Topology

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Part I

Set Thoery

Chapter 1

Axioms of Zermelo-Fraenkel

The axioms of Zermelo-Fraenkel (ZF) admits one kind of objects, namely **sets**. Some times we also call a set of sets a **family** of sets. We introduce the informal notion of **class**, describe the objects that satify a fomula. A class that is not a set is called a **proper class**.

The axioms of ZF together with the axiom of choice include:

Axiom 1.0.1 (ZFC)**.**

1. **Axiom of Extensionality**: *If X and Y have the same elements, then $X = Y$.*
2. **Axiom of Pairing**: *For any a and b there exists a set $\{a, b\dagger\}$ that contains exactly a and b.*
3. **Axiom Schema of Separation**: *If $\phi(u, p)$ is a property with parameter p, then for any X and p, there exists a set $Y = \{u \in X | \phi(u, p)\}$ that contains all those $u \in X$ that have the property ϕ.*
4. **Axiom of Union**: *For any X there exists a set $Y = \bigcup X$, the union of all elements of X.*
5. **Axiom of Power Set**: *For any X there is exists a set $Y = P(X)$, the set of all subsets of X.*
6. **Axiom of Infinity**: *There exists an infinite set.*
7. **Axiom of Schema of Replacement**: *If a class F is a function, then for any X there exists a set $Y = F(X) = \{F(x)|x \in X\}$.*
8. **Axiom of Regularity**: *Every nonempty X has an element disjoint from X.*
9. **Axiom of Choice**: *Every family of nonempty sets has a choice function.*

The Axioms 1-7 are explained in detail as follows.

1.1 Axiom of Extensionality

Axiom 1.1.1. *If X and Y have the same elements, then $X = Y$:*

$$\forall u(u \in X \leftrightarrow u \in Y) \to X = Y \tag{1.1}$$

1.2 Axiom of Pairing

Axiom 1.2.1. *For any a and b there exists a set c that contains exactly a and b:*

$$\forall a \forall b \exists c \forall x (x \in c \leftrightarrow x = a \vee x = b) \tag{1.2}$$

By extensionality, the set c is unique.

Definition 1.2.2. *We can define the* **pair**

$$\{a, b\} = \text{the unique } c \text{ such that } \forall x (x \in c \leftrightarrow x = a \vee x = b) \tag{1.3}$$

The singleton $\{a\}$ is the set $\{a, a\}$.

Definition 1.2.3 (Kuratowski)**.** *We can define the* **ordered pair**

$$(a, b) = \{\{a\}, \{a, b\}\} \tag{1.4}$$

We can further define n-tuples $(a_1, \ldots, a_n) = (\ldots((a_1, a_2), a_3) \ldots, a_n)$

Theorem 1.2.4. *Let $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ be two n-tuples.*

$$(a_1, \ldots, a_n) = (b_1, \ldots, b_n) \leftrightarrow (a_1 = b_1 \wedge \cdots \wedge a_n = b_n) \tag{1.5}$$

Proof. If $(a, b) = (c, d)$, then $\{a\} = \{c\}$ or $\{a\} = \{c, d\}$. If $\{a\} = \{c, d\}$, then $a = c = d$ and $\{a, b\} = \{c\} = \{c, d\}$. So $a = b = c = d$. If $\{a\} = \{c\}$, then $\{a, b\} = \{c, b\} = \{c, d\}$. So $b = d$. In conclusion $(a, b) = (c, d) \rightarrow a = b \wedge c = d$. When $n > 2$, use induction. □

1.3 Axiom Schema of Separation

Axiom 1.3.1. *Let $\phi(u, p)$ be a formula. For any X and p, there exists a set $Y = \{u \in X | \phi(u, p)\}$:*

$$\forall X \forall p \exists Y \forall u (u \in Y \leftrightarrow u \in X \wedge \phi(u, p)) \tag{1.6}$$

Definition 1.3.2. *We define the* **intersection** *of X and Y as*

$$X \cap Y = \{u \in X | u \in Y\} \tag{1.7}$$

Clearly $X \cap Y = Y \cap X$

Definition 1.3.3. *Let C be a nonempty class of sets, we have the set called the intersection of C*

$$\bigcap C = \{u \in X_0 | u \in X \text{ for every } X \in C\} \tag{1.8}$$

where $X_0 \in C$. Clearly $\bigcap C$ is independent on the choice of X_0.

Definition 1.3.4. *We define the* **difference** *of X and Y as*

$$X - Y = \{u \in X | u \notin Y\} \tag{1.9}$$

1.4 Axiom of Union

Axiom 1.4.1. *For any X there exists a set $Y = \bigcup X$, the union of elements of X:*

$$\forall X \exists Y \forall u (u \in Y \leftrightarrow (\exists z \in X) u \in z) \tag{1.10}$$

Now we can define

$$X \cup Y = \bigcup \{X, Y\},\ X \cup Y \cup Z = (X \cup Y) \cup Z,\ \ldots \tag{1.11}$$

and

$$\{a_1, \ldots, a_n\} = \{a_1\} \cup \cdots \cup \{a_n\} \tag{1.12}$$

Definition 1.4.2. *We define the* **symmetric difference** *of X and Y as*

$$X \triangle Y = (X - Y) \cup (Y - X) \tag{1.13}$$

1.5 Axiom of Power Set

Definition 1.5.1. *A set U is a* **subset** *of X, written as $U \subseteq X$, if*

$$\forall z (z \in U \to z \in X) \tag{1.14}$$

If $U \subseteq X$ and $U \neq X$, then U is a **proper subset** *of X.*

Axiom 1.5.2. *For any X there is exists a set Y:*

$$\forall X \exists Y \forall u (u \in Y \leftrightarrow u \subseteq X). \tag{1.15}$$

Clearly such set is unique for any X. It's called the **power set** *of X, denoted by $P(X)$.*

Definition 1.5.3. *The* **Cartesian product** *of X and Y is the set*

$$X \times Y = \{(x, y) \in PP(X \cup Y) | x \in X \wedge y \in Y\} \tag{1.16}$$

Similarly let

$$X_1 \times X_2 \times \cdots \times X_n = \{(x_1, \ldots, x_n) \in \underbrace{P \ldots P}_{n}(X_1 \cup \cdots \cup X_n) | x_1 \in X_1 \wedge \cdots \wedge x_n \in X_n\} \tag{1.17}$$

and let

$$X^n = \underbrace{X \times \cdots \times X}_{n} \tag{1.18}$$

Definition 1.5.4. *We define the* **disjoint union** *of two (not necessarily disjoint) sets A and B to be*

$$A \sqcup B = (A \times 0) \cup (B \times 1) \tag{1.19}$$

Definition 1.5.5. *A* **relation** *R on set X is a subset R of $X \times X$. We use xRy to represent that $(x, y) \in R$.*

Definition 1.5.6. *A relation $\sim$ on X is called*

1. **reflective** *iff $\forall x \in X(x \sim x)$;*
2. **transitive** *iff $\forall x, y, z \in X(x \sim y \wedge y \sim z \to x \sim z)$;*
3. **symmetric** *iff $\forall x, y \in X(x \sim y \leftrightarrow y \sim x)$;*
4. **antisymmetric** *iff $\forall x, y \in X(x \sim y \wedge y \sim x \to x = y)$.*

Definition 1.5.7. *A relation $\sim$ on X is called an* **equivalence relation** *if it's reflective, transitive and symmetric.*

Definition 1.5.8. *Let $\sim$ be an equivalence relation on X. We define the* **equivalence class** *on X to be $X/\sim = \{S \in P(X) | (S \neq \emptyset) \wedge ((\forall b \in X)((\exists a \in S) a \sim b \to b \in S)) \wedge ((\forall a, b \in S) a \sim b)\}$. Cleary $X/\sim$ is a family of mutually disjoint sets and $\bigcup(X/\sim) = X$. For each $S \in (X/\sim)$ and each $s \in S$, we may denote S by $[s]$.*

Definition 1.5.9. *A* **function***(also called a* **map***) f from a set X to a set Y is a subset S of $X \times Y$, such that $\forall x \in X \exists y \in Y((x, y) \in S)$ and $\forall x \in X((x, y) \in S \wedge (x, z) \in S \to y = z)$. For each $x \in X$ we use $f(x)$ to represent the element in Y such that $(x, f(x)) \in S$. X is called the* **domain** *of f. We use $f(X)$ to represent the set $\{y \in Y | \exists x \in X(f(x) = y)\}$, called the* **image** *of f.*

Definition 1.5.10. *Let f be a function from X to Y. If $f(X) \subseteq S \subseteq Y$, then f can be viewed as a function from X to S. Let $T \subseteq X$. f* **restricted on** *T is defined as $f|_T = f \cap (T \times Y)$.*

Definition 1.5.11. *Let f be a bijective function from X to Y. f is* **injective** *if $f(x) = f(y) \to x = y$. f is* **surjective** *if $f(X) = Y$. f is* **bijective** *if its injective and surjective.*

Definition 1.5.12. *Let f be a bijective function from X to Y. We define the* **inverse** *of f as $f^{-1} = \{(a, b) \in Y \times X | (b, a) \in f\}$.*

Definition 1.5.13. *We denote the set of all functions from A to B by B^A.*

We can also define relation and function on classes.

Definition 1.5.14. *We define an* **indexed set** *$\{X_i | i \in I\}$ to be a function X from the index set I to a family of sets. $X(i)$ is written as X_i. We define $\bigcup_{i \in I} X_i = \bigcup X(I)$, and $\bigcap_{i \in I} X_i = \bigcap X(I)$. We define $\bigsqcup_{i \in I} X_i = \bigcup_{i \in I}(X_i \times i)$*

1.6 Axiom of Infinity

Axiom 1.6.1. *There exists a set.*

Definition 1.6.2. *Let a be a set (which exists). We define the empty set as*

$$\emptyset = \{x \in a | x \neq x\} \tag{1.20}$$

By extensionality, the empty set is unique.

Definition 1.6.3. *A set S is* **inductive** *iff*

$$\emptyset \in S \land (\forall x \in S) x \cup \{x\} \in S \tag{1.21}$$

Axiom 1.6.4. *There exists an inductive set.*

1.7 Axiom Schema of Replacement

Axiom 1.7.1. *If a class F is a function, then for every set X, $F(X)$ is a set:*

$$\forall p(\forall x \forall y \forall z(\phi(p, x, y) \land \phi(p, x, z) \to y = z) \to \forall X \exists Y \forall y(y \in Y \leftrightarrow (\exists x \in X)\phi(x, y))) \tag{1.22}$$

where $\phi(p, x, y)$ is a formula with parameter p.

Chapter 2

Ordering

Definition 2.0.1. *A* **pre-order** $\leq$ *is a reflective and transitive binary relation.*

Definition 2.0.2. *A* **partial order** $\leq$ *is a pre-order that is antisymmetric.*

Definition 2.0.3. *A* **linear order** $\leq$ *on* X *is a partial order that* $\forall x, y \in X : x \leq y \vee y \leq x$.

Definition 2.0.4. *Let* X *be a linearly ordered set, and* $a < b$ *belong to* X. *We define the intervals between* a *and* b *as:*

$$(a, b) = \{x \in X | a < x < b\} \tag{2.1}$$
$$[a, b) = \{x \in X | a \leq x < b\} \tag{2.2}$$
$$(a, b] = \{x \in X | a < x \leq b\} \tag{2.3}$$
$$[a, b] = \{x \in X | a \leq x \leq b\} \tag{2.4}$$

Furthermore, we define

$$(a, \infty) = \{x \in X | x > a\} \tag{2.5}$$
$$[a, \infty) = \{x \in X | x \geq a\} \tag{2.6}$$
$$(-\infty, a) = \{x \in X | x < a\} \tag{2.7}$$
$$(-\infty, a] = \{x \in X | x \leq a\} \tag{2.8}$$

Definition 2.0.5. *Let* X *be a partially ordered set.*

1. a *is the* **largest** *element in* X *iff* $(\forall x \in X) x \leq a$.

2. a *is the* **least** *element in* X *iff* $(\forall x \in X) a \leq x$.

3. a *is a* **maximal** *element in* X *iff* $(\nexists x \in X) a < x$.

4. a *is a* **minimal** *element in* X *iff* $(\nexists x \in X) x < a$.

Definition 2.0.6. *Let* X *be a partially ordered set, and* $S \subseteq X$.

1. a *is an* **upper bound** *of* S *iff* $\forall x \in X : x \leq a$.

2. *a is a* **lower bound** *of S iff $\forall x \in X : a \leq x$.*

3. *a is the* **supremum** *of S iff its the least upper bound of S.*

4. *a is the* **infimum** *of S iff its the greatest lower bound of S.*

Definition 2.0.7. *Let f be a map form a partial ordered set X to a partial ordered set Y. If $a \leq b \to f(a) \leq f(b)$, then f is called* **order-preserving***(or* **weakly increasing** *or* **monotone***).*

Definition 2.0.8. Poset *is the category whose objects are partially ordered sets and whose morphisms are monotone maps.*

Definition 2.0.9. *A* **well-order** *$\leq$ is linear order such that every non-empty subset has a smallest element.*

From now on, we assume the morphisms between well-ordered sets to be those in the category **Poset**.

Lemma 2.0.10. *Let X be a well-ordered set and $f : X \to X$ be an injective morphism. Then $f(x) \geq x$.*

Proof. Let $S = \{x \in X | f(x) < x\}$. If S is nonempty, let s be the least element in S. Since $f(s) < s$, $f(f(s)) < f(s)$. Then $f(s) \in S$, a contradiction. □

Corollary 2.0.11. *Let X be a well-ordered set in and $f : X \to X$ be an isomorphism. Then $f = id$.*

Definition 2.0.12. *Let X be a well-ordered set and $u \in X$. Then $X_{<u} = \{x \in X | x < u\}$ is called an* **initial segment** *of X given by u.*

Lemma 2.0.13. *Let X be a well-ordered set in. X is not isomorphic to an initial segment of X.*

Theorem 2.0.14. *Let X and Y be well-ordered sets. Then one of the following three cases holds:*

1. *X is isomorphic to Y.*

2. *X is isomorphic to an initial segment of Y.*

3. *An initial segment of X is isomorphic to Y.*

Proof. We denote Let $F = \{(x, y) \in X \times Y | X_{<x}$ is isomorphic to $Y_{<y}\}$. From Lem.2.0.13, $(x_1, y) = (x_2, y) \to x_1 = x_2$. If $(x, y) \in F$, $(x', y') \in F$ and $x < x'$, let h be an isomorphism form $X_{<x'}$ to $Y_{<y'}$. Then it's easy to see that $h|_{X_{<x}}$ is an isomorphism form $X_{<x}$ to $Y_{<h(x)}$. So $(x, h(x)) \in F$. So $y = h(x) < y'$.

Let $X' = \{x \in X | (\exists y \in Y)(x, y) \in F\}$ and $Y' = \{y \in Y | (\exists x \in X)(x, y) \in F\}$. It's easy to see that F is an isomorphism from X' to Y'. If $X = X'$ and $Y = Y'$, then the 1st case holds. If $X \neq X'$ and $Y = Y'$, let $u = \inf(X - X')$. It's easy to see that $X' = X_{<u}$. Thus the 3rd case holds. If $X = X'$ and $Y \neq Y'$, similarly the 2nd case holds. If $X \neq X'$ and $Y \neq Y'$, let $X' = X_{<u}$ and $Y' = Y_{<v}$. Thus $(u, v) \in F$, which leads to a contradiction. □

This theorem shows that any two well-ordered sets can be compared.

Definition 2.0.15. *Let A and B be two partially ordered sets. The* **lexicographic order** $\leq_{lex}$ *on* $A \times B$ *is defined as*

$$(a,b) \leq_{lex} (c,d) \leftrightarrow a < c \vee (a = c \wedge b \leq d) \tag{2.9}$$

Definition 2.0.16. *Let A be a linearly ordered set. The* **canonical order** $\leq_{can}$ *on $A \times A$ is defined as*

$$(a,b) \leq_{can} (c,d) \leftrightarrow \max(a,b) < \max(c,d) \vee (\max(a,b) = \max(c,d) \wedge (a,b) \leq_{lex} (c,d)) \tag{2.10}$$

Lemma 2.0.17. *If A and B are two linearly ordered sets, $\leq_{lex}$ is a linear order on $A \times B$. If A and B are two well-ordered sets, $\leq_{lex}$ is a well-order on $A \times B$.*

Lemma 2.0.18. *If A is a well-ordered set, $\leq_{can}$ is a well-order on $A \times A$.*

Chapter 3

Ordinal Numbers

Definition 3.0.1. *A set T is* **transitive** *if every element of T is a subset of T.*

Definition 3.0.2. *A set is an* **ordinal number** *if it is transitive and (strictly) well-ordered by $\in$. The class of all ordinal numbers are called* **Ord***.*

Lemma 3.0.3.

1. *$0 = \emptyset$ is an ordinal.*
2. *Let α be an ordinal, $\gamma \in \beta \in \alpha \to \gamma \in \alpha$.*
3. *Let α be an ordinal, $\alpha \notin \alpha$.*
4. *An element of an ordinal is an ordinal.*
5. *Let $\alpha \neq \beta$ be ordinals. $\alpha \subseteq \beta \leftrightarrow \alpha \in \beta$.*
6. *Let $\alpha \neq \beta$ be ordinals. Either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.*
7. *A transitive set of ordinals is an ordinal.*

Proof.

1. Clearly.
2. Clearly.
3. Since α is strictly well-ordered by $\in$.
4. Let A be an ordinal and $a \in A$. For each $x \in a$, $y \in x \wedge x \in a \to y \in a$. So $x \subseteq a$. Since $a \subseteq A$, clearly a is well-ordered by $\in$.
5. If $\alpha \subset \beta$, let γ be the least element in $\beta - \alpha$. $x \in \gamma \in \beta \Rightarrow x \in \beta \wedge x < \gamma \Rightarrow x \in \alpha$. So $\gamma \subseteq \alpha$. If $\exists x \in \alpha - \gamma$, then $x > \gamma$ ($x \neq \gamma$ since $\gamma \notin \alpha$). So $\gamma \in x \in \alpha \to \gamma \in \alpha$, a contradiction. So $\alpha = \gamma$. The proof can be better understood with Fig.3.1.

6. Let $\gamma = \alpha \cap \beta$. Clearly γ is an ordinal. We have $\gamma = \alpha$ or $\gamma = \beta$. Otherwise, $\gamma \in \alpha \wedge \gamma \in \beta \Rightarrow \gamma \in \gamma$, which leads to a contradiction.

7. It follows form 3,4.

□

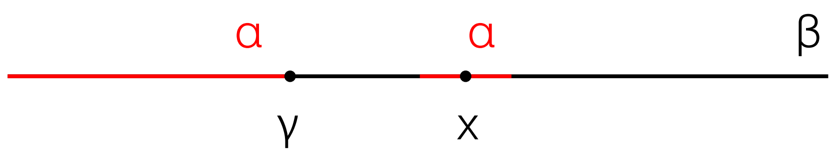

Figure 3.1: Relation of α, β, γ and x.

Definition 3.0.4. *Let α and β be ordinals. We define $\alpha < \beta$ iff $\alpha \in \beta$.*

Corollary 3.0.5. *$<$ is a (strict) linear ordering of the class Ord.*

Lemma 3.0.6. *If C is a nonempty class of ordinals, then $\bigcap C$ is the least element of C. If C is a nonempty set of ordinals, then $\bigcup C$ is the supremum of C.*

Corollary 3.0.7. *Each set of ordinals has an upper bound in Ord.*

Lemma 3.0.8. *Ord is a proper class.*

Proof. Otherwise, $Ord \in Ord$. □

Lemma 3.0.9. *Let α be an ordinal, $\alpha \cup \{\alpha\}$ is the least ordinal larger that α.*

Definition 3.0.10. *For each ordinal α, we define $\alpha + 1 = \alpha \cup \{\alpha\}$.*

Theorem 3.0.11. *Every well-ordered set is isomorphic to a unique ordinal.*

Proof. For each well-ordered set X. Let $X' = \{x \in X | \exists \text{ an ordinal } \alpha(X_{<x} \text{ is isomorphic to } \alpha)\}$. For each $x \in X'$, let $F(x)$ be the ordinal isomorphic to $X_{<x}$, which is unique due to Lem.2.0.13. By the replacement axiom, $F(X')$ is a set. Let γ be the least ordinal strictly larger than all of $F(X')$. If $X' \neq X$, let x_0 be the least element in $X - X'$. It's easy to see that F is an isomorphism form $X_{<x_0}$ to γ. So $x_0 \in X'$, a contradiction. So $X' = X$. It's easy to see that F is an isomorphism form X to γ. □

Definition 3.0.12. *The* **order type** *of a well-ordered set is the ordinal it is isomorphic to.*

3.1 Successor Ordinal and Limit Ordinal

Definition 3.1.1. *If $\alpha = \beta + 1$, then α is called a* **successor ordinal**. *Otherwise α is called an* **limit ordinal**.

Lemma 3.1.2. *There exists an inductive ordinal.*

Proof. Let S be an inductive set. Let $S' = \{s \in S | s \text{ is an ordinal }\}$. S' is nonempty since $\emptyset \in S'$. It's easy to see that S' is inductive. Let $\alpha = \bigcup S'$. It's easy to see that α is an inductive ordinal. $\square$

Lemma 3.1.3. *An ordinal is an inductive ordinal iff it's a nonzero limit ordinal.*

Proof. Let α be an inductive ordinal. If $\alpha = \beta+1$, then it's east to see $\alpha = \beta+1 \in \alpha$, a contradiction.

Let α be a nonzero limit ordinal. For each $\beta < \alpha$, $\beta + 1 < \alpha$. $\square$

Corollary 3.1.4. *There exists a nonzero limit ordinal.*

Definition 3.1.5. *We denote the least nonzero limit ordinal ω(or $\mathbb{N}$). The ordinals less that ω (elements of $\mathbb{N}$) are called* **finite ordinals** *or* **natural numbers**. *The ordinals larger than or equal to ω are called* **infinite ordinals**. *Specifically,*

$$0 = \emptyset,\ 1 = 0 + 1,\ 2 = 1 + 1,\ \ldots \tag{3.1}$$

Lemma 3.1.6. *Let α be an infinite ordinal, the order type of $\alpha \times \alpha$ with canonical order is α.*

Proof. Let $\Gamma(\alpha)$ be the order type of $\alpha \times \alpha$. Clearly $\Gamma(\alpha) \geq \alpha$ and $\Gamma(\omega) = \omega$. Let β be the least ordinal (if exists) in $\{\alpha | \Gamma(\alpha) > \alpha\}$. Let f be the isomorphism $\beta \times \beta \mapsto \Gamma(\beta)$. It's easy to see that $f(\gamma, \gamma) = \Gamma(\gamma) = \gamma$ for each $\omega \leq \gamma < \beta$. So $f(\gamma, \gamma') \leq \max(\gamma, \gamma') < \beta$ for all $\omega \leq \gamma, \gamma' < \beta$. So $\Gamma(\beta) = f(\beta \times \beta) \leq \beta$, a contradiction. $\square$

Definition 3.1.7. *A* **sequence** *is a function f whose domain is the set $\mathbb{N}$. An* **α-sequence** *is a function f whose domain is an ordinal α. $f(\beta)$ in a sequence is usually denoted by f_β.*

Definition 3.1.8. *Let a_ξ be a monotone α-sequence. For each $\beta < \alpha$ define*

$$\lim_{\xi \to \beta} = \sup\{a_\xi | \xi < \beta\} \tag{3.2}$$

3.2 Transfinite induction

Theorem 3.2.1. *Let C be a class of ordinals and assume that*

1. $0 \in C$.
2. $\alpha \in C$ *implies that* $\alpha + 1 \in C$.
3. *If α is a nonzero limit ordinal, $\beta \in C$ for all $\beta < \alpha$ implies that $\alpha \in C$.*

Then C is the class of all ordinals.

Proof. If not, consider the least ordinal not in C. $\square$

3.3 Ordinal Arithmetic

Definition 3.3.1. *(Addition) For all ordinal number* α

1. $\alpha + 0 = \alpha$
2. $\alpha + (\beta + 1) = (\alpha + \beta) + 1$ *for each* β
3. $\alpha + \beta = \lim_{\gamma \to \beta}(\alpha + \gamma)$ *for each limit* β

Definition 3.3.2. *(Multiplication) For all ordinal number* α

1. $\alpha \cdot 0 = 0$
2. $\alpha \cdot (\beta + 1) = \alpha \cdot \beta + 1$ *for each* β
3. $\alpha \cdot \beta = \lim_{\gamma \to \beta}(\alpha \cdot \gamma)$ *for each limit* β

Definition 3.3.3. *(Exponentiation) For all ordinal number* α

1. $\alpha^0 = 1$
2. $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$ *for each* β
3. $\alpha^\beta = \lim_{\gamma \to \beta} \alpha^\gamma$ *for each limit* β

Lemma 3.3.4.

1. $\forall\alpha\forall\beta\forall\gamma(\beta < \gamma \to \beta + \alpha < \gamma + \alpha)$
2. $\forall\alpha\forall\beta(\alpha < \beta \to \exists!\gamma(\alpha + \gamma = \beta))$
3. $\forall\alpha\forall\beta\forall\gamma(\alpha > 0 \wedge \beta < \gamma \to \alpha \cdot \beta < \alpha \cdot \gamma)$
4. $\forall\alpha\forall\gamma(\alpha > 0 \to \exists!\beta\exists!\rho(\rho < \alpha \wedge \gamma = \alpha \cdot \beta + \rho))$
5. $\forall\alpha\forall\beta\forall\gamma(\alpha > 1 \wedge \beta < \gamma \to \alpha^\beta < \alpha^\gamma)$

Lemma 3.3.5.

1. $\forall\alpha\forall\beta\forall\gamma(\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma)$
2. $\forall\alpha\forall\beta\forall\gamma(\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma)$

3.4 Cardinality and Uncountable Ordinal

Definition 3.4.1. *Two sets X, Y are called* **equipotent** *if there's is a bijection map between them.*

Theorem 3.4.2 (Cantor-Bernstein)**.** *If there's an injective map from X to Y, and there's also an injective map from Y to X, then X are Y are equipotent.*

Proof. Let $f_1 : X \mapsto Y$ and $f_2 : Y \mapsto X$ be injective maps. We define by induction for all $n \in \mathbb{N}$:

$$\begin{aligned} A_0 &= A &(3.3)\\ A_{n+1} &= f_2 \circ f_1(A_n) &(3.4)\\ B_0 &= f_2(B) &(3.5)\\ B_{n+1} &= f_2 \circ f_1(B_n) &(3.6)\end{aligned}$$

It's easy to see that $A_0 \supseteq B_0 \supseteq A_1 \supseteq B_1 \supseteq \cdots$.

Define $g : X \mapsto Y$ as follows

$$g(x) = \begin{cases} f_1(x) & x \in A_n - B_n \text{ for some } n \\ f_2^{-1}(x) & \text{otherwise} \end{cases} \tag{3.7}$$

It's easy to see that g is a bijection. □

Corollary 3.4.3. *Let $\alpha < \beta$ be two equipotent ordinals. Then for any γ such that $\alpha \leq \gamma \leq \beta$, α is equipotent to γ.*

Lemma 3.4.4. *Let α be an finite ordinal. Then α is the only ordinal that is equipotent to α.*

Proof. It's easy to prove that for any finite ordinal α, α is not equipotent to $\alpha + 1$.

For any finite ordinal α, let β any ordinal that is larger that α and is equipotent to α. If such β exists, clearly $\beta > \alpha + 1 > \alpha$. So α is equipotent to $\alpha + 1$, a contradiction. So there's no ordinal lager than α that is equipotent to α. Symmetrically, there's no ordinal smaller than α that is equipotent to α. □

Definition 3.4.5. *A set is called* **finite** *if it's equipotent to a finite ordinal, otherwise it's called* **infinite***. A set is called* **countable** *if it's equipotent to ω. A set is called* **at most countable** *if it's finite or countable, otherwise it's called* **uncountable***.*

Sometimes we simply abbreviate "at most countable" by "countable".

Theorem 3.4.6 (Hartogs)**.** *Let α be an ordinal, there's a least ordinal $\beta > \alpha$ and not equipotent to α.*

Proof. Let W be $\{$well orderings of $a \subset \alpha\}$. W is a set since it's a subset of $P(\alpha \times \alpha)$. Let β be the set of order types of well-orderings in W.

First we prove that β is an ordinal $> \alpha$ and not equipotent to α. For each $\gamma \in \beta$, let $f : b \mapsto \gamma$ be an isomorphism, where $b \subseteq \alpha$. For each $\delta \in \gamma$, $\delta \subset \gamma$. Then $f \cap f^{-1}(\delta) \times \delta$ is an isomorphism from $f^{-1}(\delta)$ to δ. So $\delta \in \beta$. So β is a transitive set of ordinals. So β is an ordinal. Clearly $\beta > \alpha$. If β is equipotent to α, then β induce a well-ordering on α. Thus $\beta \in \beta$, a contradiction.

Next we prove that β is the least ordinal with this property. For each γ such that $\alpha < \gamma < \beta$, there's an injective map $\gamma \mapsto \alpha$ and an injective map $\alpha \mapsto \gamma$. So γ and α are equipotent. □

Corollary 3.4.7. *There exists a least uncountable ordinal, denoted as ω_1. We define $\Omega = \omega_1 + 1$*

Chapter 4

Axiom of Regularity

Axiom 4.0.1. *Every nonempty set S has an element disjoint from S:*

$$\forall S(S \neq \emptyset \to (\exists x \in S) S \cap x = \emptyset) \tag{4.1}$$

This axiom can be reformulated as: every nonempty set has an $\in$-minimal element.

Lemma 4.0.2. *Let S be a set. $S \notin S$, and therefore $S \neq \{S\}$.*

Proof. Consider $\{S\}$. $\square$

Lemma 4.0.3. *For every set S there exists a smallest transitive set $TC(S) \supseteq S$ called the* **transitive closure** *of S.*

Proof. We define by induction

$$S_0 = S,\ S_{n+1} = \bigcup S_n \tag{4.2}$$

and let $TC(S) = \bigcup_n S_n$ $\square$

Lemma 4.0.4. *Every nonempty class C has an $\in$-minimal element.*

Proof. Let $S \in C$ be arbitrary. If S is not an $\in$-minimal element of C, $(\exists S' \in C) S' \in S$. Let $X = TC(S) \cap C$. Then $S' \in X \neq \emptyset$. By the axiom of regularity, there is $x \in X$ such that $x \cap X = \emptyset$. Since $x \subseteq TC(S)$, $x \subseteq TC(S) - C$. So $x \cap C = \emptyset$. Hence x is a minimal element of C. $\square$

Corollary 4.0.5. *It's impossible to have a sequence S_i of sets such that $S_0 \ni S_1 \ni S_2 \ni \cdots$.*

Definition 4.0.6. *We define by induction that*

$$V_0 = \emptyset \tag{4.3}$$
$$V_{\alpha+1} = P(V_\alpha) \tag{4.4}$$
$$V_\alpha = \bigcup_{\beta<\alpha} V_\beta \text{ for limit } \alpha \tag{4.5}$$

For example

$$
\begin{aligned}
V_1 &= \{\emptyset\} = 1 && (4.6)\\
V_2 &= \{\emptyset, \{\emptyset\}\} = 2 && (4.7)\\
V_3 &= \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\} = 3 \cup \{\{1\}\} && (4.8)
\end{aligned}
$$

Lemma 4.0.7.

1. *Each V_α is transitive.*
2. $\alpha < \beta \to V_\alpha \subseteq V_\beta$.
3. $\alpha \subseteq V_\alpha$.

Lemma 4.0.8. *For every x there is α such that $x \in V_\alpha$*

Proof. Let C be the class of x that are not in any V_α. If C is nonempty, then C has an $\in$-minimal element x . For each t in some V_α, we define $rank(t) = \inf\{\alpha | t \in V_\alpha\}$. Then $rank(x)$ is a set of ordinals, with supremum λ. It's easy to see that $x \subseteq V_\lambda$. So $x \in V_{\lambda+1}$. Thus C is empty. □

Definition 4.0.9. *For each set x, we define the* **rank** *of x to be* $\inf\{\alpha | x \in V_\alpha\}$.

Chapter 5

Axiom of Choice

Axiom 5.0.1. *Every family S of nonempty sets has a choice function $S \mapsto \bigsqcup S$:*

$$\forall X \in S(f(X) \in X) \tag{5.1}$$

Lemma 5.0.2. *Let A be a set and F be a choice function on nonempty subsets of A. If*

1. *We have $X_0 \neq \emptyset$*
2. *$f(\alpha)$ is defined as $f(\alpha) = F(X_\alpha)$ if $X_\alpha \neq \emptyset$*
3. *X_α is defined by a definitive rule if $f(\beta)$ is defined and $\notin X_\alpha$ for $\beta < \alpha$.*

We can define by induction $X_\alpha \in A$ for $\alpha \leq \theta$, and $f(\alpha)$ for $\alpha < \theta$, where $X_\theta = \emptyset$, and f is injective.

Proof. If such θ doesn't exist, Ord would be a subset of A. □

Theorem 5.0.3. *The axiom of choice is equivalent to*

Well-ordering principle: *Every set can be well-ordered.*

Zorn's lemma: *A nonempty partially ordered set in which every chain has an upper bound has a maximal element.*

Proof.

Axiom of choice → Well-ordering principle:

Let A be a nonempty set, and F be a choice function on nonempty subsets of A. We define a function f as $f(\alpha) = F(A - \{f(\beta)|\beta < \alpha\})$ if $A \neq \{f(\beta)|\beta < \alpha\}$. By Lem.5.0.2, there exists an α such that $f : \alpha \mapsto A$ and $A = \{f(\beta)|\beta < \alpha\}$. It's easy to see that f is bijective. So A can be well-ordered according to the well-ordering on α.

Well-ordering principle → Axiom of choice:

We well-order $\bigcup S$ and define $f(X) = \inf\{x \in \bigcup S|x \in X\}$.

Axiom of choice → Zorn's lemma:

Let P be a nonempty partially ordered set. Let F be a choice function on nonempty subsets of P. We define a function f as $f(\alpha) = F(\{p \in P|(\forall\beta < \alpha)p > f(\beta)\})$ if $(\exists p \in P)(\forall\beta < \alpha)p > f(\beta)$.

By Lem.5.0.2, there exists an α such that $f : \alpha \mapsto P$ and $(\nexists p \in P)(\forall \beta < \alpha) p > f(\beta)$. $\{f(\beta) | \beta < \alpha\}$ is a chain in P, with an upper bound q. It's easy to see that q is the maximal element in P.

Zorn's lemma $\rightarrow$ Axiom of choice:

Let S be a family of sets, and $P = \{f | f$ is a choice function on some $Z \subseteq S\}$. P is partially ordered by $\subseteq$. For each chain P' in P, it's easy to see that $\bigcup P' \in P$ is the upper bound of P'. Use the Zorn's lemma, P has a maximal element F. It's easy to see that F is a choice function on S. $\square$

Chapter 6

Cardinal Numbers

Definition 6.0.1. *An ordinal α is called a* **cardinal number** *if α is not isomorphic to β for all $\beta < \alpha$.*

Definition 6.0.2. *For each set X, $|X|$ is defined to be the least ordinal that X is equipotent to. Clearly $|X|$ is a cardinal number. $|X|$ is well-defined since X can be well-ordered.*

Lemma 6.0.3. *X are Y are equipotent iff $|X| = |Y|$. There is an injective map from X to Y iff $|X| \leq |Y|$.*

Lemma 6.0.4. $|X| < |P(X)|$.

Proof. If not, let $f : X \mapsto P(X)$ be a bijection. Let $Y = \{x \in X | x \notin f(x)\}$. $Y \notin f(X)$. □

Lemma 6.0.5. *For each cardinal α, there's a cardinal strictly larger than α.*

Definition 6.0.6. *For each ordinal α, we define α^+ to be the least cardinal strictly larger than α.*

Lemma 6.0.7. *Let X be a set of cardinals, then $\sup X$ (in Ord) is a cardinal.*

6.1 Finite and Infinite Cardinals

Definition 6.1.1. *A cardinal is called a* **finite cardinal** *iff it's a finite ordinal. A cardinal is called an* **infinite cardinal** *iff it's a infinite ordinal.*

Lemma 6.1.2. *Let α be an ordinal, then $|\alpha| = |\alpha + 1|$ iff α is infinite.*

Proof. If α is infinite, the injective map $f : (\alpha + 1) \mapsto \alpha$ is defined as

$$f(\alpha) = 0 \quad (6.1)$$
$$f(\beta) = \beta + 1 \ (\beta < \alpha) \quad (6.2)$$

□

Corollary 6.1.3. *Each finite ordinal is a finite cardinal. Each infinite cardinal is a limit ordinal.*

Definition 6.1.4. *We define*

$$\aleph_0 = \omega_0 = \omega \tag{6.3}$$
$$\aleph_{\alpha+1} = \omega_{\alpha+1} = \aleph_\alpha^+ \tag{6.4}$$
$$\aleph_\alpha = \omega_\alpha = \sup\{\omega_\beta | \beta < \alpha\} \text{ if } \alpha \text{ is a limit ordinal} \tag{6.5}$$

We usually use $\aleph_\alpha$ to refer to cardinal numbers, and use ω_α to refer to ordinal numbers.

Lemma 6.1.5. $\alpha < \beta \leftrightarrow \aleph_\alpha < \aleph_\beta$.

Lemma 6.1.6. *Each infinite cardinal number is some $\aleph_\alpha$.*

Proof. It's easy to see that $\aleph_\alpha \geq \alpha$ for each α, since $\alpha \to \aleph_\alpha$ is a morphism in **Poset**. So $\forall$ cardinal $\kappa \exists$ ordinal $\alpha(\aleph_\alpha > \kappa)$. Let $\gamma = \inf\{\beta | \aleph_\beta \geq \kappa\}$. Then it's easy to see that $\kappa = \aleph_\gamma$. □

Definition 6.1.7. *An infinite cardinal $\aleph_\alpha$ is called a* **successor cardinal** *iff α is a successor ordinal. An infinite cardinal $\aleph_\alpha$ is called a* **limit cardinal** *iff α is a limit ordinal.*

6.2 Cardinal Arithmetic

Definition 6.2.1. *Let $|A| = \kappa$ and $B = \lambda$. We define*

1. $\kappa + \lambda = |A \sqcup B|$.
2. $\kappa \cdot \lambda = |A \times B|$.
3. $\kappa^\lambda = |A^B|$.

Note that the cardinal numbers have the same order but (possibly) different arithmetic with the ordinal numbers.

Theorem 6.2.2.

$$\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha \tag{6.6}$$

Proof. Clearly $\aleph_\alpha \leq \aleph_\alpha \cdot \aleph_\alpha$. From Lem.3.1.6, we have $\aleph_\alpha \cdot \aleph_\alpha \leq \aleph_\alpha$. □

Corollary 6.2.3. *Let κ and λ be two non-zero cardinals, one of which is infinite. Then $\kappa + \lambda = \kappa \cdot \lambda = \max\{\kappa, \lambda\}$*

Proof. $\max\{\kappa, \lambda\} \leq \kappa + \lambda \leq \kappa \cdot \lambda \leq \max\{\kappa, \lambda\} \cdot \max\{\kappa, \lambda\} = \max\{\kappa, \lambda\}$ □

Definition 6.2.4. *Let $\{\kappa_i | i \in I\}$ be an indexed set of cardinal numbers. We define*

$$\sum_{i \in I} \kappa_i = \left| \bigsqcup_{i \in I} \kappa_i \right| \tag{6.7}$$

Lemma 6.2.5. *Let $\{\kappa_i | i \in I\}$ be an indexed set of cardinal numbers, and $|I|$ or $\sup\{\kappa_i\}$ be infinite. Then*

$$\sum_{i \in I} \kappa_i = |I| \cdot \sup\{\kappa_i\} \tag{6.8}$$

Proof. Note that $\sup\{\kappa_i\}$ is a cardinal. It's easy to see that there's a injective map from $\sum_{i\in I}\kappa_i$ to $|I|\cdot\sup\{\kappa_i\}$. On the other hand, it's easy to see that $|I|\leq\sum_{i\in I}\kappa_i$ and $\sup\{\kappa_i\}\leq\sum_{i\in I}\kappa_i$ (since $\forall i(\kappa_i\leq\sum_{i\in I}\kappa_i)$). □

Lemma 6.2.6. *Let $\{\kappa_{ij}|i\in I, j\in J_i\}$ be an indexed set of ordinal numbers.*

$$\sup\{\sup\{\kappa_{ij}|j\in J_i\}|i\in I\}=\sup\{\kappa_{ij}|i\in I, j\in J_i\} \tag{6.9}$$

Lemma 6.2.7. *Let κ_λ be an α-sequence, where α is a infinite cardinal. Then $\sum_{\lambda<\alpha}\kappa_\lambda=\lim_{\beta\to\alpha}\sum_{\lambda\leq\beta}\kappa_\lambda$.*

Proof. $\lim_{\beta\to\alpha}\sum_{\lambda\leq\beta}\kappa_\lambda=\lim_{\beta\to\alpha}(|\beta|\cdot\sup\{|\kappa_\lambda||\lambda\leq\beta\})=\max(\alpha,\sup\{|\kappa_\lambda||\lambda<\alpha\})=\sum_{\lambda<\alpha}\kappa_\lambda$. □

Lemma 6.2.8. $|P(X)|=2^{|X|}$

Corollary 6.2.9. $\kappa<2^\kappa$, $\kappa^+\leq 2^\kappa$

Lemma 6.2.10.

1. $(\kappa^\lambda)^\theta=\kappa^{\lambda\cdot\theta}$
2. $\kappa^\lambda\cdot\kappa^\theta=\kappa^{\lambda+\theta}$
3. $\lambda>\theta\to\kappa^\lambda\geq\kappa^\theta$
4. $\lambda>\theta\to\lambda^\kappa\geq\theta^\kappa$

Lemma 6.2.11. *If $2\leq\kappa\leq\lambda$ and λ is infinite, then $\kappa^\lambda=2^\lambda$.*

Proof.

$$2^\lambda\leq\kappa^\lambda\leq(2^\kappa)^\lambda=2^{\kappa\cdot\lambda}=2^\lambda \tag{6.10}$$

□

6.3 Cofinality

Definition 6.3.1. *A function $A\mapsto B$ between two pre-ordered set is called* **cofinal** *in β iff*

$$\forall\beta\in B\exists\alpha\in A(f(\alpha)\geq\beta) \tag{6.11}$$

Definition 6.3.2. *Let $\alpha>0$ be a limit ordinal. We define $cf(\alpha)$ to be the least β such that there's a cofinal map $\beta\mapsto\alpha$.*

$cf(\alpha)$ is the least ordinal that resembles the interval $(\alpha-\epsilon,\alpha)$.

Lemma 6.3.3.

1. *$cf(\alpha)$ is a limit ordinal.*

2. $cf(\alpha) \leq \alpha$.

3. $cf(cf(\alpha)) = cf(\alpha)$

Proof. 3. The composition of cofinal functions is cofinal. □

Definition 6.3.4. *Let $\alpha > 0$ be a limit ordinal. α is called* **regular** *if $cf(\alpha) = \alpha$. Otherwise α is called* **singular**.

Lemma 6.3.5. *Let $\alpha > 0$ be a limit ordinal. $cf(\alpha)$ is a regular cardinal.*

Lemma 6.3.6. *Let α, β be infinite cardinals. There's a cofinal function $f : \alpha \mapsto \beta$ iff there is a function $g : \alpha \mapsto \beta$ such that $\sum_{\theta<\alpha} g(\theta) = \beta$.*

Proof. Let $f : \alpha \mapsto \beta$ be cofinal, define

$$g(\theta) = \begin{cases} f(0) & \alpha = 0 \\ \bigcup_{\beta\leq\theta+1} f(\beta) - \bigcup_{\beta\leq\theta} f(\beta) & \theta > 0 \end{cases} \tag{6.12}$$

Clearly $\sum_{\theta<\alpha} g(\theta) = |\bigsqcup_{\theta<\alpha} g(\theta)| = |\bigcup_{\theta<\alpha} f(\theta)| = \beta$.

Let $g : \alpha \mapsto \beta$ be a function such that $\sum_{\theta<\alpha} g(\theta) = \beta$. We define $f(\theta) = \sum_{\lambda\leq\theta} g(\lambda)$. From Lem. 6.2.6, we have $\sup\{f(\theta)|\theta < \alpha\} = \beta$. So f is cofinal. □

Corollary 6.3.7. *Let κ be an infinite cardinal. $cf(\kappa) = \inf\{\theta \in Ord|\exists$ a θ-sequence κ_ν of cardinals $\kappa_\nu < \kappa$ with $\kappa = \sum_{\nu<\theta} \kappa_\nu\}$.*

Corollary 6.3.8. *An infinite successor cardinal is regular.*

Proof. Let κ be an infinite cardinal. There exists a $cf(\kappa^+)$-sequence κ_ν of cardinals $\kappa_\nu < \kappa^+$ with $\kappa^+ = \sum_{\nu<cf(\kappa^+)} \kappa_\nu = \max(cf(\kappa^+), \sup\{\kappa_\nu\})$. Since $\kappa_\nu < \kappa^+$, $\kappa_\nu \leq \kappa$. So $\sup\{\kappa_\nu\} \leq \kappa < \kappa^+$. We must have $cf(\kappa^+) = \kappa^+$. □

The classification of cardinal numbers is shown in Fig. 6.1.

6.4 Cardinal Exponentiation

Theorem 6.4.1 (König)**.** *If κ is an infinite cardinal, then $\kappa < \kappa^{cf(\kappa)}$.*

Proof. It's easy to see that $\kappa \leq \kappa^{cf(\kappa)}$. If $\kappa = \kappa^{cf(\kappa)}$, let f be the bijection between them. Let g be a cofinal map $cf(\kappa) \mapsto \kappa$. Define $h : cf(\kappa) \mapsto \kappa$ as

$$h(\xi) = \inf(\kappa - \{f(\alpha)(\xi)|\alpha < g(\xi)\}) \tag{6.13}$$

$h(\xi)$ is well-defined since $|\{f(\alpha)(\xi)|\alpha < g(\xi)\}| \leq |g(\xi)| < \kappa$. $h \notin f(\kappa)$ since $\forall\alpha\exists\xi(h(\xi) \neq f(\alpha)(\xi))$. □

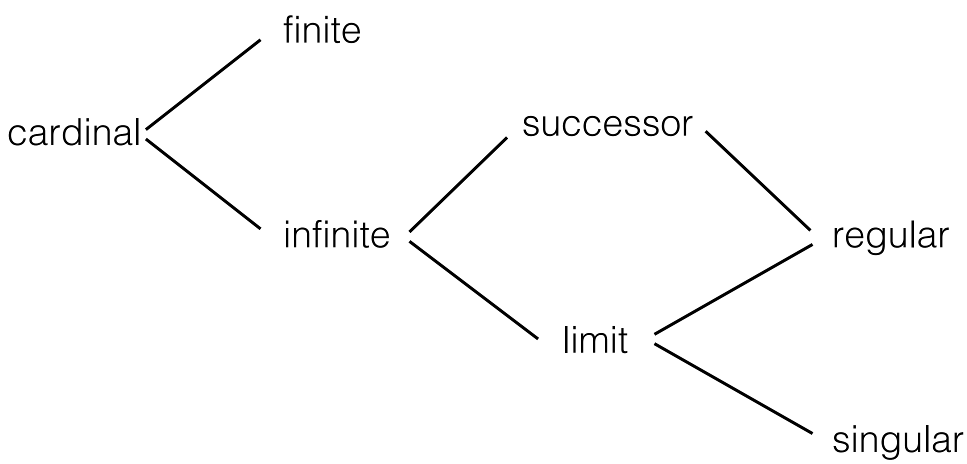

Figure 6.1: Classification of cardinal numbers.

Lemma 6.4.2. *Let κ be an infinite cardinal and $\lambda < cf(\kappa)$. Then*

$$\kappa^\lambda = \kappa \cdot \sup_{\theta<\kappa} \theta^\lambda \tag{6.14}$$

Proof. If $\lambda < cf(\kappa)$, $(\forall f : \lambda \mapsto \kappa)\exists(\theta < \kappa) f(\kappa) \subseteq \theta$. So $\kappa^\lambda = |\cup_{\alpha<\kappa} \alpha^\lambda| \leq |\sqcup_{\alpha<\kappa} \alpha^\lambda| = |\sqcup_{\alpha<\kappa} |\alpha|^\lambda| = \sum_{\alpha<\kappa} |\alpha|^\lambda = \kappa \cdot \sup_{\theta<\kappa} \theta^\lambda$ where α are ordinals and θ are cardinals. On the other hand $\kappa \leq \kappa^\lambda$ and $\sup_{\theta<\kappa} \theta^\lambda \leq \kappa^\lambda$. ☐

Corollary 6.4.3. *We have the* **Hausdorff formula** $\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_{\alpha+1} \cdot \aleph_\alpha^{\aleph_\beta}$ *for all α and β.*

Proof. When $\beta \leq \alpha$, use Lem.6.4.2. When $\beta > \alpha$, the Hausdorff formula holds because $\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} = 2^{\aleph_\beta} > \aleph_{\alpha+1}$. ☐

Theorem 6.4.4. *Let λ be an infinite ordinal. Then for all infinite cardinals κ,*

1. *if $\kappa \leq \lambda$ then $\kappa^\lambda = 2^\lambda$,*
2. *if there exists some $\mu < \kappa$ such that $\mu^\lambda \geq \kappa$, then $\kappa^\lambda = \mu^\lambda$,*
3. *if $\kappa > \lambda$ and $\mu^\lambda < \kappa$ for all $\mu < \kappa$, then*
   a. *$\kappa^\lambda = \kappa$ if $cf(\kappa) > \lambda$,*
   b. *$\kappa^\lambda = \kappa^{cf(\kappa)}$ if $cf(\kappa) \leq \lambda$.*

Proof. In the case 3, using the assumption, we have $\sup_{\alpha<\kappa} |\alpha|^\lambda = \kappa$

In the case 3.a: $\kappa^\lambda = \max(\kappa, \sup_{\alpha<\kappa} |\alpha|^\lambda) = \kappa$.

In the case 3.b: κ must be a limit cardinal. It's easy to see that $\kappa^\lambda \geq \kappa^{cf(\kappa)}$. We only need to prove that $\kappa^\lambda \leq \kappa^{cf(\kappa)}$. Let $h : cf(\kappa) \mapsto \kappa$ be a cofinal map. To each $f : \lambda \mapsto \kappa$ and each $\beta < cf(\kappa)$ we associate a function $f_\beta : \lambda \mapsto \kappa$:

$$f_\beta(\alpha) = \min(f(\alpha), h(\beta)) \tag{6.15}$$

The map $f \mapsto (f_\beta)$ is a injective map from κ^λ to $(\bigcup_{\alpha<\kappa} \alpha^\lambda)^{cf(\kappa)}$. So $\kappa^\lambda \leq |\bigcup_{\alpha<\kappa} \alpha^\lambda|^{cf(\kappa)} = \kappa^{cf(\kappa)}$. Since $\kappa \leq |\bigcup_{\alpha<\kappa} \alpha^\lambda| \leq |\bigsqcup_{\alpha<\kappa} \alpha^\lambda| = \max(\kappa, \sup_{\alpha<\kappa} |\alpha|^\lambda) = \kappa$. ☐

6.5 Continuum Hypothesis

Axiom 6.5.1. *The* **continuum hypothesis** *is the statement* $2^{\aleph_0} = \aleph_1$.

Axiom 6.5.2. *The* **generalized continuum hypothesis***(GCH) is the statement* $2^{\aleph_\alpha} = \aleph_{\alpha+1}$.

Theorem 6.5.3. *If GCH holds and κ and λ are infinite cardinals, then*

1. *if $\kappa \leq \lambda$ then $\kappa^\lambda = \lambda^+$,*
2. *if $cf(\kappa) \leq \lambda < \kappa$ then $\kappa^\lambda = \kappa^+$,*
3. *if $\lambda < cf(\kappa)$ then $\kappa^\lambda = \kappa$.*

Proof. When $\lambda < \kappa$, $\kappa \leq \kappa^\lambda \leq (2^\kappa)^\lambda = 2^\kappa = \kappa^+$. If $cf(\kappa) \leq \lambda$, then $\kappa < \kappa^{cf(\kappa)} \leq \kappa^\lambda$. So $\kappa^\lambda = \kappa^+$. If $\lambda < cf(\kappa)$, then $\kappa^\lambda = \kappa \cdot \sup\{|\alpha|^\lambda | \alpha < \kappa\} \leq \kappa \cdot \sup\{(2^{|\alpha|})^\lambda | \alpha < \kappa\} = \kappa \cdot \sup\{(|\alpha| \cdot \lambda)^+ | \alpha < \kappa\} = \kappa$. $\square$

Part II

General topology

Chapter 7

Topological Space

Definition 7.0.1. *Let X be a set and $\mathcal{T}$ be a family of subsets of X. $(X, \mathcal{T})$ is called a* **topological space** *if*

O1 any union of elements in $\mathcal{T}$ belongs to $\mathcal{T}$.

O2 any finite intersection of elements in $\mathcal{T}$ belongs to $\mathcal{T}$.

O3 $\emptyset$ and X belong to $\mathcal{T}$.

Elements of $\mathcal{T}$ are called **open subsets** *of X. We also call $\mathcal{T}$ a topology on X, and call $(X, \mathcal{T})$ X with topology $\mathcal{T}$.*

Definition 7.0.2. *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X. $\mathcal{T}_1$ is called* **finer(stronger)** *than $\mathcal{T}_2$ if $\mathcal{T}_1 \supseteq \mathcal{T}_2$. $\mathcal{T}_1$ is called* **coarser(weaker)** *than $\mathcal{T}_2$ if $\mathcal{T}_1 \subseteq \mathcal{T}_2$.*

Definition 7.0.3. *Let $(X, \mathcal{T})$ be a topological space, and S be a subset of X. S is called* **closed** *if $X - S$ (denoted by $S^\neg$) is open.*

Lemma 7.0.4. *Let $\mathcal{F}$ be the family of closed subsets of X. Then*

C1 any finite union of elements in $\mathcal{F}$ belongs to $\mathcal{F}$,

C2 any intersection of elements in $\mathcal{F}$ belongs to $\mathcal{F}$,

C3 $\emptyset$ and X belong to $\mathcal{F}$,

and furthermore

C4 $S \subseteq X$ is open if $S^\neg$ is closed.

Let $\mathcal{F}$ be a family of subsets of X that satisfies C1-C3. Then $\mathcal{F}$ is the family of closed subsets of a topology space, whose open sets are defined by C4.

7.1 Base and subbase

Definition 7.1.1. *Let $(X, \mathcal{T})$ be a topological space, and $\mathcal{B}$ be a subfamily of $\mathcal{T}$. $\mathcal{B}$ is called a* **base** *if $(\forall x \in X)(\forall S \in \mathcal{T})x \in S \to (\exists B \in \mathcal{B})x \in B \subseteq S$.*

Lemma 7.1.2. *Let X be a set and $\mathcal{B}$ be a base of $(X, \mathcal{T})$. Then*

B1 $(\forall x \in X)(\exists B \in \mathcal{B})x \in B$

B2 $(\forall B_1, B_2 \in \mathcal{B})(\forall x \in B_1 \cap B_2)(\exists B_3 \in \mathcal{B})x \in B_3 \subseteq B_1 \cap B_2$

and furthermore

B3 $S \subseteq X$ *is open if* $(\forall x \in S)(\exists B \in \mathcal{B})x \in B \subseteq S$.

Let $\mathcal{B}$ be a family of subsets of X that satisfies B1 and B2. Then $\mathcal{B}$ is the base of a topology space, and open sets are defined by B3.

Definition 7.1.3. *Let $(X, \mathcal{T})$ be a topological space, and $\mathcal{B}_x$ be a subfamily of $\mathcal{T}$ that contains x. $\mathcal{B}_x$ is called a* **local base** *at x if $(\forall S \in \mathcal{T})x \in S \to (\exists B \in \mathcal{B}_x)x \in B \subseteq S$.*

Lemma 7.1.4. *Let $(X, \mathcal{T})$ be a topological space, and for each x, $\mathcal{B}_x$ be a local base at x. Then*

LB1 $(\forall B \in \mathcal{B}_x)x \in B$,

LB2 $(\forall B_1, B_2 \in \mathcal{B}_x)(\exists B_3 \in \mathcal{B}_x)B_3 \subseteq B_1 \cap B_2$,

LB3 $(\forall B \in \mathcal{B}_x)(\forall y \in B)(\exists B' \in \mathcal{B}_y)B' \subseteq B$.

and furthermore

LB4 $S \subseteq X$ *is open if* $(\forall x \in S)(\exists B \in \mathcal{B}_x)x \in B_x \subseteq S$.

$\forall x \in X$ let $\mathcal{B}_x$ be a family of subsets of X that satisfies LB1-LB3. Then $\mathcal{B}_x$ is the local base of a topology space, and open sets are defined by LB4.

Lemma 7.1.5. *Let $(X, \mathcal{T})$ be a topological space. If $\forall x \in X$ we have a local base $\mathcal{B}_x$, then $\bigcup_{x \in X} \mathcal{B}_x$ is a base.*

Definition 7.1.6. *Let $(X, \mathcal{T})$ be a topological space, and $\mathcal{S}$ be a subfamily of $\mathcal{T}$. $\mathcal{S}$ is called a* **subbase** *if the collection of all finite intersections of $\mathcal{S}$ is a base.*

Lemma 7.1.7. *Let X be a set and $\mathcal{S}$ be a family of subsets of X. $\mathcal{S}$ is a subbase for a topology on X with the collection of all finite intersections of $\mathcal{S}$ as its base.*

7.2 Neighborhoods

Definition 7.2.1. *Let $(X, \mathcal{T})$ be a topological space, $x \in X$ and $S \subseteq X$. S is called the* **neighborhood** *at x if $\exists A \in \mathcal{T} : x \in A \subseteq S$. The family of neighborhoods at x is called the neighborhood system at x, denoted by $\mathcal{N}_x$.*

Theorem 7.2.2. *Let $(X, \mathcal{T})$ be a topological space, and $\forall x \in X$, $\mathcal{N}_x$ be a neighborhood system at x. Then*

N1 $\forall N \in \mathcal{N}_x : x \in N$,

N2 $\forall N_1, N_2 \in \mathcal{N}_x : N_1 \cap N_2 \in \mathcal{N}_x$,

N3 $\forall N \in \mathcal{N}_x \exists N' \in \mathcal{N}_x \forall y \in N' : N \in \mathcal{N}_y$.

N4 $\forall N \in \mathcal{N}_x : N \subseteq N' \Rightarrow N' \in \mathcal{N}_x$,

and furthermore

N5 $S \subseteq X$ *is open if* $\forall x \in S \exists N \in \mathcal{N}_x : x \in N_x \subseteq S$.

$\forall x \in X$ let $\mathcal{N}_x$ be a family of subsets of X that satisfies N1-N4. Then $\mathcal{N}_x$ is the neighborhood system of a topology space, and open sets are defined by N5.

Definition 7.2.3. *Let $(X, \mathcal{T})$ be a topological space, $\mathcal{NB}_x$ be a subfamily of $\mathcal{N}_x$. $\mathcal{NB}_x$ is called the* **neighborhood base** *at x if $\forall A \in \mathcal{T} : x \in A \exists B \in \mathcal{NB}_x : x \in B \subseteq A$. Elements of a neighborhood base are called* **basic neighborhoods**.

Theorem 7.2.4. *Let $(X, \mathcal{T})$ be a topological space. A local base at $x \in X$ is a neighborhood base.*

Theorem 7.2.5. *Let $(X, \mathcal{T})$ be a topological space, and $\forall x \in X : \mathcal{NB}_x$ be a neighborhood base at x. Then*

NB1 $\forall N \in \mathcal{NB}_x : x \in N$,

NB2 $\forall N_1, N_2 \in \mathcal{NB}_x \exists N_3 \in \mathcal{NB}_x : N_3 \subseteq N_1 \cap N_2$,

NB3 $\forall N \in \mathcal{NB}_x \exists N' \subseteq N \forall y \in N' \exists N'' \in \mathcal{NB}_y : N'' \subseteq N$.

and furthermore

NB4 $S \subseteq X$ *is open if* $\forall x \in S \exists N \in \mathcal{NB}_x : x \in N_x \subseteq S$.

$\forall x \in X$ let $\mathcal{NB}_x$ be a family of subsets of X that satisfies NB1-NB3. Then $\mathcal{NB}_x$ is the neighborhood base of a topology space, and open sets are defined by NB4.

Theorem 7.2.6. *Let $(X, \mathcal{T})$ be a topological space, $E \subseteq X$ Then*

1. *E is open iff E contains a basic neighborhood of each of its points.*
2. *E is closed iff each $x \notin E$ has a basic neighborhood disjoint from E.*
3. $E^- = \{x \in X |$ *each basic neighborhood of x meets E*$\}$
4. $E^\circ = \{x \in X |$ *some basic neighborhood of x is contained in E*$\}$

7.3 Closure and Interior

Definition 7.3.1. *Let $(X, \mathcal{T})$ be a topological space and $S \subseteq X$. The* **closure** *of S, denoted by S^- or $Cl_X(S)$, is defined to be the intersection of all closed subsets that contains S. The* **interior** *of S, denoted by S° or $Int_X(S)$, is defined to be the union of all open subsets that is contained in S. The* **frontier** *of S, denoted by $Fr_X(S)$, is defined to be $S^- - S^\circ$. The* **boundary** *of S, denoted by $\partial_X(S)$, is defined to be $S - S^\circ$.*

Lemma 7.3.2. *Let A and B be subsets of topological space X*

$$
\begin{aligned}
A \subseteq B \quad &\Rightarrow \quad A^- \subseteq B^- \wedge A^\circ \subseteq B^\circ && (7.1)\\
A^\circ \quad &= \quad A^{\neg - \neg} && (7.2)\\
A^{\circ -} \quad &= \quad A^{\circ - \circ -} && (7.3)\\
A^{- \circ} \quad &= \quad A^{- \circ - \circ} && (7.4)\\
(A \cup B)^- \quad &= \quad A^- \cup B^- && (7.5)\\
(A \cap B)^\circ \quad &= \quad A^\circ \cap B^\circ && (7.6)
\end{aligned}
$$

where $\neg$ denotes complementation.

Proof. Since for any closed S we have $S^{\circ -} \subseteq S$, we have $A^{\circ - \circ -} \subseteq A^{\circ -}$. Since for any open S we have $S \subseteq S^{- \circ}$, we have $A^\circ \subseteq A^{\circ - \circ}$. So $A^{\circ -} \subseteq A^{\circ - \circ -}$. So $A^{\circ -} = A^{\circ - \circ -}$. Similarly $A^{- \circ} = A^{- \circ - \circ}$. ☐

The concepts of closure, interior and frontier are illustrated in Fig. 7.1.

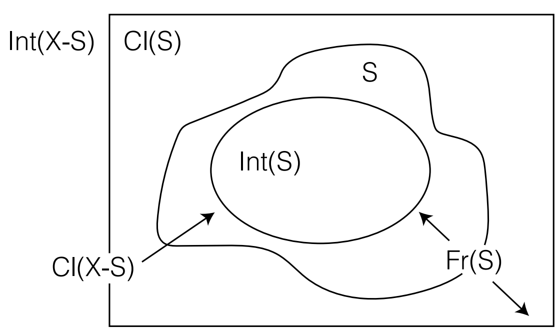

Figure 7.1: Illustration of the closure, interior and frontier of a set S.

Corollary 7.3.3. *Let X be a space and $S \subseteq X$. There're at most 14 distinct sets one can get from X by applying the operations of closure and complement, namely:*

$$
\begin{aligned}
&A,\ A^{\neg},\ A^{\neg -},\ A^{\neg - \neg},\ A^{\neg - \neg -},\ A^{\neg - \neg - \neg},\ A^{\neg - \neg - \neg -},\ A^{\neg - \neg - \neg - \neg},\\
&A^{-},\ A^{- \neg},\ A^{- \neg -},\ A^{- \neg - \neg},\ A^{- \neg - \neg -},\ A^{- \neg - \neg - \neg} && (7.7)
\end{aligned}
$$

Proof. An example that gives the 14 distinct sets is:

$$A \quad = \quad (-1, 0) \cup (0, 1) \cup (\mathbb{Q} \cap (1, 2)) \cup \{3\} \tag{7.8}$$

$$
\begin{aligned}
A^{\neg} &= (-\infty,-1]\cup\{0\}\cup([1,2]-\mathbb{Q})\cup[2,3)\cup(3,\infty) && (7.9)\\
A^{\neg-} &= (-\infty,-1]\cup\{0\}\cup[1,\infty) && (7.10)\\
A^{\neg-\neg} &= (-1,0)\cup(0,1) && (7.11)\\
A^{\neg-\neg-} &= [-1,1] && (7.12)\\
A^{\neg-\neg-\neg} &= (-\infty,-1)\cup(1,\infty) && (7.13)\\
A^{\neg-\neg-\neg-} &= (-\infty,-1]\cup[1,\infty) && (7.14)\\
A^{\neg-\neg-\neg-\neg} &= (-1,1) && (7.15)\\
A^{-} &= [-1,2]\cup\{3\} && (7.16)\\
A^{-\neg} &= (-\infty,-1)\cup(2,3)\cup(3,\infty) && (7.17)\\
A^{-\neg-} &= (-\infty,-1]\cup[2,\infty) && (7.18)\\
A^{-\neg-\neg} &= (-1,2) && (7.19)\\
A^{-\neg-\neg-} &= [-1,2] && (7.20)\\
A^{-\neg-\neg-\neg} &= (-\infty,-1)\cup(2,\infty) && (7.21)
\end{aligned}
$$

The preceding lemma tells us that there's no more. □

Lemma 7.3.4 (Kuratowski)**.** *The operation $A \to A^-$ in a topological space X has the following properties:*

K1 $A \subseteq A^-$,

K2 $A^{--} = A^-$,

K3 $(A\cup B)^- = A^- \cup B^-$,

K4 $\emptyset^- = \emptyset$,

and furthermore

K5 *A is closed in X if $A = A^-$.*

If we have a set X and a map $A \to A^-$ for each $A \subseteq X$ that satisfies K1-K4. Then X becomes a topology space if the closed sets are defined by K5. The map $A \to A^-$ in X coincides with the one we began with.

Proof. K3 $\to ((A \subseteq B) \to B^- = (A\cup(B-A))^- = A^-\cup(B-A)^-) \to ((A\subseteq B)\to(A^-\subseteq B^-))$ □

Lemma 7.3.5. *The operation $A \to A^\circ$ in a topological space X has the following properties:*

I1 $A^\circ \subseteq A$,

I2 $A^{\circ\circ} = A^\circ$,

I3 $(A\cap B)^\circ = A^\circ \cap B^\circ$,

I4 $X^\circ = X$,

and furthermore

I5 A is open in X if $A = A^\circ$.

If we have a set X and a map $A \to A^\circ$ for each $A \subseteq X$ that satisfies I1-I4. Then X becomes a topology space if the open sets are defined by I5. The map $A \to A^\circ$ in X coincides with the one we began with.

Definition 7.3.6. *Let S be a subset of X. x is a* **limit point** *of S iff for each neighborhood N of x, $(N - \{x\}) \cap S \neq \emptyset$.*

Lemma 7.3.7. *Let $(X, \mathcal{T})$ be a topological space, $x \in X$ and $S \subseteq X$. $S^- = S \cup \{$ all the limit points of $S\}$.*

Definition 7.3.8. *Let S be a closed subset of X, and $T \subseteq S$. T is called* **dense** *in S if $T^- = S$.*

7.4 Subspace

Definition 7.4.1. *Let $(X, \mathcal{T})$ be a topological space and $Y \subseteq X$. The topological space $(Y, \{S \cap Y | S \in \mathcal{T}\})$ is called* **a subspace** *of $(X, \mathcal{T})$.*

Theorem 7.4.2. *Let Y be a subspace of a topological space X, then*

1. *$H \subseteq Y$ is open in A iff $H = G \cap A$ where G is open in X.*
2. *$H \subseteq Y$ is closed in A iff $H = G \cap A$ where G is closed in X.*
3. *Let $H \subseteq Y$. Then $Cl_Y(H) = Y \cap Cl_X(H), Int_Y(H) = Y \cap Int_X(H), Fr_Y(H) = Y \cap Fr_X(H)$.*
4. *Let $x \subseteq Y$. If $\mathcal{B}_x$ is a neighborhood base(local base) at x in X, then $\{B \cap Y | B \in \mathcal{B}_x\}$ is a neighborhood base(local base) at x in Y.*
5. *If $\mathcal{B}$ is a base(subbase) for X, then $\{B \cap Y | B \in \mathcal{B}\}$ is a base(subbase) for Y*

7.5 Metric Spaces

Definition 7.5.1. *Let X be a set and $\rho : X \times X \mapsto \mathbb{R}$ be a map. ρ is called a* **pseudometric** *on X if for all $x, y \in X$:*

1. $\rho(x, y) \geq 0$
2. $\rho(x, x) = 0$
3. $\rho(x, y) = \rho(y, x)$
4. $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$

ρ is called a **metric** *on X if it's a pseudometric and $\rho(x, y) = 0 \Rightarrow x = y$.*

Definition 7.5.2. *We define the* **ball** *in X centered at x as $B(x,r) = \{y \in X | \rho(x,y) < r\}$.*

Definition 7.5.3. *Let X be a set with a pseudometric ρ. X is called a* **pseudometric space** *if it has the topology with $\{B(x,r)|r > 0\}$ as a local base at x.*

Definition 7.5.4. *A* **metric space** *is a pseudometric space whose pseudometric is a metric.*

Lemma 7.5.5. *Let X be a pseudometric space with pseudometric ρ, $\sim$ be the equivalence relation on X defined by $x \sim y \leftrightarrow \rho(x,y) = 0$. Let $X^* = X/\sim$ be the equivalence classes. We define a map $\rho^* : X^* \times X^* \mapsto \mathbb{R}$ as $\rho^*([x],[y]) = \rho(x,y)$. ρ^* is well-defined and is a metric on X^*.*

7.6 Examples

Example 7.6.1. *We define the* **discrete topology** *on a set X as the family of all subsets of X.*

Example 7.6.2. *We define the* **trivial topology** *on a set X as $\{\emptyset, X\}$.*

Example 7.6.3. *Let X be a infinite set, we define the* **cofinite topology** *on X by $\{S \in \mathcal{T} || X - S| < \aleph_0 \vee S = \emptyset\}$.*

Example 7.6.4. *We define the* **usual topology** *on $\mathbb{R}^n$ as the metric topology with the metric $\rho(x,y) = \sqrt{\sum_i (x_i - y_i)^2}$.*

Example 7.6.5. *Let X be a linearly ordered set, we define the* **order topology** *on X to be the one with the subbase $\{(-\infty, a), (a, \infty) | a \in X\}$.*

Example 7.6.6. *We define the* **radial plane** *as the real plane with the topology such that a local neighborhood base at x is $\mathcal{N}_x = \{S \subset \mathbb{R}^2 | S$ contains an open line segment through x in each direction$\}$. See Fig. 7.2.*

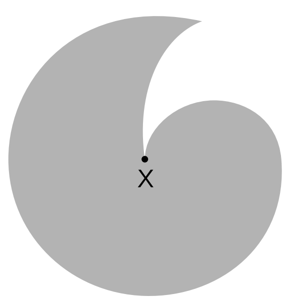

Figure 7.2: Example of neighborhood at x of radial plane.

Example 7.6.7. *We define the* **Sorgenfrey line** *as the real line with the topology such that a local base at x contains $[x,y)$ for all $y > x$.*

Example 7.6.8. *We define the* **Moore plane** *as the topology on the upper half plane $\{(x,y) \in \mathbb{R}^2 | y \geq 0\}$, such that: At (x,y) where $y > 0$, a neighborhood base contains 2-balls in the upper half plane centered at (x,y). At $(x,0)$, a neighborhood base contains sets of the form $\{(x,0)\} \cap A$, where A is a 2-ball in the upper half plane tangent to the x-axis at $(x,0)$. See Fig. 7.3.*

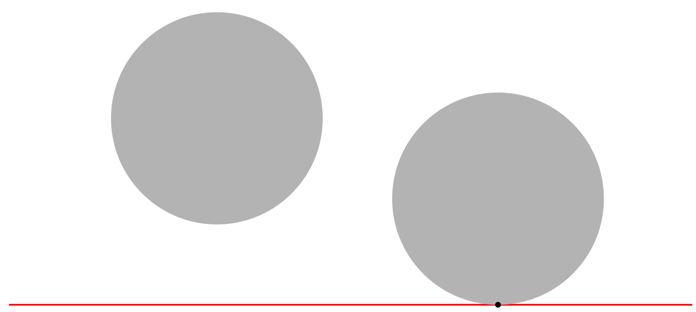

Figure 7.3: Example of neighborhoods of Moore plane.

Chapter 8

Product and Quotient Spaces

8.1 Continuous Function

Definition 8.1.1. *Let X and Y be topological spaces and $f : X \mapsto Y$. f is said to be* **continuous at** *$x \in X$ if $\forall$ neighborhood V of $f(x) \exists$ neighborhood U of $x : f(U) \subseteq V$.*

Definition 8.1.2. *Let X and Y be topological spaces and $f : X \mapsto Y$. f is called a* **continuous function** *if it satisfies the equivalent conditions:*

1. *$\forall x \in X : f$ is continuous at x.*
2. *$H \subseteq Y$ is open $\Rightarrow f^{-1}(H)$ is open.*
3. *$H \subseteq Y$ is closed $\Rightarrow f^{-1}(H)$ is closed.*
4. *$\forall H \subseteq X : f(H^-) \subseteq f(H)^-$.*
5. *$\forall H \subseteq Y : f^{-1}(H)^- \subseteq f^{-1}(H^-)$.*
6. *$\forall H \subseteq X : f(H^\circ) \supseteq f(H)^\circ$.*
7. *$\forall H \subseteq Y : f^{-1}(H)^\circ \supseteq f^{-1}(H^\circ)$.*

Theorem 8.1.3. *Let f be a continuous function from X to Y, and U be a subspace of X. Then $f|_U$ is continuous.*

Definition 8.1.4. *Let X and Y be topological spaces and $f : X \mapsto Y$ be a bijective map. f is called a* **homeomorphism** *if both f and f^{-1} are continuous.*

Theorem 8.1.5. *Let X and Y be topological spaces and $f : X \mapsto Y$ be a bijective map. The following are equivalent*

1. *f is a homeomorphism.*
2. *$H \subseteq Y$ is open $\Leftrightarrow f^{-1}(H)$ is open.*
3. *$H \subseteq Y$ is closed $\Leftrightarrow f^{-1}(H)$ is closed.*

4. $\forall H \subseteq X : f(H^-) = f(H)^-$.

5. $\forall H \subseteq Y : f^{-1}(H)^- = f^{-1}(H^-)$.

Theorem 8.1.6. *Let f be a homeomorphism from X to Y, and U be a subspace of X. Then $f|_U$ is homeomorphism from U to $f(U)$.*

Definition 8.1.7. *Let f be a continuous function from X to Y. If f is a homeomorphism from X to $f(X)$, then f is called an* **embedding**.

Definition 8.1.8. *Let f be a map from X to Y. f is called an* **open map** *if it maps open subsets to open subsets. f is called a* **closed map** *if it maps closed subsets to closed subsets.*

Example 8.1.9. *For topological spaces X and Y, let $C(X,Y)$ denote the collection of all continuous functions from X to Y. Especially, we use $C(X)$ to denote $C(X,\mathbb{R})$, and $C^*(X)$ to denote all bounded functions in $C(X)$. It's easy to see that $C(X)$ and $C^*(X)$ are algebras over $\mathbb{R}$. Moreover, $C^*(X)$ is a normed linear space with the norm $\|f\| = sup_{x\in X}|f(x)|$.*

8.2 Product Spaces and Weak Topologies

Definition 8.2.1. *Let X be a set and X_α be a topological spaces with $f_\alpha : X \mapsto X_\alpha$ for each $\alpha \in A$. The* **weak topology** *induced on X by $\{f_\alpha|\alpha \in A\}$ is the weakest topology on X making each f_α continuous.*

Theorem 8.2.2. *Let X be a set and X_α be a topological spaces with $f_\alpha : X \mapsto X_\alpha$ for each $\alpha \in A$. The weak topology induced on X by $\{f_\alpha|\alpha \in A\}$ is the one with the subbase $\{f_\alpha^{-1}(U_\alpha)|\alpha \in A, U_\alpha$ open in $X_\alpha\}$*

Theorem 8.2.3. *Let X_α be topological spaces. Let X be a set with weak topology induced by $f_\alpha : X \mapsto X_\alpha$ for each $\alpha \in A$. Let Y be a topological space. A map $f : Y \mapsto X$ is continuous iff $f_\alpha \circ f$ is continuous for each $\alpha \in A$.*

Definition 8.2.4. *Let X_α be a family of topological spaces where $\alpha \in A$, and $\prod_{\alpha\in A} X_\alpha$ be their Cartesian product. We define $\pi_\alpha : \prod_{\alpha\in A} X_\alpha \mapsto X_\alpha$ as $\pi_\alpha(x) = x_\alpha$. Then the weakest topology induced on $\prod_{\alpha\in A} X_\alpha$ by $\{\pi_\alpha|\alpha \in A\}$ is called the* **product topology**. *With this topology, $\prod_{\alpha\in A} X_\alpha$ is called the* **product space**.

Theorem 8.2.5. *Let X_α be a family of spaces where $\alpha \in A$, then their product space is the direct product of X_α in the category* **Top**.

Theorem 8.2.6. *Let X_α be a family of spaces where $\alpha \in A$, and $\prod_{\alpha\in A} X_\alpha$ be their product space. Then $\pi_\alpha : \prod_{\alpha\in A} X_\alpha \mapsto X_\alpha$ is an open map for each $\alpha \in A$.*

Theorem 8.2.7. *A map $f : X \mapsto \prod_\alpha X_\alpha$ is continuous iff $\pi_\alpha \circ f$ is continuous for each $\alpha \in A$.*

Definition 8.2.8. *Let X be a space and X_α be spaces, with $f_\alpha : X \mapsto X_\alpha$ for each $\alpha \in A$. The* **evaluation map** *$e : X \mapsto \prod_\alpha X_\alpha$ is defined by $e(x) = \bar{x}$, where $\bar{x}_\alpha = f_\alpha(x)$.*

Theorem 8.2.9. *Let X be a space and X_α be spaces, with $f_\alpha : X \mapsto X_\alpha$ for each $\alpha \in A$. Then the evaluation map $e : X \mapsto \prod_\alpha X_\alpha$*

1. *is continuous iff f_α is continuous for each $\alpha \in A$.*
2. *is injective iff $\forall x \neq y \in X \exists \alpha \in A : f_\alpha(x) \neq f_\alpha(y)$.*
3. *is an embedding iff it's injective and X has the weak topology induced by f_α.*

Proof. 1: $f_\alpha = \pi_\alpha \circ e$

3: If e is injective and X has the weak topology induced by f_α, it's easy to see that e is continuous. We only need to prove that e is open (from X to $e(X)$), which only needs to be tested on a subbase of X: $\{f_\alpha^{-1}(U_\alpha) | \alpha \in A, U_\alpha \text{ open in } X_\alpha\}$. However $e(f_\alpha^{-1}(U_\alpha)) = e(X) \cap \pi_\alpha^{-1}(U_\alpha)$, which is open in $e(X)$.

If e is an embedding, then e is injective. It's easy to see that X has the weak topology induced by e. Since $\prod_\alpha X_\alpha$ has the weak topology induced by $\{\pi_\alpha\}$. X has the weak topology induced by $\{f_\alpha\} = \{\pi_\alpha \circ e\}$. □

Theorem 8.2.10. *Let X be a pseudometric space with pseudometric $\rho : X \times X \mapsto \mathcal{R}$. ρ is a continuous function.*

8.3 Coproduct Spaces and Strong Topologies

Definition 8.3.1. *Let X be a set and X_α be a topological spaces with $f_\alpha : X_\alpha \mapsto X$ for each $\alpha \in A$. The* **strong topology** *induced on X by $\{f_\alpha | \alpha \in A\}$ is the strongest topology on X making each f_α continuous.*

Theorem 8.3.2. *Let X be a set and X_α be a topological spaces with $f_\alpha : X_\alpha \mapsto X$ for each $\alpha \in A$. The strong topology induced on X by $\{f_\alpha | \alpha \in A\}$ is the one with the open sets $\{S \subseteq X | \forall \alpha \in A : f_\alpha^{-1}(S)$ is open in $X_\alpha\}$*

Theorem 8.3.3. *Let X_α be topological spaces. Let X be a set with strong topology induced by $f_\alpha : X_\alpha \mapsto X$ for each $\alpha \in A$. Let Y be a topological space. A map $f : X \mapsto Y$ is continuous iff $f \circ f_\alpha$ is continuous for each $\alpha \in A$.*

Corollary 8.3.4. *Let f be a function from X to Y, and U_α be a family of open subspaces of X that covers X. If $\forall \alpha : f|_{U_\alpha}$ is continuous, then f is continuous.*

Proof. X has the strong topology induced by the inclusion map $\{U_\alpha \mapsto X\}$. □

Definition 8.3.5. *Let X_α be a family of topological spaces where $\alpha \in A$, and $\coprod_{\alpha \in A} X_\alpha$ be their disjoint union. We define $\iota_\alpha : X_\alpha \mapsto \coprod_{\alpha \in A} X_\alpha$ as $\iota_\alpha(x_\alpha) = x_\alpha$. Then the strong topology induced on $\coprod_{\alpha \in A} X_\alpha$ by $\{\iota_\alpha | \alpha \in A\}$ is called the* **coproduct topology**. *With this topology, $\coprod_{\alpha \in A} X_\alpha$ is called the* **coproduct space**.

Theorem 8.3.6. *Let X_α be a family of topological spaces where $\alpha \in A$, then their coproduct space is the direct sum of X_α in the category* **Top**.

Theorem 8.3.7. *Let X_α be a family of topological spaces where $\alpha \in A$, and $\coprod_{\alpha \in A} X_\alpha$ be their coproduct space. Then $\iota_\alpha : X_\alpha \mapsto \coprod_{\alpha \in A} X_\alpha$ is an open map for each $\alpha \in A$.*

8.4 Quotient map and Quotient Spaces

Definition 8.4.1. *Let $f : X \mapsto Y$ be a surjective map from a topological space X to a set Y. The* **quotient topology** *on Y induced by f is the coproduct topology.*

Lemma 8.4.2. *Let X be a set and X_α be a topological spaces with $f_\alpha : X_\alpha \mapsto X$ for each $\alpha \in A$. We define a map $f : \coprod_{\alpha\in A} X_\alpha \mapsto X$ by $f(x_\alpha) = f_\alpha(x_\alpha)$. Then X has the strong topology induced by (f_α) iff it has the quotient topology induced by f.*

Definition 8.4.3. *Let X and Y be topological spaces with an surjective map $f : X \mapsto Y$. f is called the* **quotient map** *if the topology on Y is the quotient topology induced by f.*

Definition 8.4.4. *Let X be a topological space and $\sim$ be an equivalence relation on X. The* **quotient space** *of X induced by $\sim$ is $X/\sim$ with the quotient topology induce by the canonical map $X \mapsto X/\sim$. If an equivalence relation on X is $a \sim b \leftrightarrow a, b \in A \subseteq X$, then the quotient space may also be written as X/A.*

Lemma 8.4.5. *Let X and Y be topological spaces with a map $f : X \mapsto Y$. We define an equivalence relation on X by $x \sim y \leftrightarrow f(x) = f(y)$. Then $f(X) \simeq X/\sim$.*

Definition 8.4.6. *We define an n-dimensional* **disk** *to be $D^n = \{x \in \mathbb{R}^n | |x| \leq 1\}$. We define $\partial D^n = \{x \in \mathbb{R}^n | |x| = 1\}$*

Lemma 8.4.7. *$S^n \simeq D^n/\partial D^n$*

Proof. Consider the map $f : D^n \mapsto S^n$

$$f(x) = (1 - 2|x|, 2\sqrt{\frac{1-|x|}{|x|}}x_0, \cdots, , 2\sqrt{\frac{1-|x|}{|x|}}x_{n-1}) \tag{8.1}$$

□

Theorem 8.4.8. *A surjective continuous map is a quotient map if it's open or closed.*

Proof. If $f : X \mapsto Y$ is a surjective continuous closed map. Let T be a set in Y such that $f^{-1}(T)$ is open. Then $Y - T = f(X - f^{-1}(T))$ is closed. So T is open. □

Example 8.4.9. *How to stick two spaces together? Let A and B be two topological spaces, we consider their coproduct space $A \sqcup B$. We can define some equivalence relation on $A \sqcup B$, and the quotient space $A \sqcup B/\sim$ would be the A and B stuck together.*

Chapter 9

Convergence

9.1 Moore-Smith Convergence

Definition 9.1.1. *A* **directed set** *is a set S with a pre-order $\leq$ such that any two elements are bounded. We say that S is* **directed by** $\leq$.

Definition 9.1.2. *Let Λ be a directed set. A* **net** (x_λ) *is a map from Λ to X.*

Definition 9.1.3. *Let (x_λ) be a net from Λ to X, and $S \subseteq X$. (x_λ) is said to be* **eventually in** S *if $\exists \lambda_0 : \lambda > \lambda_0 \to x_\lambda \in S$. (x_λ) is said to be* **frequently in** S *if $\forall \lambda_0 \exists \lambda > \lambda_0 : x_\lambda \in S$. It's easy to see that "not eventually in S" is equivalent to "frequently in $X - S$", and "not frequently in S" is equivalent to "eventually in $X - S$"*

Definition 9.1.4. *Let (x_λ) be a net from Λ to X. (x_λ) is said to* **converges to** x_0 *(written as $x_\lambda \to x_0$) if it's eventually in each neighborhood of x_0.*

Definition 9.1.5. *Let (x_λ) be a net from Λ to X. x_0 is said to be a* **cluster point** *of (x_λ) if the net is frequently in each neighborhood of x_0.*

Example 9.1.6. *Let X be a topological space and $x \in X$. Let Λ be a neighborhood base at x. Then Λ with the order relation $U_1 \leq U_2$ iff $U_1 \supseteq U_2$ forms a directed set. If we pick a $x_U \in U$ for each $U \in \Lambda$ (using AC), we result in a net (x_U) that converges to x.*

Definition 9.1.7. *Let (x_λ) be a net from Λ to X. A net (x'_μ) from M to X is a* **subnet** *of (x_λ) if there's an increasing cofinal function $\phi : M \mapsto \Lambda$ such that $x'_\mu = x_{\phi(\mu)}$.*

Theorem 9.1.8. *Let (x_λ) be a net from Λ to X frequently in $E \subseteq X$. Then there is a subnet of (x_λ) eventually in $E \subseteq X$.*

Theorem 9.1.9. *Let (x_λ) be a net from Λ to X eventually in $E \subseteq X$. Then all of its subnets are eventually in $E \subseteq X$.*

Theorem 9.1.10. *If a net from Λ to X converges to x, then all of its subnets converge to x.*

Theorem 9.1.11. *A net from Λ to X has x as a cluster point iff it has a subnet that converges to x.*

Proof. Let (x_λ) be a net $\Lambda \mapsto X$. We define $\Lambda' = \{(\lambda, U) \in \Lambda \times P(X) | x_\lambda \in U$ is a neighborhood of $x\}$. Λ' is directed by the pre-order: $(\lambda, U) \leq (\lambda', U')$ iff $\lambda \leq \lambda'$ and $U \supseteq U'$. We define $\theta : \Lambda' \mapsto \Lambda$ as $\theta(\lambda, U) = \lambda$. Then $(x_{\theta(\lambda,U)})$ is a subnet that converges to x. □

Theorem 9.1.12. *Let X be a topological space and $E \subseteq X$. Then $x \in E^-$ iff there is a net from Λ to E that converges to x.*

This theorem together with Lem. 7.3.4 tell us that the topology of a space is determined by the convergence of nets in it.

Theorem 9.1.13. *Let $f : X \mapsto Y$. Then f is continuous at $x_0 \in X$ iff for each net $x_\lambda \to x_0$ we have $f(x_\lambda) \to f(x_0)$.*

Proof. f is continuous at $x_0 \in X$
iff $(\forall$ open $U \ni f(x_0))x_0 \in f^{-1}(U)^\circ$
iff $(\forall$ open $U \ni f(x_0))$ and $\forall x_\lambda \to x_0$, (x_λ) is eventually in $f^{-1}(U)$.
iff $(\forall$ open $U \ni f(x_0))$ and $\forall x_\lambda \to x_0$, $f(x_\lambda)$ is eventually in U.
iff $\forall x_\lambda \to x_0, f(x_\lambda) \to f(x_0)$. □

Definition 9.1.14. *Let (x_λ) be a net from Λ to X. (x_λ) is said to be an* **ultranet** *iff for each $E \subseteq X$, (x_λ) is either eventually in E or eventually in $X - E$.*

Theorem 9.1.15. *Every subnet of an ultranet is an ultranet.*

Theorem 9.1.16. *If an ultranet has x as a cluster point, then it converges to x.*

9.2 Filters

Definition 9.2.1. *A* **filter** *$\mathcal{F}$ on a set X is a nonempty collection of nonempty subsets of S such that*

1. *if $F_1, F_2 \in \mathcal{F}$, then $F_1 \cap F_2 \in \mathcal{F}$*
2. *if $F_1 \in \mathcal{F}$ and $F_1 \subseteq F_2$, then $F_2 \in \mathcal{F}$*

Definition 9.2.2. *Let $\mathcal{F}$ be a filter on X. A subcollection $\mathcal{C}$ of $\mathcal{F}$ is called a* **filter base** *iff $(\forall F \in \mathcal{F})(\exists C \in \mathcal{C})C \subseteq F$.*

Lemma 9.2.3. *Let $\mathcal{C}$ be a filter base of the filter $\mathcal{F}$. Then $(\forall C_1, C_2 \in \mathcal{C})(\exists C \in \mathcal{C})C \subseteq C_1 \cap C_2$.*

Lemma 9.2.4. *Let $\mathcal{C}$ be a collection of nonempty subsets of X. $\mathcal{C}$ is a filter base for some filter iff $(\forall C_1, C_2 \in \mathcal{C})(\exists C \in \mathcal{C})C \subseteq C_1 \cap C_2$. If $\mathcal{C}$ satisfies this condition, then its a filter base of the filter $\mathcal{F} = \{F \subseteq X | (\exists C \in \mathcal{C})C \subseteq F\}$. We also call $\mathcal{F}$ the filter generated by $\mathcal{C}$.*

Definition 9.2.5. *Let $\mathcal{F}$ be a filter on X. A subcollection $\mathcal{C}$ of $\mathcal{F}$ is called a* **filter subbase** *iff all finite intersections of $\mathcal{C}$ is a filter base of $\mathcal{F}$.*

Lemma 9.2.6. *Let $\mathcal{C}$ be a collection of nonempty subsets of X. $\mathcal{C}$ is a filter subbase for some filter iff each finite intersection of $\mathcal{C}$ is nonempty.*

Definition 9.2.7. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two filters on X. We say $\mathcal{F}_1$ is* **finer** *that $\mathcal{F}_2$ iff $\mathcal{F}_1 \supseteq \mathcal{F}_2$*

Definition 9.2.8. *Let $\mathcal{F}$ be a filter on X. We say $\mathcal{F}$ is* **fixed** *iff $\bigcap F \neq \emptyset$. We say $\mathcal{F}$ is* **free** *iff $\bigcap F = \emptyset$.*

Example 9.2.9. *Let X be a topological space and $x \in X$. The neighborhood $\mathcal{U}_x$ of x is a filter called the* **neighborhood filter***.*

Example 9.2.10. *Let X be a set and $S \subseteq X$. Then $\{T \subseteq X | S \subseteq T\}$ is a filter called the* **principle filter** *at S.*

Example 9.2.11. *Let X be a set. The* **Fréchet filter** *on X is defined as the collection of cofinite subsets in X. It's easy to see that the Fréchet filter is free.*

Definition 9.2.12. *Let $\mathcal{F}$ be a filter on a topological space X. $\mathcal{F}$ is said to* **converge to** *x (written as $\mathcal{F} \to x$) iff $\mathcal{F}$ is finer than the neighborhood filter $\mathcal{U}_x$ at x.*

Definition 9.2.13. *Let $\mathcal{F}$ be a filter on a topological space X. $\mathcal{F}$ is said to have x as a* **cluster point** *iff $\forall F \in \mathcal{F} : x \in F^-$.*

Theorem 9.2.14. *Let $\mathcal{F}$ be a filter on X. $\mathcal{F}$ has x as a cluster point iff there's a filter finer than $\mathcal{F}$ that converges to x.*

Theorem 9.2.15. *Let X be a topological space and $E \subseteq X$. Then $x \in E^-$ iff there is a filter that contains E and converges to x.*

This theorem together with Lem. 7.3.4 tell us that the topology of a space is determined by the convergence of filters in it.

Definition 9.2.16. *Let $\mathcal{F}$ be a filter on X, and $f : X \mapsto Y$. We define $f(\mathcal{F})$ as a filter in Y with filter base $\{f(F) | F \in \mathcal{F}\}$.*

Theorem 9.2.17. *Let $f : X \mapsto Y$. Then f is continuous at $x_0 \in X$ iff for each filter $\mathcal{F} \to x_0$ we have $f(\mathcal{F}) \to f(x_0)$.*

Proof. f is continuous at $x_0 \in X$
iff $(\forall \text{ open } U \ni f(x_0)) x_0 \in f^{-1}(U)^\circ$
iff $(\forall \text{ open } U \ni f(x_0)) f^{-1}(U) \in \mathcal{U}_{x_0}$
iff $(\forall \text{ open } U \ni f(x_0)) U \in f(\mathcal{U}_{x_0})$
iff $f(\mathcal{U}_{x_0}) \to f(x_0)$
iff $\forall \mathcal{F} \to x_0$ we have $f(\mathcal{F}) \to f(x_0)$. □

Definition 9.2.18. *Let $\mathcal{F}$ be a filter on X. $\mathcal{F}$ is said to be an ultrafilter iff there is no filter strictly finer than $\mathcal{F}$.*

Theorem 9.2.19. *Let $\mathcal{F}$ be a filter on X. $\mathcal{F}$ is an ultrafilter iff for each $E \subseteq X$, $E \in \mathcal{F}$ or $X - E \in \mathcal{F}$.*

Theorem 9.2.20. *Every filter is contained in some ultrafilter.*

Proof. Let $\mathcal{F}$ be a filter. Let FT be the set of all filters of X that contains $\mathcal{F}$. FT is directed by $\subseteq$. For each chain C in FT, $\bigcup C$ is a upper bound of C. So by Zorn's lemma, there's a maximal element in FT, which is an ultrafilter that contains $\mathcal{F}$. □

Theorem 9.2.21. *If an ultrafilter has x as a cluster point, then it converges to x.*

Theorem 9.2.22. *A fixed untrafilter is a principle filter at some one-point set.*

9.3 Correspondence between Nets and Filters

Definition 9.3.1. *Let (x_λ) be a net on X. We define* **the filter generated by** (x_λ) *as the family of sets that (x_λ) is eventually in. It has a filter base $\{\{x_\gamma|\gamma > \lambda\}\lambda \in \Lambda\}$.*

Definition 9.3.2. *Let $\mathcal{F}$ be a filter on X, and let $\Lambda = \{(x, F)|x \in F \in \mathcal{F}\}$. Λ is directed by the relation $(x_1, F_1) \leq (x_2, F_2)$ iff $F_1 \supseteq F_2$. We define* **the net based on** $\mathcal{F}$ *by the map $f : \Lambda \mapsto X$ where $f(x, F) = x$.*

Note that this direct set is not partially ordered by $\leq$ (if X contains at least 2 points).

Theorem 9.3.3. *Let (x_λ) be a net on X. Let (x_λ) generates a filter $\mathcal{F}$. Then*

1. *(x_λ) is eventually in a set $S \subseteq X$ iff $S \in \mathcal{F}$.*
2. *(x_λ) is frequently in a set $S \subseteq X$ iff $X - S \notin \mathcal{F}$, iff $\forall F \in \mathcal{F} : S \cap F \neq \emptyset$.*
3. *(x_λ) converges to $x \in X$ iff $\mathcal{F}$ converges to x.*
4. *(x_λ) has $x \in X$ as a cluster point iff $\mathcal{F}$ has x as a cluster point.*
5. *(x_λ) is an ultranet iff $\mathcal{F}$ is an ultrafilter.*
6. *A subnet of (x_λ) generate a filter finer than $\mathcal{F}$.*
7. *Let $\mathcal{F}'$ be a filter finer than $\mathcal{F}$. Then $\forall F \in \mathcal{F}' : (x_\lambda)$ is frequently in F.*

Theorem 9.3.4. *Let $\mathcal{F}$ be a filter on X, and (x_λ) be a net based on $\mathcal{F}$. Then (x_λ) generates $\mathcal{F}$.*

Lemma 9.3.5. *Let (x_λ) be a net on X and $\mathcal{F}$ be a filter on X such that $\forall F \in \mathcal{F} : (x_\lambda)$ is frequently in F. Then (x_λ) has a subnet $(x'_{\lambda'})$ such that $\forall F \in \mathcal{F} : (x'_{\lambda'})$ is eventually in F.*

We can give a proof by a slight generalization on the proof of the Thm. 9.1.11, which treats the special case that $\mathcal{F} = \mathcal{U}_x$.

Proof. We define $\Lambda' = \{(\lambda, F) \in \Lambda \times \mathcal{F}|x_\lambda \in F\}$. Λ' is directed by the partial order: $(\lambda, F) \leq (\lambda', F')$ iff $\lambda \leq \lambda'$ and $F \supseteq F'$. We define $\theta : \Lambda' \mapsto \Lambda$ as $\theta(\lambda, F) = \lambda$. Then $(x_{\theta(\lambda,F)})$ is a subnet that is eventually in each $F \in \mathcal{F}$. □

Corollary 9.3.6. *Let (x_λ) be a net on X that generates a filter $\mathcal{F}$, and $\mathcal{F}'$ be a filter finer than $\mathcal{F}$. Then (x_λ) has a subnet that generates a filter finer than $\mathcal{F}'$.*

Corollary 9.3.7 (Kelley)**.** *Every net has a subnet that is an ultranet.*

Proof. Let (x_λ) be the net, which generates a filter $\mathcal{F}$. Let $\mathcal{F}'$ be an ultrafilter finer than $\mathcal{F}$. Then (x_λ) has a subnet $(x'_{\lambda'})$ that generates a filter finer than $\mathcal{F}'$. However, since $\mathcal{F}'$ is an ultrafilter, $(x'_{\lambda'})$ generates $\mathcal{F}'$. So $(x'_{\lambda'})$ is an ultranet. □

9.4 Sequential Space

We have shown that the topology of a space is determined by the convergence of nets/filters in it. We may define a space whose topology space is determined by the convergence of sequences in it.

Definition 9.4.1. *Let X be a topological space. X is a* **sequential space** *iff for each $E \subseteq X$, E is closed iff for each converging sequence in E converges to a point in E.*

Definition 9.4.2. *Let X be a topological space. X is a* **Fréchet-Urysohn space** *iff for each $E \subseteq X$, $x \in E^-$ iff there is a sequence in E that converges to x.*

Theorem 9.4.3. *A space is a Fréchet-Urysohn space if and only if every subspace is a sequential space.*

Proof. Let X be a Fréchet-Urysohn space, and S be a subspace. For each subset E of S. 1. If E is closed, let $E = E' \cap S$ where E' is closed in X. Let (x_n) be a sequence in E that converges to x in S. It's easy to see that (x_n) converges to x in X. So $x \in E'^- = E'$ in X. So $x \in E$. 2. If for each sequence in E that converges to x we have $x \in E$. We define E_1 to be E^- in S and E_2 to be E^- in X. Clearly $E_1 = E_2 \cap S$. For each $y \in E_1 \subseteq E_2$. Let (x_n) be a sequence in E that converges to y. So $E = E_1$ is closed in S.

Let X be a space such that every subspace is a sequential space. For each $E \subseteq X$. Consider the subspace $S = E \cup \{x\}$. Since S is a sequential space, $x \in E^- \leftrightarrow E$ is not closed in $S \leftrightarrow$ there's a sequence in E that converges to x. □

Corollary 9.4.4. *The subspace of a Fréchet-Urysohn space is a Fréchet-Urysohn space.*

Theorem 9.4.5. *Let X be a sequential space, and $f : X \mapsto Y$. Then f is continuous at $x_0 \in X$ iff for each sequence $(x_n) \to x_0$ we have $f(x_n) \to f(x_0)$.*

Proof. f is continuous at $x_0 \in X$
iff $(\forall \text{ open } U \ni f(x_0)) x_0 \in f^{-1}(U)^\circ$
iff $(\forall \text{ open } U \ni f(x_0))$ and $\forall x_n \to x_0$, (x_n) is eventually in $f^{-1}(U)$.
iff $(\forall \text{ open } U \ni f(x_0))$ and $\forall x_n \to x_0$, $f(x_n)$ is eventually in U.
iff $\forall x_n \to x_0, f(x_n) \to f(x_0)$. □

Chapter 10

Separation and Countability

In the first part of this chapter we propose some conditions of separation, that describe how separate two points or two closed sets are in a topological space. In the second part we propose some conditions of countability, that describe the countability of the basis of a topological space. The stricter condition a topological space satisfies, the better properties it will enjoy.

10.1 T_0, T_1 and Hausdorff(T_2) Spaces

Definition 10.1.1. *A T_0* **space***, as shown in Fig.10.1, is a topological space such that for each two different points x and y, there is an open set that contains one and not the other.*

Definition 10.1.2. *A T_1* **space***, as shown in Fig.10.2, is a topological space such that for each two different points x and y, each has a neighborhood that doesn't contain the other.*

Definition 10.1.3. *A topological space is a T_1 space iff each one point set is closed.*

Definition 10.1.4. *A topological space X is a T_1 space iff for each $x \in X$, the union of all neighborhoods at x is $\{x\}$.*

Theorem 10.1.5. *The closed image of a T_1 space is T_1.*

Definition 10.1.6. *A T_2* **space***, also called a* **Hausdorff space***, as shown in Fig.10.3, is a topological space such that for each two different points x and y, there are two disjoint open sets $S \ni x$ and $T \ni y$.*

Theorem 10.1.7. *A topological space is a Hausdorff space iff each net or filter only converges to at most one point.*

Proof. Let X be a topological space that is not a T_2 space. Let $x, y \in X$ be two different points such that $\forall S \in \mathcal{U}_x, T \in \mathcal{U}_y : S \cap T \neq \emptyset$. (Remember that $\mathcal{U}_x$ is the neighborhood filter at x.) Let $\mathcal{F} = \{S \cap T | S \in \mathcal{U}_x, T \in \mathcal{U}_y\}$. It's easy to see that $\mathcal{F}$ is a filter that contains $\mathcal{U}_x$ and $\mathcal{U}_y$. So $\mathcal{F}$ converges to both x and y. $\square$

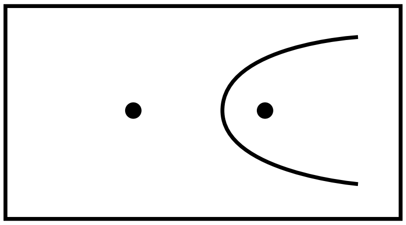

Figure 10.1: T_0 space

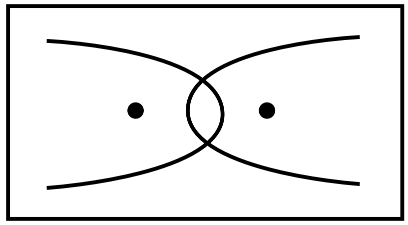

Figure 10.2: T_1 space

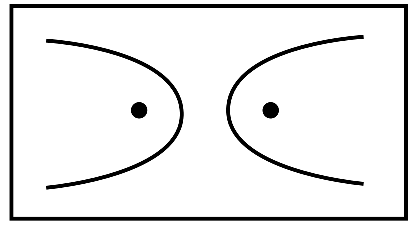

Figure 10.3: T_2 space

Definition 10.1.8. *A* **Urysohn space***, also called a* $T_{2\frac{1}{2}}$ **space***, is a topological space such that for each two different points x and y, there are two disjoint open sets $S \ni x$ and $T \ni y$ such that $S^- \cap T^- = \emptyset$.*

Theorem 10.1.9. *A Urysohn space is a T_2 space. A T_2 space is a T_1 space. A T_1 space is a T_0 space.*

Theorem 10.1.10. *The subspace of a $T_0/T_1/T_2/$ Urysohn space is $T_0/T_1/T_2/$ Urysohn.*

Theorem 10.1.11. *The product space is a $T_0/T_1/T_2/$ Urysohn space iff each factor space is $T_0/T_1/T_2/$ Urysohn.*

Lemma 10.1.12. *Let X be a Hausdorff space, then the diagonal $\Delta(X) = \{(x, x)|x \in X\}$ is closed in $X \times X$.*

Theorem 10.1.13. *Let Y be a Hausdorff space, and $f : X \mapsto Y$ and $f' : X \mapsto Y$ be two continuous maps from X to Y that coincide on a dense subset of X. Then $f = f'$.*

Proof. We define a continuous map $g : X \mapsto Y \times Y$ by $g(x) = (f(x), f'(x))$. $g^{-1}(\Delta(Y))$ is dense and closed in X. So $g^{-1}(\Delta(Y)) = X$. □

10.2 Regular Spaces

Definition 10.2.1. *A space X is a* **regular space***, as shown in Fig.10.4, iff for each closed set $A \subseteq X$ and each $b \notin A$, there are two disjoint open sets $S \supseteq A$ and $T \ni b$. A space X is a T_3* **space** *iff its T_1 and regular.*

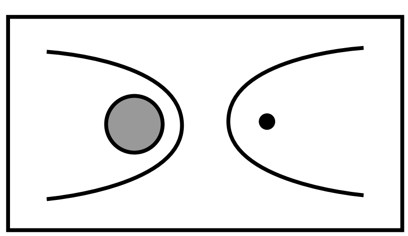

Figure 10.4: Regular space

Theorem 10.2.2. *A space X is a regular space iff for each open set $A \subseteq X$ and each $x \in A$, there exists an open set $B \ni x$ such that $B^- \subseteq A$.*

This means that regularity is a local property.

Theorem 10.2.3. *A T_0 regular space is T_3.*

Definition 10.2.4. *A space X is a* **completely regular space** *iff for each closed set $A \subseteq X$ and each $x \notin A$, there exists a continuous function $f : X \mapsto I$ such that $f(x) = 0$ and $f(A) = 1$. A space X is a* **Tychonoff space**, *also called a $T_{3\frac{1}{2}}$ space, iff its T_1 and completely regular.*

Theorem 10.2.5. *A Tychonoff space is a T_3 space. A T_3 space is a Urysohn space. A completely regular space is a regular space.*

Theorem 10.2.6. *The subspace of a regular/T_3/completely regular/Tychonoff space is regular/T_3/completely regular/Tychonoff.*

Theorem 10.2.7. *The product space is a regular/T_3/completely regular/Tychonoff space iff each factor space is regular/T_3/completely regular/Tychonoff.*

Theorem 10.2.8. *A space X is completely regular iff it has the weak topology induced by $C^*(X)$.*

Proof. Let X be a completely regular space. Let $\mathcal{B}$ be a basis of X. For each $x \in X$ and each $B \in \mathcal{B}$ such that $x \in B$. It's easy to see that there exists a function $f \in C^*(X)$ such that $f(x) = 0$ and $f(X - B) = 1$. So $x \in f^{-1}(-\infty, 1/2) \subseteq B$. So $f^{-1}(U)$ for all $f \in C^*(X)$ and open sets $U \subseteq \mathbb{R}$ form a base of X. So X has the weak topology induced by $C^*(X)$.

Let X be a space with the weak topology induced by $C^*(X)$. $\mathcal{S} = \{f^{-1}(U)|f \in C^*(X), U = (-\infty, a)$ or (a, ∞) for some $a \in \mathbb{R}\}$ form a subbase of X. Actually $\mathcal{S} = \{f^{-1}(0, \infty)|f \in C^*(X)\}$. Let $\mathcal{B}$ be the base generated by $\mathcal{S}$. For each closed set $A \subseteq X$ and each $x \notin A$, there exists $B \in \mathcal{B}$ such that $b \in B$ and $B \cap A = \emptyset$. Let $B = \cap_i f_i^{-1}(0, \infty)$. Let $f = \prod_i f_i$. It's easy to see that $f \in C^*(X)$, $B = f^{-1}(0, \infty)$, and $f(X - B) = 0$. So $f(x) \neq 0$ and $f(A) = 0$. So X is completely regular. □

Corollary 10.2.9. *A space is a Tychonoff space iff it can be embedded in a cube $\prod I_a$ (product of copies of the unit interval I).*

Proof. Use Thm. 8.2.9. □

10.3 Normal Spaces

Definition 10.3.1. *A space X is a* **normal space**, *as shown in Fig.10.5, iff for each two disjoint closed sets A and each B, there are two disjoint open sets $S \supseteq A$ and $T \supseteq B$. A space X is a T_4* **space** *iff its T_1 and normal.*

Theorem 10.3.2. *A space X is a normal space iff for each open set $A \subseteq X$ and each closed set $B \subseteq A$, there exists an open set $C \supseteq B$ such that $C^- \subseteq A$.*

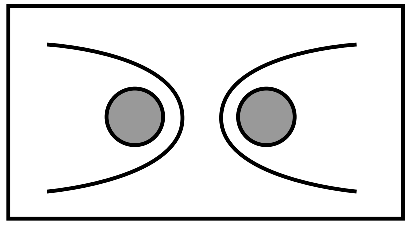

Figure 10.5: Normal space

Theorem 10.3.3 (Urysohn). *A space X is a normal space iff for each two disjoint closed sets A and each B, there exists a continuous function $f : X \mapsto I$ such that $f(A) = 0$ and $f(B) = 1$. Such a function is called a* **Urysohn function** *for A and B.*

Proof. Let X be a normal space. We define $U_0 = A$ and $U_1 = X - B$. We have $U_0^- \subseteq U_1$. We define $U_{\frac{n}{2^m}}$ for $m > 0, 0 \leq n \leq 2^m$ by induction: with U_a and U_b $(a < b)$ we define $U_{\frac{a+b}{2}}$ as an open set such that $U_a^- \subseteq U_{\frac{a+b}{2}} \subseteq U_{\frac{a+b}{2}}^- \subseteq U_b$. We define $f : X \mapsto \mathbb{R}$ as $f(x) = \inf\{r | x \in U_r\}$. It's easy to see that f is continuous, $f(A) = 0$ and $f(B) = 1$. □

Corollary 10.3.4. *Every T_4 space is Tychonoff.*

Theorem 10.3.5 (Tietze). *X is a normal space iff for each closed sets $A \subseteq X$ and each continuous map $f : A \mapsto \mathbb{R}$, there is an extension of f to all of X.*

Proof. We prove $\Rightarrow$. Since $[-1, 1]$ is homeomorhpic to $\mathbb{R}$, we only need to prove that each continuous map $f : A \mapsto [-1, 1]$ can be extended to all of X. Let $g_1 = f$. We define f_i, g_i, A_i and B_i by induction:

1. $A_i = g_i^{-1}([\frac{2^{i-1}}{3^i}, \frac{2^{i-1}}{3^{i-1}}])$ and $B_i = g_i^{-1}([-\frac{2^{i-1}}{3^{i-1}}, -\frac{2^{i-1}}{3^i}])$.
2. f_i be the Urysohn function $X \mapsto [-\frac{2^{i-1}}{3^i}, \frac{2^{i-1}}{3^i}]$ such that $f_i(A_i) = \frac{2^{i-1}}{3^i}$ and $f_i(B_i) = -\frac{2^{i-1}}{3^i}$.
3. $g_{i+1} = g_i - f_i|_A$. It's easy to see that $g_{i+1}(x) \in [-\frac{2^i}{3^i}, \frac{2^i}{3^i}]$

Since $\{f_i\}$ converges uniformly, $F = \sum_i f_i$ is continuous. It's easy to see that F is an extension of f to all of X. □

Note: we use f_i to approach f bit by bit while remaining controlled at $X - A$.

Corollary 10.3.6. *Let X be a normal space. For each closed sets $A \subseteq X$, each open $U \supseteq A$, and each continuous map $f : A \mapsto \mathbb{R}$, there is an extension of f to all of X such that supp$(f) \subseteq U$.*

Theorem 10.3.7. *The closed subspace of a normal/T_4 is normal/T_4.*

Definition 10.3.8. *A space X is called* **completely normal** *each subspace of X is normal. A T_1 completely normal space is called a T_5 space.*

Theorem 10.3.9. *A space X is completely normal iff for each pair of subsets of X, named A and B, such that $A \cap B^- = A^- \cap B = \emptyset$, there are two disjoint open sets that contain A and B respectively.*

Proof. Let X be completely normal. Consider the open subspace $Y = X - A^- \cap B^-$. $A^- \cap Y$ and $B^- \cap Y$ are disjoint closed sets in Y, and are contained in disjoint open sets A' and B'. A' and B' are disjoint open sets that contain A and B respectively.

Let X be a space that for each pair of subsets that satisfy the requirement in the theorem, there are two disjoint open sets that contain each of them respectively. For each subspace S of X, and each disjoint closed sets A and B of S. In X, it's easy to see that $A \cap B^- = A^- \cap B = \emptyset$. So There are two disjoint open sets in X that contain A and B respectively. So S is normal. □

Definition 10.3.10. *A space X is called* **perfectly normal** *if for each pair of disjoint closed sets A and B, there exists a continuous function $f : X \mapsto I$ such that $A = f^{-1}(0)$ and $B = f^{-1}(1)$. A T_1 perfectly normal space is called a T_6 space.*

Theorem 10.3.11. *A space X is perfectly normal iff*

1. *for each closed set A, there exists a continuous function $f : X \mapsto I$ such that $A = f^{-1}(0)$;*

2. *each closed set is a countable intersection of open sets.*

Proof. Clearly perfect normality $\to 1 \to 2$.

$2 \to 1$: Let $A = \cup_n U_n$ be a closed set with all U_n open. For each U_n, define a Urysohn function $f_n : X \mapsto I$ such that $f_n(A) = 0$ and $f_n(X - U_n) = 1$. Define $f = \sum_n f_n/2^n$. Then $A = f^{-1}(0)$.

$1 \to$ perfect normality: Let A and B be two disjoint closed sets, and $A = f^{-1}(0)$ and $B = g^{-1}(0)$. Then $f/(f + g)$ works. □

Theorem 10.3.12. *The subspace of a perfectly normal space is perfectly normal.*

Corollary 10.3.13. *A perfectly normal space is completely normal. A completely normal space is normal.*

Lemma 10.3.14. *Each pseudo-metric space is perfectly normal. Each metric space is T_6. $T_6 \to T_5 \to T_4$.*

10.4 Shrinking Lemma

Definition 10.4.1. *Let X be a space. A collection of subsets of X is called* **locally finite** *iff each $x \in X$ has a neighborhood that intersects only finitely may of the sets in the collection. A collection of subsets of X is called* **discrete** *iff each $x \in X$ has a neighborhood that intersects at most one set in the collection.*

Lemma 10.4.2. *Let X be a normal space with an open cover (U_1, U_2). There exists an open set V_1 such that $V_1^- \subseteq U_1$ and (V_1, U_2) covers X.*

Proof. $X - U_2$ and $X - U_1$ are disjoint closed sets, covered by open sets V_1 and V_2 respectively. V_1 is what we want. □

Lemma 10.4.3. *Let X be a normal space with an open cover $(U_1, \ldots, U_n)$. There exists an open cover $(V_1, \ldots, V_n)$ such that $\forall i(V_i^- \subseteq U_i)$.*

Proof. By induction. □

Lemma 10.4.4. *Let X be a normal space with a locally finite open cover $\{U_i\}_{i\in I}$. There exists an open cover $\{V_i\}_{i\in I}$ such that $\forall i \in I(V_i^- \subseteq U_i)$.*

Proof. We consider the set S of pairs $(J, \mathcal{V})$ where $J \subseteq I$ and $\mathcal{V} = \{V_i\}_{i\in I}$ is an open cover of X such that

1. $i \in J \to V_i^- \subseteq U_i$;
2. $i \notin J \to V_i = U_i$.

We equip the set S with a partial order $(J_1, \mathcal{V}_1) \le (J_2, \mathcal{V}_2) \leftrightarrow (J_1 \subseteq J_2) \wedge (\forall i \in J_1(V_{1i} = V_{2i}))$. It's easy to see that each chain has a maximal element. By Zorn's lemma, there's a maximal element $(J, \mathcal{V})$. It's easy to see that $J = I$. □

10.5 Countability

Definition 10.5.1. *A space is called* **1st countable** *if each point has a countable local base.*

Example 10.5.2. *A pseudo-metric space is 1st countable.*

Lemma 10.5.3. *Let X be a 1st countable space. At each $x \in X$ we can find an countable local base $\{B_n | n \in \mathbb{N}\}$ such that $B_1 \supseteq B_2 \supseteq \cdots$.*

Corollary 10.5.4. *A 1st countable space is a Fréchet-Urysohn space, and hence a sequential space.*

Theorem 10.5.5. *A 1st countable space is a Hausdorff space iff each sequence only converges to at most one point.*

Lemma 10.5.6. *A product of 1st countable spaces is 1st countable iff each factor space is 1st countable and all but countably many of the factor spaces are trivial.*

Proof. Let $\prod_{\alpha\in A} X_\alpha$ be 1st countable. Then clearly each X_α is 1st countable. Let $A' = \{\alpha \in A | X_\alpha$ is non-trivial$\}$. We can find $(x_\alpha) \in \prod_{\alpha\in A} X_\alpha$ such that $\forall \alpha \in A'$ there exists a non-trivial open set $U_\alpha \ni x_\alpha$. Let $\mathcal{B}$ be a local base at (x_α). For each $B \in \mathcal{B}$ we can find $B \supseteq \prod_{\alpha\in A_B} U(B)_\alpha \times \prod_{\alpha\notin A_B} X_\alpha \ni x$, where $U(B)_\alpha$ is an non-trivial open set in X_α. Clearly A_B is finite. Let $A'' = \bigcup_B A_B$. A'' is countable. It's easy to see that $A'' \subseteq A'$. If $A'' \neq A'$, let $\beta \in (A' - A'')$, and U_β be a non-trivial open set in X_β. Then $U_\beta \times \prod_{\alpha\in A,\alpha\neq\beta} X_\alpha$ is an open set in $\prod_{\alpha\in A} X_\alpha$ but contains no base in $\mathcal{B}$, a contradiction. So $A' = A''$ is (at most) countable. □

Definition 10.5.7. *A space is called* **2nd countable** *if it has a countable base. Clearly a 2nd countable space is a 1st countable space*

Definition 10.5.8. *A space is called* **separable** *if it has a countable dense subset.*

Definition 10.5.9. *Let X be a topological space. A family $\mathcal{A}$ of subsets of X is called a* **cover** *of X if the union of $\mathcal{A}$ is X. A subfamily of $\mathcal{A}$ is called a* **subcover** *if its union is X. An* **open cover** *is a cover of open sets.*

Definition 10.5.10. *A space is called a* **Lindelöf space** *if each open cover of it has a countable subcover.*

Theorem 10.5.11. *A regular Lindelöf space is normal.*

Proof. Let A and B be two disjoint closed sets in a regular Lindelöf space X. For each $a \in A$ we choose an open set (using AC) $U_a \ni a$ such that $U_a^- \cap B = \emptyset$. Since $\{U_a\}$ covers A, we can find a countable (assumed to be infinite) subcover $\{U_n | n \in \mathbb{N}\}$, such that $A \subseteq \bigcup U_n$. Similarly we have $B \subseteq \bigcup V_n$. However, at present, $(\bigcup U_n) \cap (\bigcup V_n) \neq \emptyset$.

We can define U'_n and V'_n by induction:

1. $U'_i = U_i - (\bigcup_{n<i} V'_i)^-$
2. $V'_i = V_i - (\bigcup_{n \leq i} U'_i)^-$

It's easy to see that $\bigcup U_n$ and $\bigcup V_n$ and disjoint open sets that contains A and B respectively. □

Theorem 10.5.12. *A 2nd countable space is a separable Lindelöf space.*

Theorem 10.5.13. *Let X be a pseudometric space. The following are equivalent:*

1. *X is 2nd countable.*
2. *X is Lindelöf.*
3. *X is separable.*

Chapter 11

Compactness

11.1 Compact Space

Definition 11.1.1. *A topological space is a* **compact space** *iff each open cover has a finite subcover.*

Lemma 11.1.2. *A topological space is a compact space iff each family of closed sets with empty intersection has a finite subfamily with empty intersection.*

Theorem 11.1.3. *Let X be a topological space. The following statements are equivalent:*

1. *X is compact*
2. *Each filter (or equivalently each net) in X has a cluster point.*
3. *Each filter (or equivalently each net) in X has a convergent finer filter(subnet).*
4. *Each ultrafilter (or equivalently each ultranet) in X converges.*

Proof. $1 \Leftrightarrow 2$:
X is compact
iff $\forall$ family of closed sets $\mathcal{F}$ such that $\bigcap_{F\in\mathcal{F}} F = \emptyset$ there's a finite subfamily $\mathcal{F}'$ such that $\bigcap_{F\in\mathcal{F}'} F = \emptyset$
iff $\forall$ family $\mathcal{F}$ such that $\bigcap_{F\in\mathcal{F}} F^- = \emptyset$ there's a finite subfamily $\mathcal{F}'$ such that $\bigcap_{F\in\mathcal{F}'} F^- = \emptyset$
iff $\forall$ family $\mathcal{F}$ such that $\bigcap_{F\in\mathcal{F}} F^- = \emptyset$ is not a filter subbase
iff $\forall$ filter $\mathcal{F}(\bigcap_{F\in\mathcal{F}} F^- \neq \emptyset)$
It's easy to see that $2 \Leftrightarrow 3 \Leftrightarrow 4$. $\square$

Theorem 11.1.4. *A compact subset of a Hausdorff space is closed.*

Theorem 11.1.5. *A closed subset of a compact space is compact.*

Theorem 11.1.6. *The continuous image of a compact space is compact.*

Theorem 11.1.7. *A non-empty product space is compact iff each factor space is compact.*

Proof. Let $X = \prod_\alpha X_\alpha$. If each X_α is compact. Let $\{(x^i_\alpha)|i \in \Lambda\}$ be a supernet in X. It's easy to see that for each α, $\{x^i_\alpha|i \in \Lambda\}$ is a supernet in X_α, and it converges, say to x_α. It's easy to see that $\{(x^i_\alpha)|i \in \Lambda\}$ converges to (x_α). □

Theorem 11.1.8. *I is compact with usual topology.*

Proof. For each open cover $\mathcal{U}$ of I, let $S = \{x|[0, x]$ is covered by finite elements of $\mathcal{U}\}$. If $\sup S < 1$, then let $\sup S \in (a, b) \subseteq U \in \mathcal{U}$, where $\sup S < b$. Then $b \in S$, a contradiction. □

Corollary 11.1.9. *A subspace of $\mathbb{R}^n$ with usual topology is compact iff it's closed and bounded.*

Lemma 11.1.10. *Disjoint compact subsets of a Hausdorff space can be separated by disjoint open sets.*

Corollary 11.1.11. *A compact Hausdorff space X is a T_4 space.*

Definition 11.1.12. *A topological space is a* **countably compact space** *iff each countable open cover has a finite subcover. Clearly a space is compact iff it's Lindelöf and countably compact.*

Lemma 11.1.13. *A topological space is a countably compact space iff each countable family of closed sets with empty intersection has a finite subfamily with empty intersection.*

Theorem 11.1.14. *A closed subset of a countably compact space is countably compact.*

Theorem 11.1.15. *A continuous real-valued function on a countable compact space is bounded.*

Theorem 11.1.16. *A space X is countably compact iff each sequence in X has a cluster point.*

We prove this theorem following the spirit of the proof of Thm. 11.1.3.

Proof. $\Rightarrow$: Let $\{x_n\}$ be a sequence. For each $n \in \mathbb{N}$, we define $S_n = \{x_m|m \geq n\}^-$. If $\bigcap_n\{S_n|n \in \mathbb{N}\} = \emptyset$, then $\exists n_0(\bigcap_n\{S_n|n < n_0\} = \emptyset)$. That means $\{x_m|m \geq n_0\}^- = \emptyset$, which is impossible. So $\bigcap_n S_n \neq \emptyset$. Any $x \in \bigcap_n S_n$ is a cluster point of $\{x_n\}$.

$\Leftarrow$: Let $\{U_n|n \in \mathbb{N}\}$ be a family of closed sets such that $\bigcap_n\{U_n|n \in \mathbb{N}\} = \emptyset$. If $(\forall n \in \mathbb{N})\bigcap_{m\leq n} U_m \neq \emptyset$, we can always choose $x_n \in \bigcap_{m\leq n} U_m$. The sequence $\{x_n\}$ has a cluster point a. Let $a \notin U_m$, then $\forall k > m(x_k \in U_m)$, a contradiction. □

Definition 11.1.17. *A topological space X is a* **sequentially compact space** *iff each sequence in X has a convergent subsequence.*

Definition 11.1.18. *A topological space X is a* **limit-point compact space** *iff each infinite subset in X has a limit point.*

Definition 11.1.19. *A space is* **locally compact** *iff each point has a compact neighborhood.*

Theorem 11.1.20. *In a locally compact Hausdorff space, each point has a compact neighborhood base.*

Proof. Let x be a point with a compact neighborhood U and a local base $\mathcal{N}_x$. U is a compact Hausdorff space, so it's T_3. For each $N \in \mathcal{N}_x$, $N \cap U$ is an open neighborhood of x, which contains a closed neighborhood N' of x in U, which is compact. It's easy to see all N's form a compact neighborhood base. □

11.2 Relationship between Different Compact Conditions

Theorem 11.2.1. *Every sequentially compact space is countable compact.*

Proof. Each sequence has a cluster point that a subsequence converges to. □

Theorem 11.2.2. *Every countable compact space is limit-point compact.*

Proof. We can construct a sequence in a infinite subset such that any two elements are different. This sequence converges to a limit point of the subset. □

Theorem 11.2.3. *A T_1 limit-point compact space is countable compact.*

Proof. For each sequence with infinite elements, we have a limit point. Since the space is T_1, the limit point has a neighborhood that excludes the first n points of the sequence for each n. So the limit point is a cluster point of the sequence. □

Theorem 11.2.4. *A sequential limit-point compact space is sequentially compact.*

Proof. Let (x_n) be a sequence with infinite elements. Let $A = \{x_n | n \in \mathbb{N}\}$ with limit point x. Clearly $A - \{x\}$ is not closed. So there's a sequence (y_n) in $A - \{x\}$ that converges. Then it's easy to find a subsequence of (x_n) that converges. □

Theorem 11.2.5. *A Lindelöf countable compact space is compact.*

Lemma 11.2.6. *Every sequentially compact metric space is 2nd countable.*

Proof. Let X be a sequentially compact metric space. For each $n \in \mathbb{N}$, pick $x_1^{[n]} \in X$, then pick $x_2^{[n]} \in (X - B(x_1^{[n]}, 1/n))$ if possible, then pick $x_3^{[n]} \in (X - B(x_1^{[n]}, 1/n) \cup B(x_2^{[n]}, 1/n))$ if possible …For each n, this process must stop in M_n steps. Otherwise we have a sequence $(x_i^{[n]})$, which has a cluster point $x^{[n]}$. $B(x^{[n]}, 1/n)$ contains some elements in $(x_i^{[n]})$, say $x_m^{[n]}$. Thus $\rho(x^{[n]}, x_m^{[n]}) < 1/n$. Thus $B(x^{[n]}, 1/n - \rho(x^{[n]}, x_m^{[n]}))$ contains no element in $(x_i^{[n]})$, a contradiction. It's easy to see that $\{B(x_i^{[n]}, 1/n) | n \in \mathbb{N}, i \leq M_n\}$ forms a countable base. □

The relationship between different compact conditions is shown in Fig. 11.1.

11.3 Compactification

Definition 11.3.1. *Let X be a topological space. A* **compactification***, written as (Y, f), is an embedding f of X to a compact space Y such that $f(X)$ is dense in Y. If Y is further Hausdorff, the compactification is called a* **Hausdorff compactification***.*

Lemma 11.3.2. *A space has a Hausdorff compactification iff it's Tychonoff.*

Definition 11.3.3. *Let X be a topological space. Let $X' = X \cup \{\infty\}$ ($\infty \notin X$). In X', let $\{X' - T | T$ is compact in $X\}$ be a neighborhood base at ∞ and the neighborhood base at any $x \in X$ is identical to X. This gives us a topology of X', with which X' is a compact space. This process is illustrated in Fig. 11.2. X' is called the* **Alexandroff extension** *of X.*

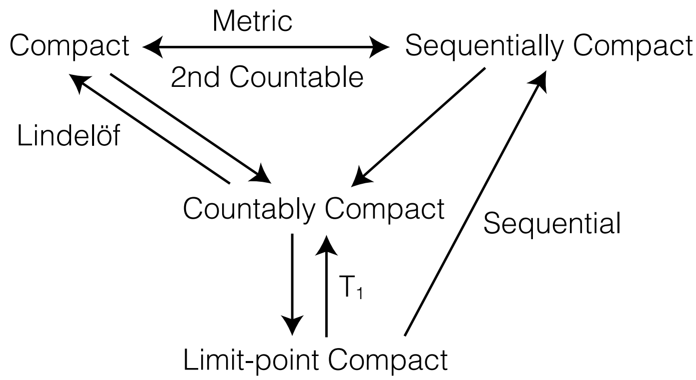

Figure 11.1: Relationship between different compact conditions.

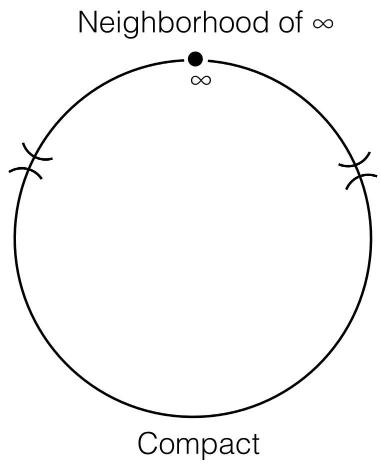

Figure 11.2: Illustration of Alexandroff extension.

Lemma 11.3.4. *Let X be a locally compact and Hausdorff but not compact space. Let X' be the Alexandroff extension of X. Then (X', ι) is a Hausdorff compactification, called* **one-point compactification**.

Corollary 11.3.5. *A locally compact Hausdorff space is Tychonoff.*

Definition 11.3.6. *Let X be a Tychonoff space. As in Col. 10.2.9, we have an embedding $e : X \mapsto \prod_{f \in C^*(X)} I_f$, where e is the evaluation map induced by $C^*(X)$. The map $X \mapsto e(X)^-$ is a Hausdorff compactification, called the* **Stone-Čech compactification**. *The space $e(X)^-$ is usually denoted by βX.*

Theorem 11.3.7. *Let K be a compact Hausdorff space, X be a Tychonoff space, $(\beta X, e)$ be the Stone-Čech compactification of X. For each continuous map $f : X \mapsto K$ be continuous, there's a unique continuous map $F : \beta X \mapsto K$ such that $F \circ e = f$.*

Proof. We already have an embedding $e : X \to \prod_{g \in C^*(X)} I_g$. Similarly we can construct an embedding $e' : K \to \prod_{g \in C^*(K)} I_g$. Using $f : X \mapsto K$, can construct a continuous function $T_f : \prod_{g \in C^*(X)} I_g \mapsto$

$\prod_{g\in C^*(K)} I_g$ by

$$(T_f(a))_g = (a)_{g\circ f},\ \forall g \in C^*(K) \tag{11.1}$$

Since $\forall x \in X \forall g \in C^*(K)$, $(T_f(e(x)))_g = (e(x))_{g\circ f} = g(f(x))$. We have $T_f \circ e = e' \circ f$. That is, the following diagram commutes.

$$\begin{tikzcd}
\prod_{g\in C^*(X)} I_g \arrow[r, "T_f"] & \prod_{g\in C^*(K)} I_g \\
X \arrow[u, "e"] \arrow[r, "f"] & K \arrow[u, "e'"']
\end{tikzcd}$$

Clearly $T_f(e(X)) \subseteq e'(K)$. Thus $T_f(\beta X) \subseteq e'(K)^- = e'(K)$. We define $F : \beta X \mapsto K$ as $F = e'^{-1} \circ T_f|_{\beta X}$. It's easy to see that $F \circ e = f$.

For uniqueness, use Thm. 10.1.13. □

Definition 11.3.8. *Let X be Tychonoff space, $\tilde{X}$ be the class of all compactifications of X. We define a equivalence relation $(K_1, h_1) \sim (K_2, h_2)$ iff there is an homeomorphism $f : K_1 \mapsto K_2$ such that $f \circ h_1 = h_2$. Let $[\tilde{X}]$ be the resulting equivalent classes. We can define a binary relation on $[\tilde{X}]$: $[(K_1, h_1)] \leq [(K_2, h_2)]$ iff there's a continuous function $F : K_2 \mapsto K_1$ such that $F \circ h_2 = h_1$.*

Lemma 11.3.9. *The binary relation $\leq$ we just defined is a well-defined partial order.*

Proof. Let $[(K_1, h_1)], [(K_2, h_2)] \in [\tilde{X}]$, $[(K_1, h_1)] \leq [(K_2, h_2)]$ with continuous function $F : K_2 \mapsto K_1$, and $[(K_2, h_2)] \leq [(K_1, h_1)]$ with continuous function $G : K_1 \mapsto K_2$. We have $G \circ F \circ h_2 = G \circ h_1 = h_2$. So $G \circ F|_{h_2(X)} = 1_{h_2(X)}$. Since $h_2(X)$ is dense in K_2, we have $G \circ F = 1$. Similarly $F \circ G = 1$. So $F = G^{-1}$ is a homeomorphism. So $[(K_1, h_1)] = [(K_2, h_2)]$. □

Theorem 11.3.10. *Let X be Tychonoff space. The Stone-Čech compactification $[(\beta X, e)]$ is the maximal element in $[\tilde{X}]$.*

11.4 Paracompactness

Definition 11.4.1. *Let $\mathcal{U}$ and $\mathcal{V}$ be two covers of X. $\mathcal{V}$ is a* **refinement** *of $\mathcal{U}$ iff $(\forall V \in \mathcal{V})(\exists U \in \mathcal{U}) V \subseteq U$.*

Definition 11.4.2. *Let X be a space. A collection of subsets of X is* **σ-locally finite** *iff it is the union of a countable family of locally finite collections of subsets of X. A collections of subsets of X is* **σ-discrete** *iff it is the union of a countable family of discrete collections of subsets of X.*

Definition 11.4.3. *A Hausdorff space X is* **paracompact** *iff each open cover of X has an open locally finite refinement.*

Theorem 11.4.4 (Michael)**.** *Let X be a T_3 space. The following are equivalent:*

1. *X is paracompact.*
2. *Each open cover of X has an open σ-locally finite refinement.*

3. *Each open cover of X has a locally finite refinement.*

4. *Each open cover of X has a closed locally finite refinement.*

Proof. $2 \to 3$: Let $\mathcal{U}$ be an open cover of X. Let $\mathcal{V} = \bigcup_n \mathcal{V}_n$ be an open σ-locally finite refinement. Let $A_n = \bigcup \mathcal{V}_n - \bigcup_{m<n}(\bigcup \mathcal{V}_m)$. Then $\{A_n | n \in \mathbb{N}\}$ is a locally finite cover of X. It's easy to see that $\{V_n \cap A_n | V_n \in \mathcal{V}_n, n \in \mathbb{N}\}$ is a locally finite refinement of $\mathcal{U}$.

$3 \to 4$: Let $\mathcal{U}$ be an open cover of X. For each $x \in X$, chose a $U_x \in \mathcal{U}$ such that $x \in U_x$. Choose an open set V_x such that $x \in V_x \subseteq V_x^- \subseteq U_x$. Then $\{V_x | x \in X\}$ is a open refinement of $\mathcal{U}$, and has a locally finite refinement $\mathcal{S}$. It's easy to see that $\{S^- | X \in \mathcal{S}\}$ is a closed locally finite refinement of $\mathcal{U}$.

$4 \to 1$: Let $\mathcal{U}$ be an open cover of X, which has a closed locally finite refinement $\mathcal{V}$. For each $x \in X$, we can choose W_x that intersects with finitely many of elements in $\mathcal{V}$. This forms an open cover $\mathcal{W} = \{W_x | x \in X\}$, which has a closed locally finite refinement $\mathcal{A}$. For each $V \in \mathcal{V}$, let $V^* = X - \bigcup\{A \in \mathcal{A} | A \cap V = \emptyset\}$. Clearly $\{V^* | V \in \mathcal{V}\}$ is an open refinement. Furthermore, for each $x \in X$, there's a neighborhood N of x that only intersects with finitely many members of $\mathcal{A}$. Since each element in $\mathcal{A}$ only intersects with finitely many members of $\mathcal{V}$. We see that N only intersects with finitely many V^*s. So $\{V^* | V \in \mathcal{V}\}$ is an open locally finite refinement of $\mathcal{U}$. □

Theorem 11.4.5 (Stone). *Every pseudometric space is paracompact.*

Proof. Let $\mathcal{U}$ be an open cover of the metric space X. For each $n \in \mathbb{N}$ and $U \in \mathcal{U}$, let $U_n = \{x \in U | \rho(x, X - U) \geq 1/2^n\}$. Let $\leq$ be a well-ordering of $\mathcal{U}$. Let $U_n^* = U_n - \bigcup_{V<U} V_{n+1}$, as shown in Fig. 11.3. It's easy to see that $\rho(U_n^*, V_n^*) \geq 1/2^{n+1}$ for each $U \neq V \in \mathcal{U}$. Let $U_n^\sim = \{x \in U | \rho(x, U_n^*) < 1/2^{n+3}\}$. $\bigcup_n \{U_n^\sim | U \in \mathcal{U}\}$ is a σ-discrete open refinement of $\mathcal{U}$. □

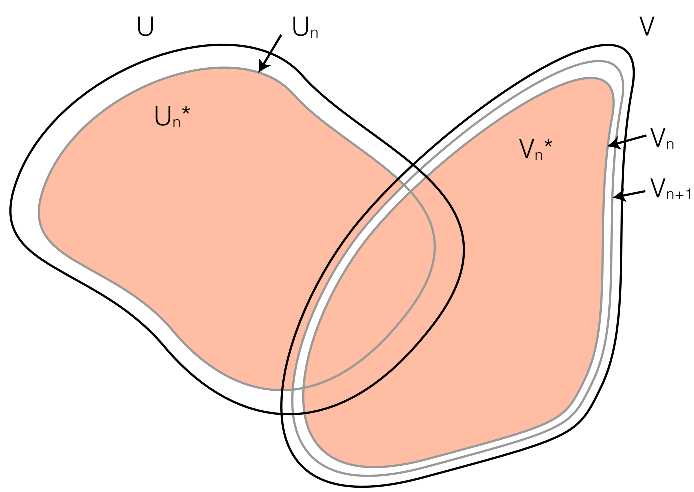

Figure 11.3: U_n^* and V_n^*, where $V < U$.

Theorem 11.4.6. *A closed subspace of a paracompact space is paracompact.*

Theorem 11.4.7. *A Hausdorff paracompact space is normal.*

Proof. Firstly, we prove the regularity. Let X be a Hausdorff paracompact space, $x \in X$ and A be a closed set not containing x. For each $a \in A$, we have two disjoint open sets $U_a \ni x$ and $V_a \ni a$. $\{X - A\} \cup \{V_a | a \in A\}$ is an open cover of X. It has an open locally finite refinement $\mathcal{V}$. Discarding the open sets contained in $X - A$, we have an open locally finite cover $\mathcal{V}'$ of A. Let N be an open neighborhood of x disjoint from A that intersects with finitely members of $\mathcal{V}'$. Since each member in $\mathcal{V}'$ doesn't contain x. We can further shrink N to N' which is disjoint from all elements of $\mathcal{V}'$. Then $\bigcup \mathcal{V}'$ is an open set that contains A and is disjoint from N'.

Secondly, we prove the normality. Let X be a Hausdorff paracompact space, A and B be two disjoint closed sets. Mimicking the first part, we have an open locally finite cover $\mathcal{U}$ of A, of which each element is disjoint from B. Similarly, for each $b \in B$ we have an open neighborhood V_b disjoint from each member of $\mathcal{U}$. Then $\bigcup \mathcal{V}'$ and $\bigcup V_b$ are disjoint open sets that contains A and B respectively. ☐

11.5 Partition of Unity

Definition 11.5.1. *Let X be a topological space, and let $\{U_i\}_{i \in I}$ be an open cover. Then a* **partition of unity** *subordinate to the cover is a set $\{f_i\}_{i \in I}$ of continuous functions $f_i : X \mapsto I$ such that*

1. $(\forall i \in I) supp(f_i) \subseteq U_i$;
2. $\{supp(f_i)\}_{i \in I}$ *is a locally finite cover;*
3. $(\forall x \in X) \sum_{i \in I} f_i(x) = 1$.

where $supp(f_i) = (f_i^{-1}((0, 1]))^-$.

Lemma 11.5.2. *Let X be a paracompact space with open cover $\mathcal{U} = \{U_i\}_{i \in I}$. Then $\mathcal{U}$ has a locally finite refinement $\{V_i\}_{i \in I}$ such that $(\forall i \in I) V_i \subseteq U_i$.*

Proof. Let $\{W_i\}_{i \in J}$ be a locally finite refinement of $\mathcal{U}$. Let $\phi : J \mapsto I$ be a choice function $W_i \subseteq U_{\phi(i)}$. Let $V_i = \bigcup_{j \in \phi^{-1}(i)} V_j$. Then $\{V_i\}_{i \in I}$ is a locally finite refinement. ☐

Lemma 11.5.3. *Let X be a paracompact Hausdorff space. For each open cover $\{U_i\}_{i \in I}$, there is a subordinate partition of unity.*

Proof. Let $\{V_i\}_{i \in I}$ be a locally finite refinement of $\{U_i\}_{i \in I}$ such that $(\forall i \in I) V_i \subseteq U_i$. By shrinking lemma, we have covers $\{W_i\}_{i \in I}$ and $\{T_i\}_{i \in I}$ such that $(\forall i \in I) T_i \subseteq T_i^- \subseteq W_i \subseteq W_i^- \subseteq V_i$. Let f_i be a Urysohn function for T_i^- and $X - W_i$, such $f_i(T_i^-) = 1$ and $f_i(X - W_i) = 0$. So $(\forall i \in I) T_i \subseteq supp(f_i) \subseteq V_i$. For each $x \in X$ let $f(x) = \sum_i f_i(x)$. It's easy to see that f_i is well-defined, non-zero and continuous. Let $f_i' = f_i / f$. Then $\{f_i'\}$ is a partition of unity. ☐

Chapter 12

Connectedness

12.1 Connectedness

Definition 12.1.1. *A space X is* **disconnected** *if there's two disjoint nonempty open sets U and V such that $X = U \cup V$. A space is* **connected** *if it's not disconnected.*

Lemma 12.1.2. *The continuous image of a connected space is connected.*

Lemma 12.1.3. *Let $S \subseteq X$ be connected. Then S^- is connected.*

Definition 12.1.4. *Two set A and B in X are* **mutually separated** *iff $A^- \cap B = A \cap B^- = \emptyset$.*

Lemma 12.1.5. *Let A and B be mutually separated sets, and $C \subseteq A \cup B$ be a connected set. Then $C \subseteq A$ or $C \subseteq B$.*

Lemma 12.1.6. *Let $\mathcal{C}$ be a family of connected subsets of X. $\bigcup \mathcal{C}$ is connected if no two members of $\mathcal{C}$ are mutually separated.*

Proof. Let $\bigcup \mathcal{C} = A \cap B$ be disconnected, where A and B are nonempty open sets in $\bigcup \mathcal{C}$. Clearly A and B are mutually disjoint in $\bigcup \mathcal{C}$. So $(\forall C \in \mathcal{C}) C \in A \vee C \in B$. Each C in A and each C' in B are mutually disjoint. $\square$

Corollary 12.1.7. *Let $\{C_n | n \in \mathbb{N}\}$ be a family of connected subsets of X. For each n, C_n and C_{n+1} are not mutually separated. Then $\bigcup_n C_n$ is connected.*

Proof. Clearly for each N, $\bigcup_{n<N} C_n$ is connected. $\square$

Lemma 12.1.8. *A product space is connected iff each factor space is connected.*

Proof. Let $X = \prod_{\alpha \in A} X_\alpha$ be disconnected. Let $X = A \cup B$ where A and B. Let U and V be two basic open sets in A and B respectively. We have a finite subset $A' = \{\alpha_n | 0 < n \leq N\} \subseteq A$ such that $U = \prod_{\alpha \in A'} U_\alpha \times \prod_{\alpha \in (A - A')} X_\alpha$ and $V = \prod_{\alpha \in A'} V_\alpha \times \prod_{\alpha \in (A - A')} X_\alpha$. We can find $u \in U$ and $v \in V$ such that $u_\alpha = v_\alpha$ for $\alpha \in (A - A')$. We define $u_n \in X$ by $u_{n,\alpha_m} = u_{\alpha_m}$ for $m < n$, $u_{n,\alpha_m} = v_{\alpha_m}$ for $m \geq n$ and $u_\alpha = v_\alpha$ for $\alpha \in (A - A')$. Clearly $u_1 = v$ and $u_{N+1} = u$. Let $u_m \in B$ and $u_{m+1} \in A$. Let $L = \{x \in X | x_{n,\alpha_i} = u_{\alpha_i}$ for $i < m$ and $x_{n,\alpha_i} = v_{\alpha_i}$ for $i > m$ and $x_\alpha = v_\alpha$ for $\alpha \in (A - A')\}$. Clearly $u_{m+1} \in A \cup L \neq \emptyset$ and $u_m \in B \cup L \neq \emptyset$. So L is disconnected. And L is homeomorphic to X_{α_m}. So there exists a disconnected factor space. $\square$

Definition 12.1.9. *Let X be a space and $x \in X$. The* **component** *of x is the union of all connected sets that contain x.*

Lemma 12.1.10. *The* **component** *of x is connected and closed.*

Definition 12.1.11. *Let X be a space. The* **components** *of X is the family of sets that are components of some $x \in X$.*

Lemma 12.1.12. *The* **components** *of X is a disjoint closed cover of X.*

Definition 12.1.13. *A space X is* **locally connected** *iff each point has a neighborhood base consisting of connected sets.*

Lemma 12.1.14. *X is locally connected iff each component of each open set is open.*

Corollary 12.1.15. *The components of a locally connected space are open and closed.*

12.2 Path-connectedness

Definition 12.2.1. *Let X be a space and $x, y \in X$. x and y are* **connected by a path** *if there's a continuous function $f : I \mapsto X$ such that $f(0) = x$ and $f(1) = y$.*

Definition 12.2.2. *A space is* **path-connected** *if any two points are connected by a path.*

Lemma 12.2.3. *Every path-connected space is connected.*

Definition 12.2.4. *A space X is* **locally path-connected** *iff each point has a neighborhood base consisting of path-connected sets.*

Theorem 12.2.5. *A connected, locally path-connected space X is path-connected.*

Proof. Let $x \in X$, and $B_x = \{y \in X | x \text{ and } y \text{ are connected by a path}\}$. It can be shown that B_x is both open and closed. So $B_x = X$. □

Example 12.2.6. *A path-connected space need not be locally path-connected. A counter example is the* **comb space**. *The comb space is a subspace of $I \times I$, defined by $C = \{(x, y) \subseteq I \times I | x = 0 \vee y = 0 \vee 1/x \in \mathbb{N}\}$. X is path-connected, but not locally path-connected at $(0, 0.5)$.*

Chapter 13

Metrizable Spaces

13.1 Metrization

Lemma 13.1.1. *Let X be a metric space with metric ρ. Let $\rho' = \min(\rho, 1)$. Then the metric space induced by ρ' is the same as X.*

Lemma 13.1.2. *A nonempty product space $\prod_\alpha M_\alpha$ is metrizable iff each M_α is metrizable and M_α is a single point for all but a countable set of indices.*

Proof. $\Rightarrow$: Use Lem. 10.5.6.

$\Leftarrow$: Let $\prod_i M_i$ be a product of countably many non-trivial metric spaces, and let $\rho_i \le 1$ be a metric of X_i. We define a metric $\rho = \sum \rho_i/2^i$, which induce a topology with the base $\{B(x,\epsilon)|x \in X, \epsilon < 1\}$. The base of the product topology is $\{\prod_{i\le n} B_i(x,\epsilon_i) \times \prod_{i>n} X_i | x \in X, \epsilon_i < 1\}$. These two bases give the same topology, since given $B = \prod_{i\le n} B_i(x,\epsilon_i) \times \prod_{i>n} X_i$ we have $B(x,\epsilon) \subseteq B$ if $\epsilon < \epsilon_i/2^i$ for all $i \le n$, and given $B(x,\epsilon)$ we have $\prod_{i\le n} B_i(x,\epsilon/2n) \times \prod_{i>n} X_i \subseteq B(x,\epsilon)$ if $\sum_{i>n} 1/2^i < \epsilon/2$. $\square$

Corollary 13.1.3. *The space I^ω with product topology is metrizable.*

Theorem 13.1.4 (Urysohn). *A 2nd countable Tychonoff space X is metrizable.*

Proof. Let $\{B_n\}$ be a countable base and $x_n \in B_n$. Let $f_n : X \mapsto I$ be a map such that $f(x_n) = 1$ and $f(X - B_n) = 0$. Using Thm. 8.2.9, we see X can be embedded in I^ω. $\square$

Theorem 13.1.5. *A space is metrizable iff it's T_3 and has a σ-locally finite base.*

Proof. Let X be a metric space with metric ρ. For each $n \in \mathbb{N}$, let $\mathcal{B}_n = \{B(x,1/n)|x \in X\}$ be an open cover of X. From Thm. 11.4.5, $\mathcal{B}_n$ has a locally finite refinement. So X has a σ-locally finite base. $\square$

13.2 Complete Metric Spaces

Theorem 13.2.1. *Every metric space M can be isometrically embedded as a dense subset of a complete space.*

Proof. Let $\mathcal{M}$ be the set of all Cauchy sequences in M. We define a pseudometric on $\mathcal{M}$ as

$$\tilde{\rho}(\{x_n\}, \{y_n\}) = \lim_{n\to\infty} \rho(x_n, y_n) \tag{13.1}$$

Since $\{\rho(x_n, y_n)\}$ is a Cauchy sequence in $\mathbb{R}$ ($|\rho(x_n, y_n) - \rho(x_m, y_m)| \le \rho(x_n, x_m) + \rho(y_n, y_m)$), the limit always exists. It's easy to see $\mathcal{M}$ is a pseudometric space. It induce a metric space $\mathcal{M}^*$ as in Lem. 7.5.5. Let $\{x^i\} = \{[\{x^i_n\}]\}$ be a Cauchy sequence in $\mathcal{M}^*$. We define N_i such that

1. $\forall i (\forall n, n' \ge N_i) \rho(x^i_n, x^i_{n'}) < \frac{1}{i}$
2. $N_1 < N_2 < \cdots$

We define $y = [\{x^i_{N_i}\}]$. Then $\rho(x^i_{N_i}, x^j_{N_j}) \le \frac{1}{i} + \frac{1}{j} + \tilde{\rho}(x^i, x^j)$. So $\{x^i_{N_i}\}$ is a Cauchy sequence in M and $y \in \mathcal{M}^*$. Since

$$\begin{aligned}
\tilde{\rho}(y, x^j) &= \lim_{i\to\infty} \rho(x^i_{N_i}, x^j_i) && (13.2)\\
&\le \rho(x^{i_0}_{N_{i_0}}, x^j_{i_0}) + \epsilon \quad (i_0 \ge N_j, i_0 \ge j) && (13.3)\\
&\le \rho(x^{i_0}_{N_{i_0}}, x^j_{N_j}) + \rho(x^j_{N_j}, x^j_{i_0}) + \epsilon && (13.4)\\
&\le \frac{1}{i_0} + \frac{2}{j} + \tilde{\rho}(x^{i_0}, x^j) + \epsilon && (13.5)
\end{aligned}$$

It's easy to see that $\tilde{\rho}(y, x^j) \to 0$. Then $x^i \to y$. So $\mathcal{M}^*$ is complete.

The embedding map $f : M \mapsto \mathcal{M}^*$ is $f(x) = [\{x, x, x, \cdots\}]$. Clearly f is an isometry. $\forall y = [\{y_i\}] \in \mathcal{M}^*$, it's easy to see that $f(y_i) \to y$ as $i \to \infty$. So $f(M)$ is dense in $\mathcal{M}^*$.

□

Chapter 14

Selected Topics

14.1 Topological Group

Definition 14.1.1. *Let G be a group and also a topological space. G is called a* **topological group** *iff the map $x \mapsto x^{-1}$ and the map $(x, y) \mapsto xy$ are continuous.*

Theorem 14.1.2. *Let G be a topological group. For each $g \in G$, the map $x \mapsto xg$ and the map $x \mapsto gx$ are automorphisms.*

Theorem 14.1.3. *Let G be a topological group, and U be a neighborhood of $g \in G$. Then Uh is a neighborhood at gh and hU is a neighborhood at hg. U^{-1} is a neighborhood at g^{-1}.*

Lemma 14.1.4. *Let G be a topological group, and U be a neighborhood of $e \in G$. Then $U^{-} \subseteq U^{-1}U$.*

Proof. Let $g \in U^{-}$. Since Ug is a neighborhood of g, $U \cap Ug \neq \emptyset$. Let $u = u'g$, then $g = u'^{-1}u \in U^{-1}U$. □

Theorem 14.1.5. *A topological group is a regular space.*

Proof. Let X be a topological group. It's enough to prove that for each neighborhood U of e, there exists a neighborhood V of e such that $V^{-} \subseteq U$. It's easy to see that there exists a neighborhood V of e such that $V = V^{-1}$ and $V^2 \subseteq U$. (First choose neighborhood T such that $T^2 \subseteq U$, which is possible since $(x, y) \mapsto xy$ is continuous. Next choose $S \subseteq T$ such that $S^{-1} \subseteq T$, which is possible since $x \mapsto x^{-1}$ is continuous. $V = S \cup S^{-1}$ satisfies the requirements.) So $V^{-} \subseteq V^{-1}V \subseteq U$. □

Theorem 14.1.6. *A topological group is a complete regular space.*

Proof. problem 2 or Willard 35F □

Theorem 14.1.7. *A T_0 topological group is a T_1 space.*

Proof. Let G be a T_0 topological group. For each $g \neq h \in G$, let $g \in gU$ such that $h \notin gU$. We have $g \notin hU^{-1}$ and $h \in hU^{-1}$. □

Corollary 14.1.8. *A T_0 topological group is a Tychonoff space.*

14.2 Manifold

Definition 14.2.1. *An n-dimensional* **manifold** *M is a 2nd countable Hausdorff space such that each point has a neighborhood that can be embedded in $\mathbb{R}^n$*

Definition 14.2.2. *Let M be an n-dimensional manifold. A* **coordinate chart** *on M is a pair (U, ϕ) where U is an open subset of M and $U \mapsto \phi(U) \in \mathbb{R}^n$ is a homeomorphism. An* **atlas** *$\{(U_\alpha, \phi_\alpha)\}$ is a family of coordinate charts such that $\{U_\alpha\}$ covers M.*

Lemma 14.2.3. *A manifold is locally compact and locally path-connected.*

Lemma 14.2.4. *A manifold has at most countably many components.*

Proof. A manifold is Lindelöf. □

Lemma 14.2.5. *A manifold is metrizable.*

Proof. Since a manifold is locally compact, it is Tychonoff. By Urysohn metrization theorem, it's metrizable. □

Corollary 14.2.6. *A manifold is perfectly normal and paracompact.*

Corollary 14.2.7. *A manifold admits partition of unity.*

Definition 14.2.8. *Let M be a manifold. Let A be a closed set in M, and $U \supseteq A$ be an open set. A* **bump function** *for a A supported in U is a continuous function $f : M \mapsto \mathbb{R}$ such that $f(A) = 1$ and $supp(f) \subseteq U$.*

Lemma 14.2.9. *Let M be a manifold. For any closed set $A \subseteq M$ and any open set $U \supseteq A$, there's a bump function for a A supported in U.*

Definition 14.2.10. *An* **exhaustion function** *for a space X is a continuous map $f : X \mapsto \mathbb{R}$ such that $f^{-1}((-\infty, c])$ is compact for each $c \in \mathbb{R}$.*

Lemma 14.2.11. *Each manifold admits an positive exhaustion function.*

Proof. Let $\{U_i\}$ be a countable base such that U_i^- is compact for each i, which exists since a manifold is 2nd countable and locally compact. Let $\{f_i\}$ be a partition of unity subordinate to $\{U_i\}$. Let $f = \sum i f_i$. It's easy to see that $f^{-1}((-\infty, c]) \subseteq f^{-1}((-\infty, \lceil c \rceil]) \subseteq \bigcup_{i \leq \lceil c \rceil} U_i^-$ is compact. □

14.3 CW Complex

Definition 14.3.1. *We can construct a* **CW complex** *X by the following procedure:*

1. *Start with a discrete set X^0, whose points are regarded as 0-cells.*
2. *Inductively, form the* **n-skeleton** *X^n from X^{n-1} by $X^n = X^{n-1} \sqcup \bigsqcup_\alpha D_\alpha^n / \sim$. For each disk we have a map $f_\alpha : \partial D_\alpha^n \mapsto X^{n-1}$. The equivalence relation is define by $x \sim f_\alpha(x)$ for $x \in \partial D_\alpha^n$. The interior of each disk D_α^n in X^n is called an* **n-cell** *e_α^n. This step is also called attaching n-cells to X^{n-1}*

3. *One can either stop at a finite stage, setting $X = X^n$. In this case the* **dimension** *of X is n. Or one can continue infinitely, setting $X = \bigcup_n X^n$. In the latter case X is given the weak topology of all projections $X \mapsto X_n$.*

Note that a CW complex can be regarded as union of cells.

Example 14.3.2. *A torus is a 2D CW complex. Its skeletons are shown in Fig. 14.1. The way to attach 2-cell to X^1 is shown in 14.2.*

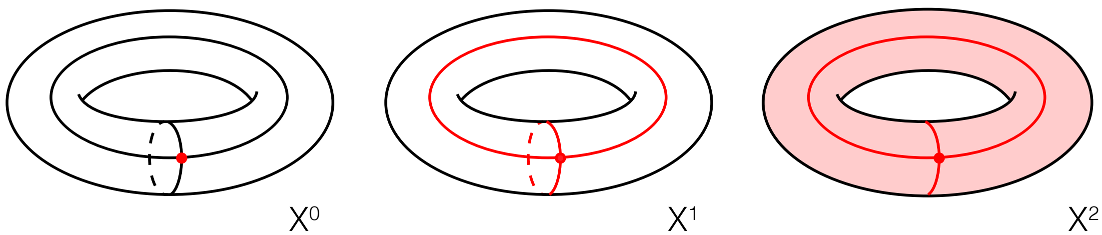

Figure 14.1: Skeletons of a torus.

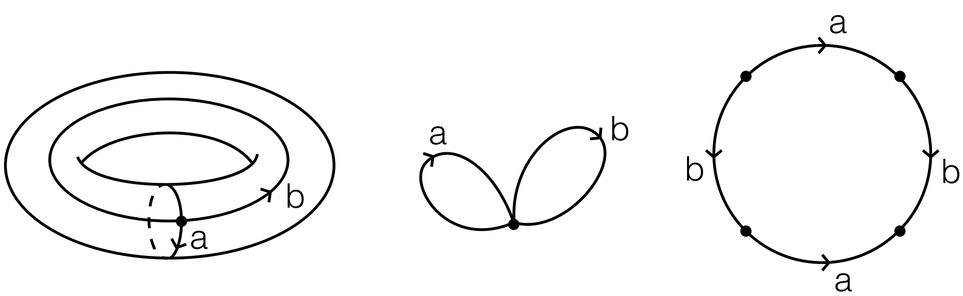

Figure 14.2: Map from S^2 to X^1 of a torus.

Example 14.3.3. *A orientable surface M_g with genius g is a surface with g holes, as shown in Fig. 14.3.*

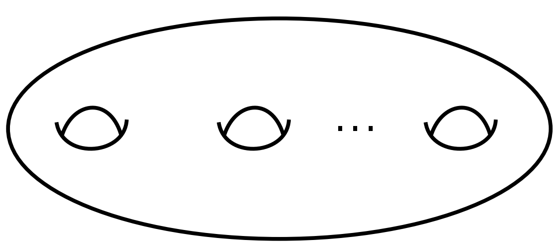

Figure 14.3: A orientable surface M_g with genius g.

Example 14.3.4. *M_g is a 2D CW complex. The way to attach 2-cell to X^1 is shown in 14.4.*

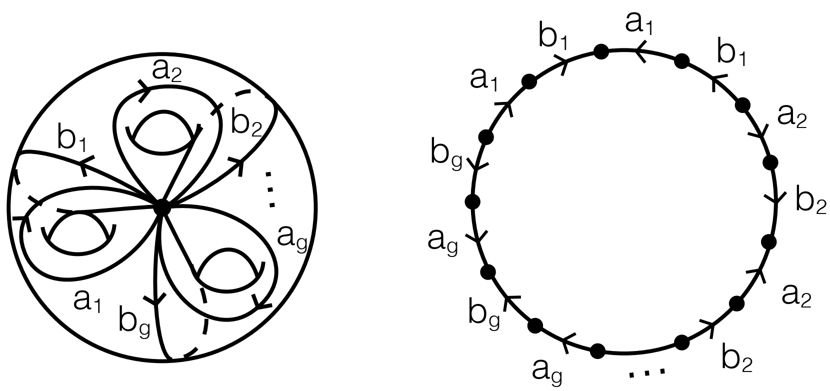

Figure 14.4: Map from S^2 to X^1 of M_g.

Part III

Homotopy Group I

Chapter 15

Basic Construction

Note: all map in this part is assumed to be continuous.

15.1 Homotopy and Homotopy type

Definition 15.1.1. *A* **retraction** *of a space X onto a subspace $A \subseteq X$ is a surjective map $f : X \mapsto A$ such that $f|_A = 1_A$.*

Definition 15.1.2. *A* **homotopy** *from $f_0 : X \mapsto Y$ to $f_1 : X \mapsto Y$ is a map $f : X \times I \mapsto Y$ such that $(\forall x \in X)(f(x, 0) = f_0(x) \wedge f(x, 1) = f_1(x))$, as shown in Fig. 15.1. Two maps f_0 and f_1 are* **homotopic** *iff there exists a homotopy that connects them, written as $f_0 \simeq f_1$.*

Definition 15.1.3. *Let $f : X \times I \mapsto Y$ be a homotopy. f is a* **homotopy relative to** *$A \subseteq X$ if $(\forall x \in A) f(x, t)$ is constant. If there exists a homotopy that connects f_0 and f_1 relative to A, we say $f_0 \simeq f_1$ rel A.*

Definition 15.1.4. *Let $f : X \mapsto A$ be a retraction. A* **deformation retraction** *of X onto A is a homotopy from 1_X to f rel A.*

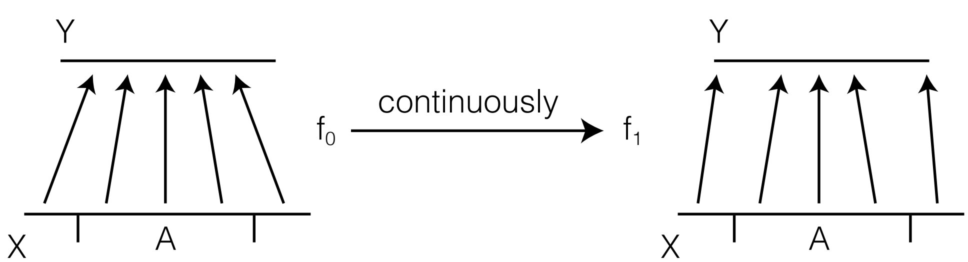

Figure 15.1: A homotopy from f_0 to f_1 relative to A.

Definition 15.1.5. *Space X and Y are said to be* **homotopy equivalent** *or to have the same* **homotopy type***, written as $X \simeq Y$, if there's a map $f : X \mapsto Y$ and a map $g : Y \mapsto X$ such that $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$.*

Definition 15.1.6. *A space that has the homotopy type of a point is called* **contractible***.*

15.2 Pointed Space

Definition 15.2.1. *A* **pointed space** *is a space with a distinguished point, the* **base point**. *Space X with based point x_0 is written as (X, x_0), abbreviated by X. Continuous maps between pointed spaces that preserve the base points are called* **based maps**.

Definition 15.2.2. *Let $\bar{X} = (X, x_0)$ and $\bar{Y} = (Y, y_0)$ be point spaces. We define there* **wedge sum** *as $\bar{X} \vee \bar{Y} = (X \sqcup Y/(x_0 \sim y_0), \overline{x_0})$*

Definition 15.2.3. *Let (X, x_0) and (Y, y_0) be point spaces, and $f_0 : X \mapsto Y$ and $f_1 : X \mapsto Y$ are based maps. A* **based point preserving homotopy** *from f_0 to f_1 is a homotopy from f_0 to f_1 rel x_0.*

Definition 15.2.4. *Two point spaces (X, x_0) and (Y, y_0) are* **homotopic equivalent**, *written as $(X, x_0) \simeq (Y, y_0)$, iff there's based maps $f : X \mapsto Y$ and $g : Y \mapsto X$ and based point preserving homotopies $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$.*

Definition 15.2.5. *Let $S^n = \{x \in \mathbb{R}^n | \sum_{i=0}^{n} x_i^2 = 1\}$ be a sphere.*

1. *The* **north hemisphere** *of S^n is $\{x \in S^n | x_0 \geq 0\}$.*
2. *The* **south hemisphere** *of S^n is $\{x \in S^n | x_0 \leq 0\}$.*
3. *The* **east hemisphere** *of S^n is $\{x \in S^n | x_1 \geq 0\}$.*
4. *The* **west hemisphere** *of S^n is $\{x \in S^n | x_1 \leq 0\}$.*
5. *The* **north pole** *of S^n is $(1, 0, \cdots, 0)$.*
6. *The* **south pole** *of S^n is $(-1, 0, \cdots, 0)$.*
7. *The* **equator** *of S^n is $\{x \in S^n | x_0 = 0\}$.*

Lemma 15.2.6. *Let S^n be a n-sphere $(n \geq 1)$ with south pole as its base point. The map $v = \iota \circ w : S^n \mapsto S^n \vee S^n$ is a based map, where $w : S^n \mapsto S^n \sqcup S^n$ defined by*

$$w(x) = (2|x_1| - 1, 2\sqrt{\frac{|x_1|}{1 + |x_1|}} x_0, 2\sqrt{\frac{|x_1|}{1 + |x_1|}} x_2, 2\sqrt{\frac{|x_1|}{1 + |x_1|}} x_3, \dots) \tag{15.1}$$

maps the east hemisphere to the 1st sphere and the west hemisphere to the 2nd sphere, and ι is the quotient map $S^n \sqcup S^n \mapsto S^n \vee S^n$. The case $n = 2$ is shown in Fig. 15.2.

Lemma 15.2.7. *Let S^n be a n-sphere $(n \geq 1)$ with south pole as its base point. The following map $i : S^n \mapsto S^n$ is a based map.*

$$i(x) = (x_0, -x_1, x_2, \dots) \tag{15.2}$$

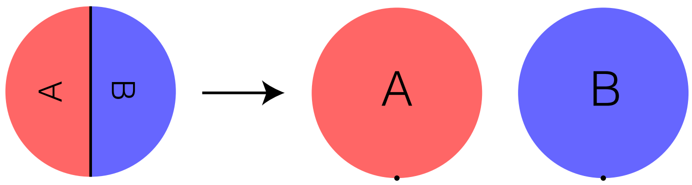

Figure 15.2: Based map from S^2 to $S^2 \vee S^2$.

15.3 Homotopy Group

Definition 15.3.1. *Let X be a space. The n-**loop** at $x_0 \in X$ is a map $f : S^n \mapsto X$, that maps the south pole to x_0.*

Definition 15.3.2. *Let f, g be two n-loops in X at x_0, we define there wedge sum $f \vee g : S^n \bigvee S^n \mapsto X$ as*

$$f \vee g(\bar{x}) = \begin{cases} f(x) & x \in 1\text{st } S^n \\ g(x) & x \in 2\text{nd } S^n \end{cases} \tag{15.3}$$

where $f \vee g$ maps the base point of $S^n \bigvee S^n$ to x_0.

Definition 15.3.3. *Let f, g be two n-loops in X at x_0. We define there composition loop $f \cdot g : S^n \mapsto X$ as*

$$f \cdot g = (f \vee g) \circ v \tag{15.4}$$

Definition 15.3.4. *Let f be an n-loops in X at x_0, we define its inverse loop $f^{-1} : S^n \mapsto X$ as*

$$f^{-1} = f \circ i \tag{15.5}$$

Definition 15.3.5. *Let X be a space. We define the constant n-loop e at x_0 to be the constant map from S^n to x_0.*

Lemma 15.3.6. *Let f be an n-loops in X at x_0. Then $f \cdot f^{-1} \simeq e$ rel the south pole.*

Proof. The homotopy map is

$$F(x, t) = f \cdot f^{-1}(\cos(\theta(1-t) + \pi t), s_1 \sin(\theta(1-t) + \pi t), s_2 \sin(\theta(1-t) + \pi t), \cdots)) \tag{15.6}$$

where $x = (\cos\theta, s_1 \sin\theta, s_2 \sin\theta, \cdots)$ and $s_1^2 + s_2^2 + \cdots = 1$. The process is shown in Fig. 15.3. $\square$

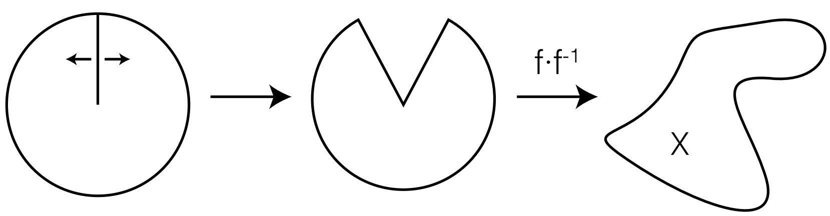

Figure 15.3: Homotopy when $0 < t < 1$.

Definition 15.3.7. *Let X be a space. The* n**-th homotopy group** *(at x_0), written as $\pi_n(X, x_0)$, is the (base point preserving) homotopy types of all n-loops at x_0, with multiplication rule $[f] \cdot [g] = [f \cdot g]$, inverse $[f]^{-1} = [f^{-1}]$ and identity $[e]$. The 1st homotopy group is also called the* **fundamental group**.

Theorem 15.3.8. *When $n \geq 2$, the n-th homotopy group is abelian. So in this case we write $f \cdot g$ as $f + g$.*

Proof. Let f, g be two n-loops in X at x_0. When $n \geq 2$, we have a homotopy $f \cdot g \simeq g \cdot f$ rel the south pole:

$$F(x,t) = \begin{cases} R_{12}(2\pi t)x & t \leq \frac{1}{2} \\ R_{02}(\pi(2t-1))R_0(\pi)x & t > \frac{1}{2} \end{cases} \tag{15.7}$$

where

$$R_{12}(\theta)(x_0, x_1, x_2, x_3, \cdots) = (x_0, x_1 \cos\theta - x_2 \sin\theta, x_2 \cos\theta + x_1 \sin\theta, x_3, \cdots) \tag{15.8}$$

and

$$R_{02}(\theta)(x_0, x_1, x_2, x_3, \cdots) = (x_0 \cos\theta - x_2 \sin\theta, x_1, x_2 \cos\theta + x_0 \sin\theta, x_3, \cdots) \tag{15.9}$$

The case $n = 2$ is shown in Fig. 15.4. □

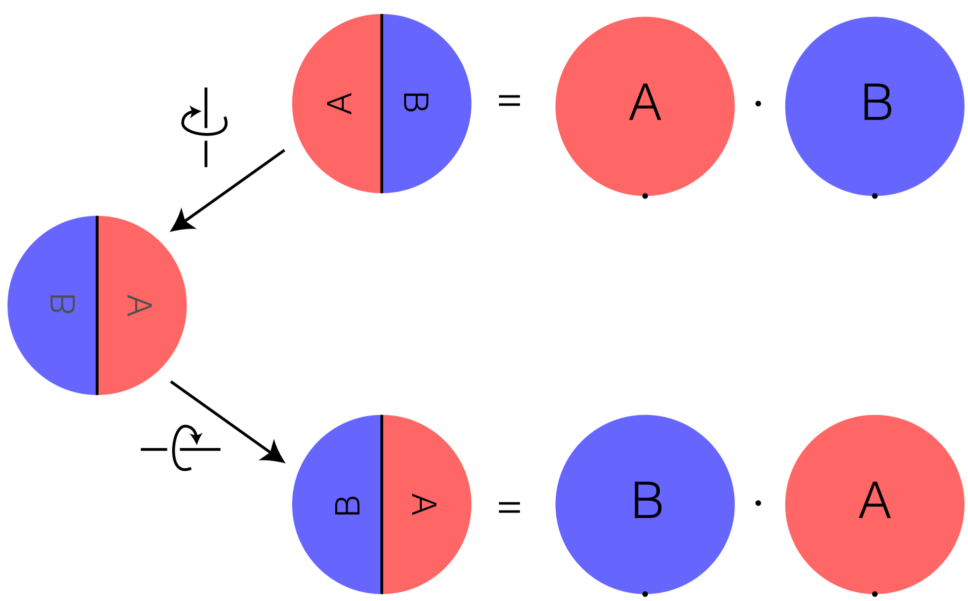

Figure 15.4: Homotopy group is abelian.

Lemma 15.3.9. *Let X_α be path-connected. Then $\pi_n(\prod_\alpha X_\alpha) \simeq \prod_\alpha \pi_n(X_\alpha)$.*

Proof. $[f]$ is mapped to f' such that $f'_i = [f_i]$. □

15.4 Change the Base Point

Definition 15.4.1. *A* **path** *in a space X from a to b is a map $f : I \mapsto X$ such that $f(0) = a$ and $f(1) = b$. When we say a homotopy between paths, we always mean a homotopy relative to $\{0, 1\}$. And for a path f we use $[f]$ to denotes its homotopy type.*

Definition 15.4.2. *Let f be a path in X. The* **inverse** *of f is the path $f^{-1}(t) = f(1 - t)$.*

Definition 15.4.3. *Let X be a path connected space. For each path γ from x_1 to x_0 and each $[l] \in \pi_n(X, x_0)$, we define $\beta_{[\lambda]} : \pi_n(X, x_0) \mapsto \pi_n(X, x_1)$ by $\beta_{[\lambda]}([l]) = [\bar{\beta}_\lambda(l)]$ where*

$$\bar{\beta}_\lambda(l)(x) = \begin{cases} l(2x_0 - 1, 2\sqrt{\frac{x_0}{1+x_0}}x_1, 2\sqrt{\frac{x_0}{1+x_0}}x_2, 2\sqrt{\frac{x_0}{1+x_0}}x_3, \dots) & x_0 > 0 \\ \lambda(x_0 + 1) & x_0 \leq 0 \end{cases} \tag{15.10}$$

The case $n = 2$ is shown in Fig. 15.5.

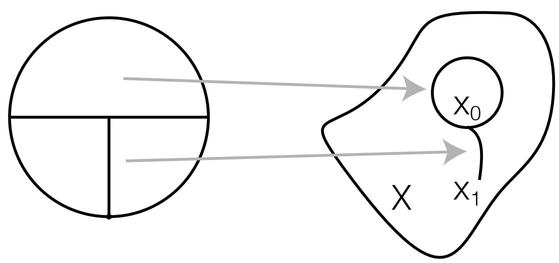

Figure 15.5: Change the base point from x_0 to x_1.

Lemma 15.4.4. *For each path λ from x_1 to x_0, $\beta_{[\lambda]}$ is a group isomorphism from $\pi_n(X, x_0)$ to $\pi_n(X, x_1)$.*

Proof. Check $\beta_{[\lambda]}\beta_{[\lambda^{-1}]} = 1$, $\beta_{[\lambda]}e = e$ and $\beta_{[\lambda]}(a + b) = \beta_{[\lambda]}a + \beta_{[\lambda]}b$. $\square$

Corollary 15.4.5. *Let X be a path connected space. Homotopy groups of X at each point are isomorphic. So we may abbreviate $\pi_n(X, x_0)$ by $\pi_n(X)$.*

Corollary 15.4.6. *For each $g \in \pi_1(X, x_0)$, β_g is a group automorphism of $\pi_n(X, x_0)$.*

Definition 15.4.7. *Let $\mathbb{Z}(\pi_1(X, x_0))$ be the group ring. We define the action of $\mathbb{Z}(\pi_1(X, x_0))$ on $\pi_n(X, x_0)$ $(n \geq 2)$ as*

$$(\sum_i n_i g_i) f = \sum_i n_i \beta_{g_i} f \tag{15.11}$$

This makes $\pi_n(X, x_0)$ a $\mathbb{Z}(\pi_1(X, x_0))$ module.

15.5 Induced Homomorphisms

Definition 15.5.1. *Let (X, x_0) and (Y, y_0) be point spaces, and $f : X \mapsto Y$ be a based map. Then f induce a homomorphism $f_* : \pi_n(X, x_0) \mapsto \pi_n(Y, y_0)$ by $f_*([l]) = [f \circ l]$.*

Lemma 15.5.2. *Let $\phi_t : X \mapsto Y$ be a base point preserving homotopy, then $\phi_{0*} = \phi_{1*}$.*

Lemma 15.5.3. *Let l be an n-loop at $x_0 \in X$, and f_t be a homotopy from X to Y. Let $p = f_t(x_0)$ be a path from $f_0(x_0)$ to $f_1(x_0)$. Then $\bar{\beta}_{p^{-1}}(f_0 \circ l) \simeq (f_1 \circ l)$. The case $n = 1$ is shown in Fig. 15.6. Thus the following diagram commutes*

$$\begin{tikzcd}
 & \pi_n(Y, f_0(x_0)) \arrow[d, "\beta_{[p^{-1}]}"] \\
\pi_n(X, x_0) \arrow[ur, "f_{0*}"] \arrow[r, "f_{1*}"] & \pi_1(Y, f_1(x_0))
\end{tikzcd}$$

Corollary 15.5.4. *Let l be an n-loop at $x_0 \in X$ and $f : X \mapsto X$ such that $f \simeq 1$ by f_t. Let $p = f_t(x_0)$ be a path from $f(x_0)$ to x_0. Then $f_* = \beta_{[p]}$.*

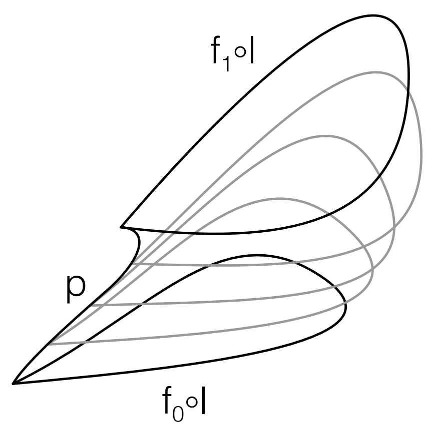

Figure 15.6: $\bar{\beta}_{p^{-1}}(f_0 \circ l) \simeq (f_1 \circ l)$.

Lemma 15.5.5. *Let $f : X \mapsto Y$ and $g : Y \mapsto X$. If $f \circ g \simeq 1$, then $f_* : \pi_n(X, x_0) \mapsto \pi_n(Y, f(x_0))$ is surjective for all $x_0 \in X$. If $g \circ f \simeq 1$, then f_* is injective.*

Proof. If $f \circ g \simeq 1$, $f_* g_* = \beta_h$ is bijective. So f_* is surjective. If $g \circ f \simeq 1$, $g_* f_* = \beta_{h'}$ is bijective. So f_* is injective. □

Corollary 15.5.6. *If a space X retract onto a subspace A. Then $i_* : \pi_n(A, x_0) \mapsto \pi_n(X, x_0)$ induced by $i : A \hookrightarrow X$ is injective. If A is a deformation retract of X, then i_* is an isomorphism.*

Chapter 16

Fundamental Group

16.1 Covering Spaces

Definition 16.1.1. *Let X be a path connected space. As illustrated in Fig. 16.1, $\tilde{X}$ is call a* **covering space** *of X if there's a map $p : \tilde{X} \mapsto X$ (called the* **covering map***) such that for each $x \in X$, there's an neighborhood U of x such that $p^{-1}(U)$ is a union of disjoint sets (named* **sheets***) in $\tilde{X}$, each of which is mapped homeomorphically onto U by p. Such U is called* **evenly covered***.*

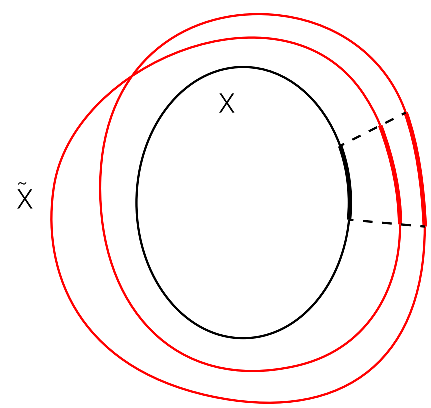

Figure 16.1: Illustration of covering space.

Definition 16.1.2. *Let $p : \tilde{X} \mapsto X$ be a covering space. For each map $f : Y \mapsto X$, $\tilde{f} : Y \mapsto \tilde{X}$* **lifts** *$f$ iff $f = p \circ \tilde{f}$. That is, the following diagram commutes.*

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

Lemma 16.1.3. *Let $p : \tilde{X} \mapsto X$ be a covering space. For each path f which is lifted to $\tilde{f}_1, \tilde{f}_2$, $f_1 = f_2$ if $\tilde{f}_1$ and $\tilde{f}_2$ agree on one point.*

Lemma 16.1.4. *Let $p : \tilde{X} \mapsto X$ be a covering space. Given a homotopy $F : Y \times I \mapsto X$ and a map $\tilde{F}_0 : Y \times \{0\} \mapsto \tilde{X}$ that lifts $F|_{Y\times\{0\}}$, then there's a homotopy $\tilde{F} : Y \times I \mapsto \tilde{X}$ lifting F such that $\tilde{F}|_{Y\times\{0\}} = \tilde{F}_0$.*

Proof. For each $y \in Y$ and each $0 \leq t \leq 1$, we have an evenly covered open basic neighborhood $N_i \times I_i \ni (y, t)$, where I_i covers I. There's a finite subcover of I_i. Then we have $0 = t_0 < \cdots < t_n = 1$ and open $N \ni y$, such that each $N \times [t_i, t_{i+1}]$ is in some evenly covered open set $U_{N,i}$. Then it's easy to construct a lift F_N of F on $N \times I$. From the last lemma we see that $F_N(y)$ is the same for each N and is unique. Thus we can define a unique lift $\tilde{F}$ of F. □

Corollary 16.1.5. *Let $p : \tilde{X} \mapsto X$ be a covering space. For each path $f : I \mapsto X$ starting at x_0 and each $\tilde{x}_0 \in p^{-1}(x_0)$, there's a unique lift $\tilde{f}$ of f starting at $\tilde{x}_0$.*

Corollary 16.1.6. *Let $p : \tilde{X} \mapsto X$ be a covering space. For each homotopy of paths $f : I \times I \mapsto X$ starting at x_0 (which means that $\forall t(f(t, 0) = x_0)$) and each $\tilde{x}_0 \in p^{-1}(x_0)$, there's a unique lift $\tilde{f}$ of f starting at $\tilde{x}_0$ which is a homotopy in $\tilde{X}$.*

Proof. Since $f(t, 1)$ is constant, $\tilde{f}(t, 1)$ must be constant. □

We can study the lift of a more general map.

Lemma 16.1.7. *Let $p : (\tilde{X}, \tilde{x}_0) \mapsto (X, x_0)$ be a covering space. Given a map $f : (Y, y_0) \mapsto (X, x_0)$ with Y path-connected and locally path-connected. Then a lift $\tilde{f} : (Y, y_0) \mapsto (\tilde{X}, \tilde{x}_0)$ of f exits iff $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.*

Proof. Assume $f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. For each $y \in Y$, let p be a path from y_0 to y. Let $q = f \circ p$ be a path in X from x_0 to $f(y)$. Let $\tilde{q}$ be the lift of q starting at $\tilde{x}_0$. We define $\tilde{f}(y) = \tilde{q}(1)$.

We first prove that $\tilde{f}(y)$ is well-defined. Let p and p' be two paths y_0 to y, and $q = f \circ p$ and $q' = f \circ p'$. Let l be the loop in $\tilde{X}$ such that $p_*([l]) = [p \circ l] = [q \cdot q'^{-1}] \in f_*(\pi_1(Y, y_0))$. Let $\tilde{q}$ and $\tilde{q}'$ be the lifts of q and q'. There's a homotopy in X from $p \circ l$ to $q \cdot q'^{-1}$, which lifts a homotopy from l to t. Then there's a homotopy in X from q to $q \cdot q'^{-1} \cdot q'$, which lifts to a homotopy from $\tilde{q}$ to $t \cdot \tilde{q}'$. Then clearly $\tilde{q}(1) = \tilde{q}'(1)$.

To see that $\tilde{f}$ is continuous, let $U \subseteq X$ be an open neighborhood of $f(y)$ having a lift $\tilde{U} \subseteq \tilde{X}$ containing $\tilde{f}(y)$ such that $p : \tilde{U} \mapsto U$ is a homeomorphism. Choose a path-connected open neighborhood V of y with $f(V) \subseteq U$. It can be proved that $\tilde{f}(V) \subseteq \tilde{U}$. □

We can also prove that the lift is unique.

Lemma 16.1.8. *Let $p : \tilde{X} \mapsto X$ be a covering space. Given a map $f : Y \mapsto X$ with Y connected. Let $\tilde{f}_1$ and $\tilde{f}_2$ be two lifts of f. If $\tilde{f}_1$ and $\tilde{f}_2$ agree on one point of Y, they must agree on all of Y.*

Proof. For each $y \in Y$, let $U \ni f(y)$ be an evenly covered open set, covered by $\{\tilde{U}_\alpha\}$. Let $\tilde{f}_1(y) \in U_1$ and $\tilde{f}_2(y) \in U_2$. There exists a neighborhood N of y such that $\tilde{f}_1(N) \in U_1$ and $\tilde{f}_2(N) \in U_2$. If $\tilde{f}_1(y) = \tilde{f}_2(y)$, then $U_1 = U_2$. It's easy to see that $(\forall y \in N)\tilde{f}_1(y) = \tilde{f}_2(y)$. If $\tilde{f}_1(y) \neq \tilde{f}_2(y)$, then $U_1 \neq U_2$. It's easy to see that $(\forall y \in N)\tilde{f}_1(y) \neq \tilde{f}_2(y)$. So the set that $\tilde{f}_1$ and $\tilde{f}_2$ agree on is both open and closed. □

Lemma 16.1.9. *Let $p : \tilde{X} \mapsto X$ be a covering space. Let a and b be two loops at $x_0 \in X$. Let $\tilde{a}$ and $\tilde{b}$ be lifts of a and b starting at $\tilde{x}_0 \in p^{-1}(x_0)$. Then $[a] = [b]$ iff $\tilde{a}(1) = \tilde{b}(1)$ and $[\tilde{a}] = [\tilde{b}]$.*

Corollary 16.1.10. *A covering map $p : (\tilde{X}, \tilde{x}_0) \mapsto (X, x_0)$ induce an injective map $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \mapsto \pi_1(X, x_0)$. And $\{\{[p \cdot q] | q$ is a path from $\tilde{x}_0$ to $a\} | a \in p^{-1}(x_0)\}$ are right cosets of $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ in $\pi_1(X, x_0)$.*

Corollary 16.1.11. *Let $p : \tilde{X} \mapsto X$ be a covering space. For each $\tilde{x} \in \tilde{X}$, let $l_{\tilde{x}}$ be a path from $\tilde{x}_0 \in p^{-1}(x_0)$ to $\tilde{x}$. Then $\pi_1(X, x_0) = \{[p \circ l_{\tilde{x}}] | \tilde{x} \in p^{-1}(x_0)\}$.*

16.2 Fundamental Group of the Circle

Lemma 16.2.1. $\pi_1(S^1) = \mathbb{Z}$

Proof. $\mathbb{R}$ is a covering space of X with covering map $p : \mathbb{R} \mapsto S^1$ defined by $p(\theta) = (\cos\theta, \sin\theta)$. Let $f_n : I \mapsto \mathbb{R}$ be $f_n(t) = (\cos(2\pi nt), \sin(2\pi nt))$. It's easy to see that $\pi_1(S^1, (1, 0)) = \{[p \circ f_n]\}$ and $[p \circ f_n] \cdot [p \circ f_m] = [p \circ f_{n+m}]$. □

Theorem 16.2.2. *Every non-constant polynomial with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$.*

Proof. Let the polynomial be $p(z) = z^n + a_1 z^{n-1} + \cdots a_n$. Define a map

$$f_{r,t}(s) = \frac{p_t(re^{2\pi is})/p_t(r)}{|p_t(re^{2\pi is})/p_t(r)|} \tag{16.1}$$

where $p_t(z) = z^n + t(a_1 z^{n-1} + \cdots a_n)$ and $r, t, s \in I$.

Fixing r, t, f is a loop in $S^1 \in \mathbb{C}$ based at 1. Supposing p has no roots, the formula is well-defined at $t = 1$. When $|z|$ is large enough, $|z^n| > |a_1 z^{n-1} + \cdots a_n|$. So $p_t(z)$ has no roots with large $|z|$, and Eqn. 16.1 is also well-defined when r is large enough.

First we fix $t = 1$ and let r goes from 0 to a very large r_0. Then we fix r and let t goes from 1 to 0. This process describe a homotopy between $f_{0,1}(s) = 1$ to $f_{r_0,0}(s) = e^{2\pi nis}$. There's a contradiction since $n > 0$. □

Lemma 16.2.3. *There's no retraction $D^2 \mapsto S^1 = \partial D^2$.*

Proof. Each loop in D^2 is homotopic to a constant loop. □

Theorem 16.2.4. *Every continuous map $h : D^2 \mapsto D^2$ has a fixed point.*

Proof. Suppose that there's no fixed point. For each $x \in D^2$, let $r(x)$ be the ray starting at $h(x)$ and passing through x. Let $f(x) \in S^1$ be the point that $r(x)$ leaves D^2. f is a retraction $D^2 \mapsto S^1 = \partial D^2$. □

Lemma 16.2.5. *Let f be a loop in S^1 such that $f(x) = -f(x + 1/2)$. Then $[f] \neq 0$.*

Proof. Let S^1 is covering by $\mathbb{R}$ with the standard covering map $p : \mathbb{R} \mapsto S^1$. Let $\tilde{f}$ lifts f. Then $\tilde{f}(x+1/2) = \tilde{f}(x)+q(x)/2$ where $q(x)$ is an odd integer. Since $q(x)$ is continuous, it must be constant. So $\tilde{f}(x + 1/2) = \tilde{f}(x) + q/2$ and thus $\tilde{f}(1) = \tilde{f}(0) + q$. So $[f] = [q] \neq 0$". □

Theorem 16.2.6. *For every continuous map $f : S^2 \mapsto \mathbb{R}^2$, there exists a pair of antipodal points x and $-x$ is S^2 with $f(x) = f(-x)$.*

Proof. If the conclusion is false, let $g(x) = (f(x) - f(-x))/|f(x) - f(-x)|$. Then g maps S^2 to S^1 and $g(x) = -g(-x)$. Let l be the loop circling the equation of the S^2. Then $q = g \circ l$ is a loop that $q(x) = -q(x + 1/2)$. Since l is homotopic to a constant loop, $[q] = 0$, a contradiction. □

16.3 The van Kampen Theorem

Theorem 16.3.1 (Serfeit, van Kampen)**.** *Let X be the union of path-connected open sets A_α each containing the base point x_0. Let each intersection $A_\alpha \cap A_\beta$ be path-connected. Then the homomorphism $\Psi : *_\alpha \pi_1(A_\alpha) \mapsto \pi_1(X)$ defined by $\Psi([l_1][l_2]\cdots) = [l_1 \cdot l_2 \cdots]$ is surjective. If in addition each intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path-connected, then the kernel of Ψ is the normal subgroup N generated by all elements of the form $i_{\alpha\beta *}(\omega) i_{\beta\alpha *}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$, and $i_{\alpha\beta} : A_\alpha \cap A_\beta \hookrightarrow A_\alpha$.*

Proof. The first part is evident. Let $l : I \mapsto X$ be a loop, then I can be covered by finite open intervals, the image of each by l lies in some A_α.

The second part needs more work. Clearly $N \in \ker \Psi$. Let $L = [l_1][l_2]\cdots$ and $L' = [l'_1][l'_2]\cdots$ such that $\Psi(l) = \Psi(l')$. There's a homotopy $F : I \times I \mapsto X$ from $l = l_1 \cdot l_2 \cdots$ to $l' = l'_1 \cdot l'_2 \cdots$. Due to the compactness of $I \times I$, one can fine $0 = s_0 < s_1 < \cdots < s_m = 1$ and $0 = t_0 < t_1 < \cdots < t_n = 1$ such that each rectangle $R_{ij} = [s_i, s_{i+1}] \times [t_i, t_{i+1}]$ is mapped to some single A_{ij} by F. We perturb the vertical edge of the rectangles slightly so that each point in $I \times I$ lies in at most three rectangles. We may assume $A_{ij} \neq A_{i,j+1}$. Otherwise we can combine multiple rectangles into one rectangle. We can also require that F maps vertical edge of the rectangles to x_0.

For each horizontal row of rectangles $I \times [t_i, t_{i+1}]$, we have $L_i = [l_{i,1}] \cdot [l_{i,2}] \cdots \in *_\alpha \pi_1(A_\alpha)$, such that $[l_{i,j}] \in \pi_1(A_{ij})$ and $l_{i,1} \cdot l_{i,2} \cdots$ is homotopic to l. We want to study the relation between L_i and L_{i+1}. For this purpose, we study the horizontal line $t = t_{i+1}$. Along this line, we can construct a combination of loops $p_{i1} \cdot p_{i2} \cdots$ which each loop p_{ij} is in both $A_{\alpha_{ij}}$, the rectangle bellow p_{ij}, and $A_{\beta_{ij}}$, the rectangle above p_{ij}. It's easy to see $L_i = \prod_j i_{\alpha_{ij}\beta_{ij}}(p_{ij})$ and $L_{i+1} = \prod_j i_{\beta_{ij}\alpha_{ij}}(p_{ij})$. Since $\prod_j i_{\alpha_{ij}\beta_{ij}}(p_{ij}) = \prod_j i_{\alpha_{ij}\beta_{ij}}(p_{ij}) i^{-1}_{\beta_{ij}\alpha_{ij}}(p_{ij}) i_{\beta_{ij}\alpha_{ij}}(p_{ij})$, it's easy to see that $L_i = L_{i+1} \cdot n$ where $n \in N$. Repeating this process, we can conclude that $L = L' \cdot n$ where $n \in N$. So $N = \ker \Psi$.

The homotopy rectangle is shown in Fig. 16.2.

□

Lemma 16.3.2. $\pi_1(\wedge_i X_i) = *_i \pi_1(X_i)$

Lemma 16.3.3. $\pi_1(S^n) = 0$ *when* $n \geq 2$.

Example 16.3.4. *In this example we calculate the fundamental group of a 2D CW complex.*

We attach a collection of 2-cells e^2_α to a path connected 1-skeleton X^1 via maps $\phi_\alpha : S^1 \mapsto X_1$. Each ϕ_α can be viewed as a loop in $X = X^2$. Let γ_α be a path from x_0 to $\phi_\alpha(0)$. Then $\eta_\alpha = \gamma_\alpha \phi_\alpha \gamma_\alpha^{-1}$ is a loop at x_0 in X.

Let's expand X to a slightly larger space X' that deformation retracts onto X by attaching rectangular strips $S_\alpha = I \times I$, with the lower edge attached to γ_α, the right edge attached along an arc in e^2_α, and all the left edges of the strips identified together, as shown in Fig. 16.3.

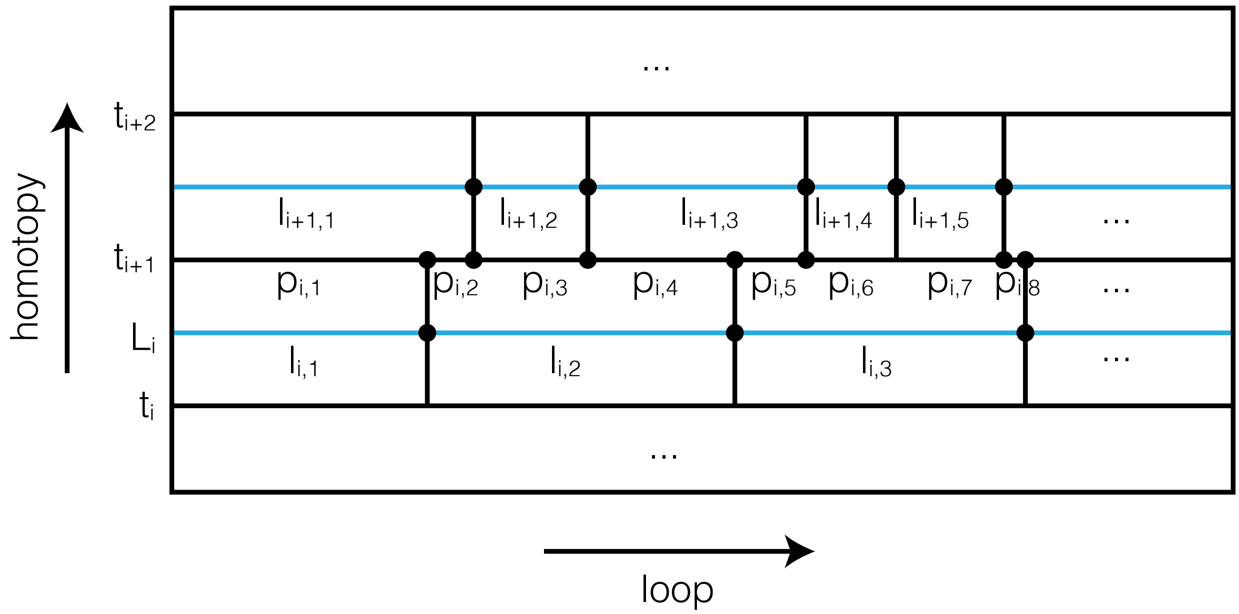

Figure 16.2: The homotopy rectangle in proving the van Kampen theorem.

Let $C = \{c_\alpha\}$ be the set of centers of e^2_α. Let $A = X' - C$ and $B = (X' - X) \cup \{x_0\}$. Then $A \cap B = (X' - X) \cup \{x_0\} - C$. It's easy to see that B is contractable. So $\pi_1(X) \simeq \pi_1(X_1)/N$, where N is the normal subgroup of $\pi_1(X)$ generated by $i(\omega)$ for $\omega \in \pi_1((X' - X) \cup \{x_0\} - C)$ and i is the homomorphism induced by $A \cap B \hookrightarrow A$. It's easy to see that $\pi_1((X' - X) \cup \{x_0\} - C)$ is a free group generated by $[l_\alpha]$, where l_α deformation retracts onto a circle in $e^2_\alpha - \{c_\alpha\}$, as shown in Fig. 16.3. So $i([l_\alpha]) = [\eta_\alpha]$. So N is the normal subgroup generated by $[\eta_\alpha]$.

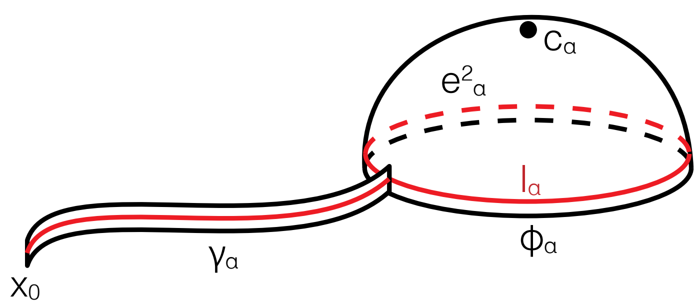

Figure 16.3: Strip attached to a CW complex.

Lemma 16.3.5. *The fundamental group of M_g is $\langle a_1, b_1, \cdots, a_g, b_g | [a_1, b_a] \cdots [a_g, b_g] \rangle$*

Lemma 16.3.6. *For every group G there is a 2D CW complex X with $\pi_1(X) = G$.*

Proof. Let $G = \langle g_\alpha | r_\beta \rangle$. We let X_1 be the wedge sum of circles representing g_αs. For each β we attach an e^2_β to X_1 according to r_β $\square$

16.4 Classification of Covering Spaces

Definition 16.4.1. *A path connected space X is* **simply connected** *iff $\pi_1(X) = 0$.*

Definition 16.4.2. *A path connected space X is* **locally simply connected** *iff each point has a simply connected local neighborhood base.*

Definition 16.4.3. *A path connected space X is* **semilocally simply connected** *iff each point has a neighborhood U such that every loop in U is contractable in X.*

Lemma 16.4.4. *A path connected space X is semilocally simply connected iff each point has a local neighborhood base containing sets in which every loop is contractable in X.*

Definition 16.4.5. *A covering space is a* **universal cover** *iff it's simply connected.*

Lemma 16.4.6. *A path-connected, locally path-connected, semilocally simply connected space X has a universal cover.*

Proof. Let x_0 be a base point in X. Let $\tilde{X}$ be $\{[\eta] | \eta$ is a path in X starting at $x_0\}$.

We need to define a topology on X. For each $x \in X$, let $\mathcal{U}_x$ be a local neighborhood base at x, containing sets in which every loop is contractable in X. For each $[\eta] \in \tilde{X}$ and each $U_{\eta(1)}$ we define a set $N(U, [\eta]) = \{[\eta \cdot \delta] | \delta$ is a path in U starting at $\eta(1)\}$. Then we define a family $\mathcal{U}_{[\eta]} = \{N(U, [\eta]) | U \in \mathcal{U}_{\eta(1)}\}$. It can be seen that $\mathcal{U}_{[\eta]}$ is a neighborhood base for a topology on X.

Next we show the map $p : \tilde{X} \mapsto X$ defined by $p([\eta]) = \eta(1)$ is a covering map. It's easy to see that p is continuous. For each $x \in X$, choose an open $U_x \in \mathcal{U}_x$. It's easy to see that $\mathcal{N}_x = \{N(U_x, [\eta]) | \eta$ is a path from x_0 to $x\}$ is a union of disjoint open sets and $\bigcup \mathcal{N}_x = p^{-1}(U_x)$.

Finally we show that $\tilde{X}$ is simply connected. For this purpose, we study the lift of a path γ starting at x_0 to a path $\tilde{\gamma}$ starting at $[x_0]$ (the homotopy type of a constant path). Let γ_t be the path defined by

$$\gamma_t(s) = \begin{cases} \gamma(s) & s < t \\ \gamma(t) & s \geq t \end{cases} \tag{16.2}$$

We define $\tilde{\gamma}(t) = [\gamma_t]$. It's easy to see that $\tilde{\gamma}$ lifts γ, as shown in 16.4. So $\tilde{X}$ is path-connected. $\tilde{X}$ is simply connected iff each loop ξ in $\tilde{X}$, $p \circ \xi$ is contractable, which is clear since $\xi(1) = [p \circ \xi] = \xi(0) = [x_0]$. □

Lemma 16.4.7. *Let X be path-connected, locally path-connected and semilocally simply connected. Then for every subgroup $H \in \pi_1(X, x_0)$ there's a covering space $p_H : X_H \mapsto X$ such that* $\operatorname{im} p_{H*} = H$.

Proof. Let $p : \tilde{X} \mapsto X$ be the universal cover of X. Choose a $\tilde{x}_0 \in p_H^{-1}(x_0)$. For each $\tilde{x} \in \tilde{X}$, we define $l_{\tilde{x}}$ to be a path from $\tilde{x}_0$ to $\tilde{x}$. We define a equivalence relation on $\tilde{X}$ by $\tilde{x} \sim \tilde{y}$ iff $p(\tilde{x}) = p(\tilde{y})$ and $[(p \circ l_{\tilde{x}}) \cdot (p \circ l_{\tilde{y}})^{-1}] \in H$. Especially $\tilde{x} \sim \tilde{x}_0$ iff $[(p \circ l_{\tilde{x}})] \in H$. We define $\tilde{X}_H = \tilde{X}/\sim$, and define $p_H : \tilde{X}_H \mapsto X$ by $p_H(\tilde{\tilde{x}}) = p(\tilde{x})$. It's easy to see that p_H is continuous. We define the natural map $q_H : \tilde{X} \mapsto \tilde{X}_H$. We have $p_H \circ q_H = 1$.

Next we prove that p_H is a covering map. For each $x \in X$, let U be an open set evenly covered by $\bigcup_\alpha U_\alpha$ in $\tilde{X}$. Let $p_\alpha : U_\alpha \mapsto U$ be the homomorphism. For each $\alpha \neq \beta$, if $(\exists u \in U)\overline{p_\alpha^{-1}(u)} = \overline{p_\beta^{-1}(u)}$, then $(\forall u \in U)\overline{p_\alpha^{-1}(u)} = \overline{p_\beta^{-1}(u)}$. Then it's easy to see that p_H is a covering map.

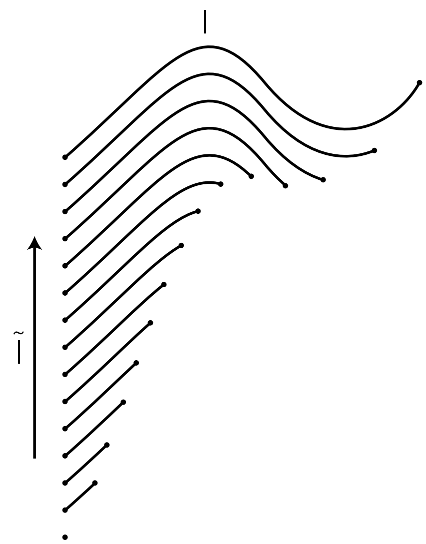

Figure 16.4: Lift of a path to a universal cover.

Finally we prove that im $p_{H*} = H$. For each loop l in X, let $\tilde{l}$ be the lift of l in $\tilde{X}$. Then $l_H = q_H \circ \tilde{l}$ is the lift of l in X_H. $[l] \in H$ iff $\tilde{l}(1) \sim \tilde{x}_0$ iff l_H is a loop in X_H iff $[l] \in$ im p_{H*}. ☐

Definition 16.4.8. *Let $p_1 : \tilde{X}_1 \mapsto X$ and $p_2 : \tilde{X}_2 \mapsto X$ be two covering spaces. An* **isomorphism** *from $\tilde{X}_1$ to $\tilde{X}_2$ is a homeomorphism $f : \tilde{X}_1 \mapsto \tilde{X}_2$ such that $p_1 = p_2 \circ f$.*

Lemma 16.4.9. *Let X be path-connected, locally path-connected and semilocally simply connected. Then two path-connected covering spaces $p_1 : \tilde{X}_1 \mapsto X$ and $p_2 : \tilde{X}_2 \mapsto X$ are isomorphic via $f : \tilde{X}_1 \mapsto \tilde{X}_2$ taking a base point $\tilde{x}_1 \in p_1^{-1}(x_0)$ to a base point $\tilde{x}_2 \in p_2^{-1}(x_0)$ iff $p_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = p_{2*}(\pi_2(\tilde{X}_2, \tilde{x}_2))$.*

Proof. We may lift $p_1 : \tilde{X}_1 \mapsto X$ to $\tilde{p}_1 : \tilde{X}_1 \mapsto \tilde{X}_2$ and $p_2 : \tilde{X}_2 \mapsto X$ to $\tilde{p}_2 : \tilde{X}_2 \mapsto \tilde{X}_1$. By the unique lifting property, $\tilde{p}_1 \circ \tilde{p}_2 = 1$ and $\tilde{p}_2 \circ \tilde{p}_1 = 1$. ☐

Corollary 16.4.10. *Let X be path-connected, locally path-connected and semilocally simply connected. Then two path-connected covering spaces $p_1 : \tilde{X}_1 \mapsto X$ and $p_2 : \tilde{X}_2 \mapsto X$ are isomorphic iff $p_{1*}(\pi_1(\tilde{X}_1))$ and $p_{2*}(\pi_2(\tilde{X}_2))$ are conjugate.*

16.5 Deck Transformations and Group Actions

Definition 16.5.1. *For a covering space $p : \tilde{X} \mapsto X$ the isomorphisms $\tilde{X} \mapsto \tilde{X}$ are called* **deck transformations***. These form a a group $G(\tilde{X})$ under composition.*

Definition 16.5.2. *A covering space $p : \tilde{X} \mapsto X$ is called* **normal** *if for each $x \in X$ and each $\tilde{x}, \tilde{x}' \in p^{-1}(x)$ there's a deck transformation taking $\tilde{x}$ to $\tilde{x}'$.*

Lemma 16.5.3. *Let $p : (\tilde{X}, \tilde{x}_0) \mapsto (X, x_0)$ be a path-connected covering space of the path-connected, locally path-connected space X. This covering space space is normal iff $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is a normal subgroup of $\pi_1(X, x_0)$*

Lemma 16.5.4. *Let $p : (\tilde{X}, \tilde{x}_0) \mapsto (X, x_0)$ be a path-connected covering space of the path-connected, locally path-connected space X. $G(\tilde{X})$ is isomorphic to $N(p_*(\pi_1(\tilde{X}, \tilde{x}_0)))/p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.*

Proof. For each $[l] \in N(p_*(\pi_1(\tilde{X}, \tilde{x}_0)))$, let $\tilde{l}$ be the lift in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{x}_1$. Then $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = p_*(\pi_1(\tilde{X}, \tilde{x}_1))$. Let $f([l])$ be the deck transformation mapping $\tilde{x}_0$ to $\tilde{x}_1$. f is a homomorphism from $N(p_*(\pi_1(\tilde{X}, \tilde{x}_0)))$ to $G(\tilde{X})$, with kernel $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. □

Definition 16.5.5. *Let G be a group. The* **action** *of G on a topological space X is a homomorphism from G to the autohomeomorphism group of X. Let X/G be the space of orbits with quotient topology. X/G is called the* **orbit space**.

Lemma 16.5.6. *Let G be a group acting on X that satisfies the condition:*

Each $x \in X$ has a neighborhood U_x such that $(\forall g \neq h \in G) g(U_x) \cap h(U_x) = \emptyset$.

The quotient map $X \mapsto X/G$ is a normal covering space of X/G. If X is path-connected, then G is the deck transformation of this covering space. If X is further locally path-connected, $\pi_1(X) = \pi_1(X/G)/G$.

Definition 16.5.7. *For each group $G = \langle g_\alpha | r_\beta \rangle$ we can define the* **Cayley graph** *of G as follows: The vertices are members of G. There's a directed edge joining g and h if there exists a generator g_α such that $gg_\alpha = h$.*

Example 16.5.8. *The Cayley graphs of $D_{12} = \langle a, b | a^{12} = 1, b^2 = 1, abab = 1 \rangle$ and $\langle a, b \rangle$ are shown in Fig. 16.5.*

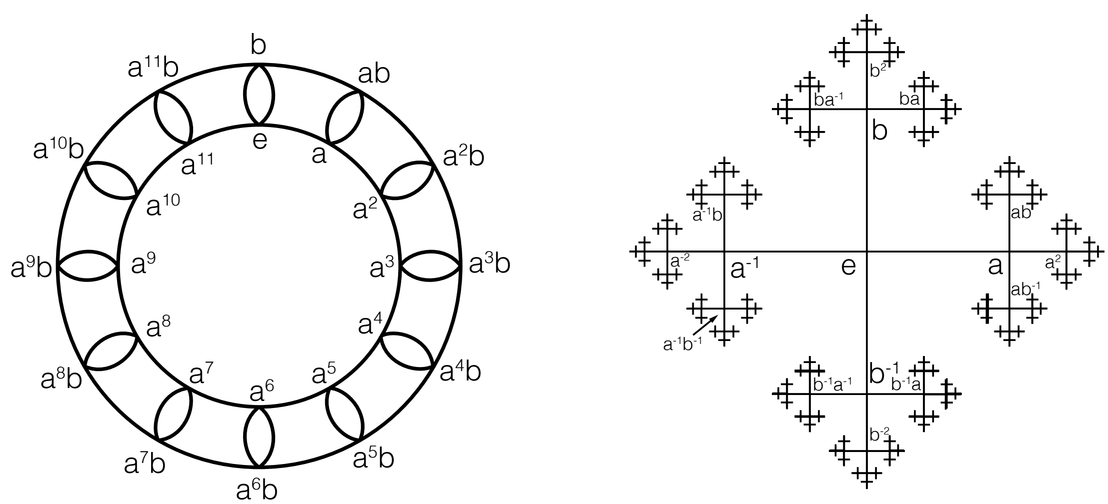

Figure 16.5: Left: Cayley graph of D_{12}. Right: Cayley graph of $\langle a, b \rangle$.

Definition 16.5.9. *For each group $G = \langle g_\alpha | r_\beta \rangle$, let C_G be the Cayley graph of G. For each $g \in G$ and each relation $r_\beta = e_1 \cdots e_n$, $g \to ge_1 \to \cdots \to (ge_1 \cdots e_{n-1}) \to g$ forms a loop, called $L(g, r_\beta)$. To each such loop we attach a S^2. The space we result in is called the* **Cayley complex** *of G.*

Lemma 16.5.10. *For each group $G = \langle g_\alpha | r_\beta \rangle$, let $\tilde{X}_G$ be the Cayley complex of G. $\tilde{X}_G$ is simply connected.*

Proof. Each loop in C_G at g can be decomposed by $P \to L(gh, r_1) \to \cdots \to L(gh, r_n) \to P^{-1}$, where P is a path from g to gh, since each $g_1 \cdots g_m = 1$ can be decomposed by $hr_1 \cdots r_n h^{-1}$. Then use the Example 16.3.4. $\square$

Lemma 16.5.11. *For each group $G = \langle g_\alpha | r_\beta \rangle$, let C_G be the Cayley graph of G, and $\tilde{X}_G$ be the Cayley complex of G. We define the action of G on vertices of C_G by $g \cdot h = gh$. This action naturally extends to an action of G on $\tilde{X}_G$. $X_G = \tilde{X}_G / G$ is just the space we define in Lem. 16.3.6.*

Part IV

Homology and Cohomology

Chapter 17

Simplicial Homology and Singular Homology

17.1 Simplicial Homology

Definition 17.1.1. *An* **n-simplex** *$(n > 0)$ denoted by and ordered n-tuple $[v_0, \ldots, v_n]$ is the region $\{\sum_i \lambda_i v_i | \lambda_i > 0, \sum_i \lambda_i = 1\}$ in $\mathbb{R}^m$ $(m \geq n)$, where v_is are affine dependent vectors (not contained in any $n-1$ dimensional subspace) in $\mathbb{R}^m$. A -1-complex is defined as an empty set.*

Definition 17.1.2. *The* **faces** *of a simplex $[v_0, \ldots, v_n]$ are simplexes $[v'_0, \ldots, v'_m]$ where $\{v'_0, \ldots, v'_m\} \in \{v_0, \ldots, v_n\}$.*

Definition 17.1.3. *An* **simplicial complex** *X is a set of simplexes with a map $f(\Delta, F) \mapsto \Delta' \in X$ for each $\Delta \in X$ and each face F of Δ, such that F and Δ' are of the same dimension and that $F \neq F' \to f(\Delta, F) \neq f(\Delta, F')$*

In the following we will not distinguish between F and $f(\Delta, F)$.

Definition 17.1.4. *Let X be a simplicial complex. We define a* **simplicial n-chain** *by*

$$\cdots \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} C_{-1}(X) = 0 \tag{17.1}$$

, where $C_n(X) = \{$the free abelian group generated by all singular n-simplexes in $X\}$, and $\partial_n : C_n(X) \mapsto C_{n-1}(X)$ is a group homomorphism defined by

$$\partial_n[v_0, \ldots, v_n] = \sum_i (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_n] \tag{17.2}$$

where $\hat{v}_i$ means that v_i is omitted.

Theorem 17.1.5. $\partial_{n-1}\partial_n = 0$

Definition 17.1.6. *Let $\cdots \to C_2(X) \to C_1(X) \to C_0(X) \to 0$ be a simplicial n-chain. We define* **simplicial n-cycles** *as $Z_n(X) = \ker \partial_n$, and* **simplicial n-boundaries** *as $B_n(X) = \operatorname{im} \partial_{n+1}$. Clearly $B_n(X) \subseteq Z_n(X)$.*

Definition 17.1.7. *Let $\cdots \to C_2(X) \to C_1(X) \to C_0(X) \to 0$ be a simplicial n-chain. We define the nth* **simplicial homology group** *of X as $H_n(X) = Z_n(X)/B_n(X)$.*

17.2 Singular Homology

Definition 17.2.1. *Let Δ_n be an n-simplex for each $n \geq -1$. A* **singular n-simplex** *in a topological space X is a continuous map $\sigma : \Delta_n \mapsto X$. Especially Δ_{-1} is the empty map $\emptyset \mapsto X$.*

Definition 17.2.2. *Let $\Delta_n = [e_1, \dots, e_n]$ and $\Delta_{n-1} = [e'_1, \dots, e'_{n-1}]$. For each i we define the ith* **face map** $\epsilon_i^n : \Delta_{n-1} \mapsto \Delta_n$ *as* $\epsilon_i^n(\sum_{j=0}^{n-1} t_j e'_j) = \sum_{j=0}^{i-1} t_j e_j + \sum_{j=i}^{n-1} t_j e_{j+1}$

Definition 17.2.3. *Let X be a topological space. We define a* **singular n-chain** *by*

$$\cdots \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} C_{-1}(X) = 0 \tag{17.3}$$

, where $C_n(X) = \{$the free abelian group generated by all singular n-simplexes in $X\}$, and $\partial_n : C_n(X) \mapsto C_{n-1}(X)$ is a group homomorphism defined by

$$\partial_n \sigma = \sum_i (-1)^i \sigma \circ \epsilon_i^n \tag{17.4}$$

Theorem 17.2.4. $\partial_{n-1}\partial_n = 0$

Definition 17.2.5. *Let $\cdots \to C_2(X) \to C_1(X) \to C_0(X) \to 0$ be a singular n-chain. We define* **singular n-cycles** *as $Z_n(X) = \ker \partial_n$, and* **singular n-boundaries** *as $B_n(X) = \mathrm{im}\ \partial_{n+1}$. Clearly $B_n(X) \subseteq Z_n(X)$.*

Definition 17.2.6. *Let $\cdots \to C_2(X) \to C_1(X) \to C_0(X) \to 0$ be a singular n-chain. We define the nth* **singular homology group** *of X as $H_n(X) = Z_n(X)/B_n(X)$.*

Part V

Homotopy Group II