MODULE - IV

Generating Functions.

An infinite power series is simply an infinite sum of terms of the form $C_n x^n$ where C_n is some constant. It can be written as,

$$\sum_{K=0}^{\infty} C_K x^K .$$

ie $C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + \cdots$

When viewed in the context of generating functions, we call such a power series as a generating series.

The generating series generate the sequence,

$C_0, C_1, C_2, C_3, C_4, \ldots$

in other words, the sequence generated by a generating series is simply the sequence of coefficients of the infinite polynomial.

Example:

We know that the power series,

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!}$$

Converges to the function e^x.

$$1, 1, \frac{1}{2}, \frac{1}{6}, \ldots\ldots$$

has generating function e^x.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$A = \sum_{k=0}^{\infty} c_k \frac{x^k}{k!}$$

Exponential generating function.

* To find a closed formula for this power series by using "multiply", "shift" and subtract" techniques.

Basic Generating functions.

* The generating function for $1, 1, 1, 1, \ldots$ is

$$\frac{1}{1-x}$$

* The generating function for $1, -1, 1, -1 \ldots$ is $\frac{1}{1+x}$

* The generating function for $1, 3, 9, 27, \ldots$ is $\frac{1}{1-3x}$

* The generating function for $0, 1, 1, 1, 1 \ldots$ is $\frac{x}{1-x}$

* The generating function for $0, 1, 3, 9, 27, \ldots$ is $\frac{x}{1-3x}$.

* The generating function for $1, 2, 3, 4, 5, \ldots$ is $\frac{1}{(1-x)^2}$

Q. The generating function for the sequence, $2, 2, 2, 2, \ldots$

Soln $2, 2, 2, 2, \ldots$ can be written as,

$$2 + 2x + 2x^2 + 2x^3 + \ldots$$

$$2(1 + x + x^2 + \ldots)$$

$$2 \times \frac{1}{1-x}$$

$$= \frac{2}{1-x}$$

Q. Find generating function for the sequence, $3, 9, 27, 81, \ldots$

Sol:- $3, 9, 27, 81, \ldots$ can be written as,

$$3 + 9x + 27x^2 + 81x^3 + \ldots$$

$$3(1 + 3x + 9x^2 + \ldots)$$

$$3 \times \frac{1}{1-3x} = \frac{3}{1-3x}$$

Q. Find generating function for the sequence, 1, 4, 9, 16 . . .

Soln. 1, 4, 9, 16 . . . . can be written as,

$1 + 4x + 9x^2 + 16x^3 + \dots$

$A = 1 + 4x + 9x^2 + 16x^3 + \dots$

$-xA = \quad x + 4x^2 + 9x^3 + \dots$

$(1-x)A = 1 + 3x + 5x^2 + 7x^3 + \dots$

$B = 1 + 3x + 5x^2 + 7x^3 + \dots$

$-xB = \quad x + 3x^2 + 5x^3 + \dots$

$(1-x)B = 1 + 2x + 2x^2 + \dots$

$(1-x)B = 1 + \frac{2x}{1-x} = \frac{1+x}{1-x}$

$B = \frac{1+x}{(1-x)^2} \Rightarrow (1-x)A = \frac{1+x}{(1-x)^2} \Rightarrow A = \frac{1+x}{(1-x)^3}$

Q. Find generating function for the sequence, 4, 4, 4, 4, ...

Soln. The sequence 4, 4, 4, 4, ... can be written as,

$4 + 4x + 4x^2 + 4x^3 + \dots$

$4(1 + x + x^2 + x^3 + \dots)$

$= 4 \times \frac{1}{1-x} = \frac{4}{1-x}$

Q. Find the generating function for the sequence, 2, 4, 6, 8, 10, ...

Sol. The sequence 2, 4, 6, 8, 10, . . . can be written as

$2+4+6x^2+8x^3+10x^4+ \dots$

$= 2(1+2+3x^2+4x^3+5x^4+ \dots)$

$= 2 \times \frac{1}{(1-x)^2}$

$= \frac{2}{(1-x)^2}$

Solution 2

The sequence, 2, 4, 6, 8, 10, ...

Can be written as,

$2+4x+6x^2+8x^3+10x^4+ \dots$

By using multiply-shift-substraction method

$A = 2+4x+6x^2+8x^3+10x^4+ \dots$

$-xA = 2x+4x^2+6x^3+8x^4+ \dots$

$(1-x)A = 2+2x+2x^2+2x^3+2x^4+ \dots$

$= 2(1+x+x^2+x^3+x^4+ \dots)$

$= 2 \times \frac{1}{1-x}$

$$(1-x)A = \frac{2}{1-x}$$

$$A = \frac{2}{(1-x)^2}$$

Q. Find generating function for the sequence,
0, 0, 0, 2, 4, 6, 8, 10, ...

Sol. The generating function for 2, 4, 6, 8, 10, . . .

$$\frac{2}{(1-x)^2}$$ [From previous solution]

Generating function for the sequence,

0, 0, 0, 2, 4, 6, 8, 10 . . . .

$$\frac{2 * x * x * x}{(1-x)^2} = \frac{2x^3}{(1-x)^2}$$

Q. Find the sequence generated by the following generating functions,

(I) $\frac{4x}{1-x}$ (II) $\frac{1}{1-4x}$ (III) $\frac{x}{1+x}$ (IV) $\frac{3x}{(1+x)^2}$

Sol: (I) $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$

$$\frac{4}{1-x} = 4 + 4x + 4x^2 + 4x^3 + \ldots$$

$$\frac{4x}{1-x} = (4 + 4x + 4x^2 + 4x^3 + \ldots)x$$

$$= 4x + 4x^2 + 4x^3 + \ldots$$

$$= 0 + 4x + 4x^2 + 4x^3 + \ldots$$

Sequence = 0, 4, 4, 4, ...

(ii) $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$

$\frac{1}{1-4x} = 1 + 4x + (4x)^2 + (4x)^3 + \dots$

$= 1 + 4x + 16x^2 + 64x^3 + \dots$

Sequence $= 1, 4, 16, 64, \dots$

(iii) $\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$

$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \dots$

$\frac{x}{1+x} = (1 - x + x^2 - x^3 + x^4 - \dots)x$

$= x - x^2 + x^3 - x^4 + x^5 - \dots$

$= 0 + x - x^2 + x^3 - x^4 + x^5 - \dots$

Sequence: $0, 1, -1, 1, -1, 1 \dots$

(iv) $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots$

$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + \dots$

$\frac{3}{(1+x)^2} = 3 - 6x + 9x^2 - 12x^3 + \dots$

$\frac{3x}{(1+x)^2} = (3 - 6x + 9x^2 - 12x^3 + \dots)x$

$= 3x - 6x^2 + 9x^3 - 12x^4 + \dots$

$= 0 + 3x - 6x^2 + 9x^3 - 12x^4 + \dots$

Sequence: $0, 3, -6, 9, -12 \dots$

Recurrence relations

A recurrence relation is an equation that recursively defines a sequence where the next term is a function of the previous terms.

eg:- Fibonacci sequence of numbers.
The sequence starts with the two immediate predecessors

$0, 1, 1, 2, 3, 5, 8, \ldots\ldots$

This sequence can be described by the relation

$a_r = a_{r-1} + a_{r-2}$

where $a_0 = 0$ and $a_1 = 1$

Linear recurrence relations with constant coefficients.

A recurrence relation of the form,

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \cdots + C_k a_{r-k} = f(r)$$

where C_i's are constants, is called linear recurrance relation with constant coefficients. Also known as k^{th} order recurrance relation.

If $f(r) = 0$, then the k^{th} order recurrence relation will be called as k^{th} order homogenous linear recurrance relation. otherwise, the k^{th} order recurrence relation will be called as k^{th} order non-homogenous linear reccurance relation

First order linear recurrence Relations with constant coefficients.

A recurrence relation of the form,

$$C_0 a_r + C_1 a_{r-1} = f(r)$$

where C_0 and C_1 are constant, and $f(r)$ is a known function, then it is called linear recurrence relation of first order with constant coefficient.

If $f(r) = 0$, then the relation is homogenous, otherwise non homogeneous.

Second order Linear Recurrence Relations with constant coefficients

A recurrence relation of the form,

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} = f(r)$$

where C_0, C_1 and C_2 are constant and $f(r)$ is a known function, then it is called linear recurrance relation of second order with constant coefficents.

If $f(r) = 0$, then the relation is homogenous otherwise non-homogenous.

Solving Linear recurrence relations

(1) Determine if recurrence relation is homogenous or non-homogenous.

(2) Determine whether or not the coefficients

are all constants.

(3) Determine what is the degree of the recurrence relation.

(4) Need to know the homogenous solution equations.

(5) Need to know the non-homogenous solution equations.

Homogeneous Solutions

For first order Recurrence relation,

A homogenous solution of linear recurrence equation with constant coefficients is of the form, $A\alpha_1^r$

where α_1 is called a characteristic root and A is a constant determined by boundary conditions.

For second order recurrence relation.

Find characteristic equation for the given recurrence relation.

Three cases may occur while finding the roots:

Case 1

If the characteristic equation factors as,

$$(r-\alpha_1)(r-\alpha_2)=0 \Rightarrow r=\alpha_1,\alpha_2$$

and it produces two distinct real roots α_1 and α_2, then

$$a_r^{(h)} = A\alpha_1^r + B\alpha_2^r$$

is the solution and where A and B are constants. To find A and B, we can use the initial conditions.

Case 2

If the characteristic equation factors as,

$$(r-\alpha_1)^2 = 0 \Rightarrow r = \alpha_1$$

and it produces single real root α_1 then,

$a_r^{(h)} = A\alpha_1^r + Br\alpha_1^r$ is the solution and where A and B are constants.

To find A and B we can use the initial conditions.

Case 3

If the characteristic equation produces two distinct real roots α_1 and α_2 in polar form. i.e

$$\alpha_1 = x + iy$$
$$\alpha_2 = x - iy.$$

Then, $a_r^{(h)} = p^r (A\cos(r\theta) + B\sin(r\theta))$

is the solution, where,

$$P = \sqrt{x^2+y^2}$$
$$\theta = \tan^{-1}(y/x)$$

To find values of constants A and B we can use the initial conditions.

Non-Homogenous Solutions

Also known as particular solution, which satisfy the recurrence equation with $f(r)$ on the right hand side.

* If,
$$f(r) = \text{constant}$$
then,
$$a_r^{(p)} = P$$

* If,
$$f(r) = r$$
then,
$$a_r^{(p)} = P_r$$

* If $f(r) = r + c$, then $a_r^{(p)} = P_1 r + P_2$, where c is a constant.

* If, $f(r) = r^2 + r + c$, then $a_r^{(p)} = P_1 r^2 + P_2 r + P_3$, where c is a constant.

* If, $f(r) = a^r$ where 'a' is a constant.
Then check if root of characteristic equation and value of 'a' are same, then,
$$a_r^{(p)} = P \cdot r^2 \cdot a^r$$
otherwise,
$$a_r^{(p)} = P \cdot a^r$$

Q Solve the recurrence relation,
$a_n = 2a_{n-1}$, where $a_0 = 3$.

Sol Given, $a_n = 2a_{n-1}$

$a_n - 2a_{n-1} = 0$

=> first order ~~rec~~ homogenous recurrence relation.

Homogenous Solution

$a_n - 2a_{n-1} = 0$

Characteristic equation: $r - 2 = 0$

$r = 2 \Rightarrow \alpha_1 = 2$.

So, solution is, $a_r^{(h)} = A\alpha_1^r$.

$a_n^{(h)} = A\alpha_1^n = A \cdot 2^n$ for some constant A.

To find A, we can use the initial condition,

$a_0 = 3$

$a_n = A \cdot 2^n$

Put $n=0$, $a_0 = A \cdot 2^0$

$3 = A \cdot 1$

$\underline{A = 3}$

$\therefore a_n^{(h)} = A \cdot 2^n = \underline{3 \cdot 2^n}$

Q. Solve the recurrence relation.

$F_n = 5F_{n-1} - 6F_{n-2}$.

where $F_0 = 1$, & $F_1 = 4$.

given,

$$F_n = 5F_{n-1} - 6F_{n-2}$$

$$F_n - 5F_{n-1} + 6F_{n-2} = 0$$

⇒ Second order homogenous recurrence relation.

Homogenous Solution

$$T_n - 5F_{n-1} + 6F_{n-2} = 0$$

Characteristic Equation: $r^2 - 5r + 6 = 0$

Characteristic Root: $r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Here, $a = 1$, $b = -5$, $c = 6$

$$r = \frac{5 \pm \sqrt{(-5)^2 - 4 \times 1 \times 6}}{2 \times 1} = \frac{5 \pm \sqrt{25 - 24}}{2}$$

$$= \frac{5 \pm 1}{2} \Rightarrow r = \frac{5+1}{2}, \frac{5-1}{2} = \underline{\underline{3, 2}}$$: Two distinct real roots.

By case 1,

$$a_r^{(h)} = A\alpha_1^r + B\alpha_2^r$$

$$F_n^{(h)} = A\alpha_1^n + B\alpha_2^n$$

$$\alpha_1 = 3, \quad \alpha_2 = 2$$

$$F_n^{(h)} = A \cdot 3^n + B \cdot 2^n$$

given, $F_0 = 1$

Put $n = 0$, $F_n^{(h)} = A \cdot 3^n + B \cdot 2^n$

$F_0 = A \cdot 3^0 + B \cdot 2^0$

$1 = A \cdot 1 + B \cdot 1$

$A + B = 1$ ——— (i)

Given,

$F_1 = 4$

$F_n^{(h)} = A \cdot 3^n + B \cdot 2^n$

Put $n = 1$,

$F_1 = A \cdot 3^1 + B \cdot 2^1$

$4 = 3A + 2B$

$3A + 2B = 4$ ——— (ii)

(i) $\times 3 \Rightarrow 3A + 3B = 3$ ——— (iii)

(iii) $-$ (ii) $\Rightarrow$

$$\begin{array}{r} 3A + 3B = 3 \quad - \\ 3A + 2B = 4 \\ \hline B = -1 \end{array}$$

(i) $A + B = 1$

$A - 1 = 1$

$A = 1 + 1 = 2$

$\therefore F_n^{(h)} = A \cdot 3^n + B \cdot 2^n = 2 \cdot 3^n - 2^n$

Q. Solve the recurrence relation,

$F_n = 10 F_{n-1} - 25 F_{n-2}$

where $F_0 = 3$ and $F_1 = 17$.

Given, $F_n = 10 F_{n-1} - 25 F_{n-2}$

$\Rightarrow$ second order homogenous recurrence relation.

Homogenous solution

$$F_n - 10F_{n-1} + 25F_{n-2} = 0$$

Characteristic equation

$$r^2 - 10r + 25 = 0$$

Characteristic Root:-

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Here, $a = 1,\ b = -10,\ c = 25$

$$r = \frac{10 \pm \sqrt{(-10)^2 - 4 \times 1 \times 25}}{1 \times 2} = \frac{10 \pm \sqrt{100 - 100}}{2}$$

$= \frac{10}{2} = \underline{\underline{5}}$: single real root.

By case 2,

$$a_r^{(h)} = A\alpha_1^r + B \cdot r \cdot \alpha_1^r$$

$$F_n^{(h)} = A\alpha_1^n + B \cdot n \cdot \alpha_1^n$$

$$F_n^{(h)} = A \cdot 5^n + B \cdot n \cdot 5^n$$

given $F_0 = 3$

$$F_n^{(h)} = A \cdot 5^n + B \cdot n \cdot 5^n$$

Put $n = 0$,

$F_0 = A \cdot 5^0 + B \cdot 0 \cdot 5^0 = A \cdot 1 + 0$

$\underline{\underline{A = 3}}$

given,

$F_1 = 17$

$F_n^{(h)} = A \cdot 5^n + B \cdot n \cdot 5^n$

Put $n = 1$,

$F_1 = A \cdot 5^1 + B \cdot 1 \cdot 5^1$

$17 = 5A + 5B$

$17 = 5 \times 3 + 5B$

$17 - 15 = 5B$

$5B = 2$

$B = \frac{2}{5}$.

$\therefore F_n^{(h)} = A \cdot 5^n + B \cdot n \cdot 5^n = \underline{\underline{3 \cdot 5^n + \frac{2}{5} \cdot n \cdot 5^n}}$

Q. Solve the recurrence relation,

$F_n = 2F_{n-1} - 2F_{n-2}$, where $F_0 = 1$ and $F_1 = 3$.

given, $F_n = 2F_{n-1} - 2F_{n-2}$

$F_n - 2F_{n-1} + 2F_{n-2} = 0$

$\Rightarrow$ Second order homogeneous relation.

Homogenous solution

$F_n - 2F_{n-1} + 2F_{n-2} = 0$

characteristic equation:

$r^2 - 2r + 2 = 0$

Characteristic Root:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Here, $a = 1, b = -2, c = 2$

$$r = \frac{2 \pm \sqrt{(-2)^2 - 4 \times 1 \times 2}}{2 \times 1} = \frac{2 \pm \sqrt{4-8}}{2}$$

$$= \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm \sqrt{4} \times \sqrt{-1}}{2}$$

$$= \frac{2 \pm 2i}{2} = \frac{2(1 \pm i)}{2} = \underline{\underline{1 \pm i}}$$; polar form

By case 3,

$\alpha_1 = 1 + i$

$\alpha_2 = 1 - i$

$x = 1, \ y = 1$

$$F_n^{(h)} = p^n (A \cos(n\theta) + B \sin(n\theta))$$

$$p = \sqrt{x^2 + y^2} = \sqrt{1^2 + 1^2} = \underline{\underline{\sqrt{2}}}$$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{1}\right)$

$= \tan^{-1}(1) = \underline{\underline{45^\circ}}$

$F_n^{(h)} = (\sqrt{2})^n (A \cos(n \times 45^\circ) + B \sin(n \times 45^\circ))$

given $f_0 = 1$

Put $n = 0$,

$F_0 = (\sqrt{2})^0 (A \cos(0 \times 45^\circ) + B \sin(0 \times 45^\circ))$

$1 = 1 \cdot (A \cos 0 + B \sin 0)$

$A \cdot 1 + B \cdot 0 = 1$

$\underline{\underline{A = 1}}$

given, $f_1 = 3$

Put $n = 1$,

$F_1 = (\sqrt{2})^1 (A \cos(1 \times 45^\circ) + B \sin(1 \times 45^\circ))$

$3 = \sqrt{2} (A \cos 45^\circ + B \sin 45^\circ)$

$3 = \sqrt{2} \left(A \times \frac{1}{\sqrt{2}} + B \times \frac{1}{\sqrt{2}}\right)$

$A + B = 3$

$1 + B = 3$

$\underline{\underline{B = 2}}$

$\therefore P_n^{(h)} = p^n (A\cos(n\theta) + B\sin(n\theta))$

$= (\sqrt{2})^n (\cos(n \times 45^\circ) + 2\sin(n \times 45^\circ))$

Q. Solve the recurrence relation,

$a_r - 5a_{r-1} + 6a_{r-2} = 1$

$\Rightarrow$ second order non-homogenous recurrence relation.

Total solution, $a_r = a_r^{(h)} + a_r^{(p)}$

Homogenous solution

$a_r - 5a_{r-1} + 6a_{r-2} = 1$

Characteristic equation:

$r^2 - 5r + 6 = 0$

Characteristic root:-

$r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Here $a = 1,\ b = -5,\ c = 6$

$r = \dfrac{5 \pm \sqrt{(-5)^2 - 4 \times 1 \times 6}}{2 \times 1} = \dfrac{5 \pm \sqrt{25 - 24}}{2}$

$r = \frac{5 \pm 1}{2}$

$r = \frac{5+1}{2}, \frac{5-1}{2} = 3, 2$: two distinct real roots.

By case 1,

$a_r^{(h)} = A\alpha_1^r + B\alpha_2^r$

$\alpha_1 = 3$

$\alpha_2 = 2$

$a_r^{(h)} = A \cdot 3^r + B \cdot 2^r$

given,

$a_2 = 278$

$a_r = A \cdot 3^r + B \cdot 2^r$

Put $n = 2$,

$a_2 = A \cdot 3^2 + B \cdot 2^2$

$278 = 9A + 4B$

$9A + 4B = 278 \longrightarrow (1)$

given,

$a_3 = 962$

Put $n = 3$,

$a_3 = A \cdot 3^3 + B \cdot 2^3$

$962 = 27A + 8B$

$$27A + 8B = 962 \quad \text{——— (ii)}$$

$$(i) \times 3 \Rightarrow 27A + 12B = 834 \quad \text{——— (iii)}$$

$$(iii) - (ii) \Rightarrow \begin{array}{r} 27A + 12B = 834 \; - \\ 27A + 8B = 962 \\ \hline 4B = -128 \end{array}$$

$$B = \frac{-128}{4}$$

$$B = -32$$

$$9A + 4B = 278 \quad \text{——— (i)}$$

$$9A + 4 \times -32 = 278$$

$$9A - 128 = 278$$

$$9A = 278 + 128 = 406$$

$$A = \frac{406}{9}$$

$$\therefore \quad a_r^{(h)} = A \cdot 3^r + B \cdot 2^r$$

$$a_r^{(h)} = \left(\frac{406}{9}\right)3^r + -32 \cdot 2^r = \left(\frac{406}{9}\right)3^r - 32 \cdot 2^r$$

Particular solution

$$a_r - 5a_{r-1} + 6a_{r-2} = 1$$

Here, $f(r) = 1 \Rightarrow$ constant

$\therefore \quad a_r^{(P)} = P$

Substituting P into the recurrence relation,

$a_r - 5a_{r-1} + 6a_{r-2} = 1$

$P - 5P + 6P = 1$

$2P = 1$

$P = \frac{1}{2}$ //

$a_r(P) = P = \frac{1}{2}$

Total solution, $a_r = a_r^{(h)} + a_r^{(P)}$

$= \left(\frac{406}{9}\right) \cdot 3^r - 32 \cdot 2^r + \frac{1}{2}$

Q. Using generating function to solve recurrence relation.

$a_K = 3a_{K-1} + 2$, where $a_0 = 1$

Given $a_K = 3a_{K-1} + 2$

let

$G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_K x^K + \dots$

$= 1 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_K x^K + \dots$

[given $a_0 = 1$]

$= 1 + (3a_0 + 2)x + (3a_1 + 2)x^2 + \dots +$
$(3a_{K-1} + 2)x^K + \dots$

[given, $a_K = 3a_{K-1} + 2$
$a_1 = 3a_0 + 2$
$a_2 = 3a_1 + 2$]

$= 1 + (3a_0 x + 3a_1 x^2 + \dots + 3a_{K-1} x^K + \dots)$
$+ (2x + 2x^2 + \dots + 2x^K + \dots)$

$= 1 + 3x(a_0 + a_1 x + \dots + a_{K-1} x^{K-1} + \dots)$
$+ 2x(1 + x + x^2 + \dots + x^{K-1} + \dots$

$= 1 + 3x\,G(x) + 2x(1 + x + x^2 + \dots + x^{K-1} + \dots)$

$G(x) = 1 + 3x\,G(x) + 2x \cdot \frac{1}{1-x}$

$G(x) = 1 + 3x\,G(x) + \frac{2x}{1-x}$

$G(x) - 3x\,G(x) = 1 + \frac{2x}{1-x}$

$G(x)(1-3x) = \frac{1 - x + 2x}{1-x}$

$G(x)(1-3x) = \frac{1+x}{1-x}$

$$G(x) = \frac{1+x}{(1-x)(1-3x)}$$ → Increasing order of x's coefficient

$$G(x) = \frac{1+x}{(1-x)(1-3x)} \quad \text{---(1)}$$

$$G(x) = \frac{1+x}{(1-x)(1-3x)}$$

By partial fraction method,

$$G(x) = \frac{A}{1-x} + \frac{B}{1-3x}$$

$$= \frac{A(1-3x) + B(1-x)}{(1-x)(1-3x)}$$

$$= \frac{A - 3Ax + B - Bx}{\cancel{(1-x)(1+3x)}\ (1-x)(1-3x)}$$

$$G(x) = \frac{(A+B) - (3A+B)x}{(1-x)(1-3x)} \quad \text{---(ii)}$$

From (I) & (II), we have

$A+B = 1$

$-(3A+B) = 1$

$3A+B = -1$ ——— (II)

(i) $\times 3 \Rightarrow 3A+3B = 3$ ——— (III)

(III) $-$ (II) $\Rightarrow$

$$\begin{array}{r} 3A+3B = 3 \quad - \\ 3A+B = -1 \\ \hline 2B = 4 \\ \underline{\underline{B = 2}} \end{array}$$

$A+B = 1$ ——— (I)

$A+2 = 1$ ———

$\underline{\underline{A = -1}}$

$$G(x) = \frac{A}{1-x} + \frac{B}{1-3x}$$

$$= \frac{-1}{1-x} + \frac{2}{1-3x}$$

$$= -1 \times \frac{1}{1-x} + 2 \times \frac{1}{1-3x}$$

$$= -1 \times \sum_{K=0}^{\infty} x^K + 2 \times \sum_{K=0}^{\infty} 3^K x^K$$

$$\left[\because \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{K=0}^{\infty} x^K\right.$$

$$\left.\frac{1}{1-3x} = 1 + 3x + 9x^2 + 27x^3 + \cdots = \sum_{K=0}^{\infty} 3^K x^K\right]$$

$$G(x) = \sum_{K=0}^{\infty} (2 \cdot 3^K - 1)\, x^K$$

Thus, solution for the given recurrence relation is

$$a_K = 2 \cdot 3^K - 1$$