Seminar Analiză Matematică

Integrale cu parametri

1 Integrale cu parametri

Exercițiul 1. Să se calculeze următoarele integrale:

1. $F(y) = \int_0^{\frac{\pi}{2}} \ln\left(y^2 - \sin^2 x\right) dx,\ y > 1$

   Soluţie Funcţia $f(x,y) = \ln\left(y^2 - \sin^2 x\right)$ este continuă pe $\left[0, \frac{\pi}{2}\right] \times (1,\infty)$, admite derivata parţială în raport cu y

$$\frac{\partial f}{\partial y} = \frac{2y}{y^2 - \sin^2 x}$$

   care este continuă pe $\left[0, \frac{\pi}{2}\right] \times (1,\infty)$. Din formula lui Leibniz avem

$$F'(y) = \int_0^{\frac{\pi}{2}} \frac{\partial f}{\partial y}(x,y)dx = 2y\int_0^{\frac{\pi}{2}} \frac{1}{y^2 - \sin^2 x}dx = 2y \int_{\left[0,\frac{\pi}{2}\right)} \frac{1}{y^2 - \sin^2 x}dx$$

   Facem schimbarea de variabilă $\tan x = u$.

$$\begin{aligned}
F'(y) &= 2y\int_0^\infty \frac{1}{y^2 - \dfrac{u^2}{1+u^2}} \cdot \frac{1}{1+u^2}du \\
&= 2y\int_0^\infty \frac{1}{y^2 + u^2(y^2-1)}du \\
&= \frac{2y}{y^2-1} \cdot \frac{\sqrt{y^2-1}}{y} \arctan \frac{u\sqrt{y^2-1}}{y}\Big|_0^\infty \\
&= \frac{\pi}{\sqrt{y^2-1}}
\end{aligned}$$

Deci $F(y) = \pi \ln\left(y + \sqrt{y^2-1}\right) + C$. Determinăm constanta C.

$$\begin{aligned}
C &= F(y) - \pi \ln\left(y + \sqrt{y^2-1}\right) \\
&= \int_0^{\frac{\pi}{2}} \ln\left(y^2 - \sin^2 x\right) dx - \pi \ln\left(y + \sqrt{y^2-1}\right) \\
&= \int_0^{\frac{\pi}{2}} \left(\ln y^2 + \ln\left(1 - \frac{\sin^2 x}{y^2}\right)\right) dx - \pi \ln\left(y + \sqrt{y^2-1}\right) \\
&= \int_0^{\frac{\pi}{2}} \ln\left(1 - \frac{\sin^2 x}{y^2}\right) dx + \pi \ln \frac{y}{y + \sqrt{y^2-1}}
\end{aligned}$$

Deoarece

$$\begin{aligned}
\left| \int_0^{\frac{\pi}{2}} \ln\left(1 - \frac{\sin^2 x}{y^2}\right) dx \right| &\le \int_0^{\frac{\pi}{2}} \left|\ln\left(1 - \frac{\sin^2 x}{y^2}\right)\right| dx \\
&\le \int_0^{\frac{\pi}{2}} \left|\ln\left(1 - \frac{1}{y^2}\right)\right| dx \\
&= \frac{\pi}{2} \ln \frac{y^2}{y^2-1} \underset{y\to\infty}{\longrightarrow} 0
\end{aligned}$$

rezultă că $\lim\limits_{y\to\infty} \int_0^{\frac{\pi}{2}} \ln\left(1 - \frac{\sin^2 x}{y^2}\right) dx = 0$. Atunci

$$\begin{aligned}
C &= \lim_{y\to\infty} \int_0^{\frac{\pi}{2}} \ln\left(1 - \frac{\sin^2 x}{y^2}\right) dx + \pi \lim_{y\to\infty} \ln \frac{y}{y + \sqrt{y^2-1}} \\
&= -\pi \ln 2.
\end{aligned}$$

Astfel

$$F(y) = \pi \ln \frac{y + \sqrt{y^2-1}}{2}.$$

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2. $F(y) = \int_0^{\pi} \ln\left(1 - 2y\cos x + y^2\right) dx,\ |y| < 1$

Soluţie Facem observaţia că $1 - 2y\cos x + y^2 = \sin^2 x + (\cos x - y)^2 > 0$ pentru $x \in [0, \pi]$ şi $y \in (-1, 1)$. Funcţia $f(x, y) = \ln\left(1 - 2y\cos x + y^2\right)$

este continuă pe $[0,\pi]\times(-1,1)$, admite derivata parţială în raport cu y

$$\frac{\partial f}{\partial y}=\frac{2y-2\cos x}{1-2y\cos x+y^2}$$

care este continuă pe $[0,\pi]\times(-1,1)$. Din formula lui Leibniz avem

$$F'(y) \;=\; \int_0^{\pi}\frac{\partial f}{\partial y}(x,y)dx=2\int_0^{\pi}\frac{y-\cos x}{1-2y\cos x+y^2}dx=2\int_{[0,\pi)}\frac{y-\cos x}{1-2y\cos x+y^2}dx$$

Facem schimbarea de variabilă $\tan\frac{x}{2}=u$. Avem

$$\begin{aligned}
F'(y) &= 2\int_0^{\infty}\frac{y-\frac{1-u^2}{1+u^2}}{1-2y\frac{1-u^2}{1+u^2}+y^2}\cdot\frac{1}{1+u^2}du\\
&= 2\int_0^{\infty}\frac{(1+y)u^2+y-1}{\left[(1+y)^2u^2+(1-y)^2\right](u^2+1)}du\\
&\underset{y\neq 0}{=} \frac{y-1}{y(y+1)}\int_0^{\infty}\frac{1}{u^2+\left(\frac{1-y}{1+y}\right)^2}du+\frac{1}{y}\int_0^{\infty}\frac{1}{u^2+1}du\\
&= \frac{y-1}{y(y+1)}\cdot\frac{1+y}{1-y}\arctan\frac{u(1+y)}{1-y}\bigg|_0^{\infty}+\frac{1}{y}\arctan u\bigg|_0^{\infty}\\
&= -\frac{\pi}{2y}+\frac{\pi}{2y}=0
\end{aligned}$$

şi $F'(0)=0$ (din continuitatea derivatei). Deci $F'(y)=0$, $(\forall)y\in(-1,1)$, prin urmare $F(y)=C$, $y\in(-1,1)$. Cum $F(0)=0$, rezultă $C=0$.
Astfel $F(y)=0$, $y\in(-1,1)$.

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3. $F(y)=\int_0^{\frac{\pi}{2}}\frac{1}{\cos x}\ln\frac{1+y\cos x}{1-y\cos x}dx$, $|y|<1$

Soluţie

Metoda 1 Integrala este doar aparent improprie. Considerăm sub integrală funcţia

$$f(x,y)=\begin{cases}\dfrac{1}{\cos x}\ln\dfrac{1+y\cos x}{1-y\cos x} & ,x\in\left[0,\dfrac{\pi}{2}\right)\\ 2y & ,x=\dfrac{\pi}{2}\end{cases},\ y\in(-1,1).$$

Funcţia f este continuă pe $\left[0, \frac{\pi}{2}\right] \times (-1, 1)$, admite derivata parţială în raport cu y

$$\frac{\partial f}{\partial y} = \begin{cases} \dfrac{2}{1 - y^2 \cos^2 x} & , x \in \left[0, \dfrac{\pi}{2}\right) \\ 2 & , x = \dfrac{\pi}{2} \end{cases} , \; y \in (-1, 1)$$

care este continuă pe $\left[0, \frac{\pi}{2}\right] \times (-1, 1)$. Din formula lui Leibniz avem

$$F'(y) = \int_0^{\frac{\pi}{2}} \frac{\partial f}{\partial y}(x, y) dx = \int_0^{\frac{\pi}{2}} \frac{2}{1 - y^2 \cos^2 x} dx = \int_{\left[0, \frac{\pi}{2}\right)} \frac{2}{1 - y^2 \cos^2 x} dx$$

Facem schimbarea de variabilă $\tan x = u$.

$$\begin{aligned} F'(y) &= \int_0^{\infty} \frac{2}{1 - y^2 \dfrac{1}{1 + u^2}} \frac{1}{1 + u^2} du \\ &= 2 \int_0^{\infty} \frac{1}{1 - y^2 + u^2} du \\ &= \frac{2}{\sqrt{1 - y^2}} \arctan \frac{u}{\sqrt{1 - y^2}} \Big|_0^{\infty} \\ &= \frac{\pi}{\sqrt{1 - y^2}} \end{aligned}$$

Deci $F(y) = \pi \arcsin y + C$. Din $F(0) = 0$, obţinem $C = 0$. Astfel

$$F(y) = \pi \arcsin y.$$

Metoda 2 Deoarece

$$\frac{1}{\cos x} \ln \frac{1 + y \cos x}{1 - y \cos x} = 2y \int_0^1 \frac{1}{1 - u^2 y^2 \cos^2 x} du$$

avem

$$F(y) = 2y \int_0^{\frac{\pi}{2}} \left(\int_0^1 \frac{1}{1 - u^2 y^2 \cos^2 x} du \right) dx.$$

Funcţia $g(x, u) = \dfrac{1}{1 - u^2 y^2 \cos^2 x}$ este continuă pe $\left[0, \frac{\pi}{2}\right] \times [0, 1]$, deci putem permuta integralele. Obţinem

$$\begin{aligned}
F(y) &= 2y\int_0^1\left(\int_0^{\frac{\pi}{2}}\frac{1}{1-u^2y^2\cos^2 x}dx\right)du \\
&\underset{\tan x=t}{=} 2y\int_0^1\left(\int_0^{\infty}\frac{1}{1-u^2y^2+t^2}dt\right)du \\
&= \pi y\int_0^1\frac{1}{\sqrt{1-u^2y^2}}du \\
&= \pi\arcsin(uy)\Big|_0^1 = \pi\arcsin y.
\end{aligned}$$

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4. $F(a,b)=\int\limits_0^1 f(x,a,b)dx$, unde

$$f(x,a,b)=\begin{cases}\cos\left(\ln\dfrac{1}{x}\right)\dfrac{x^b-x^a}{\ln x} & , x\in(0,1)\\ 0 & , x=0\\ b-a & , x=1\end{cases}\quad ; 0<a<b.$$

Soluţie Pentru $x\in(0,1)$ avem

$$\begin{aligned}
f(x,a,b) &= \cos\left(\ln\frac{1}{x}\right)\frac{x^b-x^a}{\ln x}\\
&= \cos\left(\ln\frac{1}{x}\right)\frac{1}{\ln x}x^y\Big|_a^b\\
&= \cos\left(\ln\frac{1}{x}\right)\frac{1}{\ln x}\int_a^b x^y\ln x\,dy\\
&= \cos\left(\ln\frac{1}{x}\right)\int_a^b x^y dy
\end{aligned}$$

Atunci

$$F(a,b)=\int_0^1\left(\int_a^b\cos\left(\ln\frac{1}{x}\right)x^y dy\right)dx.$$

Deoarece funcţia

$$g(x,y)=\begin{cases}\cos\left(\ln\dfrac{1}{x}\right)x^y & , x\in(0,1]\\ 0 & , x=0\end{cases}\quad ; y\in[a,b]$$

este continuă pe $[0,1] \times [a,b]$, putem permuta integralele. Obţinem

$$\begin{aligned} F(a,b) &= \int_a^b \left(\int_0^1 \cos\left(\ln \frac{1}{x}\right) x^y dx \right) dy \\ &\underset{\ln \frac{1}{x} = u}{=} \int_a^b \left(\int_0^\infty \cos u e^{-(y+1)u} du \right) dy \end{aligned}$$

Calculăm integrala din paranteză, aplicând de două ori formula de integrare prin părţi.

$$\begin{aligned} I(y) &= \int_0^\infty \cos u e^{-(y+1)u} du = \int_0^\infty (\sin u)' e^{-(y+1)u} du \\ &= \sin u e^{-(y+1)u} \Big|_0^\infty + (y+1) \int_0^\infty \sin u e^{-(y+1)u} du \\ &= -(y+1) \int_0^\infty (\cos u)' e^{-(y+1)u} du \\ &= -(y+1) \cos u e^{-(y+1)u} \Big|_0^\infty - (y+1)^2 \int_0^\infty \cos u e^{-(y+1)u} du \\ &= (y+1) - (y+1)^2 I(y), \end{aligned}$$

deci

$$I(y) = \frac{y+1}{1+(y+1)^2}.$$

Atunci

$$\begin{aligned} F(a,b) &= \int_a^b \frac{y+1}{1+(y+1)^2} dy \\ &= \frac{1}{2} \ln\left(1+(y+1)^2\right) \Big|_a^b \\ &= \frac{1}{2} \ln \frac{1+(b+1)^2}{1+(a+1)^2}. \end{aligned}$$

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