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An electrical network analogue of vibrations of a linear lattice

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Abstract The article describes experiments, suitable for undergraduate laboratory work, using low-pass transmission lines composed of discrete inductors and capacitors to simulate several aspects of linear lattice vibrations. Advantages compared with the use of mechanical systems of masses and springs are suggested.

Résumé Cet article présente des expériences de simulation de divers aspects des vibrations dans les réseaux linéaires, susceptibles de constituer des travaux pratiques de Maîtrise. Ces expériences font appel à des lignes de transmission passe-bas, composées de capacités et d'autoinductances discrètes. On discute des avantages possibles de cette technique par comparaison avec l'utilisation de systèmes mécaniques formés de masses et de ressorts.

1 Introduction

For many years we have used both the electrical transmission lines described here, and a mechanical system of the kind described by Runk *et al* (1963), in undergraduate laboratory work related to lattice vibrations. Advantages of the electrical system include

(i) easier adaption to simulate various aspects of lattice dynamics,

(ii) easier identification of the resonant modes on account of sharpness in frequency, and yet virtually instantaneous damping on detuning from resonance, associated with the use of frequencies in the MHz range, and

(iii) the possibility of more completely quantitative studies.

After a brief survey of some important relationships we shall describe the construction of suitable electrical networks and present some results to illustrate their use.

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2 The analytical basis

Equivalent mechanical and electrical *unit oscillators* are indicated in figure 1. Assuming that the equal springs in the mechanical oscillator obey Hooke's law, with force constant 2α, the differential equations for the longitudinal displacement ξ and for the current I in the electrical circuit are

$$M\frac{d^2\xi}{dt^2}+4\alpha\xi=0; \qquad L\frac{d^2I}{dt^2}+\frac{4}{C}I=0.$$

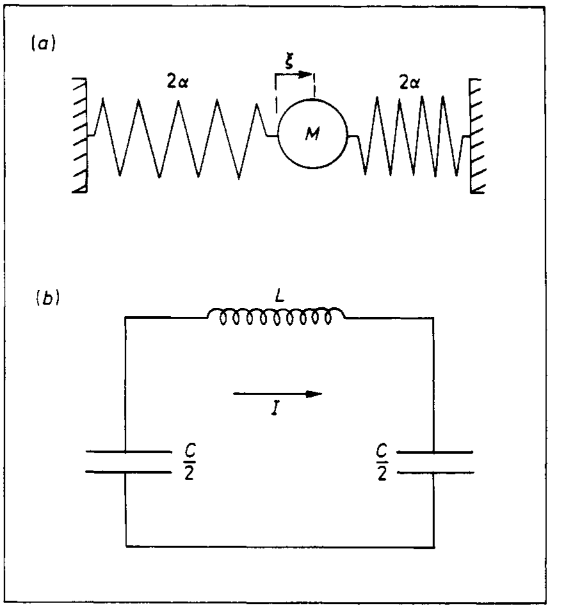


Figure 1 (*a*) Mechanical and (*b*) electrical unit oscillators.

Hence if I is regarded as analogous to ξ, the inductance L and reciprocal capacitance $1/C$ are analogous to mass M and force constant α respectively. A linear array of N identical unit oscillators, as shown in figure 2, corresponds to a monatomic

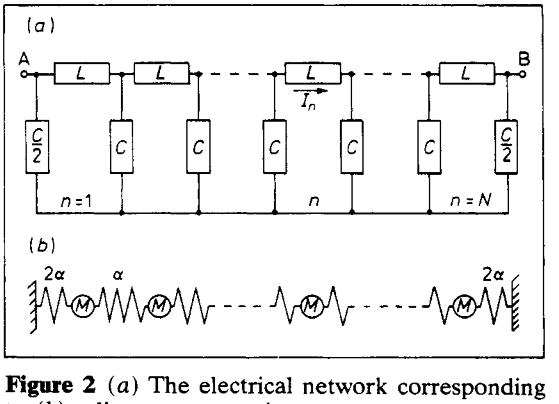


Figure 2 (*a*) The electrical network corresponding to (*b*) a linear monatomic array.

linear lattice with its ends fixed as in the single unit of figure 1(*a*). Such an array possesses N normal modes with currents in the inductors given by

$$I_n = A \sin \omega_r t \sin[(n-\tfrac{1}{2})k_r] \qquad n = 1, 2, \ldots, N \qquad (1)$$

where $\omega_r = \omega_c \sin \frac{1}{2}k_r$ with $r = 1, 2, \ldots, N$.

The angular frequency $\omega_c = 2/(LC)^{1/2}$ corresponds to an upper limit, or *cut-off* frequency, for which $k_r = k_N = \pi$.

Equation (1) satisfies the same fixed-end boundary conditions as does the mechanical array shown in figure 2(*b*) provided that

$$k_r = r\pi/N$$

or

$$r\lambda_r/2 = N, \qquad \lambda_r = 2\pi/k_r.$$

By electrically connecting the end points A and B in figure 2(*a*) the array effectively becomes a closed ring and the end conditions are replaced by the periodicity condition

$$I_{n+N} = I_n.$$

If N is even there are then $N/2$ normal modes with $k_r = 2r\pi/N$ or $r\lambda_r = N$, $r = 0, 1, \ldots, N/2$; and the currents are given by

$$I_n = A \sin \omega_r t \sin(nk_r + \phi).$$

The arbitrary phase constant ϕ becomes determined when the system is coupled in some particular way to a signal generator and when a specific unit is identified as $n = 1$.

An analogue of a diatomic linear lattice can be constructed in an obvious way with alternate unit oscillators having different inductances L_1 and L_2. Each *unit cell*, consisting of a neighbouring pair of such oscillators, is designated by an integer n and for an even number N of unit cells connected in the closed ring configuration the normal mode currents are

$$I_{2n} = A \sin \omega_r t \sin(2nk_r + \phi) \qquad n = 1, 2, \ldots, N$$
$$I_{2n-1} = B \sin \omega_r t \sin[(2n-1)k_r + \phi]$$

where $k_r = r\pi/N$; $r = 1, 2, \ldots, N/2$.

Two modes are associated with each value of k_r and have angular frequencies given by

$$\omega_r^2 = (\omega_1^2 + \omega_2^2)\left[1 \pm \left(1 - 4\frac{\omega_1^2\omega_2^2}{\omega_1^2+\omega_2^2}\sin^2 k_r\right)^{1/2}\right] \qquad (2)$$

where $\omega_1^2 = 1/L_1C$, $\omega_2^2 = 1/L_2C$.

The amplitude factors are in the ratio

$$\frac{A}{B} = -\frac{2\cos k_r}{\omega_r^2/\omega_1^2 - 2} = -\frac{\omega_r^2/\omega_2^2 - 2}{2\cos k_r} \qquad (3)$$

the inductances L_1 and L_2 being associated respectively with the unit oscillators numbered $2n$ and $2n-1$, and with the amplitude factors A and B.

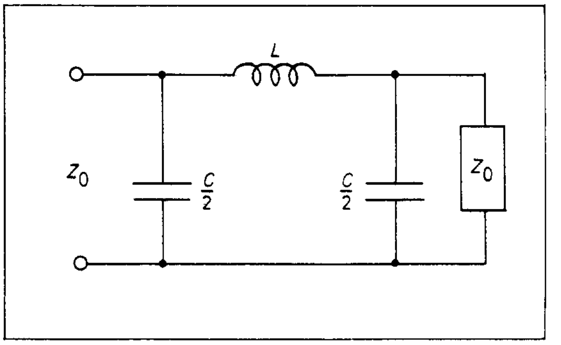


Figure 3 Illustrating the concept of characteristic impedance.

The presence of an impurity atom can be simulated by inserting a unit oscillator with values L' and $\frac{1}{2}C'$ of inductance and capacitance, both differing in the general case from those in the units of a uniform array. In particular, a light impurity in a monatomic lattice can give rise to a localised oscillation mode with a frequency above the cut-off value ω_c of the uniform lattice. In the electrical analogue the frequency of a localised mode can be calculated by making use of the concept of characteristic impedance Z_0. This is defined as that impedance which, when connected across the output terminals of a single unit oscillator, makes the input impedance also equal to Z_0. It then follows that the input impedance of a linear array of any number of units, finally terminated by the characteristic impedance, or of an infinitely long array, is similarly equal to Z_0. By reference to figure 3 it is readily found that

$$Z_0^2 = \frac{L/C}{1-\omega^2/\omega_c^2} \qquad (\omega_c^2 = 4/LC)$$

and hence that Z_0 is resistive for $\omega < \omega_c$ and takes the form $1/\mathrm{i}\omega C_0$ for $\omega > \omega_c$ where $C_0 = \frac{1}{2}C(1-\omega_c^2/\omega^2)^{1/2}$.

When the impurity unit shown in figure 4(*a*) is inserted into an otherwise uniform monatomic array of infinite length it is equivalent, at frequencies $\omega > \omega_c$, to the circuit shown in figure 4(*b*). In

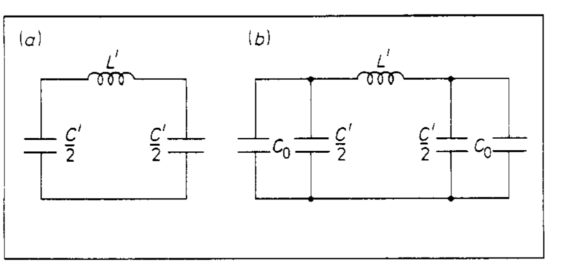


Figure 4 (*a*) An impurity oscillator and (*b*) the equivalent circuit for local mode oscillation.

practice a small number of units on either side of the impurity effectively constitutes an infinite length since a disturbance due to oscillation of the impurity above the cut-off frequency is rapidly attenuated along the array. If ω_0 denotes the resonant angular frequency, $2/(L'C')^{1/2}$, of the impurity unit as an isolated oscillator, the local mode angular frequency ω_L is given by

$$\omega_L^2 = 2/L'(C_0 + \tfrac{1}{2}C')$$

with C_0 evaluated at $\omega = \omega_L$. Hence

$$\omega_L^2 = \omega_0^2 \left[1 + \frac{C}{C'}\left(1 - \frac{\omega_c^2}{\omega_L^2}\right)^{1/2}\right]^{-1}$$

or, if $C' = C$, representing equal force constants for the two types of interatomic bond,

$$\omega_L^2 = \omega_0^4/(2\omega_0^2 - \omega_c^2). \tag{4}$$

3 Construction of electrical arrays

We have used the method of construction indicated in figure 5, each inductor consisting of a close-wound single layer coil of about 50 turns of 26 SWG enamelled wire on an insulating rod about 12 mm in diameter. It is convenient to provide each coil with its own pair of capacitors ($\frac{1}{2}C$ each) so that individual units or any section of the array can be easily isolated. With $\frac{1}{2}C \sim 100$ pF the resonant frequency of one isolated unit is about 5 MHz ($2/\pi(LC)^{1/2}$).

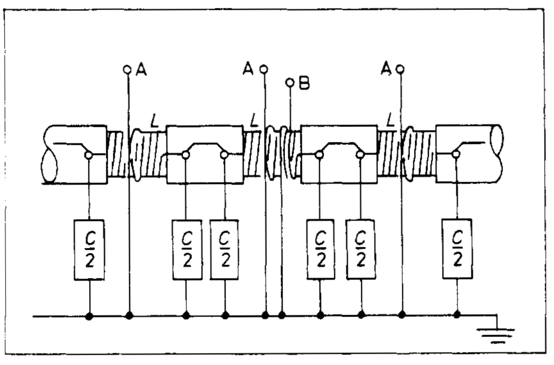


Figure 5 Construction of an electrical array. A Pick-up loop outputs. B Inputs from signal generator and reference waveform.

Each coil has a single turn of wire wound over it to act as a loosely coupled pick-up for measuring the currents I_n and one of the coils has an additional secondary winding of two or three turns for coupling the system to a signal generator. One beam of a double beam oscilloscope is used to display the signal applied to this input winding and the other to display, in turn, the voltages developed in the pick-up loops. The normal modes are then readily found as sharply increased oscilloscope amplitudes and comparison with the input waveform allows the phases as well as the relative peak-to-peak amplitudes of the currents I_n to be determined. A considerable advantage of using the closed ring, as compared with the fixed-end, configuration is that it is immaterial which unit is coupled to the signal generator. With a diatomic system it is in fact not possible to maintain accurate fixed-end conditions over all the modes without adjusting the terminating capacitors or springs.

4 Results

The nature of results obtainable is now illustrated with reference to normal mode measurements on a diatomic array and to measurements on local modes in a monatomic array.

In the diatomic case each of six unit cells consisted of a pair of coils with a 2 : 1 ratio of close-wound turns, each coil being provided with a pair of capacitors of value $\frac{1}{2}C = 82$ pF. Oscilloscope traces clearly showed the expected equality or opposition of the phases of the currents I_n, and measurements of peak-to-peak amplitudes with due account of the phase are shown in figure 6. The difference between acoustic and optical modes is clearly indicated. The characteristic angular frequencies ω_1 and ω_2 (see equation (2)) were measured by isolating one of each type of inductor, connecting a capacitor C (164 pF) across it and determining the resonance frequencies, with the results

$$\omega_1/2\pi = 2.10_5\ \text{MHz}, \qquad \omega_2/2\pi = 1.26_5\ \text{MHz}$$

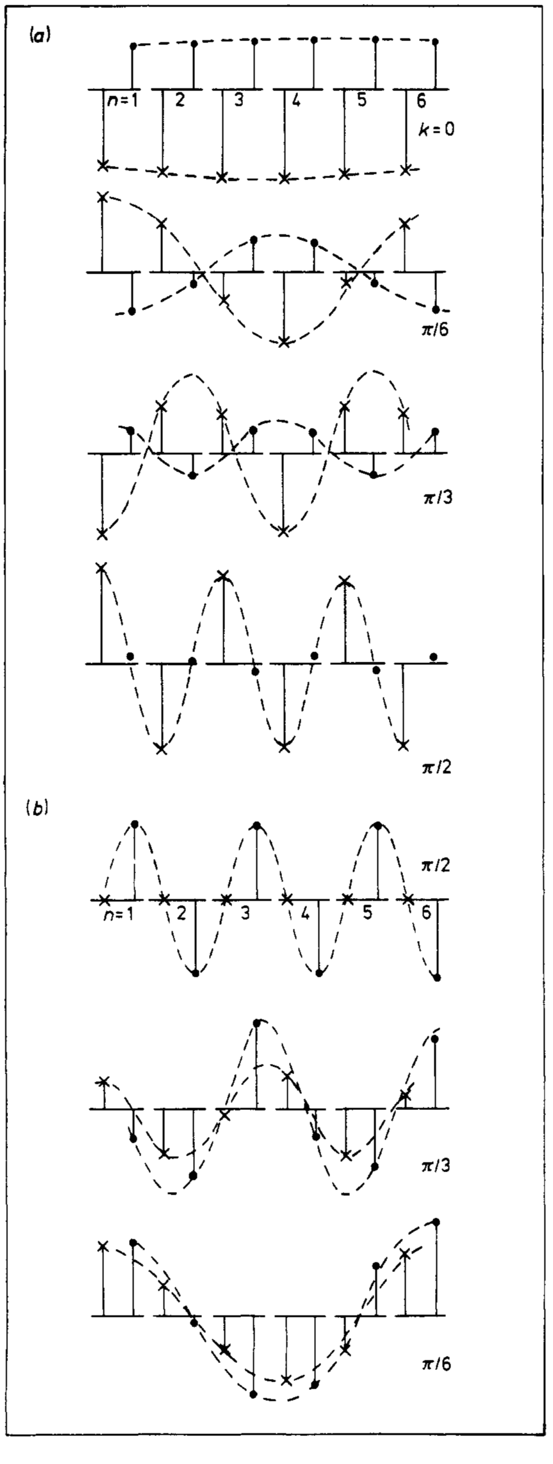
corresponding to $L_2/L_1 = 2.8$.

To evaluate the relative sensitivities of the pick-up loops on the two types of inductor a new unit oscillator was temporarily constructed using one of each coil in series, with the capacitors $\frac{1}{2}C$ connected from each extreme end to earth. When resonance of this unit was excited the same current passed through the inductors and measurement of the pick-up voltages gave the required ratio. The larger coils (L_2) gave an amplitude 1.4 times that given by the smaller coils.

Numerical results of measurements and calculations based on equations (2) and (3) are summarised in table 1 and the dispersion curve obtained from them is shown in figure 7.

In the local mode experiments several unit oscillators with resonance frequencies above the cut-off value were constructed and inserted in turn into the centre of a monatomic array consisting of twelve units. The coupling capacitors of the impurity units were equal to those of the monatomic array. Measurements of the resonance frequencies were made by coupling the impurity unit to the signal generator and also to the oscilloscope as already described and the results expressed as ratios ω_0/ω_c

and ω_L/ω_c, referring to the isolated units and the associated local modes respectively. Figure 8 gives a comparison between these measurements and calculations from equation (4).



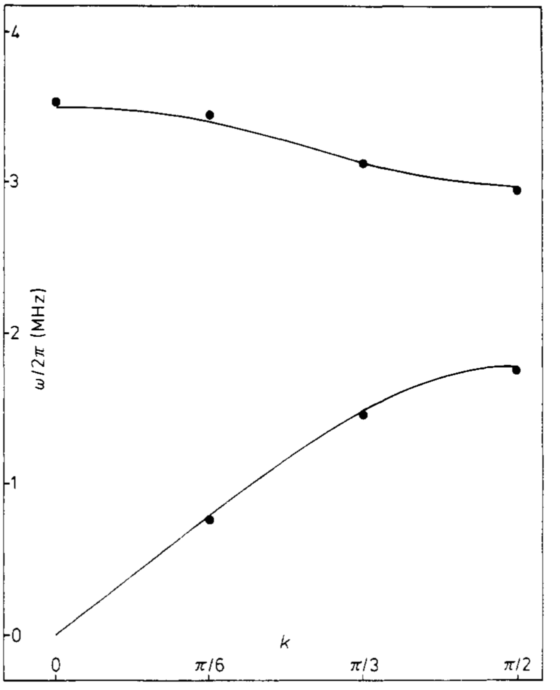


Figure 7 Dispersion curves for the diatomic array. The curves are calculated from equation (2) and the points represent measurements.

Table 1 Measured (M) and calculated (C) resonance frequencies $\omega/2\pi$ and amplitude ratios A/B.

		$\omega/2\pi$ (MHz)		A/B	
k		M	C	M†	C
0	Optic	3.53	3.47	−2.5	−2.8
	Acoustic	—	0	—	—
$\pi/6$	Optic	3.43	3.39	−2.7	−3.0
	Acoustic	0.76	0.78	1.0	0.94
$\pi/3$	Optic	3.12	3.14	−4.0	−4.2
	Acoustic	1.45	1.47	0.80	0.65
$\pi/2$	Optic	2.92	2.98	−v. large	$-\infty$
	Acoustic	1.80	1.79	~0	0

† Corrected for the measured sensitivity ratio.

Figure 6 Normal mode standing waves for a diatomic array of six unit cells: (*a*) optic modes; (*b*) acoustic modes. Measured currents in the smaller and larger inductors are indicated by × and ● respectively and the broken curves are sketched approximate sine waves. Note that the $k = \pi/2$ modes can be excited only by coupling the input to a smaller inductor (optic branch) or a larger inductor (acoustic branch).

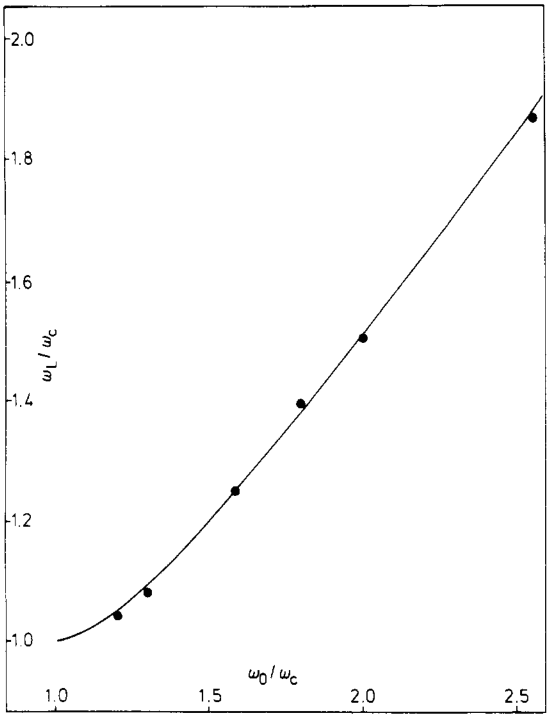


Figure 8 Local mode frequencies. The curve is calculated from equation (4) and the points represent measurements.

Reference

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Nuclear statistics with a 'poor man's detector': a laboratory experiment

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Abstract An undergraduate laboratory experiment on the statistical fluctuations of nuclear reactions is described. The aim is to familiarise students with the use of solid-state nuclear track detectors, using Cf^{252} as a source of spontaneous fission fragments and ordinary microscope slide glass (soda glass) as a detector, used in a geometry $<2\pi$. A simple method for exposure of the detector is given. The results indicate that the observed distribution of tracks, counted using a graticule fitted in the eyepiece of the microscope, is in good agreement with the expected Poisson distribution. However, for larger average values of the counts the Poisson and Gaussian distributions tend to coalesce.

Résumé Cet article décrit une expérience de Travaux Pratiques, (niveau Maîtrise), portant sur les fluctuations statistiques des réactions nucléaires, et visant à familiariser les étudiants avec l'emploi de détecteurs de traces nucléaires en matériaux solides. On utilise du Cf^{252} comme source de fragments de fission spontanée, et une lamelle de microscope ordinaire, (en verre sodique), comme détecteur, (avec un angle solide de détection $<2\pi$). On décrit également une méthode simple permettant l'exposition au rayonnement du détecteur. Les résultats indiquent que la distribution de traces qui est observée au microscope, par dénombrement à travers un réticule adapté à l'oculaire, s'accorde bien à la distribution de Poisson attendue. Cependant, quand le nombre des traces augmente, la distribution de Poisson tend à se confondre avec une distribution gaussienne.

1 Introduction

The subject of statistics is often fascinating to students as well as teachers. This is not simply because

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