Linear equations : iterative methods

Gauss-Jordon and LU quickly become unmanageable $\rightarrow$ increasing matrix size + increasing number of 0 entries (sparse matrix) – pivoting is a costly affair.

Best strategy is *iterative method* : Jacobi & Gauss-Seidel

• **Jacobi method :** works when A has no zeros in the main diagonal. It decomposes, but not factorises, A into a diagonal D and L, U

$$\mathrm{A} = \mathrm{D}+(\mathrm{L}+\mathrm{U}) = \begin{pmatrix} a_{00} & 0 & \cdots & 0 \\ 0 & a_{11} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n-1\,n-1} \end{pmatrix} + \begin{pmatrix} 0 & a_{01} & \cdots & a_{0\,n-1} \\ a_{10} & 0 & \cdots & a_{2\,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1\,0} & a_{n-1\,1} & \cdots & 0 \end{pmatrix}$$

Then the solution can be approached as

$$\mathrm{A}\cdot\mathrm{x} = \mathrm{b} \Rightarrow \Big[\mathrm{D} + (\mathrm{L}+\mathrm{U})\Big]\cdot\mathrm{x} = \mathrm{b} \Rightarrow \mathrm{x} = \mathrm{D}^{-1}\Big[\mathrm{b} - (\mathrm{L}+\mathrm{U})\cdot\mathrm{x}\Big]$$

Above expression evaluated iteratively for x until $\boxed{||\mathrm{x}^{k+1} - \mathrm{x}^{k}|| < \epsilon}$,

$$\mathrm{x}^{(k+1)} = \mathrm{D}^{-1}\left(\mathrm{b} - (\mathrm{L}+\mathrm{U})\mathrm{x}^{(k)}\right) \quad \Rightarrow \quad x_i^{(k+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{j\neq i} a_{ij}x_j^{(k)}\right)$$

Jacobi method is fairly fast but computation of $x_i^{(k+1)}$ element requires all other elements, we cannot overwrite $x_i^{(k)}$ with $x_i^{(k+1)}$.

Sufficient condition for the method to converge is that the matrix A is strictly diagonally dominant,

$$|a_{ii}| \gtrsim \sum_{j \neq i} |a_{ij}|$$

Although Jacobi method is relatively slower to converge to solutions, its main advantage is that it can be easily parallelized and hence can handle large matrices, which reduces both memory requirement for matrix storage and time taken.

As an example solve the following,

$$\begin{array}{rcr} 4x - y + z &=& 7 \\ -2x + y + 5z &=& 15 \\ 4x - 8y + z &=& -21 \end{array} \Rightarrow \mathrm{A} = \begin{pmatrix} 4 & -1 & 1 \\ -2 & 1 & 5 \\ 4 & -8 & 1 \end{pmatrix}$$

Above A is not diagonally dominant but swapping eqn. 2 with eqn. 3 *i.e.* row 2 with row 3 of A will make it so.

According to the formula above

$$\begin{aligned} x &= \frac{7+y-z}{4} & & x^{(k+1)} &= \frac{7+y^{(k)}-z^{(k)}}{4} \\ y &= \frac{21+4x+z}{8} & \Rightarrow & y^{(k+1)} &= \frac{21-4x^{(k)}-z^{(k)}}{8} \\ z &= \frac{15+2x-y}{5} & & z^{(k+1)} &= \frac{15+2x^{(k)}-y^{(k)}}{5} \end{aligned}$$

Start with initial guess $(x^{(0)}, y^{(0)}, z^{(0)}) = (0, 0, 0)$.

(k)	$x^{(k)}$	$y^{(k)}$	$z^{(k)}$
0	0	0	0
1	1.75	2.625	3.0
2	1.656	3.875	3.175
⋮	⋮	⋮	⋮
6	1.9995	3.996	2.9987

Try the following without making diagonally dominant

$$\begin{aligned} 2x+3y+z &= 13 \\ x+y+z &= 6 \\ 3x+2y+2z &= 15 \end{aligned} \qquad x=3, y=2, z=1$$

Gauss-Seidel method

Gauss-Seidel method is defined by the relation

$$A = L_\star + U \;\rightarrow\; \begin{pmatrix} a_{00} & 0 & \cdots & 0 \\ a_{10} & a_{11} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1\,1} & a_{n-1\,2} & \cdots & a_{n-1\,n-1} \end{pmatrix} + \begin{pmatrix} 0 & a_{01} & \cdots & a_{1\,n-1} \\ 0 & 0 & \cdots & a_{2\,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

The solution is obtained through the iterative process

$$L_\star x^{(k+1)} = b - U x^{(k)}$$

Iterative process continues until the changes $\boxed{||x^{(k+1)} - x^{(k)}|| < \epsilon}$

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=0}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n-1} a_{ij} x_j^{(k)} \right), \quad i = 0, 1, \ldots, n-1$$

Gauss-Seidel requires only one storage vector for $x^{(k)}$ as against two in Jacobi, since computation of $x^{(k+1)}$ uses elements already calculated $x^{(k+1)}$ and those $x^{(k)}$ that are already used in the previous iteration.

Convergence is only guaranteed if the matrix is either *strictly diagonally dominant* or *symmetric and positive definite*.

Let us try to understand Gauss-Seidel method through an explicit calculation in a 4×4 matrix

$$
\text{RHS} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix} - \begin{pmatrix} 0 & a_{01} & a_{02} & a_{03} \\ 0 & 0 & a_{12} & a_{12} \\ 0 & 0 & 0 & a_{23} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_0^{(k)} \\ x_1^{(k)} \\ x_2^{(k)} \\ x_3^{(k)} \end{pmatrix}
$$

$$
= \begin{pmatrix} b_0 - a_{01}x_1^{(k)} - a_{02}x_2^{(k)} - a_{03}x_3^{(k)} \\ b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} \\ b_2 - a_{23}x_3^{(k)} \\ b_3 \end{pmatrix} = b_i - \sum_{j=i+1}^{n-1} a_{ij}x_j^{(k)}
$$

$$
\text{LHS} = \begin{pmatrix} a_{00} & 0 & 0 & 0 \\ a_{10} & a_{11} & 0 & 0 \\ a_{20} & a_{21} & a_{22} & 0 \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_0^{(k+1)} \\ x_1^{(k+1)} \\ x_2^{(k+1)} \\ x_3^{(k+1)} \end{pmatrix}
$$

$$
= \begin{pmatrix} a_{00}x_0^{(k+1)} \\ a_{10}x_0^{(k+1)} + a_{11}x_1^{(k+1)} \\ a_{20}x_0^{(k+1)} + a_{21}x_1^{(k+1)} + a_{22}x_2^{(k+1)} \\ a_{30}x_0^{(k+1)} + a_{31}x_1^{(k+1)} + a_{32}x_2^{(k+1)} \, a_{33}x_3^{(k+1)} \end{pmatrix} = a_{ii}x_i^{(k+1)} + \sum_{j=0}^{i-1} a_{ij}x_j^{(k+1)}
$$

Equating LHS with RHS

$$\begin{aligned}
a_{00}x_0^{(k+1)} &= b_0 - a_{01}x_1^{(k)} - a_{02}x_2^{(k)} - a_{03}x_3^{(k)} \\
a_{10}x_0^{(k+1)} a_{11}x_1^{(k+1)} &= b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)} \\
a_{20}x_0^{(k+1)} + a_{21}x_1^{(k+1)} + a_{22}x_2^{(k+1)} &= b_2 - a_{23}x_3^{(k)} \\
a_{30}x_0^{(k+1)} + a_{31}x_1^{(k+1)} + a_{32}x_2^{(k+1)} a_{33}x_3^{(k+1)} &= b_3 \\
\\
a_{ii}x_i^{(k+1)} + \textstyle\sum_{j=0}^{i-1} a_{ij}x_j^{(k+1)} &= b_i - \textstyle\sum_{j=i+1}^{n-1} a_{ij}x_j^{(k)}
\end{aligned}$$

Thus we see, $x_i^{(k)}$ is used only once and can be replaced with $x_i^{(k+1)}$. More explicitly

$$\begin{aligned}
x_0^{(k+1)} &= \frac{1}{a_{00}}\left(b_0 - a_{01}x_1^{(k)} - a_{02}x_2^{(k)} - a_{03}x_3^{(k)}\right) \\
x_1^{(k+1)} &= \frac{1}{a_{11}}\left(b_1 - a_{10}x_0^{(k+1)} - a_{12}x_2^{(k)} - a_{13}x_3^{(k)}\right) \\
x_2^{(k+1)} &= \frac{1}{a_{22}}\left(b_2 - a_{20}x_0^{(k+1)} - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)}\right) \\
x_3^{(k+1)} &= \frac{1}{a_{33}}\left(b_3 - a_{30}x_0^{(k+1)} - a_{31}x_1^{(k+1)} - a_{32}x_2^{(k+1)}\right) \\
i.e.\ \ x_i^{(k+1)} &= \frac{1}{a_{ii}}\left(b_i - \sum_{j=0}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n-1} a_{ij}x_j^{(k)}\right)
\end{aligned}$$

Solve the following,

$$\begin{aligned} 4x_1 + x_2 - x_3 + x_4 &= -2 \\ x_1 + 4x_2 - x_3 - x_4 &= -1 \\ -x_1 - x_2 + 5x_3 + x_4 &= 0 \\ x_1 - x_2 + x_3 + 3x_4 &= 1 \end{aligned}$$