \lambda_2 \leq \cdots$. With this definition, it's easy to see $|n_0 n_1 \dots\rangle$ is different for different $\{n_i\}$ and forms a orthogonal-normalized basis of the Fock space, and we call it **particle number basis**. Especially, we express $|vac\rangle$ as $|0\cdots\rangle$ or $|0\rangle$.

3.2 Creation & Annihilation Operators

We can define the **creation** and **annihilation operators** to simplify operators in Fock space.

The creation operator in defined as

$$a_i^\dagger |n_0 n_1 \dots n_i \dots\rangle = \sqrt{n_i+1}\xi^{\sum_{j<i} n_i} |n_0 n_1 \dots n_i + 1 \dots\rangle \tag{3.38}$$

where $\xi = \pm 1$ for boson/fermion.

Then we have

$$|n_0 n_1 \dots\rangle = \frac{(a_0^\dagger)^{n_0}}{\sqrt{n_0!}} \frac{(a_1^\dagger)^{n_1}}{\sqrt{n_1!}} \cdots |0\rangle \tag{3.39}$$

The annihilation operator is defined as the conjugation of the creation operator. We have

$$a_i |n_0 n_1 \dots n_i \dots\rangle = \begin{cases} \sqrt{n_i}\xi^{\sum_{j<i} n_i} |n_0 n_1 \dots n_i - 1 \dots\rangle & (n_i > 0) \\ 0 & (n_i = 0) \end{cases} \tag{3.40}$$

Physically, $a_i^\dagger / a_i$ means to add/remove a particle of the ith state.

We have the following **commuting/anti-commuting relations**

$$[a_i, a_i]_\xi = [a_i^\dagger, a_i^\dagger]_\xi = 0, \ [a_i, a_i^\dagger]_\xi = \delta_{ij} \tag{3.41}$$

where $\xi = \pm$ for boson/fermion.

We prove the following lemma

Lemma 3.2.1.

$$a_\lambda^\dagger S_\pm |\lambda_1 \lambda_2 \dots\rangle = \sqrt{N+1} S_\pm |\lambda \lambda_1 \lambda_2 \dots\rangle \tag{3.42}$$

Proof. For boson

$$\begin{aligned} a_\lambda^\dagger S_+ |\lambda_1 \lambda_2 \dots\rangle &= a_\lambda^\dagger S_+ |\lambda_{g(1)} \lambda_{g(2)} \dots\rangle && (3.43) \\ &= a_\lambda^\dagger \sqrt{\frac{\prod_i n_i!}{N!}} |n_1 n_2 \dots\rangle && (3.44) \\ &= \sqrt{n_\lambda + 1} \sqrt{\frac{\prod_i n_i!}{N!}} |n_1 n_2 \dots n_\lambda + 1 \dots\rangle && (3.45) \\ &= \sqrt{N+1} S_+ |\lambda_{g(1)} \lambda_{g(2)} \dots \lambda \dots\rangle && (3.46) \\ &= \sqrt{N+1} S_+ |\lambda \lambda_1 \lambda_2 \dots\rangle && (3.47) \end{aligned}$$

where $\lambda_{g(1)} \leq \lambda_{g(2)} \leq \cdots$.

For fermion

$$
\begin{aligned}
a_\lambda^\dagger S_-|\lambda_1\lambda_2\dots\rangle &= a_\lambda^\dagger \mathrm{sgn}(g) S_-|\lambda_{g(1)}\lambda_{g(2)}\dots\rangle && (3.48)\\
&= a_\lambda^\dagger \mathrm{sgn}(g)\sqrt{\frac{\prod_i n_i!}{N!}}|n_1 n_2\dots\rangle && (3.49)\\
&= (-1)^{\sum_{\lambda'<\lambda} n_{\lambda'}}\sqrt{n_\lambda+1}\mathrm{sgn}(g)\sqrt{\frac{\prod_i n_i!}{N!}}|n_1 n_2\dots n_\lambda+1\dots\rangle && (3.50)\\
&= (-1)^{\sum_{\lambda'<\lambda} n_{\lambda'}}\mathrm{sgn}(g)\sqrt{N+1}S_-|\lambda_{g(1)}\lambda_{g(2)}\dots\lambda\dots\rangle && (3.51)\\
&= \mathrm{sgn}(g)\sqrt{N+1}S_-|\lambda\lambda_{g(1)}\lambda_{g(2)}\dots\rangle && (3.52)\\
&= \sqrt{N+1}S_-|\lambda\lambda_1\lambda_2\dots\rangle && (3.53)
\end{aligned}
$$

where similarly, $\lambda_{g(1)} \le \lambda_{g(2)} \le \cdots$. □

Sometimes we may want to change the basis of the single particle Hilbert space $\mathbb{H}$. Suppose $|\mu\rangle$s and $|\nu\rangle$s are two set of orthogonal-normalized basis. It would be nice if we can relate $a_\lambda/a_\lambda^\dagger$ with $a_\mu/a_\mu^\dagger$. Since $S_\pm$ is independent of the basis of $\mathbb{H}$, $\mathrm{sym}(\mathbb{H}^{\otimes N})/\mathrm{asm}(\mathbb{H}^{\otimes N})$ are the same for two basis. Thus Fock space for two basis are the same. Using the Lemma 3.2.1, we can prove the following theorem

Theorem 3.2.2.

$$a_\mu^\dagger = a_\lambda^\dagger\langle\lambda|\mu\rangle \tag{3.54}$$

Proof. Since the Fock space for two basis are the same, $a_\lambda^\dagger$ and $a_\mu^\dagger$ live in the same space. We only need to prove that their actions on a set of basis vectors are the same. That is, we only need to prove

$$a_\mu^\dagger|n_\mu\rangle = a_\lambda^\dagger\langle\lambda|\mu\rangle|n_\mu\rangle \tag{3.55}$$

Suppose $N = \sum n_\mu$,

$$
\begin{aligned}
\mathrm{RHS} &= a_\lambda^\dagger\langle\lambda|\mu\rangle\sqrt{\frac{N!}{\prod_i n_i!}}S_\pm|\mu_1\dots\mu_N\rangle \quad (\mu_1\le\cdots\le\mu_N) && (3.56)\\
&= a_\lambda^\dagger\langle\lambda|\mu\rangle\sqrt{\frac{N!}{\prod_i n_i!}}S_\pm\langle\lambda_1|\mu_1\rangle\cdots\langle\lambda_N|\mu_N\rangle|\lambda_1\dots\lambda_N\rangle && (3.57)\\
&= \sqrt{\frac{(N+1)!}{\prod_i n_i!}}S_\pm\langle\lambda|\mu\rangle\langle\lambda_1|\mu_1\rangle\cdots\langle\lambda_N|\mu_N\rangle|\lambda\lambda_1\dots\lambda_N\rangle && (3.58)\\
&= \sqrt{\frac{(N+1)!}{\prod_i n_i!}}S_\pm|\mu\mu_1\dots\mu_N\rangle && (3.59)\\
&= \xi^{\sum_{\mu'<\mu} n_{\mu'}}\sqrt{\frac{(N+1)!}{\prod_i n_i!}}S_\pm|\mu_1\dots\mu\dots\mu_N\rangle && (3.60)\\
&= \sqrt{n_\mu+1}\xi^{\sum_{\mu'<\mu} n_{\mu'}}|n_1\dots n_\mu+1\dots\rangle && (3.61)\\
&= \mathrm{LHS} && (3.62)
\end{aligned}
$$

□

And obviously we have

$$a_\mu = a_\lambda \langle \mu|\lambda\rangle \tag{3.63}$$

3.3 Second Quantization of Operators

For a one-body operator O in $\mathbb{H}$ defined by

$$O = O_{\mu\nu}|\mu\rangle\langle\nu| \tag{3.64}$$

Then the total O operator in $\mathbb{H}^{\otimes N}$ is

$$O_N = \sum_{i=1}^{N} O_{\mu\nu}|\mu_i\rangle\langle\nu_i| \tag{3.65}$$

where $|\mu_i\rangle\langle\nu_i|$ is short for

$$\prod_{j=1}^{i-1} id \otimes |\mu\rangle\langle\nu| \otimes \prod_{j=i+1}^{N} id \tag{3.66}$$

O_N is symmetric in the sense that

$$\begin{aligned}
P(g)O_N P(g^{-1}) &= \sum_{i=1}^{N} O_{\mu\nu}P(g)|\mu_i\rangle\langle\nu_i|P(g^{-1}) && (3.67)\\
&= \sum_{i=1}^{N} O_{\mu\nu}|\mu_{g(i)}\rangle\langle\nu_{g(i)}| && (3.68)\\
&= \sum_{i=1}^{N} O_{\mu\nu}|\mu_i\rangle\langle\nu_i| && (3.69)\\
&= O_N && (3.70)
\end{aligned}$$

Thus

$$S_\pm O_N = O_N S_\pm \tag{3.71}$$

Thus

$$O_N \mathrm{sym/asm}(\mathbb{H}^{\otimes N}) = O_N S_\pm \mathbb{H}^{\otimes N} = S_\pm O_N \mathbb{H}^{\otimes N} \subset S_\pm \mathbb{H}^{\otimes N} = \mathrm{sym/asm}(\mathbb{H}^{\otimes N}) \tag{3.72}$$

that is, $\mathrm{sym}(\mathbb{H}^{\otimes N})/\mathrm{asm}(\mathbb{H}^{\otimes N})$ are invariant spaces of O_N.

We can defined an operator O in the Fock space which acts on N particle states like O_N. Then

$$\begin{aligned}
O|n_\lambda\rangle &= O_N|n_\lambda\rangle \quad (N = \sum n_\lambda) && (3.73)\\
&= O_N S_\pm \sqrt{\frac{N!}{\prod n_\lambda!}}|\lambda_1 \dots \lambda_N\rangle \quad (\lambda_1 \le \dots \le \lambda_N) && (3.74)\\
&= S_\pm \sqrt{\frac{N!}{\prod n_\lambda!}} O_N|\lambda_1 \dots \lambda_N\rangle && (3.75)
\end{aligned}$$

$$
\begin{aligned}
&= S_{\pm}\sqrt{\frac{N!}{\prod n_\lambda!}}\sum_{i=1}^{N} O_{\lambda_i'\lambda_i}|\lambda_1\ldots\lambda_i'\ldots\lambda_N\rangle && (3.76)\\
&= \sum_{i=1}^{N}\sum_{\lambda_i'\neq\lambda_i}\sqrt{\frac{n_{\lambda_i'}+1}{n_{\lambda_i}}}O_{\lambda_i'\lambda_i}\xi^{n(\lambda_i,\lambda_i')}|n_1\ldots n_{\lambda_i'}+1\ldots n_{\lambda_i}-1\ldots\rangle + \\
&\quad \sum_{i=1}^{N} O_{\lambda_i\lambda_i}|n_1\ldots\rangle && (3.77)\\
&= \sum_{n_\lambda>0}\sum_{\lambda'\neq\lambda}\sqrt{n_{\lambda'}+1}\sqrt{n_\lambda}O_{\lambda'\lambda}\xi^{n(\lambda,\lambda')}|n_1\ldots n_{\lambda'}+1\ldots n_\lambda-1\ldots\rangle + \\
&\quad \sum_{n_\lambda>0} n_\lambda O_{\lambda\lambda}|n_1\ldots\rangle && (3.78)
\end{aligned}
$$

where $n(\lambda,\lambda')$ is the number of particles with states that lie strictly between λ and λ'.

We have

$$
a_{\lambda'}^\dagger a_\lambda|n_\lambda\rangle = \sqrt{n_{\lambda'}+1}\sqrt{n_\lambda}\xi^{n(\lambda,\lambda')}|n_1\ldots n_{\lambda'}+1\ldots n_\lambda-1\ldots\rangle \quad (n_\lambda>0,\lambda'\neq\lambda) \tag{3.79}
$$

Thus

$$
O|n_\lambda\rangle = O_{\lambda'\lambda}a_{\lambda'}^\dagger a_\lambda|n_\lambda\rangle \tag{3.80}
$$

Since this equation holds for all $|n_\lambda\rangle$, we finally get

$$
O = O_{\mu\nu}a_\mu^\dagger a_\nu \tag{3.81}
$$

For a two-body operator in $\mathbb{H}^{\otimes N}$

$$
O_N = \frac{1}{2}\sum_{i\neq j} O_{\alpha\beta\mu\nu}|\alpha_i\beta_j\rangle\langle\mu_i\nu_j| \tag{3.82}
$$

We can define the operator O in the Fock space in the same manner, and similarly

$$
O = \frac{1}{2}O_{\alpha\beta\mu\nu}a_\alpha^\dagger a_\beta^\dagger a_\nu a_\mu \tag{3.83}
$$

Example For a free Hamiltonian with only one-body term $T_{\mu\nu}|\mu\rangle\langle\nu|$, its second-quantized form is

$$
H = T_{\mu\nu}a_\mu^\dagger a_\nu \tag{3.84}
$$

To diagonal a second quantized free Hamiltonian means to find a_i such that

$$
H = E_i n_i \tag{3.85}
$$

Suppose we can diagonalize the H matrix by a unitary matrix U

$$
UTU^{-1} = T' = \mathrm{diag}(E_1, E_2, \ldots) \tag{3.86}
$$

Define $\tilde{a}$ to be

$$\tilde{a}_\mu = U_{\mu\nu} a_\nu \tag{3.87}$$

Then

$$H = T'_{\mu\nu} \tilde{a}^\dagger_\mu \tilde{a}_\nu = E_\mu \tilde{n}_\mu \tag{3.88}$$

Clearly H is diagonal over the particle number basis of $\tilde{a}/\tilde{a}^\dagger$.

For a Hamiltonian with two-body interaction, it's general form is

$$H = T_{\mu\nu} a^\dagger_\mu a_\nu + \frac{1}{2} U_{\alpha\beta\mu\nu} a^\dagger_\alpha a^\dagger_\beta a_\nu a_\mu \tag{3.89}$$

There's in general no easy way to diagonalize this many-body Hamiltonian strictly rather than the exact diagonalization method. But its ground state can be more easily derived approximately by the **Hatree-Fock method**. For N electrons, the ground state is obtained by minimizing the total energy of the state of the form

$$\prod_i \Big(\sum_\mu c_{i\mu} a^\dagger_\mu \Big) |0\rangle \tag{3.90}$$

where $c_{i\mu}$s are variational parameters.

Chapter 4

Green's Function for Many-body System

4.1 Green's Function

N-point Green's function is defined by

$$G(i_1, t_1, \ldots, i_n, t_n) = (-i)\langle\Omega|T[O(i_1, t_1)\cdots O(i_n, t_n)]|\Omega\rangle \tag{4.1}$$

The simplest kind of Green's function is the transition amplitude from the ith state to the jth state after time t:

$$\begin{aligned} G(j, t_j; i, t_i) &= (-i)\langle\Omega|T[a_j(t_j)a_i^\dagger(t_i)]|\Omega\rangle &(4.2)\\ &= (-i)e^{iE_0|t_j-t_i|}[\langle j|U(t_i \to t_j)]|i\rangle\theta(t_j - t_i) + \xi\langle\tilde{i}|U(t_j \to t_i)]|\tilde{j}\rangle\theta(t_i - t_j)] &(4.3)\end{aligned}$$

where $\xi = \pm 1$ for boson/fermion, $|0\rangle$ is the ground state, $a(t)$ is in Heisenberg picture, and $|i\rangle = a_i^\dagger|0\rangle$ and $|\tilde{i}\rangle = a_i|0\rangle$.

For non-interacting system, $a_i|0\rangle = 0$. So

$$G(j, t_j; i, t_i) = (-i)e^{iE_0|t_j-t_i|}\langle j|U(t_i \to t_j)]|i\rangle\theta(t_j - t_i) \tag{4.4}$$

which returns to the old result.

4.2 Propagator of Free System

A propagator is a two-point Green's function with field operators.

4.2.1 Free Fermions

The Hamiltonian reads

$$H = \sum_{k\sigma}\epsilon_k c_{k\sigma}^\dagger c_{k\sigma} - \mu N = \sum_{k\sigma}\tilde{\epsilon}_k c_{k\sigma}^\dagger c_{k\sigma} \tag{4.5}$$

where $\tilde{\epsilon}_k = \epsilon_k - \mu$

We have

$$|\Omega\rangle = \prod_{\tilde{\epsilon}_k<0,\sigma} c^\dagger_{k\sigma}|0\rangle \tag{4.6}$$

and

$$c_{k\sigma}(t) = e^{-i\tilde{\epsilon}_k t} c_{k\sigma} \tag{4.7}$$

$$\begin{aligned}
& iG_{\sigma_j\sigma_i}(x_j,t_j;x_i,t_i) && (4.8)\\
= & \langle\Omega|T[c_{k\sigma_j}(t_j)c^\dagger_{k\sigma_i}(t_i)]|\Omega\rangle && (4.9)\\
= & \langle\Omega|c_{\sigma_j}(x_j,t_j)c^\dagger_{\sigma_i}(x_i,t_i)|\Omega\rangle\theta(t_f-t_i) - \langle\Omega|c^\dagger_{\sigma_i}(x_i,t_i)c_{\sigma_j}(x_j,t_j)|\Omega\rangle\theta(t_i-t_f) && (4.10)\\
= & \int\frac{d^3p_j}{(2\pi)^3}\int\frac{d^3p_i}{(2\pi)^3}e^{ip_j\cdot x_j-ip_i\cdot x_i}\big[\langle\Omega|c_{\sigma_j}(p_j,t_f)c^\dagger_{\sigma_i}(p_i,t_i)|\Omega\rangle\theta(t_f-t_i)\\
& -\langle\Omega|c^\dagger_{\sigma_i}(p_i,t_i)c_{\sigma_j}(p_j,t_j)|\Omega\rangle\theta(t_i-t_f)\big] && (4.11)\\
= & \int\frac{d^3p}{(2\pi)^3}e^{ip\cdot(x_j-x_i)}\big[\langle\Omega|c_{\sigma_j}(p,t_j)c^\dagger_{\sigma_i}(p,t_i)|\Omega\rangle\theta(t_j-t_i)\\
& -\langle\Omega|c^\dagger_{\sigma_i}(p,t_i)c_{\sigma_j}(p,t_j)|\Omega\rangle\theta(t_i-t_f)\big] && (4.12)\\
= & \int\frac{d^3p}{(2\pi)^3}e^{ip\cdot(x_j-x_i)}e^{-i\tilde{\epsilon}_p(t_j-t_i)}\big[\langle\Omega|c_{p\sigma_j}c^\dagger_{p\sigma_i}|\Omega\rangle\theta(t_f-t_i) - \langle\Omega|c^\dagger_{p\sigma_i}c_{p\sigma_j}|\Omega\rangle\theta(t_i-t_f)\big] && (4.13)\\
= & \int\frac{d^3p}{(2\pi)^3}e^{ip\cdot(x_j-x_i)}e^{-i\tilde{\epsilon}_p(t_j-t_i)}\big[\theta(\tilde{\epsilon}_p)\theta(t_f-t_i)-\theta(-\tilde{\epsilon}_p)\theta(t_i-t_f)\big]\delta_{\sigma_j\sigma_i} && (4.14)
\end{aligned}$$

So

$$\begin{aligned}
iG_{\sigma_j\sigma_i}(p,t_j-t_i) &= e^{-i\tilde{\epsilon}_p(t_j-t_i)}\big[\theta(\tilde{\epsilon}_p)\theta(t_j-t_i)-\theta(-\tilde{\epsilon}_p)\theta(t_i-t_j)\big]\delta_{\sigma_j\sigma_i} && (4.15)\\
&= \frac{1}{-2\pi i}\int d\omega e^{-i\omega(t_j-t_i)}\frac{\delta_{\sigma_j\sigma_i}}{\omega-\tilde{\epsilon}_p(1+i\delta)} && (4.16)
\end{aligned}$$

where $0<\delta\ll 1$.

So

$$G_{\sigma_j\sigma_i}(p,\omega) = \frac{\delta_{\sigma_j\sigma_i}}{\omega-\tilde{\epsilon}_p(1+i\delta)} \tag{4.17}$$

Let's check the validity of Eqn. 4.16. There're 4 different cases:

1. $t_f-t_i<0$, $\tilde{\epsilon}_p>0$. The pole has a negative imaginary part. We take the C_1 contour in Fig. 4.1 (a).
$$\int_{-\infty}^{\infty}d\omega e^{-i\omega(t_j-t_i)}\frac{\delta_{\sigma_j\sigma_i}}{\omega-\tilde{\epsilon}_p(1+i\delta)} = \int_{C_1}d\omega e^{-i\omega(t_j-t_i)}\frac{\delta_{\sigma_j\sigma_i}}{\omega-\tilde{\epsilon}_p(1+i\delta)} = 0 \tag{4.18}$$

2. $t_f-t_i>0$, $\tilde{\epsilon}_p>0$. The pole has a negative imaginary part. We take the C_2 contour in Fig. 4.1 (a).
$$\int_{-\infty}^{\infty}d\omega e^{-i\omega(t_j-t_i)}\frac{\delta_{\sigma_j\sigma_i}}{\omega-\tilde{\epsilon}_p(1+i\delta)} = \int_{C_2}d\omega e^{-i\omega(t_j-t_i)}\frac{\delta_{\sigma_j\sigma_i}}{\omega-\tilde{\epsilon}_p(1+i\delta)} = -2\pi i e^{-i\tilde{\epsilon}_p(t_j-t_i)} \tag{4.19}$$

3. $t_f - t_i < 0$, $\tilde{\epsilon}_p < 0$. The pole has a positive imaginary part. We take the C_1 contour in Fig. 4.1 (b).

$$\int_{-\infty}^{\infty} d\omega e^{-i\omega(t_j - t_i)} \frac{\delta_{\sigma_j\sigma_i}}{\omega - \tilde{\epsilon}_p(1 + i\delta)} = \int_{C_2} d\omega e^{-i\omega(t_j - t_i)} \frac{\delta_{\sigma_j\sigma_i}}{\omega - \tilde{\epsilon}_p(1 + i\delta)} = 2\pi i e^{-i\tilde{\epsilon}_p(t_i - t_j)} \tag{4.20}$$

4. $t_f - t_i > 0$, $\tilde{\epsilon}_p < 0$. The pole has a positive imaginary part. We take the C_2 contour in Fig. 4.1 (b).

$$\int_{-\infty}^{\infty} d\omega e^{-i\omega(t_j - t_i)} \frac{\delta_{\sigma_j\sigma_i}}{\omega - \tilde{\epsilon}_p(1 + i\delta)} = \int_{C_2} d\omega e^{-i\omega(t_j - t_i)} \frac{\delta_{\sigma_j\sigma_i}}{\omega - \tilde{\epsilon}_p(1 + i\delta)} = 0 \tag{4.21}$$

In conclusion

$$\int_{-\infty}^{\infty} d\omega e^{-i\omega(t_j - t_i)} \frac{\delta_{\sigma_j\sigma_i}}{\omega - \tilde{\epsilon}_p(1 + i\delta)} = (-2\pi i) e^{-i\tilde{\epsilon}_p(t_j - t_i)} [\theta(\tilde{\epsilon}_p)\theta(t_j - t_i) - \theta(-\tilde{\epsilon}_p)\theta(t_i - t_j)] \delta_{\sigma_j\sigma_i} \tag{4.22}$$

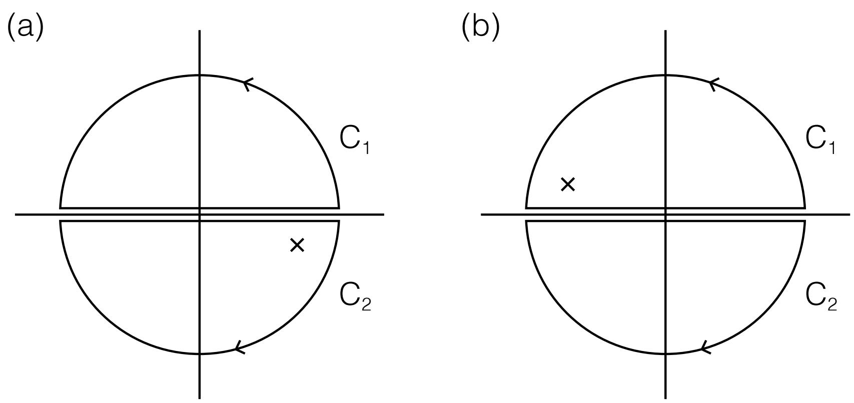


Figure 4.1: Contour integral for fermionic system. (a): the case $\tilde{\epsilon}_p > 0$. (b): the case $\tilde{\epsilon}_p < 0$

4.2.2 Free Bosons

The Hamiltonian reads

$$H = \sum_{k\sigma} \omega_k a_k^\dagger a_k \tag{4.23}$$

We have

$$|\Omega\rangle = a_0^{\dagger N} |0\rangle \tag{4.24}$$

and

$$a_k(t) = e^{-i\omega_k t} a_k \tag{4.25}$$

Let $\phi = a + a^\dagger$, we traditionally define

$$iG(x_j, t_j; x_i, t_i) = \langle \Omega | T[\phi(x_j, t_j)\phi(x_i, t_i)] | \Omega \rangle \tag{4.26}$$

So

$$
\begin{aligned}
& iG(x_j,t_j;x_i,t_i) && (4.27)\\
= & \langle\Omega|a(x_j,t_j)a^\dagger(x_i,t_i)|\Omega\rangle\theta(t_f-t_i)+\langle\Omega|a^\dagger(x_j,t_j)a(x_i,t_i)|\Omega\rangle\theta(t_f-t_i)\\
& +\langle\Omega|a(x_i,t_i)a^\dagger(x_j,t_j)|\Omega\rangle\theta(t_i-t_j)+\langle\Omega|a^\dagger(x_i,t_i)a(x_j,t_j)|\Omega\rangle\theta(t_i-t_j) && (4.28)\\
= & \int\frac{d^3p_j}{(2\pi)^3}\int\frac{d^3p_i}{(2\pi)^3}e^{ip_j\cdot x_j-ip_i\cdot x_i}\big[\langle\Omega|a(p_j,t_j)a^\dagger(p_i,t_i)|\Omega\rangle\theta(t_j-t_i)+\langle\Omega|a^\dagger(p_j,t_j)a(p_i,t_i)|\Omega\rangle\\
& \theta(t_j-t_i)+\langle\Omega|a(p_i,t_i)a^\dagger(p_j,t_j)|\Omega\rangle\theta(t_i-t_j)+\langle\Omega|a^\dagger(p_i,t_i)a(p_j,t_j)|\Omega\rangle\theta(t_i-t_j)\big] && (4.29)\\
= & \int\frac{d^3p}{(2\pi)^3}e^{ip\cdot(x_j-x_i)}\big[\langle\Omega|a(p,t_j)a^\dagger(p,t_i)|\Omega\rangle\theta(t_j-t_i)+\langle\Omega|a^\dagger(p,t_j)a(p,t_i)|\Omega\rangle\\
& \theta(t_j-t_i)+\langle\Omega|a(p,t_i)a^\dagger(p,t_j)|\Omega\rangle\theta(t_i-t_j)+\langle\Omega|a^\dagger(p,t_i)a(p,t_j)|\Omega\rangle\theta(t_i-t_j)\big] && (4.30)\\
= & \int\frac{d^3p}{(2\pi)^3}e^{ip\cdot(x_j-x_i)}\big[\langle\Omega|a(p)a^\dagger(p)|\Omega\rangle e^{-i\omega_p(t_j-t_i)}\theta(t_j-t_i)+\langle\Omega|a^\dagger(p)a(p)|\Omega\rangle e^{-i\omega_p(t_i-t_j)}\\
& \theta(t_j-t_i)+\langle\Omega|a(p)a^\dagger(p)|\Omega\rangle e^{-i\omega_p(t_i-t_j)}\theta(t_i-t_j)+\langle\Omega|a^\dagger(p)a(p)|\Omega\rangle e^{-i\omega_p(t_j-t_i)}\theta(t_i-t_j)\big] && (4.31)\\
= & \int\frac{d^3p}{(2\pi)^3}e^{ip\cdot(x_j-x_i)}\big[e^{-i\omega_p(t_j-t_i)}\theta(t_j-t_i)+e^{-i\omega_p(t_i-t_j)}\theta(t_i-t_j)\big]+N && (4.32)
\end{aligned}
$$

Let's suppose $N=0$. Then

$$
iG(x_j-x_i,t_j-t_i)=\int\frac{d^3p}{(2\pi)^3}e^{ip\cdot(x_j-x_i)}\big[e^{-i\omega_p(t_j-t_i)}\theta(t_j-t_i)+e^{-i\omega_p(t_i-t_j)}\theta(t_i-t_j)\big] \tag{4.33}
$$

So

$$
\begin{aligned}
iG(p,t_j-t_i) &= \big[e^{-i\omega_p(t_j-t_i)}\theta(t_j-t_i)+e^{-i\omega_p(t_i-t_j)}\theta(t_i-t_j)\big] && (4.34)\\
&= \frac{1}{-2\pi i}\int d\omega e^{-i\omega(t_j-t_i)}\frac{1}{\omega-\omega_p+i\delta}+\frac{1}{2\pi i}\int d\omega e^{-i\omega(t_j-t_i)}\frac{1}{\omega+\omega_p-i\delta} && (4.35)\\
&= \frac{1}{-2\pi i}\int d\omega e^{-i\omega(t_j-t_i)}\Big(\frac{1}{\omega-\omega_p+i\delta}-\frac{1}{\omega+\omega_p-i\delta}\Big) && (4.36)\\
&= \frac{1}{-2\pi i}\int d\omega e^{-i\omega(t_j-t_i)}\frac{2\omega_p}{\omega^2-(\omega_p-i\delta)^2} && (4.37)\\
&= \frac{1}{-2\pi i}\int d\omega e^{-i\omega(t_j-t_i)}\frac{2\omega_p}{\omega^2-\omega_p^2+i\delta} && (4.38)
\end{aligned}
$$

where $0<\delta\ll 1$. We take the C_1 contour in Fig. 4.2 when $t_f-t_i<0$, abd C_2 contour when $t_f-t_i>0$
So

$$
G(p,\omega)=\frac{2\omega_p}{\omega^2-\omega_p^2+i\delta} \tag{4.39}
$$

4.3 Different Green's Functions

We use Lehmann spectral representation to study relation between different two-point Green's functions.

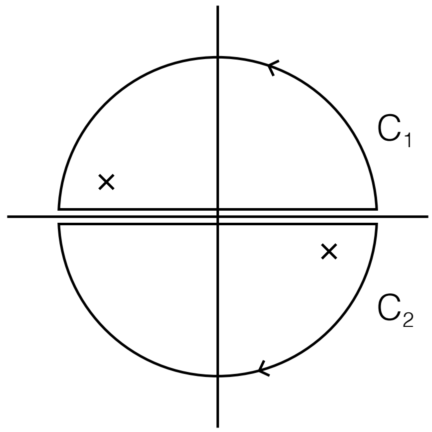


Figure 4.2: Contour integral for bosonic system

4.3.1 Feynman Propogator

$$
\begin{aligned}
iG(t) &= \langle T[O_2(t)O_1(0)]\rangle && (4.40)\\
&= \langle O_2(t)O_1(0)\rangle\theta(t) + \eta\langle O_1(0)O_2(t)\rangle\theta(-t) && (4.41)\\
&= \sum_n \left[\langle\Omega|O_2(t)|n\rangle\langle n|O_1(0)|\Omega\rangle\theta(t) + \eta\langle\Omega|O_1(0)|n\rangle\langle n|O_2(t)|\Omega\rangle\theta(-t)\right] && (4.42)\\
&= \sum_n \left[e^{-itE_n}\theta(t)O_n + \eta e^{itE_n}\theta(-t)O'_n\right] && (4.43)\\
&= \int \frac{d\omega}{-2\pi i}e^{-it\omega}\sum_n\left[\frac{O_n}{\omega - E_n + i\delta} - \frac{\eta O'_n}{\omega + E_n - i\delta}\right] && (4.44)\\
G(\omega) &= \sum_n\left[\frac{O_n}{\omega - E_n + i\delta} - \frac{\eta O'_n}{\omega + E_n - i\delta}\right] && (4.45)\\
& && (4.46)
\end{aligned}
$$

On the real axis,

$$
\begin{aligned}
\Re G(\omega) &= \sum_n\left[P\frac{O_n}{\omega - E_n} - P\frac{\eta O'_n}{\omega + E_n}\right] && (4.47)\\
\Im G(\omega) &= \sum_n\left[(-\pi)O_n\delta(\omega - E_n) - \eta\pi O'_n\delta(\omega + E_n)\right] && (4.48)\\
&= -\pi\sum_n\left[O_n\delta(\omega - E_n) + \eta O'_n\delta(\omega + E_n)\right] && (4.49)
\end{aligned}
$$

4.3.2 Retarded Green's Function

$$
iG_R(t) = \langle[O_2(t), O_1(0)]_\eta\rangle\theta(t) \qquad (4.50)
$$

$$\begin{aligned}
&= \big[\langle O_2(t)O_1(0)\rangle - \eta\langle O_1(0)O_2(t)\rangle\big]\theta(t) && (4.51)\\
&= \sum_n \big[\langle\Omega|O_2(t)|n\rangle\langle n|O_1(0)|\Omega\rangle - \eta\langle\Omega|O_1(0)|n\rangle\langle n|O_2(t)|\Omega\rangle\big]\theta(t) && (4.52)\\
&= \sum_n \big[e^{-itE_n}O_n - \eta e^{itE_n}O'_n\big]\theta(t) && (4.53)\\
&= \int \frac{d\omega}{-2\pi i}e^{-it\omega}\sum_n\Big[\frac{O_n}{\omega-E_n+i\delta} - \frac{\eta O'_n}{\omega+E_n+i\delta}\Big] && (4.54)\\
G_R(\omega) &= \sum_n\Big[\frac{O_n}{\omega-E_n+i\delta} - \frac{\eta O'_n}{\omega+E_n+i\delta}\Big] && (4.55)
\end{aligned}$$

On the real axis,

$$\begin{aligned}
\Re G_R(\omega) &= \sum_n\Big[P\frac{O_n}{\omega-E_n} - P\frac{\eta O'_n}{\omega+E_n}\Big] && (4.56)\\
\Im G_R(\omega) &= \sum_n\big[(-\pi)O_n\delta(\omega-E_n) - \eta(-\pi)O'_n\delta(\omega+E_n)\big] && (4.57)\\
&= -\pi\sum_n\big[O_n\delta(\omega-E_n) - \eta O'_n\delta(\omega+E_n)\big] && (4.58)
\end{aligned}$$

For boson ($\eta = 1$), then $n = 0$ terms vanish. So the summation should be $\sum_{n\neq 0}$.

4.3.3 Advanced Green's Function

$$\begin{aligned}
iG_A(t) &= -\langle[O_2(t), O_1(0)]_\eta\rangle\theta(-t) && (4.59)\\
&= -\big[\langle O_2(t)O_1(0)\rangle - \eta\langle O_1(0)O_2(t)\rangle\big]\theta(-t) && (4.60)\\
&= -\sum_n \big[\langle\Omega|O_2(t)|n\rangle\langle n|O_1(0)|\Omega\rangle - \eta\langle\Omega|O_1(0)|n\rangle\langle n|O_2(t)|\Omega\rangle\big]\theta(-t) && (4.61)\\
&= -\sum_n \big[e^{-itE_n}O_n - \eta e^{itE_n}O'_n\big]\theta(-t) && (4.62)\\
&= \int \frac{d\omega}{-2\pi i}e^{-it\omega}\sum_n\Big[\frac{O_n}{\omega-E_n-i\delta} - \frac{\eta O'_n}{\omega+E_n-i\delta}\Big] && (4.63)\\
G_A(\omega) &= \sum_n\Big[\frac{O_n}{\omega-E_n-i\delta} - \frac{\eta O'_n}{\omega+E_n-i\delta}\Big] && (4.64)
\end{aligned}$$

On the real axis,

$$\begin{aligned}
\Re G_A(\omega) &= \sum_n\Big[P\frac{O_n}{\omega-E_n} - P\frac{\eta O'_n}{\omega+E_n}\Big] && (4.65)\\
\Im G_A(\omega) &= \sum_n\big[\pi O_n\delta(\omega-E_n) - \eta\pi O'_n\delta(\omega+E_n)\big] && (4.66)
\end{aligned}$$

$$= \pi \sum_n \left[O_n \delta(\omega - E_n) - \eta O'_n \delta(\omega + E_n) \right] \quad (4.67)$$

For boson ($\eta = 1$), then $n = 0$ terms vanish. So the summation should be $\sum_{n \neq 0}$.

4.3.4 Relation between Different Green's Functions

$$\begin{aligned} G_A(\omega) &= G(\omega \to \omega - i\delta) && (4.68) \\ G_R(\omega) &= G(\omega \to \omega + i\delta) && (4.69) \end{aligned}$$

On the real axis,

$$\begin{aligned} \Re G(\omega) &= \Re G_A(\omega) = \Re G_R(\omega) && (4.70) \\ \Im G(\omega) &= \Im G_R(\omega) H(\omega) = -\Im G_A(\omega) H(\omega) && (4.71) \end{aligned}$$

where H is the Heaviside step function.

4.3.5 Spectral Function

Let's define

$$\begin{aligned} A(\omega) &= 2\Im G_A(\omega) && (4.72) \\ &= i(G_R(\omega) - G_A(\omega)) && (4.73) \\ &= 2\pi \sum_n \left[O_n \delta(\omega - E_n) - \eta O'_n \delta(\omega + E_n) \right] && (4.74) \end{aligned}$$

We have

$$\begin{aligned} \int \frac{d\omega}{2\pi} A(\omega) &= O_n - \eta O'_n && (4.75) \\ &= \sum_n \left[\langle \Omega | O_2 | n \rangle \langle n | O_1 | \Omega \rangle - \eta \langle \Omega | O_1 | n \rangle \langle n | O_2 | \Omega \rangle \right] && (4.76) \\ &= \langle [O_2, O_1]_\eta \rangle && (4.77) \end{aligned}$$

So $\int \frac{d\omega}{2\pi} A(\omega) = 1$ if O_2 and O_1 is a canonical pair.
We can reconstruct the Green's function just from $A(\omega)$ (on real axis) by the contour integral method.
Since $A(\omega) = i(G_R(\omega) - G_A(\omega))$

$$\begin{aligned} \int \frac{d\omega}{2\pi} (n_\eta^- \theta(\tau) + n_\eta^+ \theta(-\tau)) A(\omega) e^{-\omega\tau} &= i \int_C \frac{d\omega}{2\pi} (n_\eta^- \theta(\tau) + n_\eta^+ \theta(-\tau)) \mathcal{G}(\omega) e^{-\omega\tau} && (4.78) \\ &= \frac{1}{\beta} \sum_n \mathcal{G}(i\omega_n) e^{-i\omega_n \tau} && (4.79) \\ &= \mathcal{G}(\tau) && (4.80) \end{aligned}$$

where C is the clockwise contour around the real axis.

4.3.6 Correlation Function

$$
\begin{aligned}
S(t) &= \langle O_2(t)O_1(0)\rangle && (4.81)\\
&= \sum_n \langle\Omega|O_2(t)|n\rangle\langle n|O_1(0)|\Omega\rangle && (4.82)\\
&= \sum_n e^{-itE_n}O_n && (4.83)\\
&= \int \frac{d\omega}{-2\pi i}e^{-it\omega}\sum_n \left[\frac{O_n}{\omega-E_n+i\delta}-\frac{O_n}{\omega-E_n-i\delta}\right] && (4.84)\\
&= \int d\omega e^{-it\omega}\sum_n O_n\delta(\omega-E_n) && (4.85)\\
S(\omega) &= 2\pi\sum_n O_n\delta(\omega-E_n) && (4.86)\\
& && (4.87)
\end{aligned}
$$

We have

$$S(\omega) = -2\theta(\omega)\Im G_R(\omega) \quad (4.88)$$

4.4 Green's Function of Interacting System, the Gell-Mann-Low Theorem

Consider a quantum field be defined by Hamiltonian $H = H_0 + H_{int}$. As long as we only consider event that happened at finite past/future, we may modify it as $H(t) = H_0 + e^{-\epsilon t}H_{int}$ and let the small variable $\epsilon \to 0$ after the calculation.

Since $\epsilon \ll 1$, $H(t)$ satisfy adiabatic evolvement theorem. So we have

$$U_H(-\infty \to 0)|0\rangle = e^{i\alpha}|\Omega\rangle \quad (4.89)$$

where $|0\rangle$ is the (unique) vacuum state of H_0 and $|\Omega\rangle$ is the (unique) vacuum state of $H(0) = H_0 + H_{int}$.

Then

$$S_H(-\infty \to 0)|0\rangle = U_H(-\infty \to 0)U_{H_0}(0 \to -\infty)|0\rangle = e^{i\theta}|\Omega\rangle \quad (4.90)$$

Then we can simplify the Green's function as

$$
\begin{aligned}
iG(x_1,\ldots,x_n) &= \langle T[O_{1,H}(x_1)\cdots O_{n,H}(x_n)]\rangle && (4.91)\\
&= \langle\Omega|ph(p)O_{p_1,H}(x_{p_1})\cdots O_{p_n,H}(x_{p_n})|\Omega\rangle && (4.92)\\
&= \langle 0|ph(p)S_H(0\to-\infty)S_H(t_{p_1}\to 0)O_{p_1,I}(x_{p_1})S_H(0\to t_{p_1})\cdots \\
&\quad S_H(t_{p_n}\to 0)O_{p_n,I}(x_{p_n})S_H(0\to t_{p_n})S_H(-\infty\to 0)|0\rangle && (4.93)\\
&= \langle 0|S_H(\infty\to-\infty)ph(p)S_H(t_{p_1}\to\infty)O_{p_1,I}(x_{p_1})S_H(t_{p_2}\to t_{p_1})\cdots \\
&\quad S_H(t_{p_n}\to t_{p_{n-1}})O_{p_n,I}(x_{p_n})S_H(-\infty\to t_{p_n})|0\rangle && (4.94)
\end{aligned}
$$

$$\stackrel{\text{formally}}{=} \langle 0|S_H(\infty \to -\infty)T[S_H(-\infty \to \infty)O_{1,I}(x_1)\cdots O_{n,I}(x_n)]|0\rangle \tag{4.95}$$

where $ph(p_i)$ is the phase factor (± 1) generated by the permutation $O_1 \ldots O_n \to O_{p_1}, \ldots, O_{p_n}$. x here are 4-vectors.

Since $|0\rangle$ is unique, $\langle 0|S_H(\infty \to -\infty) = \langle 0|e^{i\phi}$. So

$$\langle 0|S_H(\infty \to -\infty) = \frac{\langle 0|}{\langle 0|S_H(-\infty \to \infty)|0\rangle} \tag{4.96}$$

So

$$iG(x_1, \ldots, x_n) = \frac{\langle 0|T[S_H(-\infty \to \infty)O_{1,I}(x_1)\cdots O_{n,I}(x_n)]|0\rangle}{\langle 0|S_H(-\infty \to \infty)|0\rangle} \tag{4.97}$$

Actually it's easy to see that

$$\langle 0|S_H(-\infty \to \infty)|0\rangle = \lim_{T\to\infty}\langle 0|S_H(-T \to T)|0\rangle = \lim_{T\to\infty} e^{iE_0 2T}\langle 0|U_H(-T \to T)|0\rangle = \lim_{T\to\infty} e^{-i\Delta E 2T + i\gamma} \tag{4.98}$$

where ΔE is the interaction energy and γ is the Berry's phase.

So we have

$$\Delta E = \lim_{T\to\infty} \frac{i}{2T} \ln\langle 0|S_H(-T \to T)|0\rangle \tag{4.99}$$

4.5 Perturbative Expansion and Wick's Theorem

We first learn how to treat $S_H(-\infty \to \infty)$ with perturbative expansion.

4.5.1 Perturbative Expansion

From Eqn. 1.38, we have the recursive formula

$$S_H(t_1 \to t_2) = I + (-i)\int_{t_1}^{t_2} d\tau_1 H_{int,I}(\tau_1) S_H(t_1 \to \tau_1) \tag{4.100}$$

Then, if H_{int} is relatively small compared to H_0, we have

$$\begin{aligned} S_H(t_1 \to t_2) &= I + (-i)\int_{t_1}^{t_2} d\tau_1 H_{int,I}(\tau_1) \\ &\quad + (-i)^2 \int_{t_1}^{t_2} d\tau_1 \int_{t_1}^{\tau_1} d\tau_2 H_{int,I}(\tau_1) H_{int,I}(\tau_2) + \ldots \end{aligned} \tag{4.101}$$

Since H_{int} is a bosonic operator, we have

$$\begin{aligned} &\int_{t_1}^{t_2} d\tau_1 \int_{t_1}^{\tau_1} d\tau_2 \cdots \int_{t_1}^{\tau_{n-1}} d\tau_n H_{int,I}(\tau_1) H_{int,I}(\tau_2)\cdots H_{int,I}(\tau_n) \\ = &\ \tfrac{1}{n!}\int_{t_1}^{t_2} d\tau_1 \int_{t_1}^{t_2} d\tau_2 \cdots \int_{t_1}^{t_2} d\tau_n T[H_{int,I}(\tau_1) H_{int,I}(\tau_2)\cdots H_{int,I}(\tau_n)] \end{aligned} \tag{4.102}$$

So formally

$$S_H(t_1 \to t_2) = I + (-i)\int_{t_1}^{t_2} d\tau_1 H_{int,I}(\tau_1) + \frac{1}{2!}T\Big[(-i)\int_{t_1}^{t_2} d\tau_1 H_{int,I}(\tau_1)\Big]^2 + \ldots \tag{4.103}$$

$$= T\Big[e^{-i\int_{t_1}^{t_2} d\tau H_{int,I}(\tau)}\Big] \tag{4.104}$$

So the Green's function becomes

$$iG(x_1,\ldots,x_n) = \frac{\langle 0|T\big[e^{-i\int_{-\infty}^{\infty} d\tau H_{int,I}(\tau)}O_{1,I}(x_1)\cdots O_{n,I}(x_n)\big]|0\rangle}{\langle 0|T\big[e^{-i\int_{-\infty}^{\infty} d\tau H_{int,I}(\tau)}\big]|0\rangle} \tag{4.105}$$

$$= \frac{\langle 0|T\big[\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\prod_{j=1}^{n}\int_{-\infty}^{\infty} d\tau_j H_{int,I}(\tau_j)O_{1,I}(x_1)\cdots O_{n,I}(x_n)\big]|0\rangle}{\langle 0|T\big[\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\prod_{j=1}^{n}\int_{-\infty}^{\infty} d\tau_j H_{int,I}(\tau_j)\big]|0\rangle} \tag{4.106}$$

Note that O_I coincides with O_H in the Heisenberg picture of H_0. So we can pretend that we are still doing the free model, and H_{int} is nothing but a special operator O. Using the old technique, the Green's function can be calculated exactly to any order.

However, instead of direct calculation, we can make our life easier by the following Wick's theorem.

4.5.2 Wick's Theorem

In this section we omit the I subscript of operators. O are always assumed to be O_I.

If we can break $O(t)$ into $O^+(t) + O^-(t)$ which satisfies:

$$O^-(t)|0\rangle = 0 \tag{4.107}$$

$$\langle 0|O^+(t) = 0 \tag{4.108}$$

$$[O_1^+(t_1), O_2^+(t_2)]_\eta = [O_1^-(t_1), O_2^-(t_2)]_\eta = 0 \tag{4.109}$$

The time ordered product of n operators is defined as

$$T[O_1(t_1)\cdots O_n(t_n)] = ph(q)O_{q_1}(t_{q_1})\cdots O_{q_n}(t_{q_n}) \tag{4.110}$$

where $t_{q_1} > \cdots > t_{q_n}$

We define the normal ordered product of n operators is defined as

$$: O_1(t_1)\cdots O_n(t_n) : = :\prod_{i=1}^{n}\sum_{\sigma=-}^{+} O_i^{\sigma}(t_i): \tag{4.111}$$

$$= \sum_{\{\sigma_i\}} :\prod_{i=1}^{n} O_i^{\sigma_i}(t_i): \tag{4.112}$$

$$= \sum_{\{\sigma_i\}} ph(p_{\{\sigma\}})\prod_{i=1}^{n} O_{p_{\{\sigma\}}(i)}^{\sigma_{p_{\{\sigma\}}(i)}}(t_{p_{\{\sigma\}}(i)}) \tag{4.113}$$

$$\tag{4.114}$$

where $p_{\{\sigma\}}$ is a permutation of 1 ...n that depends on $\{\sigma\}$ such that $\sigma_{p_{\{\sigma\}}(1)} = \cdots = \sigma_{p_{\{\sigma\}}(m)} = +$ and $\sigma_{p_{\{\sigma\}}(m+1)} = \cdots = \sigma_{p_{\{\sigma\}}(n)} = -$. This is well-defined because of 4.109.

Then we define Wick contraction as

$$\begin{aligned} \wick{\c O_1(t_1) \c O_2(t_2)} &= \begin{cases} [O_1^-(t_1), O_2^+(t_2)]_\eta & (t_1 > t_2) \\ (-1)^\eta [O_2^-(t_2), O_1^+(t_1)]_\eta & (t_1 < t_2) \end{cases} && (4.115) \\ \wick{\c O_1(t_1) \prod_i O_i(t_i) \c O_2(t_2)} &= (-1)^{\sum_i \eta_i \eta_1} \prod_i O_i(t_i) \wick{\c O_1(t_1) \c O_2(t_2)} && (4.116) \end{aligned}$$

The definition is readily generalized to multiple Wick contractions in a product: just do Wick contractions in sequence using 4.116.

It's easy to see that

$$\wick{\c O_1(t_1) \c O_2(t_2)} = (-1)^\eta \wick{\c O_2(t_2) \c O_1(t_1)} \tag{4.117}$$

The Wick's theorem states that

Theorem 4.5.1 (Wick)**.**

$$T[O_1(t_1) \cdots O_n(t_n)] =: \text{all possible contractions of } O_1(t_1) \cdots O_n(t_n) : \tag{4.118}$$

where Wick contracted operators are not taken into consideration in normal ordering.

Proof. We prove the theorem by induction on the number of operators.

First we observe

$$\begin{aligned} T[O_1(t_1)] &= O_1(t_1) && (4.119) \\ T[O_1(t_1)O_2(t_2)] &= \wick{\c O_1(t_1) \c O_2(t_2)} + :O_1(t_1)O_2(t_2): && (4.120) \\ & && (4.121) \end{aligned}$$

So the theorem is valid for product of 1 and 2 operators. Assume the theorem is valid for product of n-1 operators, we need to prove

$$T[O_1(t_1) \cdots O_n(t_n)] =: \text{all possible contractions of } O_1(t_1) \cdots O_n(t_n) : \tag{4.122}$$

Assume t_m is the largest among $t_1 \dots t_n$

$$\begin{aligned} T[O_1(t_1) \cdots O_n(t_n)] &= (-1)^{\sum_{i<m} \eta_i \eta_m} O_m(t_m) T\Big[\prod_{j \neq m} O_j(t_j)\Big] && (4.123) \\ &= : \text{all possible contractions of } \prod_j O_j(t_j) \text{ such that } O_m(t_m) \text{ is uncontracted} \\ &\quad \text{and has a plus sign} : + (-1)^{\sum_{i<m} \eta_i \eta_m} O_m^-(t_m) : \text{all possible contractions} \\ &\quad \text{of} \prod_{j \neq m} O_j(t_j) : && (4.124) \end{aligned}$$

It's not hard to see that

$$
\begin{aligned}
&(-1)^{\sum_{i<m}\eta_i\eta_m} O_m^-(t_m) : \text{all possible contractions of } \prod_{j\neq m} O_j(t_j) : && (4.125)\\
= &: \text{all possible contractions of } \prod_j O_j(t_j) \text{ such that } O_m(t_m) \text{ is uncontracted and has a}\\
&\text{minux sign} : + : \text{all possible contractions of } \prod_j O_j(t_j) \text{ with } O_m(t_m) \text{ contracted} : && (4.126)
\end{aligned}
$$

So

$$
T[O_1(t_1)\cdots O_n(t_n)] =: \text{all possible contractions of } O_1(t_1)\cdots O_n(t_n) : \tag{4.127}
$$

$\square$

For example

$$
\begin{aligned}
T[O_1(t_1)O_2(t_2)O_3(t_3)] &= :\overbracket{O_1(t_1)O_2}(t_2)O_3(t_3): + :\overbracket{O_1(t_1)O_2(t_2)O_3}(t_3): +\\
&\quad :O_1(t_1)\overbracket{O_2(t_2)O_3}(t_3): + :O_1(t_1)O_2(t_2)O_3(t_3): && (4.128)\\
T[O_1(t_1)O_2(t_2)O_3(t_3)O_4(t_4)] &= :\overbracket{O_1(t_1)O_2}(t_2)\overbracket{O_3(t_3)O_4}(t_4): + :\overbracket{O_1(t_1)O_2(t_2)O_3}(t_3)O_4(t_4): +\\
&\quad :\overbracket{O_1(t_1)\overbracket{O_2(t_2)O_3}(t_3)O_4}(t_4): + :\overbracket{O_1(t_1)O_2}(t_2)O_3(t_3)O_4(t_4): +\\
&\quad :\overbracket{O_1(t_1)O_2(t_2)O_3}(t_3)O_4(t_4): + :\overbracket{O_1(t_1)O_2(t_2)O_3(t_3)O_4}(t_4): +\\
&\quad :O_1(t_1)\overbracket{O_2(t_2)O_3}(t_3)O_4(t_4): + :O_1(t_1)\overbracket{O_2(t_2)O_3(t_3)O_4}(t_4): +\\
&\quad :O_1(t_1)O_2(t_2)\overbracket{O_3(t_3)O_4}(t_4): + :O_1(t_1)O_2(t_2)O_3(t_3)O_4(t_4): && (4.129)
\end{aligned}
$$

Obviously we have

$$
\begin{aligned}
iG^0_{12}(t_1,t_2) &= \langle 0|T[O_1(t_1)O_2(t_2)]|0\rangle && (4.130)\\
&= \langle 0|\overbracket{O_1(t_1)O_2}(t_2)|0\rangle && (4.131)
\end{aligned}
$$

where G^0 means Green's function of free system

We call operators simple if they satisfy

$$
\overbracket{O_1(t_1)O_2}(t_2)|0\rangle = |0\rangle\langle 0|\overbracket{O_1(t_1)O_2}(t_2)|0\rangle \tag{4.132}
$$

Then

$$
\begin{aligned}
\langle 0|T[O_1(t_1)\cdots O_n(t_n)]|0\rangle &= \langle 0| : \text{all full contractions of } O_1(t_1)\cdots O_n(t_n) : |0\rangle && (4.133)\\
&= \sum_{\{p\}} ph(p)\prod_i \langle 0|\overbracket{O_{p_{i,1}}(t_{p_{i,1}})O_{p_{i,2}}}(t_{p_{i,2}})|0\rangle && (4.134)\\
&= \sum_{\{p\}} ph(p)\prod_i iG_{p_{i,1}p_{i,2}}(t_{p_{i,1}},t_{p_{i,2}}) && (4.135)
\end{aligned}
$$

where full contractions means every operator is Wick contracted, $\{p_i\}$ is pairs that 1 ...n can be divided into and $ph(p)$ is some phase factor decided by $\{p_i\}$.

For example, if all operators are fermionic operators

$$iG^0_{1234}(t_1,t_2,t_3,t_4) = \langle 0|T[O_1(t_1)O_2(t_2)O_3(t_3)O_4(t_4)]|0\rangle \tag{4.136}$$

$$= iG^0_{12}(t_1,t_2)iG^0_{34}(t_3,t_4) - iG^0_{13}(t_1,t_3)iG^0_{24}(t_2,t_4) + iG^0_{14}(t_1,t_4)iG^0_{23}(t_2,t_3) \tag{4.137}$$

Chapter 5

Functional Field Integral

In this chapter, the space is always discretized into $3N$ points

5.1 Path Integral of the Scalar Field

Let $\hat{\phi}(x)$ ($x \in \mathbb{R}^3$) be a bosonic field operator. That is, one with eigen-states $|\phi(x)\rangle$, such that

$$\hat{\phi}(x)|\phi(x)\rangle = \phi(x)|\phi(x)\rangle \tag{5.1}$$

and

$$1 = \int d\phi(x)|\phi(x)\rangle\langle\phi(x)| \tag{5.2}$$

where the $|\phi(x)\rangle\langle\phi(x)|$ integrated over all $\phi(x)$.

Let the Hamiltonian be

$$H = \int d^3x \frac{1}{2}\pi^2 + V(\phi) \tag{5.3}$$

Then

$$\begin{aligned}
\langle\phi_j|U(t_i \to t_i + \Delta t)|\phi_i\rangle &= \langle\phi_j|e^{-iH\Delta t}|\phi_i\rangle && (5.4)\\
&= \int \frac{d\pi}{(2\pi)^{3N}}\langle\phi_j|\pi\rangle\langle\pi|e^{-iH\Delta t}|\phi_i\rangle && (5.5)\\
&= \int \frac{d\pi}{(2\pi)^{3N}} e^{i\int d^3x\pi\phi_j} e^{-i(\frac{\pi^2}{2}+V(\pi_i))\Delta t} e^{-i\int d^3x\pi\phi_i} && (5.6)\\
&= \int \frac{d\pi}{(2\pi)^{3N}} e^{i(\pi\frac{\Delta\phi}{\Delta t}-\frac{\pi^2}{2m}-V(\phi_i))\Delta t} && (5.7)\\
&= \left(\sqrt{\frac{1}{2\pi i\Delta t}}\right)^{3N} e^{i(\frac{1}{2}(\frac{\Delta\phi}{\Delta t})^2-V(\phi_i))\Delta t} && (5.8)\\
&= \left(\sqrt{\frac{1}{2\pi i\Delta t}}\right)^{3N} e^{iL(\phi_i,\partial_0\phi_i)\Delta t} && (5.9)
\end{aligned}$$

So

$$\begin{aligned}
\langle\phi_j|U(t_i\to t_j)|\phi_i\rangle &= \langle\phi_j|U(t_N\to t_j)|\phi_N\rangle\cdots\langle\phi_2|U(t_1\to t_2)|x_1\rangle\langle\phi_1|U(t_i\to t_1)|\phi_i\rangle && (5.10)\\
&= \int\prod d\phi_i\Big(\sqrt{\frac{m}{2\pi i\Delta t}}\Big)^{V(N+1)} e^{\sum_n iL(\phi_n,\partial_0\phi_n,t_n)\Delta t} && (5.11)\\
&= \Big(\sqrt{\frac{m}{2\pi i\Delta t}}\Big)^{V}\int\mathcal{D}\Big(\Big(\sqrt{\frac{m}{2\pi i\Delta t}}\Big)^{V}\phi\Big)e^{i\int_{t_i}^{t_j}dtL(\phi,\partial_0\phi,t)} && (5.12)
\end{aligned}$$

Besides, let $O_i(\phi)$ be a function of ϕ, then

$$\langle\phi_j|U(t_j,0)T[\prod_n O_H(t_n)]U(0,t_i)|\phi_i\rangle = C\int\mathcal{D}\phi\prod_n O(\phi(t_n))e^{i\int_{t_i}^{t_j}dtL(\phi,\partial_0\phi,t)} \quad (5.13)$$

where $t_i < t_n < t_j$

5.2 Path Integral of the Spinor Field

Let $\hat\psi(x)$ ($x\in\mathbb{R}^3$) be a fermionic field operator. Let

$$\begin{aligned}
|\eta\rangle &= e^{\int dx-\eta(x)\hat\psi^\dagger(x)}|0\rangle && (5.14)\\
\langle\bar\eta| &= \langle 0|e^{\int dx\bar\eta(x)\hat\psi(x)} && (5.15)
\end{aligned}$$

be the corresponding coherent states.

Then

$$\begin{aligned}
\langle\bar\eta_j|U(t_i\to t_i+\Delta t)|\eta_i\rangle &= \langle\bar\eta_j|e^{-iH\Delta t}|\eta_i\rangle && (5.16)\\
&\simeq \langle\bar\eta_j|(1-iH\Delta t)|\eta_i\rangle && (5.17)\\
&= e^{\int dx\bar\eta_j\eta_i-iH(\bar\eta_j,\eta_i)\Delta t} && (5.18)
\end{aligned}$$

So

$$\begin{aligned}
\langle\bar\eta|U(t_i\to t_j)|\eta\rangle &= \int\prod_i(d\bar\eta_i d\eta_i)\langle\bar\eta|U(t_N\to t_j)|\eta_N\rangle e^{-\int dx\bar\eta_N(x)\eta_N(x)}\cdots\langle\bar\eta_2|U(t_1\to t_2)|\eta_1\rangle\\
&\quad e^{-\int dx\bar\eta_1(x)\eta_1(x)}\langle\bar\eta_1|U(t_i\to t_1)|\eta\rangle && (5.19)\\
&\simeq \int\prod_i(d\bar\eta_i d\eta_i)\prod_i e^{-\int dx\bar\eta_i\Delta\eta_i-iH(\bar\eta_i,\eta_i)\Delta t} && (5.20)\\
&\simeq \int\mathcal{D}(\bar\eta,\eta)e^{i\int_{t_i}^{t_j}dt(i\int dx\bar\eta\dot\eta-H(\bar\eta,\eta))} && (5.21)\\
&= \int\mathcal{D}(\bar\eta,\eta)e^{i\int_{t_i}^{t_j}dtL} && (5.22)
\end{aligned}$$

Besides, let $O_i(\psi,\psi^\dagger)$ be a function of ψ in normal order, then

$$\langle\bar\eta|U(t_j,0)T[\prod_n O_H(t_n)]U(0,t_i)|\eta\rangle = C\int\mathcal{D}\phi\prod_n O(\bar\eta(t_n),\eta(t_n))e^{i\int_{t_i}^{t_j}dtL} \quad (5.23)$$

where $t_i < t_n < t_j$

5.3 Path Integral Representation of the Green's Function

We have

$$\langle j|U(t_j,0)T[\prod_n O_H(t_n)]U(0,t_i)|i\rangle = C\int \mathcal{D}\phi \prod_n O(\phi(t_n))e^{i\int_{t_i}^{t_j} dtL} \tag{5.24}$$

We can insert $1 = |n\rangle\langle n|$ in energy representation

$$\langle j|U(t_j,0)T[\prod_n O_H(t_n)]U(0,t_i)|i\rangle = \langle j|U(t_j,0)|m\rangle\langle m|T[\prod_n O_H(t_n)]|n\rangle\langle n|U(0,t_i)|i\rangle \tag{5.25}$$

We may complexify t, and integrate t along the path $-T(1-i\epsilon)$ to $T(1-i\epsilon)$. That is

$$\langle j|U(T(1-i\epsilon),0)|m\rangle\langle m|T[\prod_n O_H(t_n)]|n\rangle\langle n|U(0,-T(1-i\epsilon))|i\rangle = C\int \mathcal{D}\phi \prod_n O(\phi(t_n))e^{i\int_{-T(1-i\epsilon)}^{T(1-i\epsilon)} dtL} \tag{5.26}$$

It's easy to see

$$\lim_{T\to\infty}\langle j|U(T(1-i\epsilon),0)|m\rangle = \lim_{T\to\infty}\langle j|U(T(1-i\epsilon),0)|\Omega\rangle\delta_{m0} \tag{5.27}$$

$$\lim_{T\to\infty}\langle n|U(0,-T(1-i\epsilon))|i\rangle = \lim_{T\to\infty}\langle\Omega|U(0,-T(1-i\epsilon))|i\rangle\delta_{n0} \tag{5.28}$$

So the LHS of (5.26) becomes

$$\lim_{T\to\infty}\langle j|U(T(1-i\epsilon),0)|m\rangle\langle m|T[\prod_n O_H(t_n)]|n\rangle\langle n|U(0,-T(1-i\epsilon))|i\rangle \tag{5.29}$$

$$= \lim_{T\to\infty}\langle j|U(T(1-i\epsilon),0)|\Omega\rangle\langle\Omega|T[\prod_n O_H(t_n)]|\Omega\rangle\langle\Omega|U(0,-T(1-i\epsilon))|i\rangle \tag{5.30}$$

$$= \langle\Omega|T[\prod_n O_H(t_n)]|\Omega\rangle \lim_{T\to\infty}\langle j|U(T(1-i\epsilon),0)|\Omega\rangle\langle\Omega|U(0,-T(1-i\epsilon))|i\rangle \tag{5.31}$$

$$= \langle\Omega|T[\prod_n O_H(t_n)]|\Omega\rangle \lim_{T\to\infty}\langle j|U(T(1-i\epsilon),0)|n\rangle\langle n|U(0,-T(1-i\epsilon))|i\rangle \tag{5.32}$$

$$= \langle\Omega|T[\prod_n O_H(t_n)]|\Omega\rangle \lim_{T\to\infty}\langle j|U(T(1-i\epsilon),-T(1-i\epsilon))|i\rangle \tag{5.33}$$

when taking the limit $T\to\infty$.

So

$$\langle\Omega|T[\prod_n O_H(t_n)]|\Omega\rangle \lim_{T\to\infty}\langle j|U(T(1-i\epsilon),-T(1-i\epsilon))|i\rangle = C\int \mathcal{D}\phi \prod_n O(\phi(t_n))e^{i\int_{-\infty(1-i\epsilon)}^{\infty(1-i\epsilon)} dtL} \tag{5.34}$$

Without any O, we have

$$\lim_{T\to\infty}\langle j|U(T(1-i\epsilon),-T(1-i\epsilon))|i\rangle = C\int \mathcal{D}\phi e^{i\int_{-\infty(1-i\epsilon)}^{\infty(1-i\epsilon)} dtL} \tag{5.35}$$

So the Green's function becomes

$$\langle\Omega|T[\prod_n O_H(t_n)]|\Omega\rangle = \frac{\int \mathcal{D}\phi \prod_n O(\phi(t_n))e^{i\int_{-\infty(1-i\epsilon)}^{\infty(1-i\epsilon)} dtL}}{\int \mathcal{D}\phi e^{i\int_{-\infty(1-i\epsilon)}^{\infty(1-i\epsilon)} dtL}} \quad (5.36)$$

$$= \frac{1}{Z}\int \mathcal{D}\phi \prod_n O(\phi(t_n))e^{i\int_{-\infty(1-i\epsilon)}^{\infty(1-i\epsilon)} dtL} \quad (5.37)$$

where $Z = \int \mathcal{D}\phi e^{i\int_{-\infty(1-i\epsilon)}^{\infty(1-i\epsilon)} dtL}$

5.4 Generating Functional Method

The generating functional of the Green's function is the path integral with source term $J(x,t)$. The Green's function can be expressed by the functional derivative of the generating functional.

5.4.1 The Free Bosonic System

For the free bosonic system, we study

$$\int \mathcal{D}\phi \prod_n \phi(x_n, t_n)e^{i\int_{-\infty(1-i\epsilon)}^{\infty(1-i\epsilon)} dtL} \quad (5.38)$$

We define the generating functional as

$$Z[J] = \int \mathcal{D}\phi e^{i\int_{-\infty(1-i\epsilon)}^{\infty(1-i\epsilon)} dt[L+\int d^3x\phi(x,t)J(x,t)]} \quad (5.39)$$

In a free system, $\int dtL = \frac{1}{2}\int d^4xd^4x'\phi(x)A(x,x')\phi(x')$ is a bilinear form of ϕ. Then

$$Z[J] = Z\int \mathcal{D}\phi e^{i\frac{1}{2}\int d^4xd^4x' iJ(x)A^{-1}(x,x')iJ(x')} \quad (5.40)$$

The average of $\prod_n \phi(x_n, t_n)$ can be evaluate by the functional derivative of $Z[J]$ by the source term.

$$\langle\Omega|T[\prod_n \phi(x_n,t_n)]|\Omega\rangle = \frac{1}{Z}\int \mathcal{D}\phi \prod_n \phi(x_n,t_n)e^{i\int_{-\infty(1-i\epsilon)}^{\infty(1-i\epsilon)} dtL} \quad (5.41)$$

$$= \frac{1}{Z}\int \mathcal{D}\phi \prod_n \frac{\delta}{i\delta J(x_n,t_n)} e^{i\int_{-\infty(1-i\epsilon)}^{\infty(1-i\epsilon)} dt\int d^3x[\mathcal{L}+\phi(x,t)J(x,t)]}\Big|_{J=0} \quad (5.42)$$

$$= \frac{1}{Z}\prod_n \frac{\delta}{i\delta J(x_n,t_n)} Z[J]\Big|_{J=0} \quad (5.43)$$

The operator $A^{-1}(x,x')$ remains unambiguous. However we can relate it with the two point Green's function

$$iG(x'-x) = \frac{1}{Z}\frac{\delta}{i\delta J(x')}\frac{\delta}{i\delta J(x)} Z[J]\Big|_{J=0} \quad (5.44)$$

$$= \quad iA^{-1}(x', x) \tag{5.45}$$

Thus the generating functional becomes

$$Z[J] = Z \int \mathcal{D}\phi e^{-\frac{i}{2}\int d^4x d^4x' J(x) G(x-x') J(x')} \tag{5.46}$$

5.4.2 The Free Fermionic System

Similarly, for fermionic system, we evaluate

$$\int \mathcal{D}(\bar{\eta}, \eta) \prod_n \eta^\sigma(x_n, t_n) e^{i\int_{-\infty(1-i\epsilon)}^{\infty(1-i\epsilon)} dt L} \tag{5.47}$$

where $\eta^\sigma = \eta$ if $\sigma = -1$ and $\eta^\sigma = \bar{\eta}$ if $\sigma = 1$.

We define the generating functional as

$$Z[\bar{J}, J] = \int \mathcal{D}\phi e^{i\int_{-\infty(1-i\epsilon)}^{\infty(1-i\epsilon)} dt[L + \int d^3x(\bar{J}(x,t)\phi(x,t) + \bar{\phi}(x,t)J(x,t))]} \tag{5.48}$$

Similarly

$$Z[\bar{J}, J] = Z \int \mathcal{D}\phi e^{i\int d^4x d^4x' i\bar{J}(x) A^{-1}(x,x') iJ(x')} \tag{5.49}$$

Time-ordered average of operator product ise functional derivative of $Z[J]$ by the source term.

$$\begin{aligned} \langle\Omega|T[\prod_n \eta^\sigma(x_n, t_n)]|\Omega\rangle &= \int \mathcal{D}(\bar{\eta}, \eta) \prod_n \eta^\sigma(x_n, t_n) e^{i\int_{-\infty(1-i\epsilon)}^{\infty(1-i\epsilon)} dt L} && (5.50) \\ &= \prod_n \frac{\delta}{\sigma i\delta J^\sigma(x_n, t_n)} Z[\bar{J}, J]\Big|_{J=\bar{J}=0} && (5.51) \end{aligned}$$

$A^{-1}(x', x)$ is related to the two point Green's function by

$$\begin{aligned} iG(x'-x) &= \frac{1}{Z}\frac{\delta}{i\delta\bar{J}(x')}\frac{\delta}{-i\delta J(x)} Z[\bar{J}, J]\Big|_{J=\bar{J}=0} && (5.52) \\ &= iA^{-1}(x', x) && (5.53) \end{aligned}$$

So the generating functional becomes

$$Z[\bar{J}, J] = Z \int \mathcal{D}\phi e^{-i\int d^4x d^4x' \bar{J}(x) G(x-x') J(x')} \tag{5.54}$$

5.4.3 Interacting System

Let $L = L_{free} + L_{int}$. We treat L_{int} as a perturbation. Then

$$\langle\Omega|T[\prod_n O(x_n)]|\Omega\rangle \tag{5.55}$$

$$= \frac{1}{Z}\int \mathcal{D}\phi \prod_n O(x_n) e^{i\int_{-\infty(1-i\epsilon)}^{\infty(1-i\epsilon)} dt L} \tag{5.56}$$

$$= \frac{\int \mathcal{D}\phi \prod_n O(x_n) e^{i\int_{-\infty(1-i\epsilon)}^{\infty(1-i\epsilon)} dt (L_{free}+L_{int})}}{\int \mathcal{D}\phi e^{i\int_{-\infty(1-i\epsilon)}^{\infty(1-i\epsilon)} dt (L_{free}+L_{int})}} \tag{5.57}$$

$$= \frac{\int \mathcal{D}\phi \prod_n O(x_n) \sum_m \frac{1}{m!}(i\int dt L_{int})^m e^{i\int_{-\infty(1-i\epsilon)}^{\infty(1-i\epsilon)} dt L_{free}}}{\int \mathcal{D}\phi \sum_m \frac{1}{m!}(i\int dt L_{int})^m e^{i\int_{-\infty(1-i\epsilon)}^{\infty(1-i\epsilon)} dt L_{free}}} \tag{5.58}$$

$$= \frac{\frac{1}{Z_0}\prod_n O(\frac{\delta}{i\delta K(x_n)}, \frac{\delta}{-i\delta J(x_n)}, \frac{\delta}{i\delta \bar{J}(x_n)}) \sum_m \frac{1}{m!}(i\int dx L_{int}(\frac{\delta}{i\delta K(x)}, \frac{\delta}{-i\delta J(x)}, \frac{\delta}{i\delta \bar{J}(x)}))^m Z_0[K,\bar{J},J]\Big|_{K=J=\bar{J}=0}}{\frac{1}{Z_0}\sum_m \frac{1}{m!}(i\int dx L_{int}(\frac{\delta}{i\delta K(x)}, \frac{\delta}{-i\delta J(x)}, \frac{\delta}{i\delta \bar{J}(x)}))^m Z_0[K,\bar{J},J]\Big|_{K=J=\bar{J}=0}} \tag{5.59}$$

where K is the bosonic source and $J, \bar{J}$ is the fermionic source.

5.5 The Wick's Theorem Rediscovered

In this section we discuss how to systemically evaluate the functional derivative of the generating functional.

Let's recall the Wick's theorem

$$\langle 0|T[O_1(t_1)\cdots O_n(t_n)]|0\rangle = \langle 0| : \text{all full contractions of } O_1(t_1)\cdots O_n(t_n) : |0\rangle \tag{5.60}$$

$$= \sum_{\{p\}} ph(p) \prod_i \langle 0| \overbrace{O_{p_{i,1}}(t_{p_{i,1}}) O}_{p_{i,2}}(t_{p_{i,2}})|0\rangle \tag{5.61}$$

$$= \sum_{\{p\}} ph(p) \prod_i iG_{p_{i,1}p_{i,2}}(t_{p_{i,1}}, t_{p_{i,2}}) \tag{5.62}$$

The functional derivative the generating functional leads to the same result.

Here we only prove the fermionic case. The general case is similar.

Let's evaluate

$$\langle\Omega|T[\prod_n \eta^\sigma(x_n,t_n)]|\Omega\rangle = \frac{1}{Z}\prod_n \frac{\delta}{\sigma i\delta J^\sigma(x_n,t_n)} Z[\bar{J},J]\Big|_{J=\bar{J}=0} \tag{5.63}$$

where p is the pairing of $(\bar{\eta}, \eta)$ in $\prod_n \eta^\sigma(x_n,t_n)$.

Each $\frac{\delta}{\sigma i\delta J^\sigma(x,t)}$ acting on $Z[\bar{J},J]$ will drop a $-\int dx' G(x'-x) J^{-\sigma}(x',t)$. Each $\frac{\delta}{\sigma i\delta J^\sigma(x,t)}$ acting on the dopped $-\int dx G(x'-x)J^\sigma(x,t)$ will result $iG(x'-x)$.

After the action of all $\frac{\delta}{\sigma i\delta J^\sigma(x,t)}$s, there should be no J^σ left to make the path integral non-zero. So each $\frac{\delta}{-i\delta J(x,t)}$ should be paired with a $\frac{\delta}{i\delta \bar{J}(x,t)}$. One should act on $Z[\bar{J},J]$ and the other act on the dropped $-\int dx G(x'-x)J^\sigma(x,t)$. They work together to result $iG(x'-x)$. Let p be the pairings. That is, $\frac{\delta}{-i\delta J(x_{p_{i,1}}, t_{p_{i,1}})}$ and $\frac{\delta}{i\delta \bar{J}(x_{p_{i,2}}, t_{p_{i,2}})}$ are paired. Then it's easy to see

$$\langle\Omega|T[\prod_n \eta^\sigma(x_n,t_n)]|\Omega\rangle = \frac{1}{Z}\prod_n \frac{\delta}{\sigma i\delta J^\sigma(x_n,t_n)} Z[\bar{J},J]\Big|_{J=\bar{J}=0} \tag{5.64}$$

$$= \sum_{\{p\}} ph(p) \frac{1}{Z} \prod_i \frac{\delta}{-i\delta J(x_{p_{i,1}}, t_{p_{i,1}})} \frac{\delta}{i\delta \bar{J}(x_{p_{i,2}}, t_{p_{i,2}})} Z[\bar{J}, J]\Big|_{J=\bar{J}=0} \tag{5.65}$$

$$= \sum_{\{p\}} ph(p) \prod_i iG(t_{p_{i,1}}, t_{p_{i,2}}) \tag{5.66}$$

Chapter 6

Green's Function at Finite Temperature

6.1 Matsubara Green's Function

Let's define the Matsubara Green's function as

$$-\mathcal{G}(x_1,\ldots,x_n) = \langle TO_1(\mathbf{x}_1,\tau_1)\cdots O_n(\mathbf{x}_n,\tau_n)\rangle_{H,\beta} = \frac{\mathrm{Tr}[\rho_H(\beta)TO_1(\mathbf{x}_1,\tau_1)\cdots O_n(\mathbf{x}_n,\tau_n)]}{\mathrm{Tr}[\rho_H(\beta)]} \tag{6.1}$$

where $0 < t_i < \beta$, $x = (\mathbf{x},\tau)$ and $O_i(\mathbf{x}_i,\tau_i) = \rho_H^{-1}(\tau_i)O(\mathbf{x}_i)\rho_H(\tau_i)$.

6.1.1 Periodicity of Green's Functions

Let's consider a two-body Matsubara Green's function.

$$-\mathcal{G}(\tau-\tau') = \langle TO_1(\tau)O_2(\tau')\rangle_{\beta H} \tag{6.2}$$

Since $0 < \tau,\tau' < \beta$, we have $-\beta < \tau-\tau' < \beta$. When $\tau-\tau' < 0$

$$\begin{aligned}
-\mathcal{G}(\tau-\tau') &= \langle \eta O_2(\tau')O_1(\tau)\rangle_{\beta H} && (6.3)\\
&= \langle \eta e^{\beta H}O_1(\tau)e^{-\beta H}O_2(\tau')\rangle_{\beta H} && (6.4)\\
&= \langle \eta O_1(\tau+\beta)O_2(\tau')\rangle_{\beta H} && (6.5)\\
&= -\eta G(\tau-\tau'+\beta) && (6.6)
\end{aligned}$$

We be define $G(\tau-\tau')$ on the real line by assuming $G(\tau-\tau')$ has a β period for boson and 2β for fermion. Then for boson

$$\mathcal{G}(\tau) = \frac{1}{\beta}\sum_n \mathcal{G}(i\omega_n)e^{-i\omega_n\tau} \tag{6.7}$$

where

$$\mathcal{G}(i\omega_n) = \int_0^\beta \mathcal{G}(\tau)e^{i\omega_n\tau} \tag{6.8}$$

and $\omega_n = \frac{2\pi n}{\beta}$.

For fermion

$$\mathcal{G}(\tau) = \frac{1}{2\beta}\sum_n \mathcal{G}'(i\omega_n)e^{-i\omega_n\tau} \tag{6.9}$$

where

$$\mathcal{G}'(i\omega_n) = \int_{-\beta}^{\beta}\mathcal{G}(\tau)e^{i\omega_n\tau} \tag{6.10}$$

and $\omega_n = \frac{\pi(2n+1)}{\beta}$.

Since $G(\tau+\beta) = -G(\tau)$, we have

$$\mathcal{G}'(i\frac{2\pi n}{\beta}) = \int_{-\beta}^{\beta}\mathcal{G}(\tau)e^{i\frac{2\pi n}{\beta}\tau} = 0 \tag{6.11}$$

So we omit $\mathcal{G}'(i\frac{2\pi n}{\beta})$.

Define

$$\mathcal{G}(i\omega_n) = \int_0^{\beta}\mathcal{G}(\tau)e^{i\omega_n\tau} \tag{6.12}$$

Then it's easy to see that $\mathcal{G}'(i\omega_n) = 2\mathcal{G}(i\omega_n)$.

So

$$\mathcal{G}(\tau) = \frac{1}{\beta}\sum_n \mathcal{G}(i\omega_n)e^{-i\omega_n\tau} \tag{6.13}$$

In conclusion

$$\mathcal{G}(\tau) = \frac{1}{\beta}\sum_n \mathcal{G}(i\omega_n)e^{-i\omega_n\tau} \tag{6.14}$$

$$\mathcal{G}(i\omega_n) = \int_0^{\beta}\mathcal{G}(\tau)e^{i\omega_n\tau} \tag{6.15}$$

where $\omega_n = \frac{2\pi n}{\beta}$ for boson and $\omega_n = \frac{\pi(2n+1)}{\beta}$ for fermion.

6.1.2 Matsubara Green's Functions of Free System

The hamiltonian reads

$$H = \sum_k \epsilon_k c_k^\dagger c_k \tag{6.16}$$

In momentum space, the particle density is

$$\langle c_k^\dagger c_k\rangle = \frac{1}{e^{\beta\epsilon_k}-\eta} = n_k \tag{6.17}$$

$$\langle c_k c_k^\dagger\rangle = 1+\eta n_k \tag{6.18}$$

Then

$$-\mathcal{G}(k,\tau) = \langle T c_k(\tau)c_k^\dagger(0)\rangle \tag{6.19}$$

$$
\begin{aligned}
&= \theta(\tau)\langle c_k(\tau)c_k^\dagger(0)\rangle + \eta\theta(-\tau)\langle c_k^\dagger(0)c_k(\tau)\rangle && (6.20)\\
&= e^{-\epsilon_k\tau}[\theta(\tau)(1+\eta n_k) + \eta\theta(-\tau)n_k] && (6.21)\\
&= e^{-\epsilon_k\tau}[\theta(\tau) + \eta n_k] && (6.22)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{G}(k, i\omega_n) &= \int_0^\beta d\tau \mathcal{G}(k,\tau)e^{i\omega_n\tau} && (6.23)\\
&= \int_0^\beta d\tau - e^{(i\omega_n - \epsilon_k)\tau}[1+\eta n_k] && (6.24)\\
&= \frac{1-e^{(i\omega_n-\epsilon_k)\beta}}{i\omega_n - \epsilon_k}[1+\eta n_k] && (6.25)\\
&= \frac{1}{i\omega_n - \epsilon_k} && (6.26)
\end{aligned}
$$

where we have used $e^{i\omega_n\beta} = \eta$.

6.2 The Contour Integral Method

Let f be a function with no pole at the imaginary axis. We sometimes want to calculate

$$\sum_n f(i\omega_n) \tag{6.27}$$

with either $\omega_n = \frac{2\pi n}{\beta}$ or $\omega_n = \frac{\pi(2n+1)}{\beta}$.

Let's introduce an auxiliary function

$$n_\eta^\pm(z) = \frac{\pm 1}{1+\eta e^{\pm\beta z}} \tag{6.28}$$

which have poles at $i\omega_n$ with residue $-\frac{1}{\beta}$.

Let's consider

$$f(z)n_\eta^\pm(z)e^{\pm\delta z} \tag{6.29}$$

Then

$$\int_C dz f(z)n_\eta^\pm(z)e^{\pm\delta z} = 0 \tag{6.30}$$

where C is the counter-clockwise contour that enclose all the singularity.

Then

$$\int_{C_{im}} dz f(z)n_\eta^\pm(z)e^{\pm\delta z} + \sum_i \int_{C_i} dz f(z)n_\eta^\pm(z)e^{\pm\delta z} = 0 \tag{6.31}$$

where C_{im} is the counter-clockwise contour that enclose the poles at the imaginary axis, and C_is are the counter-clockwise contours that enclose the singularities off the imaginary axis.

Then

$$\frac{1}{\beta}\sum_n f(i\omega_n)e^{\pm i\omega_n\delta} = \sum_i \int_{C_i} \frac{dz}{2\pi i} f(z)n_\eta^\pm(z)e^{\pm\delta z} \tag{6.32}$$

6.3 Different Green's Functions

We use Lehmann spectral representation to study relation between different Green's funtions.

6.3.1 Matsubara Green's Function

$$\begin{aligned}
-\mathcal{G}(\tau) &= \langle T[O_2(\tau)O_1(0)]\rangle && (6.33)\\
&= \langle O_2(\tau)O_1(0)\rangle\theta(\tau) + \eta\langle O_1(0)O_2(\tau)\rangle\theta(-\tau) && (6.34)\\
&= \frac{1}{Z}\sum_{mn}\big[e^{-\beta E_m}\langle m|O_2(\tau)|n\rangle\langle n|O_1(0)|m\rangle\theta(\tau) + \eta e^{-\beta E_n}\langle n|O_1(0)|m\rangle\langle m|O_2(\tau)|n\rangle \\
&\quad \theta(-\tau)\big] && (6.35)\\
&= \frac{1}{Z}\sum_{mn} e^{-\tau(E_n-E_m)}\big[e^{-\beta E_m}\theta(\tau) + \eta e^{-\beta E_n}\theta(-\tau)\big]O_{mn} && (6.36)\\
\mathcal{G}(i\omega_s) &= \int_0^\beta \mathcal{G}(\tau)e^{i\omega_s\tau} && (6.37)\\
&= -\frac{1}{Z}\sum_{mn}\int_0^\beta e^{\tau(i\omega_s-(E_n-E_m))}e^{-\beta E_m}O_{mn} && (6.38)\\
&= -\frac{1}{Z}\sum_{mn}\frac{e^{\beta(i\omega_s-(E_n-E_m))}-1}{i\omega_s-(E_n-E_m)}e^{-\beta E_m}O_{mn} && (6.39)\\
&= \frac{1}{Z}\sum_{mn}\frac{e^{-\beta E_m}-\eta e^{-\beta E_n}}{i\omega_s-(E_n-E_m)}O_{mn} && (6.40)
\end{aligned}$$

For boson ($\eta = 1$), when $i\omega_s = 0$,

$$\mathcal{G}(i\omega_s) = \frac{1}{Z}\sum_{m\neq n}\frac{e^{-\beta E_m}-e^{-\beta E_n}}{E_m-E_n}O_{mn} - \frac{\beta}{Z}\sum_m e^{-\beta E_m}O_{mm} \tag{6.41}$$

6.3.2 Finite Temperature Feynman Propogator

$$\begin{aligned}
iG(t) &= \langle T[O_2(t)O_1(0)]\rangle && (6.42)\\
&= \langle O_2(t)O_1(0)\rangle\theta(t) + \eta\langle O_1(0)O_2(t)\rangle\theta(-t) && (6.43)\\
&= \frac{1}{Z}\sum_{mn}\big[e^{-\beta E_m}\langle m|O_2(t)|n\rangle\langle n|O_1(0)|m\rangle\theta(t) + \eta e^{-\beta E_n}\langle n|O_1(0)|m\rangle\langle m|O_2(t)|n\rangle \\
&\quad \theta(-t)\big] && (6.44)\\
&= \frac{1}{Z}\sum_{mn} e^{-it(E_n-E_m)}\big[e^{-\beta E_m}\theta(t) + \eta e^{-\beta E_n}\theta(-t)\big]O_{mn} && (6.45)
\end{aligned}$$

$$= \frac{i}{Z}\int\frac{d\omega}{(2\pi)}\sum_{mn}e^{-it\omega}\Big[\frac{e^{-\beta E_m}}{\omega-(E_n-E_m)+i\delta}-\frac{\eta e^{-\beta E_n}}{\omega-(E_n-E_m)-i\delta}\Big]O_{mn} \tag{6.46}$$

$$G(\omega) = \frac{1}{Z}\sum_{mn}\Big[\frac{e^{-\beta E_m}}{\omega-(E_n-E_m)+i\delta}-\frac{\eta e^{-\beta E_n}}{\omega-(E_n-E_m)-i\delta}\Big]O_{mn} \tag{6.47}$$

$$\Re G(\omega) = \frac{1}{Z}\sum_{mn}P\frac{e^{-\beta E_m}-\eta e^{-\beta E_n}}{\omega-(E_n-E_m)}O_{mn} \tag{6.48}$$

$$\Im G(\omega) = \frac{1}{Z}\sum_{mn}\big[e^{-\beta E_m}(-\pi)\delta(\omega-(E_n-E_m))-\eta e^{-\beta E_n}\pi\delta(\omega-(E_n-E_m))\big]O_{mn} \tag{6.49}$$

$$= -\frac{\pi}{Z}\sum_{mn}e^{-\beta E_m}(1+\eta e^{-\beta\omega})\delta(\omega-(E_n-E_m))O_{mn} \tag{6.50}$$

6.3.3 Finite Temperature Retarded Green's Function

$$iG_R(t) = \langle[O_2(t),O_1(0)]_\eta\rangle\theta(t) \tag{6.51}$$

$$= \big[\langle O_2(t)O_1(0)\rangle-\eta\langle O_1(0)O_2(t)\rangle\big]\theta(t) \tag{6.52}$$

$$= \frac{1}{Z}\sum_{mn}\big[e^{-\beta E_m}\langle m|O_2(t)|n\rangle\langle n|O_1(0)|m\rangle-\eta e^{-\beta E_n}\langle n|O_1(0)|m\rangle\langle m|O_2(t)|n\rangle\big]\theta(t) \tag{6.53}$$

$$= \frac{1}{Z}\sum_{mn}e^{-it(E_n-E_m)}\big[e^{-\beta E_m}-\eta e^{-\beta E_n}\big]\theta(t)O_{mn} \tag{6.54}$$

$$= \frac{i}{Z}\int\frac{d\omega}{(2\pi)}\sum_{mn}e^{-it\omega}\frac{e^{-\beta E_m}-\eta e^{-\beta E_n}}{\omega-(E_n-E_m)+i\delta}O_{mn} \tag{6.55}$$

$$G_R(\omega) = \frac{1}{Z}\sum_{mn}\frac{e^{-\beta E_m}-\eta e^{-\beta E_n}}{\omega-(E_n-E_m)+i\delta}O_{mn} \tag{6.56}$$

$$\Re G_R(\omega) = \frac{1}{Z}\sum_{mn}P\frac{e^{-\beta E_m}-\eta e^{-\beta E_n}}{\omega-(E_n-E_m)}O_{mn} \tag{6.57}$$

$$\Im G_R(\omega) = \frac{1}{Z}\sum_{mn}(e^{-\beta E_m}-\eta e^{-\beta E_n})(-\pi)\delta(\omega-(E_n-E_m))O_{mn} \tag{6.58}$$

$$= -\frac{\pi}{Z}\sum_{mn}e^{-\beta E_m}(1-\eta e^{-\beta\omega})\delta(\omega-(E_n-E_m))O_{mn} \tag{6.59}$$

For boson ($\eta=1$), then $n=m$ terms vanish. So the summation should be $\sum_{m\neq n}$.

6.3.4 Finite Temperature Advanced Green's Function

$$iG_A(t) = -\langle[O_2(t),O_1(0)]_\eta\rangle\theta(-t) \tag{6.60}$$

$$= -\big[\langle O_2(t)O_1(0)\rangle-\eta\langle O_1(0)O_2(t)\rangle\big]\theta(-t) \tag{6.61}$$

$$
\begin{aligned}
&= -\frac{1}{Z}\sum_{mn}\left[e^{-\beta E_m}\langle m|O_2(t)|n\rangle\langle n|O_1(0)|m\rangle - \eta e^{-\beta E_n}\langle n|O_1(0)|m\rangle\langle m|O_2(t)|n\rangle\right] \\
&\quad \theta(-t) && (6.62) \\
&= -\frac{1}{Z}\sum_{mn} e^{-it(E_n-E_m)}\left[e^{-\beta E_m} - \eta e^{-\beta E_n}\right]\theta(-t)O_{mn} && (6.63) \\
&= \frac{i}{Z}\int\frac{d\omega}{(2\pi)}\sum_{mn} e^{-it\omega}\frac{e^{-\beta E_m}-\eta e^{-\beta E_n}}{\omega-(E_n-E_m)-i\delta}O_{mn} && (6.64) \\
G_A(\omega) &= \frac{1}{Z}\sum_{mn}\frac{e^{-\beta E_m}-\eta e^{-\beta E_n}}{\omega-(E_n-E_m)-i\delta}O_{mn} && (6.65) \\
\Re G_A(\omega) &= \frac{1}{Z}\sum_{mn}P\frac{e^{-\beta E_m}-\eta e^{-\beta E_n}}{\omega-(E_n-E_m)}O_{mn} && (6.66) \\
\Im G_A(\omega) &= \frac{1}{Z}\sum_{mn}(e^{-\beta E_m}-\eta e^{-\beta E_n})\pi\delta(\omega-(E_n-E_m))O_{mn} && (6.67) \\
&= \frac{\pi}{Z}\sum_{mn}e^{-\beta E_m}(1-\eta e^{-\beta\omega})\delta(\omega-(E_n-E_m))O_{mn} && (6.68)
\end{aligned}
$$

For boson ($\eta = 1$), then $n = m$ terms vanish. So the summation should be $\sum_{m\neq n}$.

6.3.5 Relation between Different Green's Functions

$$
\begin{aligned}
G_A(\omega) &= \mathcal{G}(i\omega_n \to \omega - i\delta) && (6.69) \\
G_R(\omega) &= \mathcal{G}(i\omega_n \to \omega + i\delta) && (6.70) \\
\Re G(\omega) &= \Re G_A(\omega) = \Re G_R(\omega) && (6.71) \\
\Im G(\omega) &= \Im G_R(\omega) \times \begin{cases} \coth\frac{\beta\omega}{2} & \text{boson} \\ \tanh\frac{\beta\omega}{2} & \text{fermion} \end{cases} && (6.72)
\end{aligned}
$$

6.3.6 Spectral Function

Let's define

$$
\begin{aligned}
A(\omega) &= 2\Im G_A(\omega) && (6.73) \\
&= i(G_R(\omega) - G_A(\omega)) && (6.74) \\
&= \frac{2\pi}{Z}\sum_{mn}(e^{-\beta E_m}-\eta e^{-\beta E_n})\delta(\omega-(E_n-E_m))O_{mn} && (6.75) \\
&= \frac{2\pi}{Z}\sum_{mn}e^{-\beta E_m}(1-\eta e^{-\beta\omega})\delta(\omega-(E_n-E_m))O_{mn} && (6.76)
\end{aligned}
$$

We have

$$
\int\frac{d\omega}{2\pi}A(\omega) = \frac{1}{Z}\sum_{mn}(e^{-\beta E_m}-\eta e^{-\beta E_n})O_{mn} \qquad (6.77)
$$

$$= \frac{1}{Z}\sum_{mn}\left[e^{-\beta E_m}\langle m|O_2|n\rangle\langle n|O_1|m\rangle - \eta e^{-\beta E_n}\langle n|O_1|m\rangle\langle m|O_2|n\rangle\right] \tag{6.78}$$

$$\tag{6.79}$$

$$= \langle [O_2, O_1]_\eta\rangle \tag{6.80}$$

So $\int \frac{d\omega}{2\pi}A(\omega) = 1$ if O_2 and O_1 is a pair of canonical operators.
We can reconstruct the Green's function just from $A(\omega)$ (on real axis) by the contour integral method. Since $A(\omega) = i(G_R(\omega) - G_A(\omega))$

$$\int \frac{d\omega}{2\pi}(n_\eta^-\theta(\tau) + n_\eta^+\theta(-\tau))A(\omega)e^{-\omega\tau} = i\int_C \frac{d\omega}{2\pi}(n_\eta^-\theta(\tau) + n_\eta^+\theta(-\tau))\mathcal{G}(\omega)e^{-\omega\tau} \tag{6.81}$$

$$= \frac{1}{\beta}\sum_n \mathcal{G}(i\omega_n)e^{-i\omega_n\tau} \tag{6.82}$$

$$= \mathcal{G}(\tau) \tag{6.83}$$

where C is the clockwise contour around the real axis.

6.3.7 Correlation Function

$$S(t) = \langle O_2(t)O_1(0)\rangle \tag{6.84}$$

$$= \frac{1}{Z}\sum_{mn}\langle m|e^{-\beta E_m}O_2(t)|n\rangle\langle n|O_1(0)|m\rangle \tag{6.85}$$

$$= \sum_{mn} e^{-it(E_n-E_m)}e^{-\beta E_m}O_{mn} \tag{6.86}$$

$$= \int \frac{d\omega}{-2\pi i}e^{-it\omega}\sum_{mn}e^{-\beta E_m}\left[\frac{O_{mn}}{\omega-(E_n-E_m)+i\delta} - \frac{O_{mn}}{\omega-(E_n-E_m)-i\delta}\right] \tag{6.87}$$

$$= \int d\omega e^{-it\omega}\sum_{nm}e^{-\beta E_m}O_{mn}\delta(\omega-(E_n-E_m)) \tag{6.88}$$

$$S(\omega) = 2\pi\sum_{mn}e^{-\beta E_m}O_{mn}\delta(\omega-(E_n-E_m)) \tag{6.89}$$

$$\tag{6.90}$$

We have

$$S(\omega) = -2\frac{1}{1-\eta e^{-\beta\omega}}\Im G_R(\omega) \tag{6.91}$$

6.4 Matsubara Green's Function for Interacting System

We have the similar Schrödigner, Heisenberg and interaction picture, with time now an imaginary number. Then similar to the real-time case, we have

$$-\mathcal{G}(x_1,\ldots,x_n) = \frac{\langle TS(0\to\beta)O_{1,I}(\mathbf{x}_1,\tau_1)\cdots O_{n,I}(\mathbf{x}_n,\tau_n)\rangle_0}{\langle TS(0\to\beta)\rangle_0} \tag{6.92}$$

We can do permutation expansion

$$
\begin{aligned}
-\mathcal{G}(x_1,\ldots,x_n) &= \frac{\left\langle T\left[e^{-\int_0^\beta d\tau H_{int,I}(\tau)}O_{1,I}(\mathbf{x}_1,\tau_1)\cdots O_{n,I}(\mathbf{x}_n,\tau_n)\right]\right\rangle_0}{\left\langle T\left[e^{-\int_0^\beta d\tau H_{int,I}(\tau)}\right]\right\rangle_0} &(6.93)\\
&= \frac{\left\langle T\left[\sum_{n=0}^\infty \frac{(-)1^n}{n!}\prod_{j=1}^n\int_0^\beta d\tau_j H_{int,I}(\tau_j)O_{1,I}(\mathbf{x}_1,\tau_1)\cdots O_{n,I}(\mathbf{x}_n,\tau_n)\right]\right\rangle_0}{\left\langle T\left[\sum_{n=0}^\infty \frac{(-1)^n}{n!}\prod_{j=1}^n\int_0^\beta d\tau_j H_{int,I}(\tau_j)\right]\right\rangle_0} &(6.94)
\end{aligned}
$$

6.4.1 The Wick's theorem

Not surprisingly, we have the same old Wick's theorem to evaluate the multi-operator traces. Let's consider

$$\left\langle T\left[O_1(\tau_1)\cdots O_n(\tau_n)\right]\right\rangle_0 = \mathrm{Tr}\left[\rho_{H_0}T\left[O_1(\tau_1)\cdots O_n(\tau_n)\right]\right] \tag{6.95}$$

We assume each $O_i(\tau_i)$ is either bosonic or fermionic. We make the following partition:

$$O_i(\tau_i) = \sum_j \alpha_{ij}a_j + \beta_{ij}a_j^\dagger \tag{6.96}$$

Then

$$\rho_{H_0}^{-1}(\beta)O_i(\tau_i)\rho_{H_0}(\beta) = \sum_j \alpha_{ij}e^{-\epsilon_j\beta}a_j + \beta_{ij}e^{\epsilon_j\beta}a_j^\dagger \tag{6.97}$$

We define

$$\bar{O}_i(\tau_i) = \sum_j \frac{\alpha_{ij}a_j}{1+\eta_i e^{-\epsilon_j\beta}} + \frac{\beta_{ij}a_j^\dagger}{1+\eta_i e^{\epsilon_j\beta}} \tag{6.98}$$

So

$$\rho_{H_0}(\beta)\bar{O}_i(\tau_i) + \eta_i\bar{O}_i(\tau_i)\rho_{H_0}(\beta) = \rho_{H_0}(\beta)O_i(\tau_i) \tag{6.99}$$

Then

$$
\begin{aligned}
\mathrm{Tr}\left[\rho_{H_0}\bar{O}_1(\tau_1)O_2(\tau_2)\cdots O_n(\tau_n)\right] &= \sum_{i>1}(-1)^{\bar\eta_1\sum_{1<j<i}\bar\eta_j}\mathrm{Tr}\left[\rho_{H_0}\cdots[\bar{O}_1(\tau_1),O_i(\tau_i)]_{(-1)^{\bar\eta_1\bar\eta_i+1}}\cdots\right]\\
&\quad +(-1)^{\bar\eta_1\sum_{j>1}\bar\eta_j}\mathrm{Tr}\left[\rho_{H_0}O_2(\tau_2)\cdots O_n(\tau_n)\bar{O}_1(\tau_1)\right] &(6.100)\\
&= \sum_{i>1}(-1)^{\bar\eta_1\sum_{1<j<i}\bar\eta_j}\mathrm{Tr}\left[\rho_{H_0}\cdots[\bar{O}_1(\tau_1),O_i(\tau_i)]_{(-1)^{\bar\eta_1\bar\eta_i+1}}\cdots\right]\\
&\quad -\eta_i\mathrm{Tr}\left[\bar{O}_1(\tau_1)\rho_{H_0}O_2(\tau_2)\cdots O_n(\tau_n)\right] &(6.101)
\end{aligned}
$$

So

$$\mathrm{Tr}\left[\rho_{H_0}O_1(\tau_1)O_2(\tau_2)\cdots O_n(\tau_n)\right] = \sum_{i>1}(-1)^{\bar\eta_1\sum_{1<j<i}\bar\eta_j}\mathrm{Tr}\left[\rho_{H_0}\cdots[\bar{O}_1(\tau_1),O_i(\tau_i)]_{(-1)^{\bar\eta_1\bar\eta_i+1}}\cdots\right] \tag{6.102}$$

When both O_1 and O_i is bosonic, then

$$[\bar{O}_1(\tau_1),O_i(\tau_i)]_{(-1)^{\bar\eta_1\bar\eta_i+1}} = [\bar{O}_1(\tau_1),O_i(\tau_i)]_- \tag{6.103}$$

$$= \sum_j \frac{\alpha_{1j}\beta_{ij}}{1-e^{-\epsilon_j\beta}} + \frac{-\beta_{1j}\alpha_{ij}}{1-e^{\epsilon_j\beta}} \tag{6.104}$$

$$= \langle O_1(\tau_1), O_i(\tau_i)\rangle \tag{6.105}$$

Similarly, when both O_1 and O_i is fermionic

$$[\bar{O}_1(\tau_1), O_i(\tau_i)]_{(-1)^{\bar{\eta}_1\bar{\eta}_i+1}} = \langle O_1(\tau_1), O_i(\tau_i)\rangle \tag{6.106}$$

When one of O_1 and O_i is bosonic and the other is fermionic, then

$$[\bar{O}_1(\tau_1), O_i(\tau_i)]_{(-1)^{\bar{\eta}_1\bar{\eta}_i+1}} = 0 = \langle O_1(\tau_1), O_i(\tau_i)\rangle \tag{6.107}$$

So in all

$$[\bar{O}_1(\tau_1), O_i(\tau_i)]_{(-1)^{\bar{\eta}_1\bar{\eta}_i+1}} = \langle O_1(\tau_1), O_i(\tau_i)\rangle \tag{6.108}$$

So

$$\langle O_1(\tau_1)\cdots O_n(\tau_n)\rangle = \sum_{i>1} \langle \overbracket{O_1(\tau_1)\cdots O_i}(\tau_n)\cdots O_n(\tau_n)\rangle \tag{6.109}$$

$$= \cdots \tag{6.110}$$

$$= \sum \text{full wick contractions} \tag{6.111}$$

where

$$\langle \overbracket{O\prod_i O_i O}'\rangle = \text{ph} \prod_i O_i \langle OO'\rangle \tag{6.112}$$

where ph is the phase factor generated when moving O to the right of $\prod_i O_i$.

Then it's easy to see that

$$\langle T[O_1(\tau_1)\cdots O_n(\tau_n)]\rangle = \sum \text{full wick contractions} \tag{6.113}$$

6.5 Path Integral at Finite Temperature

As in the zero temperature case, we can transform the Matsubara Green's function into path integral. For bosonic system, we have

$$\mathrm{Tr}[e^{-\beta H} T[\prod_n O_{n,H}(\tau_n)]] = C \int \mathcal{D}(\phi, \eta, \bar{\eta}, \dots) \prod_n O_n(\tau_n) e^{-\int_0^\beta d\tau L_E(\phi,\eta,\bar{\eta},\dots)} \tag{6.114}$$

where $0 < \tau_n < \beta$

So

$$\langle T[\prod_n O_{n,H}(\tau_n)]\rangle = \frac{\int \mathcal{D}(\phi,\psi,\bar{\psi},\dots)\prod_n O_n(\tau_n) e^{-\int_0^\beta d\tau L_E(\phi,\psi,\bar{\psi},\dots)}}{\int \mathcal{D}(\phi,\psi,\bar{\psi},\dots) e^{-\int_0^\beta d\tau L_E(\phi,\psi,\bar{\psi},\dots)}} \tag{6.115}$$

Similarly, we can define the generating functional as

$$Z[K, J, \bar{J}, \dots] \tag{6.116}$$

$$= \int \mathcal{D}(\phi, \psi, \bar{\psi}, \dots) e^{-\int_0^\beta d\tau [L_E + \int d^3x \phi(x,\tau) K(x,\tau) + \int d^3x (\bar{J}(x,\tau)\psi(x,\tau) + \bar{\psi}(x,\tau) J(x,\tau)) + \cdots]} \tag{6.117}$$

$$= \int \mathcal{D}(\phi, \psi, \bar{\psi}, \dots) e^{-\int_0^\beta d\tau \int d^3x \int_0^\beta d\tau' \int d^3x' [\frac{1}{2} K(x,\tau) \mathcal{G}_\phi(x-x', t-t') K(x',\tau') + \bar{J}(x,\tau) \mathcal{G}_\psi(x-x', t-t') J(x',\tau') + \cdots]} \tag{6.118}$$

Then

$$\int \mathcal{D}(\phi, \psi, \bar{\psi}, \dots) \prod_n O(x_n) e^{-\int_0^\beta d\tau L_E} \tag{6.119}$$

$$= \frac{1}{Z[K, J, \bar{J}, \dots]} \int \mathcal{D}(\phi, \psi, \bar{\psi}, \dots) \prod_n O(\frac{\delta}{-\delta K(x_n)}, \frac{\delta}{\delta J(x_n)}, \frac{\delta}{-\delta \bar{J}(x_n)}) Z[K, J, \bar{J}, \dots]|_{K=J=\bar{J}=\dots=0} \tag{6.120}$$

Similarly, for interacting system, let $L_E = L_{E,free} + L_{E,int}$. We treat $L_{E,int}$ as a perturbation. Then

$$\langle T[\prod_n O(x_n)] \rangle \tag{6.121}$$

$$= \frac{\int \mathcal{D}(\phi, \psi, \bar{\psi}, \dots) \prod_n O_n(x_n) e^{-\int_0^\beta d\tau L_E(\phi, \psi, \bar{\psi}, \dots)}}{\int \mathcal{D}(\phi, \psi, \bar{\psi}, \dots) e^{-\int_0^\beta d\tau L_E(\phi, \psi, \bar{\psi}, \dots)}} \tag{6.122}$$

$$= \frac{\int \mathcal{D}(\phi, \psi, \bar{\psi}, \dots) \prod_n O_n(x_n) \sum_m \frac{1}{m!}(-\int dt L_{E,int})^m e^{-\int_0^\beta d\tau L_{E,free}}}{\int \mathcal{D}(\phi, \psi, \bar{\psi}, \dots) \sum_m \frac{1}{m!}(-\int dt L_{E,int})^m e^{-\int_0^\beta d\tau L_{E,free}}} \tag{6.123}$$

$$= \frac{\frac{1}{Z_0} \prod_n O_n(\frac{\delta}{-\delta K(x_n)}, \frac{\delta}{\delta J(x_n)}, \frac{\delta}{-\delta \bar{J}(x_n)}) \sum_m \frac{1}{m!}(-\int dx L_{E,int}(\frac{\delta}{-\delta K(x)}, \frac{\delta}{\delta J(x)}, \frac{\delta}{-\delta \bar{J}(x)}))^m Z_0[K, \bar{J}, J]\Big|_{K=J=\bar{J}=0}}{\frac{1}{Z_0} \sum_m \frac{1}{m!}(-\int dx L_{E,int}(\frac{\delta}{-\delta K(x)}, \frac{\delta}{\delta J(x)}, \frac{\delta}{-\delta \bar{J}(x)}))^m Z_0[K, \bar{J}, J]\Big|_{K=J=\bar{J}=0}} \tag{6.124}$$

We can similarly rediscover the Wick's theorem.

Chapter 7

Feynmann Diagram

Wick's theorem simplifies the calculation of Green's function greatly. However, there's still some multiplicity in finding all possible full Wick contractions. Beyond, the physical picture of full Wick contractions remains unclear. Feynman introduced in 1948 the graphical expression of a full Wick contraction

7.1 Basic Construction

We consider nth order perturbation term appears in the Green's function

$$\Big\langle T\Big[\frac{(-i)^n}{n!}\Big[\int d\tau H_{int}(\tau)\Big]^n O_1(t_1)\cdots O_n(t_n)\Big]\Big\rangle \tag{7.1}$$

where Os are simple and H_{int} is the sum of products of simple operators:

$$H_{int} = \sum_i \prod_j O_j^{[i]} \tag{7.2}$$

So we only need consider

$$\Big\langle T\Big[\prod_i \frac{(-i)^{n_i}}{n_i!}\Big[\int d\tau \prod_{j=1}^{n_i} O_j^{[i]}(\tau)\Big]^{n_j} O_1(t_1)\cdots O_n(t_n)\Big]\Big\rangle \tag{7.3}$$

where $\sum_i n_i = n$.

By super-selection rule, term in H_{int} contains even number of fermionic simple operators.

We can translate each full Wick contraction to labeled Feynman diagram as:

- Each interaction term $\prod_{j=1}^{n_i} O_j^{[i]}$ is mapped to a vertex or a process labeled by its order, with indices labeled by operator type and order that correspond to operator $O_j^{[i]}$s. Note that vertex type can always be inferred from the operator type labels. We may omit the order label of a vertex or an operator if it's the only one of its type.

- Each operator in 7.3 with time not integrated (that is, not part of H_{int}) is mapped to an open end labeled by operator type and time as.

- Each Wick contraction is mapped to a link between the two objects that the two Wick contracted operators mapped to. We call the link propagator. If the two operators are of different types, the propagator has a direction. For Dirac fermion, the direction of the propagator is from $\bar{\psi}$ to ψ.
- Each point has a definite space-time coordinate, and each line between to points is a process that connects them.

For example,

$$H_{int} = aab + abb \tag{7.4}$$

The expectation

$$\left\langle (-i)^2 \int d\tau a(\tau)a(\tau)b(\tau) \int d\tau a(\tau)b(\tau)b(\tau)a(t_1)b(t_2) \right\rangle \tag{7.5}$$

(the coefficient 2 comes from binomial coefficient) is mapped to labeled Feynman diagram in Fig. 7.1.

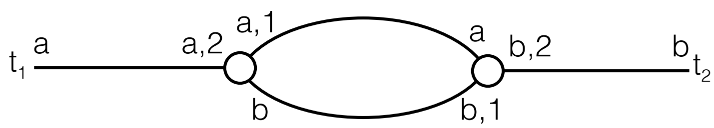


Figure 7.1: Labeled Feynman diagram.

The expectation

$$\left\langle \frac{(-i)^2}{3!} \int d\tau a(\tau)a(\tau)b(\tau) \int d\tau a(\tau)a(\tau)b(\tau) \int d\tau a(\tau)a(\tau)b(\tau) \int d\tau a(\tau)b(\tau)b(\tau)a(t_1)b(t_2) \right\rangle \tag{7.6}$$

is mapped to labeled Feynman diagram in Fig. 7.2.

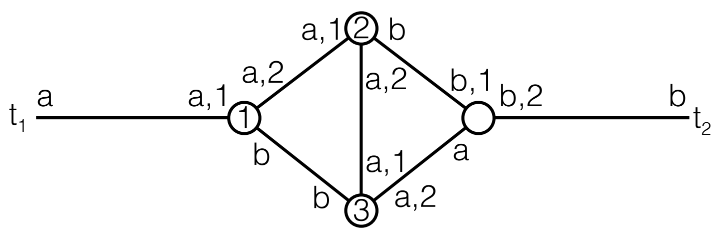


Figure 7.2: Another labeled Feynman diagram.

If we drop the label of order, we get the commonly known Feynman diagram. For example, the labeled Feynman diagram in Fig. 7.2 becomes the diagram in Fig. 7.3. If we use different type of line (with an optional arrow) in the diagram to identify different type of propogator, we may even drop the label of type, and get the most concise form of Feynman diagram shown in Fig. 7.4.

It's easy to prove that there's a 1-1 correspondence of labeled Feynman diagram and full Wick contraction. But different Wick contractions may give the same unlabeled Feynman diagrams. We will prove that different Wick contractions that give the same unlabeled Feynman diagram give the same result. We will calculate the multiplicity of Wick contractions that give the same unlabeled Feynman diagram. First we have

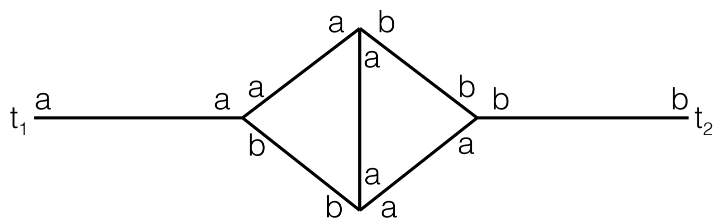


Figure 7.3: Unlabeled Feynman diagram.

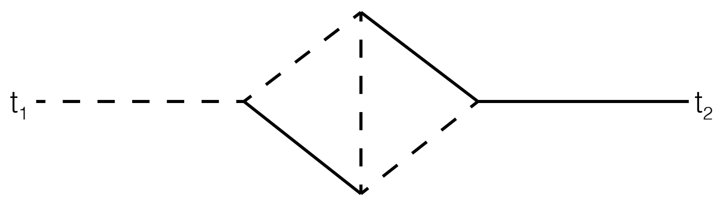


Figure 7.4: Feynman diagram in concise form, with solid line between b and b and dashed line between a and a.

Theorem 7.1.1. *Let A and B be two different labeled Feynman diagram that gives the same unlabeled Feynman diagram. Then A can be transformed into B by composite of:*

- *Permutation of the order of identical vertexes.*
- *Internal symmetry action of each vertex.*

Proof. Obvious. □

Theorem 7.1.2. *Let A and B be two different full Wick contractions that gives the same unlabeled Feynman diagram. Then A have the same value.*

Proof. From last theorem we know A can be transformed into B by composite of:

- Permutation of interaction terms.
- Internal symmetry action of each interaction term.

From definition of Wick contraction, it's easy to see that commuting operators will generate a phase factor in the natural way. Since interaction terms is bosonic operator (due to superselection law), the first operation will generate no phase factor. Since there is no identical fermionic operators inside a vertex due to Pauli's exclusion principle, the second operation will also generate no phase factor. □

The operations talked above forms a group that acts on the space of labeled Feynman diagrams. So the multiplicity of each unlabeled Feynman diagram is the order of the orbit that equals

$$\frac{|G|}{|S|} = \frac{\prod_i n_i!|G_i|^{n_i}}{\text{symmetric factor}} \tag{7.7}$$

where n_i is the repetition of the interaction term i and G_i is the internal symmetry group of the interaction term i. The order of stabilizer S of the action is also called symmetric factor. It can be defined as: the number of operations that leave the labeled Feynman diagrams invariant.

From now on, by saying Feynman diagrams, we mean unlabeled Feynman diagrams (in concise form if possible).

First we write a general local interaction Hamiltonian as

$$H_{int}(t) = \sum_i \frac{g_i}{|G_i|} H_{int}^{[i]}(t) \tag{7.8}$$

Our purpose is to calculate

$$\sum_n \Big\langle T\Big[\frac{(-i)^n}{n!}\Big[\int dt H_{int}^{[i]}(t)\Big]^n O_1(x_1)\cdots O_n(x_n)\Big]\Big\rangle \tag{7.9}$$

As we have discussed, we may only calculate

$$\Big\langle T\Big[\prod_i \frac{1}{n_i!}\Big[-i\int dt \frac{g_i}{|G_i|} H_{int}^{[i]}(t)\Big]^{n_i} O_1(t_1)\cdots O_n(t_n)\Big]\Big\rangle \tag{7.10}$$

We define the value of a Feynman diagram as the sum of all full Wick contractions in Eqn. 7.9 to give the Feynman diagram. Eqn. 7.9 is the sum of Eqn. 7.11. Obviously, each Feynman diagram only comes from a specific set of $\{n_i\}$. So the value of a Feynman diagram is just sum of all full Wick contractions in Eqn. 7.11 with the specific set of $\{n_i\}$. We denote by M the full wick contraction of

$$\Big\langle T\Big[\prod_i \Big[-i\int dt g_i H_{int}^{[i]}(t)\Big]^{n_i} O_1(t_1)\cdots O_n(t_n)\Big]\Big\rangle \tag{7.11}$$

that gives the chosen Feynman diagram.

It's easy to see any full wick contraction of 7.11 that gives the chosen Feynman diagram is

$$\frac{M}{\prod_i n_i! |G_i|^{n_i}} \tag{7.12}$$

Multiply with the multiplicity we derived, we finally know that the value of a Feynman diagram is

$$\frac{M}{\text{symmetric factor}} \tag{7.13}$$

It's easy to see that Eqn. 7.9 is the sum value of all Feynman diagrams. Obviously we can calculate the symmetric factor purely from the Feynman diagrams. If we can devise a method to calculate M purely from the Feynman diagrams, we can forget everything of: interaction picture, Gell-Mann-Low thm, Wick theorem and so on, and calculate the Green's function purely from Feynman diagrams.

7.2 Feynman Rule

The Feynman rule is set of rules to translate from a Feynman diagram to an equation, whose value is the M we introduced in the last section.

From Wick's theorem, we can translate a propagator to a two point Green's function. The rule is:

- The type of the Green's functions is determined by the types of operators at two ends of the propagator.

- The first time parameter of the Green's function is time at the starting end (vertex has a dummy time variable to be integrated later). We may choose either end to be the starting end if the propagator is not directed.

- The second time parameter of the Green's function is time at the ending end.

With these two point Green's functions, we may get M by adding $-ig_i$ and taking integrals on dummy time at each vertex (i determined by vertex type). However an overall phase factor ± 1 generating from the permutation of the operators is left unknown. In case of bosons, the overall phase is just 1. In case of Majorana fermions, we refer to [1]. Here we consider the situation that some operators are Dirac fermions.

7.2.1 Overall Phase Factor of Dirac Fermions

It's reasonable to assume that for each Dirac operator a there's only one type of Dirac operator b such that

$$G_{ab}(t_1, t_2) \neq 0 \tag{7.14}$$

We may label b as $\bar{a}$ (or possibly its derivatives). Furthermore, it's crucial to see that Dirac fermionic operators come in pairs in each interaction term at the same time. So in a Feynmann diagram, Dirac fermionic lines form several paths. Each path starts and ends at free ends of the diagram, or forms a loop. There's no bifurcation along the path.

We consider a full contraction of

$$\prod_i \Big[-i \int dt g_i H_{int}^{[i]}(t) \Big]^{n_i} \psi(t_1)\psi(t_2)\cdots\bar{\psi}(t_n) \tag{7.15}$$

Let the interaction term be $H_{int}^{[i]} = \bar{\psi}\psi\cdots$.

After necessary permutation of the interaction terms, a fermionic loop corresponds to

$$\cdots\bar{\psi}\psi\bar{\psi}\psi\cdots\bar{\psi}\psi\cdots \tag{7.16}$$

This gives a contribution of -1 factor.

Then we consider the contribution of fermionic paths. We may drop all the fermionic loops.

After necessary permutation of the interaction terms, a fermionic path with two ends $\psi(t_i)$ and $\bar{\psi}(t_j)$ corresponds to

$$\cdots\bar{\psi}\psi\bar{\psi}\psi\cdots\bar{\psi}\psi\cdots\psi(t_i)\cdots\bar{\psi}(t_j)\cdots \tag{7.17}$$

We can move the involved interaction part to the right of $\psi(t_i)$, becoming

$$\langle T[\cdots\psi(t_i)\bar{\psi}\psi\bar{\psi}\psi\cdots\bar{\psi}\psi\cdots\bar{\psi}(t_j)\cdots]\rangle \tag{7.18}$$

Let's take a closer look at this contraction. We may drop or the intermediate operators, and contract $\psi(t_i)$ and $\bar{\psi}(t_j)$ directly. This action will cause no extra sign. Let's do this operation for all fermionic paths, and drop all bosonic operators. We result in a full contraction of

$$\psi(t_1)\psi(t_2)\cdots\bar{\psi}(t_n) \tag{7.19}$$

The sign contribution of all fermionic paths of the original full contraction is just the sign of this simplified full contraction.

In conclusion

1. A fermionic loop gives a -1 factor.
2. All fermionic paths gives the same factor as the simplified full contraction. Especially, if there's only two free ends, the fermionic paths gives no extra factor.

7.2.2 Feynman Rule, an Example

Let's consider a system with Lagragian

$$L = L_0 - \int d^3\mathbf{x} U(\mathbf{x})\bar{\psi}(x)\psi(x) - \frac{1}{2}\int d^3\mathbf{x} d^3\mathbf{x}' V(\mathbf{x}-\mathbf{x}')\bar{\psi}(x)\bar{\psi}(x')\psi(x')\psi(x) - \int d^3\mathbf{x} g(\mathbf{x}-\mathbf{x}')\bar{\psi}(x)\psi(x)\phi(x') \tag{7.20}$$

where

$$L_0 = \int dx(\bar{\psi}(x)\partial_0\psi(x) - \frac{1}{2m}\partial_i\bar{\psi}(x)\partial_i\psi(x) - \mu\bar{\psi}(x)\psi(x)) + \int dx(\frac{1}{2}(\partial_0\phi(x))^2 - \frac{1}{2}M_{ij}(x)\partial_i\phi(x)\partial_j\phi(x)) \tag{7.21}$$

is the free Lagragian of electrons and phonons. Coulomb interaction and electron-phonon interaction is considered.

The space-time Feynmann rule is given in Tab. 7.1.

We can transform the Feynmann rule into momentum-energy space.

For each propogator we have

$$G(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x-y)} G(p) \tag{7.22}$$

where p is directed from y-vertex to x-vertex, and the we choose the Minkowski metric $(-1,1,1,1)$.

For Coulomb interaction vertex we have

$$\int dt \int d^3\mathbf{x} d^3\mathbf{y} V(\mathbf{x}-\mathbf{y}) = \int dt \int d^3\mathbf{x} d^3\mathbf{y} \int \frac{d^3\mathbf{q}}{(2\pi)^3} e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})} V(\mathbf{q}) \tag{7.23}$$

where q is directed from y-vertex to x-vertex.

For potential vertex we have

$$\int d^4x U(\mathbf{x}) = \int d^4x \int \frac{d^3\mathbf{q}}{(2\pi)^3} e^{i\mathbf{q}\cdot\mathbf{x}} U(\mathbf{q}) \tag{7.24}$$

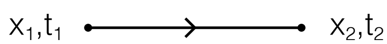
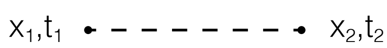
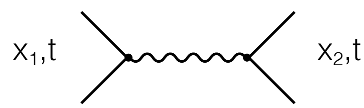
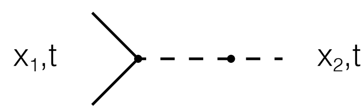
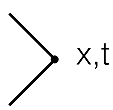
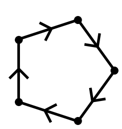
Component	Rule
	$iG(x_2 - x_1, t_2 - t_1)$
	$iD(x_2 - x_1, t_2 - t_1)$
	$-i\int dtdx_1dx_2V(x_1 - x_2)$
	$-i\int dtdx_1dx_2g(x_1 - x_2)$
	$-i\int dtdxU(x)$
	$(-1)\cdot(2S+1)$

Table 7.1: Space-time Feynmann rule for fermion-phonon system at zero temperature.

where q is directed inwards.

For electron-phonon interaction vertex we have

$$\int dt \int d^3\mathbf{x}d^3\mathbf{y}g(\mathbf{x}-\mathbf{y}) = \int dt \int d^3\mathbf{x}d^3\mathbf{y} \int \frac{d^3\mathbf{q}}{(2\pi)^3} e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{y})} g(\mathbf{q}) \tag{7.25}$$

where q is directed from y-vertex to x-vertex.

Finally we have many

$$\int d^4x e^{i\sum_i \pm p_i\cdot x} = (2\pi)^4\delta(\sum_i \pm p_i) \tag{7.26}$$

where $\pm$ is for the inwards/outwards momentum direction. This ensures the conservation of the momentum. We do such integral on interaction vertex. So on each free end, there's a $e^{ip\cdot x_i}$ left. This will result in $(2\pi)^4\delta(p_i - p)$ after Fourier transformation of the whole thing.

Another thing to notice is that to keep the order of the interaction term unchanged under time-ordering, we should modify the interaction term into

$$\begin{aligned} &\int d^3\mathbf{x}U(\mathbf{x})\bar{\psi}(\mathbf{x},t+\delta)\psi(\mathbf{x},t) - \frac{1}{2}\int d^3\mathbf{x}d^3\mathbf{x}'V(\mathbf{x}-\mathbf{x}')\bar{\psi}(\mathbf{x},t+3\delta)\bar{\psi}(\mathbf{x}',t+2\delta)\psi(\mathbf{x}',t+\delta)\psi(\mathbf{x},t) \\ &- \int d^3\mathbf{x}g(\mathbf{x}-\mathbf{x}')\bar{\psi}(\mathbf{x},t+\delta)\psi(\mathbf{x},t)\phi(\mathbf{x}',t) \end{aligned} \tag{7.27}$$

This leads to some $e^{i\delta E}$ factor in momentum-energy Feynmann rule.

The momentum-energy Feynmann rule is given in Tab. 7.2.

Similarly, at finite temperature, we have the Feynmann rule for Matsubara Green's function. The space-time Feynmann rule is given in Tab. 7.3. And the momentum-energy Feynmann rule is given in Tab. 7.4.

7.3 Cancelation of Vacuum Diagrams

$$\langle T[\prod_n O_H(t_n)]\rangle = \frac{\sum \text{diagrams with ends } O(t_1)\cdots O(t_n)}{\sum \text{diagrams with no ends}} \tag{7.28}$$

A Feynmann diagram with no ends is also called a vacuum diagram.

Let's label the connected vacuum diagrams by V_i. Each vacuum diagram can be uniquely labeled by a series (n_i), indicating that the diagram contains n_i V_is. The symmetrical factor of the vacuum diagram labeled by (n_i) is

$$\prod_i n_i! S_i^{n_i} \tag{7.29}$$

where S_i is the symmetrical factor of V_i.

Let F_i be the value of V_i and let $\tilde{F}_i$ be the value of V_i with the symmetrical value not divided. Then the value of the vacuum diagram labeled by (n_i) is

$$F_{(n_i)} = \frac{\prod_i \tilde{F}_i^{n_i}}{\prod_i n_i! S_i^{n_i}} = \prod_i \frac{F_i^{n_i}}{n_i!} \tag{7.30}$$

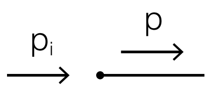
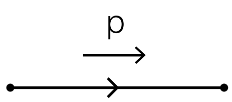
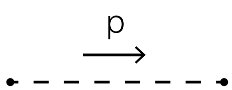
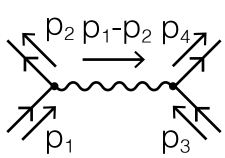
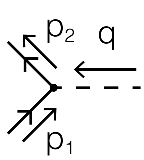
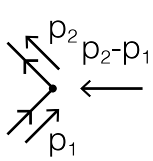
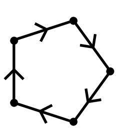
Component	Rule
	$(2\pi)^4\delta(p_i - p)$
	$i\int \frac{d^4p}{(2\pi)^4} G(p)$
	$i\int \frac{d^4p}{(2\pi)^4} D(p)$
	$-iV(\mathbf{p_1} - \mathbf{p_2})(2\pi)^4\delta^4(p_1 - p_2 + p_3 - p_4)e^{-i\delta E_3}e^{i2\delta E_4}e^{i3\delta E_2}$
	$-ig(\mathbf{q})(2\pi)^4\delta(p_1 - p_2 + q)e^{i\delta E_2}$
	$-iU(\mathbf{p_2} - \mathbf{p_1})(2\pi)\delta(E_2 - E_1)e^{i\delta E_2}$
	$(-1)\cdot(2S+1)$

Table 7.2: Momentum-energy Feynmann rule for fermion-phonon system at zero temperature.

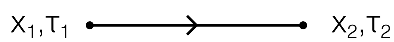
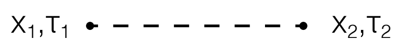
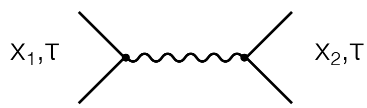
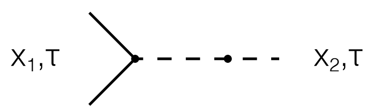
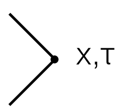
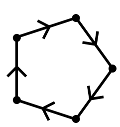
Component	Rule
	$-\mathcal{G}(x_2 - x_1, t_2 - t_1)$
	$-\mathcal{D}(x_2 - x_1, t_2 - t_1)$
	$-\int d\tau dx_1 dx_2 V(x_1 - x_2)$
	$-\int d\tau dx_1 dx_2 g(x_1 - x_2)$
	$-\int d\tau dx U(x)$
	$(-1) \cdot (2S + 1)$

Table 7.3: Space-time Feynmann rule for fermion-phonon system at finite temperature.

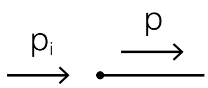
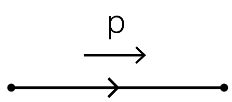
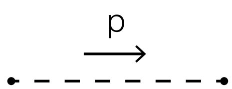
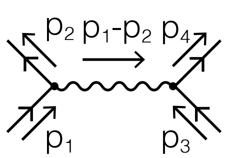
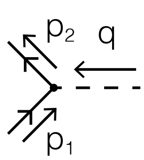
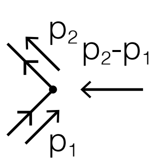
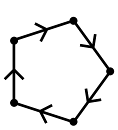
Component	Rule
	$(2\pi)^3\delta(\mathbf{p_i}-\mathbf{p})\delta_{\omega_{i,n}-\omega_n}$
	$-\int \frac{d^3p}{(2\pi)^3}\sum_n \frac{1}{\beta}\mathcal{G}(\mathbf{p},i\omega_n)$
	$-\int \frac{d^3p}{(2\pi)^3}\sum_n \frac{1}{\beta}\mathcal{D}(\mathbf{p},i\omega_n)$
	$-V(\mathbf{p_1}-\mathbf{p_2})(2\pi)^3\delta^3(\mathbf{p_1}-\mathbf{p_2}+\mathbf{p_3}-\mathbf{p_4})$ $\beta\delta_{\omega_{n_1}-\omega_{n_2}+\omega_{n_3}-\omega_{n_4}}e^{-i\delta\omega_{n_3}}e^{i2\delta\omega_{n_4}}e^{i3\delta\omega_{n_2}}$
	$-g(\mathbf{q})(2\pi)^3\delta(\mathbf{p_1}-\mathbf{p_2}+\mathbf{q})\beta\delta_{\omega_{n_1}-\omega_{n_2}+\nu_n}e^{i\delta\omega_{n_2}}$
	$-U(\mathbf{p_2}-\mathbf{p_1})\beta\delta_{\omega_{n_2}-\omega_{n_1}}e^{i\delta\omega_{n_2}}$
	$(-1)\cdot(2S+1)$

Table 7.4: Momentum-energy Feynmann rule for fermion-phonon system at finite temperature.

So

$$\begin{aligned}\sum \text{vacuum diagrams} &= \sum_{(n_i)} \prod_i \frac{F_i^{n_i}}{n_i!} &(7.31)\\ &= \prod_i \sum_{n_i} \frac{F_i^{n_i}}{n_i!} &(7.32)\\ &= \prod_i e^{F_i} &(7.33)\\ &= e^{\sum_i F_i} &(7.34)\\ &= e^{\sum \text{connected vacuum diagrams}} &(7.35)\end{aligned}$$

Similarly

$$\begin{aligned}\sum \text{diagrams with ends } O(t_1)\cdots O(t_n) &= \sum \text{connected diagrams with ends } O(t_1)\cdots O(t_n)\cdot \\ &\quad e^{\sum \text{connected vacuum diagrams}} &(7.36)\end{aligned}$$

By connected diagrams with ends $O(t_1)\cdots O(t_n)$, we mean each component of the diagram is connected to at least one of the ends. Although the diagram may not be connected.

So

$$\langle T[\prod_n O_H(t_n)]\rangle = \sum \text{connected diagrams with ends } O(t_1)\cdots O(t_n) \tag{7.37}$$

7.4 Linked-cluster Theorem

When $T=0$, since

$$\Delta E = \lim_{T\to\infty} \frac{i}{2T} \ln\langle 0|S_H(-T\to T)|0\rangle \tag{7.38}$$

We have

$$\begin{aligned}\frac{\Delta E}{V} &= \frac{i}{2VT} \ln \sum \text{vacuum diagrams} &(7.39)\\ &= \frac{i}{2VT} \sum \text{connected vacuum diagrams} &(7.40)\end{aligned}$$

So connected vacuum diagrams per space-time volume is the density of the interaction energy.

At finite temperature, we have

$$\begin{aligned}-\Delta F &= \ln \mathrm{Tr}[e^{-\beta H}] - \ln \mathrm{Tr}[e^{-\beta H_0}] &(7.41)\\ &= \ln \mathrm{Tr}\langle e^{\beta H_0} e^{-\beta H}\rangle_0 &(7.42)\\ &= \ln \mathrm{Tr}\langle S(0\to\beta)\rangle_0 &(7.43)\\ &= \ln \sum \text{vacuum diagrams} &(7.44)\\ &= \sum \text{connected vacuum diagrams} &(7.45)\end{aligned}$$

So

$$\frac{\Delta F}{V} = \frac{-1}{V\beta} \sum \text{connected vacuum diagrams} \tag{7.46}$$

Chapter 8

Linear Response Theory and Dissipation-Fluctuation Theorem

8.1 Linear Response Theory

We consider a system coupling to external field $f(x,t)$. Let the Hamiltonian be

$$H(t) = H_0 + H_{cp}(t) = H_0 - \int dx A(x) f(x,t) \tag{8.1}$$

Here we use H_0 to denote the interaction system with no external field. We treat H_0 as a free Hamiltonian and $H_{cp}(t)$ as the interaction term.

We assume the external field to vanish in the infinite past. Then

$$|\Omega(0)\rangle = e^{i\phi} S(0,-\infty)|0\rangle \tag{8.2}$$

So

$$\begin{aligned}
\langle\Omega|A_H(x,t)|\Omega\rangle &= \langle 0|S(-\infty,0)A_H(x,t)S(0,-\infty)|0\rangle && (8.3)\\
&= \langle 0|S(-\infty,t)A_I(x,t)S(t,-\infty)|0\rangle && (8.4)\\
&\simeq \langle 0|A_I(x,t) - i\int_{-\infty}^{t} dt'[A_I(x,t), H_{cp,I}(t')]|0\rangle && (8.5)\\
&= \langle 0|A_I(x,t) + i\int_{-\infty}^{t} dt'[A_I(x,t), \int dx' A_I(x',t')f(x',t')]|0\rangle && (8.6)
\end{aligned}$$

So

$$\langle\Omega|\Delta A_H(x,t)|\Omega\rangle = i\int_{-\infty}^{t} dt' \int dx' \langle 0|[\Delta A_I(x,t), \Delta A_I(x',t')]|0\rangle f(x',t') \tag{8.7}$$

where $\Delta A(x,t) = A(x,t) - \langle 0|A_I(x,t)|0\rangle$.

Define

$$\chi(x-x',t-t') = i\langle 0|[\Delta A_I(x,t), \Delta A_I(x',t')]|0\rangle\theta(t-t') \tag{8.8}$$

Then

$$\Delta A(x,t) = \langle \Omega | \Delta A_H(x,t) | \Omega \rangle = i \int dt' dx' \chi(x-x', t-t') f(x', t') \tag{8.9}$$

In momentum space

$$\Delta A(p) = \chi(p) f(p) \tag{8.10}$$

Let's define

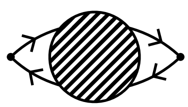
$$\chi^T(x-x', t-t') = i \langle 0 | T[\Delta A_I(x,t) \Delta A_I(x',t')] | 0 \rangle \tag{8.11}$$

As shown in Sect. 4.3, we have

$$\chi(\mathbf{p}, \omega) = \chi^T(\mathbf{p}, \omega + i\delta) \tag{8.12}$$

Now we return to our old familiar style. We continue to treat H_0 as an interacting Hamiltonian with ground state $|\Omega\rangle$. We can express $\chi^T(x-x', t-t')$ by Feynmann diagrams

$$\begin{aligned}
-i\chi^T(x-x', t-t') &= \langle \Omega | T[\Delta A(x,t) \Delta A(x',t')] | \Omega \rangle && (8.13)\\
&= \langle \Omega | T[A(x,t), A(x',t')] | \Omega \rangle - \langle \Omega | A(x,t) | \Omega \rangle \langle \Omega | A(x',t') | \Omega \rangle && (8.14)\\
&= \sum \text{diagrams connected to } x \text{ or } x' - \sum \text{diagrams connected to } x \\
&\quad \sum \text{diagrams connected to } x' && (8.15)\\
&= \sum \text{diagrams connected to } x \text{ and } x' && (8.16)
\end{aligned}$$

$= \text{x}$  x' (8.17)

One may check the correctness of the symmetrical factor.

8.2 Linear Response at Finite Temperature

At finite temperature, we study the density matrix in the interaction picture.
We assume the external field to vanish in the infinite past. Then

$$\begin{aligned}
i\rho_I(t) &= \int_{-\infty}^{t} d\tau [H_{int,I}(\tau), \rho_I(\tau)] + \rho_I(-\infty) && (8.18)\\
&= \int_{-\infty}^{t} d\tau [H_{int,I}(\tau), \rho_0] + \rho_0 + \mathcal{O}(H_{int,I}^2) && (8.19)
\end{aligned}$$

where ρ_0 is the density matrix of the free system.
So

$$\begin{aligned}
\langle A_H(x,t) \rangle &= \mathrm{Tr}[A_I(x,t) \rho_I(t)] && (8.20)\\
&= -i \mathrm{Tr}[A_I(x,t) \int_{-\infty}^{t} dt' [H_{int,I}(t'), \rho_0]] + \mathrm{Tr}[A_I(t) \rho_0] && (8.21)\\
&= -i \int_{-\infty}^{t} dt' \mathrm{Tr}[[A_I(x,t), H_{int,I}(t')] \rho_0] + \mathrm{Tr}[A_I(t) \rho_0] && (8.22)
\end{aligned}$$

$$= -i\int_{-\infty}^{t} dt' \langle [A_I(x,t), H_{int,I}(t')]\rangle_0 + \langle A_I(t)\rangle_0 \tag{8.23}$$

$$= -i\int_{-\infty}^{t} dt' \langle [A_I(x,t), \int dx' A_I(x',t') f(x',t')]\rangle_0 + \langle A_I(t)\rangle_0 \tag{8.24}$$

So

$$\langle \Delta A_H(x,t)\rangle = i\int_{-\infty}^{t} dt' \int dx' \langle [\Delta A_I(x,t), \Delta A_I(x',t')]\rangle_0 f(x',t') \tag{8.25}$$

where $\Delta A(x,t) = A(x,t) - \langle A_I(x,t)\rangle_0$.

Define

$$\chi(x-x', t-t') = i\langle [\Delta A_I(x,t), \Delta A_I(x',t')]\rangle_0 \theta(t-t') \tag{8.26}$$

Then

$$\Delta A(x,t) = \langle \Delta A_H(x,t)\rangle = i\int dt' dx' \chi(x-x', t-t') f(x',t') \tag{8.27}$$

In momentum space

$$\Delta A(p) = \chi(p) f(p) \tag{8.28}$$

Let's define

$$\chi^M(x-x', \tau-\tau') = -\langle 0|T[\Delta A_I(x,\tau)\Delta A_I(x',\tau')]|0\rangle \tag{8.29}$$

where τ and τ' are imaginary time.

As shown in Sect. 6.3, we have

$$\chi(\mathbf{p},\omega) = \chi^M(\mathbf{p}, i\omega_n \to \omega + i\delta) \tag{8.30}$$

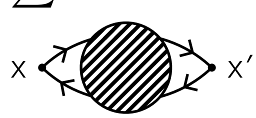
Now we return to our old familiar style. We continue to treat H_0 as an interacting Hamiltonian with ground state $|\Omega\rangle$. We can express $\chi^M(x-x', t-t')$ by Feynman diagrams

$$-\chi^M(x-x', \tau-\tau') = \langle \Omega|T[\Delta A(x,\tau)\Delta A(x',\tau')]|\Omega\rangle \tag{8.31}$$

$$= \langle \Omega|T[A(x,\tau), A(x',\tau')]|\Omega\rangle - \langle \Omega|A(x,\tau)|\Omega\rangle\langle \Omega|A(x',\tau')|\Omega\rangle \tag{8.32}$$

$$\begin{aligned} = &\sum \text{diagrams connected to } x \text{ or } x' - \sum \text{diagrams connected to } x \\ &\sum \text{diagrams connected to } x' \end{aligned} \tag{8.33}$$

$$= \sum \text{diagrams connected to } x \text{ and } x' \tag{8.34}$$

$=$ 

(8.35)

One may check the correctness of the symmetrical factor.

8.3 Dissipation-Fluctuation Theorem

We define the correlation of to operators $A(x)$ and $A(x')$ to be

$$S(x) = \langle \Delta A(x)\Delta A_2(x')\rangle \tag{8.36}$$

When $T = 0$, from Sect. 4.3 we have

$$S(\mathrm{p}, \omega) = 2\theta(\omega)\Im\chi(\mathrm{p}, \omega) \tag{8.37}$$

At finite temperature, from Sect. 6.3 we have

$$S(\mathrm{p}, \omega) = \frac{2}{1 - \eta e^{-\beta\omega}}\Im\chi(\mathrm{p}, \omega) \tag{8.38}$$

These are the quantum dissipation-fluctuation theorem.

Chapter 9

Boson System

Chapter 10

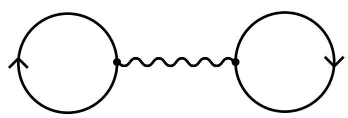
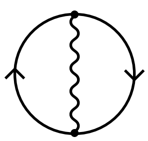
Fermion System

10.1 Hartee-Fock Approximation

Let's consider a fermionic system with Coulomb interaction.

10.1.1 Interaction Energy

To 1st order, interaction energy density is

$$\frac{\Delta E}{V} = \frac{i}{VT}\Big[\, p$$  $$q \;-\; p$$  $$q \,\Big] \tag{10.1}$$

$$\begin{aligned} &= \frac{i(2\pi)^4\delta(0)}{VT}\frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}e^{i(p_0+q_0)\delta}[(-(2S+1))^2(-iV(0))+ \\ &\quad (-(2S+1))(-iV(\mathbf{p}-\mathbf{q}))](iG(p))(iG(q)) \qquad (10.2) \\ &= \frac{1}{2}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{q}}{(2\pi)^3}[(2S+1)^2V(0)-(2S+1)V(\mathbf{p}-\mathbf{q})]f_{\mathbf{p}}f_{\mathbf{q}} \qquad (10.3)\end{aligned}$$

where

$$f_{\mathbf{p}} = -i\int\frac{d\omega}{(2\pi)}e^{i\omega\delta}G(p) = -iG(\mathbf{p},0^-) = -\langle T[c_p(t)c_p^\dagger(t+\delta)]\rangle = \langle c_p^\dagger c_p\rangle = \theta(\epsilon_F-\epsilon_{\mathbf{p}}) \tag{10.4}$$

The first term $\frac{1}{2}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{q}}{(2\pi)^3}(2S+1)^2V(0)f_{\mathbf{p}}f_{\mathbf{q}}$ cancels with the positive charged background, as we'll show in Jellium model. So the interaction energy is

$$\begin{aligned}\frac{E_{int}}{V} &= -\frac{1}{2}\int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{q}}{(2\pi)^3}(2S+1)V(\mathbf{p}-\mathbf{q})f_{\mathbf{p}}f_{\mathbf{q}} \qquad (10.5) \\ &= -\frac{1}{2}\int_{|\mathrm{p}|,|\mathrm{q}|<k_F}\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{d^3\mathbf{q}}{(2\pi)^3}(2S+1)\frac{e^2}{\epsilon_0(|\mathrm{p}-\mathrm{q}|^2)} \qquad (10.6) \\ &= -\rho\frac{3}{4\pi}\frac{e^2k_f}{4\pi\epsilon_0} \qquad (10.7)\end{aligned}$$

where $\rho = (2S+1)k_F^3/(6\pi^2)$ is the density of the electrons.

10.1.2 Exchange Correlation

Then let's consider the equal-time density correlation function

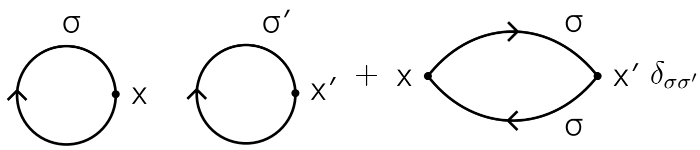
$$C_{\sigma\sigma'}(\mathbf{x}-\mathbf{x}') = \langle\Omega| : n_\sigma(\mathbf{x}) n_\sigma(\mathbf{x}') : |\Omega\rangle \tag{10.8}$$

It is related to the interaction energy by

$$\Delta E = \langle\Omega|\frac{1}{2}\int d^3\mathbf{x} d^3\mathbf{x}' V(\mathbf{x}-\mathbf{x}')\bar{\psi}(x)\bar{\psi}(x')\psi(x')\psi(x)|\Omega\rangle = \frac{1}{2}\sum_{\sigma,\sigma'}\int d^3\mathbf{x} d^3\mathbf{x}' V(\mathbf{x}-\mathbf{x}') C_{\sigma\sigma'}(\mathbf{x}-\mathbf{x}') \tag{10.9}$$

As usual, we have to the 0th order

$$C_{\sigma\sigma'}(\mathbf{x}-\mathbf{x}') = \langle\Omega|T[\bar{\psi}_\sigma(\mathbf{x}, t+3\delta)\bar{\psi}_{\sigma'}(\mathbf{x}', t+2\delta)\psi_{\sigma'}(\mathbf{x}', t+\delta)\psi_\sigma(\mathbf{x}, t)]|\Omega\rangle \tag{10.10}$$

$=$  (10.11)

$$= ((-1)\cdot iG(0,0^-))^2 + (-1)\cdot iG(\mathbf{x}-\mathbf{x}', 0^-)\cdot iG(\mathbf{x}'-\mathbf{x}, 0^-)\delta_{\sigma\sigma'} \tag{10.12}$$

$$= \rho_0^2 + G(\mathbf{x}-\mathbf{x}', 0^-)G(\mathbf{x}'-\mathbf{x}, 0^-)\delta_{\sigma\sigma'} \tag{10.13}$$

where

$$\rho_0 = -iG(0,0^-) = -\langle T[c_x(t)c_x^\dagger(t+\delta)]\rangle = \langle c_x c_x\rangle = \frac{k_F^3}{6\pi^2} \tag{10.14}$$

We have

$$\begin{aligned} G(\mathbf{x}-\mathbf{x}', 0^-) &= i\int_{|\mathbf{p}|<k_F}\frac{d^3\mathbf{p}}{(2\pi)^3}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')} && (10.15)\\ &= \frac{i}{(2\pi)^2}\int_0^{k_F}dp_r\int_0^\pi d\theta p_r^2\sin\theta e^{ip_r|\mathbf{x}-\mathbf{x}'|\cos\theta} && (10.16)\\ &= \frac{i}{2\pi^2 r^3}(\sin(k_F r) - k_F r\cos(k_F r)) && (10.17)\\ &= \frac{3i\rho_0}{(k_f r)^3}(\sin(k_F r) - k_F r\cos(k_F r)) && (10.18) \end{aligned}$$

where $r = |\mathbf{x}-\mathbf{x}'|$ and $\mathbf{p} = (p_r\sin\theta\cos\phi, p_r\sin\theta\sin\phi, p_r\cos\theta)$, with $\mathbf{x}-\mathbf{x}'$ as the direction of the z-axis.

Finally

$$C_{\sigma\sigma'}(r) = \rho_0^2[1 - 9\frac{(\sin(k_F r) - k_F r\cos(k_F r))^2}{(k_f r)^6}\delta_{\sigma\sigma'}] \tag{10.19}$$

The function between $C_{\sigma\sigma}/\rho_0^2$ and $k_F r$ is plotted in Fig. 10.1. At $r = 0$, $C_{\sigma\sigma}$ goes to 0 due to Pauli's exclusion principle.

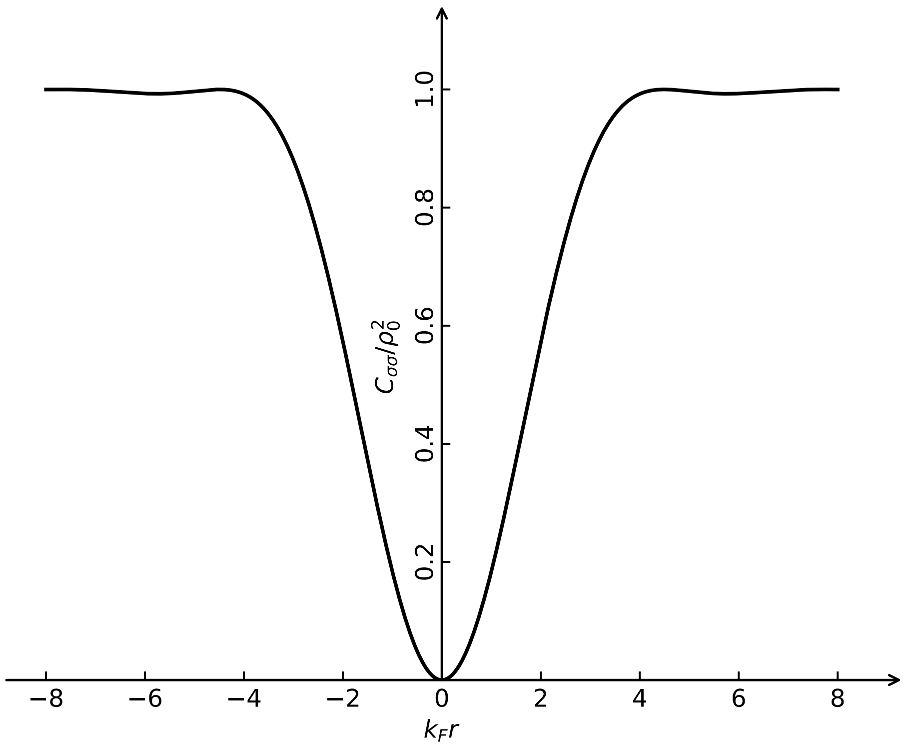


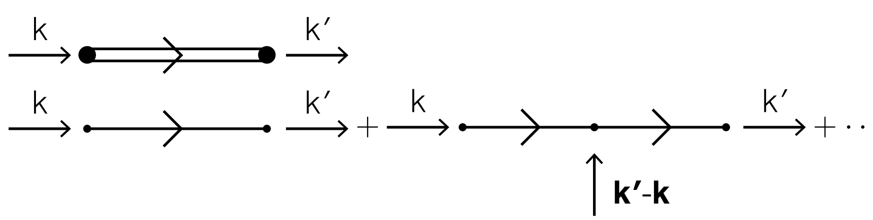
Figure 10.1: Equal-time density correlation function of the free electron system.

10.2 Potential Scattering

Let's consider a fermionic system in an attractive potential $U(x)$.
The Green's function is

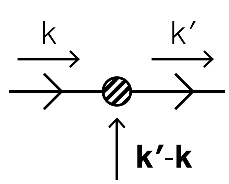
$$iG(k, k') \quad = \tag{10.20}$$

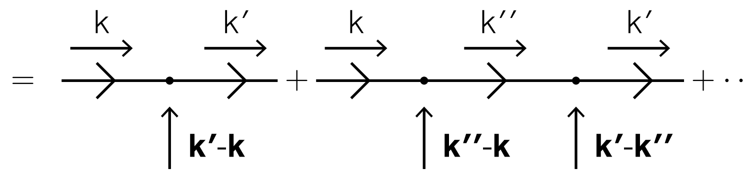
$$= \quad + \cdots \tag{10.21}$$



Let's define the renormalized interaction vertex

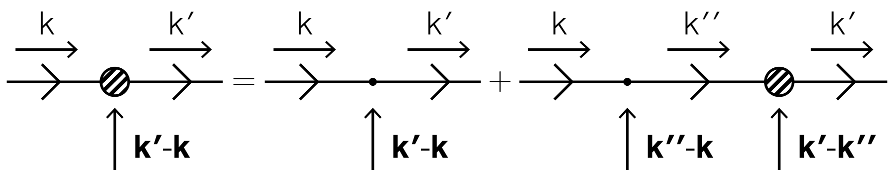
$$-it(\mathbf{k}, \mathbf{k}', k_0)(2\pi)\delta(k_0 - k_0') \quad = \tag{10.22}$$





(10.23)

We have



(10.24)

Then

$$t(\mathbf{k},\mathbf{k}',\omega) = U(\mathbf{k}'-\mathbf{k}) + \int \frac{d^3\mathbf{k}''}{(2\pi)^3} U(\mathbf{k}''-\mathbf{k}) G(\mathbf{k}'',\omega) t(\mathbf{k}'',\mathbf{k}',\omega) \tag{10.25}$$

In the case of delta-potential, we have

$$U(\mathbf{k}'-\mathbf{k}) = U \tag{10.26}$$

Then it's easy to see that $t(\mathbf{k},\mathbf{k}',\omega)$ doesn't depend on $\mathbf{k},\mathbf{k}'$.
So

$$t(\omega) = U + U \int \frac{d^3\mathbf{k}''}{(2\pi)^d} G(\mathbf{k}'',\omega) t(\omega) \tag{10.27}$$

and

$$\int \frac{d^3\mathbf{k}''}{(2\pi)^d} G(\mathbf{k}'',\omega) = \int_{-\mu}^{\Lambda} d\epsilon N(\epsilon) \frac{1}{\omega - \epsilon + i\delta \mathrm{sgn}\epsilon} \tag{10.28}$$

Let's discuss the 2 dimensional case, $N(\epsilon) = N_0 = \pi k_F^2/\mu$. Then

$$\int_{-\mu}^{\Lambda} d\epsilon N(\epsilon) \frac{1}{\omega - \epsilon + i\delta \mathrm{sgn}\epsilon} = -N_0 \ln\left[\frac{\omega - \Lambda}{\omega + \Lambda}\right] \simeq -N_0 \ln\left[\frac{-\Lambda}{\omega + \Lambda}\right] \tag{10.29}$$

Then

$$t(\omega) = \frac{U}{1 + UN_0 \ln\left[\frac{\Lambda}{-(\omega+\Lambda)}\right]} \tag{10.30}$$

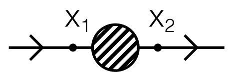
If $U < 0$, $t(\omega)$ has a pole at $-\mu - \Lambda e^{-\frac{1}{|U|N_0}}$. This is an example of the bound state.

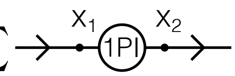
10.3 Self Energy

Let's consider a translational invariant system. The energy and momentum is conserved. In this system let's consider a particle a.

A 1 particle irreducible(1PI) diagram for a is defined as a connected diagram with exactly an incoming and an outgoing a indices, and can not be broken into two connected diagrams by a cut on each $a - \bar{a}$ bond.

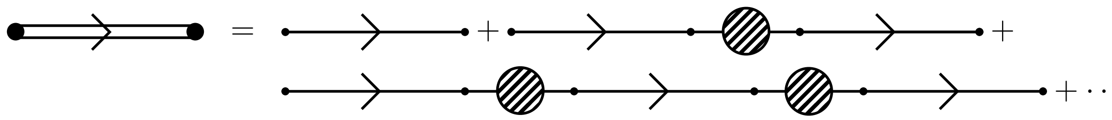
We define the self energy Σ of a as the sum of all different 1PIs:

$$-i\Sigma(x_2 - x_1) =$$ 

$$= \sum_{1PI}$$ 

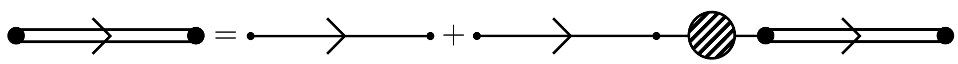
(10.31)

Then the propogator of a can be expressed as


(10.32)

You may check that we have the correct symmetric factor.

So we have the Dyson's equation


(10.33)

Let's view G,G^0 and Σ as operators, then

$$iG = iG_0 + iG_0 \cdot (-i)\Sigma \cdot iG_0 \tag{10.34}$$

So

$$G = \frac{1}{G_0^{-1} - \Sigma} \tag{10.35}$$

If the system is translational invariant, then $\Sigma(p, p') = \Sigma(p)I_{p,p'}$. So

$$G(p) = \frac{1}{G_0^{-1}(p) - \Sigma(p)} \tag{10.36}$$

Normally $G_0^{-1}(p)$ is real with an infinitesimal imaginary part, and $\Sigma(p)$ is small. For electron Green's function, let's break $\Sigma(p)$ into its real and imaginary part. Then

$$G(p) = \frac{1}{\omega - \tilde{e}_{\mathbf{p}} - \Re[\Sigma(p)] - i\Im[\Sigma(p)]} \tag{10.37}$$

This has a pole $\omega_0 = (\xi_p, \Im[\Sigma(\mathbf{p}, \omega_0)]) = (\tilde{e}_{\mathbf{p}} + \Re[\Sigma(\mathbf{p}, \omega_0)], \Im[\Sigma(\mathbf{p}, \omega_0)])$. The real-time Green's function is

$$G(t, \mathrm{p}) = \int \frac{d\omega}{2\pi} e^{-i\omega} G(p) \sim \theta(t\omega_0) e^{-i\xi_p t} e^{-|Z_{\mathbf{p}}\Im[\Sigma(\mathbf{p},\omega_0)]|t} \tag{10.38}$$

where $Z_{\mathbf{p}}$ is the residue at the pole

So $\Im[\Sigma(\mathbf{p}, \omega_0)]$ determines the decay rate of the field ψ_p. The life time is $1/|Z_{\mathbf{p}}\Im[\Sigma(\mathbf{p}, \omega_0)]|$. Besides, $\tilde{e}_{\mathbf{p}} + \xi_p$ is the renormalized energy.

10.4 Magnetic Susceptibility of Free Electron System

The free electron system interact with external magnetic field by the Hamiltonian

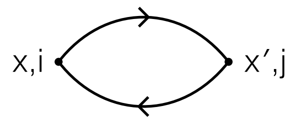
$$-\int \mu_B S(x) \cdot B(x,t) \tag{10.39}$$

where

$$S^i(x) = c^\dagger_\alpha \sigma^i_{\alpha\beta} c_\beta \tag{10.40}$$

Then

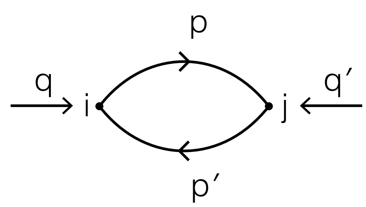
$$-i\chi^T_{ij}(x-x', t-t') = \mu_B^2$$



(10.41)

Then

$$\chi^T_{ij}(q)(2\pi)^4\delta(q+q') = i\mu_B^2$$



(10.42)

$$\begin{aligned}
&= i\mu_B^2 \int dp dp' e^{i\delta p_0 + i\delta p_0'} Tr[\sigma^j \sigma^i](-1) iG(p) iG(p') \delta(p-q-p') \\
&\quad \delta(p+q'-p') && (10.43) \\
\chi^T_{ij}(q) &= i\mu_B^2 2\delta_{ij} \int dp e^{2i\delta p_0 - i\delta q_0} G(p)G(p-q) && (10.44) \\
&= i\mu_B^2 2\delta_{ij} \int \frac{dp}{(2\pi)^4} e^{2i\delta p_0 - i\delta q_0} \frac{1}{p_0 - \epsilon_{\mathbf{p}} + i\delta \mathrm{sgn}\epsilon_{\mathbf{p}}} \frac{1}{p_0 - q_0 + \epsilon_{\mathbf{p-q}} + i\delta \mathrm{sgn}\epsilon_{\mathbf{p-q}}} && (10.45) \\
&= \mu_B^2 2\delta_{ij} \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{f_{\mathbf{p-q}} - f_{\mathbf{p}}}{q_0 - (\epsilon_{\mathbf{p-q}} - \epsilon_{\mathbf{p}}) + i\delta \mathrm{sgn}(\epsilon_{\mathbf{p-q}} - \epsilon_{\mathbf{p}})} && (10.46)
\end{aligned}$$

Then

$$\begin{aligned}
\chi_{ij}(q) &= 2\delta_{ij}\mu_B^2 \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{f_{\mathbf{p-q}} - f_{\mathbf{p}}}{q_0 - (\epsilon_{\mathbf{p-q}} - \epsilon_{\mathbf{p}}) + i\delta} && (10.47) \\
&= 2\delta_{ij}\mu_B^2 \Big[P\int \frac{d\mathbf{p}}{(2\pi)^3} \frac{f_{\mathbf{p-q}} - f_{\mathbf{p}}}{q_0 - (\epsilon_{\mathbf{p-q}} - \epsilon_{\mathbf{p}})} - i \int \frac{d\mathbf{p}}{(2\pi)^3} (f_{\mathbf{p-q}} - f_{\mathbf{p}}) \delta(q_0 - (\epsilon_{\mathbf{p-q}} - \epsilon_{\mathbf{p}})) \Big] && (10.48)
\end{aligned}$$

10.5 Jellium Model

Part II

Relativistic World

Chapter 11

Classical Field Theory

A classical field is a smooth map $\phi : M \to F$ from the base manifold M, which is usually 3+1D space-time, to some fiber space F. Intuitively, as illustrated in Fig 11.1, there's a field quantity at every space-time point. Common fibre spaces includes $\mathbb{R}$, $\mathbb{C}$, $\mathbb{R}^N$, $\mathbb{C}^N$, spinor...

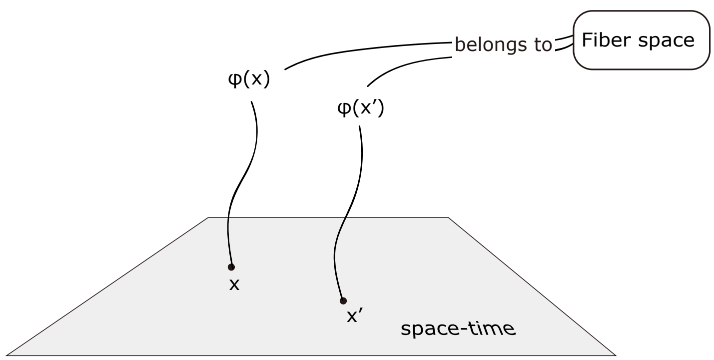


Figure 11.1: Classical field

A field $\phi(x) = \phi(\vec{x}, t) = \phi_t(\vec{x})$ can be regarded as a function of space, with t as a parameter. As t increases, the deformation of $\phi_t(\vec{x})$ is called the evolution of the field. Here we study fields whose evolution are determined by the law of mechanics. Such fields must satisfy a set of equations. This is called the equation of motion(EOM). A field that satisfies the EOM is said to be on-shell (abbreviated for "on the mass shell", meaning that after Fourier transformation, the non-vanishing term ϕ_k satisfies $k^2 = -m^2$), while a field that may not satisfy the EOM is said to be off-shell.

11.1 Lagrangian Formalism

In a dynamical system of classical field ϕ that has n components, there exist a real scaler field $\mathcal{L}[\phi, \partial_\mu \phi, x^\mu]$ called Lagrangian density. Lagrangian density is a functional of the ϕ and its first derivatives, and may also depend on x^μ explicitly. The form of Lagrangian density is decided by the nature of the dynamic system, and is considered known here. We define the action S in region Ω by the integral of the Lagrangian density:

$$S_\Omega = \int_\Omega d^4x \mathcal{L}[\phi, \partial_\mu \phi, x^\mu] \tag{11.1}$$

The core idea of the theory of classical fields is that:

An field with fixed value at $\partial\Omega$ (the boundary of Ω) is on-shell if and only if it satisfies the Hamilton's principle: the variation of the action in Ω according to the variation of the field is 0.

In other words, the action in Ω is stationary at on-shell fields. This leads to:

$$\begin{aligned}
\delta S_\Omega &= \int_\Omega d^4x \delta\mathcal{L}[\phi, \partial_\mu\phi, x^\mu] && (11.2)\\
&= \int_\Omega d^4x \Big[\frac{\partial \mathcal{L}}{\partial \phi_i}\delta\phi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_i)}\delta(\partial_\mu\phi_i)\Big] && (11.3)\\
&= \int_\Omega d^4x \Big[\frac{\partial \mathcal{L}}{\partial \phi_i}\delta\phi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_i)}\partial_\mu(\delta\phi_i)\Big] && (11.4)\\
&= \int_\Omega d^4x \Big[\frac{\partial \mathcal{L}}{\partial \phi_i}\delta\phi_i + \partial_\mu\Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_i)}\delta\phi_i\Big) - \Big(\partial_\mu\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_i)}\Big)\delta\phi_i\Big] && (11.5)\\
&= \int_\Omega d^4x \Big[\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_i)}\Big]\delta\phi_i + \int_{\partial\Omega} dS_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_i)}\delta\phi_i && (11.6)\\
&= \int_\Omega d^4x \Big[\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_i)}\Big]\delta\phi_i && (11.7)\\
&= 0 && (11.8)
\end{aligned}$$

Note that variation commutes with derivative because $\delta\partial_\mu\phi_i = \partial_\mu\phi_i' - \partial_\mu\phi_i = \partial_\mu(\phi_i' - \phi_i) = \partial_\mu\delta\phi_i$, and that $\delta\phi_i$ vanishes at $\partial\Omega$.

Since $\delta\phi_i$ is arbitrary (as long as it's small), the condition

$$\int_\Omega d^4x \Big[\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_i)}\Big]\delta\phi_i = 0 \tag{11.9}$$

is equivalent to

$$\frac{\partial \mathcal{L}}{\partial \phi_i} = \partial_\mu\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_i)} \tag{11.10}$$

in Ω.

This is called the Euler-Lagrange equation. It is just the EOM in the theory of classical fields. According to theory of partial differential equation, there is always a solution with given boundary condition ($\phi_i(x)$ at $\partial\Omega$).

11.2 Hamiltonian Formalism

For the moment lets forget $\dot\phi_i = \frac{\partial\phi_i}{\partial t}$ and treat ϕ_i and $\dot\phi_i$ in $\mathcal{L}[\phi, \partial_\mu\phi, x^\mu]$ as independent degrees of freedom.

We define the canonical momentum as

$$\pi_i(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i(x)} = \pi_i[\phi, \partial_\alpha \phi, \dot{\phi}, x^\mu] \tag{11.11}$$

where ∂_α is the spatial partial differentiation.

Thus we can express $\dot{\phi}_i$ as a functional of ϕ, $\partial_\alpha\phi$, π and x^μ. In Hamiltonian formalism, we use ϕ_i and π_i as independent degrees of freedom.

First we define the Hamiltonian density as the Legendre transformation of $\mathcal{L}$

$$\mathcal{H}[\phi, \partial_\alpha\phi, \pi, x^\mu] = \left[\pi_i\dot{\phi}_i - \mathcal{L}[\phi, \partial_\alpha\phi, \dot{\phi}, x^\mu]\right]\Big|_{\dot{\phi}=\dot{\phi}[\phi,\partial_\alpha\phi,\pi,x^\mu]} \tag{11.12}$$

We have following relations

$$\begin{aligned}
\frac{\partial \mathcal{H}}{\partial \phi_i} &= \pi_j \frac{\partial \dot{\phi}_j}{\partial \phi_i} - \frac{\partial \mathcal{L}}{\partial \phi_i} - \frac{\partial \mathcal{L}}{\partial \phi_{j,t}} \frac{\partial \dot{\phi}_j}{\partial \phi_i} = -\frac{\partial \mathcal{L}}{\partial \phi_i} && (11.13)\\
\frac{\partial \mathcal{H}}{\partial \pi_i} &= \dot{\phi}_i + \pi_j \frac{\partial \dot{\phi}_j}{\partial \pi_i} - \frac{\partial \mathcal{L}}{\partial \dot{\phi}_j} \frac{\partial \dot{\phi}_j}{\partial \pi_i} = \dot{\phi}_i && (11.14)\\
\frac{\partial \mathcal{H}}{\partial(\partial_\alpha\phi_i)} &= \pi_j \frac{\partial \dot{\phi}_j}{\partial(\partial_\alpha\phi_i)} - \frac{\partial \mathcal{L}}{\partial(\partial_\alpha\phi_i)} - \frac{\partial \mathcal{L}}{\partial \dot{\phi}_j} \frac{\partial \dot{\phi}_j}{\partial(\partial_\alpha\phi_i)} = -\frac{\partial \mathcal{L}}{\partial(\partial_\alpha\phi_i)} && (11.15)
\end{aligned}$$

Up to now, we have not determined the on-shell condition of π. However, we may impose one

$$\dot{\phi}_i = \partial_t\phi_i \Leftrightarrow \pi_i = \left.\frac{\partial \mathcal{L}}{\partial \dot{\phi}_i(x)}\right|_{\dot{\phi}_i=\partial_t\phi_i} \tag{11.16}$$

The Euler-Lagrange equation can be reformulated into

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \phi_i} &= \partial_\alpha \frac{\partial \mathcal{L}[\phi, \partial_\alpha\phi, \dot{\phi}, x^\mu]}{\partial(\partial_\alpha\phi_i)} + \partial_t \frac{\partial \mathcal{L}[\phi, \partial_\alpha\phi, \dot{\phi}, x^\mu]}{\partial(\dot{\phi}_i)} && (11.17)\\
\dot{\phi}_i &= \partial_t\phi_i && (11.18)
\end{aligned}$$

Use ϕ and π as independent variables, we have the Hamilton's equations

$$\begin{aligned}
\partial_t\phi_i &= \frac{\partial \mathcal{H}}{\partial \pi_i} && (11.19)\\
\partial_t\pi_i &= -\frac{\partial \mathcal{H}}{\partial \phi_i} + \partial_i \frac{\partial \mathcal{H}}{\partial(\partial_i\phi_i)} && (11.20)
\end{aligned}$$

11.2.1 Generalized Hamilton's Principle

We can derive the Hamilton's equations from variational principle. We define a generalized Lagrangian density $\mathcal{L}'[\phi, \partial_\alpha\phi, \pi, x]$ as

$$\mathcal{L}'[\phi, \partial_\alpha\phi, \pi, x^\mu] = \pi_i\partial_t\phi_i - \mathcal{H}[\phi, \partial_i\phi, \pi, x^\mu] = \pi_i(\partial_t\phi_i - \dot{\phi}_i) + \mathcal{L}[\phi, \partial_\alpha\phi, \dot{\phi}, x^\mu] \tag{11.21}$$

where the independent fields are ϕ and π. We show that:

Lemma 11.2.1. *The on-shell ϕ and π is the stationary path for the action*

$$S'_\Omega = \int_\Omega d^4x \mathcal{L}'[\phi, \partial_\alpha\phi, \pi, x] \tag{11.22}$$

Proof. We require S'_Ω to be stationary while varying π_i, and get (note that we do not require $\delta\pi = 0$ at $\partial\Omega$)

$$\frac{\delta S'_\Omega}{\delta \pi_i} = \frac{\partial \mathcal{L}'}{\partial \pi_i} = \partial_t\phi_i - \dot{\phi}_i = 0 \tag{11.23}$$

This is our new on-shell condition for π.

Take (11.23) into (11.21), we get our good old Lagrange density again:

$$\mathcal{L}'[\phi, \partial_\alpha\phi, \pi, x^\mu]\big|_{\dot{\phi}=\partial_t\phi} = \mathcal{L}[\phi, \partial_\mu\phi, x^\mu] \tag{11.24}$$

This procedure is sometimes called the inverse of the Legendre transformation. Take (11.24) to (11.22) and we get

$$S'_\Omega\big|_{\dot{\phi}=\partial_t\phi} = S_\Omega \tag{11.25}$$

We further require S'_Ω to be stationary while varying ϕ. Obviously we get the Euler-Lagrange equation again. □

Since ϕ and π are independent variables of the functional S'_Ω, we may get the EOM by the so-called generalized Hamilton's principle

$$\delta S'_\Omega = 0 \tag{11.26}$$

where variation is taken on $\pi(x)$ and $\phi(x)$.

This leads the the Hamilton's equations again

$$\begin{aligned} \partial_t\phi_i &= \dot{\phi}_i = \frac{\partial\mathcal{H}}{\partial\pi_i} && (11.27) \\ \partial_t\pi_i &= -\frac{\partial\mathcal{H}}{\partial\phi_i} + \partial_i\frac{\partial\mathcal{H}}{\partial(\partial_i\phi_i)} && (11.28) \end{aligned}$$

11.2.2 Dynamical Quantity

We can define the Hamiltonian from the Hamiltonian density

$$H(t) = \int d^3x \mathcal{H}(x) \tag{11.29}$$

Using (A.14), we have the following relations

$$\begin{aligned} \frac{\delta H}{\delta\pi_i(x)} &= \frac{\partial\mathcal{H}}{\partial\pi_i}(x) && (11.30) \\ \frac{\delta H}{\delta\phi_i(x)} &= \Big[\frac{\partial\mathcal{H}}{\partial\phi_i} - \partial_i\frac{\partial\mathcal{H}}{\partial(\partial_i\phi_i)}\Big](x) && (11.31) \end{aligned}$$

Thus the Hamilton's equations can be re-expressed as

$$\partial_t\phi_i(x) = \frac{\delta H}{\delta\pi_i(x)} \tag{11.32}$$

$$\partial_t\pi_i(x) = -\frac{\delta H}{\delta\phi_i(x)} \tag{11.33}$$

A dynamical quantity $Q(t)$ is the integral over space of some local functional $\mathcal{Q}[\phi, \partial_i\phi, \pi, \partial_i\pi, x^\mu]$

$$Q(t) = \int d^3x \mathcal{Q}(x) \tag{11.34}$$

Clearly Hamiltonian is a dynamical quantity.

11.2.3 Poisson Bracket

Let A and B be two dynamical quantity, their Poisson bracket is defined as

$$\{A(t), B(t)\} = \int d^3x \Big[\frac{\delta A}{\delta\phi_i(x)}\frac{\delta B}{\delta\pi_i(x)} - \frac{\delta A}{\delta\pi_i(x)}\frac{\delta B}{\delta\phi_i(x)}\Big] \tag{11.35}$$

It is also a dynamical quantity

As in classical mechanics, Poisson bracket of dynamical quantities forms a Lie algebra called Poisson algebra.

It's easy to check the following identity

$$\{\phi_i(\vec{x}, t), \pi_j(\vec{y}, t)\} = \delta^{(3)}(\vec{x} - \vec{y})\delta_{ij} \tag{11.36}$$

For on-shell field, the evolution of a dynamical quantity Q is totally determined. It's EOM is

$$\begin{aligned}
\frac{dQ}{dt} &= \lim_{\delta t\to 0}\frac{1}{\delta t}\Big[\int d^3x\mathcal{Q}[\phi'(x), \partial_i\phi'(x), \pi'(x), \partial_i\pi'(x), x'^\mu] - \int d^3x\mathcal{Q}[\phi(x'), \partial_i\phi(x'), \\
&\quad \pi(x'), \partial_i\pi(x'), x^\mu]\Big] + \lim_{\delta t\to 0}\frac{1}{\delta t}\Big[\int d^3x\mathcal{Q}[\phi'(x), \partial_i\phi'(x), \pi'(x), \partial_i\pi'(x), x^\mu] \\
&\quad - \int d^3x\mathcal{Q}[\phi(x), \partial_i\phi(x), \pi(x), \partial_i\pi(x), x^\mu]\Big] && (11.37) \\
&= \int d^3x\frac{\partial\mathcal{Q}}{\partial t} + \lim_{\delta t\to 0}\frac{1}{\delta t}\int d^3x\Big[\frac{\delta\mathcal{Q}}{\delta\phi_i(x)}(\phi_i'(x) - \phi_i(x)) + \frac{\delta\mathcal{Q}}{\delta\pi_i(x)}(\pi_i'(x) - \pi_i(x))\Big] && (11.38) \\
&= \frac{\partial Q}{\partial t} + \int d^3x\Big[\frac{\delta\mathcal{Q}}{\delta\phi_i(x)}\partial_t\phi_i(x) + \frac{\delta\mathcal{Q}}{\delta\pi_i(x)}\partial_t\pi_i(x)\Big] && (11.39) \\
&= \frac{\partial Q}{\partial t} + \int d^3x\Big[\frac{\delta\mathcal{Q}}{\delta\phi_i(x)}\frac{\delta H}{\delta\pi_i(x)} - \frac{\delta\mathcal{Q}}{\delta\pi_i(x)}\frac{\delta H}{\delta\phi_i(x)}\Big] && (11.40) \\
&= \frac{\partial Q}{\partial t} + \{Q, H\} && (11.41)
\end{aligned}$$

where $x'^\mu = x^\mu + \delta t\delta_0^\mu$, $\phi_i'(x) = \phi_i(x')$ and $\pi_i'(x) = \pi_i(x')$. During the derivation we have used (A.7) and the Hamilton's equations.

11.3 Symmetry

A symmetry operation (T, P) is a diffeomorphism T of the base manifold together with an invertible field transformation P such that for any subregion ω of the base manifold Ω, we have

$$\int_\omega d^4x\mathcal{L}[\phi, \partial_\mu\phi, x^\mu] = \int_{\omega'} d^4x\mathcal{L}[\phi', \partial_\mu\phi', x^\mu] \tag{11.42}$$

where $\omega' = T\omega$ and $\phi' = P(T\phi)$, and T acts on ϕ by $(T\phi)(x) = \phi(T^{-1}x)$. From Hamilton's principle, it's clear that if ϕ is on-shell on ω, ϕ' is also on-shell on ω' with boundary condition ϕ' at $\partial\omega'$.

If $T \neq id$, the symmetry operation is called space-time symmetry. Otherwise it is called internal symmetry.

We impose two symmetry operations (T, P) and (T', P') successively:

$$\int_\omega d^4x\mathcal{L}[\phi, \partial_\mu\phi, x^\mu] = \int_{\omega'} d^4x\mathcal{L}[\phi', \partial_\mu\phi', x^\mu] = \int_{\omega''} d^4x\mathcal{L}[\phi'', \partial_\mu\phi'', x^\mu] \tag{11.43}$$

where

$$\omega'' = T'\omega' = (T' \circ T)\omega \tag{11.44}$$

$$\phi'' = (P' \circ T')\phi' = (P' \circ T' \circ P \circ T)\phi = (P' \circ T' \circ P \circ T'^{-1})((T' \circ T)\phi) \tag{11.45}$$

This shows that $(T' \circ T, P' \circ T' \circ P \circ T'^{-1})$ is also a symmetry operation. So symmetry operations form a monoid under composition

$$(T', P') \circ (T, P) = (T' \circ T, P' \circ T' \circ P \circ T'^{-1}) \tag{11.46}$$

while the unit element is (id, id). It's easy to see that $(T, P)^{-1} = (T^{-1}, T^{-1}P^{-1}T)$, so symmetry operations form a group. This is called the symmetry group of a dynamical system. A dynamical system is called to have discreet/continuous symmetry if its symmetry group is a discreet/Lie group.

Example We define the symmetry operation (T_θ, P_θ) on a vector field $A^\mu(x)$ parameterized by θ as

$$T_\theta \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{11.47}$$

$$P_\theta \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} \tag{11.48}$$

As illustrated in Fig 11.2, this is a continuous symmetry, and is the subgroup of the symmetry group of the system with Lagrangian density

$$\mathcal{L} = (\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) + \alpha A^\mu A_\mu \tag{11.49}$$

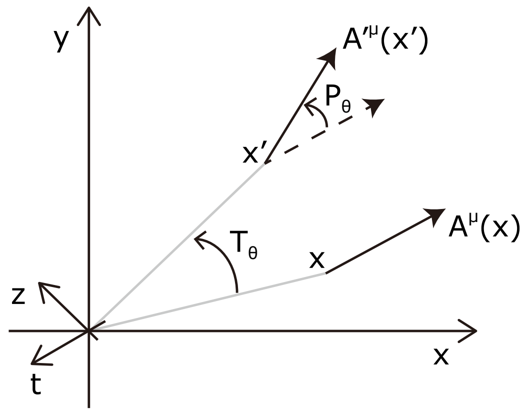


Figure 11.2: Continous symmetry operation

Example Consider a Lagrangian density of a scalar field ϕ

$$\mathcal{L} = \phi^2 + \partial^\mu\phi\partial_\mu\phi + \sum_\mu e^{-(x^\mu)^2} \tag{11.50}$$

Its symmetry group is a discreet group mapping axes to axes.

11.3.1 Noether's Theorem

Noether's theorem states that if a system has a continuous symmetry G, then each one-parameter subgroup of G corresponds to a conserved current $j^\mu[\phi, \partial_\mu\phi, x^\mu]$ such that $\partial_\mu j^\mu = 0$ when ϕ is on-shell.

Suppose the one-parameter subgroup is (T_ϵ, P_ϵ) parameterized by ϵ, with Lie algebra $(\mathcal{T}, \mathcal{P})$. The infinitesimal symmetry operation is

$$(T_\epsilon, P_\epsilon) = (id + \epsilon\mathcal{T}, id + \epsilon\mathcal{P}) \tag{11.51}$$

which act on x and ϕ as

$$x \longrightarrow x' = T_\epsilon x = x + \epsilon\mathcal{T}x = x + \delta x \tag{11.52}$$

$$\phi(x) \longrightarrow \phi'(x) = P_\epsilon\phi(T_\epsilon^{-1}x) = \phi(x - \epsilon\mathcal{T}x) + \epsilon\mathcal{P}\phi(x) = \phi(x) + \delta\phi(x) \tag{11.53}$$

Then

$$\delta x^\mu = \epsilon(\mathcal{T}x)^\mu \tag{11.54}$$

$$\delta\phi_i(x) = \phi_i(x - \epsilon\mathcal{T}x) - \phi_i(x) + \epsilon(\mathcal{P}\phi)_i(x) \tag{11.55}$$

$$= -\epsilon\partial_\mu\phi_i(x)(\mathcal{T}x)^\mu + \epsilon(\mathcal{P}\phi)_i(x) \tag{11.56}$$

From the definition of symmetry, we have

$$\int_\omega d^4x\mathcal{L}[\phi, \partial_\mu\phi, x^\mu] = \int_{\omega'} d^4x\mathcal{L}[\phi', \partial_\mu\phi', x^\mu] \tag{11.57}$$

$$
\begin{aligned}
&= \int_\omega d^4x \mathcal{L}[\phi', \partial_\mu \phi', x^\mu] + \int_{\partial\omega} dS_\mu \delta x^\mu \mathcal{L}[\phi', \partial_\mu \phi', x^\mu] && (11.58) \\
&= \int_\omega d^4x \Big[\mathcal{L}[\phi', \partial_\mu \phi', x^\mu] + \partial_\mu \big[\delta x^\mu \mathcal{L}[\phi', \partial_\mu \phi', x^\mu]\big]\Big] && (11.59)
\end{aligned}
$$

So

$$
\begin{aligned}
0 &= \int_\omega d^4x \Big[\mathcal{L}[\phi', \partial_\mu \phi', x^\mu] - \mathcal{L}[\phi, \partial_\mu \phi, x^\mu] + \partial_\mu \big[\delta x^\mu \mathcal{L}[\phi', \partial_\mu \phi', x^\mu]\big]\Big] && (11.60) \\
&= \int_\omega d^4x \Big[\frac{\partial \mathcal{L}}{\partial \phi_i}\delta\phi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\partial_\mu \delta\phi_i + \partial_\mu(\delta x^\mu \mathcal{L})\Big] && (11.61) \\
&= \int_\omega d^4x \Big[\Big(\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\Big)\delta\phi_i + \partial_\mu\Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\delta\phi_i + \delta x^\mu \mathcal{L}\Big)\Big] && (11.62)
\end{aligned}
$$

Since ω is arbitrary, the integrand must be zero everywhere

$$
\Big(\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\Big)\delta\phi_i + \partial_\mu\Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\delta\phi_i + \delta x^\mu \mathcal{L}\Big) = 0 \tag{11.63}
$$

We define the conserved current as

$$
j^\mu = \frac{1}{\epsilon}\Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\delta\phi_i + \delta x^\mu \mathcal{L}\Big) \tag{11.64}
$$

It's easy to see when ϕ is on-shell, $\partial_\mu j^\mu = 0$.
We take (11.54) and (11.56) into (11.64), and get the final expression of the conserved current

$$
\begin{aligned}
j^\mu &= \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\partial_\nu \phi_i (\mathcal{T}x)^\nu - (\mathcal{T}x)^\mu \mathcal{L}\Big) - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}(\mathcal{P}\phi)_i && (11.65) \\
&= \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\partial_\nu \phi_i - \mathcal{L}\delta^\mu_\nu\Big)(\mathcal{T}x)^\nu - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}(\mathcal{P}\phi)_i && (11.66)
\end{aligned}
$$

We further define the conserved charge Q in region Ω as

$$
Q_\Omega = \int_\Omega d^3x j^0 \tag{11.67}
$$

When ϕ is on-shell, the time derivative of Q_Ω is

$$
\begin{aligned}
\frac{dQ_\Omega}{dt} &= \int_\Omega d^3x \partial_0 j^0 && (11.68) \\
&= -\int_\Omega d^3x \partial_i j^i && (11.69) \\
&= -\int_{\partial\Omega} dS_i j^i && (11.70)
\end{aligned}
$$

This is often interpreted as the growth of the total charge inside a region is the same as the amount of charge that flows inside. This also describes the physics of incompressible flow.

We say that two conserved currents are **equivalent** if they give the same conserved charge up to a boundary term.

With a conserved current j, we can construct a conserved current

$$j'^{\mu} = j^{\mu} + \partial_\nu k^{\mu\nu} \tag{11.71}$$

where $k^{\mu\nu} = -k^{\nu\mu}$.

From

$$Q'_\Omega = \int_\Omega d^3x j'^0 = \int_\Omega d^3x(j^0 + \partial_\nu k^{0\nu}) = Q_{1\Omega} + \int_{\partial\Omega} dS_\nu k^{0\nu} \tag{11.72}$$

we see that j and j' are equivalent.

11.3.2 Change of Action by an Arbitrary Spatial Difference

For the future proof of Ward identity, we will study a slightly modified transformation:

$$\begin{aligned} x &\longrightarrow x' = T_\epsilon x = x + \epsilon(x)\mathcal{T}x = x + \delta x && (11.73)\\ \phi(x) &\longrightarrow \phi'(x) = P_\epsilon \phi(T_\epsilon^{-1}x) = \phi(x - \epsilon(x)\mathcal{T}x) + \epsilon(x)\mathcal{P}\phi(x) = \phi(x) + \delta\phi(x) && (11.74)\end{aligned}$$

So

$$\begin{aligned} \delta x^\mu &= \epsilon(x)(\mathcal{T}x)^\mu && (11.75)\\ \delta\phi_i(x) &= \phi_i(x - \epsilon(x)\mathcal{T}x) - \phi_i(x) + \epsilon(x)(\mathcal{P}\phi)_i(x) && (11.76)\\ &= -\epsilon(x)\partial_\mu\phi_i(x)(\mathcal{T}x)^\mu + \epsilon(x)(\mathcal{P}\phi)_i(x) && (11.77)\end{aligned}$$

By the definition of symmetry, if $\epsilon(x)$ is a small const, $\delta S = 0$. However if $\epsilon(x)$ is non-const,

$$\begin{aligned} \delta S_\epsilon &= \int_{\omega'} d^4x \mathcal{L}[\phi', \partial_\mu\phi', x^\mu] - \int_\omega d^4x \mathcal{L}[\phi, \partial_\mu\phi, x^\mu] && (11.78)\\ &= \int_\omega d^4x \Big[\Big(\frac{\partial\mathcal{L}}{\partial\phi_i} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\Big)\delta\phi_i + \partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\delta\phi_i + \delta x^\mu\mathcal{L}\Big)\Big] && (11.79)\\ &= \int_\omega d^4x[A\epsilon(x) - j^\mu\partial_\mu\epsilon(x)] && (11.80)\end{aligned}$$

where A is just the LHS of (11.63) and is 0 (even for off-shell field).

So

$$\delta S_\epsilon = -\int_\omega d^4x j^\mu \partial_\mu \epsilon(x) \tag{11.81}$$

Consider two equivalent currents j^μ and j'^μ such that

$$j'^{\mu} = j^{\mu} + \partial_\nu k^{\mu\nu} \tag{11.82}$$

where $k^{\mu\nu} = -k^{\nu\mu}$.

From

$$-\int_\omega d^4x(j^\mu + \partial_\nu k^{\mu\nu})\partial_\mu\epsilon(x) \tag{11.83}$$

$$= -\int_\omega d^4x j^\mu \partial_\mu \epsilon(x) - \int_\omega d^4x \partial_\nu k^{\mu\nu} \partial_\mu \epsilon(x) \tag{11.84}$$

$$= -\int_\omega d^4x j^\mu \partial_\mu \epsilon(x) - \int_\omega d^4x \partial_\mu (\partial_\nu k^{\mu\nu} \epsilon(x)) - \int_\omega d^4x \partial_\mu \partial_\nu k^{\mu\nu} \epsilon(x) \tag{11.85}$$

$$= -\int_\omega d^4x j^\mu \partial_\mu \epsilon(x) - \int_{\partial\omega} dS_\mu \partial_\nu k^{\mu\nu} \epsilon(x) \tag{11.86}$$

we see that the two equivalent currents give the same δS_ϵ up to a boundary term.

11.3.3 Poincaré Symmetry

Poincaré group P is the group isometry transformation of $\mathbb{R}^4$, whose element $T_{\Lambda,a}$ is parameterized by Lorentz transformation Λ and translation a^μ, and acts on x^μ as

$$T_{\Lambda,a} x = \Lambda x + a \tag{11.87}$$

Due to the principle of special relativity, a relativistic field must have Poincaré symmetry. Denote the symmetry group of the field as S. That is, relative to each neighborhood $N(T_{\Lambda,a})$ of P, there exists a smooth S-valued function $(T_{\Lambda',a'}, P_{\Lambda',a'})$ as function of $T_{\Lambda',a'} \in N(T_{\Lambda,a})$.

Here we make two asumptions:

1. $(T_{1,a}, 1)$ is a symmetry operation.
2. P commutes with T when acting on fields. So $(T_{\Lambda,a}, P_{\Lambda,a})(T_{\Lambda',a'}, P_{\Lambda',a'}) = (T_{\Lambda,a}T_{\Lambda',a'}, P_{\Lambda,a}P_{\Lambda',a'})$

If $(T_{\Lambda,a}, P_{\Lambda,a}) \in S$ then

$$(T_{\Lambda,0}, P_{\Lambda,a}) = (T_{1,a}, 1)^{-1}(T_{\Lambda,a}, P_{\Lambda,a}) \in S \tag{11.88}$$

If furthermore $(T_{\Lambda,0}, P_{\Lambda,0}) \in S$ then

$$(1, P_{\Lambda,0}P^{-1}_{\Lambda,a}) = (T_{\Lambda,0}, P_{\Lambda,0})(T_{\Lambda,0}, P_{\Lambda,a})^{-1} \in S \tag{11.89}$$

Define

$$G = \{g | (1, g) \text{ is symmetry}\} \tag{11.90}$$

Obviously G is a group. And we have shown that

$$(T_{\Lambda,a}, P_{\Lambda,a}) \in S \wedge (T_{\Lambda,0}, P_{\Lambda,0}) \in S \Rightarrow P_{\Lambda,0}P^{-1}_{\Lambda,a} \in G \tag{11.91}$$

Then

$$(T_{\Lambda,a}, P_{\Lambda,a}) \in S \wedge (T_{\Lambda,0}, P_{\Lambda,0}) \in S \Rightarrow (T_{\Lambda,a}, P_{\Lambda,0}) = (1, P_{\Lambda,0}P^{-1}_{\Lambda,a})(T_{\Lambda,a}, P_{\Lambda,a}) \in S \tag{11.92}$$

Since there must be some $(T_{\Lambda,a}, P_{\Lambda,a}) \in S$, we have

$$(T_{\Lambda,0}, P_{\Lambda,0}) \in S \Rightarrow (T_{\Lambda,a}, P_{\Lambda,0}) \in S \tag{11.93}$$

That means, we may choose a smooth S-valued function $(T_{\Lambda',a'}, P_{\Lambda',0})$ in each $(\Lambda', a') \in (N(\Lambda), \mathbb{R}^4)$ where $N(\Lambda)$ is a neighborhood of Λ, as long as $P_{\Lambda',0}$ is smooth in $N(\Lambda)$

So there is a one-one correspondence between the possible piece-wise smooth $(T_{\Lambda,a}, P_{\Lambda,a})$ and the possible piece-wise smooth $P_{\Lambda,0}$ together the possible G. So it's enough to just study the Lorentz symmetry $(T_\Lambda, P_\Lambda) = (T_{\Lambda,0}, P_{\Lambda,0})$. Clearly the possible piece-wise smooth $P_\Lambda = P_{\Lambda,0}$ is just the cover space of $L_+^\uparrow$. The only continuous cover space of $L_+^\uparrow$ is $L_+^\uparrow$ and $SL(2,\mathbb{C})$. Thus, the possible fields classified by Lorentz symmetry include:

1. Scalar field, vector field, tensor field ... whose Lorentz symmetry is (T_Λ, P_Λ), where P_Λ is isomorphic to $L_+^\uparrow$.

2. Spinor field ... whose Lorentz symmetry is $(T_\Lambda, \pm P_\Lambda)$, where $\pm P_\Lambda$s is isomorphic to $SL(2,\mathbb{C})$. The relationship between Λ and P_Λ is illustrated in Fig. 11.3.

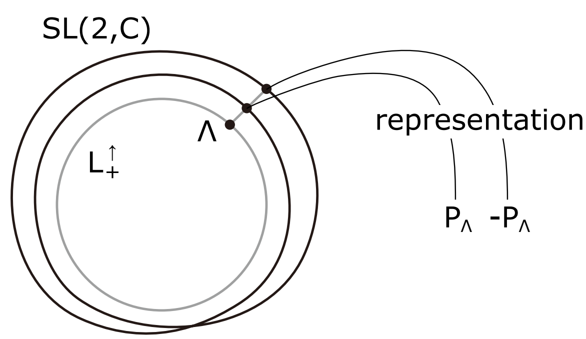


Figure 11.3: The relationship between Λ and P_Λ.

11.3.4 Conserved Current of Poincaré Symmetry

The infinitesimal transformation of translation is (ϵ^α, id). Its Lie algebra is $(\delta^\alpha, 0)$, where

$$(\delta^\alpha x)^\mu = \eta^{\alpha\mu} \tag{11.94}$$

And the conserved currents are

$$\begin{aligned} j^{\mu,\alpha} &= \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\partial_\nu \phi_i - \mathcal{L}\delta^\mu_\nu\Big)(\delta^\alpha x)^\nu & (11.95)\\ &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\partial^\alpha \phi_i - \mathcal{L}\eta^{\mu\alpha} & (11.96)\end{aligned}$$

We define the energy-momentum tensor $T^{\mu\nu}$ as

$$T^{\mu\nu} = j^{\mu,\nu} \tag{11.97}$$

with conserved charges as

$$P^\mu = \int d^3x T^{0\mu} \tag{11.98}$$

$$= \int d^3x \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_i)} \partial^\mu \phi_i - \mathcal{L}\eta^{0\mu} \tag{11.99}$$

$$= \int d^3x \pi_i \partial^\mu \phi_i - \mathcal{L}\eta^{0\mu} \tag{11.100}$$

which are called momentum 4-vector.

It can be easily checked that the current of any symmetry can be expressed as

$$j^\mu = T^\mu_{\ \nu}(\mathcal{T}x)^\nu - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}(\mathcal{P}\phi)_i \tag{11.101}$$

The term $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}(\mathcal{P}\phi)_i$ is annoying and can be eliminated for space-time symmetry (in the case of equivalent conserved current) in some cases.

The infinitesimal transformation of rotation(including boost) is $(\Lambda_\epsilon^{\alpha\beta}, P_{\Lambda_\epsilon^{\alpha\beta}})$. Its Lie algebra is $(\mathcal{M}^{\alpha\beta}, \mathcal{P}^{\alpha\beta})$, where

$$(\mathcal{M}^{\alpha\beta}x)^\mu = \eta^{\alpha\mu}x^\beta - \eta^{\beta\mu}x^\alpha \tag{11.102}$$

And the conserved currents are

$$j^{\mu,\alpha\beta} = \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\partial_\nu \phi_i - \mathcal{L}\delta^\mu_\nu\Big)(\mathcal{M}^{\alpha\beta}x)^\nu - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}(\mathcal{P}^{\alpha\beta}\phi)_i \tag{11.103}$$

$$= \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\partial_\nu \phi_i - \mathcal{L}\delta^\mu_\nu\Big)(\eta^{\alpha\nu}x^\beta - \eta^{\beta\nu}x^\alpha) - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}(\mathcal{P}^{\alpha\beta}\phi)_i \tag{11.104}$$

$$= (T^{\mu\alpha}x^\beta - T^{\mu\beta}x^\alpha) - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}(\mathcal{P}^{\alpha\beta}\phi)_i \tag{11.105}$$

which we may also call $J^{\mu\alpha\beta}$

The conserved charge of $J^{\mu\alpha\beta}$ is

$$M^{\mu\nu} = \int d^3x j^{0,\mu\nu} \tag{11.106}$$

which are called angular momentum tensors.

We define the Belinfante tensor as

$$B^{\mu\nu\rho} = \frac{1}{2}\Big[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}(\mathcal{P}^{\nu\rho}\phi)_i + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi_i)}(\mathcal{P}^{\rho\mu}\phi)_i - \frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi_i)}(\mathcal{P}^{\mu\nu}\phi)_i\Big] \tag{11.107}$$

We assume $\mathcal{P}^{\mu\nu} = -\mathcal{P}^{\nu\mu}$, which leads to $B^{\mu\nu\rho} = -B^{\nu\mu\rho}$.

We define the Belinfante energy-momentum tensor $T_B^{\mu\nu}$ as

$$T_B^{\mu\nu} = T^{\mu\nu} + \partial_\rho B^{\rho\mu\nu} \tag{11.108}$$

Since $B^{\rho\mu\nu}$ is anti-symmetric in ρ and μ, $T_B^{\mu\nu}$ and $T^{\mu\nu}$ are equivalent.

Applying ∂_μ to (11.105), and we have

$$T^{\alpha\beta} - T^{\beta\alpha} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}(\mathcal{P}^{\alpha\beta}\phi)_i \tag{11.109}$$

So

$$T_B^{\mu\nu} - T_B^{\nu\mu} = 0 \tag{11.110}$$

, that is, $T_B^{\mu\nu}$ is symmetric.

We may define

$$J_B^{\mu\nu\rho} = T_B^{\mu\nu} x^\rho - T_B^{\mu\rho} x^\nu \tag{11.111}$$

We have

$$\begin{aligned}
J_B^{\mu\nu\rho} &= T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu + (x^\rho \partial_\alpha B^{\alpha\mu\nu} - x^\nu \partial_\alpha B^{\alpha\mu\rho}) && (11.112)\\
&= T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu + \partial_\alpha (x^\rho B^{\alpha\mu\nu} - x^\nu B^{\alpha\mu\rho}) - (B^{\rho\mu\nu} - B^{\nu\mu\rho}) && (11.113)\\
&= T^{\mu\nu} x^\rho - T^{\mu\rho} x^\nu - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} (\mathcal{P}^{\alpha\beta}\phi)_i + \partial_\alpha (x^\rho B^{\alpha\mu\nu} - x^\nu B^{\alpha\mu\rho}) && (11.114)\\
&= J^{\mu\nu\rho} + \partial_\alpha (x^\rho B^{\alpha\mu\nu} - x^\nu B^{\alpha\mu\rho}) && (11.115)
\end{aligned}$$

Since $x^\rho B^{\alpha\mu\nu} - x^\nu B^{\alpha\mu\rho}$ is anti-symmetric in α and μ, $J^{\mu\nu\rho}$ and $J_B^{\mu\nu\rho}$ are equivalent.

11.3.5 Scale Invariance and It's Currents

The scale transformation is (T_λ, P_λ) that acts on x and ϕ as

$$\begin{aligned}
T_\lambda x &= \lambda x && (11.116)\\
P_\lambda \phi &= \lambda^{-\Delta} \phi && (11.117)
\end{aligned}$$

It's Lie algebra is $(1, -\Delta)$. And the conserved current is

$$j^\mu = T^\mu{}_\nu x^\nu + \Delta \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \phi_i \tag{11.118}$$

which we may also call D^μ.

Define the virial of ϕ as

$$V^\mu = \frac{\partial \mathcal{L}}{\partial(\partial^\rho \phi_i)} (\eta^{\mu\rho} + \mathcal{P}^{\mu\rho}) \phi_i \tag{11.119}$$

Assume that

$$V^\mu = \partial_\alpha \sigma^{\alpha\mu} \tag{11.120}$$

and also assume the dimension of the space-time is $d > 2$.

Define

$$\begin{aligned}
\sigma_+^{\mu\nu} &= \frac{1}{2}(\sigma^{\mu\nu} + \sigma^{\nu\mu}) && (11.121)\\
X^{\lambda\rho\mu\nu} &= \frac{2}{d-2}\{\eta^{\lambda\rho}\sigma_+^{\mu\nu} - \eta^{\lambda\mu}\sigma_+^{\rho\nu} - \eta^{\lambda\nu}\sigma_+^{\rho\mu} + \eta^{\mu\nu}\sigma_+^{\lambda\rho} + \frac{1}{d-1}(\eta^{\lambda\rho}\eta^{\mu\nu} - \eta^{\lambda\mu}\eta^{\rho\nu})\sigma_{+\alpha}^{\alpha}\} && (11.122)
\end{aligned}$$

We define

$$T_X^{\mu\nu} = T^{\mu\nu} + \partial_\rho B^{\rho\mu\nu} + \frac{1}{2}\partial_\lambda \partial_\rho X^{\lambda\rho\mu\nu} \tag{11.123}$$

It's easy to see that

$$\partial_\lambda \partial_\rho \partial_\mu X^{\lambda\rho\mu\nu} = 0 \tag{11.124}$$

So $T_X^{\mu\nu}$ is a conserved current. It's easy to see that $T_X^{\mu\nu}$ and $T^{\mu\nu}$ are equivalent and give the same δS_ϵ.

Define

$$J_X^{\mu\nu\rho} = T_X^{\mu\nu} x^\rho - T_X^{\mu\rho} x^\nu \tag{11.125}$$

It's easy to see that $J_X^{\mu\nu\rho}$ and $J^{\mu\nu\rho}$ are equivalent and give the same δS_ϵ.

Define

$$D_X^\mu = T^\mu_{X\nu} x^\nu \tag{11.126}$$

We have

$$\begin{aligned}
D_X^\mu &= T^\mu{}_\nu x^\nu + \partial_\rho B^{\rho\mu}{}_\nu x^\nu + \frac{1}{2}\partial_\lambda \partial_\rho X^{\lambda\rho\mu}{}_\nu x^\nu && (11.127)\\
&= T^\mu{}_\nu x^\nu + \partial_\rho (B^{\rho\mu}{}_\nu x^\nu) - B^{\rho\mu}{}_\rho + \frac{1}{2}\partial_\lambda \partial_\rho X^{\lambda\rho\mu}{}_\nu x^\nu && (11.128)\\
&= T^\mu{}_\nu x^\nu + \partial_\rho (B^{\rho\mu}{}_\nu x^\nu) - \frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi_i)} (\mathcal{P}^\mu{}_\rho \phi)_i + \frac{1}{2}\partial_\lambda \partial_\rho X^{\lambda\rho\mu}{}_\nu x^\nu && (11.129)\\
&= D^\mu - \Delta \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \phi_i + \partial_\rho (B^{\rho\mu}{}_\nu x^\nu) - \frac{\partial \mathcal{L}}{\partial(\partial_\rho \phi_i)} (\mathcal{P}^\mu{}_\rho \phi)_i + \frac{1}{2}\partial_\lambda \partial_\rho X^{\lambda\rho\mu}{}_\nu x^\nu && (11.130)\\
&= D^\mu - V^\mu + \partial_\rho (B^{\rho\mu}{}_\nu x^\nu) + \frac{1}{2}\partial_\lambda \partial_\rho X^{\lambda\rho\mu}{}_\nu x^\nu && (11.131)\\
&= D^\mu + \partial_\rho \left(-\sigma^{\rho\mu} + B^{\rho\mu}{}_\nu x^\nu + \frac{1}{2}\partial_\lambda X^{\lambda\rho\mu}{}_\nu x^\nu - \frac{1}{2} X^{\lambda\rho\mu}{}_\lambda\right) && (11.132)
\end{aligned}$$

So D_X^μ and D^μ are equivalent and give the same δS_ϵ.

It can also be checked that $T_X^{\mu\nu} = T_X^{\nu\mu}$ and $T^\mu_{X\mu} = 0$.

11.3.6 Symmetry in Hamiltonian Formalism

When we are dealing with on-shell field, we can interchange freely between Lagrangian formalism and Hamiltonian formalism. We can transform j^0 given in (11.66) into Hamiltonian formalism

$$\begin{aligned}
j^0 &= \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_i)} \partial_\nu \phi_i (\mathcal{T}x)^\nu - \mathcal{L}(\mathcal{T}x)^0 - \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi_i)} (\mathcal{P}\phi)_i && (11.133)\\
&= \pi_i \partial_\nu \phi_i (\mathcal{T}x)^\nu - \mathcal{L}(\mathcal{T}x)^0 - \pi_i (\mathcal{P}\phi)_i && (11.134)\\
&= \pi_i \partial_j \phi_i (\mathcal{T}x)^j + \mathcal{H}(\mathcal{T}x)^0 - \pi_i (\mathcal{P}\phi)_i && (11.135)
\end{aligned}$$

Actually this can be viewed as a definition of j^0 in Hamiltonian formalism, even for off-shell fields.

And the conserved charge in Hamiltonian formalism is

$$Q = \int d^3x [\pi_i \partial_j \phi_i (\mathcal{T}x)^j + \mathcal{H}(\mathcal{T}x)^0 - \pi_i (\mathcal{P}\phi)_i] \tag{11.136}$$

As deduced in Appendix B, we have

$$\{Q(\mathcal{L}_1), Q(\mathcal{L}_2)\} = -Q([\mathcal{L}_1, \mathcal{L}_2]) \tag{11.137}$$

That is, the Poisson bracket of two conserved charges is the (minus) conserved charge generated by $[\mathcal{L}_1, \mathcal{L}_2]$. If we view Q as a map from Lie algebra of the symmetry group to conserved charges, this also means that $-Q$ is a homomorphism between two Lie algebras.

In Hamiltonian formalism,

$$P_0 = H,\ P_i = \int d^3x \pi_i \partial_\mu \phi_i \tag{11.138}$$

It can be easily shown that

$$\begin{aligned} \frac{\delta P_\mu}{\delta \phi_i(x)} &= -\partial_\mu \pi_i(x) && (11.139) \\ \frac{\delta P_\mu}{\delta \pi_i(x)} &= \partial_\mu \phi_i(x) && (11.140) \end{aligned}$$

So for a quantity $Q(\phi, \pi, x^\mu)$ in Hamiltonian formalism, we have

$$\begin{aligned} \frac{dQ}{dx^\mu} &= \partial_\mu Q + \frac{\delta Q}{\delta \phi_i} \partial_\mu \phi_i + \frac{\delta Q}{\delta \pi_i} \partial_\mu \pi_i && (11.141) \\ &= \partial_\mu Q + \frac{\delta Q}{\delta \phi_i} \frac{\delta P_\mu}{\delta \pi_i} - \frac{\delta Q}{\delta \pi_i} \frac{\delta P_\mu}{\delta \phi_i} && (11.142) \\ &= \partial_\mu Q + \{Q, P_\mu\} && (11.143) \end{aligned}$$

From last section we have

$$\begin{aligned} \{M^{\mu\nu}, M^{\rho\sigma}\} &= \eta^{\nu\rho} M^{\mu\sigma} - \eta^{\mu\rho} M^{\nu\sigma} - \eta^{\sigma\mu} M^{\rho\nu} + \eta^{\sigma\nu} M^{\rho\mu} && (11.144) \\ \{P^\mu, M^{\rho\sigma}\} &= \eta^{\mu\rho} P^\sigma - \eta^{\mu\sigma} P^\rho && (11.145) \\ \{P^\mu, P^\nu\} &= 0 && (11.146) \end{aligned}$$

We can use (11.143) to testify (11.145) and (11.146).

11.4 Klein-Gordon Scalar Field

A real Klein-Gordon scalar field is a real field ϕ with n components, together with the Lagrangian density

$$\mathcal{L} = -\frac{1}{2} \partial^\mu \phi_i \partial_\mu \phi_i - \frac{1}{2} m^2 \phi_i^2 \tag{11.147}$$

The EOM is

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \phi_i} &= \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} && (11.148) \\ -m^2 \phi_i &= -\partial^\mu \partial_\mu \phi_i && (11.149) \\ (\partial^\mu \partial_\mu - m^2) \phi_i &= 0 && (11.150) \end{aligned}$$

(11.150) is called the Klein-Gordon equation. It has plane wave solution:

$$\phi_i(x) = \mathrm{Re}[a_i e^{ik \cdot x}] \tag{11.151}$$

where $k^2 = -m^2$. Here we can see that the momentum of plane wave solutions really lie on the mass shell. Actually we can have different k for each i, but such field can be expressed as combination of fields in the form of (11.151).

The canonical momentum for ϕ_i is

$$\pi_i = \frac{\partial \mathcal{L}}{\partial \phi_{i,t}} = \phi_{i,t} \tag{11.152}$$

The Hamiltonian density is

$$\mathcal{H} = \pi_i \phi_{i,t} - \mathcal{L} = \frac{1}{2}\pi_i^2 + \frac{1}{2}(\partial_j \phi_i)^2 + \frac{1}{2}m^2\phi_i^2 \tag{11.153}$$

$$\begin{aligned} H &= \int dx^3 \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 && (11.154)\\ &= \int \frac{dp^3}{(2\pi)^3}\frac{1}{2}\pi(p)\pi(-p) + \frac{1}{2}(p^2+m^2)\phi(p)\phi(-p) && (11.155) \end{aligned}$$

Let $a(p) = \sqrt{\frac{p^2+m^2}{2}}\phi(p) + i\sqrt{\frac{1}{2(p^2+m^2)}}\pi(p)$, then $a^\dagger(p) = \sqrt{\frac{p^2+m^2}{2}}\phi(-p) - i\sqrt{\frac{1}{2(p^2+m^2)}}\pi(-p)$

So

$$H = \int \frac{dp^3}{(2\pi)^3}\sqrt{p^2+m^2}(a^\dagger(p)a(p) + \frac{1}{2}[a(p), a^\dagger(p)]) \tag{11.156}$$

The infinitesimal symmetry operation of Poincaré symmetry is (ϵ^α, id) for infinitesimal translation and $(\Lambda_\epsilon^{\alpha\beta}, id)$ for infinitesimal rotation, with Lie algebra $(\delta^\alpha, 0)$ and $(\mathcal{M}^{\alpha\beta}, 0)$. The conserved currents and conserved charges for Poincaré symmetry are

$$\begin{aligned} T^{\mu\nu} &= -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\partial^\nu \phi_i + \mathcal{L}\eta^{\mu\nu} && (11.157)\\ &= \partial^\mu \phi_i \partial^\nu \phi_i + \mathcal{L}\eta^{\mu\nu} && (11.158)\\ P^0 &= \int d^3x T^{00} = \int d^3x((\partial_0\phi_i)^2 - \mathcal{L}) && (11.159)\\ P^i &= \int d^3x T^{0i} = -\int d^3x \partial_0\phi_j \partial_i \phi_j && (11.160)\\ M^{\mu\nu} &= \int d^3x (T^{0\mu}x^\nu - T^{0\nu}x^\mu) && (11.161)\\ &= \int d^3x(\partial^0\phi_i\partial^\mu\phi_i x^\nu - \partial^0\phi_i\partial^\nu\phi_i x^\mu + \mathcal{L}\eta^{0\mu}x^\nu - \mathcal{L}\eta^{0\nu}x^\mu) && (11.162)\\ &= \int d^3x[\partial^0\phi_i(x^\nu\partial^\mu - x^\mu\partial^\nu)\phi_i + \mathcal{L}(\eta^{0\mu}x^\nu - \eta^{0\nu}x^\mu)] && (11.163) \end{aligned}$$

For plane wave solution (11.151), we have

$$T^{\mu\nu} = \frac{1}{2}k_i^\mu k_i^\nu a_i^2 - (\frac{1}{2}k_i^\mu k_i^\nu + m^2\eta^{\mu\nu})a_i^2\cos(2k_i \cdot x) \tag{11.164}$$

$$P^\mu = \frac{1}{2}k_i^0 k_i^\mu a_i^2 \delta^{(3)}(0) \tag{11.165}$$
$$M^{\mu\nu} = 0 \tag{11.166}$$

Here we meet our first example of internal symmetry, a $O(n)$ symmetry. The symmetry operation is (id, O), where O is an orthogonal transformation in $\mathbb{R}^n$, which acts on ϕ as

$$(O\phi)_i = O_{ij}\phi_j \tag{11.167}$$

The Lie algebra is $(0, \mathcal{O}^{ij})$, where

$$(\mathcal{O}^{ij}\phi)_k = \delta_{ik}\phi_j - \delta_{jk}\phi_i \tag{11.168}$$

and i, j run over the label of components of ϕ, that is, $1 \sim n$.

The conserved currents and conserved charges are

$$j^{\mu,ij} = \partial^\mu\phi_k(\mathcal{O}^{ij}\phi)_k = \phi_j\partial^\mu\phi_i - \phi_i\partial^\mu\phi_j \tag{11.169}$$
$$O^{ij} = \int d^3x(\phi_j\partial^0\phi_i - \phi_i\partial^0\phi_j) = \int d^3x(\phi_i\partial_0\phi_j - \phi_j\partial_0\phi_i) \tag{11.170}$$

Similar discussion can be applied to the complex Klein-Gordon scalar field which is a complex field ϕ with n components, with the Lagrangian density

$$\mathcal{L} = -\partial^\mu\phi_i^*\partial_\mu\phi_i - m^2\phi_i^*\phi_i \tag{11.171}$$

11.5 Electromagnetic Field

The electromagnetic field is a real vector field A_μ, together with the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{11.172}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field-strength tensor, which is related to $\vec{E}$ and $\vec{B}$ by

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix} \tag{11.173}$$

that is, $F_{0i} = E_i$ and $F_{ij} = -\epsilon_{ijk}B_k$.

A_μ is also called the vector potential.

By definition, $F_{\mu\nu}$ is an anti-symmetric tensor, and we have the Bianchi identity

$$\partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} + \partial_\mu F_{\nu\rho} = 0 \tag{11.174}$$

The EOM is

$$\begin{aligned}
\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} &= \frac{\partial \mathcal{L}}{\partial A_\nu} = 0 && (11.175)\\
&= -\frac{1}{2}\partial_\mu\Big(F^{\alpha\beta}\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_\nu)}\Big) && (11.176)\\
&= -\frac{1}{2}\partial_\mu\Big(F^{\alpha\beta}\frac{\partial(\partial_\alpha A_\beta - \partial_\beta A_\alpha)}{\partial(\partial_\mu A_\nu)}\Big) && (11.177)\\
&= -\frac{1}{2}\partial_\mu(F^{\alpha\beta}(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha)) && (11.178)\\
&= -\partial_\mu F^{\mu\nu} && (11.179)
\end{aligned}$$

So that

$$\partial_\mu F^{\mu\nu} = 0 \qquad (11.180)$$

It's easy to show that (11.174) and (11.180) are equivalent to the Maxwell's equation in vacuum. The canonical momentum for A_μ is

$$\begin{aligned}
\pi^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} && (11.181)\\
&= -\frac{1}{2}F^{\alpha\beta}\frac{\partial F_{\alpha\beta}}{\partial(\partial_0 A_\mu)} && (11.182)\\
&= -\frac{1}{2}F^{\alpha\beta}\frac{\partial(\partial_\alpha A_\beta - \partial_\beta A_\alpha)}{\partial(\partial_0 A_\mu)} && (11.183)\\
&= -\frac{1}{2}F^{\alpha\beta}(\delta^0_\alpha\delta^\mu_\beta - \delta^0_\beta\delta^\mu_\alpha) && (11.184)\\
&= -F^{0\mu} && (11.185)
\end{aligned}$$

The Hamiltonian density is

$$\begin{aligned}
\mathcal{H} &= \pi^\mu\partial_0 A_\mu - \mathcal{L} && (11.186)\\
&= \frac{1}{2}\pi^\mu\pi_\mu + \pi^\mu\partial_\mu A_0 + \frac{1}{4}F^{ij}F_{ij} && (11.187)
\end{aligned}$$

Under the gauge transformation of the vector potential A_μ defined as

$$A_\mu \longrightarrow A'_\mu = A_\mu + \partial_\mu\epsilon \qquad (11.188)$$

the field strength $F_{\mu\nu}$ keeps invariant, since

$$F_{\mu\nu} \longrightarrow F'_{\mu\nu} = F_{\mu\nu} + \partial_\mu\partial_\nu\epsilon - \partial_\nu\partial_\mu\epsilon = F_{\mu\nu} \qquad (11.189)$$

Since A_μ may transforms differently at different x, this is a local transformation, in contract to the space-time transformation which is a global one. Obviously the Lagrangian keeps invariant after the transformation, as with the action. Thus this transformation is a symmetry operation of a local internal symmetry, called gauge symmetry. Unfortunately, this symmetry doesn't provide us with any useful information. It's easy to show that the conserved current is $F^{\mu\nu}\epsilon_\nu$, where ϵ_ν is an arbitrary vector field. The conservation of this current is equivalent to the EOM.

We usually consider A_μs that differ by a gauge transformation to be physically equivalent. We say they are the differ by a choice of gauge. Thus the gauge transformation defined above gives physically equivalent fields with different choices of gauge.

The degrees of freedom for different A_μs to be physically equivalent is called the gauge degrees of freedom, while the remaining degrees of freedom after identifying physically equivalent fields is called the physical degrees of freedom. If we reduce the gauge degrees of freedom by putting some constraint on A_μ (yet doesn't constrain the physical degrees of freedom), we are called to be fixing the gauge by the constraint, or to be choosing the gauge that reflects the constraint. These concepts are illustrated by Fig 11.4.

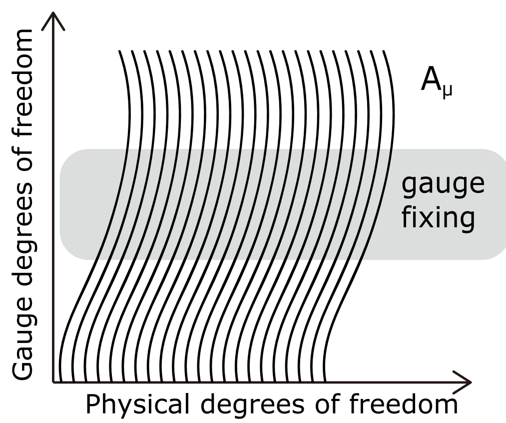


Figure 11.4: Concepts of gauge field. Every line means all physically equivalent A_μs.

Such field is called the gauge field. Potential field for electromagnetism is an example of U(1) gauge field, a term that we will explain later.

We are here to get the plane wave solutions for (11.180) with gauge fixed. Firstly we choose the Lorenz gauge which requires

$$\partial^\mu A_\mu = 0 \tag{11.190}$$

Thus (11.180) becomes

$$\begin{aligned}
\partial^\mu F_{\mu\nu} &= 0 &(11.191)\\
\partial^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) &= 0 &(11.192)\\
\partial^\mu\partial_\mu A_\nu - \partial_\nu\partial^\mu A_\mu &= 0 &(11.193)\\
\partial^\mu\partial_\mu A_\nu &= 0 &(11.194)
\end{aligned}$$

Equation (11.194) has plane wave solution

$$A_\mu = \mathrm{Re}[\alpha_\mu e^{ik\cdot x}] \tag{11.195}$$

with $k^2 = 0$ and the gauge condition is $k \cdot \alpha = 0$.

The gauge has not be totally fixed. We can impose a further gauge transformation. But we have to ensure we still have Lorenz gauge. That is, ϵ in (11.188) should satisfies

$$\partial^\mu \partial_\mu \epsilon = 0 \tag{11.196}$$

whose plane wave solution is

$$\epsilon = \mathrm{Re}[\beta e^{ik\cdot x}] \tag{11.197}$$

Thus the gauge transformation brings A_μ to

$$A'_\mu = A_\mu + \partial_\mu \epsilon = \mathrm{Re}[(\alpha_\mu + i\beta k_\mu)e^{ik\cdot x}] \tag{11.198}$$

We can fix such gauge freedom by requiring $A_0 = 0$. This is called the Coulomb gauge. The plane wave solution becomes

$$A_\mu = \mathrm{Re}[\alpha_\mu e^{ik\cdot x}] \tag{11.199}$$

with $k^2 = 0$, $\alpha_0 = 0$ and $\vec{\alpha}$ and $\hat{k}$ are orthogonal normalized vectors.

And at every k, the number of independent plane waves is 2, since there are only 2 directions perpendicular to $\vec{k}$. We choose two real normalized vectors perpendicular to $\vec{k}$ to be $\vec{\alpha_1}$ and $\vec{\alpha_2}$. Thus an arbitrary field with momentum k is

$$\begin{aligned} A_\mu &= \alpha_{\lambda\mu}\mathrm{Re}[a_\lambda e^{ik\cdot x}] \quad (\lambda = 1, 2) & (11.200)\\ &= \alpha_{\lambda\mu} A_\lambda \cos(k\cdot x + \phi_\lambda) \quad (\phi_\lambda \in [0, 2\pi)) & (11.201)\end{aligned}$$

where $a_\lambda = A_\lambda e^{i\phi_\lambda}$,$\alpha_{\lambda 0} = 0$ and $\vec{\alpha_\lambda}$ and $\hat{k}$ are real orthogonal normalized vectors.

At a point $\vec{x_0}$, A_μ evolves with time as

$$A_\mu(t) = \alpha_{\lambda\mu} A_\lambda \cos(-k_0 t + \theta + \phi_\lambda) \tag{11.202}$$

where $\theta = \vec{k}\cdot\vec{x_0}$. The trace of $A_\mu(t)$ determine the polarization of the light

If $|\phi_1 - \phi_2| = 0/\pi$, then the trace of $A_\mu(t)$ forms a line segment. In this case we call the electromagnetic wave to be linear polarized.

Otherwise, express the field as $A_\mu(t) = \alpha_{\lambda\mu} x_\lambda(t)$, the trace of $A_\mu(t)$ is described by the equation

$$\left(\frac{x_1}{A_1}\right)^2 + \left(\frac{x_2}{A_2}\right)^2 - 2\frac{x_1}{A_1}\frac{x_2}{A_2}\cos(\phi_1 - \phi_2) = \sin^2(\phi_1 - \phi_2) \tag{11.203}$$

This is a circle iff $|\phi_1 - \phi_2| = \frac{\pi}{2}/\frac{3\pi}{2}$ and $A_1 = A_2$. In this case we call the electromagnetic wave to be circular polarized.

Otherwise the trace is an oval. In this case we call the electromagnetic wave to be elliptical polarized.

The three types of polarization is illustrated in Fig 11.5.

The general field strength is

$$\begin{aligned} E_i &= \alpha_{\lambda i}\mathrm{Re}[ik_0 a_\lambda e^{ik\cdot x}] & (11.204)\\ B_i &= -\alpha_{\lambda k}\mathrm{Re}[i\epsilon_{ijk}k_j a_\lambda e^{ik\cdot x}] & (11.205)\end{aligned}$$

This shows that $\vec{E}$ and $\vec{B}$ are always transverse waves, since gauge fixing doesn't put any constraint on $F_{\mu\nu}$. Field strength of a linear polarized plane wave at a moment is illustrated in Fig 11.6.

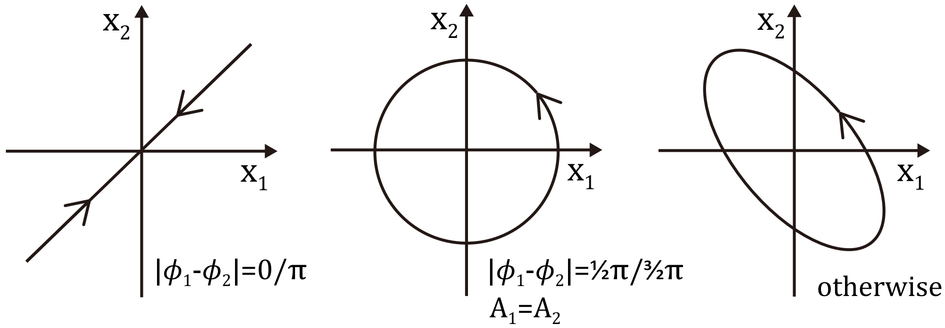


Figure 11.5: Traces of end point of A_μ for a linear, circular and elliptical polarized wave at some $\vec{x}$

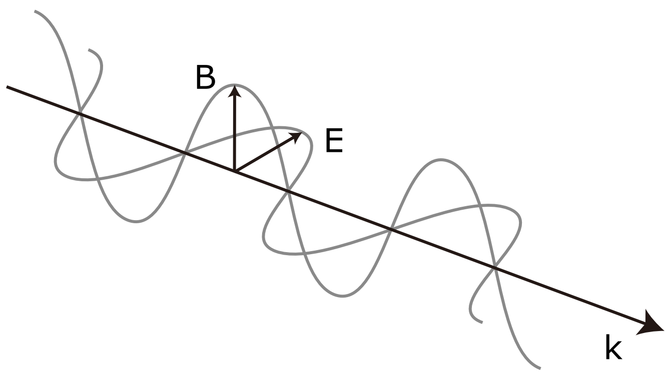


Figure 11.6: Field strength of a linear polarized plane wave at some t

Finally we work out the conserved currents and conserved charges for electromagnetic field.

The infinitesimal symmetry operation of Poincaré symmetry is (ϵ^α, id) for infinitesimal translation and $(\Lambda_\epsilon^{\alpha\beta}, P_\epsilon^{\alpha\beta})$ for infinitesimal rotation, with Lie algebra $(\delta^\alpha, 0)$ and $(\mathcal{M}^{\alpha\beta}, \mathcal{P}^{\alpha\beta})$, where

$$(\mathcal{P}^{\alpha\beta}A)^\mu = \eta^{\alpha\mu}A^\beta - \eta^{\beta\mu}A^\alpha \tag{11.206}$$

The energy-momentum tensor is

$$\begin{aligned} T^{\mu\nu} &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\rho)}\partial^\nu A_\rho - \mathcal{L}\eta^{\mu\nu} && (11.207)\\ &= -F^{\mu\rho}\partial^\nu A_\rho - \mathcal{L}\eta^{\mu\nu} && (11.208)\\ &= -F^{\mu\rho}F^\nu{}_\rho - \mathcal{L}\eta^{\mu\nu} - F^{\mu\rho}\partial_\rho A^\nu && (11.209)\end{aligned}$$

We have

$$\begin{aligned} B^{\mu\nu\rho} &= \frac{1}{2}\Big[\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\alpha)}(P^{\nu\rho}A)_\alpha + \frac{\partial\mathcal{L}}{\partial(\partial_\nu A_\alpha)}(P^{\rho\mu}A)_\alpha - \frac{\partial\mathcal{L}}{\partial(\partial_\rho A_\alpha)}(P^{\mu\nu}A)_\alpha\Big] && (11.210)\\ &= -\frac{1}{2}\Big[F^{\mu\alpha}(\delta^\nu_\alpha A^\rho - \delta^\rho_\alpha A^\nu) + F^{\nu\alpha}(\delta^\rho_\alpha A^\mu - \delta^\mu_\alpha A^\rho) - F^{\rho\alpha}(\delta^\mu_\alpha A^\nu - \delta^\nu_\alpha A^\mu)\Big] && (11.211)\\ &= -\frac{1}{2}\Big[F^{\mu\nu}A^\rho - F^{\mu\rho}A^\nu + F^{\nu\rho}A^\mu - F^{\nu\mu}A^\rho - F^{\rho\mu}A^\nu + F^{\rho\nu}A^\mu\Big] && (11.212)\end{aligned}$$

$$= -F^{\mu\nu}A^{\rho} \tag{11.213}$$

So

$$\begin{aligned} T_B^{\mu\nu} &= T^{\mu\nu} - \partial_\rho(F^{\rho\mu}A^{\nu}) & (11.214)\\ &= -F^{\mu\rho}F^{\nu}_{\ \ \rho} - \mathcal{L}\eta^{\mu\nu} - \partial_\rho F^{\rho\mu}A^{\nu} & (11.215)\end{aligned}$$

For on-shell field, we have

$$T_B^{\mu\nu} = -F^{\mu\rho}F^{\nu}_{\ \ \rho} - \mathcal{L}\eta^{\mu\nu} \tag{11.216}$$

And the 4-momentum is

$$\begin{aligned} P_0 &= \int d^3x T_{B\ 0}^{\ 0} = \int d^3x(F^{0\rho}F_{0\rho} - \mathcal{L}) = \frac{1}{2}\int d^3x(|E|^2 + |B|^2) & (11.217)\\ P_i &= \int d^3x T_{B\ i}^{\ 0} = -\int d^3x F^{0\rho}F^{i}_{\ \rho} = \int d^3x \epsilon_{ijk}E_jB_k = \int d^3x(E\times B)_i & (11.218)\end{aligned}$$

We have the conserved currents of Lorentz symmetry as

$$j^{\mu,\alpha\beta} = T_B^{\mu\alpha}x^{\beta} - T_B^{\mu\beta}x^{\alpha} \tag{11.219}$$

And the angular momentum tensors are

$$M^{\mu\nu} = \int d^3x j^{0,\mu\nu} = \int d^3x(T_B^{0\mu}x^{\nu} - T_B^{0\nu}x^{\mu}) \tag{11.220}$$

For on-shell field, we have

$$\begin{aligned} M^{0i} &= -M^{i0} & (11.221)\\ &= \int d^3x(T_B^{00}x^i - T_B^{0i}x^0) & (11.222)\\ &= \int d^3x\big[-\frac{1}{2}(|E|^2+|B|^2)x^i - \epsilon_{ijk}E_jB_kx^0\big] & (11.223)\\ M_{0i} &= -M_{i0} = \int d^3x\big[\frac{1}{2}(|E|^2+|B|^2)x^i + \epsilon_{ijk}E_jB_kx^0\big] & (11.224)\\ M_{ij} &= M^{ij} = \int d^3x(T'^{0i}x^j - T'^{0j}x^i) & (11.225)\\ &= \int d^3x(\epsilon_{ilk}x^j - \epsilon_{jlk}x^i)E_lB_k & (11.226)\end{aligned}$$

The Lagrangian of scalar QED Lagrangian density reads

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - D^{\mu}\phi_i^* D_\mu\phi_i - m^2\phi_i^*\phi_i \tag{11.227}$$

where ϕ is a one-component complex scalar field and $D_\mu = \partial_\mu + ieA_\mu$. With $\phi(x)\to e^{i\theta(x)}\phi(x)$, we still have the gauge symmetry.

11.6 Gauge Field

For a compact semi-simple Lie group G with Lie algebra $\mathfrak{g}$. A G-gauge field, also called a Yang-Mills field, is a value field $A_\mu(x)$ whose value lies in $\mathfrak{g}$ for each μ. We may also express $A_\mu(x)$ as

$$A_\mu(x) = A_{i\mu}(x) L_i \tag{11.228}$$

where L_is form a basis of $\mathfrak{g}$. Actually we regard $A_{i\mu}$s as the independent degrees of freedom.

Mostly, instead of working with the Lie algebra itself, it's more convenient for us to work with its linear representation. So we work with a linear representation T of $\mathfrak{g}$, and express the field as

$$A_\mu(x) = A_{i\mu}(x) T^i \in T(\mathfrak{g}) \tag{11.229}$$

where $T^i = T(L_i)$.

We require that killing form T_{ij} defined by

$$T_{ij} = \mathrm{Tr}[T^i T^j] \tag{11.230}$$

is non-degenerate. This requirement is always satisfied by some linear representation, for example, the adjoint representation.

The structure constant for $\mathfrak{g}$ is c_{ijk}, thus

$$[T^i, T^j] = c_{ijk} T^k \tag{11.231}$$

From the identity

$$\begin{aligned}
\mathrm{Tr}[[T^i, T^j]T^k] &= \mathrm{Tr}[T^i T^j T^k - T^j T^i T^k] && (11.232)\\
&= \mathrm{Tr}[T^j T^k T^i - T^k T^j T^i] && (11.233)\\
&= \mathrm{Tr}[[T^j, T^k]T^i] && (11.234)
\end{aligned}$$

we can get

$$c_{ijl} T_{lk} = c_{jkn} T_{ni} \tag{11.235}$$

We define field-strength tensor $F_{\mu\nu}$ as

$$\begin{aligned}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] && (11.236)\\
&= (\partial_\mu A_{i\nu} - \partial_\nu A_{i\mu}) T^i + c_{ijk} A_{i\mu} A_{j\nu} T^k && (11.237)\\
&= F_{i\mu\nu} T^i && (11.238)
\end{aligned}$$

where

$$F_{i\mu\nu} = \partial_\mu A_{i\nu} - \partial_\nu A_{i\mu} + c_{jki} A_{j\mu} A_{k\nu} \tag{11.239}$$

Obviously $F_{\mu\nu}$ is a Lie algebra valued tensor field and is anti-symmetric.

The free Lagrangian is defined as

$$\mathcal{L} = -\frac{1}{4} \mathrm{Tr}[F^{\mu\nu} F_{\mu\nu}] \tag{11.240}$$

The EOM is

$$
\begin{aligned}
\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_{i\nu})} &= \frac{\partial \mathcal{L}}{\partial A_{i\nu}} && (11.241)\\
-\frac{1}{2}\partial_\mu \mathrm{Tr}\Big[F^{\alpha\beta}\frac{\partial F_{\alpha\beta}}{\partial(\partial_\mu A_{i\nu})}\Big] &= -\frac{1}{2}\mathrm{Tr}\Big[F^{\alpha\beta}\frac{\partial F_{\alpha\beta}}{\partial A_{i\nu}}\Big] && (11.242)\\
-\frac{1}{2}\partial_\mu \mathrm{Tr}\Big[F^{\alpha\beta}T^j\frac{\partial(\partial_\alpha A_{j\beta}-\partial_\beta A_{j\alpha})}{\partial(\partial_\mu A_{i\nu})}\Big] &= -\frac{1}{2}\mathrm{Tr}\Big[F^{\alpha\beta}c_{ljk}T^k\frac{\partial A_{l\alpha}A_{j\beta}}{\partial A_{i\nu}}\Big] && (11.243)\\
-\partial_\mu \mathrm{Tr}[F^{\mu\nu}T^i] &= -\mathrm{Tr}[F^{\nu\alpha}T^k]c_{ijk}A_{j\alpha} && (11.244)\\
-\partial_\mu F_j^{\mu\nu}T_{ij} &= -F_l^{\nu\mu}c_{ijk}A_{j\mu}T_{kl} && (11.245)\\
\partial_\mu F_i^{\mu\nu}T_{ij} + F_l^{\mu\nu}c_{jlk}A_{j\mu}T_{ki} &= 0 && (11.246)\\
\partial_\mu F_i^{\mu\nu} + F_l^{\mu\nu}c_{jli}A_{j\mu} &= 0 && (11.247)\\
\partial_\mu F^{\mu\nu} + [A_\mu, F^{\mu\nu}] &= 0 && (11.248)\\
D_\mu F^{\mu\nu} &= 0 && (11.249)
\end{aligned}
$$

where $D_\mu = \partial_\mu + [A_\mu, \quad]$.

It's also easy to check the Bianchi identity

$$D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} + D_\rho F_{\mu\nu} = 0 \tag{11.250}$$

Suppose the Lie group G is parameterized by a_i and $U(a)$ is the representation of $g(a) \in G$ corresponding to that of $\mathfrak{g}$. We can construct a field of group representation $U(a(x))$ using vector fields $a_i(x)$. For a $U(a(x))$, we define the gauge transformation of A_μ relative to it as

$$A_\mu \longrightarrow A'_\mu = U^{-1}A_\mu U + U^{-1}\partial_\mu U \tag{11.251}$$

Meanwhile, the field-strength tensor transforms as

$$
\begin{aligned}
F_{\mu\nu} \longrightarrow F'_{\mu\nu} &= \partial_\mu A'_\nu - \partial_\nu A'_\mu + [A'_\mu, A'_\nu] && (11.252)\\
&= -U^{-1}\partial_\mu U U^{-1}A_\nu U + U^{-1}\partial_\mu A_\nu U + U^{-1}A_\nu\partial_\mu U - U^{-1}\partial_\mu U U^{-1}\partial_\nu U + \\
&\quad U^{-1}\partial_\mu\partial_\nu U + (U^{-1}A_\mu U + U^{-1}\partial_\mu U)(U^{-1}A_\nu U + U^{-1}\partial_\nu U) - (\mu \leftrightarrow \nu) && (11.253)\\
&= U^{-1}\partial_\mu A_\nu U + U^{-1}A_\nu\partial_\mu U + U^{-1}A_\mu A_\nu U + U^{-1}A_\mu\partial_\nu U + U^{-1}\partial_\mu\partial_\nu U \\
&\quad -(\mu \leftrightarrow \nu) && (11.254)\\
&= U^{-1}\partial_\mu A_\nu U + U^{-1}A_\mu A_\nu U - (\mu \leftrightarrow \nu) && (11.255)\\
&= U^{-1}F_{\mu\nu}U && (11.256)
\end{aligned}
$$

So the Lagrangian (11.240) keeps invariant. This is the symmetry operation for gauge symmetry.

Our electromagnetic field in last section is just a $U(1)$ gauge field, with one dimensional representation of $u(1)$ Lie algebra, which is $\mathbb{R}$.

11.6.1 Non-abelian instanton

In this section we assume that the metric is Euclindean.

A self-dual field is a field such that

$$F_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \tag{11.257}$$

and an anti-self-dual field is a field such that

$$F_{\mu\nu} = -\epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \tag{11.258}$$

A self-dual or anti-self-dual field is automatically a solution of the EOM, because, if we start from the Bianchi identity

$$\begin{aligned}
0 &= D_\mu F_{\nu\rho} + D_\nu F_{\rho\mu} + D_\rho F_{\mu\nu} && (11.259)\\
&= \pm\epsilon_{\nu\rho\sigma\lambda} D_\mu F^{\sigma\lambda} + \epsilon_{\rho\mu\sigma\lambda} D_\nu F^{\sigma\lambda} + \epsilon_{\mu\nu\sigma\lambda} D_\rho F^{\sigma\lambda} && (11.260)\\
&= \pm(\epsilon_{\nu\rho\sigma\lambda} D_\mu + \epsilon_{\rho\mu\sigma\lambda} D_\nu + \epsilon_{\mu\nu\sigma\lambda} D_\rho) F^{\sigma\lambda} && (11.261)\\
0 &= \epsilon^{\mu\nu\rho\lambda}(\epsilon_{\nu\rho\sigma\lambda} D_\mu + \epsilon_{\rho\mu\sigma\lambda} D_\nu + \epsilon_{\mu\nu\sigma\lambda} D_\rho) F^{\sigma\lambda} && (11.262)\\
&= -6(\delta^\mu_\sigma D_\mu - \delta^\nu_\sigma D_\nu + \delta^\rho_\sigma D_\rho) F^{\sigma\lambda} && (11.263)\\
&= -6 D_\sigma F^{\sigma\lambda} && (11.264)\\
D_\mu F^{\mu\nu} &= 0 && (11.265)
\end{aligned}$$

we will get the EOM.

This kind of solution of a gauge field is an example of instanton. It's topologically non-trivial, thus stable. It describes the tunneling between two topologically different vacuums. In quantum field theory, it appears in the path integral as the leading quantum corrections to the classical behavior of a system.

The action becomes

$$\begin{aligned}
S &= -\int d^4x \frac{1}{4} \mathrm{Tr}[F^{\mu\nu} F_{\mu\nu}] && (11.266)\\
&= -\int d^4x \frac{1}{4} \mathrm{Tr}[\pm\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}] && (11.267)\\
&= -\int d^4x \frac{1}{4} \mathrm{Tr}[\pm F \wedge F] && (11.268)\\
&= 4\pi^2 c && (11.269)
\end{aligned}$$

where $F = F_{\mu\nu} dx^\mu \wedge dx^\nu$.

For $SU(N)$ gauge field, c is always an integer, called Chern number. It's a topology invariant, and contains the topological information of the field.

Example For SU(2) gauge field with spin-1/2 representation, a BPST instanton [2] is an self-dual solution of the form

$$A_\mu = f(r) U(x)^{-1} \partial_\mu U(x) \tag{11.270}$$

where $r = \sqrt{(x^\mu)^2}$ and

$$\begin{aligned}
U(x) &= \frac{x^\mu}{r} \sigma_\mu \in SU(2) && (11.271)\\
\sigma_\mu &= (1, i\sigma_1, i\sigma_2, i\sigma_3) && (11.272)
\end{aligned}$$

We define

$$\begin{aligned}
\overline{\sigma_\mu} &= (1, -i\sigma_1, -i\sigma_2, -i\sigma_3) && (11.273)\\
\sigma_{\mu\nu} &= \overline{\sigma_\mu}\sigma_\nu - \overline{\sigma_\nu}\sigma_\mu && (11.274)
\end{aligned}$$

It's easy to check

$$\sigma_\mu\overline{\sigma_\nu} + \sigma_\nu\overline{\sigma_\mu} = \overline{\sigma_\mu}\sigma_\nu + \overline{\sigma_\nu}\sigma_\mu = 2\delta_{\mu\nu} \qquad (11.275)$$

We define the boundary condition: $f(r) \to 1$ as $r \to \infty$.
We have

$$\begin{aligned}
U^{-1} &= \frac{x^\mu}{r}\overline{\sigma_\mu} && (11.276)\\
U^{-1}\partial_\mu U &= -\frac{x^\mu}{r^2} + U^{-1}\frac{\sigma_\mu}{r} && (11.277)\\
\sigma_\mu U^{-1}\sigma_\nu - \sigma_\nu U^{-1}\sigma_\mu &= \frac{x^\rho}{r}(\sigma_\mu\overline{\sigma_\rho}\sigma_\nu - \sigma_\nu\overline{\sigma_\rho}\sigma_\mu) && (11.278)\\
&= \frac{x^\rho}{r}(2\delta_{\mu\rho}\sigma_\nu - \sigma_\rho\overline{\sigma_\mu}\sigma_\nu - 2\delta_{\nu\rho}\sigma_\mu + \sigma_\rho\overline{\sigma_\nu}\sigma_\mu) && (11.279)\\
&= 2\frac{x^\mu\sigma_\nu - x^\nu\sigma_\mu}{r} - U\sigma_{\mu\nu} && (11.280)\\
\partial_\mu(U^{-1}\partial_\nu U) - \partial_\nu(U^{-1}\partial_\mu U) &= -[U^{-1}\partial_\mu U, U^{-1}\partial_\nu U] && (11.281)
\end{aligned}$$

Then the field strength tensor is

$$\begin{aligned}
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] && (11.282)\\
&= \partial_\mu f U^{-1}\partial_\nu U - \partial_\nu f U^{-1}\partial_\mu U + (f^2 - f)[U^{-1}\partial_\mu U, U^{-1}\partial_\nu U] && (11.283)\\
&= \partial_\mu f\Big(-\frac{x^\nu}{r^2} + U^{-1}\frac{\sigma_\nu}{r}\Big) - \partial_\nu f\Big(-\frac{x^\mu}{r^2} + U^{-1}\frac{\sigma_\mu}{r}\Big) + (f^2 - f)\Big(U^{-1}\frac{\sigma_\mu}{r}U^{-1}\frac{\sigma_\nu}{r}\\
&\quad -U^{-1}\frac{\sigma_\nu}{r}U^{-1}\frac{\sigma_\mu}{r}\Big) && (11.284)\\
&= \Big(\partial_\nu f\frac{x^\mu}{r^2} - \partial_\mu f\frac{x^\nu}{r^2}\Big) + U^{-1}\Big(\partial_\mu f\frac{\sigma_\nu}{r} - \partial_\nu f\frac{\sigma_\mu}{r}\Big) + U^{-1}(f^2 - f)\Big(\frac{\sigma_\mu}{r}U^{-1}\frac{\sigma_\nu}{r}\\
&\quad -\frac{\sigma_\nu}{r}U^{-1}\frac{\sigma_\mu}{r}\Big) && (11.285)\\
&= \frac{1}{r^3}[U^{-1}(x^\mu\sigma_\nu - x^\nu\sigma_\mu)(r\partial_r f + 2(f^2 - f)) - (f^2 - f)r\sigma_{\mu\nu}] && (11.286)
\end{aligned}$$

It's not hard to show that $\sigma_{\mu\nu} = \epsilon_{\mu\nu\rho\lambda}\sigma^{\rho\lambda}$. Thus $F_{\mu\nu}$ is self-dual if

$$r\partial_r f + 2(f^2 - f) = 0 \qquad (11.287)$$

A solution that satisfy the boundary condition is

$$f = \frac{r^2}{r^2 + \lambda^2} \qquad (11.288)$$

where λ is a paramter that specifies the size of the instanton.

The field strength is

$$F_{\mu\nu} = \frac{\lambda^2}{(r^2+\lambda^2)^2}\sigma_{\mu\nu} \tag{11.289}$$

It's not hard to get

$$\sigma^{\mu\nu}\sigma_{\mu\nu} = -48\mathbb{I} \tag{11.290}$$

Thus it's Chern number is

$$\begin{aligned} -\frac{1}{16\pi^2}\int d^4x \mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}] &= \frac{6}{\pi^2}\int d^4x \frac{\lambda^4}{(r^2+\lambda^2)^4} && (11.291)\\ &= 6\int dx \frac{2x^3\lambda^4}{(r^2+\lambda^2)^4} && (11.292)\\ &= 1 && (11.293)\end{aligned}$$

11.7 Spinor

Reference of this section is [3].

11.7.1 Connection between $SL(2,\mathbb{C})$ and $L_+^\uparrow$

We define σ_μ and $\bar{\sigma}_\mu$ as

$$\sigma_0 = \bar{\sigma}_0 = I,\ \bar{\sigma}_i = -\sigma_i \tag{11.294}$$

and σ_is as Pauli matrices.

It's easy to check that

$$Tr(\sigma_\mu\bar{\sigma}_\nu) = -2\eta_{\mu\nu} \tag{11.295}$$

Let x^μ be a 4-vector, it's easy to check that

$$x^2 = x^\mu x_\mu = -\det(x^\mu\sigma_\mu) \tag{11.296}$$

We define the homomorphism $\Lambda : SL(2,\mathbb{C}) \to L_+^\uparrow$ by

$$M\sigma_\mu M^\dagger = \Lambda(M)^\nu{}_\mu\sigma_\nu \tag{11.297}$$

Use (11.295), we have

$$\Lambda(M)_{\nu\mu} = -\frac{1}{2}Tr[M\sigma_\mu M^\dagger\bar{\sigma}_\nu] \tag{11.298}$$

Clearly $L(M)_{\nu\mu} \in \mathbb{R}$.

Then

$$\begin{aligned} M\sigma_\mu M^\dagger x^\mu &= \Lambda(M)^\nu{}_\mu\sigma_\nu x^\mu && (11.299)\\ \det(M\sigma_\mu M^\dagger x^\mu) &= \det(\Lambda(M)^\nu{}_\mu\sigma_\nu x^\mu) && (11.300)\\ &= (\Lambda(M)^\nu{}_\mu x^\mu)^2 && (11.301)\\ &= \det(M)\det(\sigma_\mu x^\mu)\det(M^\dagger) && (11.302)\end{aligned}$$

$$= \quad x^2 \tag{11.303}$$

So $\Lambda(M)^\nu{}_\mu$ is an isometry map of $\eta_{\mu\nu}$, that is, $\Lambda(M)^\nu{}_\mu \in L$. Since $SL(2,\mathbb{C})$ is connected, $\Lambda(M)^\nu{}_\mu \in L_+$. Since $\Lambda(M)^0{}_0 = -\eta^{00}\frac{1}{2}Tr[M\sigma_0 M^\dagger\bar{\sigma}_0] = \frac{1}{2}Tr[MM^\dagger] = \frac{1}{2}|Tr[M]|^2 > 0$, $\Lambda(M)^\nu{}_\mu \in L_+^\uparrow$. It can be proved that Λ is epic, since $L_+^\uparrow$ can be generated by finite rotation and finite boost, which is mapped from $e^{a\sigma_i}$ and $e^{ia\sigma_i}$.

We may give an explicit expression of Λ^{-1}. That is, for $\Lambda^\nu{}_\mu$ we find M such that

$$M\sigma_\mu M^\dagger = \Lambda^\nu{}_\mu\sigma_\nu \tag{11.304}$$

First observe that

$$\sigma_\mu M\bar{\sigma}^\mu = -2Tr[M] \tag{11.305}$$

Thus (11.304) leads to

$$-2MTr[M^\dagger] = \Lambda^\nu{}_\mu\sigma_\nu\bar{\sigma}^\mu \tag{11.306}$$

Taking determinate at both sides, we have

$$(2Tr[M^\dagger])^2 \quad = \quad \det(\Lambda^\nu{}_\mu\sigma_\nu\bar{\sigma}^\mu) \tag{11.307}$$

$$2Tr[M^\dagger] \quad = \quad \pm[\det(\Lambda^\nu{}_\mu\sigma_\nu\bar{\sigma}^\mu)]^{\frac{1}{2}} \tag{11.308}$$

So

$$M = \pm\frac{\Lambda^\nu{}_\mu\sigma_\nu\bar{\sigma}^\mu}{[\det(\Lambda^\nu{}_\mu\sigma_\nu\bar{\sigma}^\mu)]^{\frac{1}{2}}} \tag{11.309}$$

From this we see $\ker\Lambda = \pm I$. So $SL(2,\mathbb{C})/\mathbb{Z}_2 = L_+^\uparrow$.

11.7.2 Weyl Spinor

$SL(2,\mathbb{C})$ has two inequivalent representations: the self-representation

$$T_l(M) = M \tag{11.310}$$

and the complex conjugate self-representation

$$T_r(M) = \bar{M} \tag{11.311}$$

where bar means complex conjugation.

We call the linear space that self-representation acts on the left-handed Weyl spinor space, and the linear space that complex conjugate self-representation acts on the right-handed Weyl spinor space.

In the left-handed Weyl spinor space, a left-handed Weyl spinor has lower index as ϕ_α. Self-representation has indices as $M_\alpha{}^\beta$. So the action of M on left-handed Weyl spinor is

$$\phi'_\alpha = M_\alpha{}^\beta\phi_\beta \tag{11.312}$$

In the right-handed Weyl spinor space, a right-handed Weyl spinor has lower index as $\bar{\phi}_{\dot{\alpha}}$. Note that the vector is barred and the index is dotted. Complex conjugate self-representation has indices as $\bar{M}_{\dot{\alpha}}{}^{\dot{\beta}}$. So the action of M on right-handed Weyl spinor is

$$\bar{\phi}'_{\dot{\alpha}} = \bar{M}_{\dot{\alpha}}{}^{\dot{\beta}}\bar{\phi}_{\dot{\beta}} \tag{11.313}$$

As already indicated by the notation, we can also treated the bar as the complex conjugation map from the left-handed Weyl spinor space to the right-handed Weyl spinor space. Generally, we define the bar map from left-handed Weyl spinor tensor to the right-handed Weyl spinor tensor as

$$\bar{A}_{\dot{\mu}\dot{\nu}\dots}{}^{\dot{\alpha}\dot{\beta}\dots} = (A_{\mu\nu\dots}{}^{\alpha\beta\dots})^* \tag{11.314}$$

Clearly

$$\overline{M_\alpha{}^\beta \phi_\beta} = \bar{M}_{\dot{\alpha}}{}^{\dot{\beta}} \bar{\phi}_{\dot{\beta}} \tag{11.315}$$

We define left-handed spinor tensor ϵ as

$$\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{\alpha\beta}, \ \epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{\alpha\beta} \tag{11.316}$$

Then it's easy to check

$$M_\mu{}^\alpha M_\nu{}^\beta \epsilon_{\alpha\beta} = \epsilon_{\mu\nu}, \ \epsilon^{\mu\nu}\epsilon_{\nu\rho} = \delta^\mu_\rho \tag{11.317}$$

So ϵ is $SL(2,\mathbb{C})$ invariant.

We can use $\epsilon^{\mu\nu}$ and $\epsilon_{\mu\nu}$ to raise and lower indices as

$$\phi^\alpha = \epsilon^{\alpha\beta}\phi_\beta, \ \phi_\alpha = \epsilon_{\alpha\beta}\phi^\beta \tag{11.318}$$

To be consistent, the transformation of left-handed Weyl spinor with upper index should satisfy

$$\phi'^\alpha = M^\alpha_{\ \beta}\phi^\beta \tag{11.319}$$

Simarly, $\bar{\epsilon}$ (bar map of ϵ) is an $SL(2,\mathbb{C})$ invariant right-handed Weyl spinor tensor.

We use $\epsilon^{\dot{\mu}\dot{\nu}}$ and $\epsilon_{\dot{\mu}\dot{\nu}}$ to raise and lower indices as

$$\bar{\phi}^{\dot{\alpha}} = \bar{\epsilon}^{\dot{\alpha}\dot{\beta}}\phi_{\dot{\beta}}, \ \bar{\phi}_{\dot{\alpha}} = \bar{\epsilon}_{\dot{\alpha}\dot{\beta}}\bar{\phi}^{\dot{\beta}} \tag{11.320}$$

And right-handed Weyl spinor transforms under $SL(2,\mathbb{C})$ transformation as

$$\bar{\phi}'^{\dot{\alpha}} = \bar{M}^{\dot{\alpha}}_{\ \dot{\beta}}\bar{\phi}^{\dot{\beta}} \tag{11.321}$$

We define the spinor index structure of σ^μ and $\bar{\sigma}^\mu$ as $\sigma^\mu_{\ \alpha\dot{\beta}}$ and $\bar{\sigma}^{\mu\dot{\alpha}\beta}$. It's easy to see from the connection between $SL(2,\mathbb{C})$ and $L^\uparrow_+$ that both of these are $SL(2,\mathbb{C})$ invariant. We have

$$\sigma^{\mu\alpha\dot{\beta}} = \bar{\sigma}^{\mu\dot{\beta}\alpha}, \ \sigma^\mu_{\ \alpha\dot{\beta}} = \bar{\sigma}^\mu_{\ \dot{\beta}\alpha} \tag{11.322}$$

As the classical counterpart of fermion operator, Weyl spinors form a Grassman algebra; that is, they anti-commute. We may write $\phi^\alpha\theta_\alpha$ as $\phi\theta$ and $\bar{\phi}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}$ as $\bar{\phi}\bar{\theta}$.

We have

$$\phi\theta = \phi^\alpha\theta_\alpha = \epsilon^{\alpha\beta}\epsilon_{\alpha\gamma}\phi_\beta\theta^\gamma = -\epsilon^{\beta\alpha}\epsilon_{\alpha\gamma}\phi_\beta\theta^\gamma = -\delta^\beta_\gamma\phi_\beta\theta^\gamma = -\phi_\alpha\theta^\alpha = \theta^\alpha\phi_\alpha = \theta\phi \tag{11.323}$$

$$\bar{\phi}\bar{\theta} = \bar{\phi}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} = \bar{\epsilon}_{\dot{\alpha}\dot{\beta}}\bar{\epsilon}^{\dot{\alpha}\dot{\gamma}}\bar{\phi}^{\dot{\beta}}\bar{\theta}_{\dot{\gamma}} = -\bar{\epsilon}_{\dot{\beta}\dot{\alpha}}\bar{\epsilon}^{\dot{\alpha}\dot{\gamma}}\bar{\phi}^{\dot{\beta}}\bar{\theta}_{\dot{\gamma}} = -\delta^{\dot{\gamma}}_{\dot{\beta}}\bar{\phi}^{\dot{\beta}}\bar{\theta}_{\dot{\gamma}} = -\bar{\phi}^{\dot{\alpha}}\bar{\theta}_{\dot{\alpha}} = \bar{\theta}_{\dot{\alpha}}\bar{\phi}^{\dot{\alpha}} = \bar{\theta}\bar{\phi} \tag{11.324}$$

$$\theta\theta = \theta^\alpha\theta_\alpha = \epsilon^{\alpha\beta}\theta_\beta\theta^\alpha = \theta_2\theta_1 - \theta_1\theta_2 = -2\theta_1\theta_2 \tag{11.325}$$
$$\bar\theta\bar\theta = \bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha} = \bar\epsilon_{\dot\alpha\dot\beta}\bar\theta^{\dot\beta}\bar\theta^{\dot\alpha} = -\theta^{\dot 2}\theta^{\dot 1} + \theta^{\dot 1}\theta^{\dot 2} = 2\theta^{\dot 1}\theta^{\dot 2} \tag{11.326}$$
$$\theta^\alpha\theta_\beta = \epsilon^{\alpha\gamma}\theta_\gamma\theta_\beta = \begin{cases} \theta_2\theta_1 & \alpha=\beta=1 \\ -\theta_1\theta_2 & \alpha=\beta=2 \\ 0 & \alpha\neq\beta \end{cases} = \frac{1}{2}(\theta\theta)\delta^\alpha_\beta \tag{11.327}$$
$$\bar\theta_{\dot\alpha}\bar\theta^{\dot\beta} = \bar\epsilon_{\dot\alpha\dot\gamma}\bar\theta^{\dot\gamma}\bar\theta^{\dot\beta} = \begin{cases} -\theta^{\dot 2}\theta^{\dot 1} & \dot\alpha=\dot\beta=1 \\ \theta^{\dot 1}\theta^{\dot 2} & \dot\alpha=\dot\beta=2 \\ 0 & \dot\alpha\neq\dot\beta \end{cases} = \frac{1}{2}(\bar\theta\bar\theta)\delta^{\dot\beta}_{\dot\alpha} \tag{11.328}$$
$$\tag{11.329}$$

We write $\theta^\alpha\sigma^\mu_{\alpha\dot\beta}\bar\phi^{\dot\beta}$ as $\theta\sigma^\mu\bar\phi$ and $\bar\theta_{\dot\alpha}\bar\sigma^{\mu\dot\alpha\beta}\phi_\beta$ as $\bar\theta\bar\sigma^\mu\phi$. We have

$$\theta\sigma^\mu\bar\phi = \theta^\alpha\sigma^\mu_{\alpha\dot\beta}\bar\phi^{\dot\beta} = \theta_\alpha\sigma^{\mu\alpha\dot\beta}\bar\phi_{\dot\beta} = \theta_\alpha\bar\sigma^{\mu\dot\beta\alpha}\bar\phi_{\dot\beta} = -\bar\phi_{\dot\beta}\bar\sigma^{\mu\dot\beta\alpha}\theta_\alpha = -\bar\phi\bar\sigma^\mu\theta \tag{11.330}$$

11.7.3 Dirac Spinor and Majorana Spinor

We define the Dirac spinor as a combination of left hand and right Weyl spinor as

$$\Psi_i = \begin{pmatrix} \phi_\alpha \\ \bar\lambda^{\dot\beta} \end{pmatrix}_i \tag{11.331}$$

That is, $\Psi \in \mathbb{C}^4$ and

$$\Psi_1 = \phi_1,\ \Psi_2 = \phi_2,\ \Psi_3 = \lambda^{\dot 1},\ \Psi_4 = \lambda^{\dot 2} \tag{11.332}$$

Under a $SL(2,\mathbb{C})$ transformation M, the transformation of Dirac spinor inherits from the transformation of Weyl spinor:

$$\Psi_i \to \Psi'_i = S_{ij}\Psi_j \tag{11.333}$$

where

$$S_{ij} = \begin{pmatrix} M_{\alpha'}{}^{\alpha} & 0 \\ 0 & \bar M^{\dot\beta}{}_{\dot\beta'} \end{pmatrix}_{ij} \tag{11.334}$$

We define the bar map of Dirac spinor as

$$\bar\Psi = \mathrm{bar}\begin{pmatrix} \phi_\alpha \\ \bar\lambda^{\dot\beta} \end{pmatrix} = \begin{pmatrix} \lambda^\beta \\ \bar\phi_{\dot\alpha} \end{pmatrix} \tag{11.335}$$

Similarly, the transformation of barred Dirac spinor is

$$\bar\Psi_i \to \bar\Psi'_i = S'_{ij}\bar\Psi_j \tag{11.336}$$

where

$$S'_{ij} = \begin{pmatrix} M^{\beta'}{}_{\beta} & 0 \\ 0 & \bar M_{\dot\alpha'}{}^{\dot\alpha} \end{pmatrix}_{ij} \tag{11.337}$$

It's easy to see that

$$S'_{ij}S_{ik} = \delta_{jk} \tag{11.338}$$

So $\bar{\Psi}\Psi = \bar{\Psi}_i\Psi_i$ is a scalar.

We define the gamma matrices as

$$\gamma^\mu_{ij} = i\begin{pmatrix} 0 & \sigma^\mu_{\alpha\dot{\beta}} \\ \bar{\sigma}^{\mu\dot{\alpha}\beta} & 0 \end{pmatrix}_{ij} \tag{11.339}$$

and

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \tag{11.340}$$

It's easy to see that

$$S'_{ij}\gamma^\mu_{ik}S_{kl} = \Lambda(M)^\nu{}_\mu\gamma^\nu_{jl} \tag{11.341}$$

So $\bar{\Psi}\gamma^\mu\Psi = \bar{\Psi}_i\gamma^\mu_{ij}\Psi_j$ is a Lorentz vector.

We define the charge conjugation map $\Psi \to \Psi^c$ as

$$\Psi^c = \begin{pmatrix} \phi_\alpha \\ \bar{\lambda}^{\dot{\beta}} \end{pmatrix}^c = \begin{pmatrix} \lambda_\beta \\ \bar{\phi}^{\dot{\alpha}} \end{pmatrix} = C\bar{\Psi} \tag{11.342}$$

where

$$C = \begin{pmatrix} \epsilon_{\beta'\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}'\dot{\alpha}} \end{pmatrix} \tag{11.343}$$

It's easy to see

$$C^2 = I,\ C\gamma^\mu C = -(\gamma^\mu)^T \tag{11.344}$$

A Dirac spinor Ψ is called Majorana spinor if $\Psi^c = \Psi$. A Majorana spinor can be express as

$$\Psi = \begin{pmatrix} \phi_\alpha \\ \bar{\phi}^{\dot{\beta}} \end{pmatrix} \tag{11.345}$$

11.8 Spinor from Clifford Algebra*

11.8.1 Clifford Algebra

Let T be the tensor algebra of a vector space V over a field $K(char \neq 2)$ with a symmetric bilinear form B. Here we require B to be non-degenerate. We define I_B to be the ideal generated by

$$\{vv - B(v,v)|v \in V\} \tag{11.346}$$

We define the Clifford algebra $Cl(V,B)$ to be the quotient algebra T/I_B.

We have

$$[(u+v)(u+v)] \;=\; [B(u+v,u+v)] \tag{11.347}$$

$$[uu]+[uv]+[vu]+[vv] = [B(u,u)]+[B(u,v)]+[B(v,u)]+[B(v,v)] \quad (11.348)$$
$$[uv]+[vu] = 2[B(u,v)] \quad (11.349)$$

For clarity we omit the bracket and write the equation above as

$$uv+vu=2B(u,v) \quad (11.350)$$

Clifford algebra $Cl(V,B)$ has the following universal property: If $f:V\to A$ is a linear map from V to an algebra A such that $f(x)f(x)=B(x,x)$, then there exist a unique algebra homomorphism g from $Cl(V,B)$ to A such that the diagram

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[d, "f"] & Cl(V,B) \arrow[dl, "g"] \\ A & \end{tikzcd} \quad (11.351)$$

commutes.

If V is an n dimensional vector space with basis $\{e_i\}$. We can prove that a basis of the Clifford algebra $Cl(V,B)$ is

$$1,\ e_i,\ e_ie_j(i<j),\ e_ie_je_k(i<j<k),\ \ldots,\ e_1e_2\cdots e_n \quad (11.352)$$

Since they are complete, we only need to prove

Lemma 11.8.1. *The Clifford algebra $Cl(V,B)$ is of 2^n dimensional.*

Proof. We can find a basis $\{e_i\}$ with respect to which B is diagonal, and $B(e_i,e_j)\in K^\times/(K^\times)^2$. We only need to prove that under this basis. □

Thus if V is a real linear space, we can find a basis $\{e_i\}$ such that

$$B(e_i,e_j)=\eta_{ij}=diag(\underbrace{-1,\ldots,-1}_{p},\underbrace{1,\ldots,1}_{q}) \quad (11.353)$$

We call this Clifford algebra $Cl_{p,q}(\mathbb{R})$. If V is a complex linear space of dim n, we can find a basis $\{e_i\}$ such that $B(e_i,e_j)=\delta_{ij}$. We call this Clifford algebra $Cl_n(\mathbb{C})$.

We define a map $\alpha:V\to Cl(V,B)$ as

$$\alpha(x)=-x \quad (11.354)$$

From the universal property, α extends to an automorphism of $Cl(V,B)$. It can be shown that $\alpha^2=1$. Thus we can define the projectors $P_\pm=(1\pm\alpha)/2$. Then

$$Cl(V,B)=P_+Cl(V,B)\oplus P_-Cl(V,B)=Cl^0(V,B)\oplus Cl^1(V,B) \quad (11.355)$$

where $Cl^0(V,B)=P_+Cl(V,B)$ and $Cl^1(V,B)=P_-Cl(V,B)$. Clearly $Cl^0(V,B)$ is a subalgebra of $Cl(V,B)$.

It can be easily shown that $Cl^0(V,B)$ has basis

$$1,\ e_ie_j(i<j),\ e_ie_je_ke_l(i<j<k<l),\ \ldots \quad (11.356)$$

and $Cl^1(V,B)$ has basis

$$e_i,\ e_ie_je_k(i<j<k),\ \dots \tag{11.357}$$

Clearly we have

$$Cl^{[i]}(V,B)Cl^{[j]}(V,B)=Cl^{[i+j]}(V,B) \tag{11.358}$$

where the bracket reads mod 2. Thus $Cl(V,B)$ is a $\mathbb{Z}_2$ graded algebra.

Let

$$\epsilon=e_1e_2\cdots e_n \tag{11.359}$$

where $n=\dim V$

Let $x\in Cl(V,B)$. By considering $[x,e_i]$, it can be proved that the center of $Cl(V,B)$ is K when $\dim V$ is even and is generated by $\{1,\epsilon\}$ when $\dim V$ is odd. Similarly, it can be proved that the center of $Cl^0(V,B)$ is K when $\dim V$ is odd and is generated by $\{1,\epsilon\}$ when $\dim V$ is even.

Define the transpose map on tensor algebra by

$$(e_1\otimes e_2\otimes\cdots\otimes e_m)^t=e_m\otimes\cdots e_2\otimes e_1 \tag{11.360}$$

Transpose map is an algebra anti-automorphism. Obviously $I_B^t=I_B$. So transpose map induce an algebra anti-automorphism on $Cl(V,B)$ defined by

$$(e_1e_2\cdots e_m)^t=e_m\cdots e_2e_1 \tag{11.361}$$

We can define the Clifford norm $\bar{B}$ by

$$\bar{B}(x,x)=x^tx \tag{11.362}$$

where $x\in Cl(V,B)$.

It can be shown that

$$\bar{B}|_V=B \tag{11.363}$$

11.8.2 Classification of Clifford Algebra

First we calculate the structure of real and complex Clifford Algebra of small dimension.

$Cl_{0,0}(\mathbb{R})\simeq\mathbb{R}$.

$Cl_{1,0}(\mathbb{R})$ is generated by $\{1,e_1\}$ where

$$1\cdot 1=1,e_1\cdot e_1=-1,1\cdot e_1=e_1\cdot 1=e_1 \tag{11.364}$$

Clearly $Cl_{1,0}(\mathbb{R})\simeq\mathbb{C}$.

$Cl_{0,1}(\mathbb{R})$ is generated by $\{1,e_1\}$ where

$$1\cdot 1=1,e_1\cdot e_1=1,1\cdot e_1=e_1\cdot 1=e_1 \tag{11.365}$$

Chose basis $\{i,j\}$ where $i=\frac{1+e_1}{2}$ and $j=\frac{1-e_1}{2}$, we have

$$i\cdot i=i,j\cdot j=j,i\cdot j=j\cdot i=0 \tag{11.366}$$

So

$$(ai + bj) \cdot (ci + dj) = abi + cdj \tag{11.367}$$

So $Cl_{0,1}(\mathbb{R}) \simeq \mathbb{R} \oplus \mathbb{R}$.

From the map

$$T(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, T(e_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, T(e_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{11.368}$$

we have $Cl_{1,1}(\mathbb{R}) \simeq M_2(\mathbb{R})$.

From the map

$$T(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, T(e_1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, T(e_2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{11.369}$$

we have $Cl_{0,2}(\mathbb{R}) \simeq M_2(\mathbb{R})$.

$Cl_{2,0}(\mathbb{R})$ is generated by $\{1, e_1, e_2, e_1e_2\}$. If we define

$$i = e_1, j = e_2, k = e_1e_2 \tag{11.370}$$

, we have

$$1 \cdot 1 = 1, i \cdot i = -1, j \cdot j = -1, k \cdot k = -1, 1 \cdot i = i \cdot 1 = i, 1 \cdot j = j \cdot 1 = j, 1 \cdot k = k \cdot 1 = k \tag{11.371}$$

$$i \cdot j = -j \cdot i = k, j \cdot k = -k \cdot j = i, k \cdot i = -i \cdot k = j \tag{11.372}$$

Clearly $Cl_{2,0}(\mathbb{R}) \simeq \mathbb{H}$.

It's easy to see $C_n(\mathbb{C})$ is the complexification of $C_{p,q}(\mathbb{R})$ where $p + q = n$. Thus

$$C_n(\mathbb{C}) \simeq C_{0,n}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \simeq C_{1,n-1}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \simeq \cdots \simeq C_{n,0}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \tag{11.373}$$

We have some elemental formula

$$M_m(K) \otimes_K M_n(K) \simeq M_{mn}(K) \tag{11.374}$$
$$M_n(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \simeq M_n(\mathbb{C}) \tag{11.375}$$
$$M_n(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{H} \simeq M_n(\mathbb{H}) \tag{11.376}$$
$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C} \oplus \mathbb{C} \tag{11.377}$$
$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \simeq M_2(\mathbb{C}) \tag{11.378}$$
$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} \simeq M_4(\mathbb{R}) \tag{11.379}$$

So

$$C_1(\mathbb{C}) \simeq C_{1,0}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C} \oplus \mathbb{C} \tag{11.380}$$
$$C_2(\mathbb{C}) \simeq C_{0,2}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \simeq M_2(\mathbb{C}) \tag{11.381}$$

Next we prove that if $V = W \oplus W^{\perp}$ and if $\dim W$ is even, then

$$Cl(V, B) \simeq Cl(W, B_W) \otimes Cl(W^{\perp}, \epsilon_W^2 B_{W^{\perp}}) \tag{11.382}$$

where ϵ_W is defined on an orthogonal basis of W, so ϵ_W^2 is treated as a number.

Proof. We define the map $f : V \to Cl(W, B_W) \otimes Cl(W^\perp, \epsilon_W^2 B_{W^\perp})$ by

$$\begin{aligned} f(w) &= w \otimes 1, w \in W && (11.383)\\ f(w^\perp) &= \epsilon_W \otimes w^\perp, w^\perp \in W^\perp && (11.384)\end{aligned}$$

This induce a surjective map $f : Cl(V, B) \to Cl(W, B_W) \otimes Cl(W^\perp, \epsilon_W^2 B_{W^\perp})$. Compare the dimension of each side, we see that this map is an isomorphism. □

So we have

$$\begin{aligned} Cl_{q+2,p}(\mathbb{R}) &\simeq Cl_{2,0}(\mathbb{R}) \otimes_{\mathbb{R}} Cl_{p,q}(\mathbb{R}) && (11.385)\\ Cl_{q,p+2}(\mathbb{R}) &\simeq Cl_{0,2(\mathbb{R})} \otimes_{\mathbb{R}} Cl_{p,q}(\mathbb{R}) && (11.386)\\ Cl_{p+1,q+1}(\mathbb{R}) &\simeq Cl_{1,1}(\mathbb{R}) \otimes_{\mathbb{R}} Cl_{p,q}(\mathbb{R}) && (11.387)\end{aligned}$$

Then

$$\begin{aligned} Cl_{p+4,q}(\mathbb{R}) &\simeq Cl_{2,0}(\mathbb{R}) \otimes_{\mathbb{R}} Cl_{q,p+2}(\mathbb{R}) && (11.388)\\ &\simeq Cl_{2,0}(\mathbb{R}) \otimes_{\mathbb{R}} Cl_{0,2}(\mathbb{R}) \otimes_{\mathbb{R}} Cl_{p,q}(\mathbb{R}) && (11.389)\\ &\simeq M_2(\mathbb{H}) \otimes_{\mathbb{R}} Cl_{p,q}(\mathbb{R}) && (11.390)\\ Cl_{p,q+4}(\mathbb{R}) &\simeq Cl_{0,2}(\mathbb{R}) \otimes_{\mathbb{R}} Cl_{q+2,p}(\mathbb{R}) && (11.391)\\ &\simeq Cl_{0,2}(\mathbb{R}) \otimes_{\mathbb{R}} Cl_{2,0}(\mathbb{R}) \otimes_{\mathbb{R}} Cl_{p,q}(\mathbb{R}) && (11.392)\\ &\simeq M_2(\mathbb{H}) \otimes_{\mathbb{R}} Cl_{p,q}(\mathbb{R}) && (11.393)\\ Cl_{p+8,q}(\mathbb{R}) &\simeq M_2(\mathbb{H}) \otimes_{\mathbb{R}} M_2(\mathbb{H}) \otimes_{\mathbb{R}} Cl_{p,q}(\mathbb{R}) && (11.394)\\ &\simeq M_{16}(\mathbb{R}) \otimes_{\mathbb{R}} Cl_{p,q}(\mathbb{R}) && (11.395)\\ Cl_{p,q+8}(\mathbb{R}) &\simeq M_2(\mathbb{H}) \otimes_{\mathbb{R}} M_2(\mathbb{H}) \otimes_{\mathbb{R}} Cl_{p,q}(\mathbb{R}) && (11.396)\\ &\simeq M_{16}(\mathbb{R}) \otimes_{\mathbb{R}} Cl_{p,q}(\mathbb{R}) && (11.397)\end{aligned}$$

We also have in complex case

$$\begin{aligned} Cl_{n+2}(\mathbb{C}) &\simeq C_{1,n+1}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} && (11.398)\\ &\simeq Cl_{1,1}(\mathbb{R}) \otimes_{\mathbb{R}} (Cl_{0,n}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}) && (11.399)\\ &\simeq Cl_{1,1}(\mathbb{R}) \otimes_{\mathbb{R}} Cl_n(\mathbb{C}) && (11.400)\\ &\simeq (Cl_{1,1}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} Cl_n(\mathbb{C}) && (11.401)\\ &\simeq M_2(\mathbb{C}) \otimes_{\mathbb{C}} Cl_n(\mathbb{C}) && (11.402)\end{aligned}$$

where we have used $A \otimes_{\mathbb{R}} B = A \otimes_{\mathbb{C}} (\mathbb{C} \otimes_{\mathbb{R}} B)$.

So we have

$$Cl_{2n}(\mathbb{C}) = M_{2^n}(\mathbb{C}), Cl_{2n+1}(\mathbb{C}) = M_{2^n}(\mathbb{C}) \oplus M_{2^n}(\mathbb{C}) \quad (11.403)$$

and the structure of $Cl_{p,q}(\mathbb{R})$ is listed in the Tab. 11.1.

$p-q \mod 8$	$Cl_{p,q}(\mathbb{R})$	N
0,6	$M_N(\mathbb{R})$	$2^{d/2}$
1,5	$M_N(\mathbb{C})$	$2^{(d-1)/2}$
2,4	$M_N(\mathbb{H})$	$2^{(d-2)/2}$
3	$M_N(\mathbb{H}) \oplus M_N(\mathbb{H})$	$2^{(d-3)/2}$
7	$M_N(\mathbb{R}) \oplus M_N(\mathbb{R})$	$2^{(d-1)/2}$

Table 11.1: Structure of $Cl_{p,q}(\mathbb{R})$, where $d = p + q$.

11.8.3 Pin Group and Spin Group

We define the Clifford group $\Gamma(V, B)$ of $Cl(V, B)$ as the invertible element of $x \in Cl(V, B)$ such that

$$\forall v \in V : \alpha(x)vx^{-1} \in V \tag{11.404}$$

This induce a map for each $x \in \Gamma(V, B)$

$$H_x(v) = \alpha(x)vx^{-1} \tag{11.405}$$

It's easy to see that H_x is a linear map.

It's easy to check that

$$\alpha(x)^T = \alpha(x^T), \; x^{-1T} = x^{T-1} \tag{11.406}$$

So

$$\begin{aligned}
B(H_x(v), H_x(v)) &= \bar{B}(H_x(v), H_x(v)) && (11.407)\\
&= H_x(v)^T H_x(v) && (11.408)\\
&= x^{-1T}v^T\alpha(x)^T\alpha(x)vx^{-1} && (11.409)\\
&= x^{-1T}v^T\alpha(x^Tx)vx^{-1} && (11.410)\\
&= x^Txv^Tvx^{-1T}x^{-1} && (11.411)\\
&= x^Txv^Tv(x^Tx)^{-1} && (11.412)\\
&= v^Tv && (11.413)\\
&= B(v, v) && (11.414)
\end{aligned}$$

So H_x is an isometry.

If $B(v, v) > 0$, we have

$$v^{-1} = \frac{v}{B(v, v)} \tag{11.415}$$

So

$$\begin{aligned}
H_v(v') &= -vv'v^{-1} && (11.416)\\
&= -\frac{vv'v}{B(v, v)} && (11.417)
\end{aligned}$$

$$= -\frac{B(v,v')v - v'vv}{B(v,v)} \tag{11.418}$$
$$= v' - \frac{B(v,v')}{B(v,v)}v \tag{11.419}$$

That is, H_v is the reflection in the v direction. From Cartan-Doieudonne theorem, $x \to H_x$ is an epimorphism from the Clifford group to the orthogonal group $O(V,B)$ (the group of all isometry transformations).

Next we study the kernel of H map

$$x \in \ker H \iff \forall v \in V : H_x(v) = v \tag{11.420}$$
$$\iff \forall v \in V : \alpha(x)vx^{-1} = v \tag{11.421}$$
$$\iff \forall v \in V : \alpha(x)v = vx \tag{11.422}$$

Split x into odd even part $x = P_+x + P_-x = x_0 + x_1$, we have

$$x \in \ker H \iff \forall v \in V : x_0v - x_1v = vx_0 + vx_1 \tag{11.423}$$

Split the RHS into odd even part, we have

$$x \in \ker H \iff \forall v \in V : x_0v = vx_0 \wedge -x_1v = vx_1 \tag{11.424}$$

It's easy to see that there's no odd element to anti-commute with all $v \in V$, so $x_1 = 0$. It's easy to see that $\forall v \in V : x_0v = vx_0 \iff \forall y \in Cl(V,B) : x_0y = yx_0$, so $x = x_0$ is in the center of $Cl(V,B)$. So $\ker H = \Gamma(V,B) \cap Z(Cl(V,B)) \cap Cl^0(V,B) = K^*$.

Thus, if $H_x = H_{v_1 \cdots v_n}$, we have $x^{-1}v_1 \cdots v_n \in K$. So any $x \in \Gamma(V,B)$ can be expressed $x = v_1 \cdots v_n$.

So we have the short exact sequence

$$1 \to K^* \to \Gamma(V,B) \to O(V,B) \to 1 \tag{11.425}$$

It can be easily proved that $\det(H_v) = -1$ if $v \in V$. So if we define the $\Gamma^0(V,B) = \Gamma(V,B) \cap Cl^0(V,B)$, we have the short exact sequence

$$1 \to K^* \to \Gamma^0(V,B) \to SO(V,B) \to 1 \tag{11.426}$$

If we define the pin group $Pin(V,B)$ as

$$Pin(V,B) = \{x \in \Gamma(V,B) | \bar{B}(x,x) = \pm 1\} \tag{11.427}$$

, we have the exact sequence

$$1 \to \mu \to Pin(V,B) \to O(V,B) \to K^\times/(K^\times)^2 \tag{11.428}$$

where $\mu = \{x \in K | x^2 = \pm 1\}$

Similarly, if we define the spin group $Spin(V,B) = Pin(V,B) \cap Cl^0(V,B)$, we have the exact sequence

$$1 \to \mu \to Spin(V,B) \to SO(V,B) \to K^\times/(K^\times)^2 \tag{11.429}$$

Mostly we care about the field such that $K^\times = (K^\times)^2$. In this case, we have short exact sequences

$$1 \to \mu \to Pin(V,B) \to O(V,B) \to 1 \tag{11.430}$$
$$1 \to \mu \to Spin(V,B) \to SO(V,B) \to 1 \tag{11.431}$$

Then we define $Spin_+(V,B) = \{x \in \Gamma(V,B) | \bar{B}(x,x) = 1\}$ and $SO_+(V,B)$ as the image of H map, so we have short exact sequences

$$1 \to \mu' \to Spin_+(V,B) \to SO_+(V,B) \to 1 \tag{11.432}$$

where $\mu' = \{x \in K | x^2 = 1\}$.

When $K = \mathbb{R}$, we denote the spin group of $Cl_{p,q}(\mathbb{R})$ as $Spin_{p,q}$ and abbreviate $Spin_{0,n}$ as $Spin_n$. We have

$$Spin_{n+} = Spin_n \tag{11.433}$$
$$Spin_{p,q+} = Spin_{q,p+} \tag{11.434}$$

There are some accidental isomorphisms when p and q are small

$$Spin_1 = O(1) = \pm 1 \tag{11.435}$$
$$Spin_2 = U(1) = SO(2) \tag{11.436}$$
$$Spin_3 = Sp(1) = SU(2) \tag{11.437}$$
$$Spin_4 = SU(2) \times SU(2) \tag{11.438}$$
$$Spin_5 = Sp(2) \tag{11.439}$$
$$Spin_6 = SU(4) \tag{11.440}$$
$$Spin_{1,1+} = GL(1,\mathbb{R}) \tag{11.441}$$
$$Spin_{2,1+} = SL(2,\mathbb{R}) \tag{11.442}$$
$$Spin_{3,1+} = SL(2,\mathbb{C}) \tag{11.443}$$
$$Spin_{4,1+} = Sp(1,1) \tag{11.444}$$
$$Spin_{5,1+} = SL(2,\mathbb{H}) \tag{11.445}$$
$$Spin_{2,2+} = SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \tag{11.446}$$
$$Spin_{3,2+} = Sp(4,\mathbb{R}) \tag{11.447}$$
$$Spin_{4,2+} = SU(2,2) \tag{11.448}$$
$$Spin_{3,3+} = SL(4,\mathbb{R}) \tag{11.449}$$

11.8.4 Gamma Matrices and Spinor

In this section we only consider Clifford algebra over a linear space of even dimension.

The map $T : Cl_{2n}(\mathbb{C}) \to M_{2^n}(\mathbb{C})$ can be realized by

$$T(1) = I = \underbrace{\sigma_0 \otimes \cdots \otimes \sigma_0}_{n} \tag{11.450}$$

$$\begin{aligned}
T(e_0) &= \Gamma^0 = \sigma_1 \otimes \underbrace{\sigma_0 \otimes \cdots \otimes \sigma_0}_{n-1} && (11.451)\\
T(e_1) &= \Gamma^1 = \sigma_2 \otimes \underbrace{\sigma_0 \otimes \cdots \otimes \sigma_0}_{n-1} && (11.452)\\
T(e_2) &= \Gamma^2 = \sigma_3 \otimes \sigma_1 \otimes \underbrace{\sigma_0 \otimes \cdots \otimes \sigma_0}_{n-2} && (11.453)\\
T(e_3) &= \Gamma^3 = \sigma_3 \otimes \sigma_2 \otimes \underbrace{\sigma_0 \otimes \cdots \otimes \sigma_0}_{n-2} && (11.454)\\
\cdots &= \cdots && (11.455)\\
T(e_{2l}) &= \Gamma^{2l} = \underbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}_{l} \otimes \sigma_1 \otimes \underbrace{\sigma_0 \otimes \cdots \otimes \sigma_0}_{n-l-1} && (11.456)\\
T(e_{2l+1}) &= \Gamma^{2l+1} = \underbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}_{l} \otimes \sigma_2 \otimes \underbrace{\sigma_0 \otimes \cdots \otimes \sigma_0}_{n-l-1} && (11.457)\\
\cdots &= \cdots && (11.458)\\
T(e_{2n-2}) &= \Gamma^{2n-2} = \underbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}_{n-1} \otimes \sigma_1 && (11.459)\\
T(e_{2n-1}) &= \Gamma^{2n-1} = \underbrace{\sigma_3 \otimes \cdots \otimes \sigma_3}_{n-1} \otimes \sigma_2 && (11.460)\\
& && (11.461)
\end{aligned}$$

It can be easily proved that T is surjective. Compare the dimension of the two spaces we see that T is an isomorphism.

Then we have complex a loyal representation of $Cl_{p,q}(\mathbb{C})(p+q=2n)$ defined by

$$\begin{aligned}
T(1) &= I && (11.462)\\
T(e_i) &= \gamma^i = \begin{cases} i\Gamma^i & i = 1, \ldots, p \\ \Gamma^i & i = p+1, \ldots, 2n \end{cases} && (11.463)
\end{aligned}$$

These are gamma matrices in Dirac basis.

We call vectors Ψ_D in the representation space V_D Dirac spinors.

Considering $Cl^0_{p,q}(\mathbb{C}) \in Cl_{p,q}(\mathbb{C})$ we have a projector

$$S_\pm = \frac{1 \pm i^{\frac{p-q}{2}}\epsilon}{2} \tag{11.464}$$

and each element in $Cl^0_{p,q}(\mathbb{C})$ commutes with $S_\pm$.

So using $S_\pm$, we can reduce T into $T_+ \oplus T_-$ over $Cl^0_{p,q}(\mathbb{C})$ into $V_D = V_+ \oplus V_-$. V_+ is a $\frac{n}{2}$-D space called left-handed Weyl spinor space, and $S_- V_D$ is a $\frac{n}{2}$-D space called right-handed Weyl spinor space.

We define the real structure of a complex linear space V an anti-linear map α in V such that $\alpha^2 = 1$. Then for each linear transformation H of V, we can define its complex conjugation $\bar{H} = \alpha H \alpha$. Especially, with a given representation T of $Cl_{2n}(\mathbb{C}) \simeq Cl^{\mathbb{C}}_{p,q}(\mathbb{R})$, we can construct its complex conjugation representation $\bar{T}$. Since $Cl_{2n}(\mathbb{C}) \simeq Cl^{\mathbb{C}}_{p,q}(\mathbb{R}) \simeq M_{2^n}(\mathbb{C})$ is simple, T and $\bar{T}$ is similar. That is, there is some C such that

$$\bar{T}(x) = CT(x)C^{-1} \tag{11.465}$$

We can require that $|\det C| = 1$
We have

$$\begin{aligned} \bar{T}(x)C &= CT(x) && (11.466)\\ T(x)\bar{C} &= \bar{C}\bar{T}(x) && (11.467) \end{aligned}$$

So

$$T(x)\bar{C}C = \bar{C}\bar{T}(x)C = \bar{C}CT(x) \tag{11.468}$$

So $\bar{C}C = aI$ where $|a| = 1$
So

$$\begin{aligned} \bar{C} &= aC^{-1} && (11.469)\\ C\bar{C} &= aI && (11.470)\\ \bar{C}C &= a^*I && (11.471) \end{aligned}$$

So $a = a^*$, so $a = \pm 1$.
Actually

$$a = \begin{cases} 1 & p+q \equiv 2,4 \mod 8 \\ -1 & p+q \equiv 0,6 \mod 8 \end{cases} \tag{11.472}$$

We define the charge conjugation as

$$\Psi^c_D = C^{-1}\bar{\Psi}_D \tag{11.473}$$

Majorana spinor is the Dirac spinor that satisfies

$$\Psi_M = \Psi^c_M \tag{11.474}$$

We have for Majorana spinors

$$a\Psi_M = a\Psi^c_M = aC^{-1}\bar{\Psi}_M = \bar{C}\bar{\Psi}_M = \overline{C\Psi_M} = \bar{\bar{\Psi}}_M = \Psi_M \tag{11.475}$$

Thus to have Majorana spinor, a needs to be 1, and $\dim V$ needs to be 2 or 4 mod 8.
When $(p,q) = (1,3)$,let

$$\begin{aligned} \Gamma^0 &= \sigma_1 \otimes \sigma_0 && (11.476)\\ \Gamma^1 &= \sigma_2 \otimes \sigma_1 && (11.477)\\ \Gamma^2 &= \sigma_2 \otimes \sigma_2 && (11.478)\\ \Gamma^3 &= \sigma_2 \otimes \sigma_3 && (11.479) \end{aligned}$$

and

$$\gamma^0 = i\Gamma^0,\ \gamma^1 = \Gamma^1,\ \gamma^2 = \Gamma^2,\ \gamma^3 = \Gamma^3 \tag{11.480}$$

We define the real structure to be

$$\alpha(\Psi) = \Gamma^0 \Psi^* \tag{11.481}$$

So

$$\bar{\gamma}^\mu = C\gamma^\mu C^{-1} = \alpha\gamma^\mu\alpha = \Gamma^0(\gamma^\mu)^*\Gamma^0 = -(\gamma^\mu)^T \tag{11.482}$$

A choice of C is

$$C = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{11.483}$$

11.9 Spinor Field

Spinor field in 3+1D space-time is a spinor bundle in E^4. Naively, this means that we have a map $\phi(x)$ from space-time to spinor space. We can construct a Lagrangian from Weyl spinor field ϕ as

$$L = a\bar{\phi}\sigma^\mu\partial_\mu\phi + m\phi\phi + m^*\bar{\phi}\bar{\phi} \tag{11.484}$$

where $m \in \mathbb{C}$.

We may redefine ϕ by absorbing a phase factor to make the mass real

$$L = a\bar{\phi}\sigma^\mu\partial_\mu\phi + m\phi\phi + m\bar{\phi}\bar{\phi} \tag{11.485}$$

where $m \in \mathbb{R}$.

We require the Lagrangian to be real, in the sense that

$$\begin{aligned} L &= L^* \qquad (11.486) \\ &= \overline{a\bar{\phi}\sigma^\mu\partial_\mu\phi} + m\phi\phi + m\bar{\phi}\bar{\phi} \qquad (11.487) \end{aligned}$$

11.10 Dirac Field

11.10.1 Gamma Matrices

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{11.488}$$

$$\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger \tag{11.489}$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{11.490}$$

$$(\gamma^5)^\dagger = (\gamma^5)^T = \gamma^5, (\gamma^5)^2 = 1, \{\gamma^5, \gamma^\mu\} = 0 \tag{11.491}$$

$$C^\dagger = C^{-1} = C^T = -C \tag{11.492}$$

$$C(\gamma^\mu)^T C^{-1} = -\gamma^\mu \tag{11.493}$$

11.10.2 Dirac Equation

$$\mathcal{L} = \bar{\Psi}(i\not{\partial} - m)\Psi \tag{11.494}$$

E-L equation leads to

$$(i\not{\partial} - m)\Psi = 0 \tag{11.495}$$

We can transform it into a form similar to the Schrödinger's equation

$$\begin{aligned} i\partial_t\Psi &= (-i\gamma^0\gamma^i\partial_i + \gamma^0 m)\Psi & (11.496)\\ &= (-i\alpha^i\partial_i + \beta^0 m)\Psi & (11.497)\\ &= (\alpha^i p^i + \beta^0 m)\Psi & (11.498) \end{aligned}$$

where $\alpha^i = \gamma^i$ and $\beta = \gamma^0$

positive energy solution

$$\psi = \int \frac{dp^4}{(2\pi)^4} u(p) e^{-ipx} \tag{11.499}$$

$$(\not{p} - m)u(p) = 0 \tag{11.500}$$

negative energy solution

$$\psi = \int \frac{dp^4}{(2\pi)^4} v(p) e^{ipx} \tag{11.501}$$

$$(\not{p} + m)v(p) = 0 \tag{11.502}$$

11.10.3 Solution to Dirac equation

In Weyl basis:

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \gamma^\mu = \begin{pmatrix} 0 & \sigma \\ \bar{\sigma} & 0 \end{pmatrix} \tag{11.503}$$

$$\begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\bar{\sigma} & -m \end{pmatrix} u(p) = 0 \tag{11.504}$$

$$\begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar{\sigma} & m \end{pmatrix} v(p) = 0 \tag{11.505}$$

use $(p\cdot\sigma)(p\cdot\bar{\sigma}) = m^2$, we have

$$u(p) = \begin{pmatrix} p\cdot\sigma\xi(p) \\ m\xi(p) \end{pmatrix} = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi'(p) \\ \sqrt{p\cdot\bar{\sigma}}\xi'(p) \end{pmatrix} \tag{11.506}$$

$$v(p) = \begin{pmatrix} p \cdot \sigma \xi(p) \\ -m\xi(p) \end{pmatrix} = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi'(p) \\ -\sqrt{p \cdot \bar{\sigma}} \xi'(p) \end{pmatrix} \tag{11.507}$$

In Dirac basis:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{11.508}$$

$$\begin{pmatrix} E - m & -\boldsymbol{p} \cdot \sigma \\ \boldsymbol{p} \cdot \sigma & -E - m \end{pmatrix} u(p) = 0 \tag{11.509}$$

$$\begin{pmatrix} E + m & -\boldsymbol{p} \cdot \sigma \\ \boldsymbol{p} \cdot \sigma & -E + m \end{pmatrix} v(p) = 0 \tag{11.510}$$

$$u(p) = \sqrt{E+m} \begin{pmatrix} \xi(p) \\ \frac{\boldsymbol{p} \cdot \sigma}{m+E} \xi(p) \end{pmatrix} \tag{11.511}$$

$$v(p) = u^c(p) = C\bar{u}^T(p) \tag{11.512}$$

$$\Psi(x) = \int \frac{dp^3}{(2\pi)^3} \frac{1}{\sqrt{2E}} a_p^s u^s(p) e^{-ipx} + b_p^{s\dagger} v^s(p) e^{ipx} \tag{11.513}$$

11.10.4 Orthonormal relation and Spin sums

$$u^{s\dagger}(p) u^r(p) = 2E\delta_{sr} \tag{11.514}$$

$$\bar{u}^s(p) u^r(p) = 2m\delta_{sr} \tag{11.515}$$

$$\sum_s u^s(p) \bar{u}^s(p) = \gamma \cdot p + m \tag{11.516}$$

$$v^{s\dagger}(p) v^r(p) = 2E\delta_{sr} \tag{11.517}$$

$$\bar{v}^s(p) v^r(p) = -2m\delta_{sr} \tag{11.518}$$

$$\sum_s v^s(p) \bar{v}^s(p) = \gamma \cdot p - m \tag{11.519}$$

11.10.5 Lorentz Symmetry

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \tag{11.520}$$

With a Lorentz space-time transformation

$$\Lambda^\mu{}_\nu = (1 - \frac{i}{2}\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta})^\mu{}_\nu \tag{11.521}$$

, we can define a transformation on Dirac spinor

$$\Lambda_{\frac{1}{2}} = (1 - \frac{i}{2}\omega_{\alpha\beta}S^{\alpha\beta}) \tag{11.522}$$

$$\Lambda^{-1}_{\frac{1}{2}}\gamma^\mu\Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu\gamma^\nu \tag{11.523}$$

When $\Psi \to \Lambda_{\frac{1}{2}}\Psi$, $\bar{\Psi} \to \bar{\Psi}\Lambda^{-1}_{\frac{1}{2}}$. Finally $L \to L$.

11.10.6 CPT

$$\hat{C}\phi(p)\hat{C}^{-1} = \eta_C C\bar{\phi}(p)^T \tag{11.524}$$

$$\hat{P}\phi(p)\hat{P}^{-1} = \eta_P\gamma_0\phi(\tilde{p}) \tag{11.525}$$

Chapter 12

Second Quantization aka. Canonical Quantization

Chapter 13

Path Integral Approach

Part III

Conformal Field Theory

Chapter 14

Conformal Field theory

In this chapter, tensor indices are raised and lowered by $\eta_{\mu\nu}$ rather than $g_{\mu\nu}$.

14.1 Conformal Transformation

Let $g_{\mu\nu}$ be the metric tensor on a d-dimensional manifold. A conformal transformation is a diffeomorphism $x \to x' = x'(x)$ such that $g'_{\mu\nu}(x') = f(x)g_{\mu\nu}(x)$. Clearly, conformal transformation is a generalization of the Poincaré transformation.

Clearly, when $d = 1$, any non-singular function $h(x)$ leads to a conformal transformation $x \to x' = h(x)$. We have $g'_{00}(x') = \frac{1}{h'(x)}g_{00}(x)$. For simplicity, we only study the conformal transformation of flat spacetime.

14.1.1 Conformal Transformation when $d \geq 3$

We start from the frame conformal to the inertial coordinates, that is, $g_{\mu\nu} = h(x)\eta_{\mu\nu}$. Under infinitesimal conformal transformation $x^\mu \to x'^\mu = x^\mu + \epsilon^\mu$, the metric tensor becomes $g'_{\mu\nu} = h'(x)\eta_{\mu\nu}$. To the 1st order of ϵ, $g'_{\mu\nu}$ can be expressed by

$$g'_{\mu\nu}(x') = g_{\mu\nu}(x) - g_{\nu\alpha}\partial_\mu\epsilon^\alpha(x) - g_{\mu\alpha}\partial_\nu\epsilon^\alpha(x) = h(\eta_{\mu\nu} - \partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu) \tag{14.1}$$

To have $g'_{\mu\nu} = h'\eta_{\mu\nu}$, we must require

$$\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu = f\eta_{\mu\nu} \tag{14.2}$$

for some f.

By taking an extra derivative ∂_ρ, permuting the indices and taking a linear combination, we have

$$2\partial_\rho\partial_\mu\epsilon_\nu = (\partial_\rho f)\eta_{\mu\nu} + (\partial_\mu f)\eta_{\nu\rho} - (\partial_\nu f)\eta_{\rho\mu} \tag{14.3}$$

Contracting with $\eta^{\rho\mu}$, we have

$$\partial^2\epsilon_\nu = (2-d)\partial_\nu f \tag{14.4}$$

Taking derivative and taking symmetric combination of the indices, we have

$$\partial^2\partial_\mu\epsilon_\nu + \partial^2\partial_\nu\epsilon_\mu = (2-d)\partial_\mu\partial_\nu f \tag{14.5}$$

Taking ∂^2 on (14.2), we have

$$\partial^2 \partial_\mu \epsilon_\nu + \partial^2 \partial_\nu \epsilon_\mu = \partial^2 f \eta_{\mu\nu} \tag{14.6}$$

Thus

$$\partial^2 f \eta_{\mu\nu} = (2-d)\partial_\mu \partial_\nu f \tag{14.7}$$

Contracting with $\eta^{\mu\nu}$, we have

$$(d-1)\partial^2 f = 0 \tag{14.8}$$

So when $d \geq 3$

$$\partial^2 f = \partial_\mu \partial_\nu f = 0 \tag{14.9}$$

The general solution is

$$f(x) = A + B_\mu x^\mu \tag{14.10}$$

Then from (14.3), we see ϵ_μ is at most quadratic in the coordinates. So the general form of ϵ_μ is

$$\epsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho \tag{14.11}$$

where the coefficients satisfy

$$b_{\mu\nu} + b_{\nu\mu} = A\eta_{\mu\nu} \tag{14.12}$$
$$2c_{\mu\nu\rho} = B_\nu \eta_{\mu\rho} + B_\rho \eta_{\mu\nu} - B_\mu \eta_{\nu\rho} \tag{14.13}$$

So

$$b_{\mu\nu} = \alpha \eta_{\mu\nu} + m_{\mu\nu} \tag{14.14}$$
$$c_{\mu\nu\rho} = (b_\nu \eta_{\mu\rho} + b_\rho \eta_{\mu\nu} - b_\mu \eta_{\nu\rho})/2 \tag{14.15}$$

where $m_{\mu\nu} = -m_{\nu\mu}$.

Finally

$$\epsilon^\mu = a'^\mu + \alpha x^\mu + m^\mu{}_\nu x^\nu + 2(b \cdot x)x^\mu - b^\mu x^2 \tag{14.16}$$

where $m^{\mu\nu} = -m^{\nu\mu}$, $a \cdot b = a^\mu b^\nu \eta_{\mu\nu}$ and $a^2 = a \cdot a$.

So the infinitesimal conformal transformation is

$$x^\mu \to x'^\mu = x^\mu + a^\mu + \alpha x^\mu + m^\mu{}_\nu x^\nu + 2(b \cdot x)x^\mu - b^\mu x^2 \tag{14.17}$$

Note that a^μ, α, $m^{\mu\rho}$ and b^μ are independent infinitesimal constants.

It's easy to see the infinitesimal transformation $x^\mu \to x'^\mu = x^\mu + a^\mu$, $x'^\mu = x^\mu + \alpha x^\mu$, and $x^\mu \to x'^\mu = x^\mu + m^\mu{}_\nu x^\nu$ correspond to space-time translation, space-time rescaling, and Lorentz transformation respectively. The only new thing is the infinitesimal transformation $x^\mu \to x'^\mu = x^\mu + 2(b \cdot x)x^\mu - b^\mu x^2$. This leads to the so-called special conformal transformation(SCT). Here we calculate finite SCT with respect to a fixed direction $\hat{b}^\mu$. That is, to find a diffeomorphism $x \to x_t(x)$ such that

$$\dot{x}^\mu = 2(\hat{b} \cdot x)x^\mu - \hat{b}^\mu x^2 \tag{14.18}$$

Define

$$y^\mu = P(x^\mu) = x^\mu / x^2 \tag{14.19}$$

Then it's easy to see

$$\dot{y}^\mu = -\hat{b}^\mu \tag{14.20}$$

So

$$y^\mu = x_0^\mu / x_0^2 - \hat{b}^\mu t \tag{14.21}$$

So

$$\begin{aligned} x^\mu &= \frac{x_0^\mu/x_0^2 - \hat{b}^\mu t}{\eta_{\mu\nu}(x_0^\mu/x_0^2 - \hat{b}^\mu t)(x_0^\nu/x_0^2 - \hat{b}^\nu t)} && (14.22) \\ &= \frac{x_0^\mu - x_0^2 \hat{b}^\mu t}{1 - 2x_0 \cdot (\hat{b}t) + x_0^2 (\hat{b}t)^2} && (14.23) \end{aligned}$$

We may define $b^\mu = \hat{b}^\mu t$. So finite SCT with respect to b^μ can be defined as

$$x^\mu \to K_b x^\mu = \frac{x^\mu - x^2 b^\mu}{1 - 2x \cdot b + x^2 b^2} \tag{14.24}$$

From the deduction above, it's easy to see

$$K_b = P T_{-b} P \tag{14.25}$$

where T_{b^μ} is space-time translation.

So

$$K_a K_b = P T_{-a} P P T_{-b} P = P T_{-(a+b)} P = K_{a+b} \tag{14.26}$$

So the subgroups of finite special conformal group is

$$\begin{aligned} T_a x^\mu &= x^\mu + a^\mu && (14.27) \\ L_\Lambda x^\mu &= \Lambda^\mu{}_\nu x^\nu && (14.28) \\ D_\lambda x^\mu &= \lambda x^\mu && (14.29) \\ K_b x^\mu &= \frac{x^\mu - x^2 b^\mu}{1 - 2x \cdot b + x^2 b^2} = x^\mu + 2(b \cdot x) x^\mu - b^\mu x^2 + o(b) && (14.30) \end{aligned}$$

We have

$$\begin{aligned} T_a T_b &= T_{a+b} && (14.31) \\ D_\lambda D_{\lambda'} &= D_{\lambda\lambda'} && (14.32) \\ L_\Lambda L_{\Lambda'} &= L_{\Lambda\Lambda'} && (14.33) \\ T_a^{-1} L_\Lambda T_a &= T_{(\Lambda-1)a} L_\Lambda && (14.34) \\ T_a^{-1} D_\lambda T_a &= T_{(\lambda-1)a} && (14.35) \\ T_a^{-1} K_b T_a &= T_{2b\cdot aa - a^2 b} K_b D_{1+2a\cdot b} L_{e^{2a^\mu b_\nu - 2b^\mu a_\nu}} + o(b) && (14.36) \\ L_\Lambda^{-1} D_\lambda L_\Lambda &= D_\lambda && (14.37) \\ L_\Lambda^{-1} K_b L_\Lambda &= K_{\Lambda^{-1} b} && (14.38) \\ D_\lambda^{-1} K_b D_\lambda &= K_{\lambda b} && (14.39) \end{aligned}$$

Note We have $T_a^{-1}K_bT_a = T_{K_ba-a}K_{-K_{K_ba}(-b)}C$, where $C0 = 0$ and $(PCP)0 = 0$. It's easy to see that $(K_ba)^2 = \frac{a^2}{1-2a\cdot b+a^2b^2}$, $(K_{K_ba}(-b))^2 = b^2(1-2a\cdot b+a^2b^2)$ and $(K_ba)\cdot(K_{K_ba}(-b)) = -2a\cdot b$. So $C(-a) = K_{K_{K_ba}(-b)}(-K_ba)$. So $(C(-a))^2 = \frac{a^2}{(1-2a\cdot b+a^2b^2)^2}$. So let $T_a^{-1}K_bT_a = T_{K_ba-a}K_{-K_{K_ba}(-b)}D_{\frac{1}{(1-2a\cdot b+a^2b^2)^2}}F$, we have $F0 = 0$, $(PFP)0 = 0$ and $(F(-a))^2 = a^2$. I can't prove but guess $F = 1 + 2a^\mu b_\nu - 2b^\mu a_\nu + o(b)$ is a Lorentz transformation.

The corresponding Lie algebra is

$$
\begin{aligned}
P_\mu &= \partial_\epsilon T_{\epsilon\delta^\mu} && (14.40)\\
M_{\mu\nu} &= \partial_\epsilon L_{1+\epsilon\omega^{\mu\nu}} && (14.41)\\
D &= \partial_\epsilon D_{1+\epsilon} && (14.42)\\
K_\mu &= \partial_\epsilon K_{\epsilon\delta^\mu} && (14.43)
\end{aligned}
$$

where δ^μ and $\omega^{\mu\nu}$ are defined by

$$(\delta^\mu)^\alpha = \eta^{\mu\alpha},\ (\omega^{\mu\nu})^{\alpha\beta} = \eta^{\mu\alpha}\eta^{\nu\beta} - \eta^{\mu\beta}\eta^{\nu\alpha} \tag{14.44}$$

We have

$$
\begin{aligned}
T_a &= 1 + a^\mu P_\mu + o(a) && (14.45)\\
L_{1+\Lambda} &= 1 + \frac{1}{2}\Lambda^{\mu\nu}M_{\mu\nu} + o(\Lambda) && (14.46)\\
D_{1+\lambda} &= 1 + \lambda D + o(\lambda) && (14.47)\\
K_b &= 1 + b^\mu K_\mu + o(b) && (14.48)
\end{aligned}
$$

We have

$$
\begin{aligned}
T_a^{-1}P_\mu T_a &= P_\mu && (14.49)\\
T_a^{-1}M_{\mu\nu}T_a &= a_\nu P_\mu - a_\mu P_\nu && (14.50)\\
T_a^{-1}DT_a &= a^\mu P_\mu && (14.51)\\
T_a^{-1}K_\mu T_a &= 2a_\mu D + 2a^\nu M_{\nu\mu} + 2a_\mu a^\nu P_\nu + a^2 P_\mu && (14.52)\\
L_\Lambda^{-1}M_{\mu\nu}L_\Lambda &= \Lambda_\mu{}^\alpha\Lambda_\nu{}^\beta M_{\alpha\beta} && (14.53)\\
L_\Lambda^{-1}DL_\Lambda &= D && (14.54)\\
L_\Lambda^{-1}K_\mu L_\Lambda &= \Lambda_\mu{}^\nu K_\nu && (14.55)\\
D_\lambda^{-1}DD_\lambda &= D && (14.56)\\
D_\lambda^{-1}K_\mu D_\lambda &= \lambda K_\mu && (14.57)\\
K_b^{-1}K_\mu K_b &= K_\mu && (14.58)
\end{aligned}
$$

We have

$$
\begin{aligned}
[P_\mu, P_\nu] &= 0 && (14.59)\\
[P_\rho, M_{\mu\nu}] &= \eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu && (14.60)\\
[D, P_\mu] &= P_\mu && (14.61)
\end{aligned}
$$

$$\begin{aligned}
[K_\mu, P_\nu] &= 2\eta_{\mu\nu} D - 2M_{\mu\nu} &(14.62)\\
[M_{\mu\nu}, M_{\rho\sigma}] &= \eta_{\sigma\nu} M_{\rho\mu} + \eta_{\sigma\mu} M_{\nu\rho} + \eta_{\rho\mu} M_{\sigma\nu} + \eta_{\rho\nu} M_{\mu\sigma} &(14.63)\\
[D, M_{\mu\nu}] &= 0 &(14.64)\\
[K_\rho, M_{\mu\nu}] &= \eta_{\rho\mu} K_\nu - \eta_{\rho\nu} K_\mu &(14.65)\\
[K_\mu, D] &= K_\mu &(14.66)\\
[K_\mu, K_\nu] &= 0 &(14.67)
\end{aligned}$$

Define J_{ab} where $a, b = -2, \ldots, d-1$ as

$$\begin{aligned}
J_{\mu\nu} &= M_{\mu\nu} &(14.68)\\
J_{-2,-1} &= D &(14.69)\\
J_{-2,\mu} &= \frac{1}{2}(P_\mu - K_\mu) &(14.70)\\
J_{-1,\mu} &= \frac{1}{2}(P_\mu + K_\mu) &(14.71)
\end{aligned}$$

and $J_{ab} = -J_{ba}$.

We may also expand $\eta_{\mu\nu}$ to diagonal tensor η_{ab} where $a, b = -2, \ldots, d-1$ by setting $\eta_{-2,-2} = -1$ and $\eta_{-1,-1} = 1$.

We have

$$[J_{ab}, J_{cd}] = \eta_{ad} J_{bc} + \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} \tag{14.72}$$

Thus special conformal algebra is actually $\mathfrak{so}(d,1)$(if space-time is Euclidean) or $\mathfrak{so}(d-1,2)$(if space-time is Minkowski).

14.1.2 Conformal Transformation when $d = 2$

The transformation $x \to x' = f(x)$ is conformal if

$$g'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x'^\nu}{\partial x^\sigma} g^{\rho\sigma} \sim g^{\mu\nu} \tag{14.73}$$

Same as before, we assume $g^{\mu\nu} \sim \eta^{\mu\nu}$ and $g'^{\mu\nu} \sim \eta^{\mu\nu}$. So we have

$$\frac{\partial x'^\mu}{\partial x^\rho} \frac{\partial x'^\nu}{\partial x^\sigma} \eta^{\rho\sigma} = h(x) \eta^{\mu\nu} \tag{14.74}$$

That is

$$\begin{aligned}
\partial_0 x'^0 \partial_0 x'^0 \eta^{00} + \partial_1 x'^0 \partial_1 x'^0 \eta^{11} &= h(x)\eta^{00} &(14.75)\\
\partial_0 x'^1 \partial_0 x'^1 \eta^{00} + \partial_1 x'^1 \partial_1 x'^1 \eta^{11} &= h(x)\eta^{11} &(14.76)\\
\partial_0 x'^0 \partial_0 x'^1 \eta^{00} + \partial_1 x'^0 \partial_1 x'^1 \eta^{11} &= 0 &(14.77)\\
\partial_0 x'^1 \partial_0 x'^0 \eta^{00} + \partial_1 x'^1 \partial_1 x'^0 \eta^{11} &= 0 &(14.78)
\end{aligned}$$

which is

$$
\begin{aligned}
(\partial_0 x'^0)^2 + (\partial_1 x'^0)^2 \frac{\eta^{11}}{\eta^{00}} &= (\partial_0 x'^1)^2 \frac{\eta^{00}}{\eta^{11}} + (\partial_1 x'^1)^2 && (14.79)\\
\partial_0 x'^1 \partial_0 x'^0 + \partial_1 x'^1 \partial_1 x'^0 \frac{\eta^{11}}{\eta^{00}} &= 0 && (14.80)
\end{aligned}
$$

The solution is

$$\partial_0 x'^0 = \partial_1 x'^1,\ \partial_0 x'^1 = -\partial_1 x'^0 \frac{\eta^{11}}{\eta^{00}} \tag{14.81}$$

or

$$\partial_0 x'^0 = -\partial_1 x'^1,\ \partial_0 x'^1 = \partial_1 x'^0 \frac{\eta^{11}}{\eta^{00}} \tag{14.82}$$

Define

$$
\begin{aligned}
z &= x^0 + i\sqrt{\frac{\eta^{00}}{\eta^{11}}} x^1 && (14.83)\\
\bar{z} &= x^0 - i\sqrt{\frac{\eta^{00}}{\eta^{11}}} x^1 && (14.84)
\end{aligned}
$$

We have

$$\partial_z z' = 0 \tag{14.85}$$

or

$$\partial_{\bar{z}} z' = 0 \tag{14.86}$$

This means that $z' = f(z)$ is holomorphic or anti-holomorphic. Clearly the transformation connected to the identity is holomorphic. So we only study this case.

We express infinitesimal holomorphic transformation $f(z)$ as

$$
\begin{aligned}
z' &= z + \sum_n c_n z^{n+1} && (14.87)\\
\bar{z}' &= \bar{z} + \sum_n \bar{c}_n \bar{z}^{n+1} && (14.88)
\end{aligned}
$$

This is an infinitesimal element of the complex Lie group(actually monoid).

Consider a generator g_n and $\bar{g}_n$

$$
\begin{aligned}
z' &= z + c_n z^{n+1} && (14.89)\\
\bar{z}' &= \bar{z} + \bar{c}_n \bar{z}^{n+1} && (14.90)
\end{aligned}
$$

The loyal representation Tg_n that acts on the field $\phi(z, \bar{z})$ is

$$\phi(z, \bar{z}) \to \phi'(z', \bar{z}') = \phi(z, \bar{z}) \tag{14.91}$$

$$\phi'(z, \bar{z}) = \phi(z, \bar{z}) - c_n z^{n+1} \partial_z \phi(z, \bar{z}) - \bar{c}_n \bar{z}^{n+1} \partial_{\bar{z}} \phi(z, \bar{z}) + o(c_n) = Tg_n \phi(z, \bar{z}) \tag{14.92}$$

We have Lie algebra of TG as

$$\begin{aligned} T_*(l_n) &= \partial_{c_n} T g_n(c_n) = -z^{n+1}\partial_z && (14.93) \\ T_*(\bar{l}_n) &= \partial_{\bar{c}_n} T g_n(c_n) = -\bar{z}^{n+1}\partial_{\bar{z}} && (14.94) \end{aligned}$$

where l_n and $\bar{l}_n$ are Lie algebras of g_n and $\bar{g}_n$.

It's easy to check that they form the Witt algebra

$$\begin{aligned} [l_n, l_m] &= (n-m)l_{n+m} && (14.95) \\ [\bar{l}_n, \bar{l}_m] &= (n-m)\bar{l}_{n+m} && (14.96) \\ [l_n, \bar{l}_m] &= 0 && (14.97) \end{aligned}$$

Next we calculate the finite transformation of $g_n(c,t)$, that is, $z \to z_t$ ($z_0 = z$) such that

$$\dot{z}_t = c z_t^{n+1} \tag{14.98}$$

The solution is

$$z_t = \begin{cases} z + ct & n = -1 \\ z e^{ct} & n = 0 \\ (z^{-n} - nct)^{-1/n} & n \neq -1, 0 \end{cases} \tag{14.99}$$

We may express the finite transformation as

$$g_n(c)z = \begin{cases} z + c & n = -1 \\ cz & n = 0 \\ (z^{-n} + c)^{-1/n} & n \neq -1, 0 \end{cases} \tag{14.100}$$

Clearly $g_n(c)$ is defined globally (on Riemannian sphere) only when $n = 0, \pm 1$. The finite transformation is

$$\begin{aligned} g_{-1}(c)z &= z + c && (14.101) \\ g_0(c)z &= cz && (14.102) \\ g_1(c)z &= \frac{z}{1 + cz} && (14.103) \end{aligned}$$

They form a group called special conformal group, with element

$$\frac{az + b}{cz + d} \tag{14.104}$$

with the restriction $ad - bc \neq 0$. The corresponding Lie algebra l_{-1}, l_0 and l_1 form a sub-algebra.

Elements in the special conformal group are diffeomorphism of Riemannian sphere onto itself. We may rescale (a, b, c, d) such that $ad - bc = 1$. Thus each group element corresponds to two (a, b, c, d)s: (a, b, c, d) and $(-a, -b, -c, -d)$. Actually, the special conformal group is just $PSL(2, \mathbb{C}) = SL(2, \mathbb{C})/\mathbb{Z}_2$. To see this, just observe that

$$\frac{a''z + b''}{c''z + d''} = \frac{a'z + b'}{c'z + d'} \frac{az + b}{cz + d} \tag{14.105}$$

leads to

$$\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix} = \pm \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{14.106}$$

14.1.3 Virasoro Algebra

The central extension of a Lie algebra $[x_i, x_j] = f_{ijk}x_k$ is to add some central elements 1_k into the algebra such that

$$\begin{aligned}
[x_i, x_j] &= f_{ijk}x_k + g_{ijk}1_k && (14.107)\\
[x_i, 1_j] &= 0 && (14.108)\\
[1_i, 1_j] &= 0 && (14.109)
\end{aligned}$$

g_{ijk} should satisfy

$$\begin{aligned}
g_{ijk} + g_{jik} &= 0 && (14.110)\\
f_{ijl}g_{lkm} + f_{jkl}g_{lim} + f_{kil}g_{ljm} &= 0 && (14.111)
\end{aligned}$$

We get the central extension of Witt algebra by adding a central element 1 (and omit 1 for clearity). The algebra becomes

$$[L_n, L_m] = (n-m)L_{n+m} + g_{nm} \tag{14.112}$$

We may redefine the algebra by

$$\tilde{L}_n = \begin{cases} L_n + \frac{g_{n0}}{n} & n \neq 0 \\ L_0 + \frac{g_{1,-1}}{2} & n = 0 \end{cases} \tag{14.113}$$

Then

$$[\tilde{L}_n, \tilde{L}_m] = (n-m)\tilde{L}_{n+m} + \tilde{g}_{nm} \tag{14.114}$$

where

$$\tilde{g}_{nm} = \begin{cases} g_{nm} - \frac{n-m}{n+m}g_{n+m,0} & n+m \neq 0 \\ g_{nm} - \frac{n-m}{2}g_{1,-1} & n+m = 0 \end{cases} \tag{14.115}$$

Thus $\tilde{g}_{1,-1} = \tilde{g}_{n0} = 0$.

We have

$$(i-j)\tilde{g}_{i+j,k} + (j-k)\tilde{g}_{j+k,i} + (k-i)\tilde{g}_{k+i,j} = 0 \tag{14.116}$$

Set $k = 0$, we have

$$(i-j)\tilde{g}_{i+j,0} + j\tilde{g}_{j,i} - i\tilde{g}_{i,j} = 0 \tag{14.117}$$

So $\tilde{g}_{i,j} = 0$ unless $i + j = 0$.

Set $k = -1, j = -i + 1$, we have

$$(2i-1)\tilde{g}_{1,-1} + (-i+2)\tilde{g}_{-i,i} + (-i-1)\tilde{g}_{i-1,-i+1} = 0 \tag{14.118}$$

So $\tilde{g}_{i,-i} = \frac{i^3-i}{6}\tilde{g}_{2,-2} = \frac{i^3-i}{12}c$ where we set $\tilde{g}_{2,-2} = \frac{c}{2}$

To sum up

$$[\tilde{L}_n, \tilde{L}_m] = (n-m)\tilde{L}_{n+m} + \frac{n^3-n}{12}c\delta_{n+m,0} \tag{14.119}$$

This is called the Virasoro algebra.

14.2 Classical Conformal Symmetry

From now on we only study the case $d = 2$.

A field is called primary of conformal dimension $(h, \bar{h})$ if it transforms under conformal transformation $z \to z'$ as

$$\phi(z, \bar{z}) \to \phi'(z', \bar{z}') = \left(\frac{\partial z'}{\partial z}\right)^{-h} \left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{-\bar{h}} \phi(z, \bar{z}) \tag{14.120}$$

We define $\Delta = h + \bar{h}$ and $s = h - \bar{h}$.

A field is call quasi-primary of conformal dimension $(h, \bar{h})$ if it transforms in this way only under special conformal transformations.

Under infinitesimal transformation $z \to z' = z + \epsilon(z)$, a primary field transforms as $\phi(z, \bar{z}) \to \phi'(z, \bar{z})$ where

$$\begin{aligned} \phi'(z', \bar{z}') &= \left(\frac{\partial z'}{\partial z}\right)^{-h} \left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{-\bar{h}} \phi(z, \bar{z}) && (14.121) \\ \phi'(z, \bar{z}) &= (1 + \partial_z \epsilon)^{-h} (1 + \partial_{\bar{z}} \bar{\epsilon})^{-\bar{h}} \phi(z - \epsilon, \bar{z} - \bar{\epsilon}) + o(\epsilon) && (14.122) \\ &= (1 - \epsilon \partial_z - \bar{\epsilon} \partial_{\bar{z}} - h(\partial_z \epsilon) - \bar{h}(\partial_{\bar{z}} \bar{\epsilon})) \phi(z, \bar{z}) + o(\epsilon) && (14.123) \end{aligned}$$

That is,

$$\delta\phi = [-\epsilon \partial_z - h \partial_z \epsilon - \bar{\epsilon} \partial_{\bar{z}} - \bar{h} \partial_{\bar{z}} \bar{\epsilon}] \phi \tag{14.124}$$

14.2.1 Traceless Energy-Momentum Tensor

In chap .. we discussed in some cases we can alternate the energy-momentum tensor to a symmetric traceless one. In this subsection we take a totally different approach.

Let $L(\eta_{\mu\nu}, \phi)$ be a system with conformal symmetry. Suppose that we can embed this system into a system $L(g_{\mu\nu}, \phi)$ defined in arbitrary metric. In the new system, the conformal transformation induce a transformation $\phi \to \phi'$ together with $g_{\mu\nu} \to g'_{\mu\nu} = h(x) g_{\mu\nu}$. We assume that the embedding is nicely chosen such that under this transformation (as usual, with arbitrary support), $\delta S = 0$.

For a system defined in arbitrary metric, we have

$$T^{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} \tag{14.125}$$

So the energy-momentum is symmetric.

The new system under conformal transformation will have

$$\delta S = \int \sqrt{-g} T^{\mu\nu} \delta g_{\mu\nu} + \delta_\phi S \tag{14.126}$$

The conformal symmetry of the original system tells us that $\delta_\phi S = 0$. Then

$$\int \sqrt{-g} T^{\mu\nu} \delta g_{\mu\nu} = \int \sqrt{-g} T^\mu_\mu \delta h = 0 \tag{14.127}$$

Since the integration is computed on an arbitrary space-time region, we have $T^\mu_\mu = 0$.

Finally we define the energy-momentum tensor of the original system to be $T^{\mu\nu}|_{g_{\mu\nu}=\eta_{\mu\nu}}$. And we result in a symmetric traceless energy-momentum tensor.

Transforming into coordinates of z and $\bar{z}$, we have

$$\begin{aligned} T_{zz} &= \frac{1}{4}(T_{00} - iT_{01} - iT_{10} - T_{11}) = \frac{1}{2}(T_{00} - iT_{01}) && (14.128) \\ T_{\bar{z}\bar{z}} &= \frac{1}{4}(T_{00} + iT_{01} + iT_{10} - T_{11}) = \frac{1}{2}(T_{00} + iT_{01}) && (14.129) \\ T_{z\bar{z}} &= \frac{1}{4}(T_{00} + iT_{01} - iT_{10} + T_{11}) = 0 && (14.130) \\ T_{\bar{z}z} &= \frac{1}{4}(T_{00} - iT_{01} + iT_{10} + T_{11}) = 0 && (14.131) \\ & && (14.132) \end{aligned}$$

Then we have

$$\partial_{\bar{z}} T_{zz} = \partial_z T_{\bar{z}\bar{z}} = 0 \qquad (14.133)$$

Thus we may define

$$T(z) = T_{zz},\ \bar{T}(\bar{z}) = T_{\bar{z}\bar{z}} \qquad (14.134)$$

Then for an infinitesimal conformal transformation $x \to x' = x + \epsilon$

$$\begin{aligned} \partial^\mu(T_{\mu\nu}\epsilon^\nu) &= (\partial^\mu T_{\mu\nu})\epsilon^\nu + T_{\mu\nu}\partial^\mu\epsilon^\nu && (14.135) \\ &= \frac{1}{2}T_{\mu\nu}(\partial^\mu\epsilon^\nu + \partial^\nu\epsilon^\mu) && (14.136) \\ &\sim \frac{1}{2}T_{\mu\nu}\eta^{\mu\nu} && (14.137) \\ &= 0 && (14.138) \end{aligned}$$

That is, $J_\mu = T_{\mu\nu}\epsilon^\nu$ is a conserved current. In complex coordinates, we have $J_z = T(z)\epsilon(z)$ and $J_{\bar{z}} = \bar{T}(\bar{z})\bar{\epsilon}(\bar{z})$.

14.3 Correlation Function

We study the quantized of conformal field. We use the conventional path-integral quantization method.

Let

$$\begin{aligned} x &\to x' && (14.139) \\ \phi(x) &\to \phi'(x) = \mathcal{F}\phi(x) && (14.140) \end{aligned}$$

be a **globally defined** symmetry transformation

Denote by X the collection $\phi_1(x_1)\cdots\phi_n(x_n)$, we have

$$\langle X(x')\rangle = \frac{1}{Z}\int \mathcal{D}\phi X(x')e^{-S} = \frac{1}{Z}\int \mathcal{D}\phi' X'(x')e^{-S'} = \frac{1}{Z}\int \mathcal{D}\phi \mathcal{F}X(x)e^{-S} = \langle \mathcal{F}X(x)\rangle \qquad (14.141)$$

14.3.1 2 Point Correlation Function

Then we have

$$\langle\phi_1(z',\bar{z}')\phi_2(w',\bar{w}')\rangle = \left(\frac{\partial z'}{\partial z}\right)^{-h_1}\left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{-\bar{h}_1}\left(\frac{\partial w'}{\partial w}\right)^{-h_2}\left(\frac{\partial \bar{w}'}{\partial \bar{w}}\right)^{-\bar{h}_2}\langle\phi_1(z,\bar{z})\phi_2(w,\bar{w})\rangle \tag{14.142}$$

We may define

$$\langle\phi_1(z,\bar{z})\phi_2(w,\bar{w})\rangle = f(z,\bar{z},w,\bar{w}) \tag{14.143}$$

So

$$f(z',\bar{z}',w',\bar{w}') = \left(\frac{\partial z'}{\partial z}\right)^{-h_1}\left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{-\bar{h}_1}\left(\frac{\partial w'}{\partial w}\right)^{-h_2}\left(\frac{\partial \bar{w}'}{\partial \bar{w}}\right)^{-\bar{h}_2} f(z,\bar{z},w,\bar{w}) \tag{14.144}$$

Under the transformation $z \to z + a$ we have

$$f(z+a,\bar{z}+\bar{a},w+a,\bar{w}+\bar{a}) = f(z,\bar{z},w,\bar{w}) \tag{14.145}$$

Taking ∂_a and $\partial_{\bar{a}}$ at $a = 0$, we have

$$\begin{aligned}(\partial_1+\partial_3)f &= 0 && (14.146)\\ (\partial_2+\partial_4)f &= 0 && (14.147)\end{aligned}$$

That is

$$\partial_{z+w} f(z,\bar{z},w,\bar{w}) = \partial_{\bar{z}+\bar{w}} f(z,\bar{z},w,\bar{w}) = 0 \tag{14.148}$$

So

$$f(z,\bar{z},w,\bar{w}) = g(z-w,\bar{z}-\bar{w}) \tag{14.149}$$

Under the transformation $z \to az$ we have

$$g(a(z-w),\bar{a}(\bar{z}-\bar{w})) = a^{-h_1-h_2}\bar{a}^{-\bar{h}_1-\bar{h}_2} g(z-w,\bar{z}-\bar{w}) \tag{14.150}$$

Taking ∂_a at $a = 1$, we have

$$\begin{aligned}(z-w)\partial_1 g &= -(h_1+h_2)g && (14.151)\\ \partial_{z-w}((z-w)^{h_1+h_2}g) &= 0 && (14.152)\\ g &= (z-w)^{-h_1-h_2}e(\bar{z}-\bar{w}) && (14.153)\end{aligned}$$

Similarly, taking $\partial_{\bar{a}}$ at $a = 1$, we have

$$g = (\bar{z}-\bar{w})^{-\bar{h}_1-\bar{h}_2}e'(z-w) \tag{14.154}$$

So

$$f = g = \frac{C_{12}}{(z-w)^{h_1+h_2}(\bar{z}-\bar{w})^{\bar{h}_1+\bar{h}_2}} \tag{14.155}$$

Finally, under the transformation $z \to \frac{z}{1+az}$ we have

$$\frac{C_{12}}{\left(\frac{z}{1+az}-\frac{w}{1+aw}\right)^{h_1+h_2}\left(\frac{\bar{z}}{1+\bar{a}\bar{z}}-\frac{\bar{w}}{1+\bar{a}\bar{w}}\right)^{\bar{h}_1+\bar{h}_2}} = (1+az)^{2h_1}(1+\bar{a}\bar{z})^{2\bar{h}_1}(1+aw)^{2h_2}(1+\bar{a}\bar{w})^{2\bar{h}_2}$$

$$\frac{C_{12}}{(z-w)^{h_1+h_2}(\bar{z}-\bar{w})^{\bar{h}_1+\bar{h}_2}} \tag{14.156}$$

So if $C_{12} \neq 0$

$$\big((1+az)(1+aw)\big)^{h_1+h_2}\big((1+\bar{a}\bar{z})(1+\bar{a}\bar{w})\big)^{\bar{h}_1+\bar{h}_2} = (1+az)^{2h_1}(1+\bar{a}\bar{z})^{2\bar{h}_1}(1+aw)^{2h_2}(1+\bar{a}\bar{w})^{2\bar{h}_2} \tag{14.157}$$

This means $h_1 = h_2$, $\bar{h}_1 = \bar{h}_2$
So we have

$$\langle\phi_1(z,\bar{z})\phi_2(w,\bar{w})\rangle = \begin{cases} \frac{C_{12}}{(z-w)^{2h}(\bar{z}-\bar{w})^{2\bar{h}}} & h_1 = h_2 = h,\ \bar{h}_1 = \bar{h}_2 = \bar{h} \\ 0 & \text{otherwise} \end{cases} \tag{14.158}$$

Since we have only used special conformal transformation, the above result holds for quasi-primary fields.

14.3.2 3 Point Correlation Function

Then we consider 3-point correlation function

$$\langle\phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\phi_3(z_3,\bar{z}_3)\rangle = f(z_1,\bar{z}_1,z_2,\bar{z}_2,z_3,\bar{z}_3) \tag{14.159}$$

Similarly, from the symmetry $x \to x + a$ we have

$$\partial_{z_1+z_2+z_3} f = \partial_{\bar{z}_1+\bar{z}_2+\bar{z}_3} f = 0 \tag{14.160}$$

Thus

$$f = g(z_{12}, z_{23}, \bar{z}_{12}, \bar{z}_{23}) \tag{14.161}$$

where $z_{12} = z_1 - z_2$, $z_{13} = z_1 - z_3$ and $z_{23} = z_2 - z_3$
Then from the symmetry $x \to ax$ we have

$$z_{12}\partial_{z_{12}} g + z_{23}\partial_{z_{23}} g = -(h_1 + h_2 + h_3)g \tag{14.162}$$

Define

$$q = (z_{12} + z_{23})^{h_3+h_1-h_2} z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} g \tag{14.163}$$

Then

$$z_{12}\partial_{z_{12}} q + z_{23}\partial_{z_{23}} q = 0 \tag{14.164}$$

$$\partial_{\ln z_{12}} q + \partial_{\ln z_{23}} q = 0 \tag{14.165}$$

So

$$q = k(\frac{z_{12}}{z_{23}}, \bar{z}_{12}, \bar{z}_{23}) \tag{14.166}$$

That is,

$$g = \frac{k(\frac{z_{12}}{z_{23}}, \bar{z}_{12}, \bar{z}_{23})}{z_{13}^{h_3+h_1-h_2} z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1}} \tag{14.167}$$

Make a similar treatment on $\bar{z}$, we have

$$g = \frac{l(\frac{z_{12}}{z_{23}}, \frac{\bar{z}_{12}}{\bar{z}_{23}})}{z_{13}^{h_3+h_1-h_2} z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} \bar{z}_{13}^{\bar{h}_3+\bar{h}_1-\bar{h}_2} \bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3} \bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1}} \tag{14.168}$$

Finally, from the symmetry $z \to \frac{z}{1+az}$ we have

$$l(\frac{z_{12}}{z_{23}}\frac{1+az_3}{1+az_1}, \frac{\bar{z}_{12}}{\bar{z}_{23}}\frac{1+\bar{a}\bar{z}_3}{1+\bar{a}\bar{z}_1}) = l(\frac{z_{12}}{z_{23}}, \frac{\bar{z}_{12}}{\bar{z}_{23}}) \tag{14.169}$$

That tells us $l = C_{123}$. So

$$\langle \phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\phi_3(z_3,\bar{z}_3)\rangle = \frac{C_{123}}{z_{13}^{h_3+h_1-h_2} z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} \bar{z}_{13}^{\bar{h}_3+\bar{h}_1-\bar{h}_2} \bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3} \bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1}} \tag{14.170}$$

The 4-point correlation function is not uniquely determined. It's general form is

$$\langle \phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\phi_3(z_3,\bar{z}_3)\phi_4(z_4,\bar{z}_4)\rangle = f(\eta,\bar{\eta}) \prod_{0\le i<j\le 4} z_{ij}^{h/3-h_i-h_j} \bar{z}_{ij}^{\bar{h}/3-\bar{h}_i-\bar{h}_j} \tag{14.171}$$

where $\eta = \frac{z_{12}z_{34}}{z_{13}z_{24}}$.

14.4 Ward Identity

We have introduced in the section of Noether's theorem that under a position-dependent transformation $\delta x = \epsilon(x)$, $\delta\phi(x) = \epsilon(x)G\phi(x)$

$$\delta S = -\int dx j^\mu \partial_\mu \epsilon(x) \tag{14.172}$$

Denote by X the collection $\phi_1(x_1)\cdots\phi_n(x_n)$, we have

$$\langle X\rangle = \frac{1}{Z}\int \mathcal{D}\phi X e^{-S} = \frac{1}{Z}\int \mathcal{D}\phi' X' e^{-S'} = \frac{1}{Z}\int \mathcal{D}\phi (X+\delta X) e^{-S-\delta S} \tag{14.173}$$

So

$$\begin{aligned}
\langle \delta X\rangle &= \frac{1}{Z}\int \mathcal{D}\phi \delta X e^{-S} && (14.174)\\
&= \frac{1}{Z}\int \mathcal{D}\phi X \delta S e^{-S} && (14.175)\\
&= -\frac{1}{Z}\int \mathcal{D}\phi X \int dx j^\mu \partial_\mu\epsilon(x) e^{-S} && (14.176)\\
&= -\int dx \Big[\frac{1}{Z}\int \mathcal{D}\phi X j^\mu e^{-S}\Big]\partial_\mu \epsilon(x) && (14.177)\\
&= -\int dx \langle X j^\mu\rangle \partial_\mu\epsilon(x) && (14.178)
\end{aligned}$$

$$= \int dx \partial_\mu \langle X j^\mu \rangle \epsilon(x) \tag{14.179}$$

Since

$$\begin{aligned} \delta\phi_i(x) &= -\epsilon(x)\partial_\mu\phi(x)(\mathcal{T}x)^\mu + \epsilon(x)(P\phi)(x) && (14.180) \\ &= \epsilon(x)(\mathcal{G}\phi)(x) && (14.181) \end{aligned}$$

where

$$\mathcal{G}\phi = -\partial_\mu\phi(\mathcal{T}x)^\mu + (\mathcal{P}\phi) \tag{14.182}$$

we have

$$\begin{aligned} \langle \delta X \rangle &= \sum_i \langle \phi_1(x_1)\cdots\delta\phi_i(x_i)\cdots\phi_n(x_n)\rangle && (14.183) \\ &= \sum_i \langle \phi_1(x_1)\cdots\epsilon(x_i)(\mathcal{G}\phi_i)(x_i)\cdots\phi_n(x_n)\rangle && (14.184) \\ &= \int dx\epsilon(x)\sum_i \delta(x-x_i)\langle \phi_1(x_1)\cdots(\mathcal{G}\phi_i)(x_i)\cdots\phi_n(x_n)\rangle && (14.185) \end{aligned}$$

So we have the Ward identity

$$\sum_i \delta(x-x_i)\langle \phi_1\cdots\mathcal{G}\phi_i\cdots\phi_n\rangle = \partial_\mu\langle X j^\mu(x)\rangle \tag{14.186}$$

Note The Ward identity work for locally defined symmetry as well, since we can choose $\epsilon(x)$ to have a local support.

Integrate both side, we have

$$\begin{aligned} \sum_{x_i\in M} \langle \phi_1\cdots\mathcal{G}\phi_i\cdots\phi_n\rangle &= \int_M dx^2\partial_\mu\langle X j^\mu(x)\rangle && (14.187) \\ &= \int_{\partial M} dS_\mu\langle X j^\mu(x)\rangle && (14.188) \end{aligned}$$

If the symmetry is defined globally, we may let ∂M to go to infinity, where the RHS is assumed to be zero. Then

$$\sum_i \langle \phi_1\cdots\mathcal{G}\phi_i\cdots\phi_n\rangle = 0 \tag{14.189}$$

This is consistent with our previous result (14.141).

If M is the region $t < t_0$, we have

$$\sum_{t_i<t_0} \langle \phi_1\cdots\mathcal{G}\phi_i\cdots\phi_n\rangle = \langle XQ(t_0)\rangle \tag{14.190}$$

In the following we study some Ward identities of common symmetry. **Strictly speaking, the expression of conserved currents $J^{\mu\nu\rho}$ and D^μ we use in the following only holds for $d > 2$. However, we assume that it also holds for $d = 2$.**

14.4.1 Ward Identity for Translation

The symmetry transformation of translation is

$$x \rightarrow x + a \tag{14.191}$$
$$\phi \rightarrow \phi \tag{14.192}$$

The Lie algebra is $(\delta^\mu, 0)$.

$$\mathcal{G}_\mu \phi_i = -\partial_\mu \phi_i \tag{14.193}$$

The current is $T^{\mu\nu}$.
So the Ward identity is

$$-\sum_i \delta(x - x_i)\langle \phi_1 \cdots \partial_\nu \phi_i \cdots \phi_n \rangle = \partial_\mu \langle X T^\mu{}_\nu(x) \rangle \tag{14.194}$$
$$-\sum_i \delta(x - x_i)\frac{\partial}{\partial x_i^\nu}\langle X \rangle = \partial_\mu \langle X T^\mu{}_\nu(x) \rangle \tag{14.195}$$

14.4.2 Ward Identity for Rotation

The symmetry transformation of rotation is

$$x \rightarrow R(\theta)x \tag{14.196}$$
$$\phi \rightarrow e^{-is\theta}\phi \tag{14.197}$$

The Lie algebra is $(-\varepsilon_{\mu\nu}, -is)$. $(\varepsilon_{12} = 1)$

$$\mathcal{G}\phi_i = \partial^\mu \phi_i \varepsilon_{\mu\nu} x^\nu - is_i \phi_i \tag{14.198}$$

The current is $-\frac{1}{2}\varepsilon_{\nu\rho}(T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu)$.
So the Ward identity is

$$\sum_i \delta(x - x_i)\langle \phi_1 \cdots (\partial^\mu \phi_i \varepsilon_{\mu\nu} x^\nu - is_i\phi_i) \cdots \phi_n \rangle \tag{14.199}$$
$$= -\frac{1}{2}\partial_\mu \langle X \varepsilon_{\nu\rho}(T^{\mu\nu}x^\rho - T^{\mu\rho}x^\nu)\rangle \tag{14.200}$$
$$= -\frac{1}{2}\langle X \varepsilon_{\nu\rho}(\partial_\mu T^{\mu\nu}x^\rho - \partial_\mu T^{\mu\rho}x^\nu)\rangle - \frac{1}{2}\langle X\varepsilon_{\nu\rho}(T^{\rho\nu} - T^{\nu\rho})\rangle \tag{14.201}$$

Since

$$\frac{1}{2}\langle X\varepsilon_{\nu\rho}(\partial_\mu T^{\mu\nu}x^\rho - \partial_\mu T^{\mu\rho}x^\nu)\rangle = \frac{1}{2}[\varepsilon_{\nu\rho}x^\rho \partial_\mu \langle X T^{\mu\nu}\rangle - \varepsilon_{\nu\rho}x^\nu \partial_\mu \langle X T^{\mu\rho}\rangle] \tag{14.202}$$
$$= -\sum_i \delta(x - x_i)\langle \phi_1 \cdots \varepsilon_{\nu\rho}x^\rho \partial^\nu \phi_i \cdots \phi_n \rangle \tag{14.203}$$

we have

$$\sum_i \delta(x-x_i)\langle\phi_1\cdots(is_i\phi_i)\cdots\phi_n\rangle = \frac{1}{2}\langle X\varepsilon_{\nu\rho}(T^{\rho\nu}-T^{\nu\rho})\rangle \quad (14.204)$$

$$-i\sum_i s_i\delta(x-x_i)\langle X\rangle = \varepsilon_{\mu\nu}\langle XT^{\mu\nu}(x)\rangle \quad (14.205)$$

Note We have made $T^{\mu\nu}$ symmetric only for on-shell field, but the path-integral deals with off-shell field. So $\epsilon_{\mu\nu}\langle XT^{\mu\nu}\rangle$ is not trivial.

14.4.3 Ward Identity for Scale Transformation

The scale transformation is

$$x \to \lambda x \quad (14.206)$$

$$\phi \to \lambda^{-\Delta}\phi \quad (14.207)$$

The Lie algebra is $(1,-\Delta)$.

$$\mathcal{G}\phi_i = -\partial_\mu\phi_i x^\mu - \Delta_i\phi_i \quad (14.208)$$

The current is $T^\mu_{\ \nu}x^\nu$.
So the Ward identity is

$$-\sum_i \delta(x-x_i)\langle\phi_1\cdots(\partial_\mu\phi_i x^\mu+\Delta_i\phi_i)\cdots\phi_n\rangle = \partial_\mu\langle XT^\mu_{\ \nu}x^\nu\rangle \quad (14.209)$$

$$= \langle X\partial_\mu T^\mu_{\ \nu}x^\nu\rangle + \langle XT^\mu_{\ \mu}\rangle \quad (14.210)$$

Since

$$\langle X\partial_\mu T^\mu_{\ \nu}x^\nu\rangle = x^\nu\partial_\mu\langle XT^\mu_{\ \nu}\rangle \quad (14.211)$$

$$= -\sum_i \delta(x-x_i)\langle\phi_1\cdots x^\nu\partial_\nu\phi_i\cdots\phi_n\rangle \quad (14.212)$$

we have

$$-\sum_i \delta(x-x_i)\langle\phi_1\cdots(\Delta_i\phi_i)\cdots\phi_n\rangle = \langle XT^\mu_{\ \mu}\rangle \quad (14.213)$$

$$-\sum_i \Delta_i\delta(x-x_i)\langle X\rangle = \langle XT^\mu_{\ \mu}(x)\rangle \quad (14.214)$$

To sum up, we have

$$\partial_\mu\langle XT^\mu_{\ \nu}(x)\rangle = -\sum_i \delta(x-x_i)\frac{\partial}{\partial x_i^\nu}\langle X\rangle \quad (14.215)$$

$$\varepsilon_{\mu\nu}\langle XT^{\mu\nu}(x)\rangle = -i\sum_i s_i\delta(x-x_i)\langle X\rangle \quad (14.216)$$

$$\langle XT^\mu_{\ \mu}(x)\rangle = -\sum_i \Delta_i\delta(x-x_i)\langle X\rangle \quad (14.217)$$

14.4.4 Ward Identity in Harmonic Coordinates

Using $\delta(x) = \frac{1}{\pi}\partial_{\bar{z}}\frac{1}{z} = \frac{1}{\pi}\partial_z\frac{1}{\bar{z}}$, the above Ward identities can be transformed into

$$2\pi\partial_z\langle T_{\bar{z}z}X\rangle + 2\pi\partial_{\bar{z}}\langle T_{zz}X\rangle = -\sum_i \partial_{\bar{z}}\frac{1}{z-w_i}\partial_{w_i}\langle X\rangle \tag{14.218}$$

$$2\pi\partial_z\langle T_{\bar{z}\bar{z}}X\rangle + 2\pi\partial_{\bar{z}}\langle T_{z\bar{z}}X\rangle = -\sum_i \partial_z\frac{1}{\bar{z}-\bar{w}_i}\partial_{\bar{w}_i}\langle X\rangle \tag{14.219}$$

$$2\pi\langle T_{\bar{z}z}X\rangle = -\sum_i \partial_{\bar{z}}\frac{1}{z-w_i}h_i\langle X\rangle \tag{14.220}$$

$$2\pi\langle T_{z\bar{z}}X\rangle = -\sum_i \partial_z\frac{1}{\bar{z}-\bar{w}_i}\bar{h}_i\langle X\rangle \tag{14.221}$$

We define $T = -2\pi T_{zz}$ and $\bar{T} = -2\pi T_{\bar{z}\bar{z}}$ and the above identities lead to

$$\partial_{\bar{z}}\Big[\langle TX\rangle - \sum_i \big[\frac{1}{z-w_i}\partial_{w_i}\langle X\rangle + \frac{h_i}{(z-w_i)^2}\langle X\rangle\big]\Big] = 0 \tag{14.222}$$

$$\partial_z\Big[\langle \bar{T}X\rangle - \sum_i \big[\frac{1}{\bar{z}-\bar{w}_i}\partial_{\bar{w}_i}\langle X\rangle + \frac{\bar{h}_i}{(\bar{z}-\bar{w}_i)^2}\langle X\rangle\big]\Big] = 0 \tag{14.223}$$

So

$$\langle TX\rangle = \sum_i \big[\frac{1}{z-w_i}\partial_{w_i}\langle X\rangle + \frac{h_i}{(z-w_i)^2}\langle X\rangle\big] + \text{reg} \tag{14.224}$$

where reg are regular terms of z.

14.4.5 Ward Identity for Arbitrary Transformation

Given an arbitrary conformal coordinate variation $\epsilon^\nu(x)$, we can write

$$\partial_\mu(\epsilon_\nu T^{\mu\nu}) = \epsilon_\nu\partial_\mu T^{\mu\nu} + \frac{1}{2}(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu)T^{\mu\nu} + \frac{1}{2}(\partial_\mu\epsilon_\nu - \partial_\nu\epsilon_\mu)T^{\mu\nu} \tag{14.225}$$

$$= \epsilon_\nu\partial_\mu T^{\mu\nu} + \frac{1}{2}\partial_\nu\epsilon^\nu T^\mu_{\ \mu} + \frac{1}{2}\varepsilon^{\alpha\beta}\partial_\alpha\epsilon_\beta\varepsilon_{\mu\nu}T^{\mu\nu} \tag{14.226}$$

So

$$\partial_\mu\langle T^{\mu\nu}(x)\epsilon_\nu(x)X\rangle = \epsilon_\nu(x)\partial_\mu\langle T^{\mu\nu}(x)X\rangle + \frac{1}{2}\partial_\nu\epsilon_\nu(x)\langle T^\mu_{\ \mu}(x)X\rangle + \frac{1}{2}\varepsilon^{\alpha\beta}\partial_\alpha\epsilon_\beta\varepsilon_{\mu\nu}\langle T^{\mu\nu}(x)X\rangle \tag{14.227}$$

$$= -\sum_i \delta(x-x_i)\Big[\epsilon^\mu(x)\frac{\partial}{\partial x_i^\mu}\langle X\rangle + \frac{is_i}{2}\varepsilon^{\mu\nu}\partial_\mu\epsilon_\nu\langle X\rangle + \frac{\Delta_i}{2}\partial_\mu\epsilon^\mu\langle X\rangle\Big] \tag{14.228}$$

$$= -\sum_i \delta(x-x_i)(\epsilon_z(x)\partial_z + \epsilon_{\bar{z}}(x)\partial_{\bar{z}} + h_i\partial_z + \bar{h}_i\partial_{\bar{z}})\langle X\rangle \tag{14.229}$$

$$= \sum_i \delta(x-x_i)\delta_{i,\epsilon}\langle X\rangle \tag{14.230}$$

where $\delta_{i,\epsilon}$ is the variation of ϕ_i in X to the order of ϵ. In the last line we have used (14.124).

Thus we can view $T^{\mu\nu}(x)\epsilon_\nu(x)$ as the current that 'generates' the conformal transformation.

Taking integral on x in M that contains the positions of all the fields in X, we have

$$\delta_\epsilon\langle X\rangle = \int_M d^2x\partial_\mu\langle T^{\mu\nu}(x)\epsilon_\nu(x)X\rangle \tag{14.231}$$

Using Gauss's theorem, we have

$$\delta_\epsilon\langle X\rangle = -\frac{1}{2\pi i}\int_{C,ccl} dz\epsilon(z)\langle TX\rangle + \frac{1}{2\pi i}\int_{C,ccl} d\bar{z}\bar{\epsilon}(\bar{z})\langle \bar{T}X\rangle \tag{14.232}$$

where $C = \partial M$ and ccl means counterclockwise and T is the time-ordering product. Here $\epsilon = \epsilon_z$ and $\bar{\epsilon} = \epsilon_{\bar{z}}$. Since ϵ is an infinitesimal conformal transformation, then $\epsilon = \epsilon(z)$ and $\bar{\epsilon} = \bar{\epsilon}(\bar{z})$.

14.5 Operator Product Expansion

Let's consider the expectation value of a product of local operators

$$\langle O_{i_1}(x_1)O_{i_2}(x_2)\cdots O_{i_n}(x_n)\rangle \tag{14.233}$$

where $x_1 \neq x_2 \neq \cdots \neq x_n$.

Let's consider two operators $O_{i_m}(x_m)O_{i_{m+1}}(x_{m+1})$. In the limit $x_m \to x_{m+1}$, we states that the operator products can be expanded as

$$O_{i_m}(x_m)O_{i_{m+1}}(x_{m+1}) = \sum_k c^k_{i_m i_{m+1}} O_k(x_{m+1}) \tag{14.234}$$

This holds when $|x_m - x_{m+1}| \ll |x_i - x_{m+1}|$ for all $i \neq m, m+1$.

Generally, we have the operator product expansion

$$O_i(x_1)O_j(x_2) = \sum_k c^k_{ij} O_k(x_2) \tag{14.235}$$

We may use $\sim$ in the operator product expansion, which means that the expansion holds up to non-sigular term.

In operator product expansion, an operator may be a product of fields at the same point. To avoid singularity, the operator is in normal order.

We define the normal order following the Wick's theorem

$$O_1\cdots O_n = \sum \text{contractions of } :O_1\cdots O_n: \tag{14.236}$$

where the contraction of each pair is the 2-point Greens function.

It's easy to see that

$$:O_1\cdots O_n: = \sum \text{subtractions of } O_1\cdots O_n \tag{14.237}$$

where the subtraction of each pair is the negative of 2-point Greens function.

Let $\mathcal{F}$ and $\mathcal{G}$ be two products of operators. We have

$$:\mathcal{F}::\mathcal{G}: = :\mathcal{F}\mathcal{G}: + \sum \text{cross} - \text{contractions of } :\mathcal{F}\mathcal{G}: \tag{14.238}$$

14.5.1 Free Bosonic System

Let's consider a free Bosonic model

$$\mathcal{L} = \frac{g}{2}\partial_u\phi \cdot \partial^\mu \phi \tag{14.239}$$

where ϕ is a Bosonic field of D components.

Let the action be Euclidean with metric $\delta_{\mu\nu}$. The Green's function satisfies

$$g(-\partial_x^2)G_{ij}(x,y) = \delta(x-y)\delta_{ij} \tag{14.240}$$

The solution is

$$\langle\phi_i(x)\phi_j(y)\rangle = G_{ij}(x,y) = -\frac{\delta_{ij}}{2\pi g}\ln|x-y| + \text{const} \tag{14.241}$$

In complex cordinates,

$$\langle\phi_i(z,\bar{z})\phi_j(w,\bar{w})\rangle = -\frac{\delta_{ij}}{4\pi g}(\ln(z-w) + \ln(\bar{z}-\bar{w})) \tag{14.242}$$

Then

$$\langle\partial_z\phi_i(z,\bar{z})\partial_w\phi_j(w,\bar{w})\rangle = -\frac{\delta_{ij}}{4\pi g}\frac{1}{(z-w)^2} \tag{14.243}$$

The energy-momentum tensor is

$$T_{\mu\nu} = g(\partial_\mu\phi\cdot\partial_\nu\phi - \frac{1}{2}\delta_{\mu\nu}\partial_\rho\phi\cdot\partial^\rho\phi) \tag{14.244}$$

which differs from our previous convention by a -1 factor.

So

$$T(z) = -2\pi g : \partial\phi\cdot\partial\phi : \tag{14.245}$$

where $\partial\phi = \partial_z\phi(z,\bar{z})$.

Then we have the OPE

$$T(z)\partial\phi^i(w) = -2\pi g : \partial\phi(z)\cdot\partial\phi(z) : \partial\phi^i(w) \tag{14.246}$$

$$= -4\pi g : \partial\phi(z)\cdot\partial\phi(z) : \partial\phi^i(w) \tag{14.247}$$

$$= \frac{\partial\phi^i(z)}{(z-w)^2} \tag{14.248}$$

$$\sim \frac{\partial\phi^i(w)}{(z-w)^2} + \frac{\partial^2\phi^i(w)}{z-w} \tag{14.249}$$

The OPE for energy-momentum tensor with itself is

$$T(z)T(w) = 4\pi^2 g^2 : \partial\phi(z)\cdot\partial\phi(z) :: \partial\phi(w)\cdot\partial\phi(w) : \tag{14.250}$$

$$= 16\pi^2 g^2 : \partial\phi(z)\cdot\partial\phi(z) :: \partial\phi(w)\cdot\partial\phi(w) : + \tag{14.251}$$

$$8\pi^2 g^2 : \partial\phi(z)\cdot\partial\phi(z) :: \partial\phi(w)\cdot\partial\phi(w) : \tag{14.252}$$

$$
\begin{aligned}
&= -4\pi g \frac{:\partial\phi(z)\cdot\partial\phi(w):}{(z-w)^2} + \frac{D/2}{(z-w)^4} && (14.253)\\
&\sim \frac{D/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} && (14.254)
\end{aligned}
$$

Similarly

$$
T(z)\bar{T}(\bar{w}) \sim 0 \tag{14.255}
$$

Generally we have

$$
T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \tag{14.256}
$$

where c is called the central charge. In this model $c = D$.

14.5.2 Transformation of the Energy-Momentum Tensor

Under an infinitesimal conformal transformation

$$
\begin{aligned}
\delta_\epsilon T(w) &= -\frac{1}{2\pi i}\int_C dz\epsilon(z)T(z)T(w) && (14.257)\\
&= -\frac{1}{12}c\partial^3\epsilon(w) - 2T(w)\partial\epsilon(w) - \epsilon(w)\partial T(w) && (14.258)
\end{aligned}
$$

The transformation of $T(w)$ under a finite conformal transformation is

$$
T(w) = \Big(\frac{dw}{dz}\Big)^{-2}\big(T(z) - \frac{c}{12}\{w;z\}\big) \tag{14.259}
$$

where we have induced the Schwarzian derivative

$$
\{w;z\} = \frac{d^3w/dz^3}{dw/dz} - \frac{3}{2}\Big(\frac{d^2w/dz^2}{dw/dz}\Big)^2 \tag{14.260}
$$

Under the global conformal map

$$
w(z) = \frac{az+b}{cz+d} \tag{14.261}
$$

the Schwarzian derivative vanishes. So T is a quasi-primary field.

14.6 Radial Quantization

A quantization starts from a classical model defined on a manifold M, where time direction is ambiguous. We first give a foliation of M by a time function $t : M \to \mathbb{R}$ which defines the time of each point in M. The spatial superplane at $t = t_0$ is $\Sigma_{t_0} = t^{-1}(t_0)$. Then we define a Hilbert space H_{t_0} at each t_0, and a unitary map $U(t_0 \to t_1) : H_{t_0} \to H_{t_1}$ commonly known as the time evolution operator. We also gives a translation between the classical quantity Q defined at t_0 and quantum operator $\hat{Q}$ in H_{t_0}.

We define the correlation function as

$$\langle T\hat{O}(x_1)\hat{O}(x_2)\cdots\hat{O}(x_n)\rangle \tag{14.262}$$

where $\hat{O}(x_i)$ is an operator defined in $H_{t(x_i)}$ and T is the time-ordering product.

The correlation function defined here should coincide with the one defined by the path-integral

$$\langle T\hat{O}(x_1)\hat{O}(x_2)\cdots\hat{O}(x_n)\rangle = \langle O(x_1)O(x_2)\cdots O(x_n)\rangle \tag{14.263}$$

14.6.1 Operator Commutator

Remember we have the Ward identity

$$\sum_{t_i<t}\langle \phi_1\cdots\mathcal{G}\phi_i\cdots\phi_n\rangle = \langle XQ(t)\rangle \tag{14.264}$$

In quantized form, it's

$$\sum_{t_i<t}\langle T\hat{\phi}_1\cdots\mathcal{G}\hat{\phi}_i\cdots\hat{\phi}_n\rangle = \langle T\hat{X}\hat{Q}(t)\rangle \tag{14.265}$$

So

$$\langle T\hat{X}\hat{Q}(t_i+\epsilon)\rangle - \langle T\hat{X}\hat{Q}(t_i-\epsilon)\rangle = \int_{C(x_i)} dx\langle T\hat{X}\hat{j}(x)\epsilon\rangle = \langle T\hat{\phi}_1\cdots\mathcal{G}\hat{\phi}_i\cdots\hat{\phi}_n\rangle \tag{14.266}$$

So

$$[\hat{Q}(t), \hat{\phi}_i(x)] = \mathcal{G}\hat{\phi}_i(x) \tag{14.267}$$

Similarly, from (14.230) we have

$$[\hat{Q}_\epsilon(t), \hat{\phi}_i(x)] = \delta_\epsilon\hat{\phi}_i(x) \tag{14.268}$$

where ϵ is an infinitesimal conformal transformation and $\hat{Q}_\epsilon(t_0) = \int_{t=t_0} dS^\mu \hat{T}_{\mu\nu}(x)\epsilon^\nu(x)$ is the generator of infinitesimal conformal symmetry.

Generally for any local operator $O(x)$, we have

$$[\hat{Q}_\epsilon(t), \hat{O}(x)] = \int_{C(x)} dy\langle T\hat{O}(x)\hat{j}(y)\epsilon\rangle = \delta_\epsilon\hat{O}(x) \tag{14.269}$$

14.6.2 Radial Ordering

First consider a CFT defined on a cylinder with x^0 as the time direction. We have $(x^0, x^1) = (x^0, x^1+2\pi)$. Under a conformal map $z = x_0 + ix_1 \to w = e^z$, as shown in Fig. 14.1, the belt $\mathbb{R}\times[0, 2\pi)$ is mapped to $\mathbb{C}-\{0\}$. The infinite past $(-\infty, x_1)$ is mapped to 0 and the infinite future (∞, x_1) is mapped to ∞.

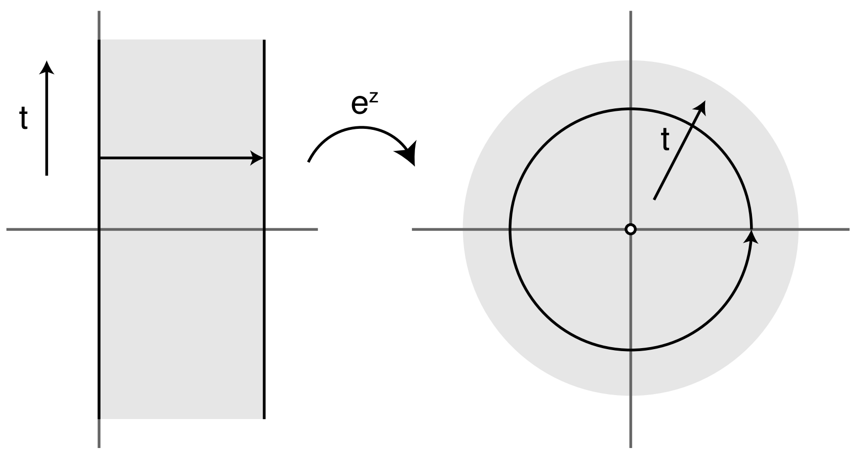


Figure 14.1: Coordinate transformation form cylinder to plane.

Appendix A

Functional derivative

Suppose $F[f]$ is a functional from function $f(x)$ to number. We define the functional derivative $\frac{\delta F}{\delta f(x)}$ by the equation

$$\delta F = \int dx \frac{\delta F}{\delta f(x)} \delta f(x) \tag{A.1}$$

By theorems of functional analysis, functional derivative always exists and is unique.

Suppose $F : (\mathbb{R}^3 \to Fb) \to K$ is a functional from fields defined on space $f : \mathbb{R}^3 \to Fb$ to K. Define functional $F' : (\mathbb{R}^4 \to Fb) \to (\mathbb{R} \to K)$ from field defined on space-time $f' : \mathbb{R}^4 \to Fb$ to function of time $F'[f'] : \mathbb{R} \to K$ as

$$F'[f'](t) = F[f'|_t] \tag{A.2}$$

where $f'|_t$ is the restriction of f' at t.

We can treat t as a parameter, and define the functional derivative of F' at given t.

$$\frac{\delta F'[f'](t)}{\delta f'(x)} = \frac{\delta F[f]}{\delta f(\vec{x})}\Big|_{f=f'|_t} \tag{A.3}$$

And we have

$$\begin{aligned}
\delta F'[f'](t) &= F'[f' + \delta f'](t) - F'[f'](t) && \text{(A.4)}\\
&= F[(f' + \delta f')|_t] - F[f'|_t] && \text{(A.5)}\\
&= \int d^3x \frac{\delta F}{\delta f(\vec{x})}\Big|_{f=f'|_t} \delta f'|_t(\vec{x}) && \text{(A.6)}\\
&= \int d^3x \frac{\delta F'}{\delta f'(x)} \delta f'(x) && \text{(A.7)}
\end{aligned}$$

Examples

1. $f(x)$ can be interpreted as taking the function value at x of the function f. So it is a functional of f to a number, and its functional derivative is

$$\frac{\delta f(x)}{\delta f(y)} = \delta(x - y) \tag{A.8}$$

2. We remember the action is defined in (11.1), which equals to (11.7). Thus its functional derivative is (the variation of ϕ is required to vanish at boundary)

$$\frac{\delta S_\Omega}{\delta\phi(x)} = \left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right] \tag{A.9}$$

3. Suppose $\mathcal{F}[f]$ is a local functional from f to a function of x. We can define the functional

$$F = \int dx\mathcal{F}[f] \tag{A.10}$$

Then

$$\frac{\delta F}{\delta f(x)} = \frac{\partial\mathcal{F}}{\partial f}(x) \tag{A.11}$$

4. Suppose $\mathcal{F}[f, \partial_i f]$ is a local functional of $f(\vec{x})$ and $\partial_i f(\vec{x})$. Define

$$F = \int d^3x\mathcal{F}[f] \tag{A.12}$$

As have explained, we can extend F to F' as a functional of $f(x)$ and itself a function of t as

$$F'[f'](t) = F[f'|_t] = \int d^3x\mathcal{F}[f'|_t] = \int d^3x\mathcal{F}'[f'](t) \tag{A.13}$$

where $\mathcal{F}$ has been extended to $\mathcal{F}'$ as a functional of $f(x)$ and $\partial_i f(x)$ in the natural way (that is, t is treated as a parameter).

Then the functional derivative of F' is

$$\frac{\delta F'}{\delta f'(x)} = \frac{\delta F}{\delta f(\vec{x})}\Big|_{f=f'|_t} = \frac{\partial\mathcal{F}[f]}{\partial f}(\vec{x})\Big|_{f=f'|_t} = \frac{\partial\mathcal{F}'[f']}{\partial f'}(x) \tag{A.14}$$

Appendix B

Poisson Bracket of Conserved Charge

The conserved charge is

$$Q = \int d^3x[\pi_i \partial_j \phi_i (\mathcal{T}x)^j + \mathcal{H}(\mathcal{T}x)^0 - \pi_i(\mathcal{P}\phi)_i] \tag{B.1}$$

Its function derivatives are

$$\begin{aligned}
\frac{\delta Q}{\delta \phi_i} &= -\partial_j(\pi_i(\mathcal{T}x)^j) + \frac{\partial \mathcal{H}}{\partial \phi_i}(\mathcal{T}x)^0 - \partial_j\Big(\frac{\partial \mathcal{H}}{\partial(\partial_j\phi_i)}(\mathcal{T}x)^0\Big) - \pi_j \frac{\partial(\mathcal{P}\phi)_j}{\partial \phi_i} && \text{(B.2)}\\
&= -\partial_j(\pi_i(\mathcal{T}x)^j) - \partial_0\pi_i(\mathcal{T}x)^0 + \partial_j\Big(\frac{\partial \mathcal{H}}{\partial(\partial_j\phi_i)}\Big)(\mathcal{T}x)^0 - \partial_j\Big(\frac{\partial \mathcal{H}}{\partial(\partial_j\phi_i)}(\mathcal{T}x)^0\Big) && \text{(B.3)}\\
&\quad -\pi_j\frac{\partial(\mathcal{P}\phi)_j}{\partial\phi_i} && \text{(B.4)}\\
&= -\partial_\mu(\pi_i(\mathcal{T}x)^\mu) + \pi_i\partial_0(\mathcal{T}x)^0 - \frac{\partial \mathcal{H}}{\partial(\partial_j\phi_i)}\partial_j(\mathcal{T}x)^0 - \pi_j\frac{\partial(\mathcal{P}\phi)_j}{\partial\phi_i} && \text{(B.5)}\\
&= -\partial_\mu(\pi_i(\mathcal{T}x)^\mu) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_i)}\partial_\mu(\mathcal{T}x)^0 - \pi_j\frac{\partial(\mathcal{P}\phi)_j}{\partial\phi_i} && \text{(B.6)}\\
\frac{\delta Q}{\delta \pi_i} &= \partial_j\phi_i(\mathcal{T}x)^j + \frac{\partial \mathcal{H}}{\partial \pi_i}(\mathcal{T}x)^0 - (\mathcal{P}\phi)_i && \text{(B.7)}\\
&= \partial_\mu\phi_i(\mathcal{T}x)^\mu - (\mathcal{P}\phi)_i && \text{(B.8)}
\end{aligned}$$

Suppose we have two conserved charge generated by two infinitesimal symmetry operation $L_{1\epsilon} = (T_{1\epsilon}, P_{1\epsilon})$ and $L_{2\delta} = (T_{2\delta}, P_{2\delta})$ whose Lie algebras are $\mathcal{L}_1 = (\mathcal{T}_1, \mathcal{P}_1)$ and $\mathcal{L}_2 = (\mathcal{T}_2, \mathcal{P}_2)$, their Poisson bracket is

$$\begin{aligned}
\{Q(\mathcal{L}_1), Q(\mathcal{L}_2)\} &= \int d^3x\Big[\frac{\delta Q(\mathcal{L}_1)}{\delta\phi_i}\frac{\delta Q(\mathcal{L}_2)}{\delta\pi_i} - \frac{\delta Q(\mathcal{L}_1)}{\delta\pi_i}\frac{\delta Q(\mathcal{L}_2)}{\delta\phi_i}\Big] && \text{(B.9)}\\
&= \int d^3x\Big[\Big(-\partial_\mu(\pi_i(\mathcal{T}_1x)^\mu) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}\partial_\mu(\mathcal{T}_1x)^0 - \pi_j\frac{\partial(\mathcal{P}_1\phi)_j}{\partial\phi_i}\Big)(\mathcal{S}_2\phi)_i\\
&\quad -(1\leftrightarrow 2)\Big] && \text{(B.10)}\\
&= \int d^3x\Big[-\partial_\mu(\pi_i(\mathcal{T}_1x)^\mu)(\mathcal{S}_2\phi)_i + \partial_\mu(\mathcal{T}_1x)^0\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_i)}(\mathcal{S}_2\phi)_i - \pi_j\frac{\partial(\mathcal{P}_1\phi)_j}{\partial\phi_i}(\mathcal{S}_2\phi)_i
\end{aligned}$$

$$
\begin{aligned}
& -(1 \leftrightarrow 2)\Big] && \text{(B.11)}\\
= & \int d^3x\Big[-\partial_\mu(\pi_i(\mathcal{T}_1x)^\mu)(\mathcal{S}_2\phi)_i + \partial_\mu(\mathcal{T}_1x)^0 j_2^\mu + \partial_\mu(\mathcal{T}_1x)^0\mathcal{L}(\mathcal{T}_2x)^\mu \\
& -\pi_j\frac{\partial(\mathcal{P}_1\phi)_j}{\partial\phi_i}(\mathcal{S}_2\phi)_i - (1\leftrightarrow 2)\Big] && \text{(B.12)}
\end{aligned}
$$

The second term is

$$
\begin{aligned}
\int d^3x\partial_\mu(\mathcal{T}_1x)^0 j_2^\mu &= \int d^3x\partial_\mu((\mathcal{T}_1x)^0 j_2^\mu) && \text{(B.13)}\\
&= \int d^3x\partial_0((\mathcal{T}_1x)^0 j_2^0) && \text{(B.14)}\\
&= \int d^3x\partial_0(\pi_i(\mathcal{T}_1x)^0(\mathcal{S}_2\phi)_i - \mathcal{L}(\mathcal{T}_1x)^0(\mathcal{T}_2x)^0) && \text{(B.15)}
\end{aligned}
$$

Take (B.15) back to (B.12)

$$
\begin{aligned}
\{Q(\mathcal{L}_1), Q(\mathcal{L}_2)\} &= \int d^3x\Big[-\partial_\mu(\pi_i(\mathcal{T}_1x)^\mu)(\mathcal{S}_2\phi)_i + \partial_0(\pi_i(\mathcal{T}_1x)^0(\mathcal{S}_2\phi)_i - \mathcal{L}(\mathcal{T}_1x)^0(\mathcal{T}_2x)^0) \\
&\quad +\partial_\mu(\mathcal{T}_1x)^0\mathcal{L}(\mathcal{T}_2x)^\mu - \pi_j\frac{\partial(\mathcal{P}_1\phi)_j}{\partial\phi_i}(\mathcal{S}_2\phi)_i - (1\leftrightarrow 2)\Big] && \text{(B.16)}\\
&= \int d^3x\Big[\pi_i(\mathcal{T}_1x)^\mu\partial_\mu(\mathcal{S}_2\phi)_i + \partial_\mu(\mathcal{T}_1x)^0\mathcal{L}(\mathcal{T}_2x)^\mu - \pi_j\frac{\partial(\mathcal{P}_1\phi)_j}{\partial\phi_i}(\mathcal{S}_2\phi)_i \\
&\quad -(1\leftrightarrow 2)\Big] && \text{(B.17)}\\
&= \int d^3x\Big[\pi_i\partial_\nu\phi_i\Big((\mathcal{T}_1x)^\mu\partial_\mu(\mathcal{T}_2x)^\nu - (1\leftrightarrow 2)\Big) - \mathcal{L}\Big((\mathcal{T}_1x)^\mu\partial_\mu(\mathcal{T}_2x)^0 \\
&\quad -(1\leftrightarrow 2)\Big) + \pi_j\Big(\frac{\partial(\mathcal{P}_1\phi)_j}{\partial\phi_i}(\mathcal{P}_2\phi)_i - (1\leftrightarrow 2)\Big)\Big] && \text{(B.18)}\\
&= \int d^3x\Big[\pi_i\partial_\nu\phi_i[\mathcal{T}_1x, \mathcal{T}_2x]^\nu - \mathcal{L}[\mathcal{T}_1x, \mathcal{T}_2x]^0 + \pi_j\Big(\frac{\partial(\mathcal{P}_1\phi)_j}{\partial\phi_i}(\mathcal{P}_2\phi)_i \\
&\quad -(1\leftrightarrow 2)\Big)\Big] && \text{(B.19)}
\end{aligned}
$$

where $[\mathcal{T}_1x, \mathcal{T}_2x]$ is the Lie bracket of two vector field.

We also have

$$
\begin{aligned}
& \int d^3x\Big(\pi_j\frac{\partial(\mathcal{P}_1\phi)_j}{\partial\phi_i}(\mathcal{P}_2\phi)_i - (1\leftrightarrow 2)\Big) && \text{(B.20)}\\
= & \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int d^3x\pi_i[(\mathcal{P}_1(\phi+\epsilon\mathcal{P}_2\phi))_i - (\mathcal{P}_1\phi)_i] - (1\leftrightarrow 2) && \text{(B.21)}\\
= & \int d^3x\pi_i\big([\mathcal{P}_1, \mathcal{P}_2]\phi\big)_i && \text{(B.22)}
\end{aligned}
$$

The Lie bracket of $\mathcal{L}_1$ and $\mathcal{L}_2$ is

$$
[\mathcal{L}_1, \mathcal{L}_2] = \frac{d}{d\epsilon}\frac{d}{d\delta}L_{2\delta}^{-1}L_{1\epsilon}L_{2\delta} \qquad \text{(B.23)}
$$

$$\begin{aligned}
&= \frac{d}{d\epsilon}\frac{d}{d\delta}(T_{2\delta}^{-1}, T_{2\delta}^{-1}P_{2\delta}^{-1}T_{2\delta})(T_{1\epsilon}, P_{1\epsilon})(T_{2\delta}, P_{2\delta}) &\text{(B.24)}\\
&= \frac{d}{d\epsilon}\frac{d}{d\delta}(T_{2\delta}^{-1}T_{1\epsilon}T_{2\delta}, T_{2\delta}^{-1}P_{2\delta}^{-1}P_{1\epsilon}T_{1\epsilon}P_{2\delta}T_{1\epsilon}^{-1}T_{2\delta}) &\text{(B.25)}\\
&= ([\mathcal{T}_1, \mathcal{T}_2], [\mathcal{P}_1, \mathcal{T}_2] + [\mathcal{T}_1, \mathcal{P}_2] + [\mathcal{P}_1, \mathcal{P}_2]) &\text{(B.26)}
\end{aligned}$$

We assume that

$$[\mathcal{P}_1, \mathcal{T}_2] = [\mathcal{T}_1, \mathcal{P}_2] = 0 \tag{B.27}$$

So

$$[\mathcal{L}_1, \mathcal{L}_2] = ([\mathcal{T}_1, \mathcal{T}_2], [\mathcal{P}_1, \mathcal{P}_2]) \tag{B.28}$$

From differential geometry

$$[\mathcal{T}_1, \mathcal{T}_2]x = -[\mathcal{T}_1 x, \mathcal{T}_2 x] \tag{B.29}$$

Thus following (B.19)

$$\begin{aligned}
\{Q(\mathcal{L}_1), Q(\mathcal{L}_2)\} &= -\int d^3x \Big[\pi_i \partial_\nu \phi_i ([\mathcal{T}_1, \mathcal{T}_2]x)^\nu - \mathcal{L}([\mathcal{T}_1, \mathcal{T}_2]x)^0 - \pi_i([\mathcal{P}_1, \mathcal{P}_2]\phi)_i\Big] &\text{(B.30)}\\
&= -Q([\mathcal{L}_1, \mathcal{L}_2]) &\text{(B.31)}
\end{aligned}$$

Appendix C

Bogliubov Transformation

The Bogliubov Transformation is a transformation of a quadratic boson Hamiltonian into one without abnormal paring (the aa and $a^\dagger a^\dagger$ terms).

We consider a Hamiltonian with 2 bosons

$$H = M(ab + b^\dagger a^\dagger) + \frac{N_1}{2}(aa^\dagger + a^\dagger a) + \frac{N_2}{2}(bb^\dagger + b^\dagger b) \tag{C.1}$$

It can be re-expressed as

$$H = \frac{1}{2}(a^\dagger \ b^\dagger \ a \ b) \begin{pmatrix} N_1 & 0 & 0 & M \\ 0 & N_2 & M & 0 \\ 0 & M & N_1 & 0 \\ M & 0 & 0 & N_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ a^\dagger \\ b^\dagger \end{pmatrix} \tag{C.2}$$

Define the following transformation

$$\begin{pmatrix} c \\ d \\ c^\dagger \\ d^\dagger \end{pmatrix} = T \begin{pmatrix} a \\ b \\ a^\dagger \\ b^\dagger \end{pmatrix} \tag{C.3}$$

T is well-defined iff it's of the form

$$\begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix} \tag{C.4}$$

And c and d satisfy canonical commuting relation iff

$$T^\dagger \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{C.5}$$

We want to use T to diagonalize the matrix in (C.2). We may try

$$T = \begin{pmatrix} u & 0 & 0 & -v \\ 0 & u & -v & 0 \\ 0 & -v & u & 0 \\ -v & 0 & 0 & u \end{pmatrix} \tag{C.6}$$

where $u, v \in \mathbb{R}$. (C.5) requires $u^2 - v^2 = 1$. It's easy to see

$$T^{-1} = \begin{pmatrix} u & 0 & 0 & v \\ 0 & u & v & 0 \\ 0 & v & u & 0 \\ v & 0 & 0 & u \end{pmatrix} \tag{C.7}$$

Then

$$H = \frac{1}{2}(c^\dagger \; d^\dagger \; c \; d) T^{\dagger -1} \begin{pmatrix} N_1 & 0 & 0 & M \\ 0 & N_2 & M & 0 \\ 0 & M & N_1 & 0 \\ M & 0 & 0 & N_2 \end{pmatrix} T^{-1} \begin{pmatrix} c \\ d \\ c^\dagger \\ d^\dagger \end{pmatrix} \tag{C.8}$$

$$= \frac{1}{2}(c^\dagger \; d^\dagger \; c \; d) \begin{pmatrix} A & 0 & 0 & C \\ 0 & B & C & 0 \\ 0 & C & A & 0 \\ C & 0 & 0 & B \end{pmatrix} \begin{pmatrix} c \\ d \\ c^\dagger \\ d^\dagger \end{pmatrix} \tag{C.9}$$

where

$$A = N_1 u^2 + N_2 v^2 + 2Muv \tag{C.10}$$
$$B = N_1 v^2 + N_2 u^2 + 2Muv \tag{C.11}$$
$$C = M(u^2 + v^2) + (N_1 + N_2)uv \tag{C.12}$$

To meet our purpose, we require $C = 0$. This, together with the constraint $u^2 - v^2 = 1$, leads to

$$u^2 = \frac{1}{2} + \frac{N_1 + N_2}{2\sqrt{(N_1 + N_2)^2 - 4M^2}} \tag{C.13}$$
$$v^2 = -\frac{1}{2} + \frac{N_1 + N_2}{2\sqrt{(N_1 + N_2)^2 - 4M^2}} \tag{C.14}$$
$$uv = -\frac{M}{\sqrt{(N_1 + N_2)^2 - 4M^2}} \tag{C.15}$$

So finally

$$H = \frac{1}{4}(N_1 - N_2 + \sqrt{(N_1 + N_2)^2 - 4M^2})(cc^\dagger + c^\dagger c) + \frac{1}{4}(N_2 - N_1 + \sqrt{(N_1 + N_2)^2 - 4M^2})(dd^\dagger + d^\dagger d) \tag{C.16}$$

Appendix D

Some differential geometry

Gauss's theorem is

$$\int_M dx \partial_\mu F^\mu = \int_{\partial M} dS_\mu F^\mu \tag{D.1}$$

where dS_μ is the norm vector at ∂M pointing outwards.

In coordinate (x^0, x^1), the metric and antisymmetry tensor is

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \epsilon_{\mu\nu} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \epsilon^{\mu\nu} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{D.2}$$

In coordinate $(z, \bar{z})$, the metric and antisymmetry tensor is

$$g_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \ g^{\mu\nu} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}, \ \epsilon_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{2}i \\ -\frac{1}{2}i & 0 \end{pmatrix}, \ \epsilon^{\mu\nu} = \begin{pmatrix} 0 & -2i \\ 2i & 0 \end{pmatrix} \tag{D.3}$$

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