Artificial Intelligence

Fuzzy Logic

Lesson 3: Fuzzy Control

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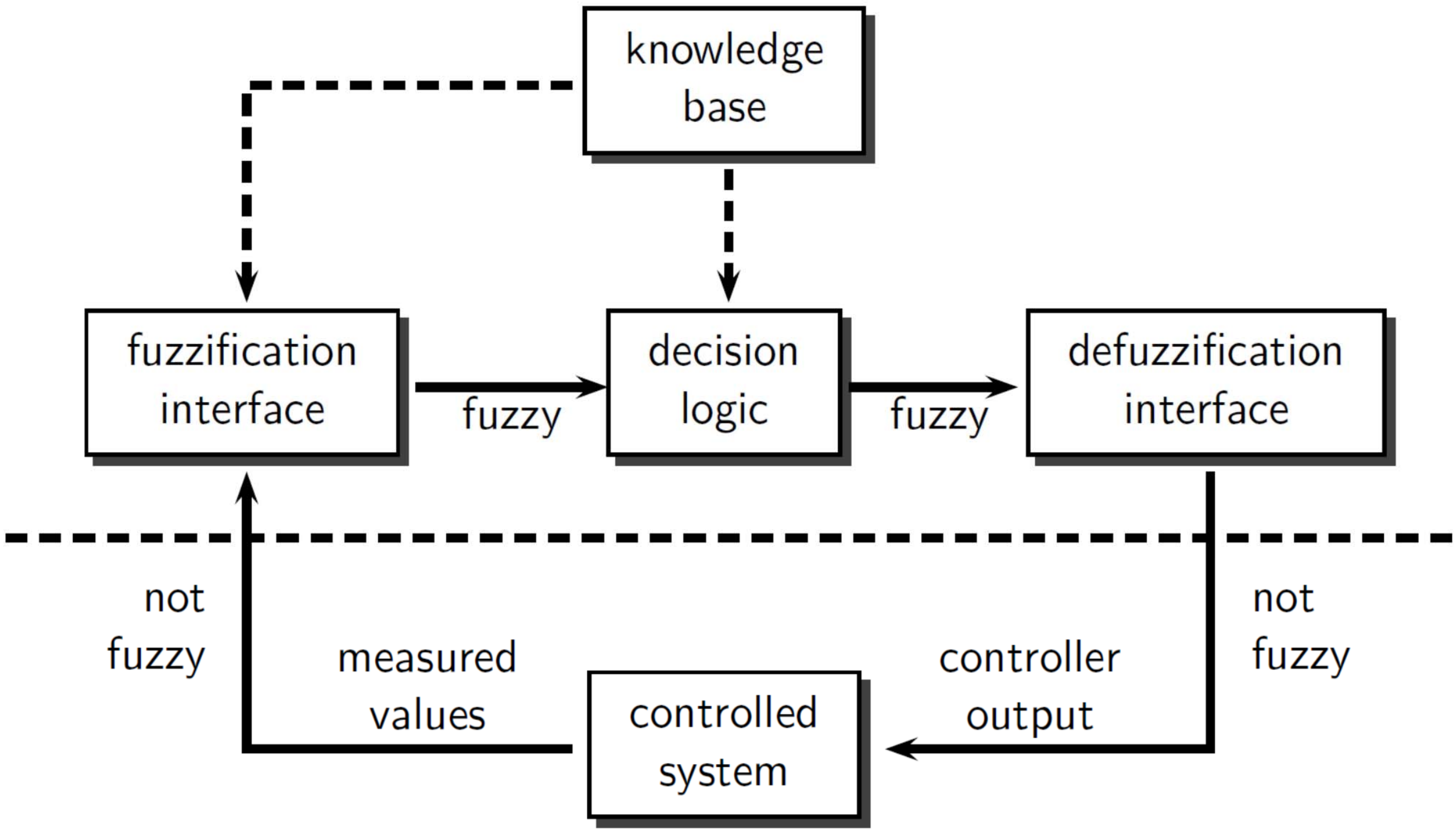
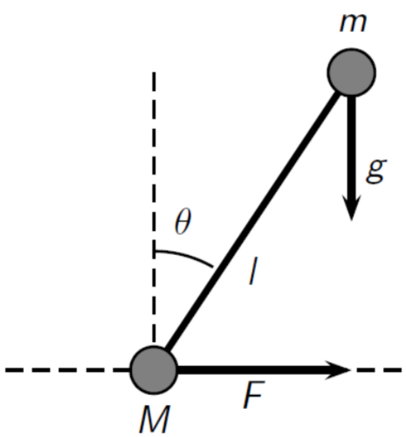
Architecture of a Fuzzy Controller



Mamdani Control

Example: Cartpole Problem (1)



- Balance an upright standing pole by moving its foot
- Lower end of pole can be moved unrestrained along horizontal axis
- Mass m at foot and mass M at head
- Influence of mass of shaft itself is negligible
- Determine force F (control variable) that is necessary to balance pole standing upright
- That is measurement of following output variables
  - angle θ of pole in relation to vertical axis,
  - change of angle, *i.e.* triangular velocity $\dot{\theta} = \frac{d\theta}{dt}$
- Both should converge to zero

Example: Cartpole Problem (2)

- Angle $\theta \in X_1 = [-90°,\ 90°]$
- Theoretically, every angle velocity $\dot{\theta}$ possible
- Extreme $\dot{\theta}$ are artificially achievable
- Assume $-45\,°/s \leq \dot{\theta} \leq 45\,°/s$ holds, i.e.

$$\dot{\theta} \in X_2 = [-45\,°/s, 45\,°/s]$$

- Absolute value of force $|F| \leq 10\mathrm{N}$
- Thus define $F \in Y = [-10\mathrm{N}, 10\mathrm{N}]$

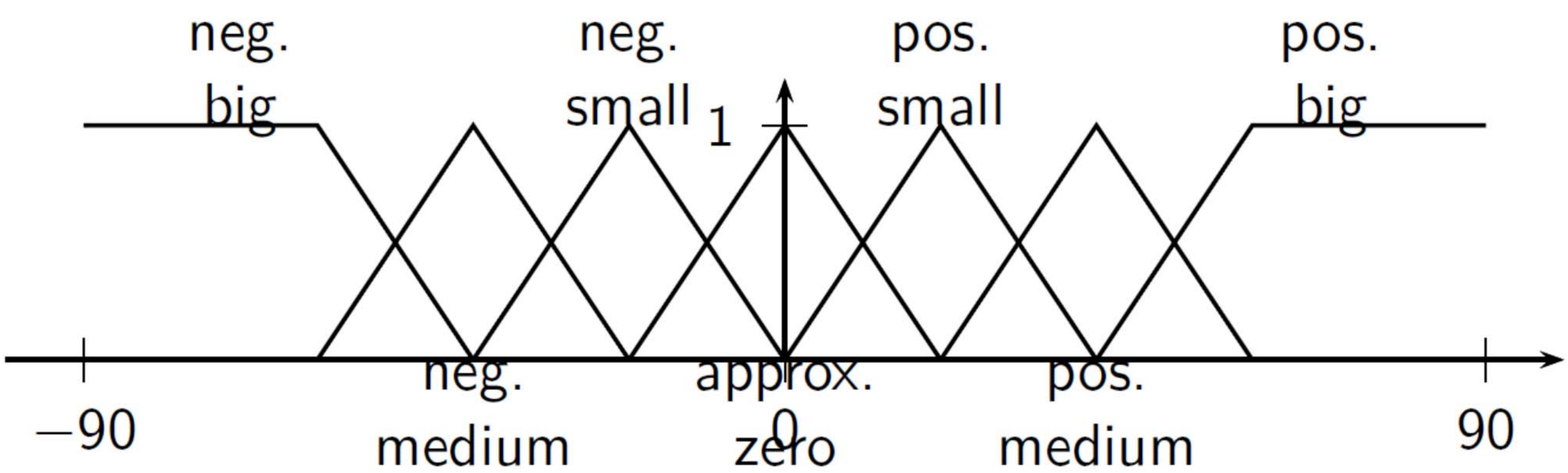
Example: Cartpole Problem (3)

- X_1 partitioned into 7 fuzzy sets
  - Support of fuzzy sets: intervals with length $1/4$ of whole range X_1
  - Similar fuzzy partitions for X_2 and Y

- Specify rules
  - if ξ_1 is $A^{(1)}$ and . . . and ξ_n is $A^{(n)}$ then η is B,

    $A^{(1)}, \ldots, A^{(n)}$ and B represent linguistic terms corresponding to $\mu^{(1)}, \ldots, \mu^{(n)}$ and μ according to $X_1, \ldots, X_n$ and Y
  - Rule base consists of k rules

Example: Cartpole Problem (4)



Example: Cartpole Problem (5)

$\dot{\theta}$ \ θ	nb	nm	ns	az	ps	pm	pb
nb			ps	pb			
nm				pm			
ns	nm		ns	ps			
az	nb	nm	ns	az	ps	pm	pb
ps				ns	ps		pm
pm				nm			
pb				nb	ns		

- 19 rules for cartpole problem, often not necessary to determine all table entries e.g.

  If θ is *approximately zero* and $\dot{\theta}$ is *negative medium* then F is *positive medium*

Definition of Table-based Control Function (1)

- Measurement $(x_1, \ldots, x_n) \in X_1 \times \ldots \times X_n$ is forwarded to decision logic
- Consider rule

  if ξ_1 is $A^{(1)}$ and . . . and ξ_n is $A^{(n)}$ then η is B
- Decision logic computes degree to $\xi_1, \ldots, \xi_n$ fulfills premise of rule
- For $1 \leq \nu \leq n$, the value $\mu^{(\nu)}(x_\nu)$ is calculated
- Combine values conjunctively by

$$\alpha = \min\{\mu^{(1)}, \ldots, \quad \mu^{(n)}\}$$

- For each rule R_r with $1 \leq r \leq n$, compute

$$\alpha_r = \min\left\{\mu_{i_{1,r}}^{(1)}(x_1), \ldots, \mu_{i_{n,r}}^{(n)}(x_n)\right\}$$

Definition of Table-based Control Function (2)

- Output of $R_r =$ fuzzy set of output values
- Thus "cutting off" fuzzy set μ_{i_r} associated with conclusion of R_r at α_r
- For input $(x_1, \ldots, x_n)$, R_r implies fuzzy set

$$\mu_{x_1,\ldots,x_n}^{\text{ouptut}(R_r)}: Y \ \rightarrow \ [0,1]$$

$$y \mapsto \min\left\{\mu_{i_{1,r}}^{(1)}(x_1), \ldots, \mu_{i_{n,r}}^{(n)}(x_n), \mu_{i_r}(y)\right\}$$

- If $\mu_{i_{1,r}}^{(1)}(x_1) = \cdots = \mu_{i_{n,r}}^{(n)}(x_n) = 1$ then $\mu_{x_1,\ldots,x_n}^{\text{ouptut}(R_r)} = \mu_{i_r}$
- If for all $\nu \in \{1, \ldots, n\}$ $\quad \mu_{i_{\nu,r}}^{(\nu)}(x_\nu) = 0$, then $\mu_{x_1,\ldots,x_n}^{\text{ouptut}(R_r)} = 0$

Combination of Rules

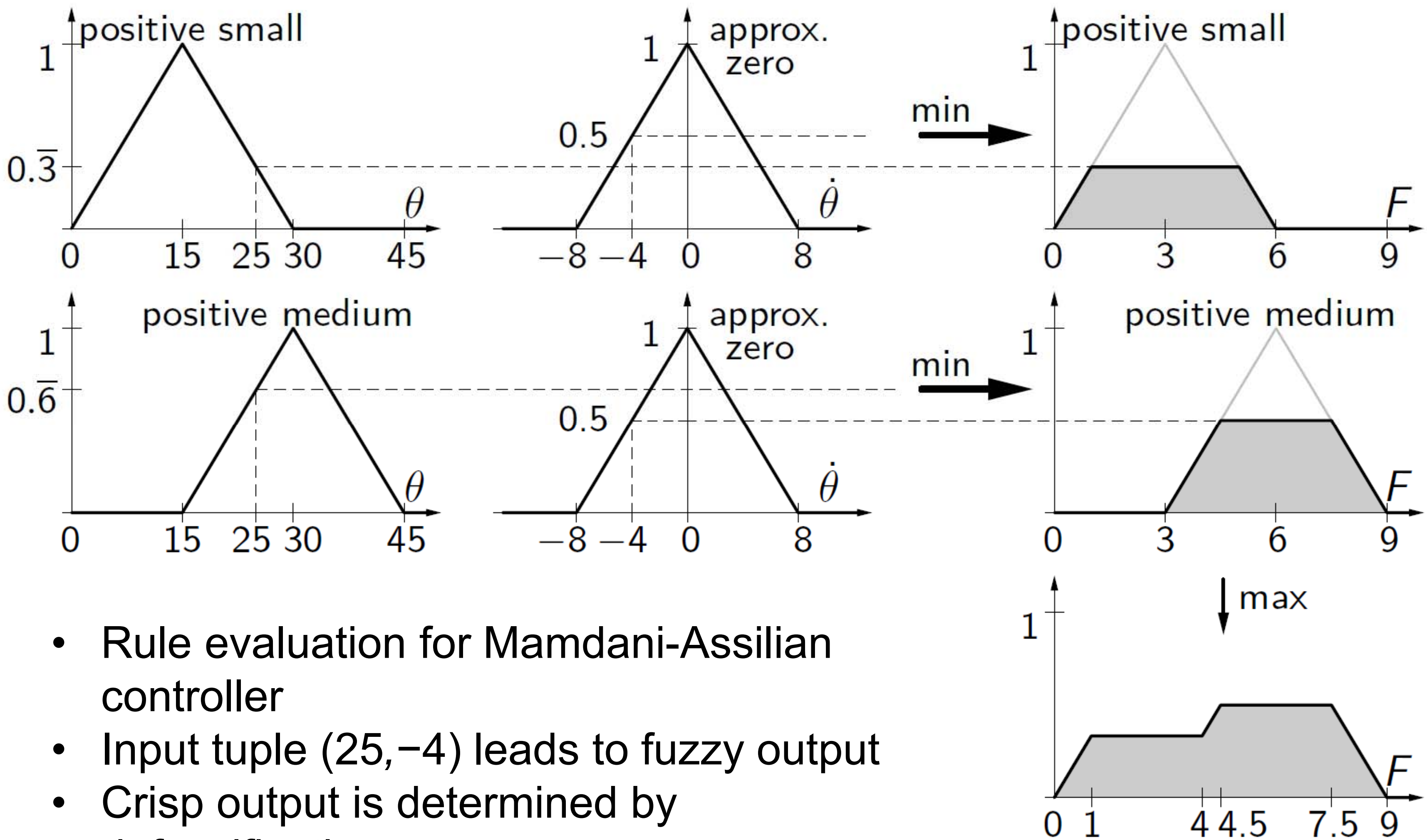
- The decision logic combines the fuzzy sets from all rules
- The **maximum** leads to the output fuzzy set

$$\mu_{x_1,\ldots,x_n}^{\text{ouptut}(R_r)}\colon Y \to [0,1]$$

$$y \mapsto \min\left\{\mu_{i_{1,r}}^{(1)}(x_1),\ldots,\mu_{i_{n,r}}^{(n)}(x_n),\mu_{i_r}(y)\right\}$$

- Then $\mu_{x_1,\ldots,x_n}^{\text{ouptut}}$ is passed to defuzzification interface

Rule Evaluation



- Rule evaluation for Mamdani-Assilian controller
- Input tuple (25,−4) leads to fuzzy output
- Crisp output is determined by defuzzification

Defuzzification

- Mapping between each $(n_1, \ldots, n_n)$ and $\mu^{\text{ouptut}}_{x_1,\ldots,x_n}$
- Output = description of output value as fuzzy set
- Defuzzification interface derives crisp value from $\mu^{\text{ouptut}}_{x_1,\ldots,x_n}$
- This step is called defuzzification
- Most common methods:
  - max criterion
  - mean of maxima
  - center of gravity

The Max Criterion Method

- Choose an arbitrary $y \in Y$ for which $\mu_{x_1,\ldots,x_n}^{\text{ouptut}}$ reaches the maximum membership value
- Advantages:
  - Applicable for arbitrary fuzzy sets
  - Applicable for arbitrary domain Y(even for $Y \neq \mathbb{R}$)
- Disadvantages:
  - Rather class of defuzzification strategies than single method
  - Which value of maximum membership?
  - Random values and thus non-deterministic controller
  - Leads to discontinuous control actions

The Mean of Maxima (MOM) Method

- Preconditions:
  1. Y is interval
  2. $Y_{\text{Max}} = \{y \in Y \mid \forall y' \in Y : \mu^{\text{ouptut}}_{x_1,\ldots,x_n}(y') \leq \mu^{\text{ouptut}}_{x_1,\ldots,x_n}(y)\}$ is non-empty and measurable
  3. Y_{Max} is set of all $y \in Y$ such that $\mu^{\text{ouptut}}_{x_1,\ldots,x_n}$ is maximal
- Crisp output value = mean value of Y_{Max}

if Y_{Max} is finite

$$\eta = \frac{1}{|Y_{\text{Max}}|} \sum_{y_i \in Y_{\text{Max}}} y_i$$

if Y_{Max} is infinite

$$\eta = \frac{\int_{y \in Y_{\text{Max}}} y \, dy}{\int_{y \in Y_{\text{Max}}} dy}$$

- MOM can lead to discontinuous control actions

Center of Gravity (COG) Method (1)

- Same preconditions as MOM method
  - η = center of gravity, i.e., center of area of $\mu_{x_1,\ldots,x_n}^{\text{ouptut}}$
  - If Y is finite, then

$$\eta = \frac{\sum_{y_i \in Y} y_i \cdot \mu_{x_1,\ldots,x_n}^{\text{ouptut}}(y_i)}{\sum_{y_i \in Y} \mu_{x_1,\ldots,x_n}^{\text{ouptut}}\ (y_i)}$$

  - If Y is infinite, then

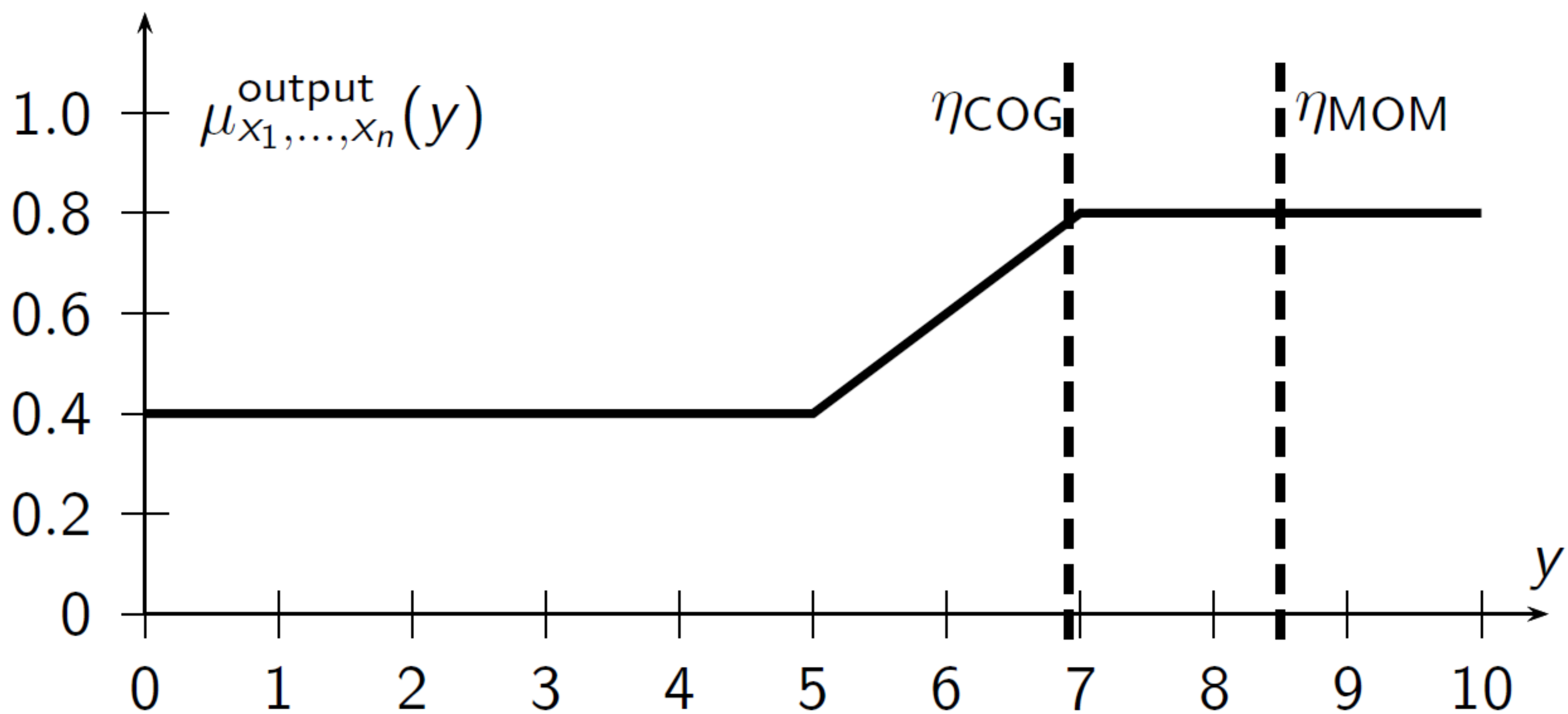
$$\eta = \frac{\int_{y_i \in Y} y \cdot \mu_{x_1,\ldots,x_n}^{\text{ouptut}}\, dy}{\int_{y_i \in Y} \mu_{x_1,\ldots,x_n}^{\text{ouptut}}\, dy}$$

Center of Gravity (COG) Method (2)

- Advantages
  - Nearly always smooth behavior
  - If certain rule dominates once, not necessarily dominating again
- Disadvantage:
  - No semantic justification
  - Long computation
  - Counterintuitive results possible
- Also called center of area (COA) method
- Take value that splits $\mu_{x_1,\ldots,x_n}^{\text{ouptut}}$ into 2 equal parts

Example

- Task: compute η_{COG} and η_{MOM} of fuzzy set shown below.
- Based on finite set $Y = 0, 1, \ldots, 10$ and infinite set $Y = [0, 10]$



Example for COG Continuous and Discrete Output Space

$$\eta_{\text{COG}} = \frac{\int_0^{10} y \cdot \mu^{\text{output}}_{x_1,\ldots,x_n}(y)\, dy}{\int_0^{10} \mu^{\text{output}}_{x_1,\ldots,x_n}(y)\, dy}$$

$$= \frac{\int_0^5 0.4y\, dy + \int_5^7 (0.2y - 0.6)y\, dy + \int_7^{10} 0.8y\, dy}{5 \cdot 0.4 + 2 \cdot \frac{0.8+0.4}{2} + 3 \cdot 0.8}$$

$$\approx \frac{38.7333}{5.6} \approx 6.917$$

$$\eta_{\text{COG}} = \frac{0.4 \cdot (0+1+2+3+4+5) + 0.6 \cdot 6 + 0.8 \cdot (7+8+9+10)}{0.4 \cdot 6 + 0.6 \cdot 1 + 0.8 \cdot 4}$$

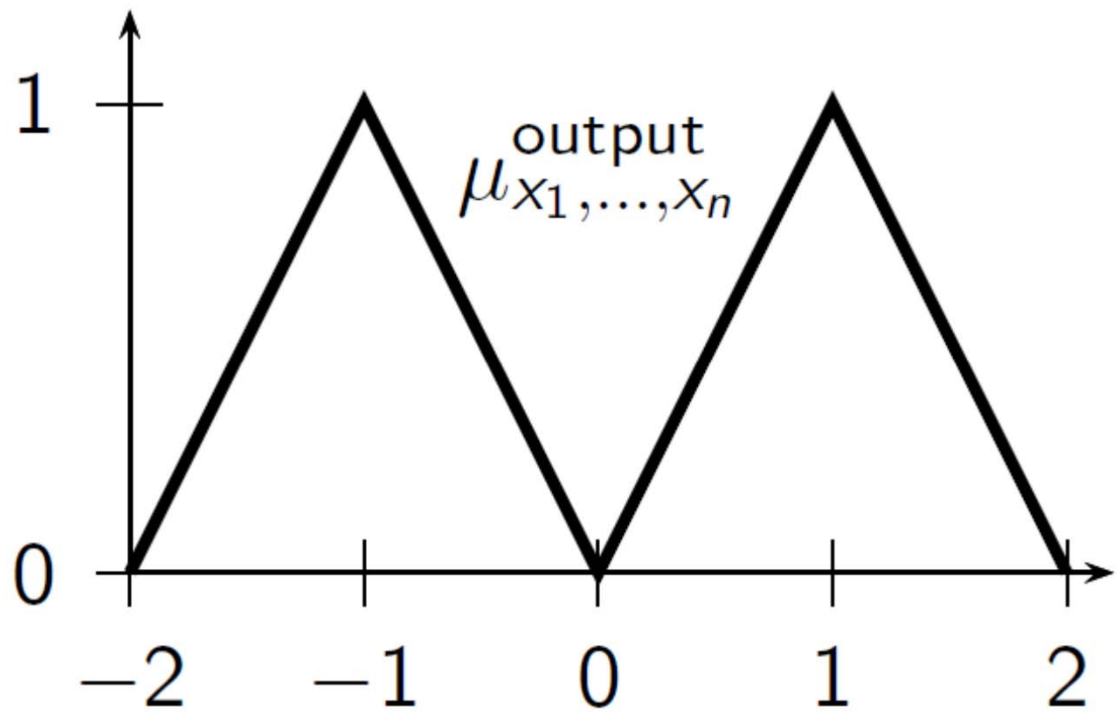
$$= \frac{36.8}{6.2} \approx 5.935$$

Example for MOM
Continuous and Discrete Output Space

$$
\begin{aligned}
\eta_{\text{MOM}} &= \frac{\int_7^{10} y \, dy}{\int_7^{10} \, dy} \\
&= \frac{50 - 24.5}{10 - 7} = \frac{25.5}{3} \\
&= 8.5
\end{aligned}
$$

$$
\begin{aligned}
\eta_{\text{MOM}} &= \frac{7 + 8 + 9 + 10}{4} \\
&= \frac{34}{4} \\
&= 8.5
\end{aligned}
$$

Problem Case for MOM and COG



- What would be the output of MOM or COG?
- Is this desirable or not?

Takagi-Sugeno Control

Takagi-Sugeno Control (1)

- Modification and extension of Mamdani controller

- Both in common: fuzzy partitions of input domain $X_1, \ldots, X_n$

- Difference to Mamdani controller
  - no fuzzy partition of output domain Y
  - controller rules $R_1, \ldots, R_k$ are given by

    R_r : **if** $\xi_1$is $A^{(1)}_{i_{1,r}}$ and . . . and ξ_n is $A^{(n)}_{i_{n,r}}$

    then $\eta_r = f_r(\xi_1, \ldots, \xi_n)$,

    $f_r : X1 \times \ldots \times X_n \to Y$
  - generally, f_r is linear, i.e. $f_r(x_1, \ldots, x_n) = a_0^r + \sum_{i=1}^{n} a_i^r \, x_i$

Takagi-Sugeno Control (2)

- For given input $(x_1, \ldots, x_n)$ and for each R_r, decision logic computes the truth value α_r of each premise, and then $\mathrm{f_r}(x_1, \ldots, x_n)$

- Analogously to Mamdani controller

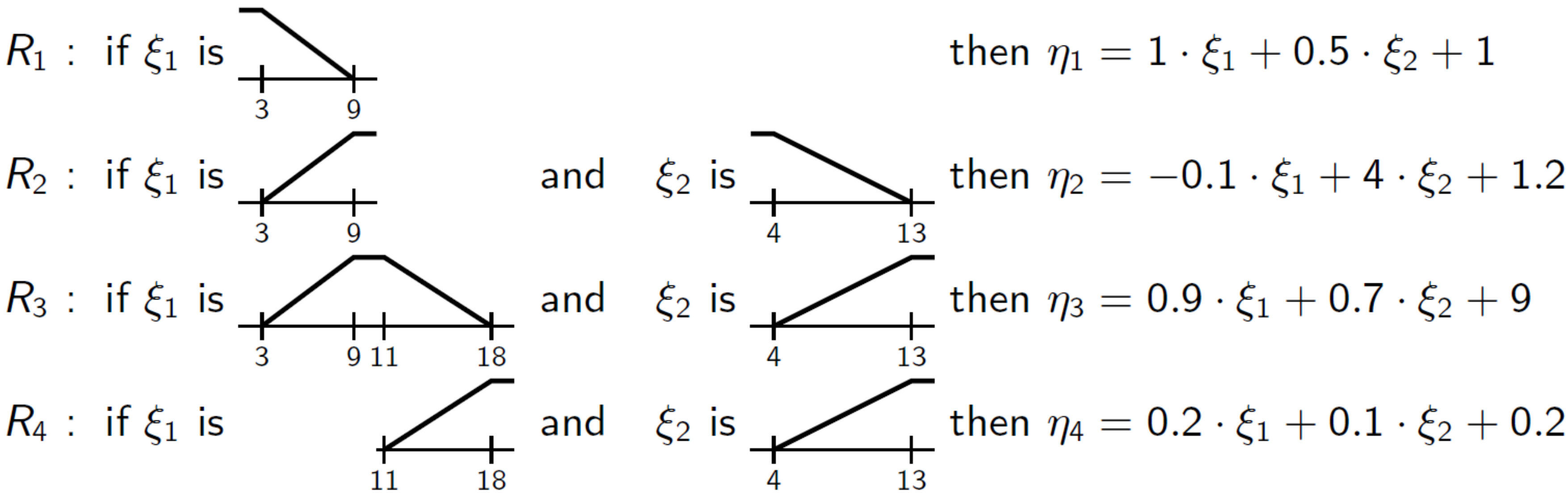
$$\alpha_r = \min\left\{\mu_{i_{1,r}}^{(1)}(x_1), \ldots, \mu_{i_{n,r}}^{(n)}(x_n)\right\}$$

- Output equals crisp control value

$$\eta = \frac{\sum_{r=1}^{k} \alpha_r \cdot f_r(x_1, \ldots, x_n)}{\sum_{r=1}^{k} \alpha_r}$$

Thus no defuzzification method necessary

Example



- If a certain clause "x_j is $A^{(1)}_{i_{1,r}}$" in rule R_r is missing, then $\mu_{i_{j,r}}(x_j) \equiv 1$ for all linguistic values $i_{j,r}$
- For instance, here x_2 in R_1, so $\mu_{i_{2,1}}(x_2) \equiv 1$ for all $i_{2,1}$

Example: Output Computation

input: $(\xi_1, \xi_2) = (6, 7)$

$$\alpha_1 = {}^1\!/_2 \wedge 1 = {}^1\!/_2 \qquad \eta_1 = 6 + {}^7\!/_2 + 1 = 10.5$$

$$\alpha_2 = {}^1\!/_2 \wedge {}^2\!/_3 = {}^1\!/_2 \qquad \eta_2 = -0.6 + 28 + 1.2 = 28.6$$

$$\alpha_3 = {}^1\!/_2 \wedge {}^1\!/_3 = {}^1\!/_3 \qquad \eta_3 = 0.9 \cdot 6 + 0.7 \cdot 7 + 9 = 19.3$$

$$\alpha_4 = 0 \wedge {}^1\!/_3 = 0 \qquad \eta_4 = 6 + {}^7\!/_2 + 1 = 10.5$$

output: $\eta = f(6, 7) = \dfrac{{}^1\!/_2 \cdot 10.5 + {}^1\!/_2 \cdot 28.6 + {}^1\!/_3 \cdot 19.3}{{}^1\!/_2 + {}^1\!/_2 + {}^1\!/_3} = 19.5$

Fuzzy Control as Similarity-Based Reasoning

Interpolation in the Presence of Fuzziness

- Both Takagi-Sugeno and Mamdani are based on heuristics
- They are used without a concrete interpretation
- Fuzzy control is interpreted as a method to specify a non-linear transition function by knowledge-based interpolation
- A fuzzy controller can be interpreted as fuzzy interpolation
- It uses the concept of *fuzzy equivalence* relations (also called *similarity relations*)

Similarity: Example

- Specification of a partial control mapping ("good control actions")

		gradient						
		-40.0	-6.0	-3.0	0.0	3.0	6.0	40.0
	-70.0	22.5	15.0	15.0	10.0	10.0	5.0	5.0
	-50.0	22.5	15.0	10.0	10.0	5.0	5.0	0.0
	-30.0	15.0	10.0	5.0	5.0	0.0	0.0	0.0
deviation	0.0	5.0	5.0	0.0	0.0	0.0	-10.0	-15.0
	30.0	0.0	0.0	0.0	-5.0	-5.0	-10.0	-10.0
	50.0	0.0	-5.0	-5.0	-10.0	-15.0	-15.0	-22.5
	70.0	-5.0	-5.0	-15.0	-15.0	-15.0	-15.0	-15.0

Interpolation of Control Table

- There might be additional knowledge available
  - Some values are “indistinguishable”, “similar” or “approximately equal”
  - Or they should be treated in a similar way

- Two problems:
  1. How to model information about similarity?
  2. How to interpolate in case of an existing similarity information?

How to Model Similarity? Equivalence Relation?

- Definition
  - Let A be a set and $\approx$ be a binary relation on A
  - $\approx$ is called an equivalence relation if and only if $\forall\, a, b, c \in A$
    1) $a \approx a$ (reflexivity)
    2) $a \approx b \leftrightarrow b \approx a$ (symmetry)
    3) $a \approx b \wedge b \approx c \rightarrow a \approx c$ (transitivity)

  - Let us try $a \approx b \Leftrightarrow |a - b| < \varepsilon$ where ε is fixed
  - $\approx$ is not transitive, $\approx$ is no equivalence relation
  - Poincaré paradox: $a \approx b,\ b \approx c,\ a \not\approx c$
  - This is counterintuitive

How to Model Similarity? Fuzzy Equivalence Relation

- Definition

  A function $E : X^2 \to [0,1]$ is called a fuzzy equivalence relation with respect to the t-norm ⊤ if it satisfies the following conditions

$$\forall x,y,z \in X \quad \begin{array}{l} 1)\ E(x,x) = 1 \text{ (reflexivity)} \\ 2)\ E(x,y) = E(y,x) \text{ (symmetry)} \\ 3)\ \top(E(x,y),E(y,z)) \leq E(x,z) \text{ (t-transitivity)} \end{array}$$

  $E(x,y)$ is the degree to which $x \approx y$ holds

  - E is also called similarity relation, t-equivalence relation, indistinguishability operator, or tolerance relation
  - Note that property 3) corresponds to the vague statement **if** $(x \approx y) \wedge (y \approx z)$ **then** $x \approx z$

Fuzzy Equivalence Relations: Example

- Let δ be a pseudo metric on X

  Furthermore $\top(a, b) = \max\{a + b - 1, 0\}$ Łukasiewicz t-norm

  Then $E_\delta(x, y) = 1 - \min\{\delta(x, y), 1\}$ is a fuzzy equivalence relation

  $\delta(x, y) = 1 - E_\delta(x, y)$ is the induced pseudo metric

  Fuzzy equivalence and distance are dual notions in this case

- Definition

  A function $E : X^2 \rightarrow [0, 1]$ is called a fuzzy equivalence relation if $\forall\, x, y, z \in X$

  1) $E(x, x) = 1$ (reflexivity)

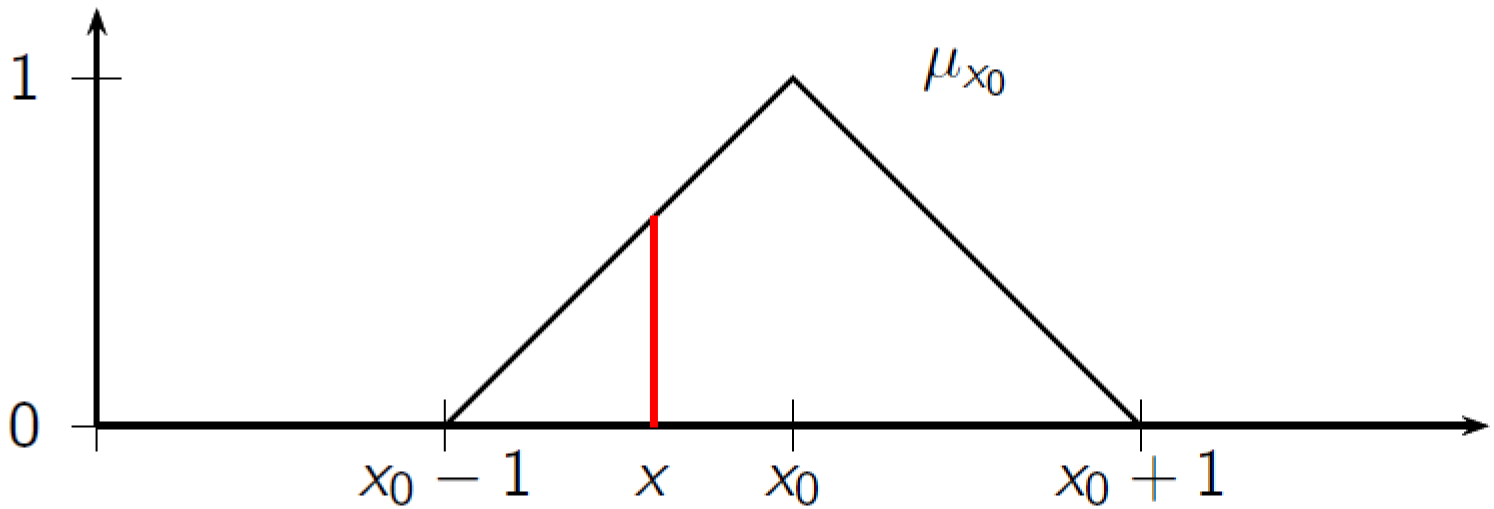
  2) $E(x, y) = E(y, x)$ (symmetry)

  3) $\max\{E(x, y) + E(y, z) - 1, 0\} \leq E(x, z)$ (Łukasiewicz transitivity)

Fuzzy Sets as Derived Concept

$\delta(x, y) = |x - y|$ metric

$E_\delta(x, y) = 1 - \min\{|x - y|, 1\}$ fuzzy equivalence relation



$\mu_{x_0} : X \to [0, 1]$

$x \mapsto E_\delta(x, x_0)$ fuzzy singleton

μ_{x_0} describes “local” similarities

Extensional Hull

- $E : \mathbb{R} \times \mathbb{R} \to [0,1], \qquad (x,y) \longmapsto 1 - \min\{|x - y|, 1\}$ is fuzzy equivalence relation w.r.t. $\top_{\text{Łuka}}$
- Definition

  Let E be a fuzzy equivalence relation on X w.r.t. $\top$

  $\mu \in F(X)$ is extensional if and only if

  $$\forall x,y \in X : \top(\mu(x), E(x,y)) \leq \mu(y)$$
- Definition

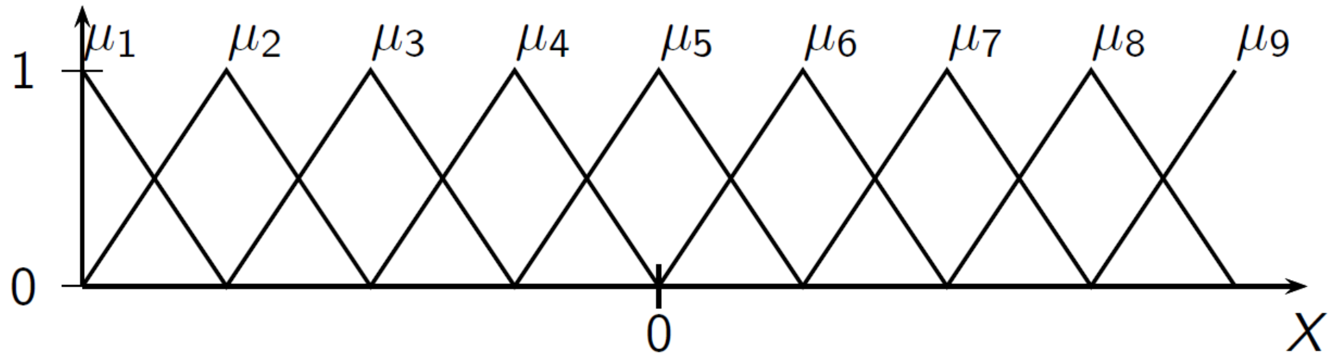
  Let E be a fuzzy equivalence relation on a set X

  The extensional hull of a set $M \subseteq X$ is the fuzzy set

  $$\mu_M : X \to [0,1], \qquad x \longmapsto \sup\{E(x,y) \mid y \in M\}$$
- The extensional hull of $\{x_0\}$ is called a singleton

Specification of Fuzzy Equivalence Relation

- Given a family of fuzzy sets that describes "local" similarities



  - There exists a fuzzy equivalence relation on X with induced singletons μ_i if and only if

$$\forall i\,, j : \sup_{x \in X} \left\{\mu_i\,(x) + \mu_j\,(x) - 1\right\} \leq \inf_{y \in X} \{1 - |\mu_i\,(y) - \mu_j\,(y)|\}$$

  - If $\mu_i\,(x) + \mu_j\,(x) \leq 1$ for $i \neq j$, then there is a fuzzy equivalence relation E on X

$$E(x, y) = \inf_{i \in I} \{1 - |\mu_i\,(x) - \mu_i\,(y)|\}$$

Necessity of Scaling (1)

- Are there other fuzzy equivalence relations on $\mathbb{R}$ than $E(x, y) = 1 - \min\{|x - y|, 1\}$?

- A fuzzy equivalence relation depends on the measurement unit, e.g.
  - Celsius: E(20 °C, 20.5 °C) = 0.5,
  - Fahrenheit: E(68 F, 68.9 F) = 0.9,
  - scaling factor for Celsius/Fahrenheit = 1.8 (F = 9/5C + 32)
- $E(x, y) = 1 - \min\{|c \cdot x - c \cdot y|, 1\}$ is a fuzzy equivalence relation

Necessity of Scaling (2)

- How to generalize scaling concept?
  - $X = [a, b]$
  - Scaling

$$c : X \to [0, \infty)$$

  - Transformation

$$f : X \to [0, \infty), x \longmapsto \int_a^x c(t)dt$$

  - Fuzzy equivalence relation

$$E : X \times Y \to [0, 1],\ (x, y) \longmapsto 1 - \min\{|f(x) - f(y)|, 1\}$$

Fuzzy Equivalence Relations: Fuzzy Control

- The imprecision of measurements is modeled by fuzzy equivalence relations $E_1, \ldots, E_n$ and F on $X_1, \ldots, X_n$ and Y, respectively

- The information provided by control expert are
  - k input-output tuples $(x_1^{(r)}, \ldots, x_n^{(r)}, \mathrm{y}^{(r)})$
  - the description of the fuzzy equivalence relations for input and output spaces, respectively

- The goal is to derive a control function $\varphi\colon X_1 \times \ldots \times X_n \rightarrow Y$ from this information

Determine Fuzzy-valued Control Functions (1)

- The extensional hull of graph of φ must be determined

- Then the equivalence relation on $X_1 \times \ldots \times X_n \times\ Y$ is

$$E((x_1, \ldots, x_n, y), (x'_1, \ldots, x'_n\ , y'))$$
$$=\ \min\{E1(x1, x'1), \ldots, E_n(x_n, x'_n), F(y, y')\}$$

Determine Fuzzy-valued Control Functions (2)

- For X_i and Y, define the sets

$$X_i^{(0)} = \{x \in X_i \mid \exists\, r \in \{1,\ldots,k\} : x = x_i^r\}$$

and

$$Y^{(0)} = \left\{y \in Y \mid \exists\, r \in \{1,\ldots,k\} : y = y^{(r)}\right\}$$

- $X_i^{(0)}$ and $Y^{(0)}$ contain all values of the r input-output tuples $(x_i^r,\ldots,x_n^r, y^{(r)})$
- For each $x_0 \in X_i^{(0)}$, singleton μ_{x_0} is obtained by

$$\mu_{x_0}(x) = E_i\,(x, x_0)$$

Determine Fuzzy-valued Control Functions (3)

- If φ is only partly given, then use $E_1, \ldots, E_n, F$ to fill the gaps of φ_0
- The extensional hull of φ_0 is a fuzzy set

$$\mu_0'(x_1', \ldots, x_n', y') = \max_{r \in \{1,\ldots,k\}} \left\{ min\{E_1(x_1^{(r)}, x_1'), \ldots, E_n(x_n^{(r)}, x_n'), F(\mathrm{y}^{(r)}, y')\} \right\}$$

- μ_0' is the smallest fuzzy set containing the graph of φ_0
- Obviously, $\mu_{\varphi_0} \leq \mu_{\varphi}$

$$\mu_{\varphi_0}^{(x_1, \ldots, x_n)} : Y \rightarrow [0, 1],$$

$$y \longmapsto \mu_{\varphi_0}(x_1, \ldots, x_n, y)$$

Reinterpretation of Mamdani Controller (1)

- For input $(x_1,...,x_n)$, the projection of the extensional hull of graph of φ_0 leads to a fuzzy set as output
- This is identical to the Mamdani controller output
- It identifies the input-output tuples of φ_0 by linguistic rules

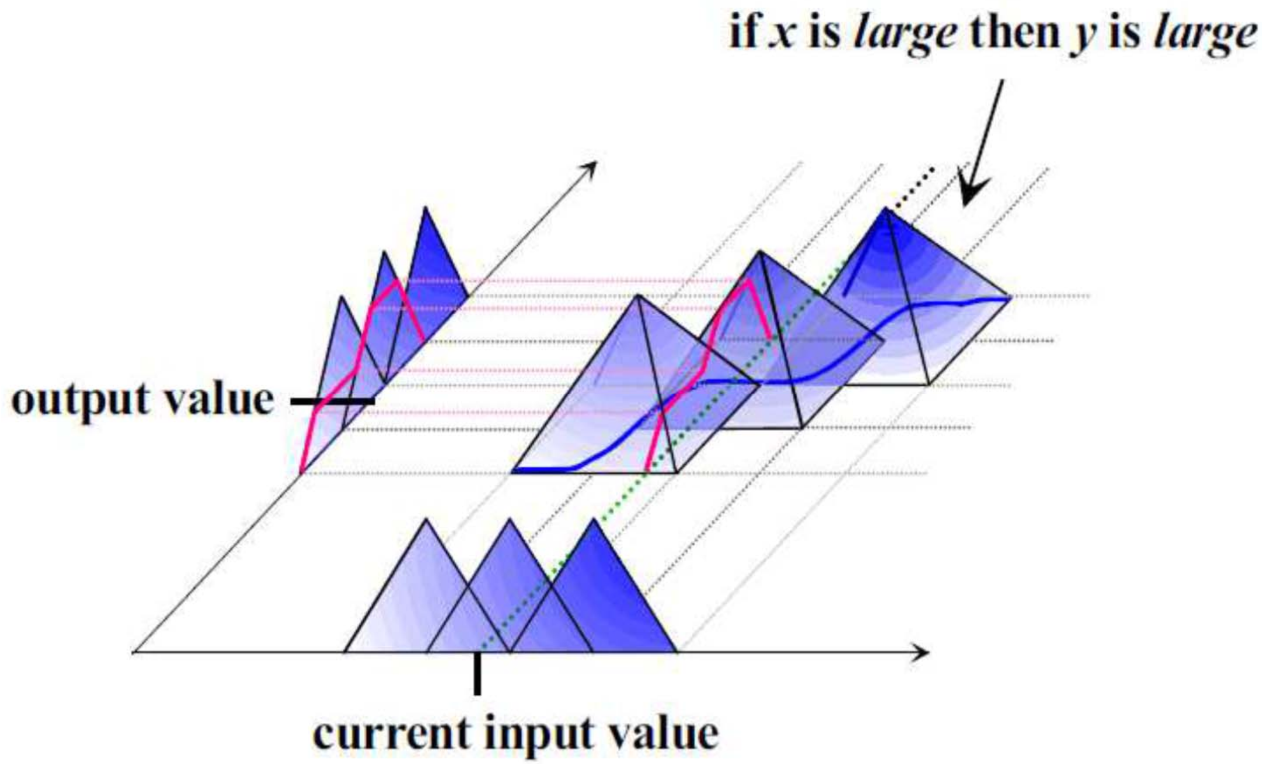
  R_r : **if** $\mathcal{X}_1$ **is** approximately $x_1^{(r)}$

  and. . .

  and $\mathcal{X}_n$ **is** approximately $x_n^{(r)}$

  then $\mathcal{Y}$ **is** $y^{(r)}$
- A fuzzy controller based on equivalence relations behaves like a Mamdani controller

Reinterpretation of Mamdani Controller (2)



- 3 fuzzy rules (specified by 3 input-output tuples)
- The extensional hull is the maximum of all fuzzy rules