MTH112M End Semester Examination MTH112M

SHREYAS GUPTA

TOTAL POINTS

21 / 40

QUESTION 1

1 Question 1 **1 / 5**

+ 0 pts Wrong answer/ Not attempted

+ 1 pts Correctly showing that limit $$\lim_{(x,y)\rightarrow (0,0)}f(x,y)$$ exist.

+ 1 pts $$\lim_{(x,y)\rightarrow (0,0)}f(x,y)= f(0,0)$$

+ 1 pts Writing correct steps like providing suitable inequalities towards showing $$\lim_{(x,y)\rightarrow (0,0)}f(x,y)$$ exist.

+ 1 pts Computation of $$f_y(0,0)$$ is correct

✓ **+ 1 pts** *Correct computation of $$f_y(x,y)$$ for some $$(x,y)\neq (0,0)$$*

+ 2 pts Correctly showing $$\lim_{(x,y)\rightarrow (0,0)}f_y(x,y)$$ does not exist

+ 5 pts Full correct

① What is your final assertion? Is $$f$$ continuous at $$(0,0)$$ or not? You have _not said anything_ clearly. No marks.

② No, $$f_y(0,0)=-1$$.

③ No, it doesn't. It approaches $$0$$ at the same rate. And in any case, this is not a precise statement.

④ What is your final assertion? Is $$f_y$$ continuous at $$(0,0)$$ or not? It is, in fact, discontinuous at $$(0,0)$$. You'll get 1 out of 3 here.

QUESTION 2

2 Question 2 **1 / 5**

+ 0 pts Wrong answer/ Not attempted

✓ **+ 1 pts** *Some steps provided to find the equation of normal*

+ 2 pts Correct equation is provided for the normal to the given curve

+ 1 pts Finding a condition to show the curve satisfies an equation of circle

+ 2 pts Correctly showing that an arbitrary point on the given curve satisfies an equation of a circle

+ 1 pts For the correct observation that $$R(t)$$ lies on a circle of constant curvature.

+ 5 pts Full correct answer

QUESTION 3

3 Question 3 **5 / 5**

+ 0 pts Wrong answer/ Not attempted

+ 1 pts The given surface is a level surface $$\left\{ (x,y,z)\in \mathbb{R}^3: f(x,y,z)= 0 \right\}$$ where $$f(x,y,z)= x^2+2xy-2y-z $$.

+ 1 pts The normal direction of the tangent plane at an arbitrary point on the given level surface is given by $$\nabla f (x,y,z)$$

+ 1 pts $$\nabla f (x,y,z)= (2x+2y, ~2x-2,~-1)$$

+ 1 pts Finding condition for the tangent plane parallel to xy-plane

+ 1 pts Computation of the required point $$(1, -1,1)$$ on the given surface is correct.

✓ **+ 5 pts** *All correct*

QUESTION 4

4 Question 4 **2 / 7**

+ 0 pts Wrong answer/ Not attempted

✓ **+ 1 pts** *Observing that $$(\frac{2}{3}, \frac{1}{3})$$ is a critical point of the given function which is in the interior of $$D$$.*

+ 1 pts Observing $$(\frac{1}{\sqrt{3}}, 0)$$ is an extremum point of $$f$$

+ 1 pts Observing that $$(0,0)$$ is an is an extremum point of $$f$$

✓ **+ 1 pts** *Observing $$(2,0)$$ is an extremum point of $$f$$*

+ 1 pts Observing that $$(0,2)$$ is an is an extremum point of $$f$$

+ 1 pts Observing that $$(1,1)$$ is an is an extremum point of $$f$$

+ 1 pts $$(2,0)$$ is a point of absolute maximum of the given function.

+ 7 pts All correct

QUESTION 5

5 Question 5 **5 / 5**

+ 0 pts Wrong answer/ Not attempted

+ 1 pts Using Green's theorem the given line integral $$\frac{1}{2}\int_C -y dx + xdy= \int\int_D (N_x - M_y) ~dx dy$$

+ 1 pts Functions $$M(x,y)$$ and $$N(x,y)$$ are given correctly to apply Green's theorem

+ 1 pts Shown that $$N_x - M_y =1$$

+ 1 pts $$\int\int_D (N_x - M_y) ~dx dy= \int\int_D ~dx dy= Area (D)$$

+ 1 pts Area(D) = $$\int_{a}^{b}f(x)~dx$$

✓ **+ 5 pts** *All correct*

QUESTION 6

6 Question 6 **3 / 6**

+ 0 pts Wrong answer/ Not attempted

✓ **+ 1 pts** *Writing $$div(F)$$ formula correctly*

✓ **+ 1 pts** *div$$(F)= 15 (x^2 +y^2)$$*

✓ **+ 1 pts** *By divergence theorem we have the required surface integral is a triple integral $$\int \int \int_D div(F) dV$$*

+ 1 pts Change of variables in the computation is correct

+ 2 pts Correct computation of the triple integral is $$160 \pi$$

+ 6 pts All correct

QUESTION 7

7 Question 7 **4 / 7**

+ 0 pts Wrong answer/ Not attempted

✓ **+ 1 pts** *By Stokes' theorem the line integral is equal to the surface integral $$\int \int_S Curl (F).\overrightarrow{n} d\sigma$$*

✓ **+ 2 pts** *Curl$$(F)= (xz, -yz, x^2+y^2)$$*

✓ **+ 1 pts** *Computation of $$\overrightarrow{n} $$ is correct*

+ 1 pts Change of variables in the computation is correct

+ 2 pts Correct computation of the surface integral is $$\int \int_S Curl (F).\overrightarrow{n} d\sigma= \frac{\pi}{2}$$

+ 7 pts All correct

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Question 7. Consider the surface $S = \{(x,y,z) \in \mathbb{R}^3 :\ z^2 = x^2 + y^2,\ 0 \leq z \leq 1\}$ and let $F(x,y,z) = (\sin x - \frac{y^3}{3},\ \cos y + \frac{x^3}{3},\ xyz)$ be the function defined on $\mathbb{R}^3$. Use Stokes' Theorem to evaluate the line integral $\int_C F.dR$ where C is the boundary curve of S. [7 Marks]

Answer :

By Stokes Theorem :

$$\int_C F \cdot dR = \iint_S (\vec{\nabla} \times \vec{F}) \circ \vec{n}\ d\sigma = \iint_S (\vec{\nabla} \times \vec{F}) \circ \vec{n} \sqrt{1 + \overset{\frac{\partial g}{\partial x}}{g_x^2} + \overset{\frac{\partial g}{\partial y}}{g_y^2}}\ dx\,dy$$

$g(x,y,z) = x^2+y^2-z^2$
$=$ level surface

~~$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ s & & \end{vmatrix} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ & & \\ \sin x - \frac{y^3}{3} & & \end{vmatrix}$~~

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin x - \frac{y^3}{3} & \cos y + \frac{x^3}{3} & xyz \end{vmatrix}$$ ~~$= (x^2+y^2)\hat{i} + (-yz)\hat{j}$~~

$$= xz\hat{i} - yz\hat{j} + (x^2+y^2)\hat{k}$$

~~$\hat{n} = \frac{\vec{\nabla} F}{\| \vec{\nabla} F \|} = \frac{\cos x\,\hat{i} - \sin y\,\hat{j} + xy\,\hat{k}}{\sqrt{\cos^2 x + \sin^2 y + x^2y^2}}$~~

level surface $\Rightarrow x^2+y^2-z^2 = 0 = g(x,y,z)$

$$\hat{n} = \frac{\vec{\nabla} g}{\|\vec{\nabla} g\|} = \frac{2x\hat{i} + 2y\hat{j} - 2z\hat{k}}{\sqrt{4(x^2+y^2+z^2)}} = \frac{x\hat{i} + y\hat{j} - z\hat{k}}{\sqrt{x^2+y^2+z^2}}$$

$$\iint_S (\vec{\nabla} \times \vec{F}) \circ \vec{n} \times \sqrt{1 + g_x^2 + g_y^2}\ dx\,dy$$

$$= \iint_S \left(xz\hat{i} - yz\hat{j} + (x^2+y^2)\hat{k}\right) \circ \left(\frac{x\hat{i} + y\hat{j} - z\hat{k}}{\sqrt{x^2+y^2+z^2}}\right) \cdot \sqrt{1+4x^2+4y^2}\ dx\,dy$$

$$-2y^2z = -2y^2(\sqrt{x^2+y^2})$$

$$= \iint_S \frac{\cancel{x^2z} - y^2z - \cancel{x^2z} - y^2z}{\sqrt{2}\sqrt{x^2+y^2}} \cdot \sqrt{1+4x^2+4y^2}\ dx\,dy$$

In polar form $\Rightarrow r \in [0,1)$ $\theta \in [0, 2\pi$

$$\iint_S -\sqrt{2}\, r^3 \sqrt{1+4r^2}\, \sin^2\theta\ dr\,d\theta = \int_0^{2\pi}\int_0^1 (-\sqrt{2}\sin^2\theta)\, r^3\sqrt{1+4r^2}\ dr\,d\theta$$

$=$

End Semester Examination
MTH112M/MTH101A-Part II

Date : 19/11/2023 | Time : 5:30 - 7:15 pm | Total Marks : 40

NAME : SHREYAS GUPTA	ROLL : 230991

INSTRUCTIONS: (Read carefully)

- Please enter your NAME in CAPITAL LETTERS and ROLL NUMBER in the space provided on EACH page.
- Only those booklets with name and roll number on every page will be graded. All other booklets will NOT be graded.
- This answer booklet has 8 pages with 7 questions. In case you have received a wrongly printed or missing pages question paper, ask for the replacement immediately.
- Answer each question ONLY in the space provided. Answers written outside the space provided for it WILL NOT be considered for grading. So remember to use space judiciously.
- For rough work, separate sheets will be provided to you. Write your name and roll number on rough sheets as well. However, they WILL NOT be collected back along with the answer booklet.
- No calculators, mobile phones, smart watches or other electronic gadgets are permitted in the exam hall.
- Notations: All notations used are as discussed in class.
- All questions are compulsory.
- Do NOT remove any of the sheets in this booklet.

Name: Shreyas Gupta Roll: 230991

Question 1. Consider $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{x^2y-y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Check whether the functions $f(x,y)$ and $f_y(x,y)$ are continuous at $(0,0)$? [2+3=5 Marks]

Answer :

(i) For continuity :

$|f(x,y) - f(0,0)| \to 0$ for $(x,y) \to (0,0)$

$\left|\frac{x^2y-y^3}{x^2+y^2} - 0\right| = \left|\frac{x^2y-y^3}{x^2+y^2}\right| \to 0$ ~~as for x=y~~ $\cancel{\Rightarrow \left|\frac{x^3-x^3}{x^2+x^2}\right|}$ ①

as $(x,y) \to (0,0)$

~~non zero~~ ~~always positive~~

(ii) $\frac{\partial(f(x,y))}{\partial y} = \frac{(x^2-3y^2)(x^2+y^2)-(2y)(x^2y-y^3)}{(x^2+y^2)^2}$

$= \frac{x^4-y^4-4x^2y^2}{(x^2+y^2)^2}$ for $(x,y) \neq (0,0)$

for $(x,y) = (0,0) \Rightarrow f_y(x,y) = 0$ ②

Checking continuity :

$|f_y(x,y) - f_y(0,0)| = \left|\frac{x^4-y^4-4x^2y^2}{(x^2+y^2)^2} - 0\right|$

Clearly, $(x^2+y^2)^2$ approaches zero slowly than $x^4-y^4-4x^2y^2$ ③

$\therefore \left|\frac{x^4-y^4-4x^2y^2}{(x^2+y^2)^2}\right| \to 0$ ④ for $(x,y) \to (0,0)$

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Question 6. Let $D = \{(x,y,z) \in \mathbb{R}^3 : 0 \le z \le 4 - x^2 - y^2\}$ and S be the boundary surface of D. Consider $F(x,y,z) = (5x^3 + z^{19},\ 5y^3 + xe^{11},\ x + \cos xy^{23})$ defined on $\mathbb{R}^3$. Evaluate the surface integral $\iint_S F.\vec{n}\, d\sigma$ where $\vec{n}$ is the outward unit normal to the surface S. [6 Marks]

By Divergence theorem:

$\iiint \operatorname{div}(F)\, dV = \iint_S F.\vec{n}\, d\sigma$

where $\operatorname{div}(F) = \vec{\nabla}.\vec{F} = \vec{\nabla}.\vec{F}(x,y,z)$

$\vec{\nabla}.\vec{F} = \frac{\partial}{\partial x}(5x^3+z^{19}) + \frac{\partial}{\partial y}(5y^3+xe^{11}) + \frac{\partial}{\partial z}(x+\cos xy^{23})$

$= 15x^2 + 15y^2 = 15(x^2+y^2)$

$\iiint 15(x^2+y^2)\ \cancel{dx\,dy\,dz}\ dV = \int_{-2}^{2}\left(\int_{-2}^{2}\left(\int_{0}^{4-x^2-y^2} 15(x^2+y^2)\,dz\right)dy\right)dx$ ✓

$= \int_{-2}^{2}\left(\int_{-2}^{2}\left[15(x^2+y^2)\left[4-x^2-y^2\right]\right]dy\right)dx$

$= 15\int_{-2}^{2}\int_{-2}^{2}(4x^2+4y^2-x^4-2x^2y^2-y^4)\,dy\,dx$ as function is odd

$= 2\times15\int_{-2}^{2}\left[4x^2y + \frac{4y^3}{3} - x^4y - \frac{2x^2y^3}{3} - \frac{y^5}{5}\right]_0^2 dx$

$= 30\int_{-2}^{2}\left(8x^2 + \frac{32}{3} - 2x^4 - \frac{16x^3}{3} - \frac{32}{5}\right)dx$

~~$= 30\int_{-2}^{2}\frac{8x^3}{3} + 64$~~ $= 30\left[\frac{8x^3}{3} + \frac{64}{15}x - \frac{2x^5}{5} - \frac{16x^3}{9}\right]_{-2}^{2}$

$= 60\left[\frac{64}{3} + \frac{128}{15} - \frac{64}{5} - \frac{128}{9}\right]$

$= 60\times64\left[\frac{1}{3} + \frac{2}{15} - \frac{1}{5} - \frac{2}{9}\right]$

$= \cancel{60}^{4}\times64\cdot\frac{2}{\cancel{45}\,3} = \frac{512}{3}$

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Question 5. Let $f:[a,b]\to\mathbb{R}$ be a real valued smooth function with $f(x)>0$ for all $x\in[a,b]$. Consider the closed region $D=\{(x,y)\in\mathbb{R}^2 : a\le x\le b,\ 0\le y\le f(x)\}$ in xy-plane. Let C be the boundary curve of D oriented counterclockwise. Prove that the line integral $\frac{1}{2}\int_C(-ydx+xdy)=\int_a^b f(x)dx$. [5 Marks]

Answer :

By Green's theorem, area enclosed by a simple closed piecewise curve = line integral along its boundary of it

$$\oint_C (M\,dx+N\,dy) = \iint_S \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$$ where $F(x,y) = \{M(x,y), N(x,y)\}$

Here $M(x,y)=x$ and $N(x,y)=f(x)$

$$\therefore \iint_S \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy = \int_a^b\int_0^{f(x)}\left(\frac{\partial f(x)}{\partial x}-\frac{\partial x}{\partial y}\right)dy\,dx$$

$$= \int_a^b\int_0^{f(x)}\left(\frac{\partial f(x)}{\partial x}dy\right)dx = \int_a^b f(x)\,dx$$ = Area of D region

Also, area enclosed by any region $= \iint_S dx\,dy = \iint_S\left(\frac{1}{2}+\frac{1}{2}\right)dx\,dy$

~~By green's theorem where~~ $-\frac{\partial M(x,y)}{\partial y}=\frac{1}{2}$ and $\frac{\partial N(x,y)}{\partial x}=\frac{1}{2}$

$\Rightarrow M=-\frac{y}{2}$ and $N=\frac{x}{2}$,

By Green's Theorem, $\iint_S dx\,dy = \int_C\left(-\frac{y}{2}dx+\frac{x}{2}dy\right) = \int_C M\,dx+N\,dy$

$$= +\frac{1}{2}\int_C(-y\,dx+x\,dy)$$

$$\iint_S dx\,dy = \int_a^b f(x)\,dx$$ = Area of region

Therefore, $\frac{1}{2}\int_C(-y\,dx+x\,dy)=\int_a^b f(x)\,dx$

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Question 2. Let $R:(a,b)\to\mathbb{R}^2$ be a smooth map with $R'(t)\neq(0,0)$ for all $t\in(a,b)$ and let C be the smooth curve in xy-plane defined by $C=\{R(t):\ t\in(a,b)\}$. Find the equation of the normal line to the curve C at $R(t)$. Prove that if all the normal lines to the curve C pass through a fixed point $(p,q)\in\mathbb{R}^2$ then the curve C is contained in a curve of constant curvature. [2+3=5 Marks]

Answer :

$R(t) = x(t)\,\hat{i} + y(t)\,\hat{j}$

$N(t) = \frac{T'(t)}{\|T'(t)\|}$; $T(t) = \frac{\frac{dR}{dt}}{\|\frac{dR}{dt}\|} = \frac{R'(t)}{\|R'(t)\|}$

$T'(t) = \frac{R''(t)}{\|R'(t)\|}$

$$N(t) = \frac{\frac{R''(t)}{\|R'(t)\|}}{\frac{\|R''(t)\|}{\|R'(t)\|}} = \frac{R''(t)}{\|R''(t)\|}$$

~~for $t=t_0$, $R(t)=p\hat{i}$~~

for $t=t_0$, ~~$R(t)=p\hat{i}+q\hat{j}$~~ $N(t_0) = p\hat{i}+q\hat{j}$

Curvature $= \frac{dT}{ds} = \frac{T'(t)}{\|\frac{dR}{dt}\|} = \frac{\frac{R''(t)}{\|R'(t)\|}}{\|\frac{dR}{dt}\|}$

$\hookrightarrow \frac{\frac{dT}{dt}}{\frac{ds}{dt}}$

$$= \frac{N(t)\times\|R''(t)\|}{(\|R'(t)\|)^2}$$

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Question 3. Let $S = \{(x, y, z) \in \mathbb{R}^3 : \ z = x^2 + 2xy - 2y\}$ be a surface in $\mathbb{R}^3$. Find a point on S at which the tangent plane is parallel to the xy plane. [5 Marks]

Answer :

$z = f(x,y)$

Tangent plane $\Rightarrow$ $z = z_0 + f_x(x-x_0) + f_y(y-y_0)$

$f_x = \frac{\partial f}{\partial x}$, $f_y = \frac{\partial f}{\partial y}$.

~~$z = x_0^2 + 2x_0y_0 - 2y_0 +$~~ Suppose tangent plane is parallel to the ~~#~~ xy plane at point (x_0, y_0, z_0)

$= (x_0, y_0, f(x_0, y_0))$

$z = \cancel{x_0^2+2}\ \underbrace{x_0^2 + 2x_0y_0 - 2y_0}_{\text{Let } k} + 2(x_0+y_0)(x-x_0) + 2(x_0-1)(y-y_0)$

$\Rightarrow$

$2(x_0+y_0)(x-x_0) + 2(x_0-1)(y-y_0) - z + k = 0$

$\vec{n}$ = Normal vector $= 2(x_0+y_0)\hat{i} + 2(x_0-1)\hat{j} - \hat{k}$

is parallel to $\hat{k}$.

$\therefore$ $\vec{n} \times \vec{k} = 0 \Rightarrow 2(x_0+y_0)(-\hat{j}) + 2(x_0-1)\hat{i} = (0,0,0)$

$2(x_0-1) = 0 \Rightarrow \boxed{x_0 = 1}$

and $-2(x_0+y_0) = 0 \Rightarrow -2(y_0+1) = 0 \Rightarrow \boxed{y_0 = -1}$

~~The point is (1, 1,~~

$z_0 = x_0^2 + 2x_0y_0 - 2y_0$

$= 1 + 2(1)(-1) - 2(-1)$

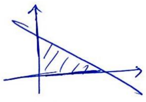
$= 1$

The point is $(1, -1, 1)$.

Name: Roll:

Question 4. Consider $D = \{(x, y) \in \mathbb{R}^2 : \ x \geq 0, y \geq 0 \text{ and } x + y \leq 2\}$. Find the absolute maximum of $f : D \longrightarrow \mathbb{R}$ defined by $f(x, y) = x^3 + y^2 - xy - x$. [7 Marks]

Answer :



For a local maxima to occur, $f_x = \frac{\partial f}{\partial x} = 0$, $f_y = \frac{\partial f}{\partial y} = 0$, for the 1st derivative ~~test.~~

For second derivative ~~test~~, $(f_{xx}^2 + f_{yy}^2 - 2f_{xy}^2) > 0$ and $f_{xx} < 0$.

$\left(f_{xx} = \frac{\partial^2 f}{\partial x^2}, f_{yy} = \frac{\partial^2 f}{\partial y^2}, f_{xy} = \frac{\partial^2 f}{\partial y \partial x}\right) \downarrow \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$

$\frac{\partial f}{\partial x} = 3x^2 - y - 1 = 0$

$\frac{\partial f}{\partial y} = \cancel{2y} \ 2y - x = 0$

$\Rightarrow$ Critical points by solving :

~~$3x^2$~~ $x = 2y$

~~$3x^2$~~ $12y^2 - y - 1 = 0$

$y = \frac{1 \pm 7}{24} \begin{cases} \frac{1}{3} \\ -\frac{1}{4} \ \times \end{cases}$ (as $y \geq 0$)

$x = \frac{2}{3} = 2y$

Only one critical point in the domain : $\left(\frac{2}{3}, \frac{1}{3}\right)$

$f_{xx} = 6x = 4$ $f_{yy} = 2$ $f_{xy} = -1$

$f_{xx}^2 + f_{yy}^2 - 2f_{xy}^2 = 16 + 4 - 2 > 0$

$= 18 > 0$

$f_{xx} = 4 > 0$ $\therefore$ minima not maxima.

Now, ~~for boun~~ maxima lies on boundary points,

for $(2,0) \rightarrow f(x,y) = 6$, $f(x,y)$ is maximum

For all points lying on $x+y=2$, $x=0$, $y=0$ trivially.,

~~as x^3 is ~~