Math 5740 Homework 3

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Exercise 1. We wrote the Euler-Lagrange equation for the function $P(y) = \int_0^1 F(y, y')\,dx$ as

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}y' - \frac{\partial F}{\partial y}\right) = 0.$$

a) For $F = F(y, y')$, prove the following identity:

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'} - F\right) = y'\left[\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y}\right].$$

b) Use the identity to show that the Euler-Lagrange equation is equivalent to

$$\frac{\partial F}{\partial y'}y' - F = C.$$

Proof

a) To show part a, note the following:

$$\begin{aligned}\frac{d}{dx}\left(\frac{\partial F}{\partial y'}y' - F\right) &= \frac{d}{dx}\frac{\partial F}{\partial y'}y' - \frac{d}{dx}F \\ &= \frac{d}{dx}\frac{\partial F}{\partial y'}y' - \frac{\partial F}{\partial y}\frac{dy}{dx} \\ &= y'\left(\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y}\right).\end{aligned}$$

b) For part b, we have that

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'} - F\right) = y'\left[\frac{d}{dx}\frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y}\right] = 0 = \frac{d}{dx}\left(\frac{\partial F}{\partial y'}y' - \frac{\partial F}{\partial y}\right).$$

We can integrate the left side with respect to x, and we see that

$$\int \frac{d}{dx}\left(\frac{\partial F}{\partial y'} - F\right)dx = 0 \to \frac{\partial F}{\partial y'} - F = C.$$

It follows then that solving the Euler-Lagrange equation is equivalent to solving for $\frac{\partial F}{\partial y'} - F = C$.

Exercise 2. Find the stationary functions of $P(y) = \int_0^1 F(y, y')\,dx$ for the following functions:

a) $F(y, y') = \frac{\sqrt{1+(y')^2}}{y}$

b) $F(y, y') = y^2 - (y')^2$

Solution

a) Using the above equality, we have that

$$\begin{aligned}
\frac{\partial F}{\partial y'} - F &= \frac{(y')^2}{y\sqrt{1+(y')^2}} \\
&= \frac{-1}{y\sqrt{1+(y')^2}} \\
&= \frac{1}{y^2\sqrt{1+(y')^2}} = C.
\end{aligned}$$

Rearranging the above equation we can get $y' = \frac{\sqrt{b-y^2}}{y}$. This is a separable equation, and yeilds the solution $y = \sqrt{b-x^2}$, where b is a constant.

b) Using the same technique as above, we have that

$$\frac{\partial F}{\partial y'} - F = -(y')^2 - y^2$$

Solving for y', we see that

$$\begin{aligned}
y' &= \sqrt{a-y^2} \rightarrow \\
1 &= \frac{y'}{a-y^2}.
\end{aligned}$$

This is also a separable equation. Integrating both sides of the above equation gives

$$\sin^{-1}(\frac{y}{b}) = x,$$

where b is a constant. Solving for y gives us $y = b\sin(x)$.

Exercise 3. To find the curve joining the point $(-1, a)$ and $(1, a)$ that yields a surface of revolution of minimum area when revolved around the x-axis, we minimize

$$P(y) = \int_{-1}^{1} 2\pi y\sqrt{1+(y')^2}\,dx,$$

subject to the constraint that $y(-1) = a = y(1)$. Solve the Euler-Lagrange equations for this functional and interpret the solutions.
Solution

$$\frac{\partial F}{\partial y'}y' - F = \frac{-2\pi y}{\sqrt{1+(y')^2}} = C.$$

Solving for y', we get that $y' = K\sqrt{4\pi^2y^2-1}$. Solving the separable differential equation gives

$$K\ln|y+\sqrt{y^2-1}| = x.$$

By manipulating this equation algebraically we can write this in the form

$$y = \frac{e^{2cx} + 1}{2e^{cx}} = \cosh{(cx)}.$$

In order to determine c, we need to evaluate the function $\cosh c(-1) = a = \cosh c$. It is important to note that the function is only defined if $1 \leq a < \infty$.