Artificial Intelligence

Fuzzy Logic

Lesson 1: Introduction to Fuzzy Logic

Vincenzo Piuri

Università degli Studi di Milano

Contents

- Motivation
- Fuzzy sets
- Representation of Fuzzy Sets
- Multi-valued Logics
- From Logic to Fuzzy Logic
- Fuzzy Set Theory
- Fuzzy Set Complement
- Fuzzy Sets Inclusion

Motivation

Motivation

- Humans use imprecise linguistic terms

  e.g., high, fat, big, small, fat, etc.

- All the complex human reasonings are based on that terms

  e.g., driving, taking financial decisions, givinig a lecture, etc.

- Classical mathematics cannot manage these terms

- How to use these terms with computers?

  FUZZY LOGIC

Imprecision

- Any notion is said to be **imprecise** when its meaning is not fixed
  - Can be in the following conditions: fully/to certain degree/not at all.
  - Gradualness (“membership gradience”) is also called *fuzziness*.

- A **proposition** is imprecise if it contains gradual predicates
  - Such propositions may be true to a certain degree, i.e. partial truth
  - E.g., in natural language

    very, rather, almost not, etc.

Imprecision Example: the Sorites Paradox

- Paradox
  - If a sand dune is small, adding one grain of sand to it leaves it small
  - A sand dune with a single grain is small
  - Hence all sand dunes are small
- Paradox comes from treatment of *small*
- Question
  - How many grains of sand has a sand dune at least?
- Formulation
  - Statement $A(n)$: "n grains of sand are a sand dune"
  - Let $d_n = T(A(n))$ denote "degree of acceptance" for $A(n)$
  - Then $0 = d_0 \leq d_1 \leq . . . \leq d_n \leq . . . \leq 1$

    can be seen as truth values of a many valued logic

Uncertainty

- Uncertainty refers to situations involving imperfect or unknown information

- Consider the notion bald
  - A man without hair on his head is bald
  - A hairy man is not bald
- Usually a man is not completely bald
- Where to set baldness/non baldness threshold?

- **Fuzzy set theory does not assume any threshold**

Difference Between Imprecision and Uncertainty

- Imprecision:
  - e.g. “Today the weather is fine.”
  - Imprecisely defined concepts neglect of details computing with words

- Uncertainty:
  - e.g. “How will the exchange rate of the dollar be tomorrow?”
  - probability, possibility

Examples of Imprecision and Uncertainty

- Uncertainty differs from imprecision
  - “This car is rather old” (imprecision)

    Lack of ability to measure or to evaluate numerical features
  - “This car was probably made in Germany” (uncertainty)

    Uncertainty about well-defined proposition made in Germany, perhaps based on statistics (random experiment)
  - “The car I chose randomly is perhaps very big” (uncertainty and imprecision)

    Lack of precise definition of notion big

    Modifier very indicates rough degree of “bigness”

Principle of Incompatibility

- As the complexity of a system increases, our ability to make precise and yet significant statements about its behavior diminishes until a threshold is reached beyond which precision and significance (or relevance) become almost mutually exclusive characteristics

- Fuzzy sets and fuzzy logic are used as mechanism for abstraction of unnecessary or too complex details

Applications of Fuzzy Systems

- Fields
  - Control Engineering
  - Approximate Reasoning
  - Data Analysis
  - Image Analysis

- Advantages
  - Use of imprecise or uncertain information
  - Use of expert knowledge
  - Robust nonlinear control
  - Time to market
  - Marketing aspects

Fuzzy sets

Fuzzy Sets

- A fuzzy set is a set of elements with a continuum of membership grades

- A **fuzzy set** μ of a set X (the universe) is a mapping

$$\mu : X \mapsto [0, 1]$$

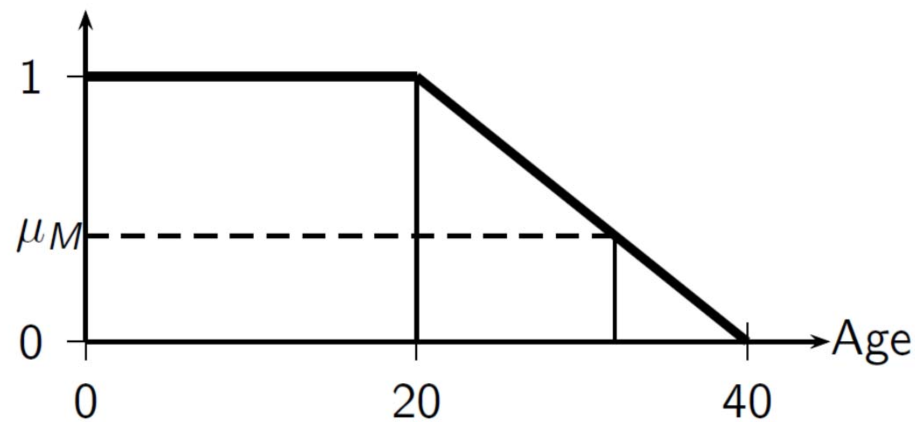
  which assigns to each element $x \in X$ a degree of membership $\mu(x)$ to the fuzzy set μ itself

Membership Functions (1)

$\mu_M(u) = 1$ reflects full membership in M

$\mu_M(u) = 0$ expresses absolute non-membership in M

- Sets can be viewed as special case of fuzzy sets where only full membership and absolute non-membership are allowed
- Such sets are called *crisp sets* or Boolean sets
- Membership degrees $0 < \mu_M < 1$ represent *partial membership*



Representing young in "a young person"

Membership Functions (2)

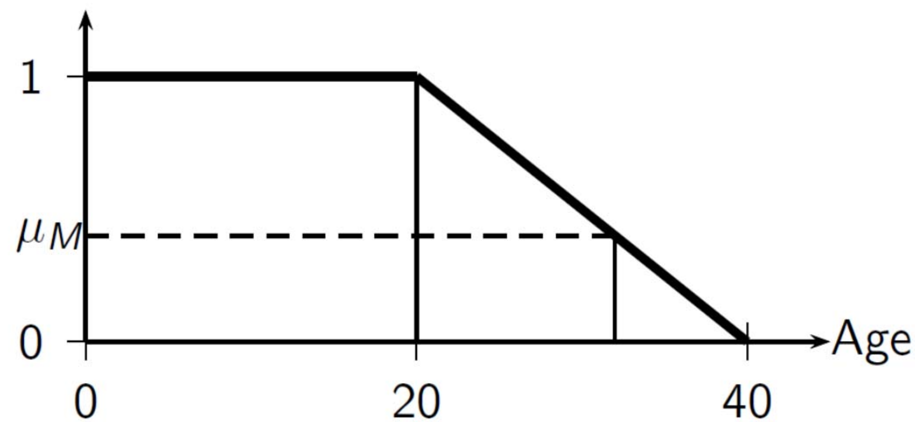
- A Membership function attached to a given linguistic description (such as young) depends on context
  - A young retired person is certainly older than young student
  - Even idea of young student depends on the user

- Membership degrees are fixed only by convention:
  - Unit interval as range of membership grades is arbitrary
  - Natural for modeling membership grades of fuzzy sets of real numbers

Membership Functions (3)

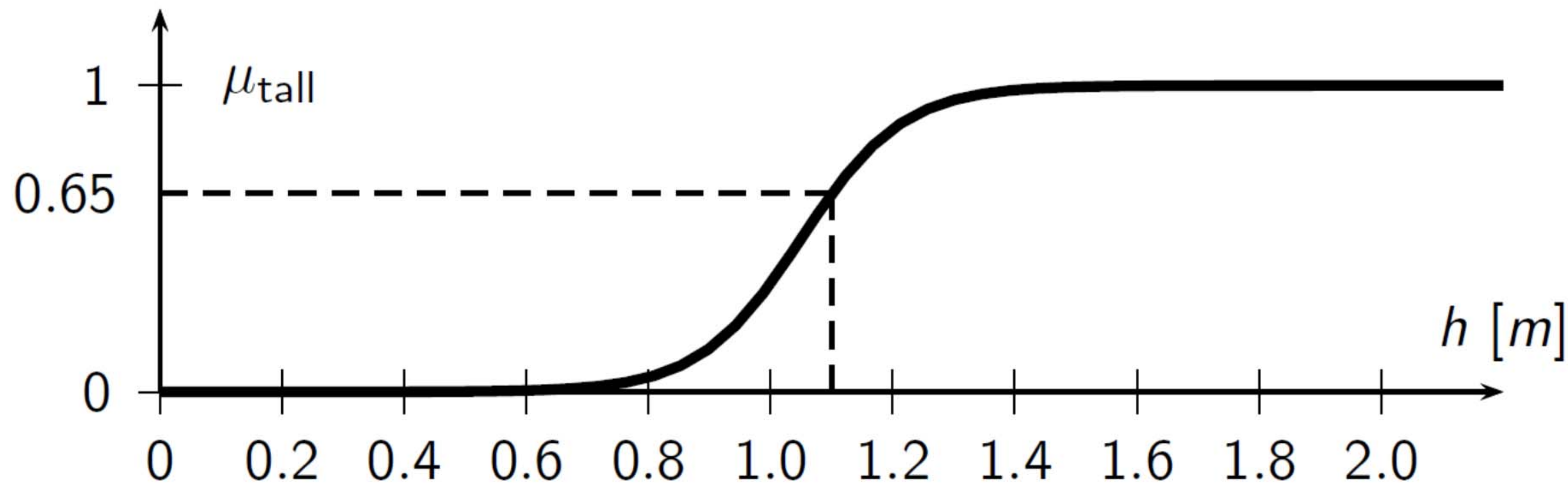
- There is no precise threshold between
  - prototypes of young and
  - prototypes of not young

- Fuzzy sets offer natural interface between linguistic and numerical representations



Representing young in “a young person”

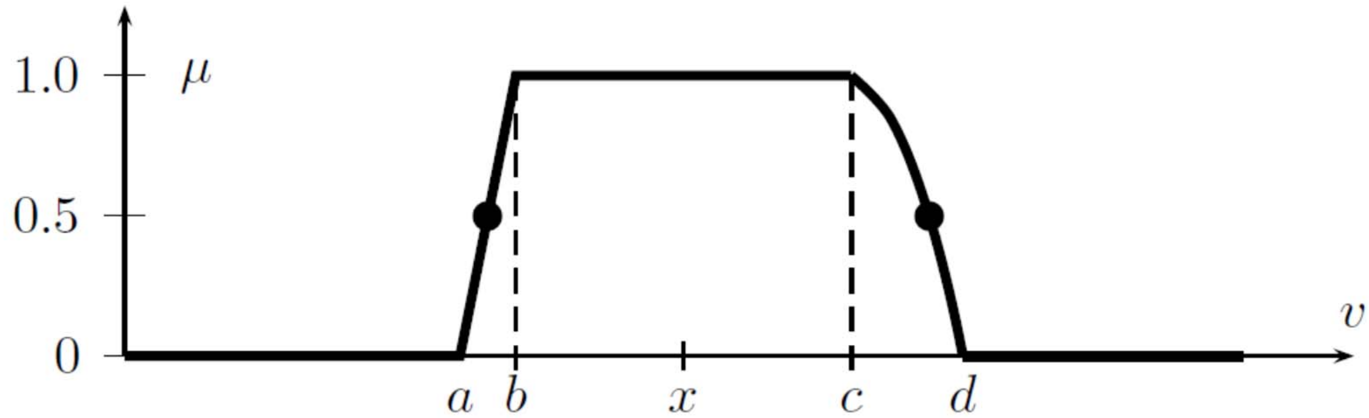
Example: Body Height of 4 Year Old Boys



Membership function for μ_{tall}

- 1.5 m is for sure tall, 0.7 m is for sure small, *but in-between*?
- Imprecise predicate *tall* modeled as sigmoid function
- *e.g.* height of 1.1 m satisfies predicate *tall* with 0.65

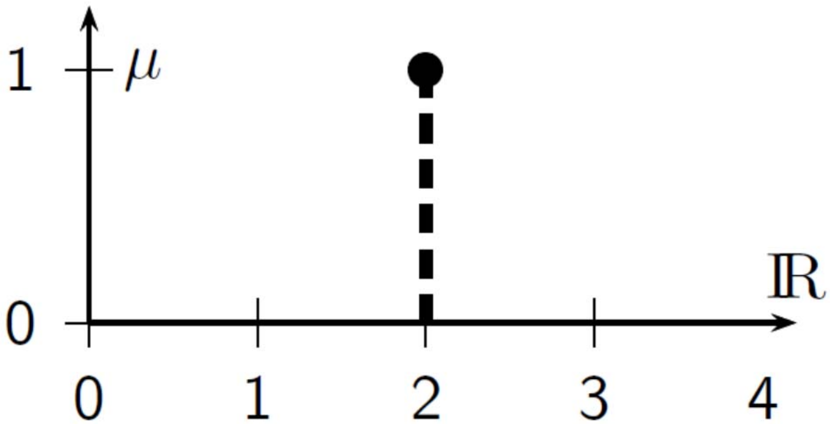
Example: Velocity of Rotating Hard Disk



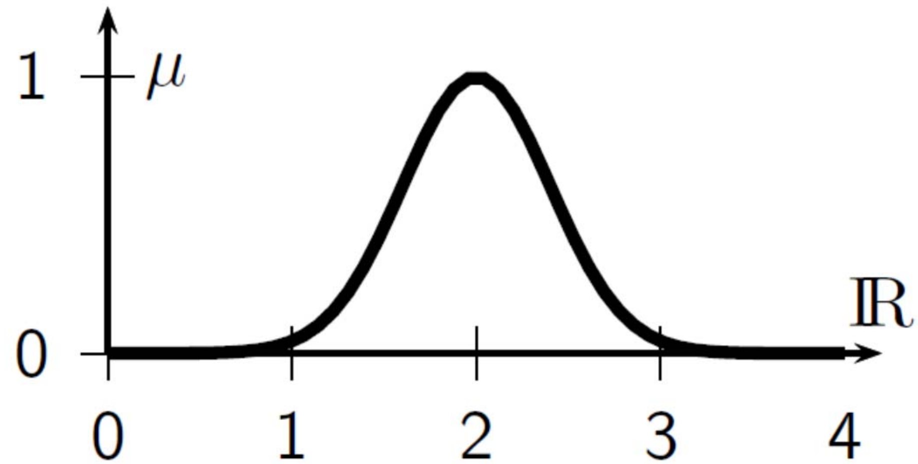
Fuzzy set μ characterizing velocity of rotating hard disk

- Let x be velocity v of rotating hard disk in rpms
- Expert's knowledge
  - "It's impossible that v drops under a or exceeds d."
  - "It's highly certain that any value between [b, c] can occur."
- Interval [a, d] is called *support* of the fuzzy set
- Interval [b, c] is denoted as *core* of the fuzzy set

Fuzzy Numbers



Exactly 2



Around 2

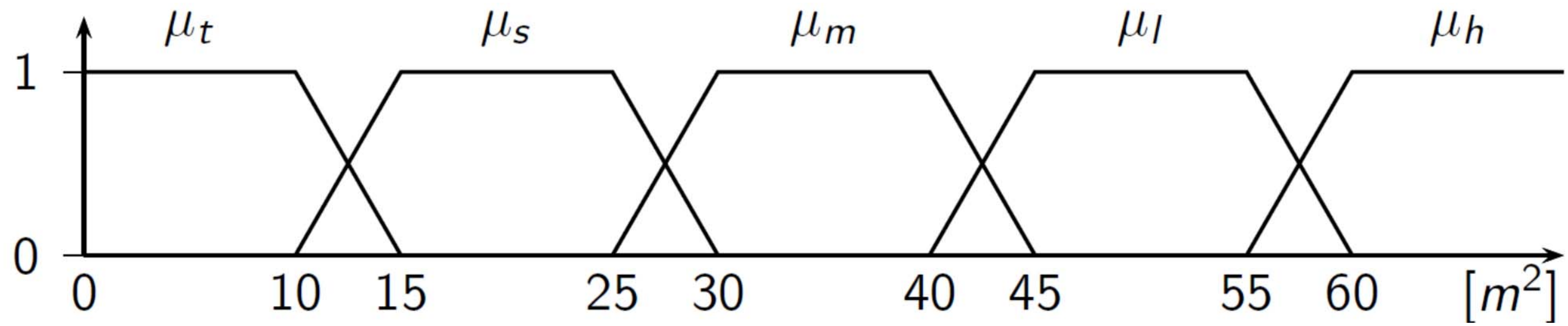
- Exact numerical numbers have membership degree 1
- Terms like around are modeled using different shapes: e.g., triangle, trapezoid, Gaussian, sigmoid, etc.

Linguistic Variables and Linguistic Values

- **Linguistic variables** represent attributes in fuzzy systems
- They assume **linguistic values**
- Linguistic values usually partition the possible values of the linguistic variables subjectively (based on human intuition)
- All linguistic values have a meaning, not a precise numerical value

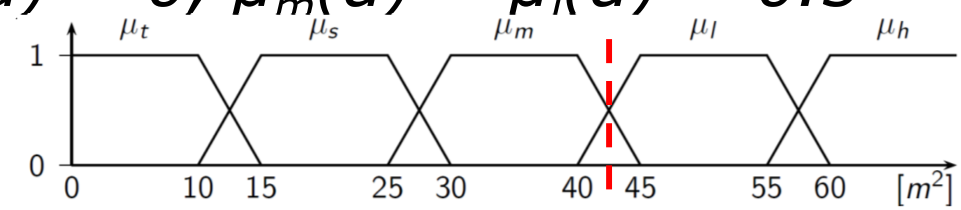
Example of Linguistic Values

- Linguistic variable *living area of a flat A*
- Linguistic values: *tiny*, *small*, *medium*, *large*, *huge*



- Every $x \in A$ has $\mu(x) \in [0, 1]$ to each value

- E.g., $a = 42.5$ m²
  $\mu_t(a) = \mu_s(a) = \mu_h(a) = 0,\ \mu_m(a) = \mu_l(a) = 0.5$



Semantics of Fuzzy Sets

- Fuzzy sets are relevant in 3 types of information-driven tasks
    1. classification and data analysis
    2. decision-making problems
    3. approximate reasoning

- These tasks exploit three semantics of membership grades
    1. similarity
    2. preference
    3. possibility

Degree of Similarity

- $\mu(u)$ is the degree of proximity of u from prototype elements of μ
- Proximity between pieces of information is modelled
- This view is used in
  - pattern classification
  - cluster analysis
  - regression
- In fuzzy control: similarity degrees are measured between current situation and prototypical ones

Degree of Preference

- μ represents both
  - set of more or less preferred objects
  - values of a decision variable X
- $\mu(u)$ represents both
  - intensity of preference in favor of object u
  - feasibility of selecting u as value of X
- Fuzzy sets represent criteria or flexible constraints
- This view is used in
  - fuzzy optimization (especially fuzzy linear programming)
  - decision analysis
- Typical applications
  - engineering design
  - scheduling problems

Degree of Possibility

- $\mu(u)$ can be viewed as:
  - degree of possibility that parameter X has value u
  - given the only information "X is μ"
- Support values are mutually exclusive and membership degrees rank these values by their possibility
- This view is used in
  - expert systems
  - artificial intelligence

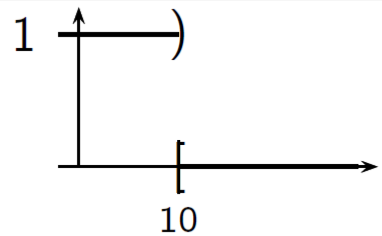
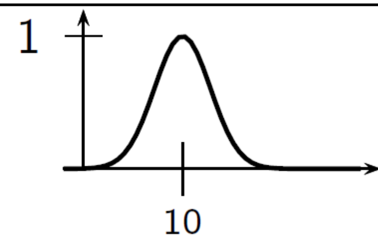
Representation of Fuzzy Sets

Definition of a "Set"

- By a *set* we mean every collection made into a whole of definite, distinct objects of our intuition or of our thought

- Properties:
  - $x \neq \{x\}$
  - If $x \in X$ and $X \in Y$, then $X \notin Y$
  - The set of all subsets of X is denoted as 2^X
  - $\emptyset$ is the empty set

Extension to a Fuzzy Set

Linguistic description		**Model**	
all numbers smaller than 10	Objective		Characteristic function of a set
all numbers almost equal to 10	Subjective		Membership function of a "fuzzy set"

- Definition

  A fuzzy set μ of $X \neq \emptyset$ is a function from the reference set X to the unit interval, i.e. $\mu : X \mapsto [0, 1]$

  $F(X)$ represents the set of all fuzzy sets of X, i.e.
  $F(X) \stackrel{\text{def}}{=} \{\mu \mid \mu : X \mapsto [0, 1]\}$

Vertical Representation

- Representation

  Fuzzy sets are described by their membership function and assigning degree of membership $\mu(x)$ to each element $x \in X$

- Example 1

  Linguistic expression "about *m*"

$$\mu_{m,d}(x) = \begin{cases} 1 - \left|\dfrac{m - x}{d}\right|, & \text{if } m - d \leq x \leq m + d \\ 0, & \text{otherwise,} \end{cases}$$

- Example 2

  Linguistic expression "approximately between *b* and *c*"

$$\mu_{a,b,c,d}(x) = \begin{cases} \dfrac{x - a}{b - a}, & \text{if } a \leq x \leq b \\ 1, & \text{if } b \leq x \leq c \\ \dfrac{x - d}{c - d}, & \text{if } c \leq x < d \\ 0, & \text{if } x < a \text{ or } x > \mathrm{d} \end{cases}$$

Horizontal Representation

- Representation

  For all membership degrees belonging to chosen subset of [0, 1], human expert lists elements of X that fulfill vague concept of fuzzy set with degree $\geq \alpha$.

  That is the horizontal representation of fuzzy sets by their $\boldsymbol{\alpha}$-**cuts**.

- Definition

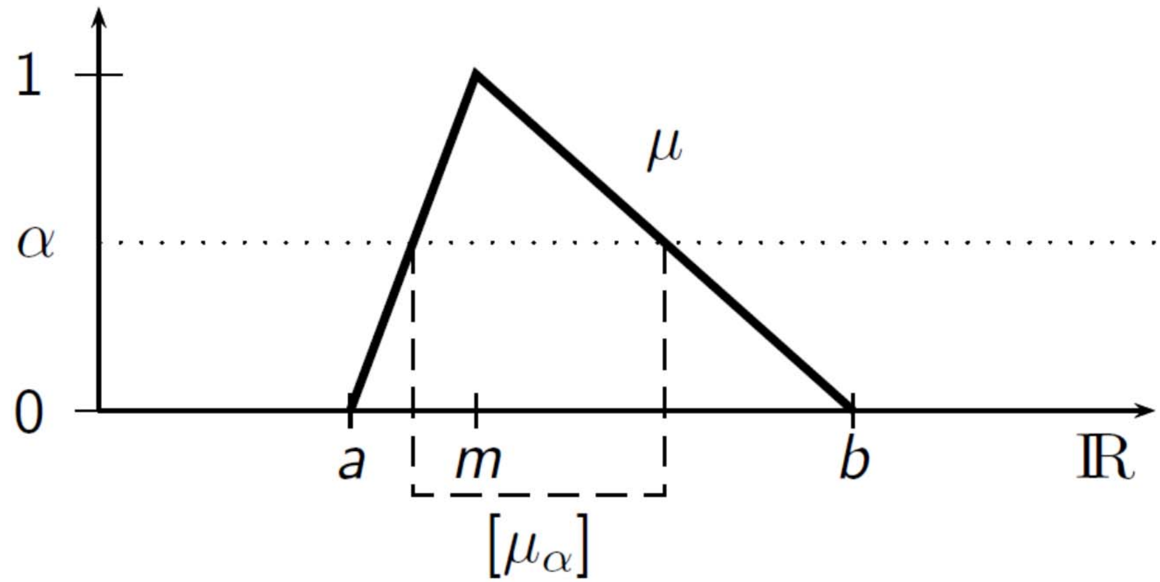
  Let $\mu \in \mathcal{F}(X)$ and $\in [0, 1]$. Then the sets

  $[\mu]_\alpha = \{x \in X \mid \mu(x) \geq \alpha\}$, $[\mu]_{\underline{\alpha}} = \{x \in X \mid \mu(x) > \alpha\}$

  are called the α-*cut* and *strict* α-*cut* of μ

Example

- Let μ be triangular function on $\mathbb{R}$



α-cut of μ can be constructed by

1. Draw an horizontal line parallel to x-axis through point $(0, \boldsymbol{\alpha})$
2. Project this section onto x-axis

$$[\mu]_\alpha = \begin{cases} [a + \alpha\,(m - a), b - \alpha\,(b - m)], & \text{if } 0 < \alpha \leq 1 \\ \mathbb{R}, & \text{otherwise} \end{cases}$$

Properties of α-custs (1)

- Any fuzzy set can be described by specifying its $\boldsymbol{\alpha}$-cuts

- Theorem

  Let $\mu \in \mathcal{F}(X)$, $\boldsymbol{\alpha} \in [0, 1]$ and $\beta \in [0, 1]$

  - $[\mu]_0 = X$,
  - $\alpha < \beta \implies [\mu]_\alpha \supseteq [\mu]_\beta$,
  - $\cap_{\alpha\,:\,\alpha < \beta} [\mu]_\alpha = [\mu]_\beta$.

Properties of α-custs (2)

- Theorem (Representation Theorem)

  Let $\mu \in \mathcal{F}(X)$, then

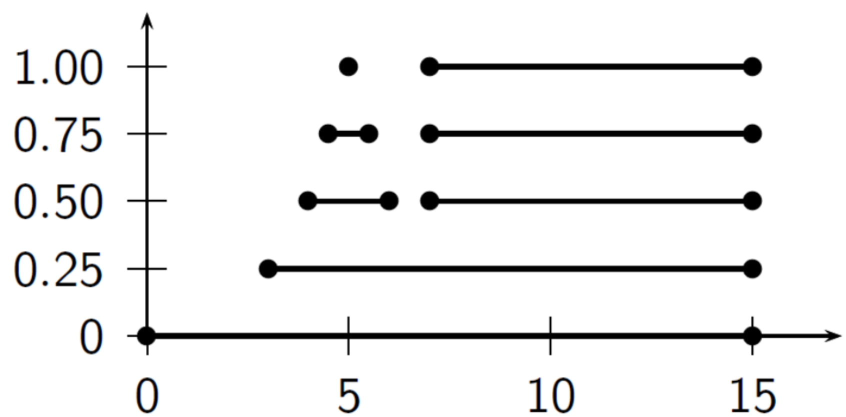
  $$[\mu]_0 = \sup_{\alpha \in [0,1]} \left\{ \min\left(\alpha, \chi_{[\mu]_\alpha}(x)\right) \right\}$$

  $$\text{where } \chi_{[\mu]_\alpha}(x) = \begin{cases} 1, & \text{if } x \in [\mu]_\alpha \\ 0, & \text{otherwise} \end{cases}$$

  - Fuzzy set can be obtained as upper envelope of its $\boldsymbol{\alpha}$-cuts
  - Simply draw $\boldsymbol{\alpha}$-cuts parallel to horizontal axis in height of $\boldsymbol{\alpha}$
  - In applications it is recommended to select finite subset $L \subseteq [0, 1]$ of relevant degrees of membership
  - They must be semantically distinguishable, i.e., fix level sets of fuzzy sets to characterize only for these levels

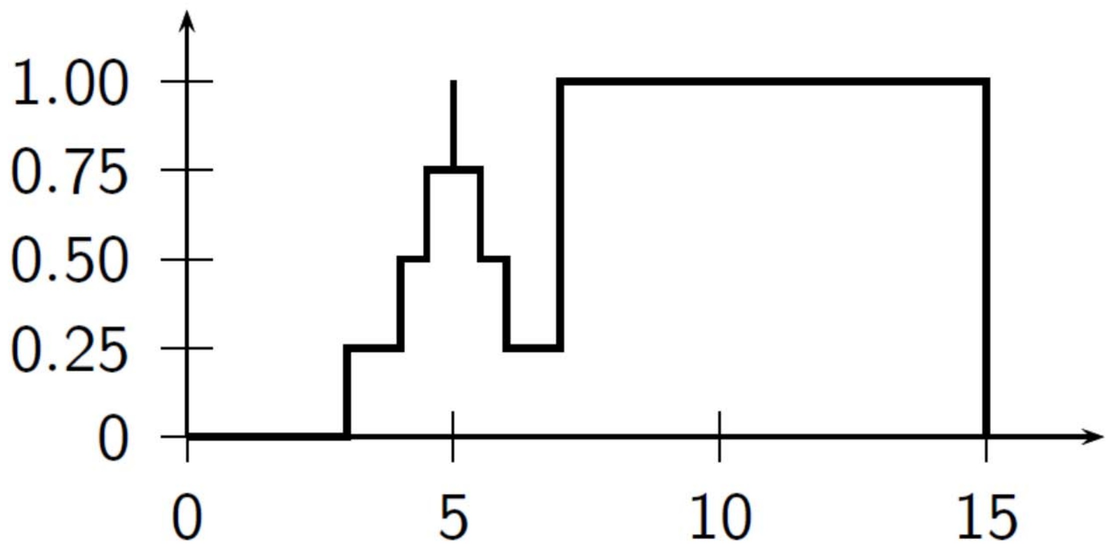
"Approximately 5 or greater than or equal to 7" (1)

- Suppose that $X = [0, 15]$
- An expert chooses $L = \{0, 0.25, 0.5, 0.75, 1\}$ and $\boldsymbol{\alpha}$ -cuts
  - $A_0 = [0, 15]$
  - $A_{0.25} = [3, 15]$
  - $A_{0.5} = [4, 6] \cup [7, 15]$
  - $A_{0.75} = [4.5, 5.5] \cup [7, 15]$
  - $A_1 = \{5\} \cup [7, 15]$



"Approximately 5 or greater than or equal to 7" (2)

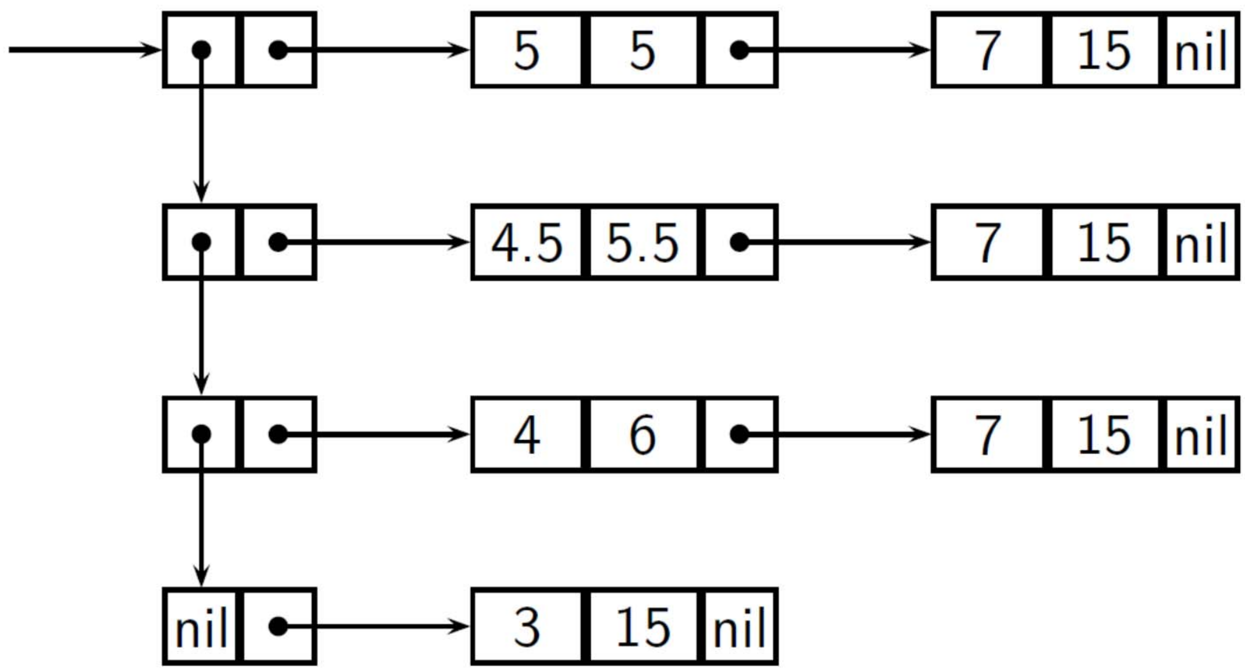
- μ_A is obtained as upper envelope of the family A of sets



- The horizontal representation is easier to process in computers
- Also, restricting the domain of x-axis to a discrete set is usually done

Horizontal Representation in the Computer

- Fuzzy sets are usually stored as chain of linear lists
- For each α-level, $\alpha \neq 0$
- A finite union of closed intervals is stored by their bounds
- This data structure is appropriate for arithmetic operators



Support and Core of a Fuzzy Set

- The **support** $S(\mu)$ of a fuzzy set $\mu \in \mathcal{F}(X)$ is the crisp set that contains all elements of X that have nonzero membership.

$$S(\mu) = [\mu]_{\underline{0}} = \{x \in X \mid \mu(x) > 0\}$$

- The **core** $C(\mu)$ of a fuzzy set $\mu \in \mathcal{F}(X)$ is the crisp set that contains all elements of X that have membership of one.

$$C(\mu) = [\mu]_1 = \{x \in X \mid \mu(x) = 1\}$$

Height of a Fuzzy Set

- The height $h(\mu)$ of a fuzzy set $\mu \in \mathcal{F}(X)$ is the largest membership grade obtained by any element in that set.

$$h(\mu) = \sup_{x \in X} \{\mu(x)\}$$

$h(\mu)$ may also be viewed as supremum of for which $[\mu]_\alpha \neq 0$

- A fuzzy set μ is called *normal*, iff $h(\mu) = 1$
It is called *subnormal*, iff $h(\mu) < 1$

Convex Fuzzy Sets

- Let X be a vector space. A fuzzy set $\mu \in \mathcal{F}(X)$ is called **fuzzy convex** if its α-cuts are convex for all $\alpha \in (0, 1]$

- The membership function of a convex fuzzy set **is not** a convex function

- Classical definition

  The membership functions are actually **concave**

Fuzzy Numbers

- μ is a fuzzy number if and only if μ is normal and $[\mu]_\alpha$ is bounded, closed, and convex $\forall\alpha \in (0, 1]$.

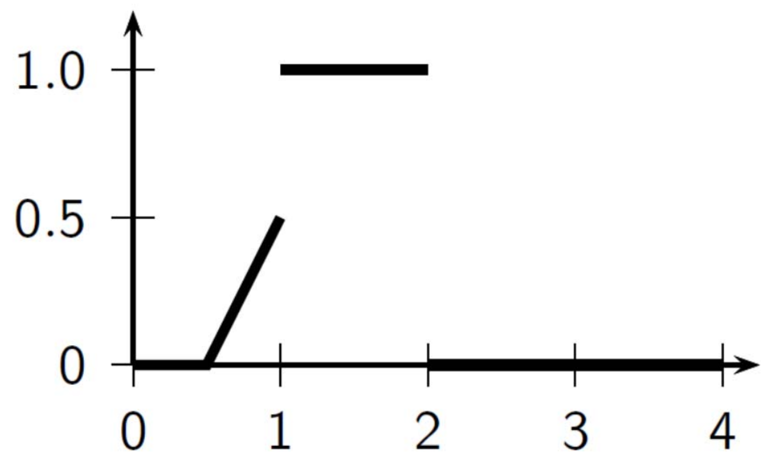
- Example

  The term *approximately* x_0 is often described by a parametrized class of membership functions, e.g.

  $\mu_1(x) = \max\{0, 1 - c_1|x - x_0|\}$ $\qquad c_1 > 0$

  $\mu_2(x) = \exp\{-c_2\|x - x_0\|_p\}$ $\qquad c_2 > 0, \qquad p > 1$

Fuzzy Numbers: Example



$$[\mu]_\alpha = \begin{cases} [1,2] & \text{if } \alpha \ \geq 0.5, \\ [0.5+\alpha, 2) & \text{if } 0 > \alpha < 0.5, \\ \mathbb{R} & \text{if } \alpha = 0 \end{cases}$$

- Upper semi-continuous functions are often convenient in applications
- In many applications (e.g. fuzzy control) the class of the functions and their exact parameters have a limited influence on the results
- In other applications (e.g. medical diagnosis) more precise membership degrees are needed

Multi-valued Logics

Set Operators

- Set operators are defined by using traditional logics operators

- Let X be universe of discourse (universal set):

  $A \cap B = \{x \in X \mid x \in A \wedge x \in B\}$

  $A \cup B = \{x \in X \mid x \in A \vee x \in B\}$

  $A^c = \{x \in X \mid x \notin A\} = \{x \in X \mid \neg(x \in A)\}$

  $A \subseteq B$ if and only if $(x \in A) \rightarrow (x \in B)$ for every $x \in X$

Aristotlelian Logic

- There are traditional, linguistic, psychological, epistemological and mathematical schools
- Aristotlelian logic can be seen as formal approach to human reasoning
- It’s still used today in Artificial Intelligence for knowledge representation and reasoning about knowledge

Classical Logic: An Overview

- Classical logic deals with propositions (either true or false)

- The propositional logic handles combination of logical variables

- Key idea: express n-ary logic functions with logic **primitives**, e.g. $\neg, \wedge, \vee, \rightarrow$

- A set of logic primitives is **complete** if any logic function can be composed by a finite number of these primitives, e.g. $\{\neg, \wedge, \vee\}$, $\{\neg, \wedge\}$, $\{\neg, \rightarrow\}$, $\{\downarrow\}$ (NOR), $\{|\}$ (NAND)

Inference Rules

- When a variable represented by logical formula is:
  - *true* for all possible truth values, i.e. it is called **tautology**
  - *false* for all possible truth values, i.e. it is called **contradiction**
- Various forms of tautologies exist to perform deductive inference, and are called inference rules:

$$\big(a \wedge (a \rightarrow b)\big) \rightarrow b \qquad (modus\ ponens)$$

$$\big(\neg b \wedge (a \rightarrow b)\big) \rightarrow \neg a \qquad (modus\ tollens)$$

$$\big((a \rightarrow b) \wedge (b \rightarrow c)\big) \rightarrow (a \rightarrow c) \qquad (hypothetical\ syllogism)$$

- e.g. modus ponens: given two true propositions a and $a \rightarrow b$ (*premises*), truth of proposition b (*conclusion*) can be inferred

Boolean Algebra

- The propositional logic based on finite set of logic variables is isomorphic to **finite set theory**

- Both of these systems are isomorphic to a finite **Boolean algebra**

- Definition: A Boolean algebra on a set B is defined as quadruple $\mathcal{B} = (B, +, \cdot, \overline{})$ where B has at least two elements (bounds) 0 and 1, + and · are binary operators on B, and $\overline{}$ is a unary operator on B for which the following properties hold

Properties of Boolean Algebras (1)

(B1) Idempotence	$a + a = a$	$a \cdot a = a$
(B2) Commutativity	$a + b = b + a$	$a \cdot b = b \cdot a$
(B3) Associativity	$(a + b) + c = a + (b + c)$	$(a \cdot b) \cdot c = a \cdot (b \cdot c)$
(B4) Absorption	$a + (a \cdot b) = a$	$a \cdot (a + b) = a$
(B5) Distributivity	$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$	$a + (b \cdot c) = (a + b) \cdot (a + c)$
(B6) Universal Bounds	$a + 0 = a, a + 1 = 1$	$a \cdot 1 = a, a \cdot 0 = 0$
(B7) Complementary	$a + \overline{a} = 1$	$a \cdot \overline{a} = 0$
(B8) Involution	$\overline{\overline{a}} = a$	
(B9) Dualization	$\overline{a + b} = \overline{a} \cdot \overline{b}$	$\overline{a \cdot b} = \overline{a} + \overline{b}$

Properties (B1)-(B4) are common to every lattice,

i.e. a Boolean algebra is a distributive (B5), bounded (B6), and complemented (B7)-(B9) lattice,

i.e. every Boolean algebra can be characterized by a partial ordering on a set, i.e. $a \leq b$ if $a \cdot b = a$ or, alternatively, if $a + b = b$

Set Theory, Boolean Algebra, Propositional Logic

- Every theorem in one theory has a counterpart in each other theory

Meaning	Set Theory	Boolean Algebra	Prop. Logic
values	2^X	B	$\mathcal{L}(V)$
"meet"/"and"	$\cap$	$\cdot$	$\wedge$
"join"/"or	$\cup$	$+$	$\vee$
"complement"/"not"	^{c}	$\overline{}$	$\neg$
identity element	X	1	1
zero element	$\emptyset$	0	0
partial order	$\subseteq$	$\leq$	$\rightarrow$

power set 2^X , set of logic variables V, set of all combinations $\mathcal{L}(V)$ of truth values of V

Every Theorem in One Theory has a Counterpart in Each Other Theory

- The Principle of Bivalence

  Every proposition is either true or false

- The Principle of Valence

  Every proposition has a truth value

Three-valued Logics

- A 2-valued logic can be extended to a 3-valued logic in several ways

		Łukasiewicz				Bochvar				Kleene				Heyting				Reichenbach			
a	b	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$	$\wedge$	$\vee$	$\rightarrow$	$\leftrightarrow$
0	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
0	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$\frac{1}{2}$	1	0	0	$\frac{1}{2}$	1	$\frac{1}{2}$
0	1	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0
$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1	1	$\frac{1}{2}$
1	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0
1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

n-valued Logics

- Various n-valued logics were developed
- Usually truth values are assigned by rational number in [0, 1]
- Key idea: uniformly divide [0, 1] into n truth values

- Definition

  The set T_n of truth values of an *n*-valued logic is defined as

$$T_n = \left\{ 0 = \frac{0}{n-1}, \frac{1}{n-1}, \frac{2}{n-1}, \dots, \frac{n-2}{n-1}, \frac{n-1}{n-1} = 1 \right\}$$

  These values can be interpreted as degree of truth

Primitives in *n*-valued Logics

- Generalization of Łukasiewicz three-valued logic
- It uses truth values in T_n and defines primitives as follows

$$\neg a = 1 - a$$

$$a \wedge b = \min(a, b)$$

$$a \vee b = \max(a, b)$$

$$a \rightarrow b = \min(1, 1 + b - a)$$

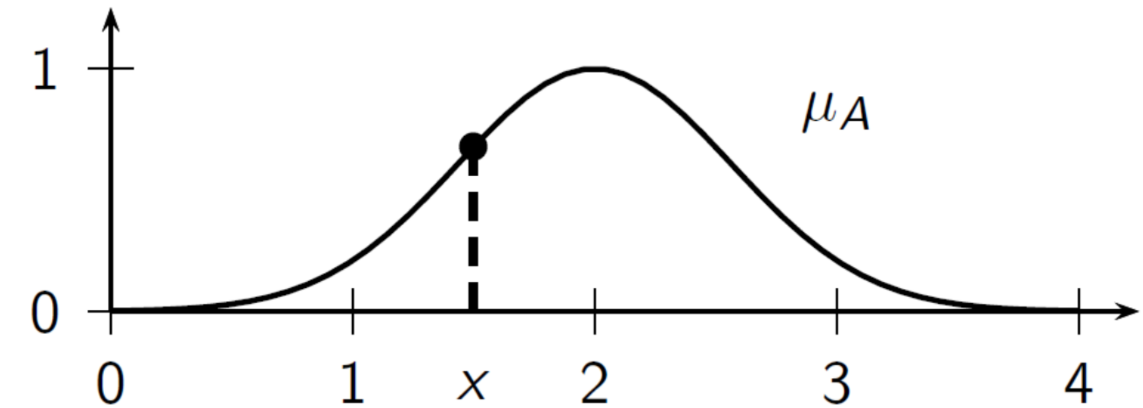
$$a \leftrightarrow b = 1 - |a - b|$$

- This *n*-valued logic is denoted by L_n
- The sequence (L_2, L_3, . . . , L_∞) contains the classical two-valued logic L_2 and an infinite-valued logic L_∞ (rational countable values T_∞)
- The infinite-valued logic L_1 is the logic with all real numbers in [0, 1] (1 = cardinality of continuum)

Fuzzy Set Theory

What does a fuzzy set represent?

- A logic with values in [0, 1]
- Consider fuzzy proposition A ("approximately two") on $\mathbb{R}$, fuzzy logic offers means to construct such



A defined by membership function μ_A, i.e. truth values $\forall x \in \mathbb{R}$

- let $x \in \mathbb{R}$ be a subject/observation
- $\mu_A(x)$is the degree of truth that x is A

Standard Fuzzy Set Operators

- Definition

We define the following algebraic operators on $\mathcal{F}(X)$

$(\mu \ \wedge \mu') \ =^{\mathrm{def}} \ \min\{\mu(x), \mu'(x)\}$ intersection ("AND")

$(\mu \ \vee \mu') \ =^{\mathrm{def}} \ \max\{\mu(x), \mu'(x)\}$ union ("OR")

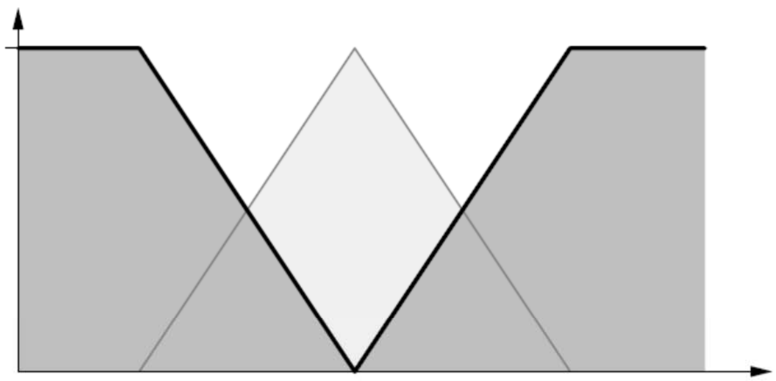
$\neg \, \mu =^{\mathrm{def}} \ 1 \ - \mu(x)$ complement ("NOT")

μ is subset of μ' if and only if $\mu \leq \mu'$

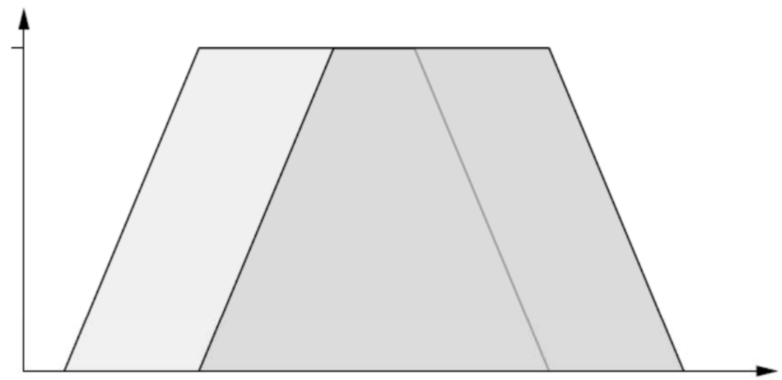
- Theorem

$(\mathcal{F}(X), \wedge, \vee, \neg)$ is a complete distributive lattice but no Boolean algebra

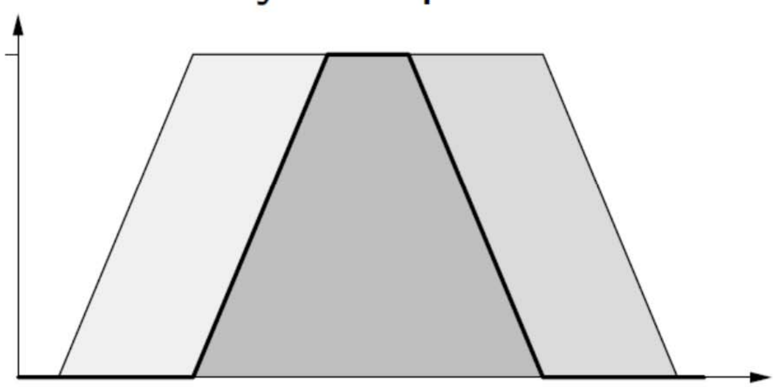
Standard Fuzzy Set Operators: Example



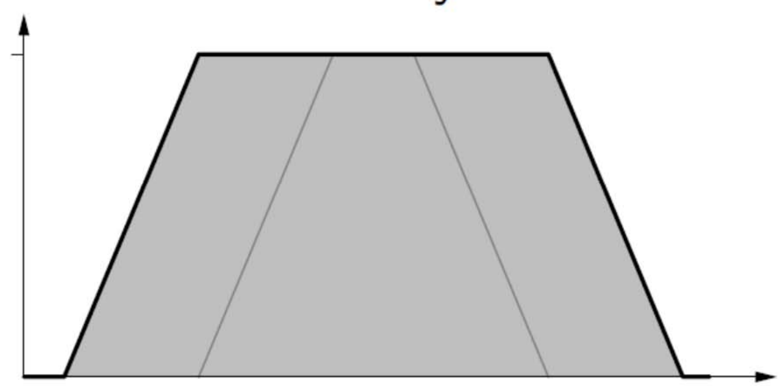
fuzzy complement



two fuzzy sets



fuzzy intersection



fuzzy union

Fuzzy Set Complement

Fuzzy Complement/Fuzzy Negation

- Definition

  Let X be a given set and $\mu \in \mathcal{F}(X)$. Then the complement $\bar{\mu}$ can be defined pointwise by $\bar{\mu}(x) :=\sim (\mu(x))$ where $\sim: [0,1] \longrightarrow [0,1]$ satisfies the conditions

  $$\sim (0) = 1, \qquad \sim (1) = 0$$

  and

  for $x, y \in [0,1]$, $x \leq y \Rightarrow \sim x \geq \sim y$ ($\sim$ is non-increasing)

- Abbreviation

  $\sim x :=\sim (x)$

Strict and Strong Negations

- Additional properties may be required

  $x, y \in [0,1], x < y \Rightarrow \sim x > \sim y$ ($\sim$ is strictly decreasing)

  $\sim$ is continuous

  $\sim\sim x = x$ for all $x \in [0,1]$ ($\sim$ is involutive)

- Definition
  - A negation is called *strict* if it is also strictly decreasing and continuous
  - A strict negation is said to be *strong* if it is involutive, too
  - $\sim x = 1 - x^2$, for instance, is strict, not strong, thus not involutive

Families of Negations

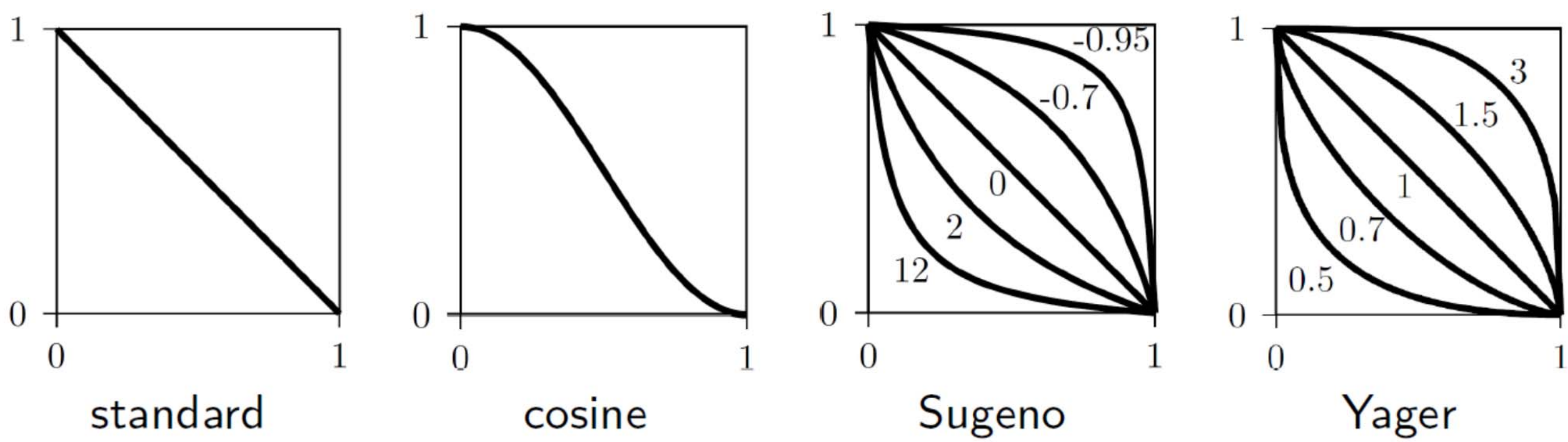
standard negation: $\sim x = 1 - x$

threshold negation: $\sim_\theta x = \begin{cases} 1 & \text{if } x \le \theta \\ 0 & \text{oherwise} \end{cases}$

Cosine negation: $\sim x = \frac{1}{2}(1 + \cos(\pi x))$

Sugeno negation: $\sim x = \dfrac{1 - x}{1 + \lambda x}, \qquad \lambda \succ 1$

Yager negation: $\sim x = \left(1 - x^\lambda\right)^{\frac{1}{\lambda}}$



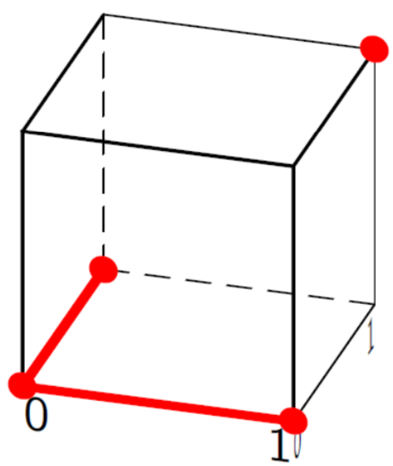
Fuzzy Set Intersection and Union

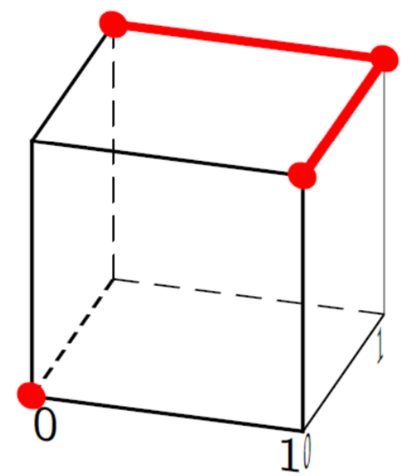
Classical Intersection and Union

- Classical set intersection represents logical conjunction
- Classical set union represents logical disjunction
- Generalization from {0, 1} to [0, 1] as follows

$x \wedge y$	0	1
0	0	0
1	0	1

$x \vee y$	0	1
0	0	1
1	1	1





Fuzzy Set Intersection and Union

Let A, B be fuzzy subsets of X, i.e. $A, B \in \mathcal{F}(X)$

Their intersection and union can be defined pointwise using

$(A \cap B)(x) = \top(A(x), B(x))$ where $\top : [0,1]^2 \rightarrow [0,1]$

$(A \cup B)(x) = \perp(A(x), B(x))$ where $\perp : [0,1]^2 \rightarrow [0,1]$

Triangular Norms and Conorms (1)

- $\top$ is a triangular norm (*t*-norm) $\Leftrightarrow \top$ satisfies conditions T1-T4
- $\perp$ is a triangular conorm (*t*-conorm) $\Leftrightarrow$ $\perp$ satisfies C1-C4

- For all $x, y \in [0,1]$, the following laws hold
  - Identity Law

    T1: $\top(x, 1) = x \ (A \cap X = A)$

    C1: $\perp(x, 0) = x \ (A \cup \emptyset = A)$
  - Commutativity

    T2: $\top(x, y) = \top(y, x) \ (A \cap B = B \cap A)$

    C2: $\perp(x, y) = \perp(y, x) \ (A \cup B = B \cup A)$

Triangular Norms and Conorms (2)

- For all $x, y \in [0, 1]$, the following laws hold
  - Associativity

    T3: $\top(x, \top(y, z)) = \top(\top(x, y), z)$ $(A \cap (B \cap C)) = ((A \cap B) \cap C)$

    C3: $\perp(x, \perp(y, z)) = \perp(\perp(x, y), z)$ $(A \cup (B \cup C)) = ((A \cup B) \cup C)$
  - Associativity

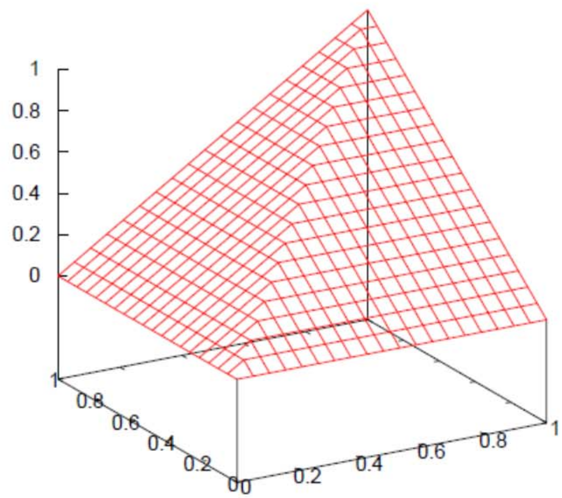
    $x \leq z$ implies

    T4: $\top(x, y) \leq \top(x, z)$

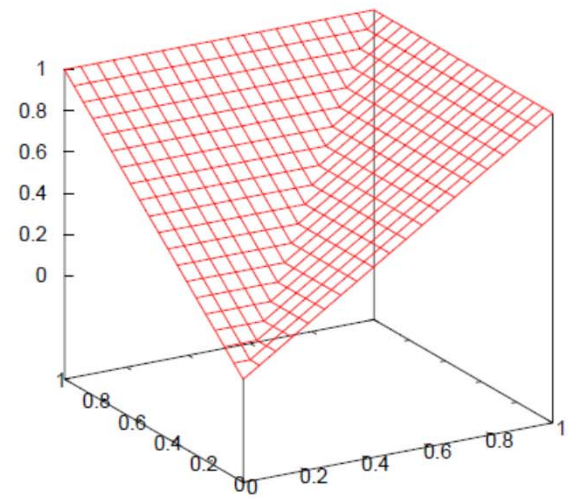
    C4: $\perp(x, y) \leq \perp(x, z)$

Minimum and Maximum (1)

- $\top_{\min}(x, y) = \min(x, y)$, $\perp_{\max}(x, y) = \max(x, y)$
- Minimum is the greatest t-norm and max is the weakest t-conorm
- $\top(x, y) \leq \min(x, y)$ and $\perp(x, y) \geq \max(x, y)$ for any $\top$ and $\perp$



$\top_{\min}$



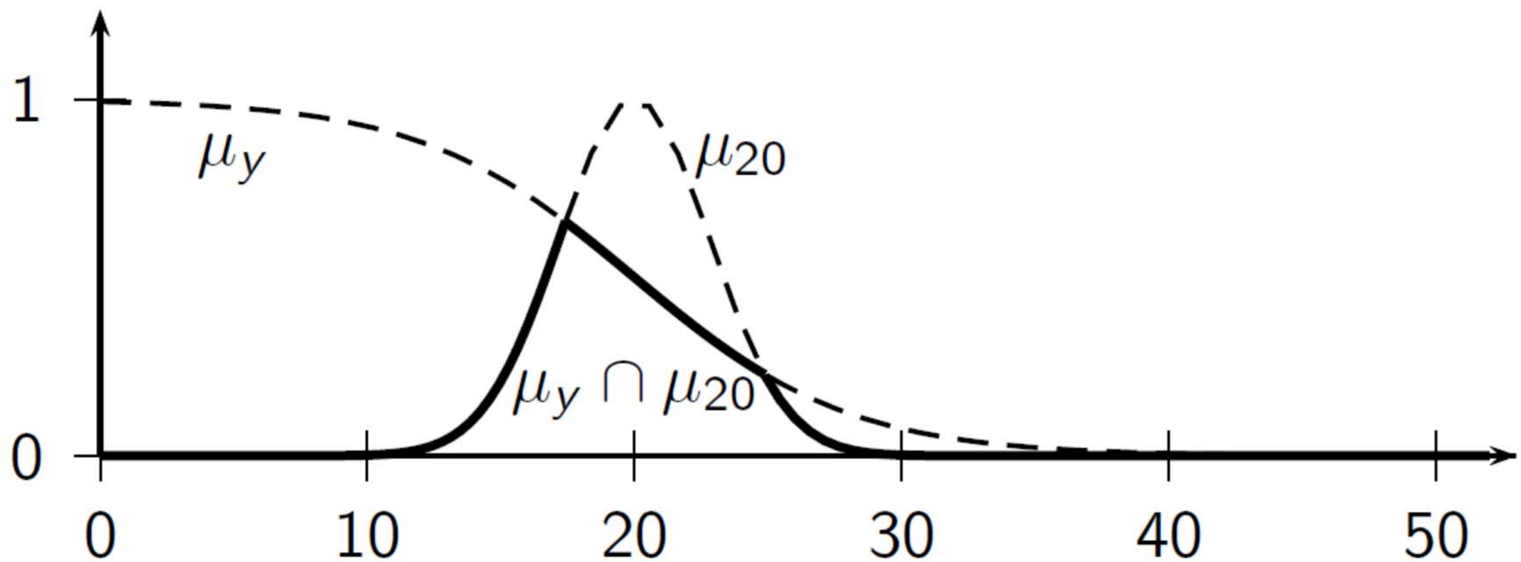
$\perp_{\max}$

Minimum and Maximum (2)

- $\top_{\min}$ and $\perp_{\max}$ can be easily processed numerically and visually

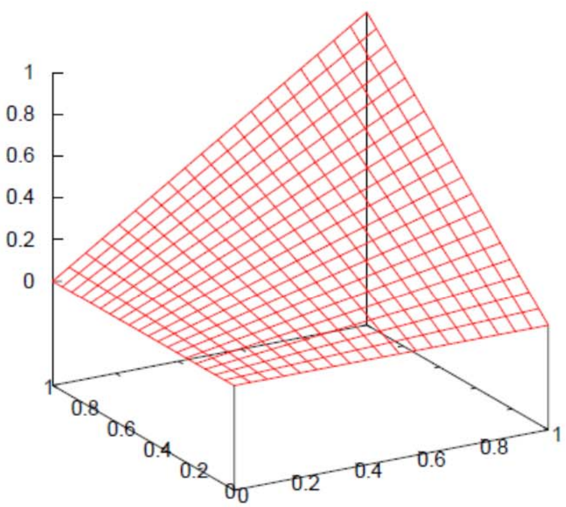
- *e.g.* linguistic values *young* and *approx. 20* described by μ_y , μ_{20}

  $\top_{\min}(\mu_y , \mu_{20})$ is shown below

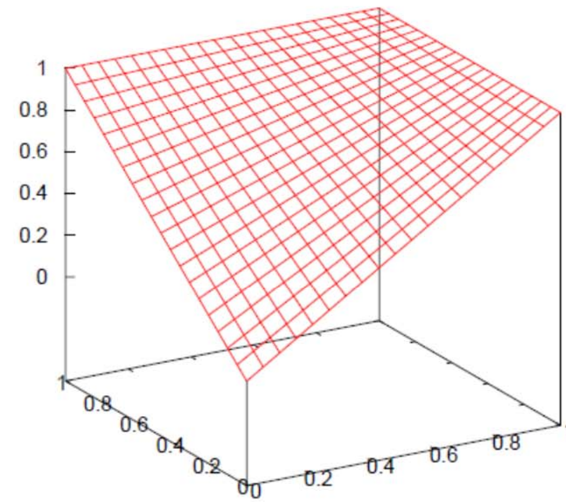


Product and Probabilistic Sum

- $\top_{\text{prod}}(x, y) = x \cdot y$, $\qquad \perp_{\text{sum}}(x, y) = x + y - x \cdot y$
- Note that use of product and its dual has nothing to do with probability theory



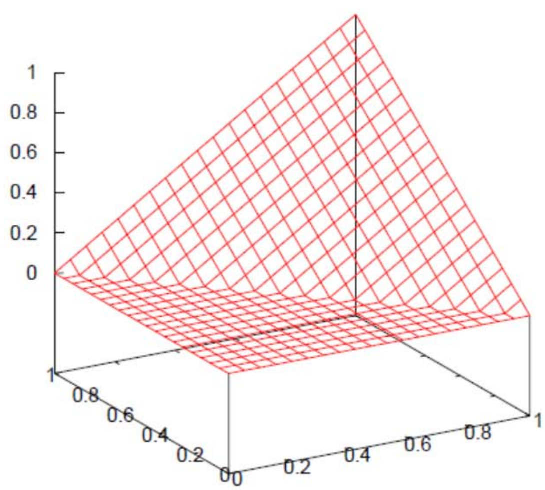
$\top_{\text{prod}}$



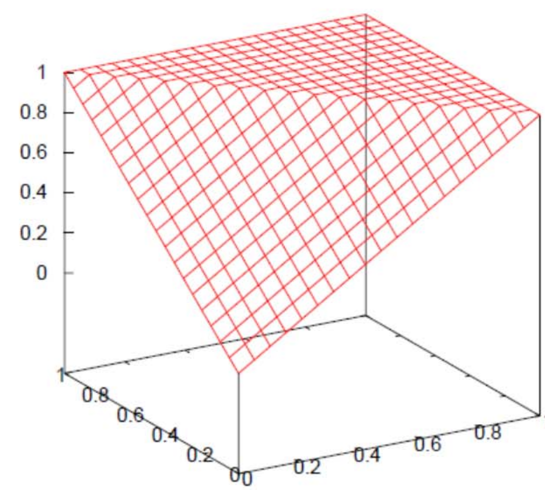
$\perp_{\text{sum}}$

Łukasiewicz t-norm and t-conorm

- $\top_{\text{Łuka}}(x, y) = \max\{0, x + y - 1\}$, $\perp_{\text{Łuka}}(x, y) = \min\{1, x + y\}$
- $\top_{\text{Łuka}}$, $\perp_{\text{Łuka}}$ are also called bold intersection and bounded sum



$\top_{\text{Łuka}}$

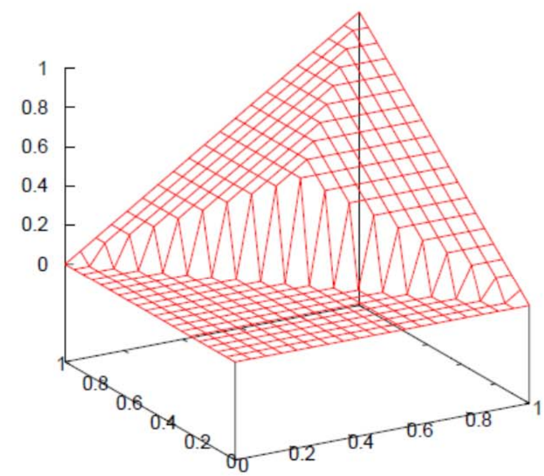


$\perp_{\text{Łuka}}$

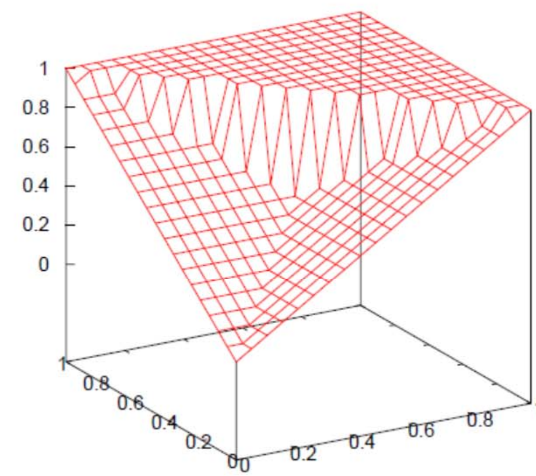
Nilpotent Minimum and Maximum

- $\top_{\min_0}(x, y) = \begin{cases} \min(x, y) & \text{if } x + y > 1 \\ 0 & \text{otherwise} \end{cases}$

  $\perp_{\max_1}(x, y) = \begin{cases} \max(x, y) & \text{if } x + y < 1 \\ 1 & \text{otherwise} \end{cases}$



$\top_{\min_0}$



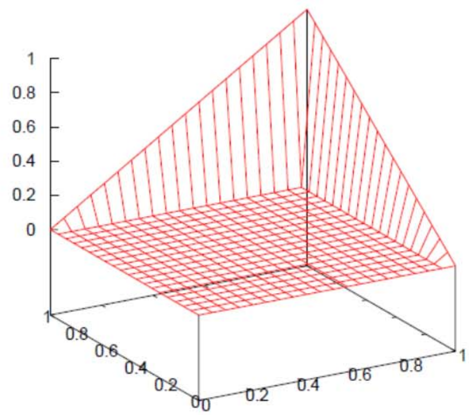
$\perp_{\max_1}$

Drastic Product and Sum

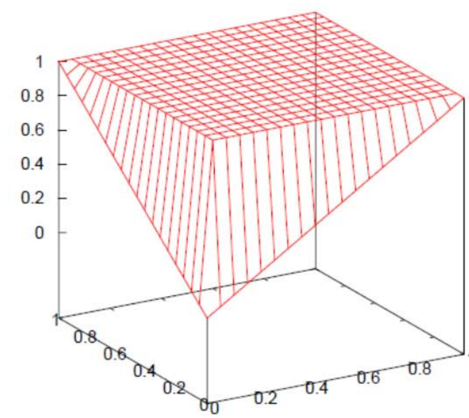
- $\top_{-1}(x, y) = \begin{cases} \min(x, y) & \text{if } \max(x + y) = 1 \\ 0 & \text{otherwise} \end{cases}$

  $\bot_{-1}\ (x, y) = \begin{cases} \max(x, y) & \text{if } \min(x + y) = 0 \\ 0 & \text{otherwise} \end{cases}$

- $\top_{-1}$ is the weakest *t*-norm, $\bot_{-1}$ is the strongest *t*-conorm
- $\top_{-1} \leq \top \leq \top_{\min}$, $\bot_{\max} \leq \bot \leq \bot_{-1}$ for any $\top$ and $\bot$

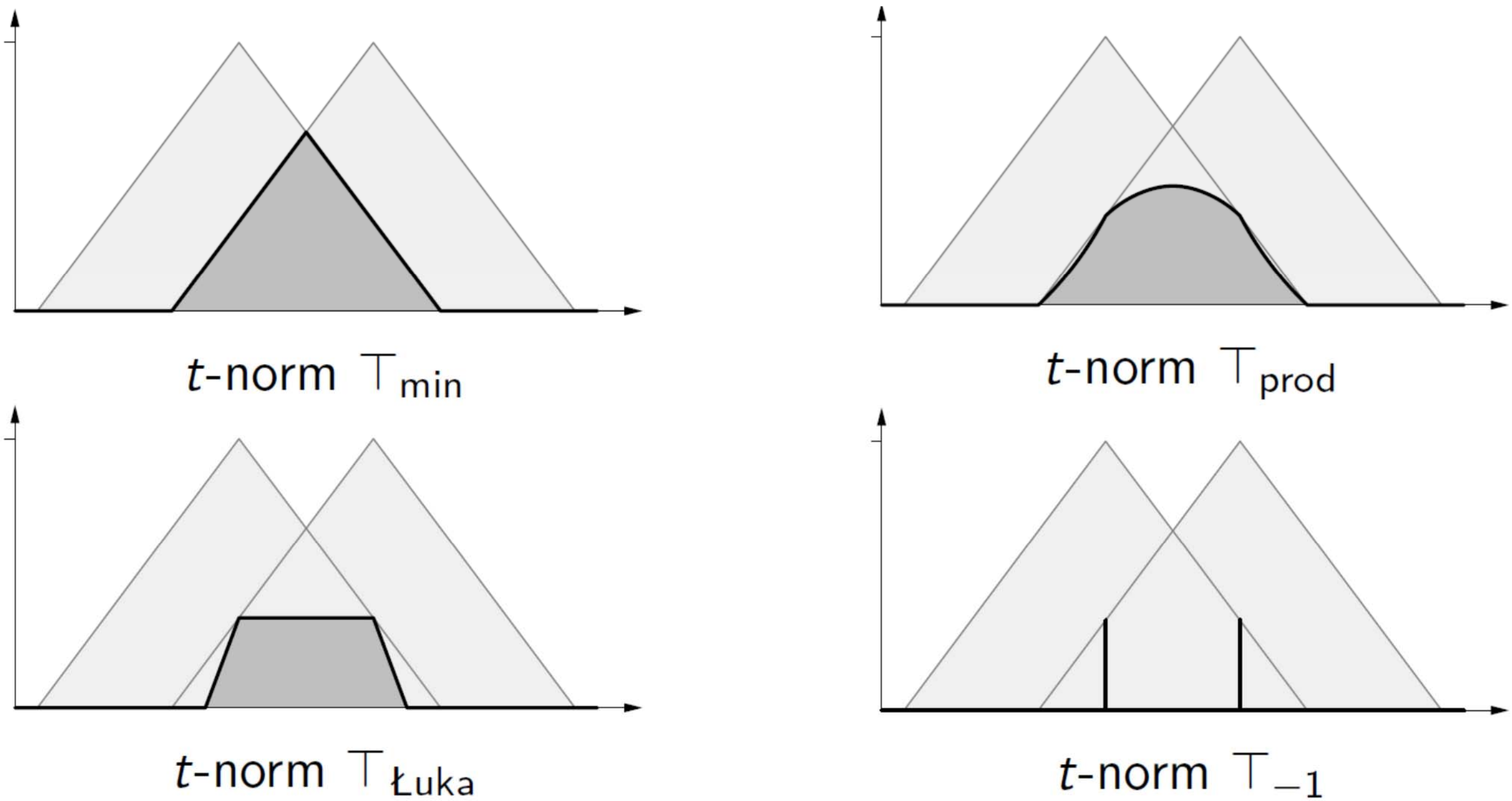


$\top_{-1}$



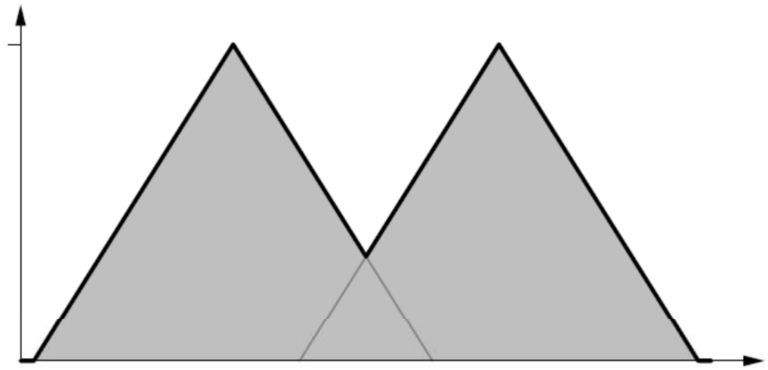
$\bot_{-1}$

Examples of Fuzzy Intersections

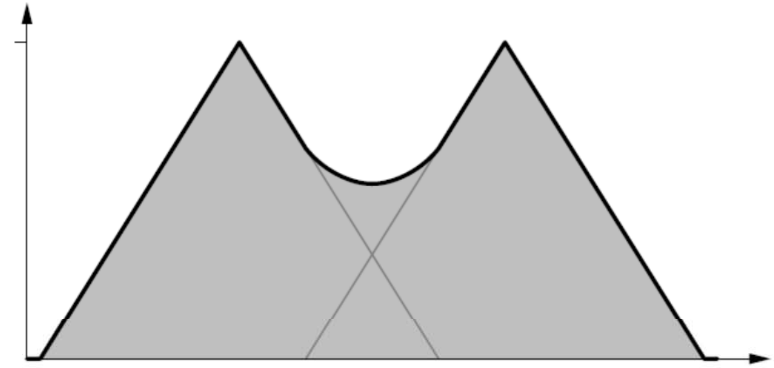


Note that all fuzzy intersections are contained within upper left graph and lower right one

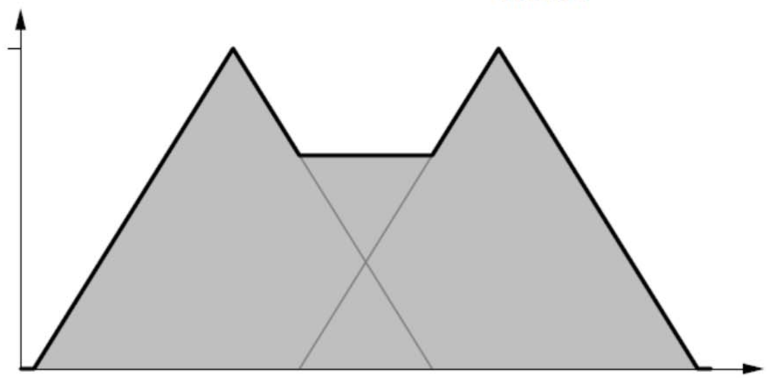
Examples of Fuzzy Unions



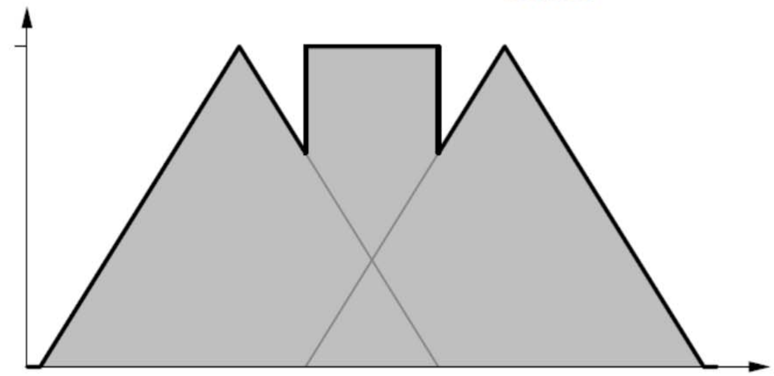
t-conorm $\perp_{\mathrm{max}}$



t-conorm $\perp_{\mathrm{sum}}$



t-conorm $\perp_{\text{Łuka}}$



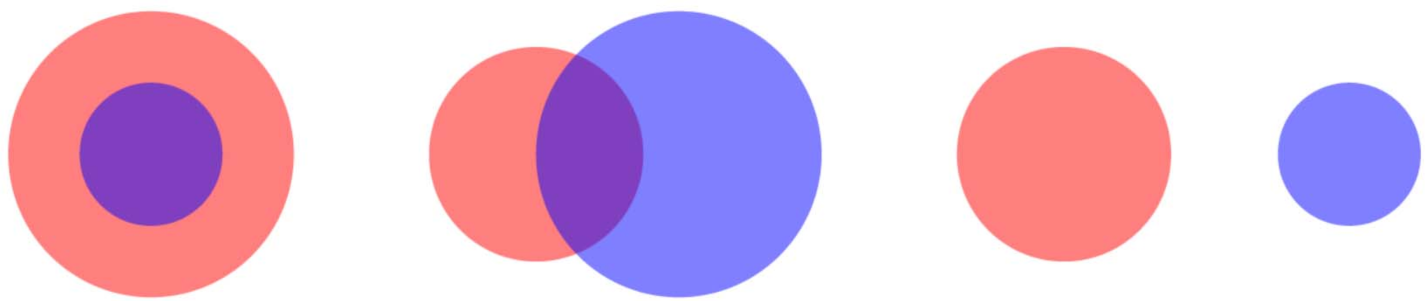
t-conorm $\perp_{-1}$

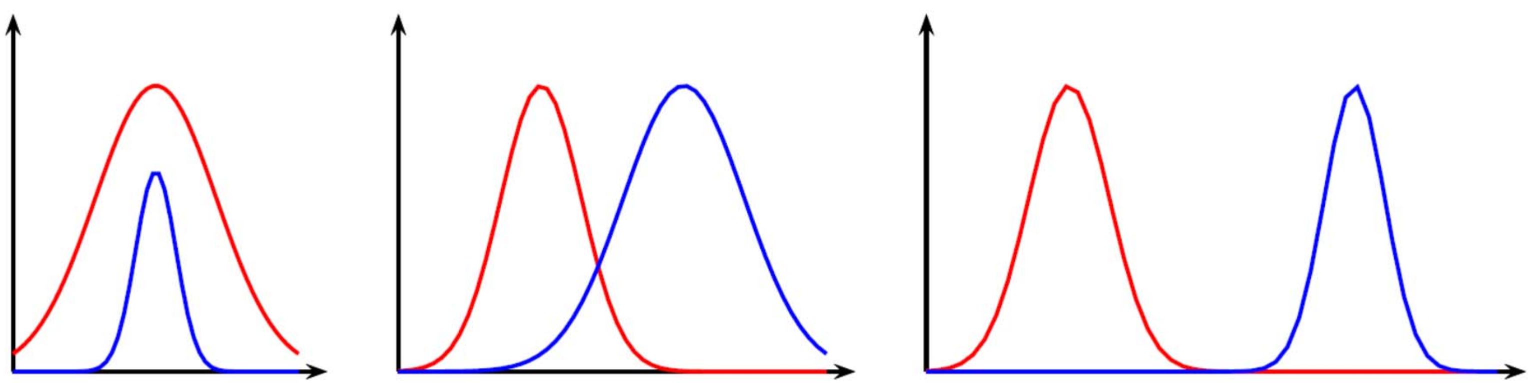
Note that all fuzzy unions are contained within upper left graph and lower right one

Fuzzy Sets Inclusion

Fuzzy Implications

- crisp: $x \in A \Rightarrow x \in B$
- fuzzy: $x \in \mu \Rightarrow x \in \mu'$





Definitions of Fuzzy Implications

- One way of defining I is to use $\forall a, b \in \{0, 1\}$
$$I(a, b) = \neg a \vee b$$
- In fuzzy logic, disjunction and negation are t-conorm and fuzzy complement, respectively, thus $\forall a, b \in [0, 1]$
$$I(a, b) = \perp (\sim a, b)$$
- Another way in classical logic is $\forall a, b \in \{0, 1\}$
$$I(a, b) = \max\{x \in \{0, 1\} \mid a \wedge x \leq b\}$$
- In fuzzy logic, conjunction represents t-norm, thus $\forall a, b \in [0, 1]$
$$I(a, b) = \sup\{x \in [0, 1] \mid \top(a, x) \leq b\}$$
- Classical definitions are equal, fuzzy extensions are not: law of absorption of negation does not hold in fuzzy logic

S-Implications

- Implications based on $I(a,b) = \perp(\sim a, b)$ are called S-implications.
- Symbol S is often used to denote t-conorms.
- Four well-known S-implications are based on $\sim a = 1 - a$

Name	$\mathrm{I(a,b)}$	$\perp \mathrm{(a,b)}$
Kleene-Dienes	$I_{\max}(a,b) = \max(1 - a, b)$	$\max(a,b)$
Reichenbach	$I_{\text{sum}}(a,b) = 1 - a + ab$	$a + b - ab$
Łukasiewicz	$I_{Ł}(a,b) = \min(1, 1 - a + b)$	$\min(1, a + b)$
largest	$I_{-1}(a,b) = \begin{cases} b, & \text{if } a=1 \\ 1 - a, & \text{if } b=0 \\ 1, & \text{otherwise} \end{cases}$	$\begin{cases} b, & \text{if } a=0 \\ a, & \text{if } b=0 \\ 1, & \text{otherwise} \end{cases}$

***R*-Implications** (1)

- $I(a,b) = \sup\{x \in [0,1] \mid \top(a,x) \leq b\}$
  leads to *R*-implications
- Symbol *R* represents close connection to residuated semigroup
- Well-known *R*-implications based on $\sim a = 1 - a$
  1. Standard fuzzy intersection leads to Gödel implication
     $$I_{\min}(a,b) = \sup\{x \mid \min(a,x) \leq b\} = \begin{cases} 1, & \text{if } a \leq b \\ b, & \text{if } a > b \end{cases}$$
  2. Product leads to Goguen implication
     $$I_{\text{prod}}(a,b) = \sup\{x \mid ax \leq b\} = \begin{cases} 1, & \text{if } a \leq b \\ b/a, & \text{if } a > b \end{cases}$$
  3. Łukasiewicz t-norm leads to Łukasiewicz implication
     $$I_{Ł}(a,b) = \sup\{x \mid \max(0, a + x - 1) \leq b\} = \min(1, 1 - a + b)$$

R-Implications (2)

Name	Formula	$\top(a,b) =$
Gödel	$I_{\min}(a,b) = \begin{cases} 1, & \text{if } a \leq b \\ b, & \text{if } a > b \end{cases}$	$\min(a,b)$
Goguen	$I_{\text{prod}}(a,b) = \begin{cases} 1, & \text{if } a \leq b \\ b/a, & \text{if } a > b \end{cases}$	ab
Łukasiewicz	$I_{Ł}(a,b) = \min(1, 1 - a + b)$	$\max(0, a + b - 1)$
largest	$I_{\mathrm{L}}(a,b) = \begin{cases} b, & \text{if } a = 0 \\ 1, & \text{otherwise} \end{cases}$	not defined

- I_{L} is actually the limit of all *R*-implications
- It serves as least upper bound
- It cannot be defined by

$$I(a,b) = \sup\{x \in [0,1] \mid \top(a,x) \leq b\}$$

QL-Implications

- Implications based on $I(a, b) = \perp(\sim a, \top(a, b))$ are called *QL*-implications (*QL* from quantum logic)

- Well-known *QL*-implications based on $\sim a = 1 - a$
  1. Standard min and max lead to Zadeh implication
  $$I_{\mathrm{Z}}(a, b) = \max[1 - a, \min(a, b)]$$
  2. The algebraic product and sum lead to
  $$I_{\mathrm{p}}(a, b) = 1 - a + a^2 b$$
  3. Using $\top_{Ł}$ and $\perp_{Ł}$ leads to Kleene-Dienes implication again
  4. Using $\top_{-1}$ and $\perp_{-1}$ leads to
  $$I_{\mathrm{q}}(a, b) = \begin{cases} b, & \text{if } a = 1 \\ 1 - a, & \text{if } a \neq 1, \quad b \neq 1 \\ 1, & \text{if } a \neq 1, \quad b = 1 \end{cases}$$

Fuzzy Logic Implications

- All I come from generalizations of the classical implication.
- They collapse to the classical implication when truth values are 0 or 1

Which Fuzzy Implication?

- Since the meaning of I is not unique, we must resolve the question: Which I should be used for calculating the fuzzy relation R?

- Hence meaningful criteria are needed

- They emerge from various fuzzy inference rules, i.e. modus ponens, modus tollens, hypothetical syllogism