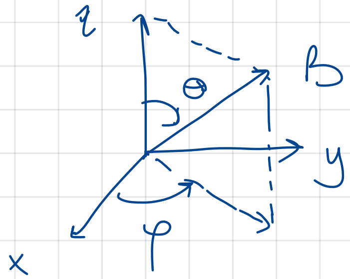

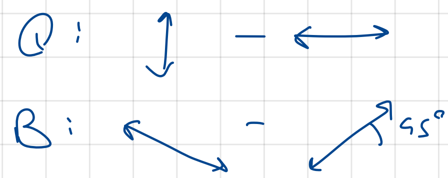

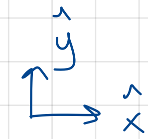

$$B_x = B \sin\theta \cos\varphi$$
$$B_y = B \sin\theta \sin\varphi$$

$$Q = B_y^2 - B_x^2$$
$$= B^2 \sin^2\theta \left(\sin^2\varphi - \cos^2\varphi\right).$$
$$= B^2 \sin^2\theta \cos 2\varphi$$

$$U = \left(\frac{B_y}{\sqrt{2}} - \frac{B_x}{\sqrt{2}}\right)^2 - \left(\frac{B_y}{\sqrt{2}} + \frac{B_x}{\sqrt{2}}\right)^2$$
$$= -2 B_x B_y$$
$$= -2B^2 \sin^2\theta \sin\varphi \cos\varphi$$
$$= -B^2 \sin^2\theta \sin 2\varphi$$

$$Q \pm iU = B^2 \sin^2\theta \left(\cos 2\varphi \mp i \sin 2\varphi\right)$$

$$I^2 = Q^2 + U^2$$
$$= B^4 \sin^4\theta \cos^2 2\varphi + B^4 \sin^4\theta \sin^2 2\varphi$$
$$I = B^2 \sin^2\theta.$$

$$\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta = 4\pi. \Rightarrow P(\hat{n}) = \frac{1}{4\pi}.$$

$$\left|\begin{array}{l}(Q+iU)(\hat n) = \sum_{\ell m} (E_{\ell m} + iB_{\ell m})\,{}_{+2}Y_{\ell m}(\hat n) \equiv \sum_{\ell m} a^{+2}_{\ell m}\,{}_{+2}Y_{\ell m} \\ (Q-iU)(\hat n) = \sum_{\ell m} (E_{\ell m} - iB_{\ell m})\,{}_{-2}Y_{\ell m}(\hat n) = \sum_{\ell m} a^{-2}_{\ell m}\,{}_{-2}Y_{\ell m}\end{array}\right.$$

$$\Rightarrow \left|\begin{array}{l} a^{+2}_{\ell m} = \int d^2\hat n\, (Q+iU)\,{}_{+2}Y_{\ell m} \\ a^{-2}_{\ell m} = \int d^2\hat n\, (Q-iU)\,{}_{-2}Y_{\ell m} = \int d^2\hat n\, (Q-iU)(-1)^\ell\,{}_{+2}Y_{\ell m}\end{array}\right.$$

$$\text{where} \left|\begin{array}{l} E_{\ell m} = \frac{1}{2}\left(a^{+2}_{\ell m} + a^{-2}_{\ell m}\right) \\ B_{\ell m} = \frac{1}{2i}\left(a^{+2}_{\ell m} - a^{-2}_{\ell m}\right). \end{array}\right.$$

$$\begin{aligned} Q \pm iU &= B^2 \sin^2\theta\,(\cos 2\varphi \mp i\sin 2\varphi). \\ &\propto \sin^2\theta\cos 2\varphi \mp i\sin^2\theta\sin 2\varphi \\ &\equiv c \mp is = \sum_{\ell m} (c_{\ell m} \mp i s_{\ell m})\,{}_{+2}Y_{\ell m}. \end{aligned}$$

$$\text{so} \left|\begin{array}{l} a^{+2}_{\ell m} \propto \int d^2\hat n\, (Q+iU)\,{}_{+2}Y_{\ell m} = c_{\ell m} - i s_{\ell m} \\ a^{-2}_{\ell m} \propto \int d^2\hat n\, (Q-iU)\,{}_{-2}Y_{\ell m} = (c_{\ell m} + i s_{\ell m})\cdot(-1)^\ell \end{array}\right.$$

$$\Rightarrow \left|\begin{array}{l} E_{\ell m} = \frac{1}{2}\left[c_{\ell m}\left(1+(-1)^\ell\right) + i s_{\ell m}\left((-1)^\ell - 1\right)\right] \\ B_{\ell m} = \frac{1}{2i}\left[c_{\ell m}\left(1-(-1)^\ell\right) - i s_{\ell m}\left(1+(-1)^\ell\right)\right] \end{array}\right.$$

$$\begin{aligned}\text{where}\quad c_{\ell m} &= \int d^2\hat n\, \sin^2\theta\cos 2\varphi\,{}_{+2}Y_{\ell m}(\hat n) \\ &= \int_0^\pi d\varphi \int_0^{2\pi} d\theta\, \sin^3\theta\cos 2\varphi\,{}_{+2}Y_{\ell m}(\theta,\varphi). \end{aligned}$$

$$s_{\ell m} = \int_0^\pi d\varphi \int_0^{2\pi} d\theta\, \sin^3\theta\sin 2\varphi\,{}_{-2}Y_{\ell m}(\theta,\varphi).$$

$$\langle c_{\ell m} c^*_{\ell' m'} \rangle = \int d^2\hat{n} \int d^2\hat{n}' \langle c(\hat{n}) c^*(\hat{n}') \rangle \, {}_{+2}Y_{\ell m}(\hat{n}) \, {}_{+2}Y^*_{\ell' m'}(\hat{n}') \, \delta(\hat{n}-\hat{n}')$$

$$= \underbrace{\langle c(\hat{n}) c^*(\hat{n}) \rangle}_{} \, \delta_{\ell\ell'} \delta_{mm'}$$

$(\hat{n})$ is a uniformly distributed random angle in the unit 2 sphere

$$\langle c^2 \rangle = \int d\hat{n} \; c^2(\hat{n}) \, P(\hat{n}) = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \left(\sin^2\theta \cos 2\varphi\right)^2 \frac{1}{4\pi} = \frac{4}{15},$$

$$\langle s^2 \rangle = \int d\hat{n} \; s^2(\hat{n}) \, P(\hat{n}) = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \left(\sin^2\theta \sin^2\varphi\right)^2 \frac{1}{4\pi} = \frac{4}{15}$$

$$\langle cs \rangle = \int d\hat{n} \; s(\hat{n}) c(\hat{n}) \, P_n = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin\theta \left(\sin^2\theta\right)^2 \cos 2\varphi \sin 2\varphi \frac{1}{4\pi} = 0$$

$$\cancel{\langle c(\hat{n}) c^*(\hat{n}) \rangle = \left\langle \sum_{\ell m} c_{\ell m} \, {}_{2+}Y_{\ell m}(\hat{n}) \sum_{\ell' m'} c^*_{\ell' m'} \, {}_{+2}Y^*_{\ell' m'}(\hat{n}) \right\rangle}$$

$$\cancel{= \sum_{\ell m} \langle c_{\ell m} c^*_{\ell m} \rangle \, Y_{\ell m}(\hat{n}) \, Y_{\ell m}(\hat{n}')} \quad \text{(impose } \delta_{\ell\ell'}\delta_{mm'}\text{)}$$

$$\cancel{= \langle c_{\ell m} c^*_{\ell m} \rangle \, \delta(\hat{n}-\hat{n}).}$$

$$\cancel{\langle c_{\ell m} c^*_{\ell m} \rangle = \int d^2\hat{n} \left(\sin^2\theta \cos 2\varphi\right)^2 \, {}_{+2}Y_{\ell m} \, {}_{+2}Y^*_{\ell m}}$$

$$\cancel{= \int_0^{\pi} d\varphi \int_0^{2\pi} d\theta \, \sin^5\theta \, \cos^2 2\varphi \, {}_{+2}Y_{\ell m} \, {}_{+2}Y^*_{\ell m}}$$

$$\cancel{= \int_0^{\pi} d\varphi \int_0^{2\pi} d\theta \, \sin^5\theta \, \cos^2 2\varphi \, \frac{2\ell+1}{4\pi} \frac{(\ell-m)!}{(\ell+m)!} \left| P_\ell^m(\cos\theta) \right|^2}$$

$$\langle c_{\ell m} c^*_{\ell' m'} \rangle = \frac{4}{15} \delta_{\ell\ell'} \delta_{mm'} \qquad \langle c_{\ell m} s^*_{\ell' m'} \rangle = 0$$

$$\langle s_{\ell m} s^*_{\ell' m'} \rangle = \frac{4}{15} \delta_{\ell\ell'} \delta_{mm'}$$

$$E_{\ell m} \propto \frac{1}{2}\left[c_{\ell m}\left(1 + (-1)^\ell\right) + i\, s_{\ell m}\left((-1)^\ell - 1\right)\right]$$

$$B_{\ell m} \propto \frac{1}{2i}\left[c_{\ell m}\left(1 - (-1)^\ell\right) - i\, s_{\ell m}\left(1 + (-1)^\ell\right)\right]$$

$$(a+ib)(a-ib) = a^2 - \cancel{iab} + \cancel{iab} + b^2$$

$$\Rightarrow \langle E_{\ell m} E^*_{\ell' m'} \rangle \propto \frac{1}{4}\left[\frac{4}{15} \left(1+(-1)^\ell\right)^2 + \frac{4}{15} \left(\left((-1)^{\ell'}-1\right)\right)^2\right]$$

$$= \frac{1}{15} \left(1+1+2(-1)^\ell + 1+1-2(-1)^{\ell'}\right) \delta_{\ell\ell'}$$

$$= \frac{4}{15} \times e^{-\frac{\ell^2\sigma^2}{8\ln 2}}, \quad \sigma = \frac{\ell_{dom}}{a\chi}$$

$$\langle B_{\ell m} B^*_{\ell' m'} \rangle \propto \left(\frac{1}{2i} \frac{1}{-2i}\right)\left[\frac{4}{15} \left(1-(-1)^\ell\right)^2 + \frac{4}{15} \left(1+(-1)^\ell\right)^2\right]$$

$$= \frac{1}{4} \frac{4}{15} \left(1+1-2(-1)^\ell + 1+1+2(-1)^\ell\right)$$

$$= \frac{4}{15} \times e^{-\frac{\ell^2\sigma^2}{8\ln 2}}$$

$$\langle E B \rangle = 0$$

$$E_{\ell m} \sim \sqrt{\frac{4}{15}}\, e^{-\frac{\ell^2\sigma^2}{16\ln 2}}\ ; \quad B_{\ell m} \sim \sqrt{\frac{4}{15}}\, e^{-\frac{\ell^2\sigma^2}{16\ln 2}}$$

Gaussian should be normalized to 1: $\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{\sigma^2}}$.

FT: $\frac{1}{\sqrt{2\pi}} e^{-k^2\sigma^2}$ where $\sigma = k_{mag}$

Limber approx: $k = (\ell+\frac{1}{2})/\chi$; $\sigma = \frac{\ell_{mag}}{a\chi} \frac{1}{\sqrt{8\ln 2}}$.

comoving wavenumber

physical distance

$$\Rightarrow \mathbb{1} \exp\left[-\frac{(\ell+\frac{1}{2})^2}{\ell_{mag}^2} \frac{1}{8\ln 2}\right]$$

This is the profile of the coherent $\vec{B}$ field domain

square root of beam

FWHM of beam

normalization in Fourier space. assuming

F.T. convention in cosmology: $\varphi(r) = \int \frac{d^3k}{(2\pi)^3} \varphi_k\, e^{ik\cdot r}$

Try again:

$$(Q\pm iU)^{obs}(\hat{n}) = T^{disc}(\hat{n})\ \sin^2\theta\,(\cos 2\varphi \mp i\sin 2\varphi)$$

$$\simeq \bar{T}\ \tau(\hat{n})\ \sin^2\theta\,(\cos 2\varphi \mp i\sin 2\varphi)$$

$$= \bar{T}\sum_{\ell m}\tau_{\ell m}\,Y_{\ell m}\ \sum_{\ell' m'}\left(c_{\ell' m'} \mp i s_{\ell' m'}\right)\ {}_{\pm 2}Y_{\ell' m'}$$

LHS = RHS, and expand each separately later.

Note, from previously, we had that
$$\begin{cases} c^{+2}_{\ell m} + i s^{+2}_{\ell m} = (-1)^{-\ell}\,(E_{\ell m} - iB_{\ell m}) \\ c^{+2}_{\ell m} - i s^{+2}_{\ell m} = (E_{\ell m} + iB_{\ell m}) \end{cases}$$

where $(-1)^{-\ell}$ is there because we considered the ${}_{+2}Y_{\ell m}$ expansion basis.

If ${}_{-2}Y_{\ell m}$ basis, I guess:
$$\begin{cases} c^{-2}_{\ell m} + i s^{-2}_{\ell m} = E_{\ell m} - iB_{\ell m} \\ c^{-2}_{\ell m} - i s^{-2}_{\ell m} = (-1)^{-\ell}\,(E_{\ell m} + iB_{\ell m}). \end{cases}$$

Also, from Wayne Hu, it seems true that:

$${}_{s_1}Y_{\ell_1 m_1}\ {}_{s_2}Y_{\ell_2 m_2} = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi}}\sum_{\ell m s}\langle \ell_1 \ell_2; m_1 m_2 | \ell_1 \ell_2; \ell m\rangle\langle \ell_1 \ell_2; -s_1 -s_2 | \ell_1 \ell_2; \ell\, {-s}\rangle\sqrt{\frac{4\pi}{2\ell+1}}\ {}_sY_{\ell m}.$$

where: $\langle \ell_1 \ell_2; m_1 m_2 | \ell_1 \ell_2; \ell m\rangle = (-1)^{m+\ell_1-\ell_2}\sqrt{2\ell+1}\begin{pmatrix}\ell_1 & \ell_2 & \ell \\ m_1 & m_2 & -m\end{pmatrix}$

$\langle \ell_1 \ell_2; -s_1 -s_2 | \ell_1 \ell_2; \ell -s\rangle = (-1)^{-s+\ell_1-\ell_2}\sqrt{2\ell+1}\begin{pmatrix}\ell_1 & \ell_2 & \ell \\ -s_1 & -s_2 & s\end{pmatrix}$

Since $s_1 = 0$, then expanding in the ${}_{+2}Y_{\ell m}$ basis $\Rightarrow$ s must be $+2$ or gives 0.

$$\Rightarrow Y_{\ell_1 m_1}\ {}_{\pm 2}Y_{\ell_2 m_2} = \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)}{4\pi\,(2\ell+1)}}\sum_{\ell m}\left[(-1)^{m+\ell_1-\ell_2}(-1)^{\ell_1-\ell_2}(2\ell+1)\begin{pmatrix}\ell_1 & \ell_2 & \ell \\ m_1 & m_2 & -m\end{pmatrix}\begin{pmatrix}\ell_1 & \ell_2 & \ell \\ 0 & \mp 2 & \pm 2\end{pmatrix}\ {}_{\pm 2}Y_{\ell m}\right]$$

$$\mathrm{LHS} = (Q \pm iU)^{obs}(\hat{n}) = \sum_{\ell m} \left(E^{obs}_{\ell m} \pm i B^{obs}_{\ell m} \right) {}_{\pm 2}Y_{\ell m} .$$

$$\mathrm{RHS} = \mp T \sum_{\ell_1 m_1} \tau_{\ell_1 m_1} Y_{\ell_1 m_1} \left(c_{\ell_2 m_2} \mp i s_{\ell_2 m_2} \right)^{\pm 2} {}_{\pm 2}Y_{\ell_2 m_2}$$

$$= \mp \sum_{\substack{\ell_1 m_1 \\ \ell_2 m_2 \\ \ell m}} \tau_{\ell_1 m_1} \left(c_{\ell_2 m_2} \mp i s_{\ell_2 m_2} \right)^{\pm 2} \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell+1)}{4\pi}} (-1)^m \begin{pmatrix} \ell & \ell_1 & \ell_2 \\ -m & m_1 & m_2 \end{pmatrix} \begin{pmatrix} \ell_2 & \ell & \ell_1 \\ \mp 2 & \pm 2 & 0 \end{pmatrix} {}_{\pm 2}Y_{\ell m}$$

relabel

$$= \mp \sum_{\substack{\ell m \\ \ell' m' \\ \ell'' m''}} \tau_{\ell'' m''} \left(c_{\ell' m'} \mp i s_{\ell' m'} \right)^{\pm 2} \sqrt{\frac{(2\ell+1)(2\ell'+1)(2\ell''+1)}{4\pi}} (-1)^m {}_{\pm 2}Y_{\ell m} \begin{pmatrix} \ell & \ell'' & \ell' \\ -m & m'' & m' \end{pmatrix} \begin{pmatrix} \ell' & \ell & \ell'' \\ \mp 2 & \pm 2 & 0 \end{pmatrix}$$

switch order of rows

$$\underbrace{\times (-1)^{\ell+\ell''+\ell'} \qquad \times (-1)^{\ell'+\ell+\ell''}}_{\text{cancel out to } 1.}$$

$$= \mp \sum_{\substack{\ell m \\ \ell' m' \\ \ell'' m''}} \tau_{\ell'' m''} \left(c_{\ell' m'} \mp i s_{\ell' m'} \right)^{\pm 2} \sqrt{\frac{(2\ell+1)(2\ell'+1)(2\ell''+1)}{4\pi}} (-1)^m {}_{\pm 2}Y_{\ell m} \begin{pmatrix} \ell & \ell' & \ell'' \\ -m & m' & m'' \end{pmatrix} \begin{pmatrix} \ell & \ell' & \ell'' \\ \pm 2 & \mp 2 & 0 \end{pmatrix}$$

RHS = LHS therefore :

$$\sum_{\ell m} \left(E^{obs}_{\ell m} \pm i B^{obs}_{\ell m} \right) {}_{\pm 2}Y_{\ell m} = \sum_{\substack{\ell m \\ \ell' m' \\ \ell'' m''}} \mp \tau_{\ell'' m''} \left(c_{\ell' m'} \mp i s_{\ell' m'} \right)^{\pm 2} (-1)^m \sqrt{\frac{(2\ell+1)(2\ell'+1)(2\ell''+1)}{4\pi}} \times$$

$$\times \begin{pmatrix} \ell & \ell' & \ell'' \\ \pm 2 & \mp 2 & 0 \end{pmatrix} \begin{pmatrix} \ell & \ell' & \ell'' \\ -m & m' & m'' \end{pmatrix} {}_{\pm 2}Y_{\ell m}$$

E, B should cover orthonormal parity states ~~to~~ for spin-2 ; these are $\frac{1}{2}\left[{}_{+2}a_{\ell m} \, {}_{+2}Y_{\ell m} \pm {}_{-2}a_{\ell m} \, {}_{-2}Y_{\ell m} \right]$.

So, really we should have the following: $(E_{\ell m})$ selects "+" branch and $(iB_{\ell m})$ selects "−"

$$E^{abs}_{\ell m} = \sum_{\substack{\ell m \\ \ell' m' \\ \ell'' m''}} \mp \tau_{\ell'' m''} (-1)^m \sqrt{\frac{(2\ell+1)(2\ell'+1)(2\ell''+1)}{4\pi}} \begin{pmatrix} \ell & \ell' & \ell'' \\ -m & m' & m'' \end{pmatrix} \times$$

$$\times \frac{1}{2} \left[\begin{pmatrix} \ell & \ell' & \ell'' \\ +2 & -2 & 0 \end{pmatrix} \left(c_{\ell' m'} - i s_{\ell' m'} \right)^{+2} {}_{+2}Y_{\ell m} + \begin{pmatrix} \ell & \ell' & \ell'' \\ -2 & +2 & 0 \end{pmatrix} \left(c_{\ell' m'} - i s_{\ell' m'} \right)^{-2} {}_{-2}Y_{\ell m} \right]$$

$$i B^{abs}_{\ell m} = \sum_{\substack{\ell m \\ \ell' m' \\ \ell'' m''}} \mp \tau_{\ell'' m''} (-1)^m \sqrt{\frac{(2\ell+1)(2\ell'+1)(2\ell''+1)}{4\pi}} \begin{pmatrix} \ell & \ell' & \ell'' \\ -m & m' & m'' \end{pmatrix} \times$$

$$\times \frac{1}{2} \left[\begin{pmatrix} \ell & \ell' & \ell'' \\ +2 & -2 & 0 \end{pmatrix} \left(c_{\ell' m'} + i s_{\ell' m'} \right)^{-2} {}_{-2}Y_{\ell m} - \begin{pmatrix} \ell & \ell' & \ell'' \\ -2 & +2 & 0 \end{pmatrix} \left(c_{\ell' m'} + i s_{\ell' m'} \right)^{+2} {}_{+2}Y_{\ell m} \right]$$

Let $W^{s s' s''}_{\ell \ell' \ell''} \equiv \sqrt{\frac{(2\ell+1)(2\ell'+1)(2\ell''+1)}{4\pi}} \begin{pmatrix} \ell & \ell' & \ell'' \\ -s & s' & s'' \end{pmatrix}$

$$E^{obs}_{\ell m} = \sum_{\substack{\ell' m' \\ \ell'' m''}} \mp \tau_{\ell'' m''} (-1)^m \begin{pmatrix} \ell & \ell' & \ell'' \\ -m & m' & m'' \end{pmatrix} W^{220}_{\ell \ell' \ell''} \left(E_{\ell' m'} + i B_{\ell' m'}\right) \frac{1}{2}\Big[\overbrace{(-1)^{\ell+\ell'+\ell''}}^{\text{from changing sign of } m\text{'s}} + \underbrace{(-1)^{\ell}}_{\text{from changing spin sign in } Y^s_{\ell m}} \overbrace{(-1)^{-\ell}}^{\text{from }-2\text{ expansion terms}} \Big]$$

$$B^{obs}_{\ell m} = \sum_{\substack{\ell' m' \\ \ell'' m''}} \mp \tau_{\ell'' m''} (-1)^m \begin{pmatrix} \ell & \ell' & \ell'' \\ -m & m' & m'' \end{pmatrix} W^{220}_{\ell \ell' \ell''} \left(E_{\ell' m'} - i B_{\ell' m'}\right) \frac{1}{2i}\Big[1 - \overbrace{(-1)^{\ell+\ell'+\ell''}}^{\text{from changing sign of } m\text{'s}} \underbrace{(-1)^{+\ell}}_{\text{from changing spin sign in } Y^s_{\ell m}} \overbrace{(-1)^{-\ell}}^{\text{from }+2\text{ expansion terms}} \Big]$$

$$\Rightarrow \left| \begin{aligned} E^{obs}_{\ell m} &= \mp \sum_{\substack{\ell' m' \\ \ell'' m''}} \tau_{\ell'' m''} \left(E_{\ell' m'} + i B_{\ell' m'}\right) e_{\ell \ell' \ell''} (-1)^m W^{220}_{\ell \ell' \ell''} \begin{pmatrix} \ell & \ell' & \ell'' \\ -m & m' & m'' \end{pmatrix} \\ i B^{obs}_{\ell m} &= \mp \sum_{\substack{\ell' m' \\ \ell'' m''}} \tau_{\ell'' m''} \left(E_{\ell' m'} - i B_{\ell' m'}\right) o_{\ell \ell' \ell''} (-1)^m W^{220}_{\ell \ell' \ell''} \begin{pmatrix} \ell & \ell' & \ell'' \\ -m & m' & m'' \end{pmatrix} \end{aligned} \right.$$

where here $\underbrace{E^{obs}_{\ell m} \perp B^{obs}_{\ell m}}_{\text{screened field}}$ and $\underbrace{B_{\ell' m'} \perp E_{\ell' m'}}_{\text{axion polarization}}$

$$T^{obs}_{\ell m} = \mp \sum_{\substack{\ell' m' \\ \ell'' m''}} \tau_{\ell'' m''} T_{\ell' m'} (-1)^m W^{000}_{\ell \ell' \ell''} \begin{pmatrix} \ell & \ell' & \ell'' \\ -m & m' & m'' \end{pmatrix}$$

For spectra it follows that

$\langle E_{\ell m} B_{\ell m} \rangle = 0$ because $e_{\ell \ell' \ell''} \times o_{\ell \ell' \ell''}$ gives 0 to Wigner 3j's

$\langle T_{\ell m} B_{\ell m} \rangle = 0$ because $W^{000}_{\ell \ell' \ell''} = 0$ for $\ell + \ell' + \ell'' = \text{odd}$

$$C_\ell^{TT} = \frac{\bar T^2}{4\pi} C_\ell^{\tau\tau} + \sum_{\ell'\ell''} C_{\ell''}^{\tau\tau} C_{\ell'}^{TT} \left(W^{000}_{\ell\ell'\ell''}\right)^2 \simeq \frac{\bar T^2}{4\pi} C_\ell^{\tau\tau}$$

There's also an anisotropic part, same as for dark photon

$$\langle E^{obs\,*}_{\ell m} E^{obs}_{LM}\rangle = \Big\langle \bar T \sum_{\substack{\ell'm'\\ \ell''m''}} \tau_{\ell''m''} \left(E_{\ell'm'} + iB_{\ell'm'}\right)^* e_{\ell\ell'\ell''} (-1)^m W^{220}_{\ell\ell'\ell''} \begin{pmatrix} \ell & \ell' & \ell'' \\ -m & m' & m''\end{pmatrix}$$

$$\bar T \sum_{\substack{L'M'\\ L''M''}} \tau_{L''M''} \left(E_{L'M'} + iB_{L'M'}\right) e_{LL'L''} (-1)^M W^{220}_{LL'L''} \begin{pmatrix} L & L' & L'' \\ -M & M' & M''\end{pmatrix}\Big\rangle$$

$$\langle E^*_{\ell'm'} E_{L'M'}\rangle = |E^2_{\ell'm'}|\, \delta_{\ell'L'}\delta_{m'M'} \quad , \quad \langle E^*_{\ell'm'} B_{L'M'}\rangle = 0$$

$$\langle E^*_{\ell m} E_{LM}\rangle^{obs} = \bar T^2 \sum_{\substack{\ell'm'\\ \ell''m''}} C^{\tau\tau}_{\ell''}\left[C^{EE}_{\ell'} + C^{BB}_{\ell'}\right] e_{\ell\ell'\ell''} (-1)^{m+M} \frac{(2\ell+1)(2\ell'+1)(2\ell''+1)}{4\pi} \begin{pmatrix}\ell&\ell'&\ell''\\-2&2&0\end{pmatrix}\begin{pmatrix}L&\ell'&\ell''\\-2&2&0\end{pmatrix}\begin{pmatrix}\ell&\ell'&\ell''\\-m&m'&m''\end{pmatrix}\begin{pmatrix}L&\ell'&\ell''\\-M&m'&m''\end{pmatrix}$$

since $\sum_{m'm''} \begin{pmatrix}\ell&\ell'&\ell''\\-m&m'&m''\end{pmatrix}\begin{pmatrix}L&\ell'&\ell''\\-M&m'&m''\end{pmatrix}(2\ell+1) = \delta_{\ell L}\delta_{mM}$, and $\bar W^{\ell\ell'\ell''}_{ss's''} = \sqrt{\frac{(2\ell'+1)(2\ell''+1)}{4\pi}} \begin{pmatrix}\ell&\ell'&\ell''\\-s&s'&s''\end{pmatrix}$

$$\langle E^*_{\ell m} E_{\ell m}\rangle^{obs} = \bar T^2 \sum_{\ell'\ell''} C^{\tau\tau}_{\ell''}\left(C^{EE}_{\ell'} + C^{BB}_{\ell'}\right) e_{\ell\ell'\ell''} \left(\bar W^{\ell\ell'\ell''}_{220}\right)^2 , \quad e_{\ell\ell'\ell''} = \frac12\left(1+(-1)^{\ell+\ell'+\ell''}\right)$$

$$\langle B^*_{\ell m} B_{\ell m}\rangle^{obs} = \bar T^2 \sum_{\ell'\ell''} C^{\tau\tau}_{\ell''}\left(C^{EE}_{\ell'} + C^{BB}_{\ell'}\right) o_{\ell\ell'\ell''} \left(\bar W^{\ell\ell'\ell''}_{220}\right)^2 , \quad o_{\ell\ell'\ell''} = \frac12\left(1-(-1)^{\ell+\ell'+\ell''}\right)$$

$$\langle \cancel{T^*_{\ell m} E_{LM}}\rangle = \Big\langle \bar T \sum_{\substack{\ell'm'\\ \ell''m''}} \tau_{\ell''m''} T_{\ell'm'} (-1)^m \sqrt{\frac{(2\ell+1)(2\ell'+1)(2\ell''+1)}{4\pi}} \begin{pmatrix}\ell&\ell'&\ell''\\0&0&0\end{pmatrix}\begin{pmatrix}\ell&\ell'&\ell''\\-m&m'&m''\end{pmatrix}$$

0

$$\bar T \sum_{\substack{L'M'\\ L''M''}} \tau_{L''M''}\left(E_{L'M'} + iB_{L'M'}\right) \cancel{e_{LL'L''}} (-1)^M \sqrt{\frac{(2L+1)(2L'+1)(2L''+1)}{4\pi}} \begin{pmatrix}L&L'&L''\\-2&2&0\end{pmatrix}\begin{pmatrix}L&L'&L''\\-M&M'&M''\end{pmatrix}\Big\rangle$$

but $\langle T_{\ell'm'} E_{L'M'}\rangle = \langle T_{\ell'm'} B_{L'M'}\rangle = 0$, uncorrelated since $E, B \propto$ random angles θ, φ and

and $\langle T_{\ell m} c_{\ell'm'}\rangle$, $\langle T_{\ell m} s_{\ell'm'}\rangle$ are zero by symmetry: $Tc, Ts \propto \int_0^{2\pi} d\theta \sin\theta \left[\sin^2\theta\right] = 0$.

Everything else is the same as for dark photon but with an extra factor of $E^2, B^2 \propto \left[\frac{\bar T}{\sqrt{4\pi}} \frac{1-e^x}{x}\right]^2$

Everything else is the same as for dark photon but with extra factor

and with $C^{EE}_{\ell'}(x) + C^{BB}_{\ell'}(x) = 2x \frac{4}{15} 4\pi\sigma^2 e^{-\ell'(\ell'+1)\sigma^2}$

Next, we want $\langle B(\hat{n})\, B(\hat{n}')\, T(\hat{n}'')\rangle$

$$\psi^{\tau\tau}(\theta) = \sum_\ell \frac{2\ell+1}{4\pi} C_\ell^{\tau\tau}\, d^\ell_{00}(\theta)$$

$$\psi^{TT}(\theta) = \sum_\ell \frac{2\ell+1}{4\pi} C_\ell^{TT}\, d^\ell_{00}(\theta)$$

$$\psi^{EE}_\pm(\theta) = \pm \sum_\ell \frac{2\ell+1}{4\pi} C_\ell^{EE}\, d^\ell_{2,\pm 2}(\theta).$$

$$\psi^{BB}_\pm(\theta) = \pm \sum_\ell \frac{2\ell+1}{4\pi} C_\ell^{BB}\, d^\ell_{2,\pm 2}(\theta).$$

$$C_\ell^{TT(scr)} = \bar{T}^2 \sum_{\ell'\ell''} C_{\ell''}^{\tau\tau}\, C_{\ell'}^{TT} \left(\bar{W}^{000}_{\ell\ell'\ell''}\right)^2$$

$$C_\ell^{EE(scr)} = \bar{T}^2 \sum_{\ell'\ell''} C_{\ell''}^{\tau\tau} \left(C_{\ell'}^{EE} + C_{\ell'}^{BB}\right) e_{\ell\ell'\ell''} \left(\bar{W}^{\ell\ell'\ell''}_{220}\right)^2$$

$$C_\ell^{BB(scr)} = \bar{T}^2 \sum_{\ell'\ell''} C_{\ell''}^{\tau\tau} \left(C_{\ell'}^{EE} + C_{\ell'}^{BB}\right) o_{\ell\ell'\ell''} \left(\bar{W}^{\ell\ell'\ell''}_{220}\right)^2$$

$$C_\ell^{TT(scr)} = \bar{T}^2 \sum_{\ell'\ell''} C_{\ell''}^{\tau\tau}\, C_{\ell'}^{TT} \left(\bar{W}^{000}_{\ell\ell'\ell''}\right)^2$$

$$\psi^{TT(scr)} = \sum_\ell \frac{2\ell+1}{4\pi} C_\ell^{TT(scr)}\, d^\ell_{00}(\theta) = \sum_{\ell\ell'\ell''} \frac{2\ell+1}{4\pi} \bar{T}^2\, C_{\ell''}^{\tau\tau}\, C_{\ell'}^{TT}\, \frac{(2\ell'+1)(2\ell''+1)}{4\pi} \begin{pmatrix} \ell & \ell' & \ell'' \\ 0 & 0 & 0 \end{pmatrix}^2 P_\ell(\cos\theta)$$

$$= \sum_{\ell\ell'\ell''} \frac{(2\ell+1)(2\ell'+1)(2\ell''+1)}{(4\pi)^2} \bar{T}^2\, C_{\ell''}^{\tau\tau}\, C_{\ell'}^{TT}\, \frac{1}{2}\int_{-1}^{1} d\cos\theta\; P^2_\ell(\cos\theta)\, P_{\ell'}(\cos\theta)\, P_{\ell''}(\cos\theta)$$

$$= \frac{1}{2}\bar{T}^2 \int_{-1}^{1} dx \sum_{\ell\ell'\ell''} \frac{(2\ell+1)(2\ell'+1)(2\ell''+1)}{(4\pi)^2} P^2_\ell(x)\, P_{\ell'}(x)\, P_{\ell''}(x)\; C_{\ell''}^{\tau\tau}\, C_{\ell'}^{TT}$$

$$= \frac{4\pi}{2}\bar{T}^2 \int_{-1}^{1} dx \left(\underbrace{\sum_\ell \frac{2\ell+1}{4\pi} P^2_\ell(x)}\right) \left(\sum_{\ell'} \frac{2\ell'+1}{4\pi} C_{\ell'}^{TT} P_{\ell'}(x)\right) \left(\sum_{\ell''} \frac{2\ell''+1}{4\pi} C_{\ell''}^{\tau\tau} P_{\ell''}(x)\right)$$

$$= 2\pi\bar{T}^2 \int_{-1}^{1} dx\; \delta(x-x)\; \psi^{TT}(x)\; \psi^{\tau\tau}(x)$$

$$= 2\pi\,\bar{T}^2\, \psi^{TT}(\theta)\, \psi^{\tau\tau}(\theta).$$

$$\psi^{EE(scr)}_\pm = \pm \sum_\ell \frac{2\ell+1}{4\pi} C_\ell^{EE(scr)}\, d^\ell_{2,\pm 2}(\theta) =$$

$$= \pm \sum_\ell \frac{2\ell+1}{4\pi} \bar{T}^2 \sum_{\ell'\ell''} C_{\ell''}^{\tau\tau} \left(C_{\ell'}^{EE} + i\, C_{\ell'}^{BB}\right) e_{\ell\ell'\ell''} \left(\bar{W}^{\ell\ell'\ell''}_{220}\right)^2 d^\ell_{2,\pm 2}(\theta).$$

$$C_\ell^{EE} = \frac{1}{2} \int_{-1}^{1} dx \left[\psi_+^{EE\,scr} + \psi_-^{EE,\,scr}\right]$$

where $\psi_+^{EE(scr)} = 2\pi\, \psi^{\tau\tau}(\theta)\, \psi^{EE}_+$

$$\psi^{EE}_+ = \sum_\ell \frac{2\ell+1}{4\pi} \left(C_\ell^{EE} + C_\ell^{BB}\right) d^\ell_{2,+2}$$