Artificial Intelligence

Fuzzy Logic

Lesson 2: Theory of Fuzzy Logic

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The Extension Principle

Motivation (1)

- How to extend $\phi : X^n \rightarrow Y$ to $\widehat{\phi}$: $\mathcal{F}(X)n \rightarrow \mathcal{F}(Y)$? From functions defined on sets to functions defined on fuzzy sets
- Let $\mu \in \mathcal{F}(\mathbb{R})$ be a fuzzy set of the imprecise concept "about 2"

  Then the degree of membership $\mu(2.2)$ can be seen as truth value of the statement "2.2 is about equal to 2"

  Let $\mu' \in \mathcal{F}(\mathbb{R})$ be a fuzzy set of the imprecise concept "old"

  Then the truth value of "2.2 is about equal 2 and 2.2 is old" can be seen as membership degree of 2.2 w.r.t. imprecise concept "about 2 and old"

Motivation (2): Operating on Truth Values

- Any triangular norm (t-norm) $\top$ can be used to represent conjunction
- Any triangular co-norm (t-conorm) $\perp$ can be used to represent disjunction
- However, now only $\top_{\min}$ and $\perp_{\max}$ will be used

- Let $\mathcal{P}$ be set of imprecise statements that can be combined by the operators "and" and "or":
  - $\mathrm{truth}: \mathcal{P} \to [0,1]$ assigns truth value $\mathrm{truth}(a)$ to every $a \in \mathcal{P}$
  - $\mathrm{truth}(a) = 0$ means a is definitely false
  - $\mathrm{truth}(a) = 1$ means a is definitely true
  - If $0 < \mathrm{truth}(a) < 1$, then only gradual truth of statement a

Motivation (3): Extension Principle

- Combination of two statements $a, b \in \mathcal{P}$
  - $\text{truth}(a \text{ and } b) = \text{truth}(a \wedge b) = \min\{\text{truth}(a), \text{truth}(b)\}$
  - $\text{truth}(a \text{ or } b) = \text{truth}(a \vee b) = \max\{\text{truth}(a), \text{truth}(b)\}$
- For infinite number of statements $a_i, i \in I$:
  - $\text{truth}(\forall i \in I : a_i) = \inf\{\text{truth}(a_i) \mid i \in I\}$
  - $\text{truth}(\exists i \in I : a_i) = \sup\{\text{truth}(a_i) \mid i \in I\}$
- This concept helps to extend $\phi : X^n \to Y$ to $\hat{\phi}: \mathcal{F}(X)^n \to \mathcal{F}(Y)$
- Crisp tuple $(x_1, \ldots, x_n)$ is mapped to crisp value $\phi(x_1, \ldots, x_n)$
- Imprecise descriptions $(\mu_1, \ldots, \mu_n)$ of $(x_1, \ldots, x_n)$ are mapped to fuzzy value $\hat{\phi}(\mu_1, \ldots, \mu_n)$

Example: How to extend the addition? (1)

- $+: \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \ (a, b) \mapsto a + b$
- Extensions to sets:

$$+: 2^{\mathbb{R}} \times 2^{\mathbb{R}} \to 2^{\mathbb{R}}$$

$$(A, B) \mapsto A + B$$

$$= \{y \mid (\exists a)(\exists b)(y = a + b) \wedge (a \in A) \wedge (b \in B)\}$$

- Extensions to fuzzy sets:

$$+: \mathcal{F}(\mathbb{R}) \times \mathcal{F}(\mathbb{R}) \to \mathcal{F}(\mathbb{R}), (\mu_1, \mu_2) \mapsto \mu_1 \oplus \mu_2$$

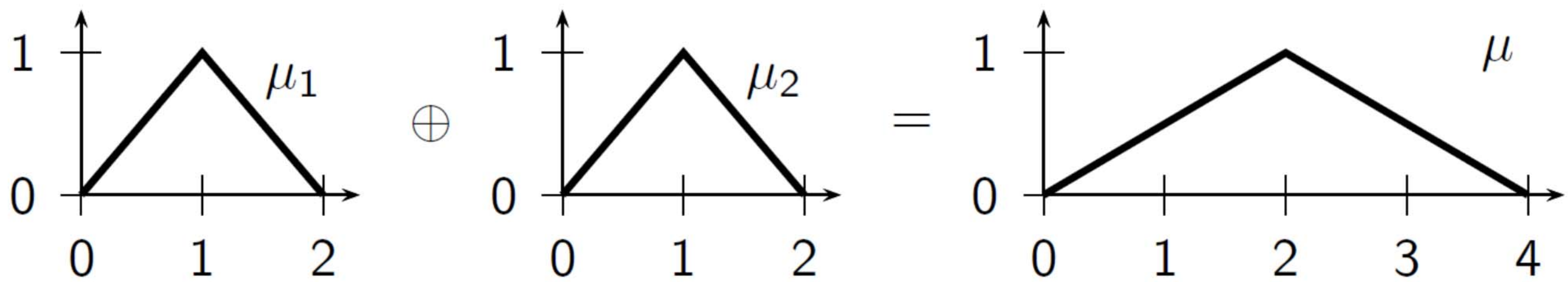
$$\mathrm{truth}(y \in \mu_1 \oplus \mu_2)$$

$$= \mathrm{truth}((\exists a)(\exists b) : (y = a + b) \wedge (a \in \mu_1) \wedge (b \in \mu_2))$$

$$= \sup_{a,b} \{\mathrm{truth}(y = a + b) \wedge \mathrm{truth}(a \in \mu_1) \wedge \mathrm{truth}(b \in \mu_2)\}$$

$$= \sup_{a,b: y=a+b} \{\min(\mu\_1(a), \mu\_2(b))\}$$

Example: How to extend the addition? (2)



- $\mu(2) = 1$ because $\mu_1(1) = 1$ and $\mu_2(1) = 1$
- $\mu(5) = 0$ because if $a + b = 5$,
  then $\min\{\mu_1(a), \mu_2(b)\} = 0$
- $\mu(1) = 0.5$ because e.g. $a = 0.5$ and $b = 0.5$

Extension to Sets

- Definition

  Let $\phi : X^n \rightarrow Y$ be a mapping

  The extension $\hat{\phi}$ of ϕ is given by

  $\hat{\phi} : [2^X]^n \rightarrow 2^Y$ with

$$\hat{\phi}(A_1, \ldots, A_n) = \{y \in Y \mid \exists (x_1, \ldots, x_n) \in A_1 \times \ldots \times A_n : \phi(x_1, \ldots, x_n) = y\}$$

Extension to Fuzzy Sets

- Definition

  Let $\phi : X^n \rightarrow Y$ be a mapping

  The extension $\hat{\phi}$ of ϕ is given by

  $\hat{\phi} : [\mathcal{F}(X)]^n \rightarrow \mathcal{F}(Y)$ with

$$\hat{\phi}(A_1, \ldots, A_n) = \sup \{\min\{\mu_1(x_1), \ldots, \mu_n(x_n)\} \mid (x_1, \ldots, x_n) \in X^n \wedge \phi(x_1, \ldots, x_n) = y\}$$

  assuming that $\sup \emptyset = 0$

Example (1)

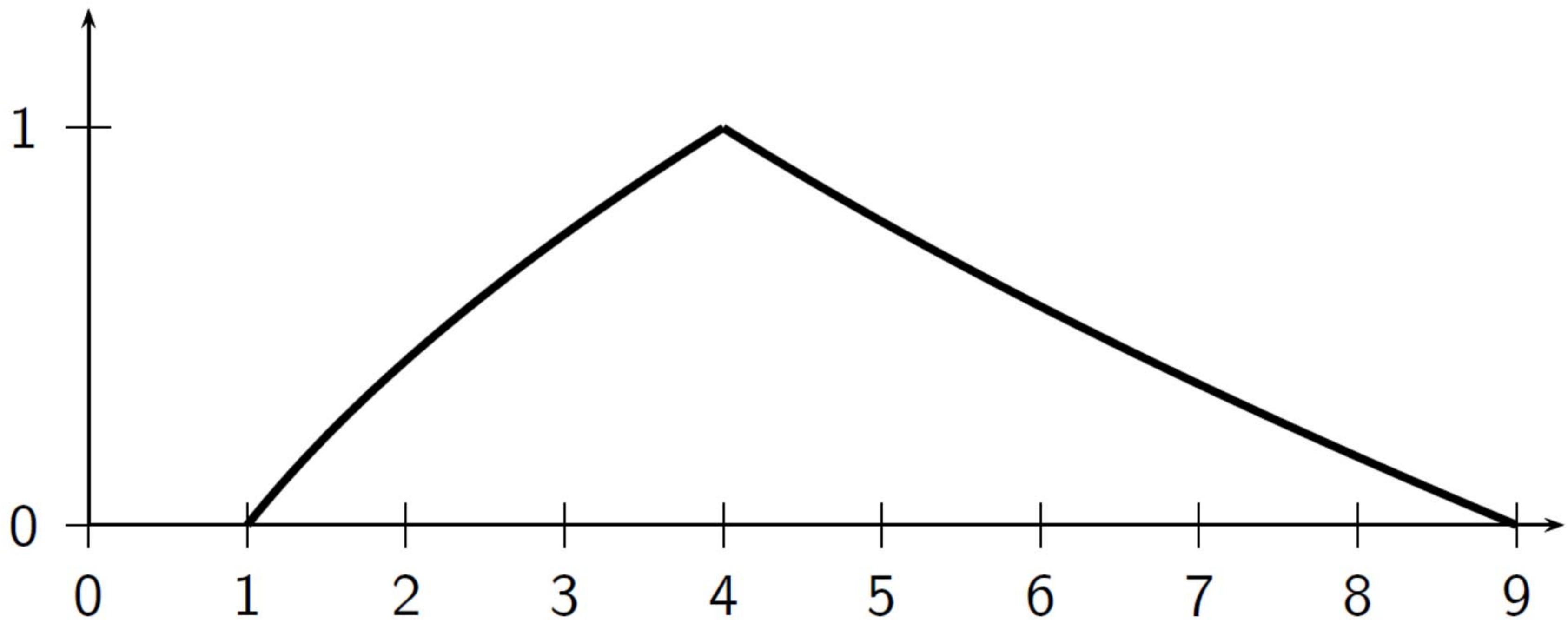
- Let fuzzy set "approximately 2" be defined as

$$\mu(x) = \begin{cases} x - 1, & \text{if } 1 \le x \le 2 \\ 3 - x, & \text{if } 2 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$$

- The extension of $\phi : \mathbb{R} \to \mathbb{R}, \; x \longmapsto x^2$ to fuzzy sets on $\mathbb{R}$ is

$$\widehat{\phi}(\mu)(y) = \sup\{\mu(x) \mid x \in \mathbb{R} \wedge x^2 = y\}$$
$$= \begin{cases} \sqrt{y} - 1, & \text{if } 1 \le y \le 4 \\ 3 - \sqrt{y}, & \text{if } 4 \le y \le 9 \\ 0, & \text{otherwise} \end{cases}$$

Example (2)



- The extension principle is taken as basis for "fuzzifying" the whole theory

Fuzzy Arithmetic

Fuzzy Sets on the Real Numbers

- There are many different types of fuzzy sets
- Consider fuzzy sets defined on set $\mathbb{R}$ of real numbers
- Membership functions of such sets, i.e.

$$\mu : \mathbb{R} \rightarrow [0, 1]$$

  clearly indicate quantitative meaning
- Such concepts may essentially characterize states of fuzzy variables
- They play important role in many applications, e.g. fuzzy control, decision making, approximate reasoning, optimization, and statistics with imprecise probabilities

Some Special Fuzzy Sets (1)

- Some special classes $\mathcal{F}(\mathbb{R})$ of fuzzy sets μ on $\mathbb{R}$

- Definition
  - Normal Fuzzy Set

    $\mathcal{F}_N(\mathbb{R}) =^{\text{def}} \{\mu \in \mathcal{F}(\mathbb{R}) \mid \exists x \in \mathbb{R} : \mu(x) = 1\}$
  - Upper Semi-Continuous Fuzzy Set

    $\mathcal{F}_C(\mathbb{R}) =^{\text{def}} \{\mu \in \mathcal{F}_N(\mathbb{R}) \mid \forall \alpha \in (0,1] : [\mu]_\alpha \text{ is compact}\}$
  - Fuzzy Intervals

    $$F_I(\mathbb{R}) =^{def} \{\mu \in \mathcal{F}_N(\mathbb{R}) \mid \forall a,b,c \in \mathbb{R} : c \in [a,b] \Rightarrow \mu(c) \geq \min\{\mu(a),\mu(b)\}\}$$

Some Special Fuzzy Sets (2)

Normal Fuzzy Set

$$\mathcal{F}_N(\mathbb{R}) \stackrel{\text{def}}{=} \{\mu \in \mathcal{F}(\mathbb{R}) \mid \exists x \in \mathbb{R} : \mu(x) = 1\}$$

- An element in $\mathcal{F}_N(\mathbb{R})$ is called normal fuzzy set:
  - It's meaningful if $\mu \in \mathcal{F}_N(\mathbb{R})$ is used as imprecise description of an existing (but not precisely measurable) variable $\subseteq \mathbb{R}$
  - In such cases it would not be plausible to assign maximum membership degree of 1 to no single real number

Some Special Fuzzy Sets (3)

Upper Semi-Continuous Fuzzy Set

$$\mathcal{F}_C(\mathbb{R}) =^{\text{def}} \{\mu \in \mathcal{F}_N(\mathbb{R}) \mid \forall\, \alpha \in (0,1] : [\mu]_\alpha \text{ is compact}\}$$

- Sets in $\mathcal{F}_C(\mathbb{R})$ are upper semi-continuous:
  - Function f is upper semi-continuous at point x_0 if values near x_0 are either close to $f\,(x_0)$ or less than

    $$f\,(x_0) \Rightarrow \lim_{x \to x_0} \sup f(x) \le f(x_0)$$
  - This simplifies arithmetic operations applied to them

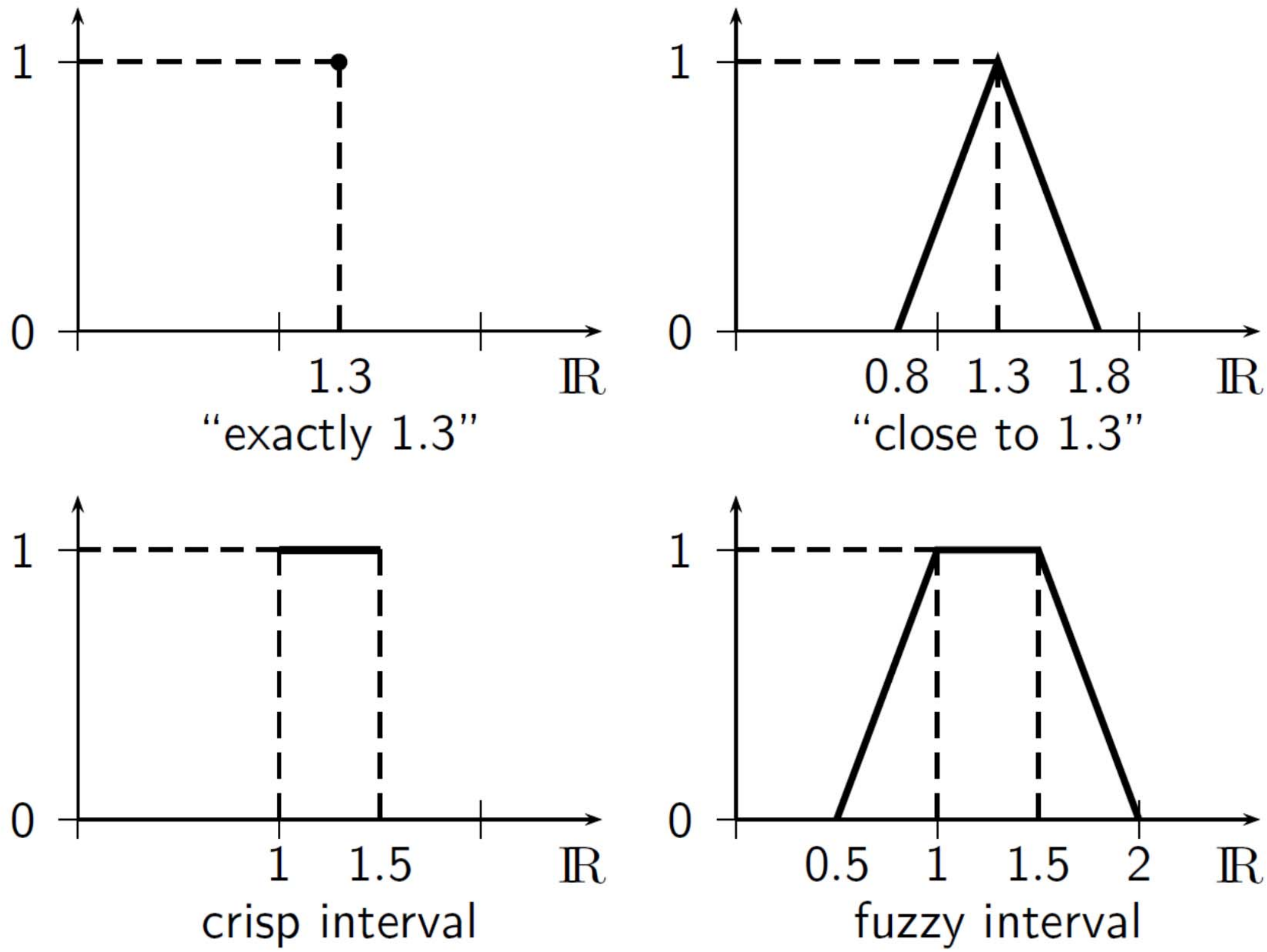
Some Special Fuzzy Sets (4)

Fuzzy Intervals

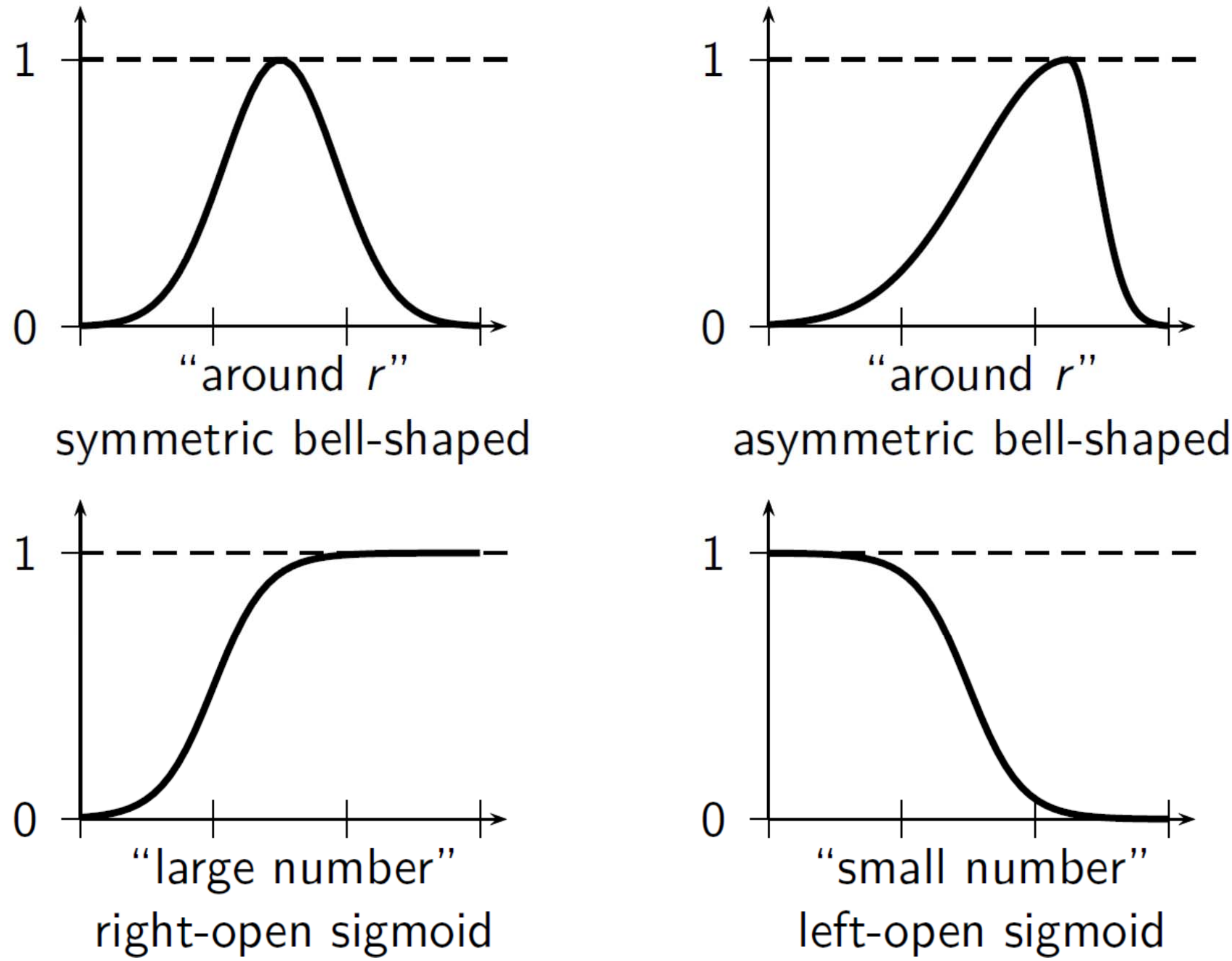
$$F_I(\mathbb{R}) =^{def} \{\mu \in \mathcal{F}_N(\mathbb{R}) \mid \forall\, a, b, c \in \mathbb{R} : c \in [a, b] \Rightarrow \mu(c) \geq \min\{\mu(a), \mu(b)\}\}$$

- Fuzzy sets in $F_I(\mathbb{R})$ are called fuzzy intervals
  - The are normal and fuzzy convex
  - Their core is a classical interval
  - $\mu \in F_I(\mathbb{R})$ for real numbers are called fuzzy numbers

Comparison of Crisp Sets and Fuzzy Sets on $\mathbb{R}$



Basic Types of Fuzzy Numbers

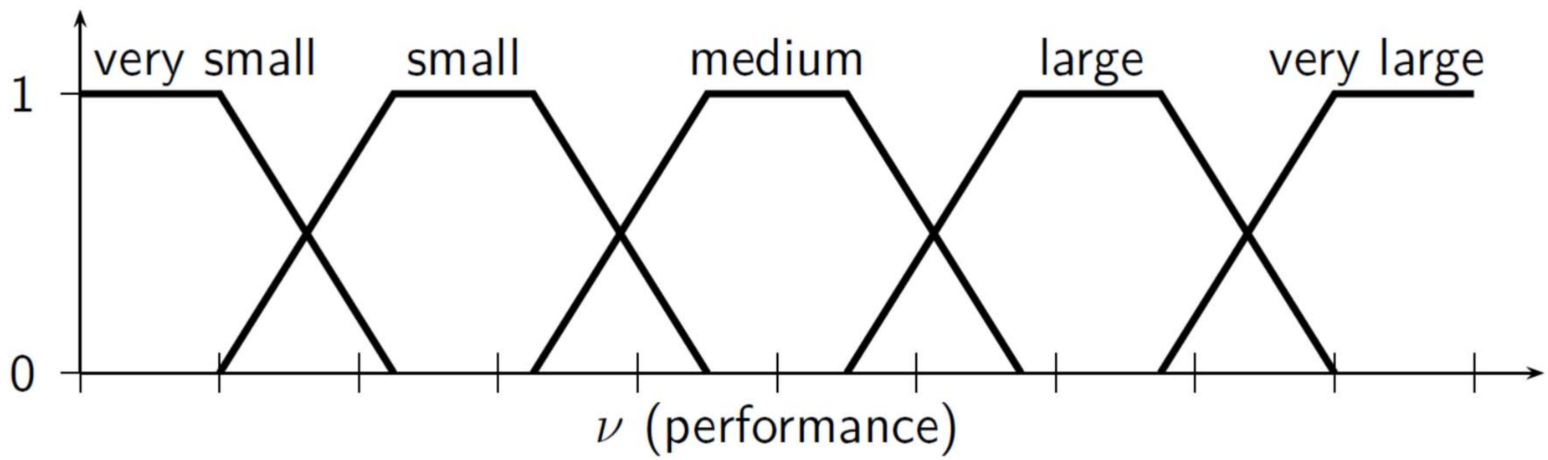


Linguistic Information Processing

Quantitative Fuzzy Variables

- The concept of a fuzzy number plays fundamental role in formulating quantitative fuzzy variables
- These are variables whose states are fuzzy numbers
- When the fuzzy quantities represent linguistic concepts, e.g. very small, small, medium, etc. then fuzzy variables are called linguistic variables
- Each linguistic variable is defined in terms of base variable which is a variable in classical sense, e.g. temperature, pressure, age
- Linguistic terms representing approximate values of base variable are captured by appropriate fuzzy numbers

Linguistic Variables



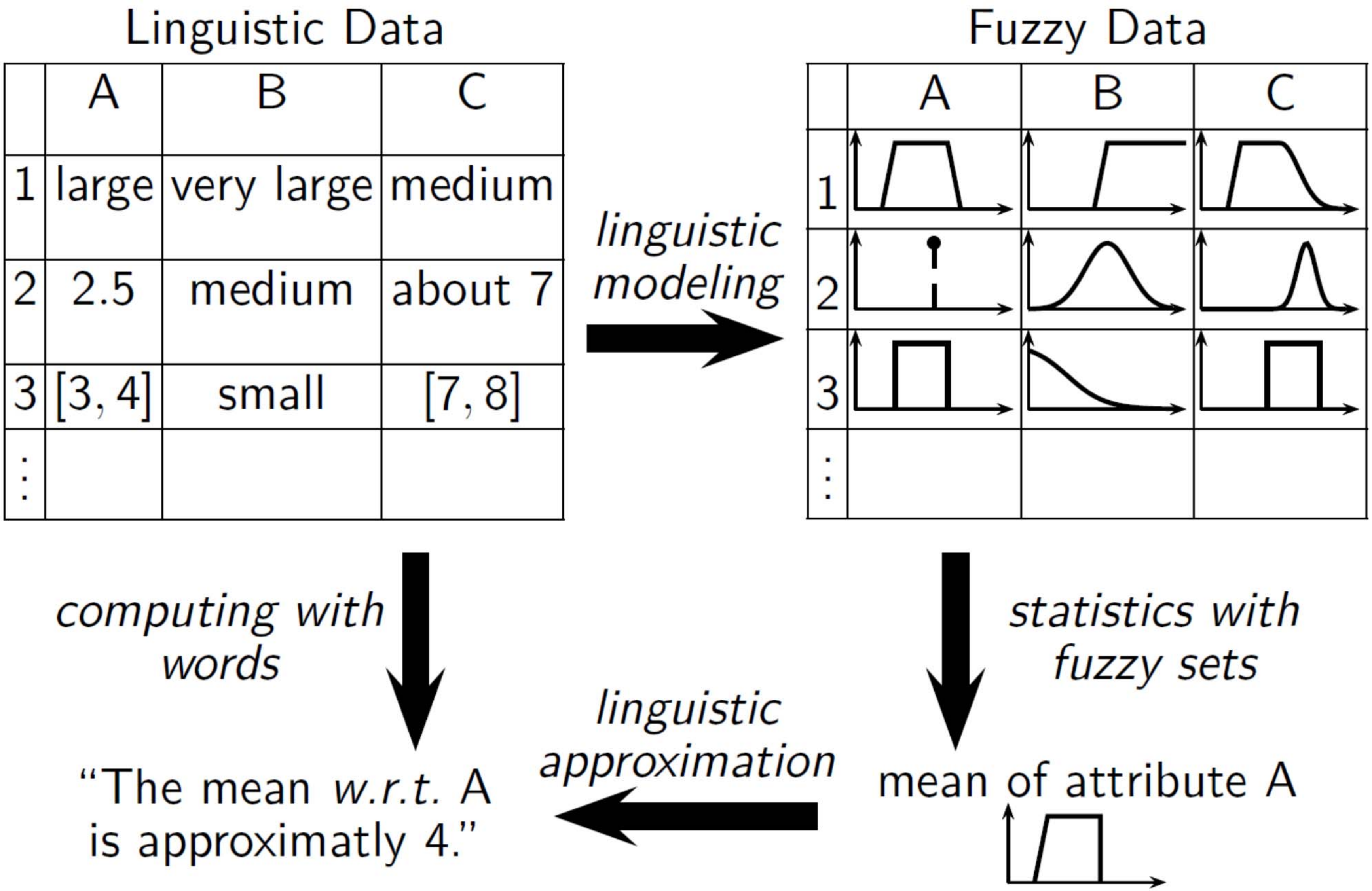
- Each linguistic variable is defined by quintuple (ν, T, X, g, m)
  - name ν of the variable
  - set T of linguistic terms of ν
  - base variable $X \subseteq \mathbb{R}$
  - syntactic rule g (grammar) for generating linguistic terms
  - semantic rule m that assigns meaning $m(t)$ to every $t \in T$, i.e. $m : T \to \mathcal{F}(X)$

Operations on Linguistic Variables

- To deal with linguistic variables, consider
  - not only set-theoretic operations
  - but also arithmetic operations on fuzzy numbers (i.e. interval arithmetic)

- e.g. statistics:
  - Given a sample = (*small*, *medium*, *small*, *large*, ...).
  - How to define mean value or standard deviation?

Analysis of Linguistic Data



Example: Application of Linguistic Data

- Consider the problem to model the climatic conditions of several towns
- A tourist may want information about tourist attractions
- Assume that linguistic random samples are based on subjective observations of selected people, e.g.
  - climatic attribute clouding
  - linguistic values cloudless, clear, fair, cloudy, . . .

Example: Linguistic Modeling by an Expert

- The attribute clouding is modeled by elementary linguistic values, e.g.

$$
\begin{aligned}
\text{cloudless} &\longmapsto \text{sigmoid}(0, -0.07) \\
\text{clear} &\longmapsto \text{Gauss}(25, 15) \\
\text{fair} &\longmapsto \text{Gauss}(50, 20) \\
\text{cloudy} &\longmapsto \text{Gauss}(75, 15) \\
\text{overcast} &\longmapsto \text{sigmoid}(100, 0.07) \\
\text{exactly)}(x) &\longmapsto \text{exact}(x) \\
\text{approx)}(x) &\longmapsto \text{Gauss}(x,\ 3) \\
\text{between}(x,\ y) &\longmapsto \text{rectangle}(x,\ y) \\
\text{approx\_between}(x,\ y) &\longmapsto \text{trapezoid}(x\ -\ 20, x, y, y\ +\ 20)
\end{aligned}
$$

where $x, y\ \in\ [0, 100]\ \subset\ \mathbb{R}$.

Example

- $\mathrm{Gauss}(a, b)$ is, e.g. a function defined by

$$f(x) = \exp\left(-\left(\frac{x - a}{b}\right)^2\right), \qquad x, a, b \in \mathbb{R}, b > 0$$

- induced language of expressions:

  <expression> := <elementary linguistic value> |

  (<expression>) |

  { not | dil | con | int } <expression> |

  <expression> { and | or } <expression>

- dil = dilatation , con = concentration

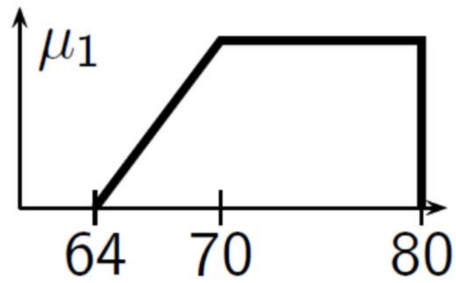
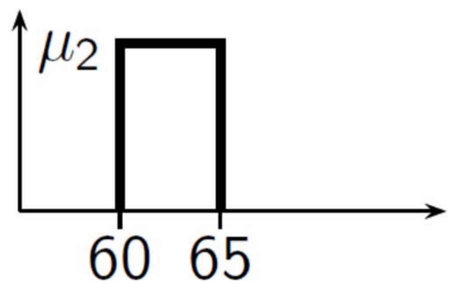
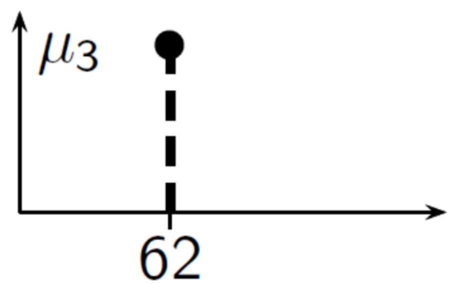
- e.g. $approx(x)$ and $cloudy$ is represented by function $\min\{\mathrm{Gauss}(x, 3), \mathrm{Gauss}(75, 15)\}$

Example – Linguistic Random Sample

Attribute	:	Clouding
Observations	:	Limassol, Cyprus
2009/10/23	:	cloudy
2009/10/24	:	dil approx_between(50, 70)
2009/10/25	:	fair or cloudy
2009/10/26	:	approx(75)
2009/10/27	:	dil(clear or fair)
2009/10/28	:	int cloudy
2009/10/29	:	con fair
2009/11/30	:	approx(0)
2009/11/31	:	cloudless
2009/11/01	:	cloudless or dil clear
2009/11/02	:	overcast
2009/11/03	:	cloudy and between(70, 80)
. . .	:	. . .
2009/11/10	:	clear

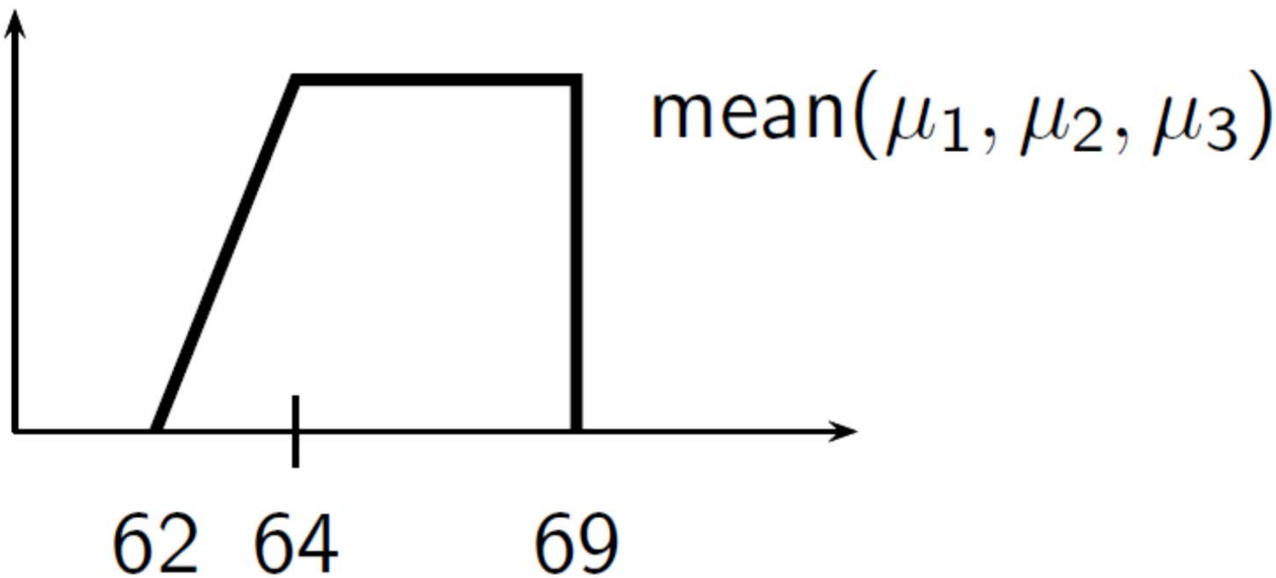
- Statistics with fuzzy sets are necessary to analyze linguistic data

Example: Linguistic Random Sample of 3 People

no.	age (linguistic data)	age (fuzzy data)
1	approx. between 70 and 80 and definitely not older than 80	
2	between 60 and 65	
3	62	

Example:
Mean Value of Linguistic Random Sample

$$mean(\mu 1, \mu 2, \mu 3) = \frac{1}{3} (\mu 1 \oplus \mu 2 \oplus \mu 3)$$



i.e. approximately between 64 and 69 but not older than 69

Efficient Operations (1)

- How to define arithmetic operations for calculating with $\mathcal{F}(\mathbb{R})$?
- Using extension principle for sum $\mu \oplus \mu'$, product $\mu \odot \mu'$ and reciprocal value $\mathrm{rec}(\mu)$ of arbitrary fuzzy sets $\mu, \mu' \in \mathcal{F}(\mathbb{R})$

$$(\mu \oplus \mu')(t) = \sup\{\min\{\mu(x_1), \mu'(x_2)\} \mid x_1, x_2 \in \mathbb{R}, x_1 + x_2 = t\}$$

$$(\mu \odot \mu')(t) = \sup\{\min\{\mu(x_1), \mu'(x_2)\} \mid x_1, x_2 \in \mathbb{R}, x_1 \cdot x_2 = t\}$$

$$\mathrm{rec}(\mu)(t) = \sup\left\{\mu(x) \mid x \in \mathbb{R} \setminus \{0\}, \frac{1}{x} = t\right\}$$

- In general, operations on fuzzy sets are much more complex (especially if vertical instead of horizontal representation is applied)
- It's desirable to reduce fuzzy arithmetic to ordinary set arithmetic
- Then, we apply elementary operations of interval arithmetic

Efficient Operations (2)

- Definition

  A family $(A_\alpha)_{\alpha \in (0,1)}$ of sets is called set representation of $\mu \in \mathcal{F}_N(\mathbb{R})$ if

  a) $0 < \alpha < \beta < 1 \Longrightarrow A_\alpha \subseteq A_\beta \subseteq \mathbb{R}$ and

  b) $\mu(t) = \sup\{\alpha \in [0,1] \,|\, t \in A_\alpha\}$

  holds where $\sup \emptyset = 0$

- Theorem

  Let $\mu \in \mathcal{F}_N(\mathbb{R})$

  The family $(A_\alpha)_{\alpha \in (0,1)}$ of sets is a set representation of μ if and only if

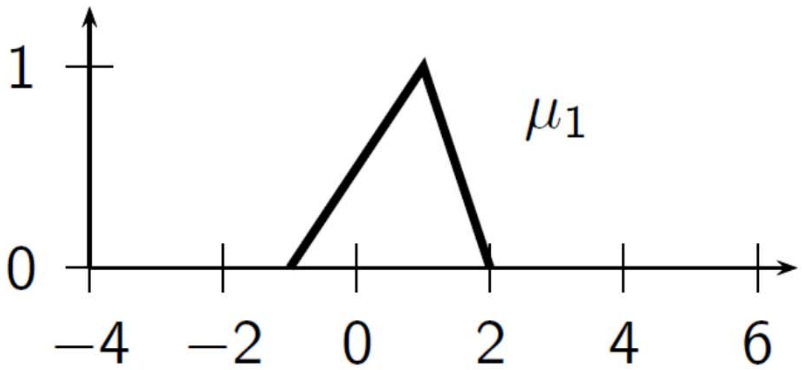
  $[\mu]_{\underline{\alpha}} = \{t \in \mathbb{R} \,|\, \mu(t) > \alpha\} \subseteq A_\alpha \subseteq \{t \in \mathbb{R} \,|\, \mu(t) \geq \alpha\} = [\mu]_\alpha$

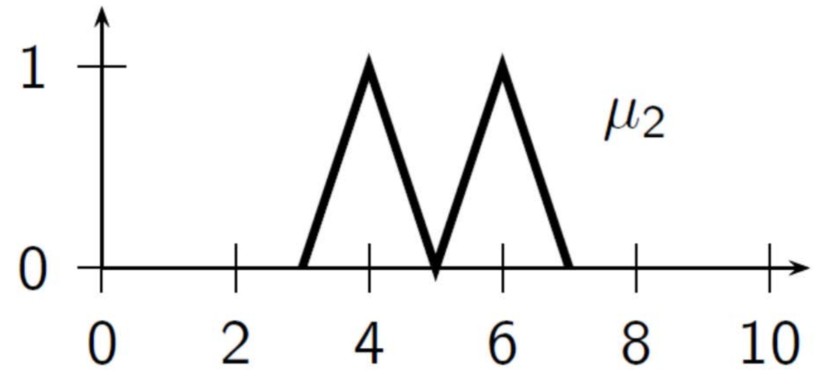
  is valid for all $\alpha \in (0,1)$

Efficient Operations

- Let $\mu_1, \mu_2, \ldots, \mu_n$ be normal fuzzy sets of $\mathbb{R}$ and $\phi : \mathbb{R}^n \to \mathbb{R}$ be a mapping. Then the following holds
  - $\forall \alpha \in [0,1) : \left[\hat{\phi}(\mu_1, \ldots, \mu_n)\right]_{\underline{\alpha}} = \phi([\mu_1]_{\underline{\alpha}}, \ldots, [\mu_n]_{\underline{\alpha}}),$
  - $\forall \alpha \in (0,1] : \left[\hat{\phi}(\mu_1, \ldots, \mu_n)\right]_{\alpha} \supseteq \phi([\mu_1]_{\alpha}, \ldots, [\mu_n]_{\alpha}),$
  - if $(A_\alpha)_{\alpha \in (0,1)}$ is a set representation of μ_i for $1 \leq i \leq n$, then $\left(\phi\left((A_1)_\alpha, \ldots, (A_n)_\alpha\right)\right)_{\alpha \in (0,1)}$ is a set representation of $\hat{\phi}(\mu_1, \ldots, \mu_n)$
- For arbitrary mapping ϕ, set representation of its extension $\hat{\phi}$ can be obtained with help of set representation $\left((A_i)_\alpha\right)_{\alpha \in (0,1)}, i = 1, 2, \ldots, n$
- It's used to carry out arithmetic operations on fuzzy sets efficiently

Example (1)



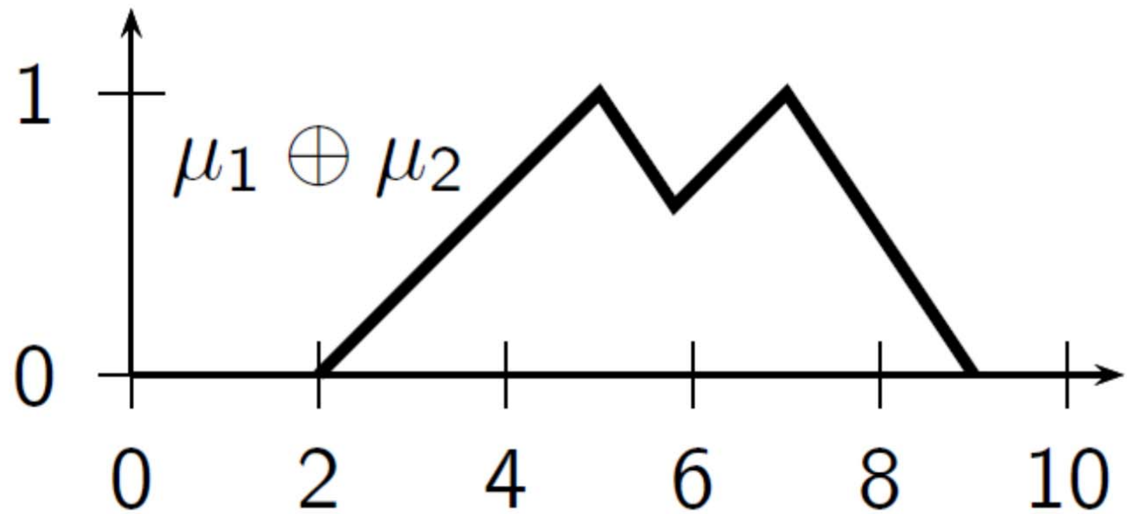


- For μ_1, μ_2, the set representations are
  - $[\mu_1]_\alpha = [2\alpha - 1, 2 - \alpha]$
  - $[\mu_2]_\alpha = [\alpha + 3, 5 - \alpha] \cup [\alpha + 5, 7 - \alpha]$
- Let $\mathrm{add}(x, y) = x + y$, then $(A_\alpha)_{\alpha \in (0,1)}$ represents $\mu_1 \oplus \mu_2$

$$A_\alpha = \mathrm{add}([\mu_1]_\alpha, [\mu_2]_\alpha) = [3\alpha + 2, 7 - 2\alpha] \cup [3\alpha + 4, 9 - 2\alpha]$$

$$= \begin{cases} [3\alpha + 2, 7 - 2\alpha] \cup [3\alpha + 4, 9 - 2\alpha], & \text{if } \alpha \geq 0.6 \\ [3\alpha + 2, 9 - 2\alpha], & \text{if } \alpha \leq 0 \end{cases}$$

Example (2)



$$(\mu 1 \oplus \mu 2)(x) = \begin{cases} \frac{x-2}{3}, & \text{if } 2 \leq x \leq 5 \\ \frac{7-x}{2}, & \text{if } 5 \leq x \leq 5.8 \\ \frac{x-4}{3}, & \text{if } 5.8 \leq x \leq 7 \\ \frac{9-x}{2}, & \text{if } 7 \leq x \leq 9 \\ 0, & \text{otherwise} \end{cases}$$

Interval Arithmetic (1)

- Determining the set representations of arbitrary combinations of fuzzy sets can be reduced very often to simple interval arithmetic
- Using fundamental operations of arithmetic leads to the following $(a, b, c, d \in \mathbb{R})$

$$[a, b] + [c, d] = [a + c, b + d]$$

$$[a, b] - [c, d] = [a - d, b - c]$$

$$[a, b] \cdot [c, d] = \begin{cases} [ac, bd], & \text{for } a \geq 0 \wedge c \geq 0 \\ [bd, ac], & \text{for } b < 0 \wedge d < 0 \\ [\min\{ad, bc\}, \max\{ad, bc\}], & \text{for } ab \geq 0 \wedge cd \geq 0 \wedge ac < 0 \\ [\min\{ad, bc\}, \max\{ac, bd\}], & \text{for } ab < 0 \vee cd < 0 \end{cases}$$

$$\frac{1}{ab} = \begin{cases} \left[\frac{1}{b}, \frac{1}{a}\right], & \text{if } 0 \notin [a, b] \\ \left[\frac{1}{b}, \infty\right) \cup \left(-\infty, \frac{1}{a}\right], & \text{if } a < 0 \wedge b > 0 \\ \left[\frac{1}{b}, \infty\right), & \text{if } a = 0 \wedge b > 0 \\ \left(-\infty, \frac{1}{a}\right], & \text{if } a < 0 \wedge b = 0 \end{cases}$$

Interval Arithmetic (2)

- In general, set representation of α-cuts of extensions $\hat{\phi}(\mu_1,\ldots,\mu_n)$ cannot be determined directly from α-cuts.
- It only works always for continuous ϕ and fuzzy sets in $\mathcal{F}_C(\mathbb{R})$
- Theorem
  - Let $\mu_1,\mu_2,\ldots,\mu_n \in \mathcal{F}_C(\mathbb{R})$ and $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous mapping
  - Then $\forall\alpha \in (0,1] : \left[\hat{\phi}(\mu_1,\ldots,\mu_n)\right]_\alpha = \phi([\mu_1]_\alpha,\ldots,[\mu_n]_\alpha)$
  - So, a horizontal representation is better than a vertical one
  - Finding $\hat{\phi}$ values is easier than directly applying the extension principle
  - However, all α-cuts cannot be stored in a computer
  - Only a finite number of α-cuts can be stored

Fuzzy Relations

Motivation

- A **crisp relation** represents presence or absence of association, interaction or interconnection between elements of ≥ 2 sets
  - This concept can be generalized to various degrees or strengths of association or interaction between elements
- A **fuzzy relation** generalizes these degrees to membership grades
  - So, a crisp relation is a restricted case of a fuzzy relation

Definition of Relation

- A relation among crisp sets $X_1,\ldots,X_n$ is a subset of $X_1 \times \ldots \times X_n$ denoted as $R(X_1,\ldots,X_n)$ or $R(X_i \mid 1 \leq i \leq n)$
- So, the relation $R(X_1,\ldots,X_n) \subseteq X_1 \times \ldots \times X_n$ is set, too

- The basic concept of sets can be also applied to relations:
  - containment, subset, union, intersection, complement

- Each crisp relation can be defined by its characteristic function

$$R(x_1,\ldots,x_n) = \begin{cases} 1, & \text{if and only if } (x\_1,\ldots,x\_n) \in R \\ 0, & \text{otherwise} \end{cases}$$

- The membership of $(X_1,\ldots,X_n\ )$ in R means that the elements of$(X_1,\ldots,X_n\ )$ are related to each other

Relation as Ordered Set of Tuples

- A relation can be written as a set of ordered tuples
- Thus $R(X_1, \ldots, X_n)$ represents *n*-dimensional membership array $\boldsymbol{R} = [r_{i_1, \ldots, i_n}]$
  - Each element of i_1 of $\boldsymbol{R}$ corresponds to exactly one member of X_1
  - Each element of i_2 of $\boldsymbol{R}$ corresponds to exactly one member of X_2
  - And so on...
- If $(x_1, \ldots, x_n) \in X_1 \times \ldots \times X_n$ corresponds to $r_{i_1, \ldots, i_n} \in \boldsymbol{R}$, then

$$r_{i_1, \ldots, i_n} = \begin{cases} 1, & \text{if and only if } (x\_1, \ldots, x\_n) \in R \\ 0, & \text{otherwise} \end{cases}$$

Fuzzy Relations

- The characteristic function of a crisp relation can be generalized to allow tuples to have degrees of membership
- Similar to the generalization of the characteristic function of a crisp set
- A fuzzy relation is a fuzzy set defined on tuples $(x_1, \ldots, x_n)$ that may have varying degrees of membership within the relation
- The membership grade indicates strength of the present relation between elements of the tuple
- The fuzzy relation can also be represented by an n-dimensional membership array

Example

- Let R be a fuzzy relation between two sets

  X = {New York City, Paris}

  and Y = {Beijing, New York City, London}
- R shall represent relational concept "very far"
- It can be represented as two-dimensional membership array

	NYC	Paris
Beijing	1	0.9
NYC	0	0.7
London	0.6	0.3

Cartesian Product of Fuzzy Sets: *n* Dimensions

- Let $n \geq 2$ fuzzy sets $A_1,\ldots,A_n$ be defined in the universes of discourse $X_1,\ldots,X_n$, respectively
- The Cartesian product of $A_1,\ldots,A_n$ denoted by $A_1 \times \ldots \times A_n$ is a fuzzy relation in the product space $X_1 \times \ldots \times X_n$
- It is defined by its membership function

$$\mu_{A_1 \times \ldots A_n}(x_1,\ldots,x_n) = \top(\mu_{A_1}(x_1),\ldots,\mu_{A_n}(x_n))$$

  whereas $x_i \in X_i, 1 \leq i \leq n$
- Usually $\top$ is the minimum (sometimes also the product)

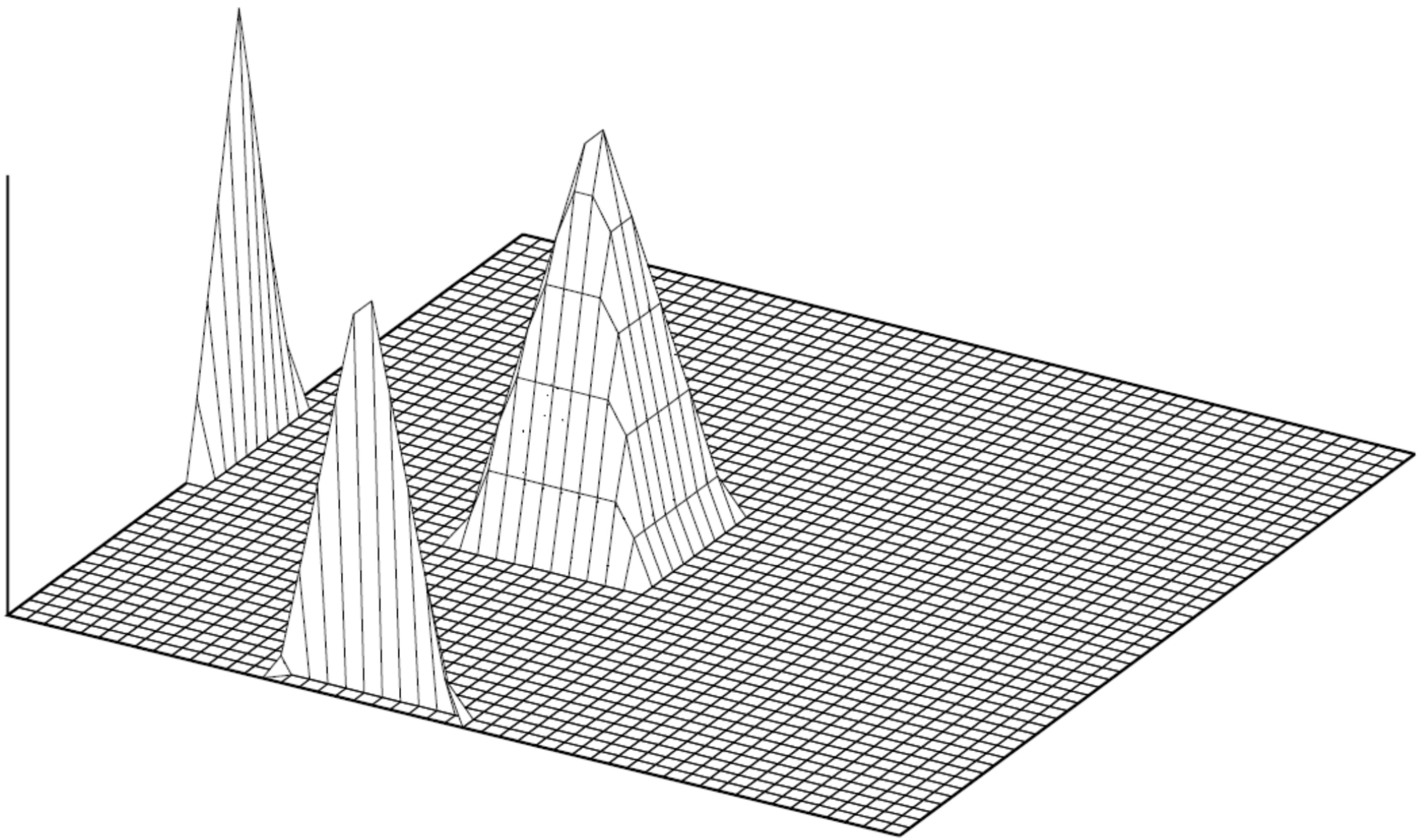
Cartesian Product of Fuzzy Sets: 2 Dimensions

- A special case of the Cartesian product is when $n = 2$

- The Cartesian product of fuzzy sets $A \in F(X)$ and $B \in F(Y)$ is a fuzzy relation $A \times B \in F(X \times Y)$ defined by

$$\mu_{A \times B}(x, y) = \top[\mu_A(x), \mu_B(y)], \forall x \in X, \forall y \in Y$$

Example: Cartesian Product in $F(X \times Y)$ with $t-\mathrm{norm} = \min$



Subsequences

- Consider the Cartesian product of all sets in the family
$$\mathcal{X} = \{X_i \mid i \in \mathbb{N}_n = \{1, 2, \ldots, n\}\}$$
- For each sequence (n-tuple)
$$\boldsymbol{x} = (x_1, \ldots, x_n) \in \times_{i \in \mathbb{N}_n} X_i$$
and each sequence (r -tuple, $r \leq n$)
$$\boldsymbol{y} = (y_1, \ldots, y_r) \in \times_{j \in J} X_j$$
where $J \subseteq \mathbb{N}_n$ and $|J| = r$
- $\boldsymbol{y}$ is called subsequence of $\boldsymbol{x}$ if and only if $y_j = x_j, \forall j \in J$
- $\boldsymbol{y} \prec \boldsymbol{x}$ denotes that $\boldsymbol{y}$ is subsequence of $\boldsymbol{x}$

Projection

- Given a relation $R(x_1, \ldots, x_n)$
- Let $[R \downarrow \mathcal{Y}]$ denote the projection of R on $\mathcal{Y}$
- It disregards all sets in X except those in the family

$$\mathcal{Y} = \{X_j \mid j \in J \subseteq \mathbb{N}_n\}$$

- Then $[R \downarrow \mathcal{Y}]$ is a fuzzy relation whose membership function is defined on the Cartesian product of the sets in

$$\mathcal{Y}[R \downarrow \mathcal{Y}](\boldsymbol{y}) = \max_{\mathbf{x} \succ \mathbf{y}} R(\boldsymbol{x}).$$

- Under special circumstances, this projection can be generalized by replacing the max operator by another t-conorm

Example

- Consider the sets $X_1 = \{0,1\}$, $X_2 = \{0,1\}$, $X_3 = \{0,1,2\}$ and the ternary fuzzy relation on $X_1 \times X_2 \times X_3$ defined as follows
- Let $R_{ij} = [R \downarrow \{X_i, X_j\}]$ and $R_i = [R \downarrow \{X_i\}]$ for all $i, j \in \{1,2,3\}$
- Using this notation, all possible projections of R are given below

$(x_1,$	$x_2,$	$x_3)$	$R(x_1,x_2,x_3)$	$R_{12}(x_1,x_2)$	$R_{13}(x_1,x_3)$	$R_{23}(x_2,x_3)$	$R_1(x_1)$	$R_2(x_2)$	$R_3(x_3)$
0	0	0	0.4	0.9	1.0	0.5	1.0	0.9	1.0
0	0	1	0.9	0.9	0.9	0.9	1.0	0.9	0.9
0	0	2	0.2	0.9	0.8	0.2	1.0	0.9	1.0
0	1	0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
0	1	1	0.0	1.0	0.9	0.5	1.0	1.0	0.9
0	1	2	0.8	1.0	0.8	1.0	1.0	1.0	1.0
1	0	0	0.5	0.5	0.5	0.5	1.0	0.9	1.0
1	0	1	0.3	0.5	0.5	0.9	1.0	0.9	0.9
1	0	2	0.1	0.5	1.0	0.2	1.0	0.9	1.0
1	1	0	0.0	1.0	0.5	1.0	1.0	1.0	1.0
1	1	1	0.5	1.0	0.5	0.5	1.0	1.0	0.9
1	1	2	1.0	1.0	1.0	1.0	1.0	1.0	1.0

Example: Detailed Calculation

- Here, only consider the projection R_{12}

$(x_1,$	$x_2,$	$x_3)$	$R(x_1, x_2, x_3)$	$R_{12}(x_1, x_2)$
0	0	0	0.4	
0	0	1	0.9	$\max[R(0,0,0), R(0,0,1), R(0,0,2)] = 0.9$
0	0	2	0.2	
0	1	0	1.0	
0	1	1	0.0	$\max[R(0,1,0), R(0,1,1), R(0,1,2)] = 1.0$
0	1	2	0.8	
1	0	0	0.5	
1	0	1	0.3	$\max[R(1,0,0), R(1,0,1), R(1,0,2)] = 0.5$
1	0	2	0.1	
1	1	0	0.0	
1	1	1	0.5	$\max[R(1,1,0), R(1,1,1), R(1,1,2)] = 1.0$
1	1	2	1.0	

Cylindric Extension

- Let $\mathcal{X}$ and $\mathcal{Y}$ denote the same families of sets as used for projection
- Let R be a relation defined on Cartesian product of sets in family $\mathcal{Y}$
- Let $[R \uparrow \mathcal{X} \backslash \mathcal{Y}]$ denote the cylindric extension of R into sets $X_1, (i \in \mathbb{N}_n)$ which are in $\mathcal{X}$ but not in $\mathcal{Y}$
- For each $\boldsymbol{x}$ with $\boldsymbol{x} \succ \boldsymbol{y}$

$$[R \uparrow \mathcal{X} \backslash \mathcal{Y}](\boldsymbol{x}) = R(\boldsymbol{y})$$

- The cylindric extension
  - produces largest fuzzy relation that is compatible with projection
  - is the least specific of all relations compatible with projection
  - guarantees that no information not included in projection is used to determine extended relation

Example

- Consider again the example for the projection
- The membership functions of the cylindric extensions of all projections are already shown in the table under the assumption that their arguments are extended to (x_1, x_2, x_3) e.g.

$$[R_{23} \uparrow \{X_1\}](0,0,2) = [R_{23} \uparrow \{X_1\}](1,0,2) = R_{23}(0,2) = 0.2$$

- In this example none of the cylindric extensions are equal to the original fuzzy relation
- This is identical with the respective projections
- Some information was lost when the given relation was replaced by any one of its projections

Cylindric Closure

- Relations that can be reconstructed from one of their projections by cylindric extension exist
- However, they are rather rare and not is more common that relation can be exactly reconstructed
  - from several of its projections (max)
  - by taking set intersection of their cylindric extensions (min)
- The resulting relation is usually called **cylindric closure**
- Let the set of projections $\{Pi \mid i \in I\}$ of a relation on $\mathcal{X}$ be given

  Then the cylindric closure $\mathrm{cyl}\{P_i\}$ is defined for each $\boldsymbol{x} \in \mathcal{X}$ as

$$\mathrm{cyl}\{P_i\}(\boldsymbol{x}) = \min_{i \in I}[P_i \uparrow \mathcal{X} \backslash \mathcal{Y}_i](\boldsymbol{x})$$

$\mathcal{Y}_i$ denotes the family of sets on which P_i is defined

Example

- The cylindric closures of three families of the projections are shown below

$(x_1,$	$x_2,$	$x_3)$	$R(x_1, x_2, x_3)$	$\mathrm{cyl}(R_{12}, R_{13}, R_{23})$	$\mathrm{cyl}(R_1, R_2, R_3)$	$\mathrm{cyl}(R_{12}, R_3)$
0	0	0	0.4	0.5	0.9	0.9
0	0	1	0.9	0.9	0.9	0.9
0	0	2	0.2	0.2	0.9	0.9
0	1	0	1.0	1.0	1.0	1.0
0	1	1	0.0	0.5	0.9	0.9
0	1	2	0.8	0.8	1.0	1.0
1	0	0	0.5	0.5	0.9	0.5
1	0	1	0.3	0.5	0.9	0.5
1	0	2	0.1	0.2	0.9	0.5
1	1	0	0.0	0.5	1.0	1.0
1	1	1	0.5	0.5	0.9	0.9
1	1	2	1.0	1.0	1.0	1.0

  - None of them is the same as the original relation R
  - So the relation R is not fully reconstructable from its projections

Binary Fuzzy Relations

Motivation and Domain

- Binary relations are significant among *n*-dimensional relations
- They are generalized mathematical functions
- On the contrary to functions from X to Y, binary relations $R(X,Y)$ may assign to each element of X two or more elements of Y
- Some basic operations on functions, e.g. inverse and composition, are applicable to binary relations as well
- Given a fuzzy binary relation $R(X,Y)$

  Its domain $\mathrm{dom}\, R$ is the fuzzy set on X whose membership function is defined for each $x \in X$ as

$$\mathrm{dom}\, R(x) = \max_{y \in Y}\{R(x,y)\}$$

  i.e. each $x \in X$ belongs to the domain of R to a degree equal to the strength of its strongest relation to any $y \in Y$

Range and Height

- The range ran of $R(X,Y)$ is a fuzzy binary relation on Y whose membership function is defined for each $y \in Y$ as

$$\mathrm{ran}\, R(y) = \max_{x \in X}\{R(x,y)\},$$

i.e. the strength of the strongest relation which each $y \in Y$ has to an $x \in X$ equals to the degree of membership of y in the range of R

- The height h of $R(X,Y)$ is a number defined by

$$h(R) = \max_{y \in Y} \max_{x \in X}\{R(x,y)\}.$$

$h(R)$ is the largest membership grade obtained by any pair $(x,y) \in R$

Representation and Inverse

- Consider e.g. the membership matrix

  $\boldsymbol{R} = [r_{xy}]$ with $r_{xy} = R(x, y)$

  Its inverse $R^{-1}(Y, X)$ of $R(X, Y)$ is a relation on $Y \times X$ defined by

  $$R^{-1}(y, x) = R(x, y), \forall x \in X, \forall y \in Y$$

  $\boldsymbol{R}^{-1} = [r_{xy}^{-1}]$ representing $R^{-1}(y, x)$ is the transpose of $\boldsymbol{R}$ for $R(X, Y)$

  $$(\boldsymbol{R}^{-1})^{-1} = \boldsymbol{R}, \forall R$$

Standard Composition

- Consider the binary relations $P(X,Y)$, $Q(Y,Z)$ with common set Y

  The standard composition of P and Q is defined as

  $(x,z) \in P \circ Q \Leftrightarrow \exists y \in Y : \{(x,y) \in P \wedge (y,z) \in Q\}$

  In the fuzzy case this is generalized by

  $[P \circ Q](x,z) = \sup_{y \in Y}\{\min\{P(x,y), Q(y,z)\}, \forall x \in X, \forall z \in Z\}$

- If Y is finite, sup operator is replaced by max, then the standard composition is also called max-min composition

Inverse of Standard Composition

- The inverse of the max-min composition follows from its definition

$$[P(X,Y) \circ Q(Y,Z)]^{-1} = Q^{-1}(Z,Y) \circ P^{-1}(Y,X)$$

- Its associativity also comes directly from its definition:

$$[P(X,Y)] \circ Q(Y,Z)] \circ R(Z,W)$$
$$= P(X,Y) \circ [Q(Y,Z) \circ R(Z,W)]$$

- Note that the standard composition is not commutative
- Matrix notation: $[r_{ij}] = [p_{ik}] \circ [q_{kj}]$
with $r_{ij} = \max_k \min(p_{ik}, q_{kj})$

Example

$$\boldsymbol{P} \circ \boldsymbol{Q} = \boldsymbol{R}$$

$$\begin{bmatrix} .3 & .5 & .8 \\ 0 & .7 & 1 \\ .4 & .6 & .5 \end{bmatrix} \circ \begin{bmatrix} .9 & .5 & .7 & .7 \\ .3 & .2 & 0 & .9 \\ 1 & 0 & .5 & .5 \end{bmatrix} = \begin{bmatrix} .8 & .3 & .5 & .5 \\ 1 & .2 & 5 & .7 \\ .5 & .4 & .5 & .5 \end{bmatrix}$$

- For instance:
  - $r_{11} = \max\{\min(p_{11}, q_{11}), \min(p_{12}, q_{21}), \min(p_{13}, q_{31})\}$
    $= \max\{\min(.3, .9), \min(.5, .3), \min(.8, 1)\} = .8$
  - $r_{32} = \max\{\min(p_{31}, q_{12}), \min(p_{32}, q_{22}), \min(p_{33}, q_{32})\}$
    $= \max\{\min(.4, .5), \min(.6, .2), \min(.5, 0)\} = .4$

Relational Join

- Relational Join yields triples (whereas composition returned pairs)

  For $P(X,Y)$ and $Q(Y,Z)$, the relational join $P * Q$ is defined by

  $$[P * Q](x,y,z) = \min\{P(x,y), Q(y,z)\}, \forall x \in X, \forall y \in Y, \forall z \in Z$$

- Then the max-min composition is obtained by aggregating the join by the maximum

  $$[P \circ Q](x,z) = \max_{y \in Y} [P * Q](x,y,z), \forall x \in X, \forall z \in Z$$

Example

- The join $S = P * Q$ of the relations P and Q has the following membership function (shown below on left-hand side)
- To convert this join into its corresponding composition $R = P \circ Q$ (shown on right-hand side)
- The two indicated pairs of $S(x, y, z)$ are aggregated using max

x	y	z	$\mu_S x, y, z$
1	a	α	.6
1	a	β	.7*
1	b	β	.5*
2	a	α	.6
2	a	β	.8
3	b	β	1
4	b	β	.4*
4	c	β	.3*

x	z	$\mu_R(x, z)$
1	α	.6
1	β	.7
2	α	.6
2	β	.8
3	β	.1
4	β	.4

- For instance,

$$R(1, \beta) = \max\{S(1, a, \beta), S(1, b, \beta)\} = \max\{.7, .5\} = .7$$

Binary Relations on a Single Set

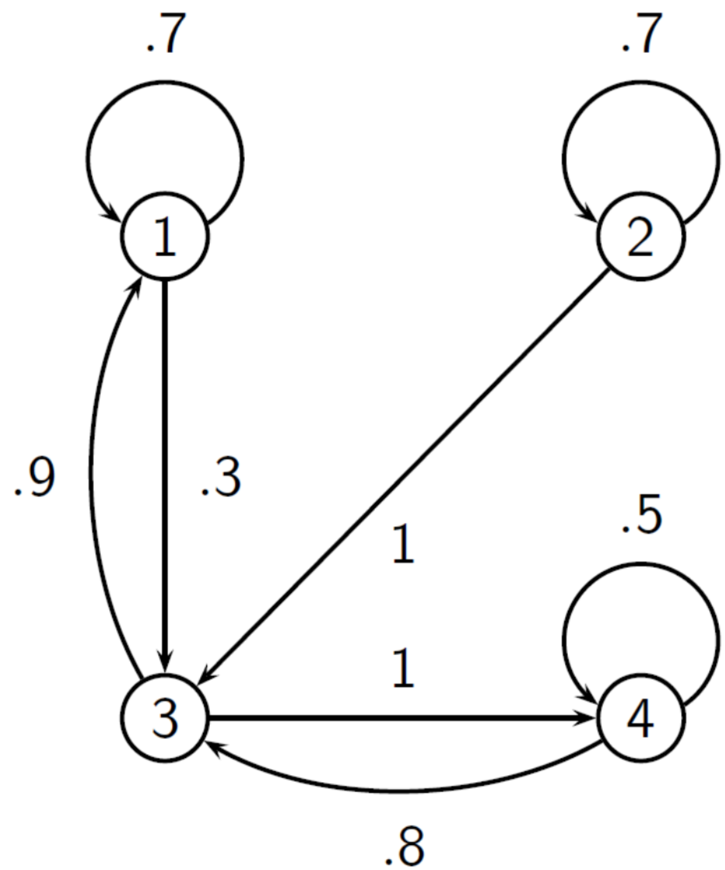
Binary Relations on a Single Set

- It is also possible to define crisp or fuzzy binary relations among elements of a single set X
- Such a binary relation can be denoted by $R(X, X)$ or $R(X^2)$ which is a subset of $X \times X = X^2$
- These relations are often referred to as directed graphs, which is also a representation of them
  - Each element of X is represented as node
  - Directed connections between nodes indicate pairs of $x \in X$ for which the grade of the membership is nonzero
  - Each connection is labeled by its actual membership grade of the corresponding pair in R

Example

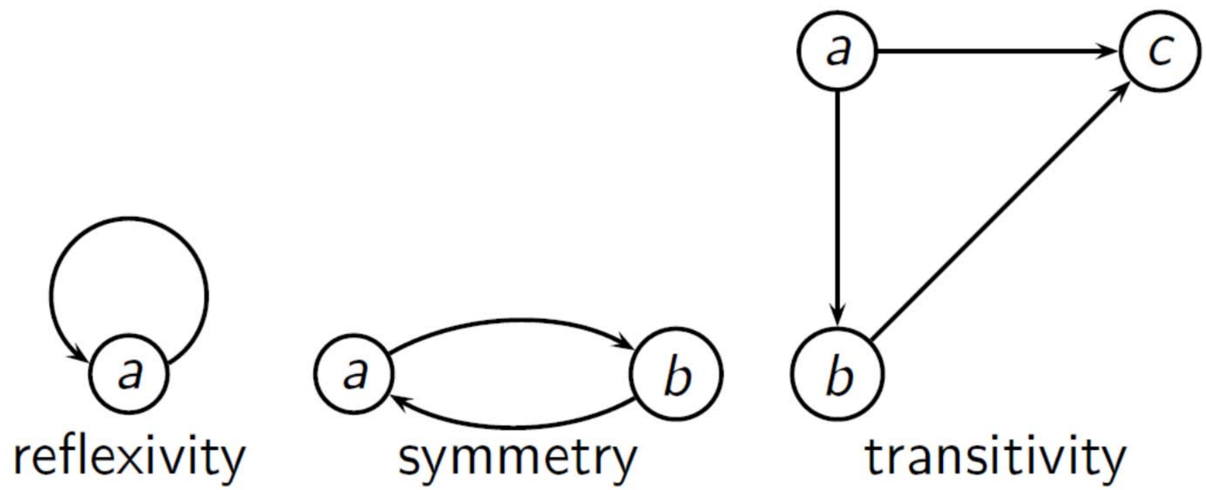
- An example of $R(X, X)$ defined on $X = \{1, 2, 3, 4\}$
- Two different representation are shown below

	1	2	3	4
1	.7	0	.3	0
2	0	.7	1	0
3	.9	0	0	1
3	0	0	.8	.5



Properties of Crisp Relations

- A crisp relation $R(X, X)$ is called
  - *reflexive* if and only if $\forall x \in X : (x, x) \in R$
  - *symmetric* if and only $if \; \forall x, y \in X : (x, y) \in R \leftrightarrow (y, x) \in R$
  - *transitive* if and only if $(x, z) \in R$ whenever both $(x, y) \in R$ and $(y, z) \in R$ for at least one $y \in X$



- All these properties are preserved under inversion of the relation

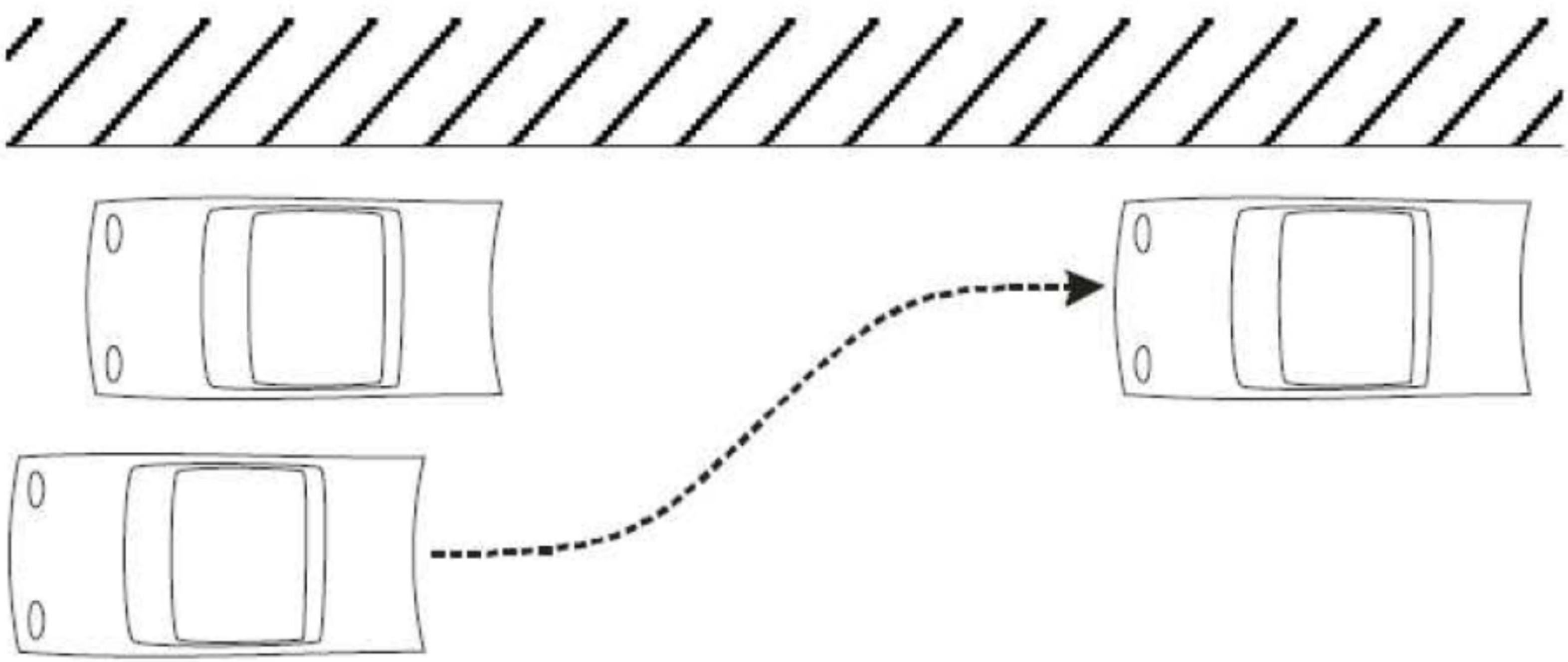
Properties of Fuzzy Relations

- These properties can be extended for fuzzy relations, by defining them in terms of the membership function of the relation
- A fuzzy relation $R(X, X)$ is called
  - *reflexive* if and only if $\forall x \in X : R(x, x) = 1$
  - *symmetric* if and only if $\forall x, y \in X : R(x, y) = R(y, x)$
  - *transitive* if it satisfies

    $$R(x, z) \geq \max_{y \in Y} \min\{R(x, y), R(y, z)\}, \ \forall (x, z) \in X^2$$
- Note that a fuzzy binary relation that is reflexive, symmetric and transitive is called fuzzy *equivalence relation*

Fuzzy Control Basics

Example – Parking a car backwards



- Questions:
  - What is the meaning of satisfactory parking?
  - Demand on precision?
  - Realization of control?

Fuzzy Control

- Biggest success of fuzzy systems in industry and commerce
- Special kind of non-linear table-based control method
- Definition of non-linear transition function can be made without specifying each entry individually
- Examples: technical systems
  - Electrical engine moving an elevator
  - Heating installation
- Goal: define certain behavior
  - Engine should maintain certain number of revolutions per minute
  - Heating should guarantee certain room temperature

Table-based Control (1)

- Control systems all share a time-dependent *output variable*
  - Revolutions per minute
  - Room temperature
- Output is controlled by *control variable*
  - Adjustment of current
  - Thermostat
- Also, *disturbance variables* influence output
  - Load of elevator, . . .
  - Outside temperature or sunshine through a window, . . .

Table-based Control (2)

- Computation of actual value incorporates both control variable measurements of current output variable ξ and change of output variable $\Delta\xi = \frac{d\xi}{dt}$

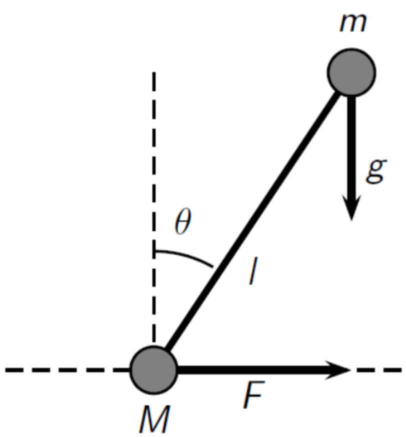
- If ξ is given in finite time intervals, then set
  $\Delta\xi(t_{n+1}) = \xi(t_{n+1}) - \xi(t_n)$.
  In this case measurement of$\Delta\xi$ not necessary

Notation

- Input variables $\xi_1,\ldots,\xi_n$, control variable η
- Measurements used to determine actual value of η
- η may specify change of η
- Assumption: ξ_i , $1 \leq i \leq n$ is value of X_i , $\eta \in Y$
- Solution: *control function* φ

$$\varphi : X_1 \times \ldots \times X_n \rightarrow Y$$
$$(x_1,\ldots,x_n) \longmapsto y$$

Example: Cartpole Problem (1)



- Balance an upright standing pole by moving its foot
- Lower end of pole can be moved unrestrained along horizontal axis
- Mass m at foot and mass M at head
- Influence of mass of shaft itself is negligible
- Determine force F (control variable) that is necessary to balance pole standing upright
- That is measurement of following output variables
  - angle θ of pole in relation to vertical axis,
  - change of angle, *i.e.* triangular velocity $\dot{\theta} = \frac{d\theta}{dt}$
- Both should converge to zero

Example: Cartpole Problem (2)

- Angle $\theta \in X_1 = [-90°,\ 90°]$
- Theoretically, every angle velocity $\dot{\theta}$ possible
- Extreme $\dot{\theta}$ are artificially achievable
- Assume $-45\,°/s \leq \dot{\theta} \leq 45\,°/s$ holds, i.e.
$$\dot{\theta} \in X_2 = [-45\,°/s, 45\,°/s]$$
- Absolute value of force $|F| \leq 10\mathrm{N}$
- Thus define $F \in Y = [-10\mathrm{N}, 10\mathrm{N}]$

Example: Cartpole Problem (3)

- Differential equation of cartpole problem

  $$(M+m)\sin^2\theta \cdot l \cdot \ddot{\theta} + m \cdot l \cdot \sin\theta\cos\theta \cdot \dot{\theta}^2 - (M+m) \cdot g \cdot \sin\theta$$
  $$= -F \cdot \cos\theta$$

- Compute $F(t)$ such that $\theta(t)$ and $\dot{\theta}(t)$ converge towards zero quickly
- Physical analysis demands knowledge about physical process

Problems of Classical Approach

- Often very difficult or even impossible to specify accurate mathematical model
- Description with differential equations is very complex
- Profound physical knowledge is needed
- Exact solution can be very difficult
- It should be possible to control a process without a physical-mathematical model

  e.g. human being knows how to ride bike without knowing existence of differential equations

Fuzzy Approach

- Simulate behavior of human who knows how to control
- That is a *knowledge-based* analysis
- Directly ask expert to perform analysis
- Then expert specifies knowledge as *linguistic rules*

  e.g. for cartpole problem:

  “If θ is approximately zero and $\dot{\theta}$ is also approximately zero, then F has to be approximately zero, too”

Fuzzy Approach: Fuzzy Partitioning (1)

- Formulate a set of linguistic rules
  - Determine linguistic terms (represented by fuzzy sets) $X_1, \ldots, X_n$ and Y is partitioned into fuzzy sets
  - Define p1 distinct fuzzy sets $\mu_1^{(1)}, \ldots, \mu_{p1}^{(1)} \in \mathcal{F}(X_1)$ on set X_1
  - Associate linguistic term with each set

Fuzzy Approach: Fuzzy Partitioning (2)

- Set X_1 corresponds to interval $[a, b]$ of real line, then $\mu_1^{(1)}, \ldots, \mu_{p1}^{(1)} \in \mathcal{F}(X_1)$ are triangular functions

$$\mu_{x_{0,\varepsilon}}\colon [\mathrm{a}, \mathrm{b}] \rightarrow [0, 1]$$

$$x \longmapsto 1 - \min\{\varepsilon \cdot |x - x_0|, 1\}$$

If $a < x_1 < \ldots < x_{p1} < b$, only $\mu_2^{(1)}, \ldots, \mu_{\mathrm{p}_1 - 1}^{(1)}$ are triangular

- Boundaries are treated differently

Fuzzy Approach: Fuzzy Partitioning (3)

- left fuzzy set:

$$\mu_1^{(1)}: [a, b] \rightarrow [0, 1]$$

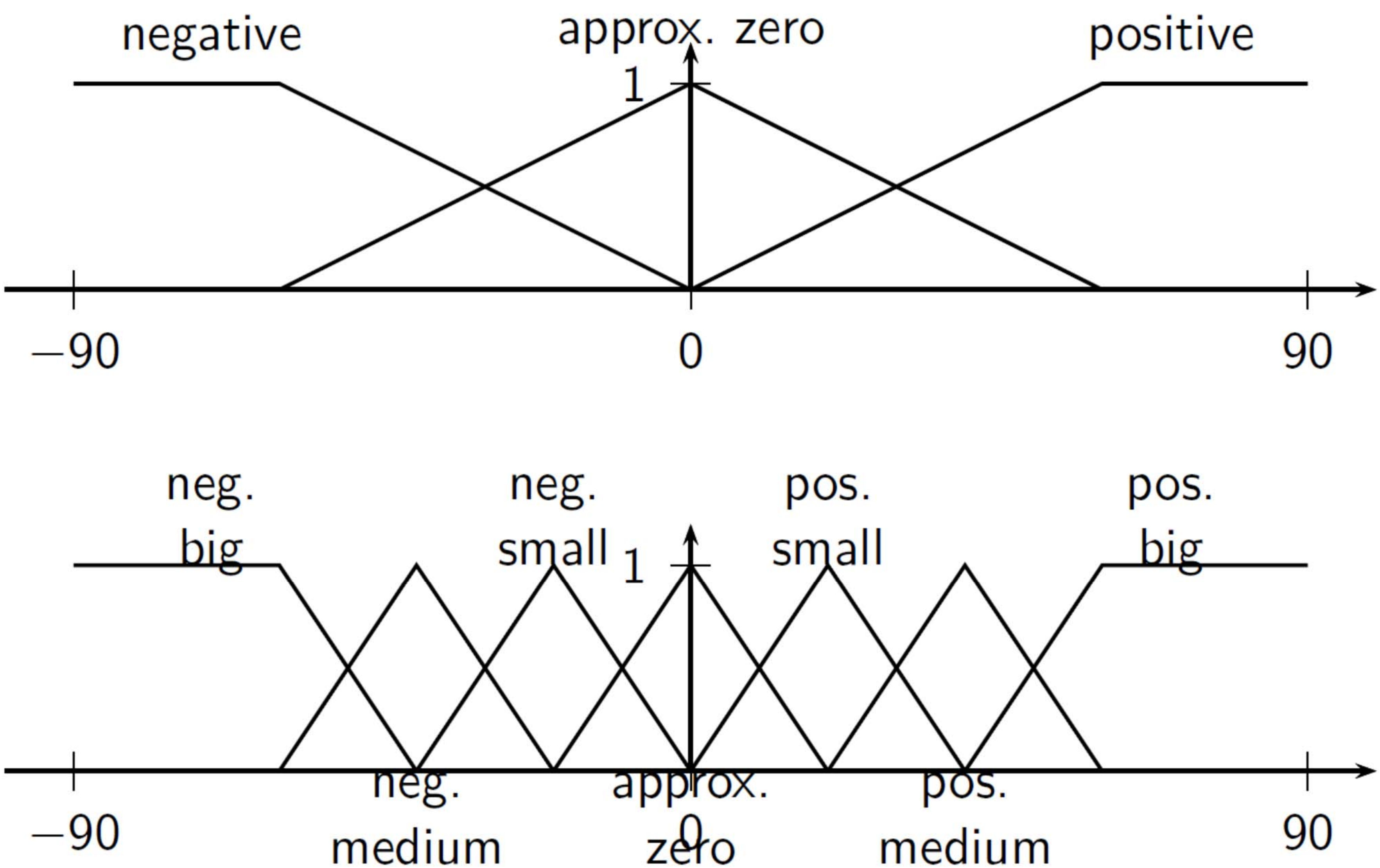
$$x \mapsto \begin{cases} 1, & \text{if } x \leq x_1 \\ 1 - \min\{\varepsilon \cdot (x - x_1), 1\}, & \text{otherwise} \end{cases}$$

- right fuzzy set:

$$\mu_{p_1}^{(1)}: [a, b] \rightarrow [0, 1]$$

$$x \mapsto \begin{cases} 1, & \text{if } x_{p_1} \leq x \\ 1 - \min\{\varepsilon \cdot (x_{p_1} - x), 1\}, & \text{otherwise} \end{cases}$$

Coarse and Fine Fuzzy Partitions



Example: Cartpole Problem (4)

- X_1 partitioned into 7 fuzzy sets
  - Support of fuzzy sets: intervals with length $1/4$ of whole range X_1
  - Similar fuzzy partitions for X_2 and Y

- Specify rules
  - if ξ_1 is $A^{(1)}$ and . . . and ξ_n is $A^{(n)}$ then η is B,

    $A^{(1)}, \dots, A^{(n)}$and B represent linguistic terms corresponding to $\mu^{(1)}, \dots, \mu^{(n)}$ and μ according to $X_1, \dots, X_n$ and Y
  - Rule base consists of *k* rules

Example: Cartpole Problem (5)

		θ						
		nb	nm	ns	az	ps	pm	pb
$\dot{\theta}$	nb			ps	pb			
	nm				pm			
	ns	nm		ns	ps			
	az	nb	nm	ns	az	ps	pm	pb
	ps				ns	ps		pm
	pm				nm			
	pb				nb	ns		

- 19 rules for cartpole problem, often not necessary to determine all table entries e.g.

  If θ is *approximately zero* and $\dot{\theta}$ is *negative medium* then F is *positive medium*

Fuzzy Approach: Challenge

- How to define function $\varphi : X \to Y$ that fits to rule set?

- Idea:
  - Represent set of rules as fuzzy relation
  - Specify desired table-based controller by this fuzzy relation

Fuzzy Relation

- Consider only crisp sets

  Solving control problem means specifying control function

  $$\varphi : X \rightarrow Y$$

  φ corresponds to relation

  $$R_\varphi = \{(x, \varphi(x)) \mid x \in X\} \subseteq X \times Y$$

  For measured input $x \in X$, control value

  $$\{\varphi(x)\} = \{x\} \circ R_\varphi$$

Fuzzy Control Rules

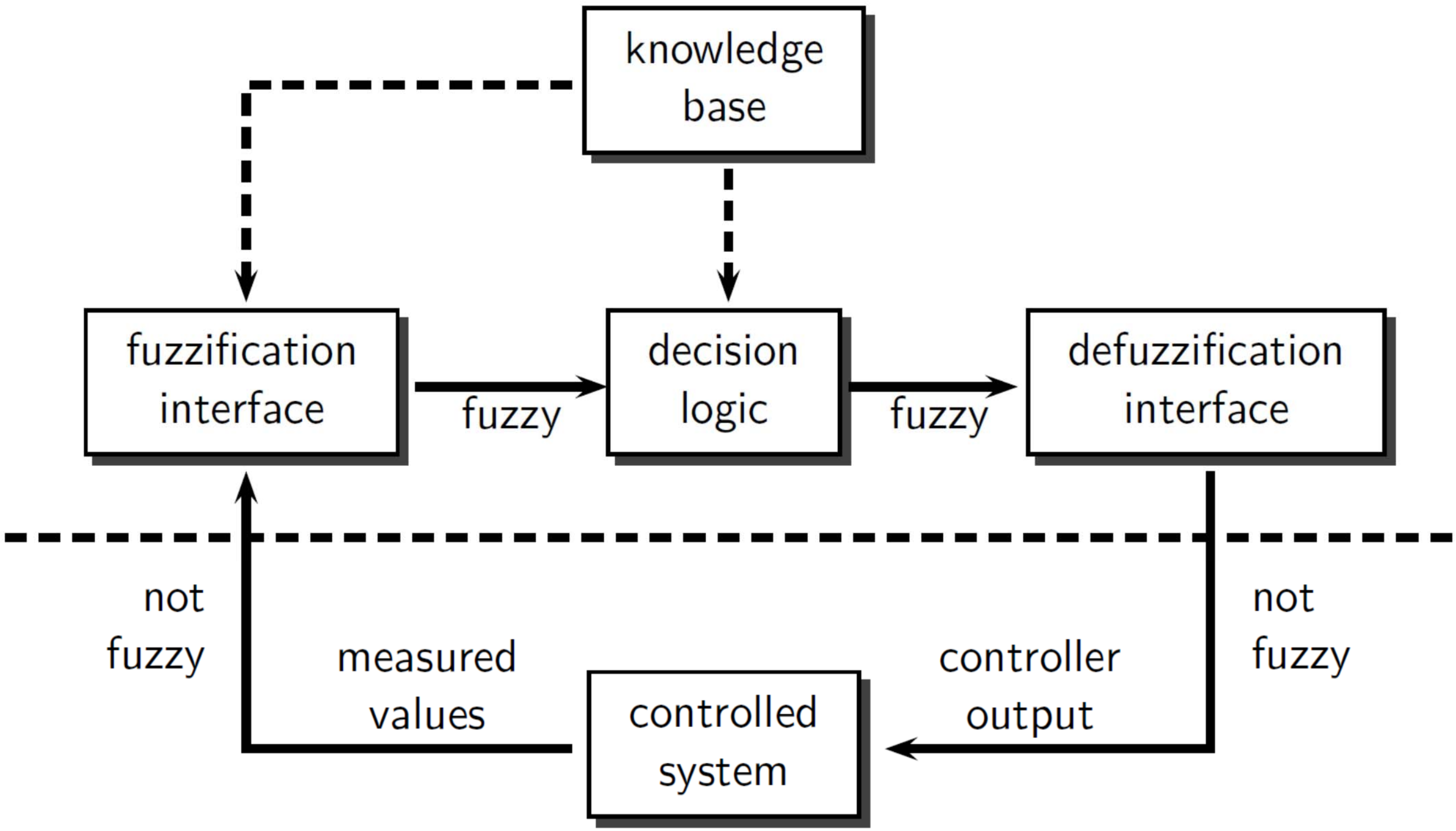
- **If** temperature is very high **and** pressure is slightly low,

  then heat change should be slightly negative

- **If** rate of descent = positive big **and** airspeed = negative big **and** glide slope = positive big,

  then rpm change = positive big **and** elevator angle change = insignificant change

Architecture of a Fuzzy Controller (1)



Architecture of a Fuzzy Controller

- *Fuzzification interface*
  - receives current input value (eventually maps it to suitable domain)
  - converts input value into linguistic term or into fuzzy set
- *Knowledge base* (consists of data base and rule base)
  - Data base contains information about boundaries, possible domain transformations, and fuzzy sets with corresponding linguistic terms
  - Rule base contains linguistic control rules
- *Decision logic* (represents processing unit)
  - computes output from measured input according to knowledge base
- *Defuzzification interface* (represents processing unit)
  - determines crisp output value

    (and eventually maps it back to appropriate domain)

Fuzzy Rule Bases

Approximate Reasoning with Fuzzy Rules

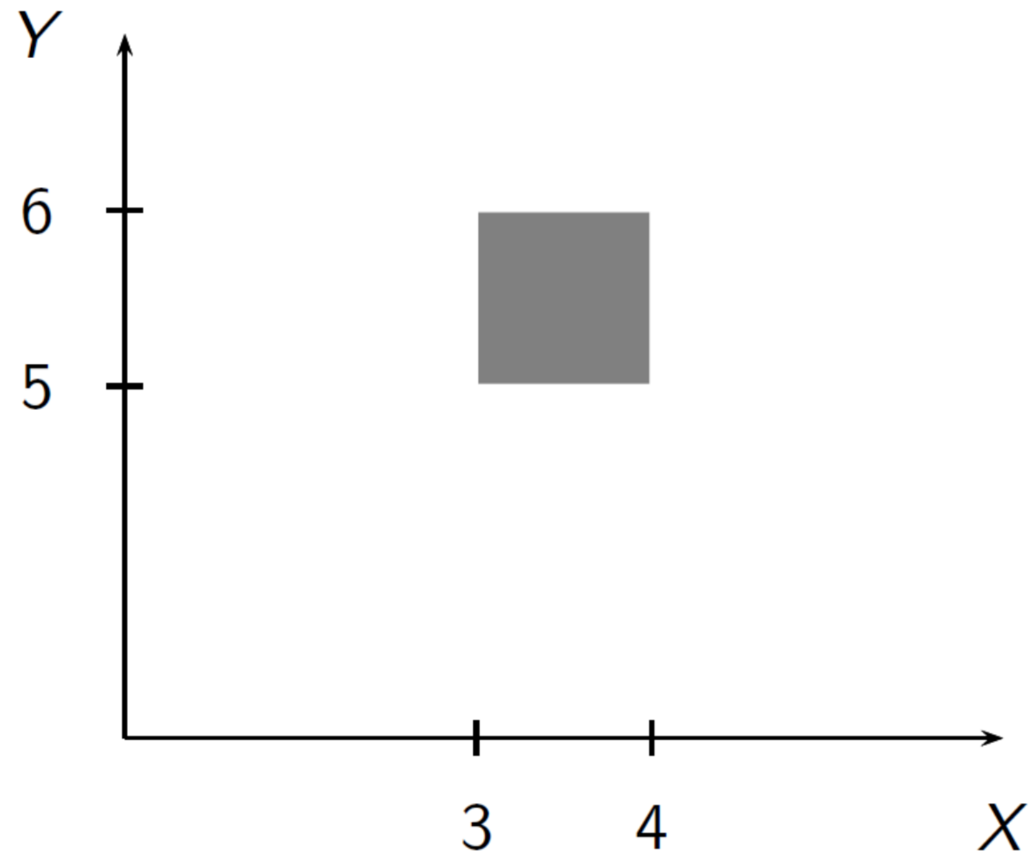
- General schema

Rule 1:	if X is M_1, then Y is N_1
Rule 2:	if X is M_2, then Y is N_2
…	…
Rule r:	if X is M_r, then Y is N_r
Fact:	X is M'
Conclusion:	Y is N'

- Given r **if-then rules** and fact "X is M'", we conclude "Y is N'".
- Typically used in **fuzzy controllers**

Approximate Reasoning: Disjunctive Imprecise Rule (1)

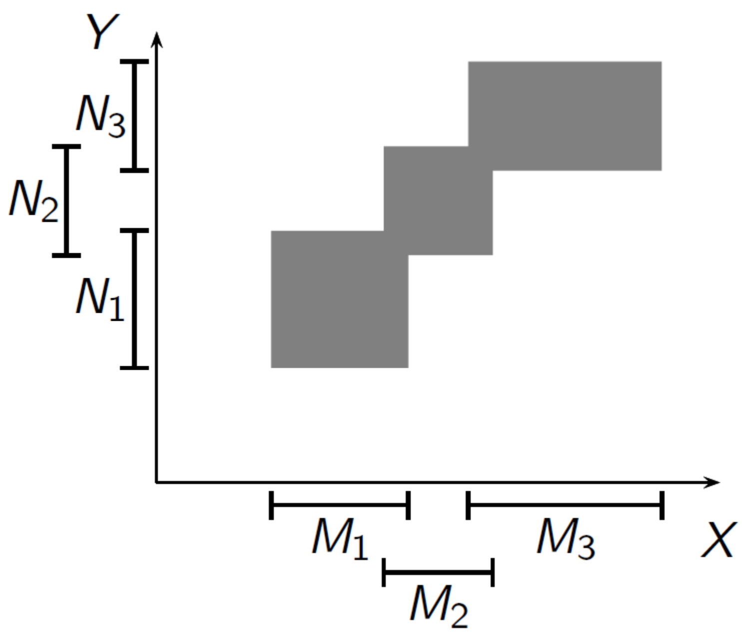
- Imprecise rule: **if** $X = [3, 4]$ **then** $Y = [5, 6]$
- Interpretation: values coming from $[3, 4] \times [5, 6]$



Approximate Reasoning: Disjunctive Imprecise Rules (2)

- Several imprecise rules
  - if $X = M_1$ then $Y = N_1$
  - if $X = M_2$ then $Y = N_2$
  - if $X = M_3$ then $Y = N_3$

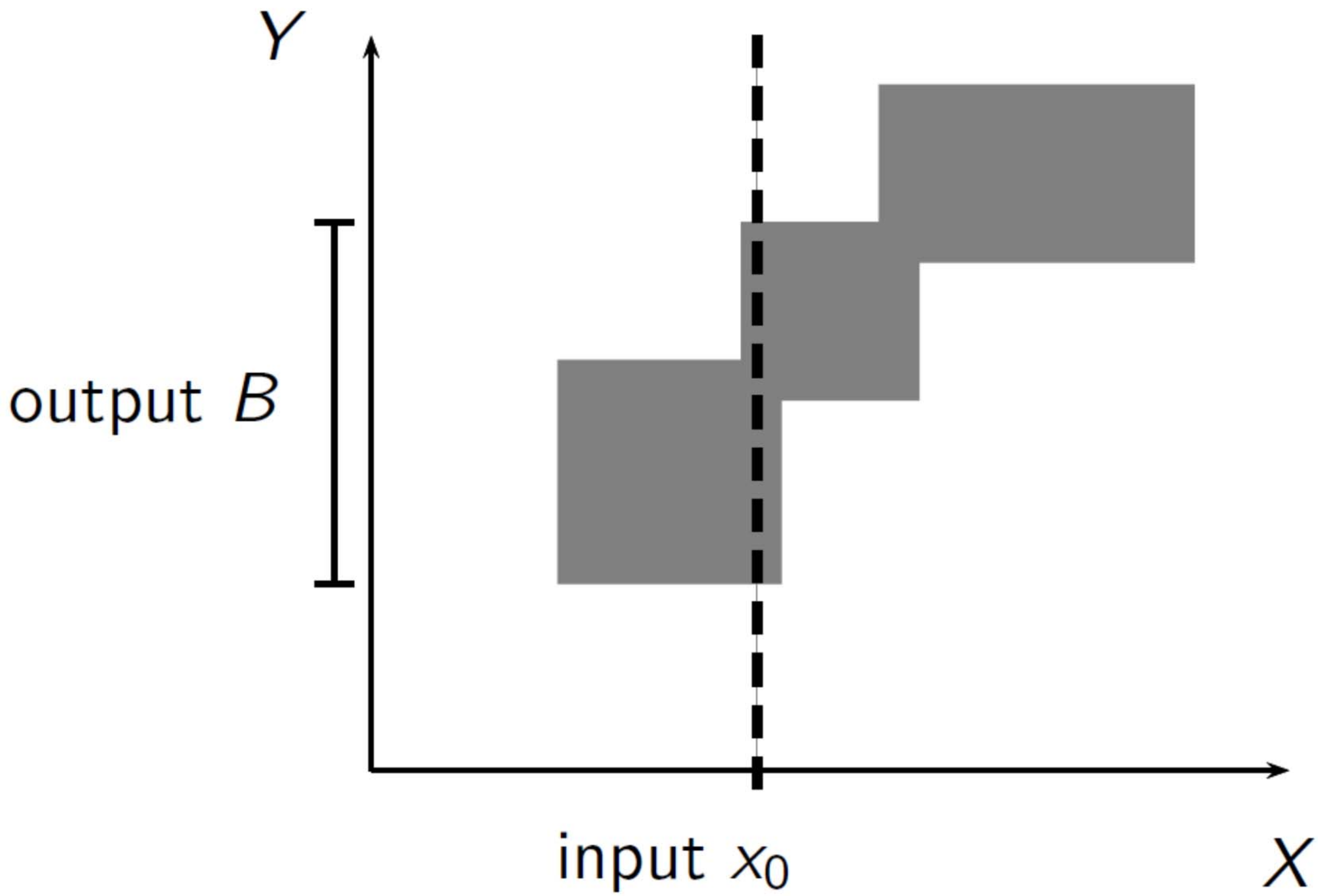
  Interpretation: rule 1 as well as rule 2 as well as rule 3 hold true



$$S = \cup_{i=1}^{r} M_i \times N_i$$

"patchwork rug" describing function's behavior as indicator function

Approximate Reasoning: Conclusion

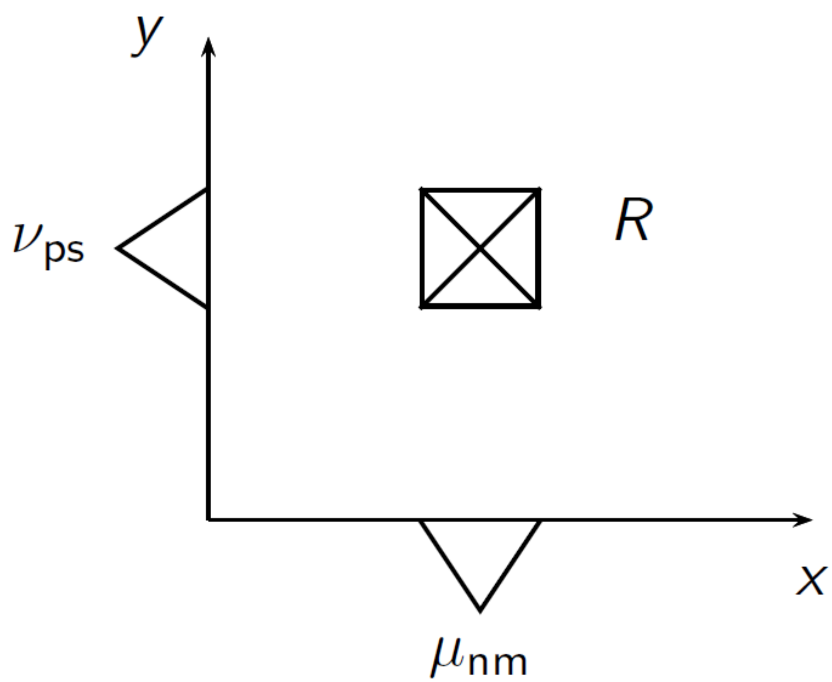


$$\{x_0\} \circ S = B$$

Approximate Reasoning: Disjunctive Fuzzy Rules (1)

one fuzzy rule:

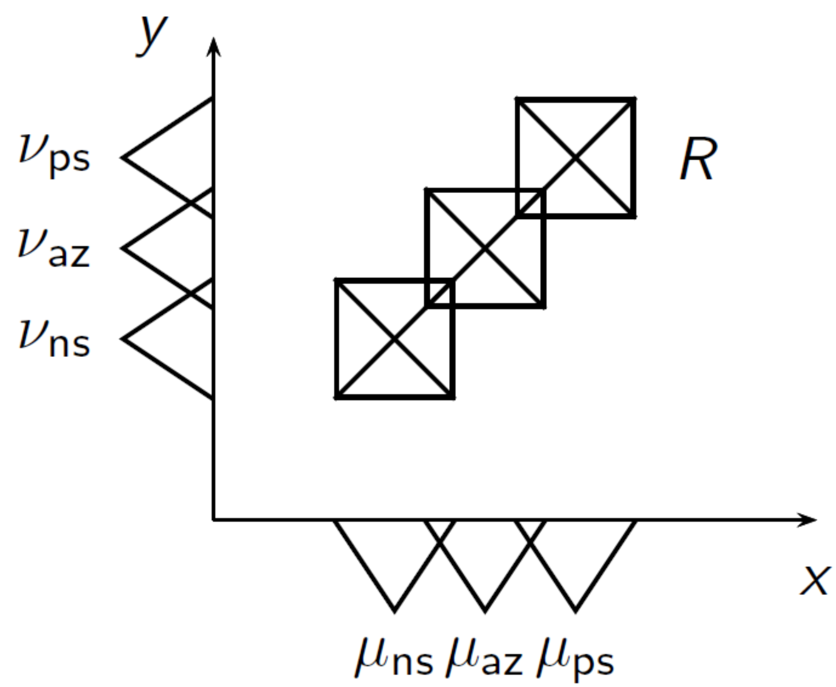
if $X = \mathrm{nm}$ **then** $Y = \mathrm{ps}$



$$R = \mu_{\mathrm{nm}} \times \nu_{\mathrm{ps}}$$

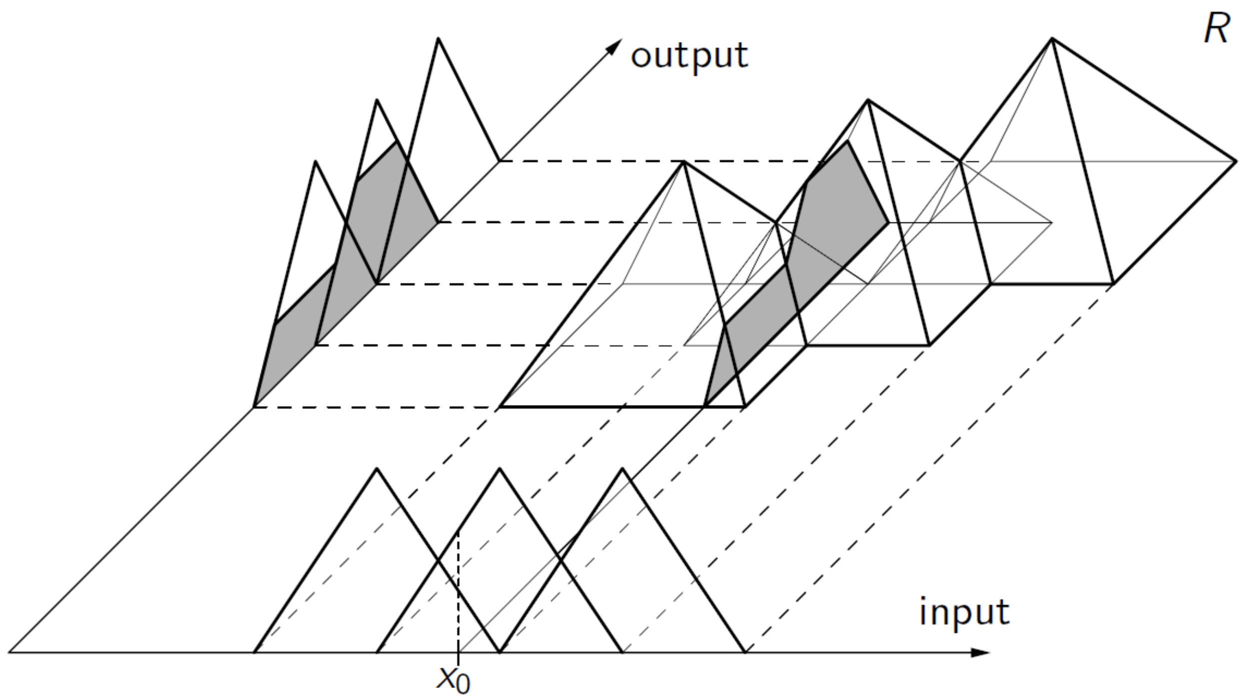
several fuzzy rules:

$\mathrm{ns} \rightarrow \mathrm{ns'}, \mathrm{az} \rightarrow \mathrm{az'}, \mathrm{ps} \rightarrow \mathrm{ps'}$



$$R = \mu_{ns} \times \nu_{ns'} \cup \mu_{az} \times \nu_{az'} \cup \mu_{ps} \times \nu_{ps'}$$

Approximate Reasoning: Disjunctive Fuzzy Rules (2)



- 3 fuzzy rules
- Every pyramid is specified by 1 fuzzy rule (Cartesian product)
- Input x_0 leads to gray-shaded fuzzy output $\{x_0\} \circ R$

Disjunctive or Conjunctive? (1)

- Fuzzy relation R employed in reasoning is obtained as follows
  - For each rule i, we determine relation R_i by

    $R_i\,(x, y) = \min[M_i\,(x), N_i\,(y)]$ for all $x \in X, y \in Y$
  - R is defined by union of R_i , i.e.

    $$R = \cup_{1 \le i \le r} R_i$$

    if-then rules are treated **disjunctive**
  - If-then rules can be also treated **conjunctive** by

    $$R = \cap_{1 \le i \le r} R_i$$

Disjunctive or Conjunctive? (2)

- Decision depends on intended use and how R_i are obtained
- For both interpretations, two possible ways of applying composition

$$B_1' = A' \circ (\cup_{1 \le i \le r} R_i) \qquad B_2' = A' \circ (\cap_{1 \le i \le r} R_i)$$

$$B_3' = \cup_{1 \le i \le r} A' \circ R_i \qquad B_4' = \cap_{1 \le i \le r} A' \circ R_i$$

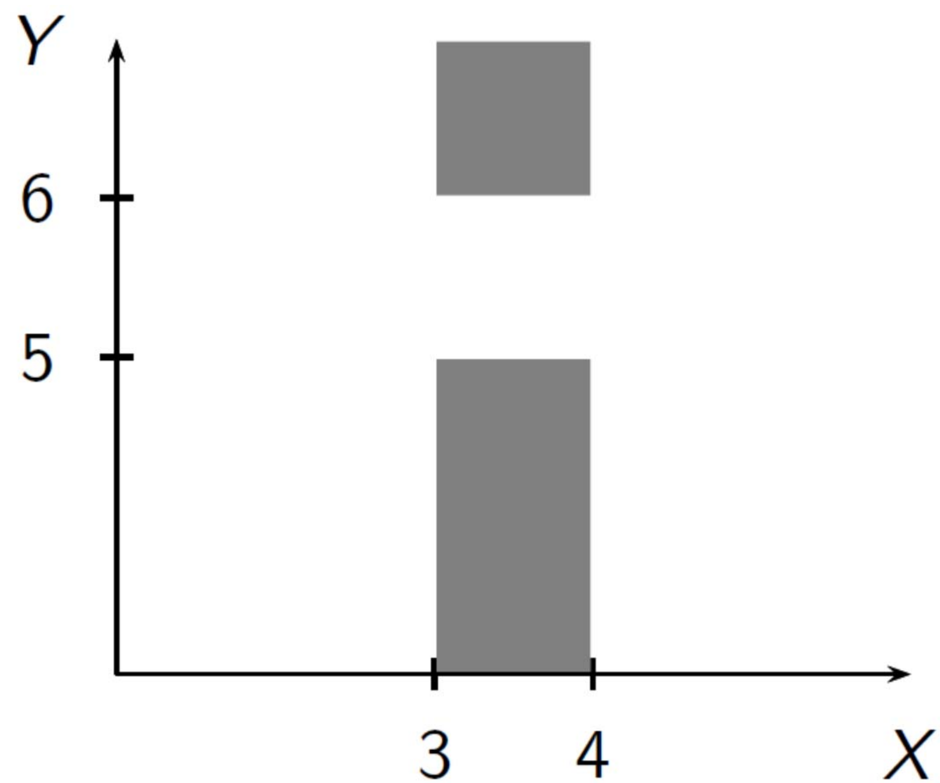
- Theorem

$$B_2' \subseteq B_4' \subseteq B_1' = B_3'$$

This holds for any continuous $\top$ used in composition

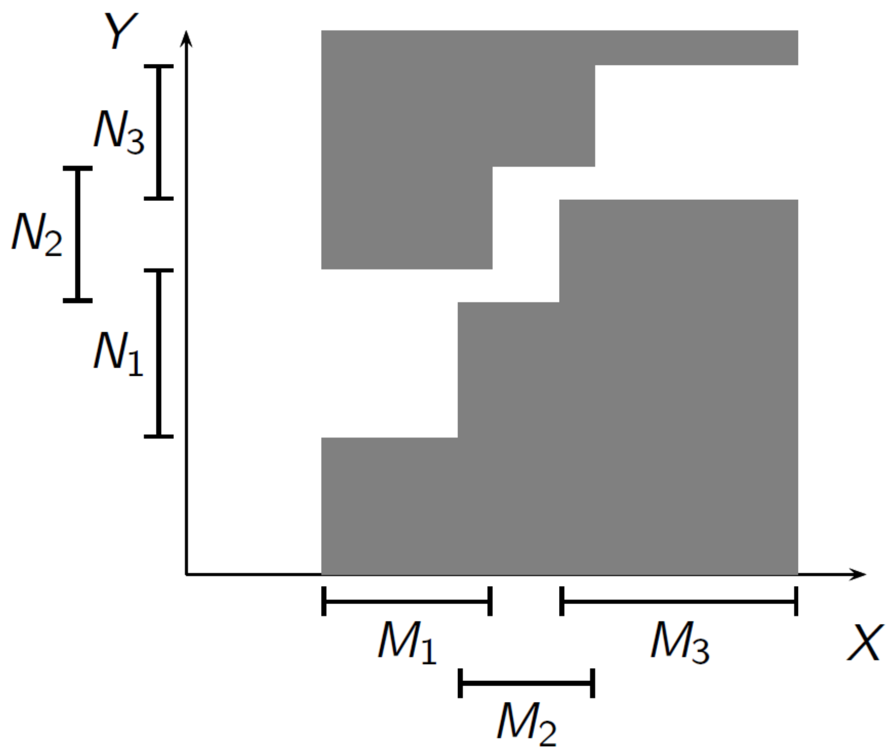
Approximate Reasoning: Conjunctive Imprecise Rules (1)

- **if** $X = [3, 4]$ **then** $Y = [5, 6]$
- Gray-shaded values are impossible, white ones are possible



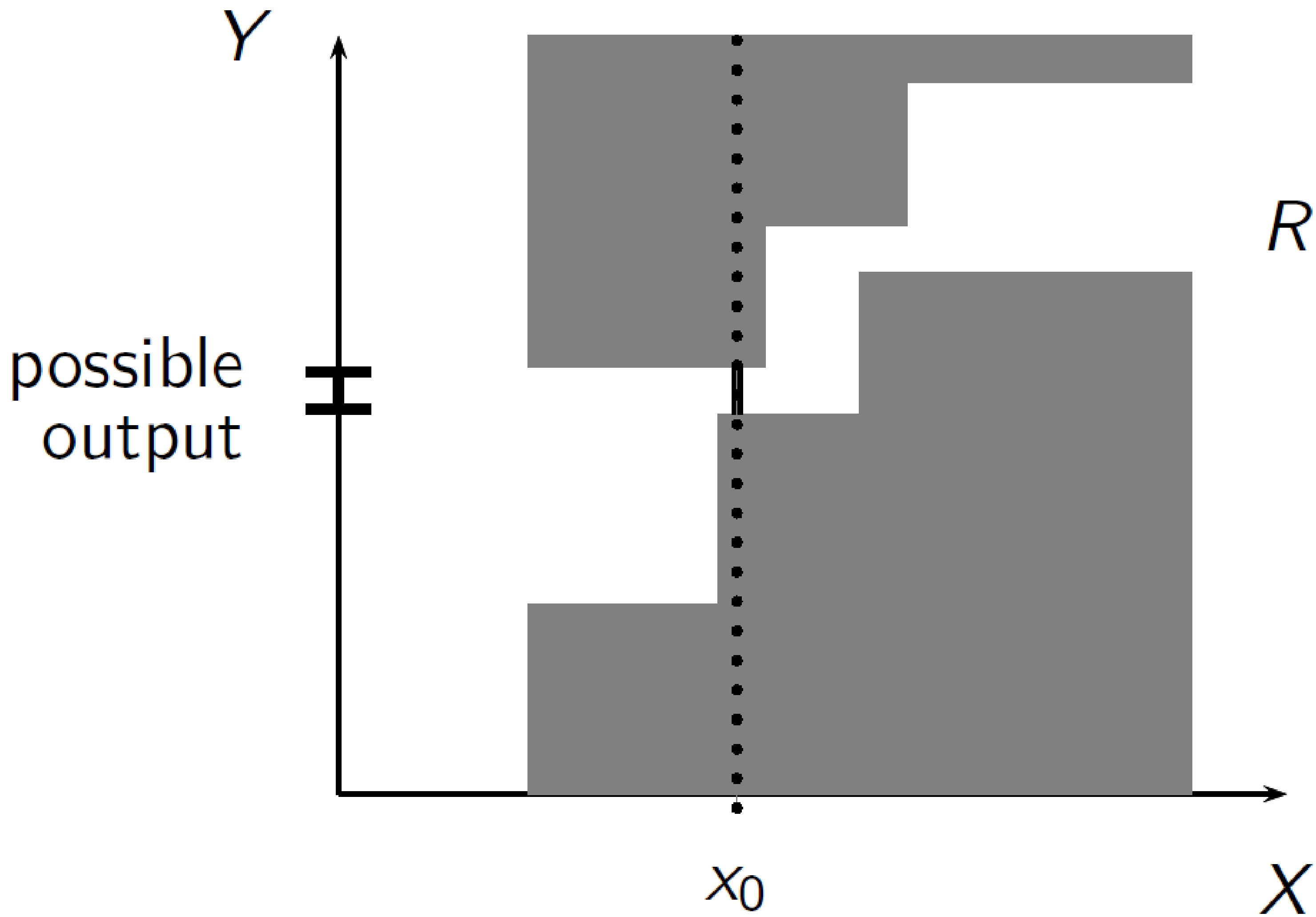
Approximate Reasoning: Conjunctive Imprecise Rules (2)

- Several imprecise rules
  - if $X = M_1$ then $Y = N_1$
  - if $X = M_2$ then $Y = N_2$
  - if $X = M_3$ then $Y = N_3$



Still possible are
$R = \cap_{i=1}^{r} (M_i \times N_i) \cup (M_i^C \times Y)$
"corridor" describing function's behavior

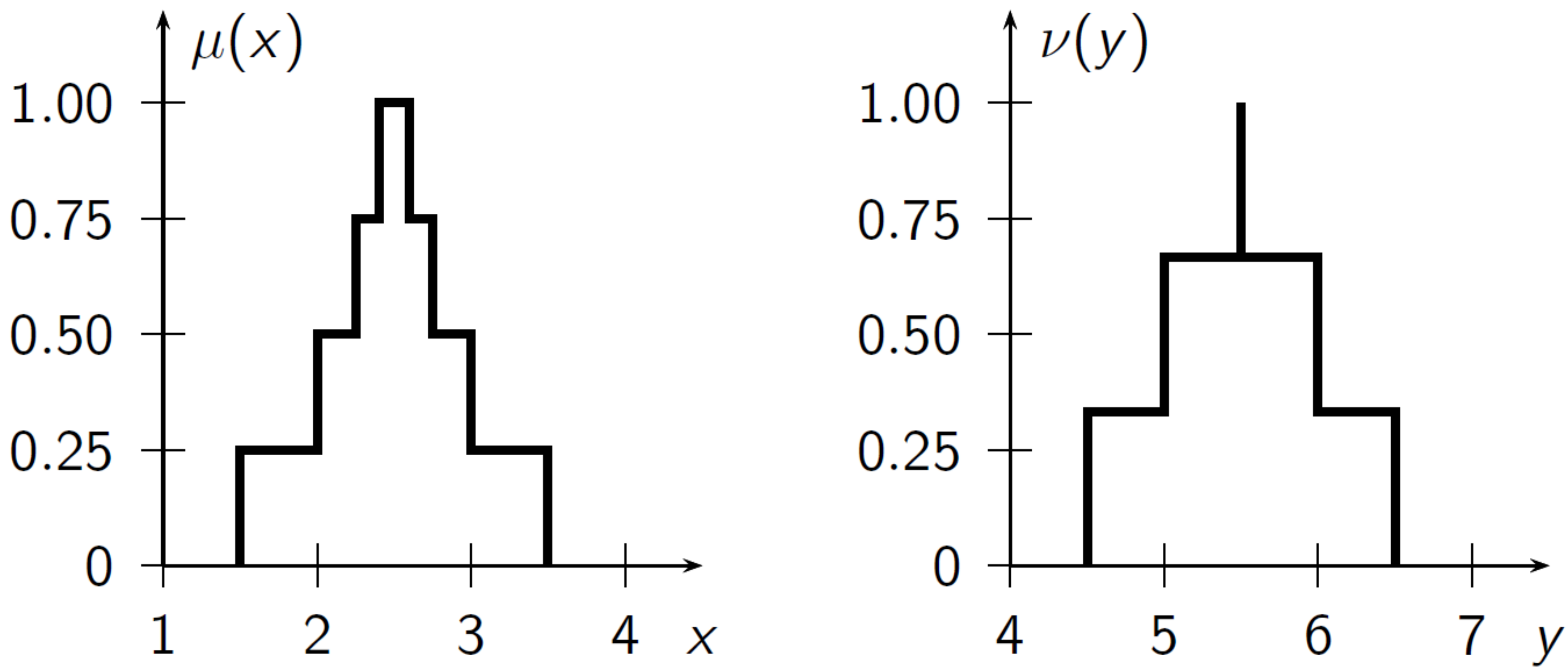
Approximate Reasoning with Crisp Input



$$\text{output} = \{x_0\} \circ R$$

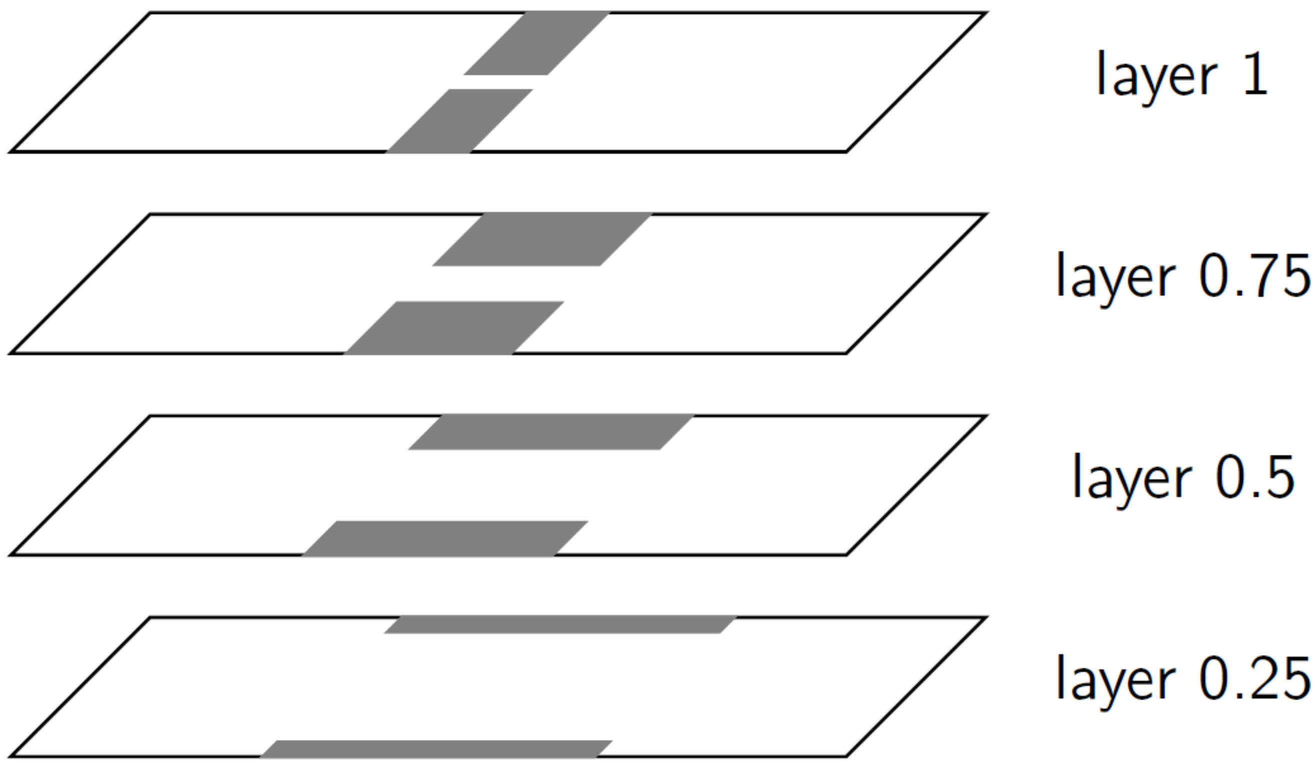
Generalization to Fuzzy Rules

- **if** X is approx. 2.5 **then** Y is approx. 5.5



Modeling a Fuzzy Rule in Layers

R_1 **: if** $X = \mu_{M_1}$ **then** $Y = \nu_{B_1}$



$$\mu_{R_1} : X \times Y \rightarrow [0,1], I(x,y) = \begin{cases} 1, & if\ \mu_{M_1}(x) \leq \nu_{B_1}(y) \\ \nu_{B_1}(y), & \text{otherwise} \end{cases}$$

Conjunctive Fuzzy Rule Base

R_1 : **if** $X = \mu_{M_1}$ **then** $Y = \nu_{B_1}$, . . . ,

R_n : **if** $X = \mu_{M_n}$ **then** $Y = \nu_{B_n}$

$$\mu_R = \min_{1 \le i \le r} \mu_{R_i}$$

Input μ_A, then output η with

$$\eta(y) = \sup_{x \in X} \min\{\mu_A(x), \mu_R(x, y)\}$$

Example: Fuzzy Relation

- Classes of cars $X = \{s, m, h\}$

  (small, medium, high quality)
- Possible maximum speeds

  $Y = \{140, 160, 180, 200, 220\}$ (in km/h)
- For any $(x, y) \in X \times Y$, fuzzy relation ϱ states possibility that maximum speed of car of class x is y

ϱ	140	160	180	200	220
s	1	.5	.1	0	0
m	0	.5	1	.5	0
h	0	0	.4	.8	1

Fuzzy Relational Equations

- Given $\mu_1, \ldots, \mu_r$ of X and $\nu_1, \ldots, \nu_r$ of Y and r rules **if** μ_i **then** ν_i
- What is a fuzzy relation ϱ that fits the rule system?
- One solution is to find a relation ϱ such that

$$\forall i \in \{1, \ldots, r\} : \nu_i = \mu_i \circ \varrho$$

$$\mu \circ \varrho : Y \to [0,1], \qquad y \longmapsto \sup_{x \in X} \min\{\mu(x), \varrho(x,y)\}$$

Solution of a Relational Equation

- Theorem
  1. Let “if A then B” be a rule with $\mu_A \in F(X)$ and $\nu_B \in F(Y)$

     The relational equation $\nu_B = \mu_A \circ \varrho$ can be solved iff the Gödel relation $\varrho_{A \circledcirc B}$ is a solution.

     $\varrho_{A \circledcirc B} : X \times Y \to [0,1]$ is defined by

     $$(x, y) \longmapsto \begin{cases} 1, & \text{if } \mu_A(x) \le \nu_B(y) \\ \nu_B(y), & \text{otherwise} \end{cases}$$

  2. If ϱ is a solution, then the set of solutions

     $R = \{\varrho_S \in \mathcal{F}(X \times Y) \mid \nu_B = \mu_A \circ \varrho_S\}$ has the following property: If $\varrho_{S'} \in R$, then $\varrho_{S'} \cup \varrho_{S''} \in R$

  3. If $\varrho_{A \circledcirc B}$ is a solution, then $\varrho_{A \circledcirc B}$ is the largest solution w.r.t. $\subseteq$

Example

$$\mu_A = \begin{pmatrix} .9 & 1 & .7 \end{pmatrix} \qquad \nu_B = \begin{pmatrix} 1 & .4 & .8 & .7 \end{pmatrix}$$

$$\varrho_{A \circledS B} = \begin{pmatrix} 1 & .4 & .8 & .7 \\ 1 & .4 & .8 & .7 \\ 1 & .4 & 1 & 1 \end{pmatrix} \qquad \varrho_1 = \begin{pmatrix} 0 & 0 & 0 & .7 \\ 1 & .4 & .8 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{array}{ccc|cccc} & & & 1 & .4 & .8 & .7 \\ & & & 1 & .4 & .8 & .7 \\ & & & 1 & .4 & 1 & 1 \\ \hline .9 & 1 & .7 & 1 & .4 & .8 & .7 \end{array} \qquad \varrho_2 = \begin{pmatrix} 0 & .4 & .8 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & .7 \end{pmatrix}$$

- $\varrho_{A \circledS B}$ largest solution, ϱ_1, ϱ_2 are two minimal solutions
- Solution space forms upper semilattice

Solution of a Set of Relational Equations

- Generalization of this result to system of r relational equations
- Theorem
  - Let $\nu_{B_i} = \mu_{A_i} \circ \varrho$ for $i = 1, \ldots, r$ be a system of relational equations
    1. There is a solution iff $\cap_{i=1}^{r} \varrho_{A_i \circledast B_i}$ is a solution
    2. If $\cap_{i=1}^{r} \varrho_{A_i \circledast B_i}$ is a solution, then this solution is the biggest solution
  - Remark: if there is no solution, then Gödel relation is often at least a good approximation

Solving a System of Relational Equations

- Sometimes it is a good choice not to use the largest but a smaller solution

  i.e. the Cartesian product $\varrho_{A\times B}(x,y) = \min\{\mu_A(x), \nu_B(y)\}$

  If a solution of the relational equation $\nu_B = \mu_A \circ \varrho$ for ϱ exists, then $\varrho_{A\times B}$ is a solution, too

- Theorem
  - Let $\mu_A \in \mathcal{F}(X)$, $\nu_B \in \mathcal{F}(Y)$. Furthermore, let $\varrho \in F(X \times Y)$ be a fuzzy relation which satisfies the relational equation $\nu_B = \mu_A \circ \varrho$
  - Then $\nu_B = \mu_A \circ \varrho_{A\times B}$ holds

Solving a System of Relational Equations by Using Cartesian product

- $\mu_{A_i} = \nu_{B_i} \circ \varrho,\ 1 \le i \le r$ can be reasonably solved with $A \times B$ by

  $$\varrho = \max\{\varrho_{A_i \times B_i} \mid 1 \le i \le r\}$$

- For crisp value $x_0 \in X$ (represented by $1_{\{x_0\}}$)

  $$v(y) = \left(1_{\{x_0\}} \circ \varrho\right)(\mathrm{y})$$
  $$= \max_{1 \le i \le r} \left\{ \sup_{x \in X} \min\left\{1_{\{x_0\}}(x), \varrho_{A_i \times B_i}(x, y)\right\}\right\}$$
  $$= \max_{1 \le i \le r} \left\{\min\left\{\mu_{A_i}(x_0), v_{B_i}(y)\right\}\right\}$$

- This solution is the Mamdani-Assilian fuzzy control