BS./BSc.

in Applied AI and Data Science

Linear algebra and numerical analysis

Module 3.5 Linear Independence

1. Linear Independence
2. Orthogonality and basis
3. Inner product
4. Orthogonal projections and matrix derivatives

Linear Independence

Definition

Dependent

- For at least one $\lambda \neq 0$ $\qquad 0 = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n, \quad \lambda \in \mathbb{R}$
- A set of vectors is dependent if at least one vector in the set can be expressed as a linear weighted combination of the other vectors in that set.

Definition

Independent

- Only when all $\lambda_i = 0$ $\qquad 0 = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n, \quad \lambda \in \mathbb{R}$
- No vector in the set is a linear combination of the others (has only the trivial solution)

That is, a set of vectors can be linearly independent or linearly dependent; it doesn't make sense to ask whether a single vector or a vector within a set, is independent.

Linear Independence

Linear dependence

$$\mathbf{0} = \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \ldots + \lambda_n\mathbf{v}_n, \quad \lambda \in \mathbb{R}$$

This may seem like a strange definition: Where does it come from, and why is the zeros vector so important?

Some rearranging, starting with subtracting specific terms from both sides of the equation, will reveal why this equation indicates dependence:

$$\lambda_1\mathbf{v}_1 = \lambda_2\mathbf{v}_2 + \ldots + \lambda_n\mathbf{v}_n$$

$$\mathbf{v}_1 = \frac{\lambda_2}{\lambda_1}\mathbf{v}_2 + \ldots + \frac{\lambda_n}{\lambda_1}\mathbf{v}_n, \quad \lambda \in \mathbb{R},\ \lambda_1 \neq 0$$

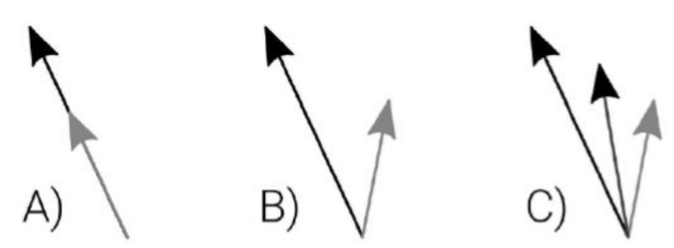
Linear Independence (Geometry)

Definition

A set of vectors is linear independent if the subspace dimensionality (its span) equals the number of vectors.

Example

- vectors spans?



Geometric sets of vectors in $\mathbb{R}^2$

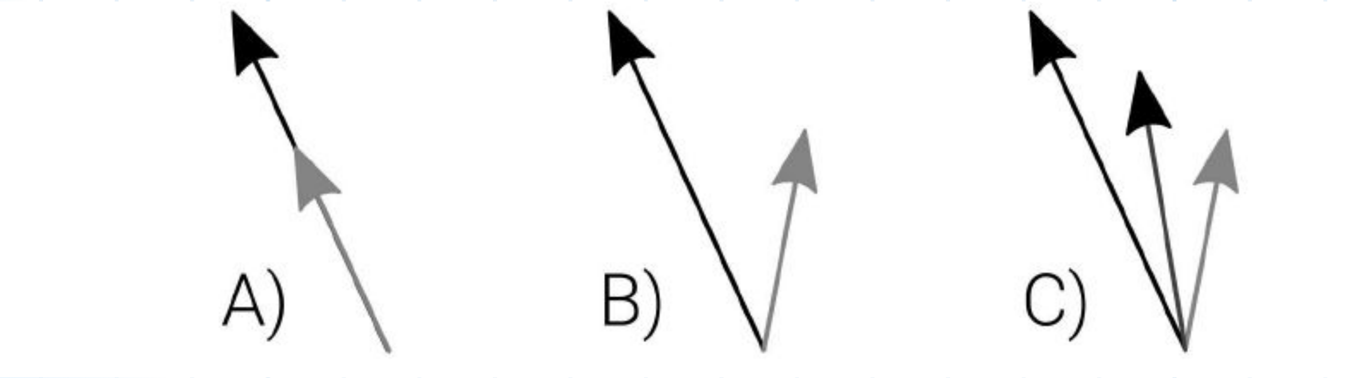
The left-hand set (panel A) contains two collinear vectors; this set is linearly dependent because you can create one vector as a scaled version of another vector.

Now consider the middle set (panel B): Again, two vectors point in different directions; it is impossible to create one vector as a scaled version of the other vector.

Linear Independence (Geometry)

Finally, consider the right-hand set (panel C): This set of three vectors in R^2 is linearly dependent because any of the vectors can be obtained by a linear combination of the other two vectors.

In this example, the middle vector can be obtained by averaging the other two vectors (summing them and scalar multiplying by = .5). But that's not just a quirk of this example. There is a theorem about



Any set of $M > N$ vectors in $\mathbb{R}^N$ is necessarily linearly dependent.

Any set of $M \leq N$ vectors in $\mathbb{R}^N$ *could be* linearly independent.

Determining whether a set is linearly dependent or independent

Before learning about matrix-based algorithms for computing whether a set is linearly independent, you can apply a 4-step procedure to determine the linear dependency of a set of vectors.

Note that this is a strategy for solving exercise problems or exam problems by hand; this method does not scale up to larger matrices and is way too time-consuming to use in real-world applications.

Step 3: If you've gotten this far, it means you need to start doing some trial-and-error educated guesswork. Start by looking for zeros in the entries of some vectors, with the knowledge that zeros in some vectors in combination with non-zero entries in corresponding dimensions in other vectors is a tip towards independence (you cannot create something from nothing, with the possible exception of the big bang).

Step 4: This is where the real educated guesswork comes in. Start by creating one element as a weighted combination of other vectors, and see whether that same weighted combination will work for the other dimensions. Consider the following set of vectors:

Determining whether a set is linearly dependent or independent

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 8 \end{bmatrix} \right\}$$

Look across the first dimension (top row of vector elements) and notice that 2 times the first vector plus 1 times the second vector produces the first element in the third vector. That same set of weights is also true for the second dimension. However, it doesn't work for the third dimension.

That proves that the third vector cannot be expressed as this linear combination of the first two. You have to repeat this process (try to find weights of M-1 vector entries that equal the Mth vector entry) for each M vector. Eventually, you will determine whether the set is linearly independent or dependent.

Linear Independence and dependence

The three vectors are given as

$$\mathbf{a}_{(1)} = [\;3 \quad 0 \quad 2 \quad 2]$$
$$\mathbf{a}_{(2)} = [-6 \quad 42 \quad 24 \quad 54]$$
$$\mathbf{a}_{(3)} = [\;21 \quad -21 \quad 0 \quad -15]$$

Although vector arithmetic easily checks this (do it!), it is not so easy to discover. We will now discuss a systematic method for determining linear independence and dependence.

Linear Independence and dependence

Algebra: A set of vectors is dependent if at least one vector can be expressed as a linear weighted combination of the other vectors in that set. Consider the following examples.

$$\{\mathbf{w}_1, \mathbf{w}_2\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \right\}. \quad \mathbf{w}_2 = 2\mathbf{w}_1$$

$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} -27 \\ 5 \\ -37 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 8 \end{bmatrix} \right\}. \quad \mathbf{v}_2 = 7\mathbf{v}_1 - 9\mathbf{v}_3$$

Both examples show dependent sets

Linear Independence and dependence of functions

Function Spaces:
A vector space can be formed by the set of all functions from a set to a field (like real numbers).

- For instance, the set of all continuous functions from the real numbers to the real numbers is a vector space.
- This is crucial in areas like differential equations and signal processing.

Functions Linearly Independent

- Let f(t) and g(t) be differentiable functions. Then they are called linearly dependent if there are nonzero constants c_1 and c_2 with

$$c_1 f(t) + c_2 g(t) = 0$$

for all t. Otherwise they are called linearly independent.

Example

Linearly dependent or independent?

- Functions $f(t) = 2\sin^2 t$ and $g(t) = 1 - cos^2 t$
- Functions $\{\sin^2 x, \cos^2 x, \cos(2x)\} \subset \mathcal{F}$

What is a basis?

A basis is the combination of span and independence: A set of vectors **{$\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_n$}** forms a basis for some subspace of $\mathbf{R}^N$ if it **(1)** spans that subspace and **(2)** is an independent set of vectors.

Geometrically, a basis is like a ruler for a space. The basis vectors tell you the fundamental units (length and direction) to measure the space they describe.

For example, the most common basis set is the familiar Cartesian axis basis vectors, which contain only 0s and 1s:

$$\mathbb{R}^2 : \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \qquad \mathbb{R}^3 : \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

This basis set is so widely used because of its simplicity: Each basis vector has a unit length, and all vectors in the set are mutually orthogonal (that is, the dot product of any vector with any other vector is zero).

Question 1

Are the following two sets dependent or independent?

$$\{\mathbf{w}_1, \mathbf{w}_2\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix} \right\}$$

$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} -27 \\ 0 \\ -37 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 9 \end{bmatrix} \right\}$$

Solution

$$\{\mathbf{w}_1, \mathbf{w}_2\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix} \right\}$$

$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} -27 \\ 0 \\ -37 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 9 \end{bmatrix} \right\}$$

Try as hard as you can and for as long as you like; you will never
be able to define any vector in each set using a linear weighted combination of the other vectors in the same set. That's easy to see in the first set: When considering only the first two rows, then $w_1 = 2w_2$. However, this weighted combination fails for the third row.

Recap

- Linear independence describes a set of vectors where no vector can be expressed as a linear combination of the others.

- In simpler terms, none of the vectors are redundant; they each contribute unique information to the space they span. Conversely, linearly dependent vectors mean that at least one vector can be formed from a combination of the remaining vectors.

- Linear independence is critical for determining a vector space dimension and constructing bases, which are minimal sets of vectors that can span the entire space.

- The span can be a subspace of a vector space.

- The concept of linear independence is vital to many scientific and engineering fields.

Coming up next...............

- Concepts of Affin Spaces.
- Orthogonal basis, Orthogonal complement.
- Inner product, Orthogonal projections, Matrix derivatives.

Thank you