Math 5740 Homework 2

Cory Rindlisbacher

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Exercise 1. We want to know how many years it would take for the pollution in a lake to reach 5% of its current level, assuming no pollution flows in. Make some simplifying assumptions to estimate this. Find this time for Lake Erie, Lake Michigan, Lake Superior, and the Great Salt Lake where their volumes, V (in cubic meters) and outflow of water r (cubic meters per day) are given in the following table:

Lake	Volume, V	Outflow of water, r
Erie	458×10^9	479×10^6
Michigan	$4,871 \times 10^9$	433×10^6
Superior	$12,221 \times 10^9$	178×10^6
Great Salt Lake	18.9×10^9	0

Clearly state your simplifying assumptions and comment on sources of error. What's different about the Great Salt Lake?

Solution
The first assumption we make is that the pollution is well mixed and evenly dispersed throughout the lake. Let P be the amount of pollution and V be the volume of water in cubic meters. Then the pollution per cubic meter is the ratio $\frac{P}{V}$. We can model the rate of change of the lake's pollution levels as being proportional to $\frac{P}{V}$ multiplied by the outflow of water, r. Thus we get

$$\frac{dP}{dt} = \frac{P}{V} r.$$

Solving the differential equation, we obtain the model

$$P = P_0 e^{\frac{rt}{V}}.$$

From this model we can determine the amount of time it will take each lake to reach 5% of its initial pollution level. Solving for t we get $t = \ln(.05)\frac{V}{r}$.

Lake	Time (days)	Time (Years)
Erie	2864	7.85
Michigan	$33,700$	92.3
Superior	$205,678$	563.5
Great Salt Lake	∞	∞

Exercise 2. Consider the one-dimensional dynamical system

$$\frac{dx}{dt} = \sin x.$$

a) What are the equilibrium solutions of this dynamical system?

b) Find the linear stability of each equilibtrium solution.

c) Draw the phase portrait two different ways. First as in the course notes, draw the phase portrait while thinking of x as lying on the real line, and then as x as a point on the circle.

Exercise 3. A model for the population growth is given by

$$\frac{dN}{dt} = f(N) = rN\left(\frac{N}{U} - 1\right)\left(1 - \frac{N}{K}\right), \qquad N(0) = N_0$$

where r, K, U are positive parameters with $U < K$.

a) Sketch the function $f(N)$ and find the equilibrium solution.

b) Find the linear stability of each equilibrium solution.

c) Draw the phase portrait and sketch some of the solutions for different N_0 (do not solve the ODE).

d) Discuss the behavior of $N(t)$ as $t \to \infty$.

e) Discuss the behavior of $N(t)$ for small N_0. This is called critical depensation.