

Math 5740 Homework 3

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February 11, 2016

Exercise 1. We wrote the Euler-Lagrange equation for the function $P(y) = \int_0^1 F(y, y') dx$ as

$$\frac{d}{dx} \left(\frac{\partial F}{\partial y'} y' - \frac{\partial F}{\partial y} \right) = 0.$$

a) For $F = F(y, y')$, prove the following identity:

$$\frac{d}{dx} \left(\frac{\partial F}{\partial y'} - F \right) = y' \left[\frac{d}{dx} \frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} \right].$$

b) Use the identity to show that the Euler-Lagrange equation is equivalent to

$$\frac{\partial F}{\partial y'} y' - F = C.$$

Proof

a) To show part a, note the following:

$$\begin{aligned} \frac{d}{dx} \left(\frac{\partial F}{\partial y'} y' - F \right) &= \frac{d}{dx} \frac{\partial F}{\partial y'} y' - \frac{d}{dx} F \\ &= \frac{d}{dx} \frac{\partial F}{\partial y'} y' - \frac{\partial F}{\partial y} \frac{dy}{dx} \\ &= y' \left(\frac{d}{dx} \frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} \right). \end{aligned}$$

b) For part b, we have that

$$\frac{d}{dx} \left(\frac{\partial F}{\partial y'} - F \right) = y' \left[\frac{d}{dx} \frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y} \right] = 0 = \frac{d}{dx} \left(\frac{\partial F}{\partial y'} y' - \frac{\partial F}{\partial y} \right).$$

We can integrate the left side with respect to x , and we see that

$$\int \frac{d}{dx} \left(\frac{\partial F}{\partial y'} - F \right) dx = 0 \rightarrow \frac{\partial F}{\partial y'} y' - F = C.$$

It follows then that solving the Euler-Lagrange equation is equivalent to solving for $\frac{\partial F}{\partial y'} y' - F = C$.

Exercise 2. Find the stationary functions of $P(y) = \int_0^1 F(y, y') dx$ for the following functions:

a) $F(y, y') = \frac{\sqrt{1+(y')^2}}{y}$

b) $F(y, y') = y^2 - (y')^2$

Solution

a) Using the above equality, we have that

$$\begin{aligned} \frac{\partial F}{\partial y'} - F &= \frac{(y')^2}{y\sqrt{1+(y')^2}} \\ &= \frac{-1}{y\sqrt{1+(y')^2}} \\ &= \frac{1}{y^2\sqrt{1+(y')^2}} = C. \end{aligned}$$

Rearranging the above equation we can get $y' = \frac{\sqrt{b-y^2}}{y}$. This is a separable equation, and yields the solution $y = \sqrt{b-x^2}$, where b is a constant.

b) Using the same technique as above, we have that

$$\frac{\partial F}{\partial y'} - F = -(y')^2 - y^2$$

Solving for y' , we see that

$$\begin{aligned} y' &= \sqrt{a-y^2} \rightarrow \\ 1 &= \frac{y'}{a-y^2}. \end{aligned}$$

This is also a separable equation. Integrating both sides of the above equation gives

$$\sin^{-1}\left(\frac{y}{b}\right) = x,$$

where b is a constant. Solving for y gives us $y = b \sin(x)$.

Exercise 5. *Proof.*

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