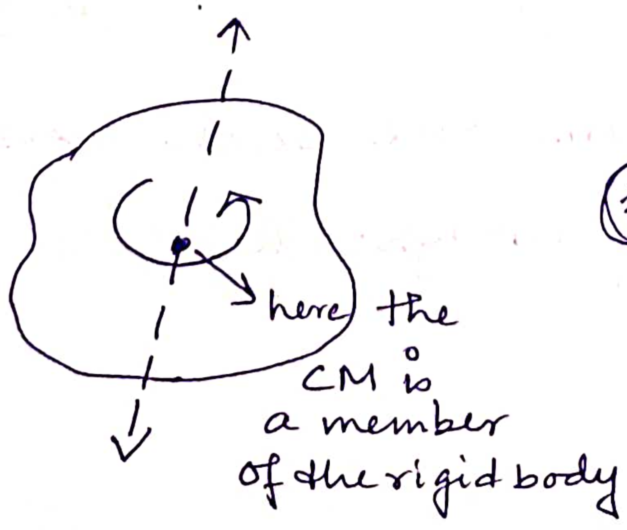
Motion of Rigid Bodies

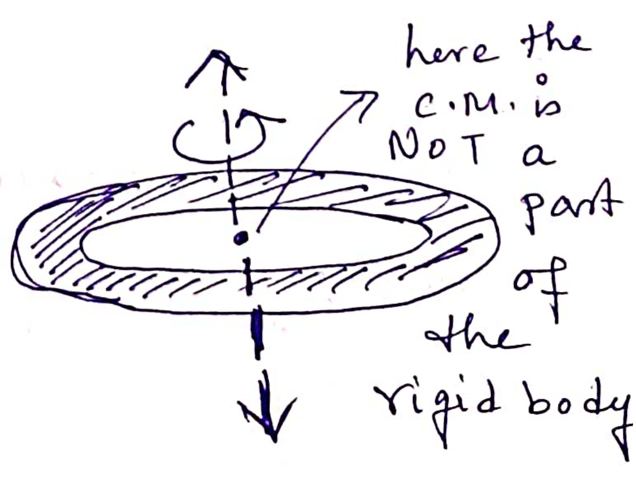
- Rigid bodies are system of particles having constant interparticle distances

- The degrees of freedom of a rigid body is = 6 as we discussed in the lecture

- 6 d.o.f. ≡ 3 translational d.o.f.
  (all the particles move)
  + 3 rotational d.o.f.
  (at least one point or a set of points passing through the rigid body do not move)

- Note that the fixed point may not be a real member of the system of particles but it represents somehow the 'society' of particles e.g. the centre of mass.



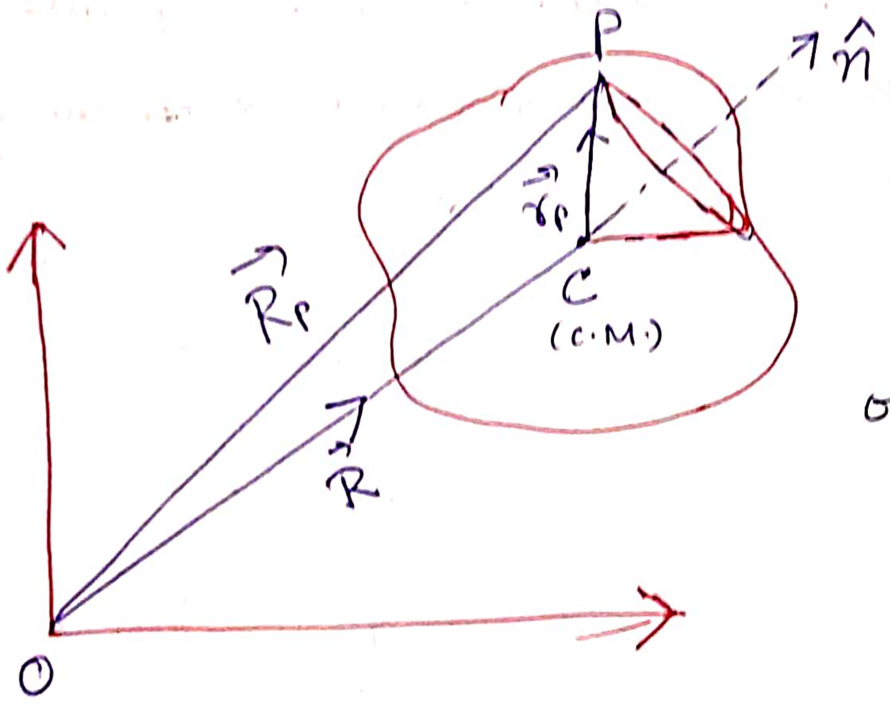


[2] Chasle's Theorem: Any general motion of a rigid body can be decomposed as the motion of a the following types:

(i) Translational motion of a representative point of the rigid body (e.g. Centre of mass)

(ii) Rotational motion w.r.t. an axis (can change with time) passing through that fixed point.

A generalized rotation can be thought to be composed of 3 elemental rotation (a rotation about a fixed axis)



Here, we assume one such axis (mentioned in (ii)) passing through the C.M. C.

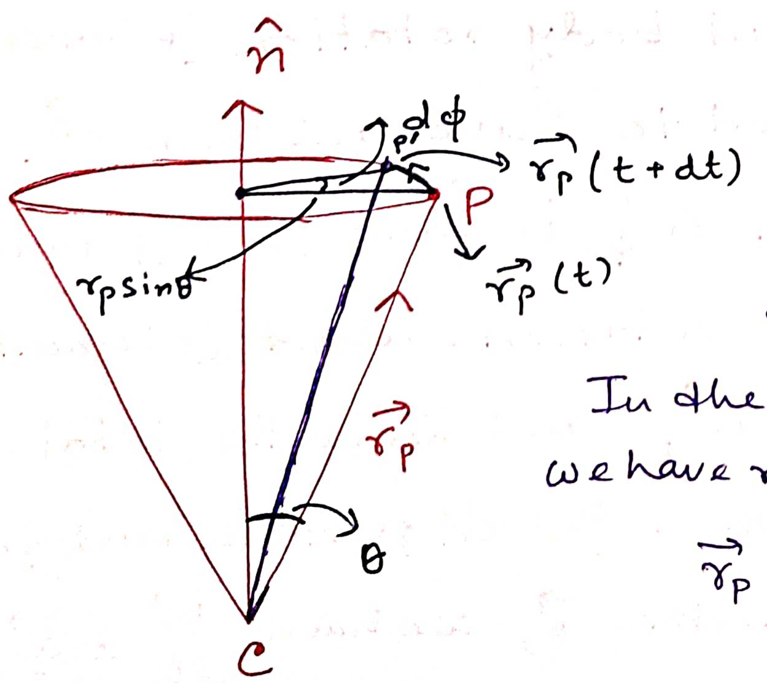
▨ Let us first take the situation where the C.M. is clamped (cannot translate) or we put an observer at C.M. with a coordinate system.

In both cases, the body will <u>experience</u> (or undergo) only rotational motion

▨ Let us do it analytically:

$$\vec{R}_p = \vec{r}_p + \vec{R} \Rightarrow \dot{\vec{R}}_p = \underbrace{\dot{\vec{r}}_p}_{\text{rotation about C.M.}} + \underbrace{\dot{\vec{R}}}_{\text{translation of C.M.}}$$

First we take the example of an elemental rotation about $\hat{n}$ by an angle ϕ



Due to rotation, the P point becomes P′ as the azimuthal angle changes by $d\phi$

In the 'almost' triangle CPP′, we have roughly, $\vec{CP} + \vec{PP'} = \vec{CP'}$

$$\vec{r}_p(t+dt) = \vec{r}_p(t) + r_p \sin\theta \, d\phi \, \hat{\phi}$$

④

$$\Rightarrow \quad \vec{r}_p(t+dt) - \vec{r}_p(t) = r_p \sin\theta \, d\phi \, \hat{\phi}$$

$$\Rightarrow \quad \frac{d\vec{r}_p}{dt} = r_p \sin\theta \frac{d\phi}{dt} \hat{\phi} = \underbrace{\left(\frac{d\phi}{dt}\hat{n}\right) \times r_p \hat{r}_p}$$

one can write that as $\angle \hat{r}_p, \hat{n} = \theta$

$$\Rightarrow \frac{d\vec{r}_p}{dt} = \vec{\omega}_\phi \times \vec{r}_p$$

(to designate the angular velocity corresponding to the elemental rotation by the angle ϕ)

∴ If the point undergoes three elemental rotations (sufficient to represent a generalized rotation) by angles ϕ, θ and ψ, the total angular velocity of the combined rotation will be $\vec{\omega} = \vec{\omega}_\varphi + \vec{\omega}_\theta + \vec{\omega}_\psi$

☑ Since, in a rigid body rotation (elemental), all particles need to have uniform $\vec{\omega}$, for the first rotation $\vec{\omega}_\varphi$ is same for all points.

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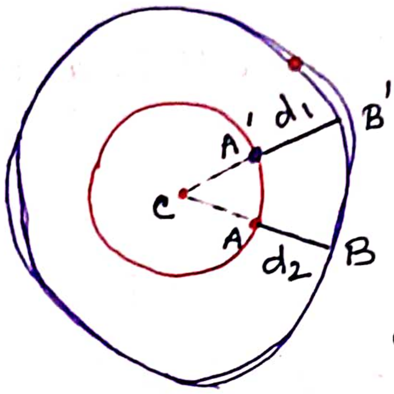
For the other two rotations ~~about~~ (elemental), similarly $\vec{\omega}_\theta$ and $\vec{\omega}_\psi$ and hence the total $\vec{\omega}$ will also be the same for all particles and for an arbitrary position vector $\vec{r}$, we have $\frac{d\vec{r}}{dt} = \underset{\text{Constant}}{\vec{\omega}} \times \vec{r}$

☑ To understand why $\vec{\omega}_\phi$ is uniform for all points, we need to take a top view of the plane of rotation:



Let us assume the rigid body is rotating and making circles in the plane of rotation with C as the centre.

Rigid body points A and B, after rotation become A' and B'.

Rigidity constraint: $AB = A'B'$

$\Rightarrow$ This is only possible when both A and B are rotated by the same angle $d\phi$ and hence have equal $\frac{d\phi}{dt} = \omega_\phi$.

☑ Coming back to our initial discussion:

we have $\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$ (with $\vec{\omega}$ being independent of $\vec{r}$)

and the total angular momentum w.r.t the centre of mass

$$\vec{L} = \int \rho \, \vec{r} \times \frac{d\vec{r}}{dt} \, d^3\vec{r}$$

$$= \int \rho \, \vec{r} \times (\vec{\omega} \times \vec{r}) \, d^3\vec{r}$$

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$$= \int \rho \left[r^2 \vec{\omega} - (\vec{r}.\vec{\omega})\vec{r} \right] d^3\vec{r}$$

Now $r^2\vec{\omega} - (\vec{r}.\vec{\omega})\vec{r}$ is a vector. We want to express it something times $\vec{\omega}$

(If we can do that then since,

linear momentum $= \vec{p} = m\vec{v}$, we can say angular momentum $= \vec{L} = \boxed{?}\,\vec{\omega}$

measure of inertia for the rotational motion)

Clearly one can understand $\boxed{?}$ cannot be a scalar. However, just like a vector can be obtained by multiplying a vector by a scalar, a vector (3×1) can be obtained by contracting a tensor (3×3) by a vector (3×1) to its right.

⚠ Maybe $\boxed{?}$ is a tensor of rank 2. (a planar matrix).

How to find that ?

For that, we first write, the i^{th} component of $\vec{L}$

$$L_i = \int \rho \left[r^2 \omega_i - \sum r_k \omega_k r_i \right] d^3\vec{r}$$

$$\Rightarrow L_i = \int \rho \Big[\overbrace{\sum_k \delta_{ik} r^2 \omega_k}^{\text{an equivalent expression for } r^2\omega_i} - \sum_k r_k \omega_k r_i \Big] d^3\vec{r}$$

$$= \sum_k \underbrace{\int \rho \big[\delta_{ik} r^2 - r_k r_i \big] d^3\vec{r}}_{\text{Call } I_{ik}} \; \omega_k$$

$$= \sum_k I_{ik}\,\omega_k \quad \text{where}$$

[the sum is over components and the integral is over the space/volume]

$$I = \int \rho \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix} d^3\vec{r}$$

(Check this carefully).

$$\Rightarrow \boxed{\vec{L} = \overline{\overline{I}} \cdot \vec{\omega}}$$

☒ Now, Linear momentum ($\vec{p}$)

= measure of inertia (m) × Linear velocity ($\vec{v}$)

Similarly $\vec{L} = \overline{\overline{I}} \cdot \vec{\omega}$ and $\overline{\overline{I}}$ can be thought to be a measure of inertia for rotational motion and is called the Moment of Inertia tensor ($\overline{\overline{I}}$)

⚠ Since, m is a scalar, $\vec{p} \parallel \vec{v}$ but $\overline{\overline{I}}$ is a tensor and hence $\vec{L}$ is NOT $\parallel$'l to $\vec{\omega}$

For an elemental rotation about z axis, evidenty $\vec{\omega} = \omega_z \hat{z}$ and then

$$\vec{L} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \omega_z \end{pmatrix}$$

(off diagonal terms) product of inertia

(diagonal terms) moments of inertia terms)

$$\vec{L} = I_{xz}\,\omega_z\,\hat{x} + I_{yz}\,\omega_z\,\hat{y} + I_{zz}\,\omega_z\,\hat{z}$$

Still $\vec{L}$ is not $\parallel$'e to $\vec{\omega}$

For a point mass moving in x-y plane, we can effectively set $z=0$ and hence $I_{xz} = 0 = I_{yz}$

but $I_{zz} = \int \rho\,(x^2+y^2)\, d^3\vec{r} \neq 0$

and therefore $\vec{L} = I_{zz}\,\omega_z\,\hat{z}$ and $\boxed{\vec{L} \parallel \vec{\omega}}$

In 3d space motion of a perfectly spherical body about 3 mutually $\perp$'r direction passing through its centre,

$$I = \int \rho \begin{pmatrix} \Phi & 0 & 0 \\ 0 & \Phi & 0 \\ 0 & 0 & \Phi \end{pmatrix} d^3\vec{r}$$

i.e. $\bar{\bar{I}} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}$

and then,

$$\vec{L} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \underset{\text{scalar}}{I}\,\vec{\omega} \Rightarrow \boxed{\vec{L} \parallel \vec{\omega}}$$

[2] Let us find the moment of inertia tensor for a uniform sphere about an axis passing through its centre.

Of course, here only (any) rotational motions about an axis (changing with time) passing through the centre of the rigid body is concerned. Also note that the centre of the sphere is the C.M. of the sphere.

(How would you show that?)

Calculate the mass moments of all the particles of the sphere (rigid) w.r.t. the centre O and we obtain

$$\int \rho \vec{r}\, d^3\vec{r} = \rho \int (r\hat{r})\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

uniform sphere

blunder as $\hat{r}$ contains θ & ϕ

$$\boxed{= 4\pi\rho \int r^3\, dr\, \hat{r}}$$

$$= \int \cancel{4\pi}\, \rho \left[r^3 dr \left(\sin\theta\cos\phi\, \hat{x} + \sin\theta\sin\phi\, \hat{y} + \cos\theta\, \hat{z} \right) \sin\theta\, d\theta\, d\phi \right]$$

$\hat{r} = \sin\theta\cos\phi\, \hat{x} + \sin\theta\sin\phi\, \hat{y} + \cos\theta\, \hat{z}$

$$= \rho \int \left[r^3 dr \sin^2\theta \, d\theta \cos\phi \, d\phi \right] \hat{x}$$

$$+ \rho \int \left[r^3 dr \sin^2\theta \, d\theta \sin\phi \, d\phi \right] \hat{y}$$

$$+ \rho \int \left[r^3 dr \cos\theta \sin\theta \, d\theta \, d\phi \right] \hat{z}$$

Taking Limit $0 \leq r \leq R$, $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$ $\rightarrow$ we get $\int \rho \vec{r} \, d^3\vec{r} = 0$.

$\Rightarrow$ The centre of the sphere is the centre of mass of it.

Now, we have to calculate, the moment of inertia tensor w.r.t an axis passing through the centre (also the c.M).

From symmetry, $I_{xx} = I_{yy} = I_{zz}$.

$$I_{zz} = \int \rho (x^2 + y^2) \, r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \int \rho (r^2 - z^2) \, r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \rho \int r^4 dr \sin\theta \, d\phi \, d\theta + \rho \int r^4 dr \cos^2\theta \, d(\cos\theta) \, d\phi$$

$$= \rho . 4\pi \frac{R^5}{5} + \rho . 2\pi \frac{R^5}{5} \left[-\frac{2}{3} \right] = \frac{4\pi \rho R^5}{5} \left[1 - \frac{1}{3} \right]$$

(with $M = \frac{4}{3}\pi R^3 \rho$) $\quad = \frac{2}{5} M R^2$

and Similarly we can write, $I_{xx} = I_{yy} = \frac{2}{5} M R^2$

and $$I_{xy} = -\int \rho \, xy \, r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= -\int \rho \, r(\cos\phi \sin\theta) . r (\sin\phi \sin\theta) \, r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= 0 \quad \left[\text{check it assuming} \int_{r=0}^{R} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \right]$$

and similarly for I_{yz} and I_{zx} and hence the moment of inertia tensor is given by

$$\overline{\overline{I}} = \begin{pmatrix} \frac{2}{5}MR^2 & 0 & 0 \\ 0 & \frac{2}{5}MR^2 & 0 \\ 0 & 0 & \frac{2}{5}MR^2 \end{pmatrix} .$$

Please note that $\overline{\overline{I}}$ is not in general a diagonal matrix.

Only when we consider the elemental rotations about particular axes, then $\overline{\overline{I}}$ becomes diagonal and the three particular directions / axes are known as Principal axes or principal directions

Relation between the total time derivatives of a vector in the C.M. frame and the body frame which is rotating w.r.t C.M:

Let us now assume two coordinate systems. For one, which we call an unprimed system, the observer is situated at C and if C is clamped, then that coordinate system is just equivalent to the lab frame. An arbitrary vector in this system can be written as : (Cartesian)

$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$ where the unit vectors are fixed in time.

For the second system, the observer is at P and hence is rotating with $\vec{\omega}$ about C. Let us assume at $t=0$, both the coordinate systems have unit vectors $\hat{i}, \hat{j}$ and $\hat{k}$ & $\hat{i}', \hat{j}'$ and $\hat{k}'$ being parallel to each other.

But with time, $\hat{i}', \hat{j}', \hat{k}'$ changes w.r.t. unprimed system and are fixed w.r.t the primed system. We also have, (Not needed but can be easy)

$\vec{A} = A_x' \hat{i}' + A_y' \hat{j}' + A_z' \hat{k}'$

As it is showed earlier that any vector $\vec{M}$ which rotates about C and is having constant magnitude,

$$\frac{d\vec{M}}{dt} = \vec{\Omega} \times \vec{M},$$ we can show

$$\frac{d}{dt}\hat{i}' = \vec{\Omega} \times \hat{i}'$$ and the same for $\hat{j}'$ and $\hat{k}'$

[all the unit vectors in primed coordinates are nothing both vectors with constant magnitude unity and rotating about an axis through C]

✻ This result is essential in deriving the Euler's Equations for rigid bodies.

We therefore have, for a vector $\vec{C}$

$$\left.\frac{d\vec{C}}{dt}\right|_{\text{lab or clamped C.M. fram}} = \left(\frac{dC_x'}{dt}\hat{i}' + \frac{dC_y'}{dt}\hat{j}' + \frac{dC_z'}{dt}\hat{k}'\right) + \left(C_x'\frac{d\hat{i}'}{dt} + C_y'\frac{d\hat{j}'}{dt} + C_z'\frac{d\hat{k}'}{dt}\right)$$

$$= \left.\frac{d\vec{C}}{dt}\right|_{\text{rotating frame}} + (\vec{\Omega} \times \vec{C})$$

(as in this frame $\hat{i}'$, $\hat{j}'$ and $\hat{k}'$ are fixed in time)

$\vec{\omega}$ of system about CM

$$\therefore \left.\frac{d\vec{C}}{dt}\right|_{\text{non rotating}} = \left.\frac{d\vec{C}}{dt}\right|_{\text{rotating}} + \vec{\Omega} \times \vec{C} \quad ✻$$