Math 5610 Homework 4

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1 **Spline versus Cubic Hermite Interpolation** Let the function

$$\begin{aligned} s(x) &= (\gamma - 1)(x^3 - x^2) + x + 1 \text{ if } x \in [0,1] \\ &= \gamma x^3 - 5\gamma x^2 + 8\gamma x - 4\gamma + 2 \text{ if } x \in [1,2] \end{aligned}$$

a) Show that s is the piecewise cubic Hermite interpolant to the data:

$$s(0) = 1, \qquad s(1) = s(2) = 2, \qquad s'(0) = 1, \qquad s'(1) = \gamma, \qquad s'(2) = 0$$

b) For what value of γ does s become a spline?

Solution Part A

We first see that

$$\begin{aligned} s'(x) &= (\gamma - 1)(3x^2 - 2x) + 1 \text{ if } x \in [0,1] \\ s'(x) &= 3\gamma x^2 - 10\gamma x + 8\gamma \text{ if } x \in [1,2] \end{aligned}$$

It follows that

$$\begin{aligned} s(0) &= (\gamma - 1)(0 - 0) + 0 + 1 = 1 \\ s(1) &= (\gamma - 1)(1 - 1) + 1 + 1 = 2 = \gamma - 5\gamma + 8\gamma - 4\gamma + 2 = s(1) \\ s(2) &= 8\gamma - 20\gamma + 16\gamma - 4\gamma + 2 = 2 \\ s'(0) &= (\gamma - 1)(0 - 0) + 1 = 1 \\ s'(1) &= (\gamma - 1)(3 - 2) + 1 = \gamma = 3\gamma - 10\gamma + 8\gamma = s'(1) \\ s'(2) &= (12\gamma - 20\gamma + 8\gamma) = 0 \end{aligned}$$

Thus, the function s interpolates the data.
Solution Part B For s to be a cubic spline, it must interpolate the data at $s''(x)$. We get that

$$\begin{aligned} s''(x) &= (\gamma - 1)(6x - 2) \text{ if } x \in [0,1] \\ s''(x) &= 6\gamma x - 10\gamma \text{ if } x \in [1,2] \end{aligned}$$

We set the two equations above equal to each other, and solve for γ when $x = 1$.

$$\begin{aligned}
(\gamma - 1)(6x - 2) &= 6\gamma x - 10\gamma, \\
6x\gamma - 6x - 2\gamma + 2 &= 6x\gamma - 10\gamma, \\
-4 - 2\gamma &= -10\gamma, \\
8\gamma &= 4, \\
\gamma &= \frac{1}{2}
\end{aligned}$$

Thus, γ must be $1/2$ in order to be a cubic spline.

2 **More on Cubic Splines** Consider the data

$$x_i : 1, 2, 3, 4$$

$$y_i : 1, 8, 27, 64$$

a Construct the cubic interpolant, i.e., find the cubic polynomial p that satisfies

$$p(x_i) = y_i$$

and draw its graph.

b Construct the interpolating natural cubic spline and draw its graph.

c Comment on your results

Proof

The cubic interpolant for the given data is $p(x) = x^3$. There are a few ways to see this. The first is to note that an interpolating polynomial is unique. In this case we have the points (x_i, x_i^3). Since $f(x) = x^3$ interpolates the data perfectly, we have that it must be the case that $p(x) = x^3$. Another way to determine this is by using the Lagrange form. Using this method, we get the following:

$$\begin{aligned}
p(x) &= \Sigma_{i=0}^{n} x_i^3 L_i(x) \\
&= \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} \\
&+ \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)}
\end{aligned}$$

Expanding and simplifying this yeilds $p(x) = x^3$.

3 **Polynomial Interpolation** Suppose you want to interpolate to the data (x_i, y_i), $i = 0, \ldots, n$ by a polynomial of degree n. Show that

$$\Sigma_{i=0}^{n} x_i^j L_i(x) = x^j \text{ for } j = 0, \ldots, n.$$

Proof
Note that the above formula is equivalent to interpolating any set of n points where each point is defined by (x_i, x_i^j) and $j \leq 0, 1, \ldots n$. Let $f(x) = x^j$. Then the interpolating polynomial for $\{(x_i, x_i^j)\}$, $i = 0, 1, \ldots, n$ is given by the following:

$$\Sigma_{i=0}^{n} x_i^j L_i(x) = \Sigma_{i=0}^{n} f(x_i) L_i(x)$$

The points are perfectly interpolated by the function $f(x)$. Since the interpolating polynomial is unique, we know that it must be the case that

$$\Sigma_{i=0}^{n} x_i^j L_i(x) = \Sigma_{i=0}^{n} f(x_i) L_i(x) = f(x) = x^j.$$

4 **The Infamous Runge-Phenomenon** It is not generally true that higher degree interpolation polynomials yield more accurate approximations. Let

$$f(x) = \frac{1}{1+x^2} \text{ and } x_j = -5 + jh, j = 0, 1, \cdots, n, h = \frac{10}{n}.$$

For

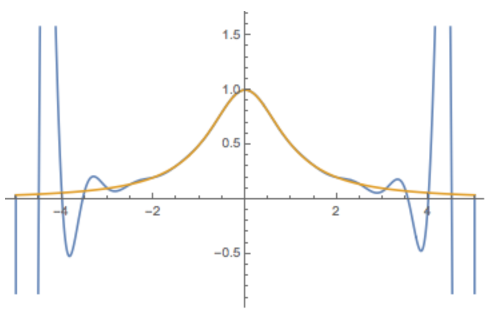
$$n = 1, 2, 3, \cdots, 20$$

plot the graph of the interpolant

$$p(x) = \Sigma_{i=0}^{n} \alpha_i x^i$$

defined by $p(x_i) = f(x_i)$.

Proof
Below are the graphs of $f(x)$ and $p_{20}(x)$. You can see that interpolating along equally spaced intervals yields polynomials that oscillate wildly toward the final end points.

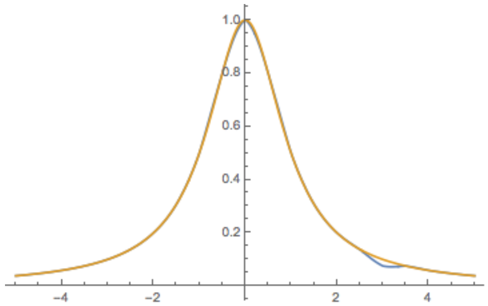


The plot is too cluttered with additional lines. However, the case where $n = 20$ is informative and shows the general issue with interpolating at equal intervals.

5 **Judicious Interpolation** Repeat the above except that you interpolate at the roots of the Chebycheff polynomials.

Proof

Here is the plot of interpolating at the roots of the Chebycheff Polynomials along with the actual graph of $f(x)$.



You can see that the interpolating polynomial is much more accurate by interpolating at the roots of the Chebycheff polynomials than equidistant points.

6 **The interpolant to symmetric data is symmetric.** Suppose you are given symmetric data

$$(x_i, y_i), i = -n, -n+1, \dots, n-1, n,$$

such that

$$x_{-i} = -x_i, y_{-1} = -y_i, i = 0, 1, \dots, n.$$

What is the required degree of the interpolating polynomial? Show that the interpolating polynomial is odd, i.e.

$$p(x) = -p(-x).$$

Proof We have that

$$p(x_i) = y_i = -y_{-i} = -p(x_{-i}) = -p(-x_i).$$

This implies that the function is odd, and that therefore the degree of $p(x)$ must be odd. Since we have $2n+2$ points (2 from the point $(0,0)$, we are looking for a degree $2n+1$ polynomial.

7 **Linear Independence of Bernstein-Bezier Basis Functions** show that, for all $n \geq 0$,

$$\sigma_{i=0}^{n} c_i \binom{n}{i} x^i (1-x)^{n-i} = 0 \to c_i = 0, i = 0, \cdots, n.$$

Proof

In order to show that the Bernstein Bezier functions are linearly independent, we first note that

$$B_{k,n}(t) = \Sigma_{i=k}^{n} (-1)^{i-k} \binom{n}{i} \binom{i}{k} t^i$$

which is simply a change of basis to the power basis. Now, assume that the Bernstein-Bezier functions are not linearly independent. Then for some set of c_i, we have that

$$
\begin{aligned}
0 &= \Sigma_{i=0}^{n} B_{i,n}(t) \\
&= c_0 \Sigma_{i=0}^{n} (-1)^i \binom{n}{i}\binom{i}{0} t^i + c_1 \Sigma_{i=1}^{n} (-1)^{i-1} \binom{n}{i}\binom{i}{1} t^i + \cdots + c_n \Sigma_{i=n}^{n} (-1)^{i-n} \binom{n}{i}\binom{i}{n} t^i \\
&= c_0 + \Sigma_{i=0}^{1} \binom{n}{1}\binom{1}{1} t^1 + \cdots + \Sigma_{i=0}^{n} \binom{n}{n}\binom{n}{n} t^n
\end{aligned}
$$

Since the power basis is linearly independent, we see that in order to have a linearly dependent set, we require that

$$
\begin{aligned}
c_0 &= 0 \\
0 &= \Sigma_{i=0}^{1} \binom{n}{1}\binom{1}{1} \\
&\vdots \\
&= \Sigma_{i=0}^{n} \binom{n}{n}\binom{n}{n}
\end{aligned}
$$

which implies that $c_0 = c_1 = \cdots = c_n = 0$. Thus, the Bernstein Bezier functions are linearly independent.

8 **Uniqueness of the interpolating polynomial** Assume you are given the data

$$x_i : 1, 2, 4, 8$$

$$y_i : 1, 2, 3, 4$$

Construct the interpolating polynomial using

- the power form
- the Lagrange form
- the Newton form

and show that they all yield the same polynomial.

Solution

Using the power method, we have the Vandermonde Matrix given by

$$
A = \begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix}
$$

With the given values for x_i, we get the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 4 & 16 & 64 \\ 1 & 8 & 64 & 512 \end{bmatrix}$$

Let $b = [1234]^T$. Then we can obtain the interpolated polynomial by solving the system $Ax = b$, where b returns the coefficients of the given interpolating polynomial. Using Mathematica to solve the system, we get the polynomial

$$p(x) = -\frac{10}{21} + \frac{7x}{4} - \frac{7x^2}{24} + \frac{x^3}{56}.$$

Using the Lagrange Form, we get the following formula:

$$p(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + 2\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} + 3\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}$$
$$+ 4\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}.$$

Substituting in the values given above for x_i and expanding the term using Mathematica, we see that we get the polynomial

$$p(x) = -\frac{10}{21} + \frac{7x}{4} - \frac{7x^2}{24} + \frac{x^3}{56}.$$

With Newton's method of divided difference, we get the equation

$$p_n(x) = f(x_0)+(x-x_0)f[x_0,x_1]+(x-x_0)(x-x_1)f[x_0,x_1,x_2]+(x-x_0)(x-x_1)(x-x_2)f[x_0,x_1,x_2,x_3]$$

which, when expanded out yields $p(x) = -\frac{10}{21} + \frac{7x}{4} - \frac{7x^2}{24} + \frac{x^3}{56}$.

9 **The Method of Undetermined Coefficients** Suppose that for some reason you wish to use a differentiation formula of the form

$$f'(a) = \frac{1}{h}(\alpha_0 f(a) + \alpha_1 f(a + \frac{h}{3}) + \alpha_2 f(a + \frac{h}{2}) + \alpha_3 f(a+h)),$$

where the αs so as to make the formula exact for polunomials of degree as high as possible. What are the αs?

Solution We expand around the points $f(a+\frac{h}{3}), f(a+\frac{h}{2}), f(a+h)$ with Taylor series and input that into the function $f'(a)$. We then group our coefficients for $f(a), f'(a), f''(a), f'''(a)$. This yields a system of four equations, where

$$(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3) = 0$$
$$h(\alpha_1/3 + \alpha_2/2 + \alpha_3) = 1$$
$$h^2(\alpha_2/18 + \alpha_2/8 + \alpha_3/2) = 0$$
$$h^3(\alpha_1/243 + \alpha_2/48 + \alpha_3/6) = 0$$

This yeilds the solution $\alpha_0 = \frac{-24}{5h}, \alpha_1 = \frac{81}{10h}, \alpha_2 = \frac{-16}{5h}, \alpha_3 = \frac{-1}{10h}$.

10 **The three term recurrence relation** Let the inner product (f, g) be defined by

$$(f, g) = \int_a^b w(x)f(x)g(x)dx.$$

Prove that the sequence of polynomisl defined by

$$\begin{aligned}
Q_n &= (x - a_n)Q_{n-1} - b_n Q_{n-2} \\
Q_0 &= 1, Q_1 = x - a_1 \\
a_n &= (xQ_{n-1}, Q_{n-1})/(Q_{n-1}, Q_{n-1}) \\
b_n &= (xQ_{n-1}, Q_{n-2})/(Q_{n-2}, Q_{n-2})
\end{aligned}$$

is orthogonal with respect to the given inner product. This proof uses the property that

$$(xf, g) = (f, xg).$$

Proof
We will show this by induction. First, note that Q_0, Q_1 are orthogonal to each other, as seen by the following:

$$\begin{aligned}
(Q_0, Q_1) &= (1, x - \frac{(x, 1)}{(1, 1)}) \\
&= \int_a^b w(x)(x - \frac{(x, 1)}{(1, 1)})dx \\
&= \int_a^b w(x)xdx - \int_a^b w(x)\frac{(x, 1)}{(1, 1)}dx \\
&= (1, x) - (1, \frac{(x, 1)}{(1, 1)}) \\
&= (1, x) - \int_a^b w(x)\frac{\int_a^b w(x)xdx}{\int_a^b w(x)} \\
&= (1, x) - (1, x) \\
&= 0.
\end{aligned}$$

For the induction, assume that any two polynomials $Q_0, Q_1, \ldots, Q_{n-1}$ are orthogonal. Then for the n^{th} case, and with $k < n$, note that

$$\begin{aligned}
(Q_n, Q_k) &= (Q_{n-1}[x - \frac{(xQ_{n-1}, Q_{n-1})}{(Q_{n-1}, Q_{n-1})}] - [\frac{(xQ_{n-1}, Q_{n-2})}{(Q_{n-2}, Q_{n-2})}]Q_{n-2}, Q_k) \\
&= (xQ_{n-1}, Q_k) - \frac{(xQ_{n-1}, Q_{n-1})}{(Q_{n-1}, Q_{n-1})}(Q_{n-1}, Q_k) - \frac{(xQ_{n-1}, Q_{n-2})}{(Q_{n-2}, Q_{n-2})}(Q_{n-2}, Q_k).
\end{aligned}$$

Note that each of the above terms has a factor of (Q_{n-i}, Q_k), with $0 < i < n$. Since we also have $k < n$, then each of these factors is 0, which means that the entire term is 0. Thus, since the above equation is 0, we have that the sequence of polynomials must be orthogonal.

11 **Recurrence Relation** Consider the inner product

$$(f,g) - \int_{-1}^{1} f(x)g(x)dx.$$

Use the recurrence relation from the previous problem to compute Q_i for $i = 0, 1, 2, 3, 4, 5$.

solution
We are already given $Q_0 = 1$. We then get that

$$Q_1 = x - \frac{\int_{-1}^{1} xdx}{\int_{-1}^{1} x^2 dx} = x$$

$$Q_2 = (x - \frac{\int_{-1}^{1} x^3 dx}{\int_{-1}^{1}} x^2 dx)x - \frac{\int_{-1}^{1} x^2 dx}{\int_{-1}^{1} dx} = x^2 - 1/3$$

$$Q_3 = x(x^2 - 1/3) - \frac{\int_{-1}^{1}(x^3 - 1/3x)xdx}{\int_{-1}^{1} x^2 dx} = x^3 - 1/3x - 3/15x = x^3 - 3/5x$$

$$Q_4 = x(x^3 - 3/5x) - \frac{\int_{-1}^{1} x(x^3 - 3/5x)(x^2 - 1/3)dx}{\int_{-1}^{1}(x^2 - 1/3)^2 dx}$$
$$= x(x^3 - 3/5x) - 9/35(x^2 - 1/3) = x^4 - \frac{6}{7}x^2 - \frac{3}{35}$$

$$Q_5 = x(x^4 - \frac{6}{7}x^2 - \frac{3}{35}) - \frac{\int_{-1}^{1} x(x^4 - \frac{6}{7}x^2 - \frac{3}{35})(x^3 - 3/5x)dx}{\int_{-1}^{1}(x^3 - 3/5x)^2 dx}$$
$$= x^5 - \frac{10}{9}x^3 - \frac{5}{21}x$$

Here, we have used the fact that a_n over the interval $[-1, 1]$ is the integral of an odd function. Because of the symmetric interval, this integral is 0.

12 **Example for Gram Schmidt Process** Use the Gram-Schmidt Process to find a basis of

$$\{1, x, e^x\}$$

that is orthonormal with respect to the inner product

$$(f,g) = \int_{0}^{1} f(x)g(x)dx.$$

Solution

We construct an orthonormal basis $V = \{v_1, v_2, v_3\}$ that spans $\{1, x, e^x\}$

by first setting $v_1 = 1$. Then we can create an orthonormal basis by setting

$$\begin{aligned} v_2 &= x - \frac{\int_0^1 x dx}{\int_0^1 dx} \\ &= x - \frac{1}{2} \end{aligned}$$

We then normalize v_2 by the following:

$$v_2 = \frac{v_2}{\sqrt{(v_2, v_2)}} = \sqrt{3}(2x-1).$$

For v_3, we follow the same procedure:

$$\begin{aligned} v_3 &= e^x - \frac{\int_0^1 e^x dx}{\int_0^1 dx} - \frac{\int_0^1 e^x(\sqrt{3}(2x-1))dx}{\int_0^1(\sqrt{3}(2x-1))^2 dx} \\ &= e^x - (e^x - 1) - (-\sqrt{3}(-3+e)) \\ &= 1 + \sqrt{3}(-3+e) \end{aligned}$$

To normalize v_3, we calculate

$$v_3 = \frac{v_3}{\sqrt{(v_3, v_3)}} = 1 + \frac{\sqrt{3}(-3+e)}{1+\sqrt{3}(-3+e)}$$

Our orthonormal basis that spans $\{1, x, e^x\}$ is given by

$$v = \{1, \sqrt{3}(2x-1), 1 + \sqrt{3}(-3+e)\}.$$