Anghelus
Irina-Carmina
10LF301

Examen fundamentele algebrice ale informaticii

6. $C_{n,k}$ - cod liniar binar $\Rightarrow C_{n,k} \leq \mathbb{Z}_2^n$

$\begin{cases} \dim C_{n,k} = k \Rightarrow G \in \mathcal{M}_{k,n}(\mathbb{Z}_2) \\ \dim C_{n,k}^{\perp} = n-k \Rightarrow H \in \mathcal{M}_{n-k,n}(\mathbb{Z}_2) \end{cases}$

$$H = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Aici $\left.\begin{matrix} H \in \mathcal{M}_{3,6}(\mathbb{Z}_2) \\ H \in \mathcal{M}_{n-k,n}(\mathbb{Z}_2) \end{matrix}\right\} \Rightarrow \begin{cases} n-k = 3 \Rightarrow k = 3 \\ n = 6 \end{cases}$

$\Rightarrow$ cod liniar $(6,3)$ peste $\mathbb{Z}_2$

$$C_{6,3}: H \cdot Y = 0 \Leftrightarrow \underbrace{\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}}_{\neq 0} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{pmatrix} = 0$$

$\Rightarrow y_1, y_2, y_3$ nec. principale

$$\Leftrightarrow \begin{cases} y_3 + y_4 + y_5 + y_6 = 0 \\ y_2 + y_4 + y_5 = 0 \\ y_1 + y_4 + y_6 = 0 \end{cases}$$

$\Rightarrow y_4 = \alpha$
$y_5 = \beta$
$y_6 = \gamma$

$$\Leftrightarrow \begin{cases} y_3 = \alpha + \beta + \gamma \\ y_2 = \alpha + \beta \\ y_1 = \alpha + \gamma \end{cases}$$

$$C_{6,3} = \{ (\alpha+\gamma, \alpha+\beta, \alpha+\beta+\gamma, \alpha, \beta, \gamma) \mid \alpha, \beta, \gamma \in \mathbb{Z}_2 \}$$

$$\Rightarrow C_{6,3} = \{ \underbrace{111100}_{v_1}, \underbrace{011010}_{v_2}, \underbrace{101001}_{v_3}]$$

$\Rightarrow \{v_1, v_2, v_3\}$ - sistem de generatori in $C_{6,3}$ (1)

Studiem liniar independența

$$\text{rang}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 3 \Rightarrow 3 \text{ vectori liniari independenți (2)}$$

$\neq 0$

Din (1), (2) $\Rightarrow \{v_1, v_2, v_3\}$ bază în $C_{6,3}$

$$\Rightarrow G = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \text{ matricea generatoare}$$

Forma cuvintelor cod: $(\alpha+\gamma, \alpha+\beta, \alpha+\beta+\gamma, \alpha, \beta, \gamma)$

Regula de codificare / decodificare este

$$\alpha\beta\gamma \xrightarrow{\text{codific}} \underline{\alpha+\gamma}\ \underline{\alpha+\beta}\ \underline{\alpha+\beta+\gamma}\ \underline{\alpha}\ \underline{\beta}\ \underline{\gamma}$$

$$111 \xrightarrow{\text{codific}} \underline{0}\ \underline{0}\ \underline{1}\ \underline{1}\ \underline{1}\ \underline{1}$$

Mesajul principal e pe pozițiile: 4, 5, 6

Poziții de control: 1, 2, 3

Tabela de simdroame.

$$H \cdot y = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \vdots \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} ; \alpha, \beta, \gamma \in \mathbb{Z}_2$$

$\Rightarrow 2^3 = 8$ simdroame

TSD:

eroare	sindrom
000000	000
100000	001
010000	010
110000	011
001000	100
000001	101
000010	110
000100	111

Recepționez $v = 110111$

Sindromul lui v:

$$H \cdot v = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow v$ nu e cuvânt cod

$v = v_{corect} + e, \quad e = 000100$

$\Rightarrow v_{corect} = 110011$

$11\underline{0011} \xrightarrow{\text{decodific}} 011$

5. $u_1 = (1,0,1,0)$, $u_2 = (0,1,1,0)$, $u_3 = (0,0,1,1)$, $u_4 = (1,1,3,1)$

$\{u_1, u_2, u_3, u_4\}$ = sistem de generatori pentru V

Studiem liniar independența

$$\text{rang} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \end{vmatrix} \xlongequal{L_3 = L_3 - L_1} \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{vmatrix} =$$

$$= (-1)^{1+1} \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{vmatrix} = 1 + 1 - 2 = 0$$

$$d = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1 \Rightarrow$$ 3 vectori liniar independenți

$\Rightarrow \{u_1, v_2, u_3\}$ bază în V

$\Rightarrow$ Matricea generatoare $G = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

$$V^{\perp} : G \cdot X = 0 \Leftrightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0 \Leftrightarrow$$

$\Rightarrow x_1, x_2, x_4$ - necunoscute principale

$\Rightarrow x_3 = \alpha$

$$\Rightarrow \begin{cases} x_1 + x_3 = 0 \\ x_2 + x_3 = 0 \\ x_3 + x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -\alpha \\ x_2 = -\alpha \\ x_4 = -\alpha \end{cases}$$

$$V^{\perp} = \{(-\alpha, -\alpha, \alpha, -\alpha) \mid \alpha \in \mathbb{R}\}$$

$$V^{\perp} = \{\alpha \underbrace{(-1, -1, 1, -1)}_{w_1} \mid \alpha \in \mathbb{R}\}$$

$\Rightarrow \{w_1\}$ sistem de generatori în $V^{\perp}$ (1)

$w_1 = (-1, -1, 1, -1) \neq (0,0,0,0) \Rightarrow w_1$ liniar independent (2)

Din (1), (2) $\Rightarrow \{w_1\}$ bază în $V^{\perp}$

$\Rightarrow \dim V^{\perp} = 1$

$B = \{w_1\} = \{e_1 = (-1,-1,1,-1)\} \quad e_1 \in \mathbb{R}^4\}$

$v = (2,2,-2,2)$

$v = a \cdot e_1$

$(2,2,-2,2) = a(-1,-1,1,-1) \Rightarrow a = -2$

$\Rightarrow v = -2\cdot(-1,-1,1,-1) = -2 \cdot e_1$

4. inelul $(\mathbb{Z}_{2021}[X], +, \cdot)$

polinoame inversabile de grad 2021?

$f = a_0 + a_1X + \dots + a_{2021}X^{2021} \in U(\mathbb{Z}_{2021}[X])$

$\Leftrightarrow \begin{cases} a_0 \in U(\mathbb{Z}_{2021}) \\ a_1, a_2, \dots, a_{2021} \in Nil(\mathbb{Z}_{2021}) \end{cases}$

$|U(\mathbb{Z}_{2021})| = \varphi(2021) = \varphi(43\cdot 47) =$

$= \varphi(43)\cdot\varphi(47) =$

$= (43^1 - 43^0)\cdot(47^1 - 47^0) =$

$= 42 \cdot 46 = 1932$

$\Rightarrow a_0$ poate lua 1932 valori $\Rightarrow$ 1932 variante

$|Nil(\mathbb{Z}_{2021})| = \frac{2021}{43\cdot 47} = 1 \Rightarrow a_1, \dots, a_{2020}$ pot lua o singură valoare

$a_{2021} \neq \hat{0} \Rightarrow$ nu există valori pt. a_{2021}

$\Rightarrow$ sunt 0 polinoame inversabile de grad 2021 în inelul $(\mathbb{Z}_{2021}[X], +, \cdot)$.

3. $\sigma \in S_{2021}$, fără puncte fixe

$$\sigma = \underbrace{\underbrace{(\ldots)}_{k}\underbrace{(\ldots)}_{k}\ldots\underbrace{(\ldots)}_{k}}_{n \text{ cicluri de lungime } k}$$

n minim $\Rightarrow$ k maxim

$n \cdot k \leq 2021$

$\Rightarrow \begin{cases} k = 2021 \\ n = 1 \end{cases} \Rightarrow$ un ciclu de lungime 2021 $\Rightarrow$

$\Rightarrow$ 2021 de puncte nu sunt fixe

$\Rightarrow \sigma = (i_1, i_2, \ldots, i_{2021})$

$$\sigma = \underbrace{(i_1, i_2)(i_2, i_3)\ldots(i_{2020}, i_{2021})}_{2020 \text{ de transpoziții}}$$

$\varepsilon(\sigma) = (-1)^{2020} = 1 \Rightarrow \sigma$ e permutare pară

$o(\sigma) = k = 2021$

1. $A = \{x \in \mathbb{N} \mid \exists a \in \mathbb{N}, x = a^2\}$

$xRy \Leftrightarrow 10 \mid x - y$

R este o relație de echivalență $\Leftrightarrow$

$\Leftrightarrow$ R este
- reflexivă $\Leftrightarrow xRx, \forall x \in \mathbb{N}$
- simetrică $\Leftrightarrow xRy$ implică $yRx, \forall x, y \in \mathbb{N}$
- tranzitivă $\Leftrightarrow xRy$ și yRz implică $xRz, \forall x, y, z \in \mathbb{N}$

$10 \mid x - x, (\forall) x \in \mathbb{N} \Rightarrow xRx \Rightarrow R$ - reflexivă

Fie $10 \mid x - y \Rightarrow 10 \mid (-1)(x-y) \Rightarrow$

$\Rightarrow 10 \mid y - x \Rightarrow xRy$ implică $yRx \Rightarrow R$ - simetrică

Fie $\left\{\begin{array}{l}10 \mid x-y \\ \text{și} \\ 10 \mid y-z\end{array}\right. \Rightarrow 10 \mid x-y+y-z \Rightarrow$

$\Rightarrow 10 \mid x-z$

$\Rightarrow xRy$ și yRz implică xRz

$\Rightarrow R$ - tranzitivă

R simetrică, reflexivă și tranzitivă

$\Rightarrow R$ e relație de echivalență

Clasele de echivalență

$b \in \mathbb{N}$, $\hat{b} = \{x \in \mathbb{N} \mid 10 \mid x-b\}$

$\Downarrow$

$x-b = 10k$

$\Rightarrow x = b+10k$

$\Rightarrow \hat{b} = \{b+10k \mid k \in \mathbb{N}\}$

$(\exists)\ c, r \in \mathbb{N}$, $b = 10c+r$, $0 \le r < 10$

$\hat{b} = \{r+10(c+k) \mid k \in \mathbb{Z}\} = \hat{r}$

La împărțirea prin 10 sunt posibile

10 resturi : $\{0, 1, \dots, 9\}$

$\Rightarrow$ $\hat{0} = \{10k \mid k \in \mathbb{N}\}$

$\hat{1} = \{1+10k \mid k \in \mathbb{N}\}$

$\vdots$

$\hat{9} = \{9+10k \mid k \in \mathbb{N}\}$

$\Rightarrow$ 10 clase de echivalență

2. $p, q, r \in L(i)$ a.î. $p^4 q^3 = q^3 p^4$

Arătați că $p^{2020} q^{2019} = q^{2019} p^{2020}$

Dacă $p^4 q^3 = q^3 p^4 \Rightarrow (\exists) r \in L(i)$, $m, n \in \mathbb{N}$

a.î. $p^4 = r^m$, $q^3 = r^n$

$$p^{2020} q^{2019} = (p^4)^{505} \cdot (q^3)^{673} =$$

$$= (r^m)^{505} \cdot (r^n)^{673} =$$

$$= r^{505m} \cdot r^{673n} =$$

$$= r^{505m + 673n}$$

$$q^{2019} \cdot p^{2020} = (q^3)^{673} \cdot (p^4)^{505} =$$

$$= (r^n)^{673} \cdot (r^m)^{505} =$$

$$= r^{n673} \cdot r^{m \cdot 505} =$$

$$= r^{673n + 505m}$$

$$\left.\begin{array}{l} p^{2020} q^{2019} = r^{505m + 673n} \\ q^{2019} \cdot p^{2020} = r^{673n + 505m} \end{array}\right\} \Rightarrow p^{2020} q^{2019} = q^{2019} \cdot p^{2020}$$