Anghelus, Examen analiză matematică
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1. $\sum_{n=1}^{\infty} n \cdot a^n, a>0$

Notăm $b_n = n \cdot a^n$

Folosind criteriul raportului D'Alembert

$$\Rightarrow \lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \lim_{n\to\infty} \frac{(n+1)\cdot a^{\cancel{n+1}}}{n\cdot \cancel{a^n}} =$$

$$= \lim_{n\to\infty} a \cdot \frac{n+1}{n} =$$

$$= \lim_{n\to\infty} a \cdot \frac{\cancel{n}(1+\overset{\to 0}{\frac{1}{n}})}{\cancel{n}} =$$

$$= a$$

$L = a \Rightarrow \begin{cases} \text{seria } \sum_{n=0}^{\infty} n\cdot a^n \text{ este convergentă pentru } 0<a<1 \\ \text{seria } \sum_{n=0}^{\infty} n\cdot a^n \text{ este divergentă pentru } a \geq 1 \end{cases}$

2. $\int_0^{\frac{1}{2}} \frac{1}{x \cdot \ln x} dx$

Notăm $f(x) = \frac{1}{x\cdot \ln x}$

funcția f nu este definită în 0

$\Rightarrow$ integrala este improprie în 0

Notăm $y > 0$

$$\Rightarrow i(y) = \int_{y}^{\frac{1}{2}} \frac{1}{x \ln x}\, dx = \int_{y}^{\frac{1}{2}} \frac{(\ln x)'}{\ln x}\, dx =$$

$$= \ln \ln x \Big|_{y}^{\frac{1}{2}} =$$

$$= \ln \ln \tfrac{1}{2} - \ln \ln y$$

$$\lim_{\substack{y \to 0 \\ y > 0}} i(y) = \lim_{\substack{y \to 0 \\ y > 0}} \left(\ln \ln \tfrac{1}{2} - \ln \ln y\right) =$$

$$= \ln \ln \tfrac{1}{2} - \lim_{\substack{y \to 0 \\ y > 0}} \ln \ln y =$$

$$= \ln \ln \tfrac{1}{2} - \ln(-\infty) =$$

$$= \ln \ln \tfrac{1}{2} - \ln(-1) - \ln(\infty) =$$

$$= -\infty \Rightarrow \text{integrala } \int_{0}^{\frac{1}{2}} \frac{1}{x \ln x}\, dx \text{ este divergentă}$$

4. $\begin{cases} f(x,y) = x^2 + y^2 \\ x - y = 1 \end{cases}$

Lagrange

$$L(x,y) = x^2 + y^2 + \lambda(x - y - 1)$$

$$\begin{cases} \frac{\partial L}{\partial x} = 0 \\ \frac{\partial L}{\partial y} = 0 \\ \frac{\partial L}{\partial \lambda} = 0 \end{cases} \Leftrightarrow \begin{cases} 2x + \lambda = 0 \\ 2y - \lambda = 0 \\ x - y = 1 \end{cases} \Leftrightarrow$$

$$D_1 = \frac{\partial^2 f}{\partial x^2} = 2 > 0$$

$$D_2 = \begin{vmatrix} \frac{\partial^2 f}{\partial x^2}(P) & \frac{\partial^2 f}{\partial x \partial y}(P) \\ \frac{\partial^2 f}{\partial y \partial x}(P) & \frac{\partial^2 f}{\partial y^2}(P) \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0$$

$\left.\begin{array}{l} D_1 > 0 \\ D_2 > 0 \end{array}\right| \Rightarrow \left(\frac{1}{2}, -\frac{1}{2}\right)$ este punct de minim pentru L și prin urmare este un punct de minim condiționat pentru f

5. $\int_C (2-y)dx + x\,dy$, $C: \begin{cases} x = t - \sin t \\ y = 1 - \cos t \end{cases}$, $t \in [0, 2\pi]$

$$I = \int_0^{2\pi} [(2-(1-\cos t))(1-\cos t) + (t-\sin t)\cdot \sin t]\,dt =$$

$$= \int_0^{2\pi} \{(2-1+\cos t)(1-\cos t) + (t-\sin t)\sin t]\,dt =$$

$$= \int_0^{2\pi} \{(1+\cos t)(1-\cos t) + (t-\sin t)\cdot\sin t]\,dt =$$

$$= \int_0^{2\pi} (1-\cos^2 t + t\sin t - \sin^2 t]\,dt =$$

$$= \int_0^{2\pi} (1-\underbrace{(\cos^2 t + \sin^2 t)}_{=1} + t\sin t)\,dt =$$

$$= \int_0^{2\pi} (1-1+t\sin t)\,dt =$$

$$= \int_0^{2\pi} t\sin t\,dt$$

$f = t \ldots f' = 1$

$g' = \sin t \ldots g = -\cos t$

$$\Rightarrow i = -\cos t \cdot t \Big|_0^{2\pi} + \int_0^{2\pi} \cos t \, dt =$$

$$= -\cos 2\pi \cdot 2\pi + \sin t \Big|_0^{2\pi} =$$

$$= -2\pi$$

3. $f(x,y) = \varphi(x+y^2, x^2y)$

Notăm $u = x+y^2$, $v = x^2y$

$$\frac{\partial f}{\partial x} = \varphi'(u,v) \cdot \left[\frac{\partial \varphi}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial \varphi}{\partial v} \cdot \frac{\partial v}{\partial x}\right) =$$

$$= \varphi(u,v)\left(\frac{\partial \varphi}{\partial u} \cdot 1 + \frac{\partial \varphi}{\partial v} \cdot 2xy\right)$$

$$\frac{\partial \varphi}{\partial y} = \varphi'(u,v)\left(\frac{\partial \varphi}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial \varphi}{\partial v} \cdot \frac{\partial v}{\partial y}\right) =$$

$$= \varphi'(u,v)\left(\frac{\partial \varphi}{\partial u} \cdot 2y + \frac{\partial \varphi}{\partial v} \cdot x^2\right)$$

$$df(x,y) = \frac{\partial f}{\partial x}(x,y)\,dx + \frac{\partial f}{\partial y}(x,y)\,dy$$

$$df(x,y) = \varphi'(u,v)\left(\frac{\partial f}{\partial u} + 2xy\frac{\partial f}{\partial v}\right)dx +$$

$$\varphi'(u,v)\left(2y\frac{\partial f}{\partial u} + x^2 \cdot \frac{\partial f}{\partial v}\right)dy$$

$$d^2 f(x,y) = \left(\frac{\partial f}{\partial x}(x,y)\,dx + \frac{\partial f}{\partial y}(x,y)\,dy\right)^2 f$$

$$d^2 f(x,y) = \frac{\partial^2 f}{\partial x^2}(x,y)\,dx^2 + 2 \cdot \frac{\partial^2 f}{\partial x \partial y}(x,y)\,dx \cdot dy +$$

$$\frac{\partial^2 f}{\partial y^2}(x,y)\,dy^2$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \left(\varphi'(u,v)\left(\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\cdot 2xy\right)\right)'_x =$$

$$= \varphi''(u,v)\left(\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\cdot 2xy\right)^2 +$$

$$\varphi'(u,v)\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\cdot 2xy + \frac{\partial f}{\partial v}\cdot 2y\right)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \left(\varphi'(u,v)\left(\frac{\partial f}{\partial u}\cdot 2y + \frac{\partial f}{\partial v}\cdot x^2\right)\right)'_y =$$

$$= \varphi''(x,y)\left(\frac{\partial f}{\partial u}\cdot 2y + \frac{\partial f}{\partial v}\cdot x^2\right)^2 +$$

$$\varphi'(u,v)\left(\frac{\partial^2 f}{\partial u^2}\cdot 2y + \frac{\partial f}{\partial u}\cdot 2 + \frac{\partial^2 f}{\partial v^2}\cdot x^2\right)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \left(\varphi'(u,v)\left(\frac{\partial f}{\partial u}\cdot 2y + \frac{\partial f}{\partial v}\cdot x^2\right)\right)'_x =$$

$$= \varphi''(u,v)\left(\frac{\partial f}{\partial u}\cdot 2y + \frac{\partial f}{\partial v}\cdot x^2\right)\left(\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\cdot 2xy\right) +$$

$$\varphi'(u,v)\left(\frac{\partial^2 f}{\partial u^2}\cdot 2y + \frac{\partial^2 f}{\partial v^2}x^2 + \frac{\partial f}{\partial v}\cdot 2x\right)$$

$$d^2 f(x,y) = \Big[\varphi''(u,v)\left(\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\cdot 2xy\right)^2 + \varphi'(u,v)\Big(\frac{\partial^2 f}{\partial u^2} +$$

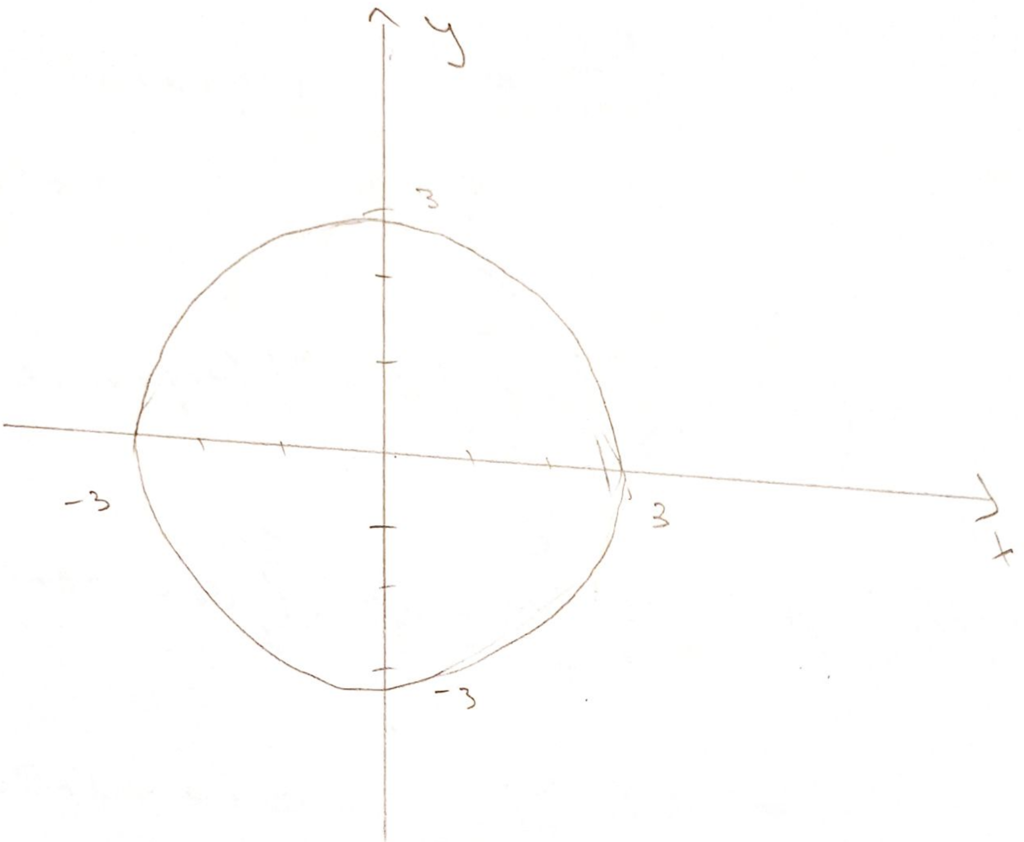
$$\frac{\partial^2 f}{\partial v^2}\cdot 2xy + \frac{\partial f}{\partial v}\cdot 2y\Big)\Big]dx^2 + 2\Big[\varphi''(u,v)\Big(\frac{\partial f}{\partial u}\cdot 2y +$$

$$\frac{\partial f}{\partial v}\cdot x^2\Big)\left(\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}\cdot 2xy\right) + \varphi'(u,v)\Big(\frac{\partial^2 f}{\partial u^2}\cdot 2y +$$

$$\frac{\partial^2 f}{\partial v^2}\cdot x^2 + \frac{\partial f}{\partial v}\cdot 2x\Big)\Big]dxdy + \Big[\varphi''(u,v)\Big(\frac{\partial f}{\partial u}\cdot 2y +$$

$$\frac{\partial f}{\partial v}\cdot x^2\Big)^2 + \varphi'(u,v)\left(\frac{\partial^2 f}{\partial u^2}\cdot 2y + \frac{\partial f}{\partial u}\cdot 2 + \frac{\partial^2 f}{\partial v^2}\cdot x^2\right)\Big]dy^2$$

6. $\iint\limits_D \ln(1+x^2+y^2)\,dx\,dy$, $(D): x^2+y^2 \le 9$



$$x = \rho\cos\theta$$
$$y = \rho\sin\theta \quad , \rho \ge 0, \theta \in [-\pi, \pi]$$

$$\left.\begin{array}{l} x^2+y^2 \le 9 \Rightarrow \rho^2 \le 9 \\ \rho \ge 0 \end{array}\right| \Rightarrow \rho \le 3 \Rightarrow (D'): \begin{cases} 0 \le \rho \le 3 \\ -\pi \le \theta \le \pi \end{cases}$$

$$I = \iint\limits_D \ln(1+x^2+y^2)\,dx\,dy =$$

$$= \iint\limits_{D'} \ln(1+\rho^2\cos^2\theta + \rho^2\sin^2\theta)\,d\rho\,d\theta =$$

$$= \iint\limits_{D'} \ln(1+\rho^2(\cos^2\theta+\sin^2\theta))\,d\rho\,d\theta =$$

$$= \iint\limits_{D'} \ln(1+\rho^2)\,d\rho\,d\theta =$$

$$= \int_{-\pi}^{\pi} \left(\int_{0}^{3} \ln(1+\rho^2)\, d\rho \right) d\theta$$

$$f = \ln(1+\rho^2) \ldots \quad f' = \frac{2\rho}{1+\rho^2}$$

$$g' = 1 \quad \ldots \quad g = \rho$$

$$I = \int_{-\pi}^{\pi} \rho \ln(1+\rho^2) \Big|_0^3 - 2 \left(\int_0^3 \frac{1+\rho^2-1}{1+\rho^2}\, d\rho \right) d\theta =$$

$$= \int_{-\pi}^{\pi} \left(3\ln 10 - 2\int_0^3 1\, d\rho + 2\int_0^3 \frac{1}{1+\rho^2}\, d\rho \right) d\theta =$$

$$= \int_{-\pi}^{\pi} 3\ln 10 - 6 + 2\operatorname{arctg} \rho \Big|_0^3 \, d\theta =$$

$$= \int_{-\pi}^{\pi} (3\ln 10 - 6 + 2\operatorname{arctg} 3)\, d\theta =$$

$$= (3\ln 10 - 6 + 2\operatorname{arctg} 3) \cdot \theta \Big|_{-\pi}^{\pi} =$$

$$= 2\pi (3\ln 10 - 6 + \operatorname{arctg} 3)$$