BS./BSc.

in Applied AI and Data Science

Linear algebra and numerical analysis

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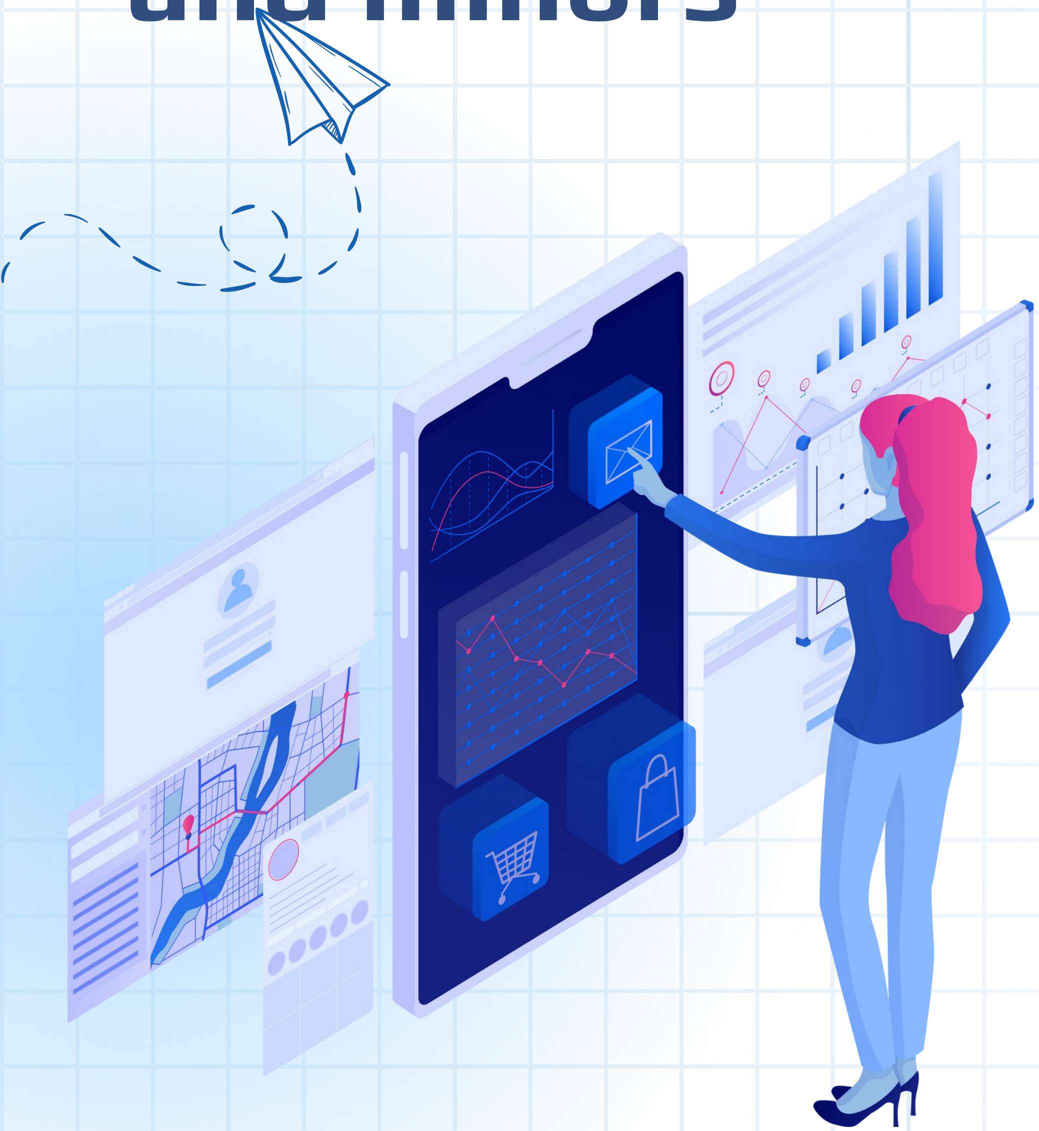
Module 2.4 Matrix inversion methods and minors



1. **Matrix inverse estimation procedure**
2. **Echelon Form:** Reducing the matrix to echelon form
3. **Matrix minors**
4. **Eigenvalues, Eigenvectors and Matrix Diagonalization**

Inverse matrix using row reduction technique

Invertible matrices have many special properties

Finding the inverse of a matrix using row reduction is a powerful and systematic method. Here's a breakdown of the process:

The Core Idea:

- The method leverages the concept of elementary row operations to transform a matrix into the identity matrix.
- Simultaneously, these same operations are applied to an identity matrix, which transforms into the inverse of the original matrix.

Quickly quelling the 2x2 matrix inverse

In general, if a matrix has an inverse, that inverse can be found happily using row reduction, though a much quicker and neater process is available for 2x2 matrices.

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & -7 \\ 2 & -3 \end{bmatrix}$$

You first compute the number that will be dividing each element: (4)(–3) –(–7)(2) = –12 + 14 = 2. The divisor is 2. Now reverse the positions of the 4 and –3, and change the –7 to 7 and the 2 to –2. Divide each element by 2.

Inverse estimation using previous procedure

Solution:

$$A^{-1} = \begin{bmatrix} \frac{-3}{2} & \frac{7}{2} \\ \frac{-2}{2} & \frac{4}{2} \end{bmatrix} = \begin{bmatrix} -\frac{3}{2} & \frac{7}{2} \\ -1 & 2 \end{bmatrix}$$

When you multiply the original matrix A times the inverse A^{-1}, the result is the 2x2 identity matrix. The result is the 2x2 identity matrix.

$$A^*A^{-1} = \begin{bmatrix} 4 & -7 \\ 2 & -3 \end{bmatrix} * \begin{bmatrix} -\frac{3}{2} & \frac{7}{2} \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4\left(-\frac{3}{2}\right)+7 & 4\left(\frac{7}{2}\right)-14 \\ 2\left(-\frac{3}{2}\right)+(-3)(-1) & 2\left(\frac{7}{2}\right)+(-3)(2) \end{bmatrix}$$

$$= \begin{bmatrix} -6+7 & 14-14 \\ -3+3 & 7-6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Finding inverses using row reduction

The straightforward rule for finding the inverses of 2 x 2 matrices is the preferred approach for matrices of that size. For larger square matrices, inverses are determined by performing additions and multiplications through row operations.

The row operations are

1. Interchange any two rows.

2. Multiply all the elements in a row by a real number (not 0).

3. Add multiples of one row's elements to another row's elements.

$$C = \begin{bmatrix} 3 & 4 & -2 & 8 \\ 5 & -6 & 1 & -3 \\ 0 & 5 & 20 & 30 \\ 1 & 6 & 3 & 3 \end{bmatrix}$$

Finding inverses using row reduction……..

Solutions:

2. Multiply all the elements in row 3 by $^1/_5$.

The notation $^1/_5\, R_3 \to R_3$ is read, "One-fifth of each element in row 3 becomes the new row 3."

$$\frac{1}{5}R_3 \to R_3 \begin{bmatrix} 1 & 6 & 3 & 3 \\ 5 & -6 & 1 & -3 \\ 0 & 1 & 4 & 6 \\ 3 & 4 & -2 & 8 \end{bmatrix}$$

Stepping through inverse matrix steps

3. Multiply all the elements in row 1 by –5, and add them to row 2.

The notation $-5R_1 + R_2 \to R_2$ is read, "Negative 5 times each element in row 1 added to the elements in row 2 produces a new row 2." Note that the elements in row 1 do not change; you just use the multiples of the elements in the operation to create a new row 2.

$$-5R_1 + R_2 \to R_2 \begin{bmatrix} 1 & 6 & 3 & 3 \\ 0 & -36 & -14 & -18 \\ 0 & 1 & 4 & 6 \\ 3 & 4 & -2 & 8 \end{bmatrix}$$

Row Reduction and Echelon Forms

We will get more practice with row reduction and, in the process, introduce two important types of matrix forms. The first one is Row echelon form (REF)

$$\begin{bmatrix} 1 & 5 & 0 & -2 & -1 & 7 & -4 \\ 0 & 2 & -2 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -9 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 5 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The above augmented matrix has the following properties:

P1. All nonzero rows are above any rows of all zeros.
P2. The leftmost nonzero entry of a row is to the right of the leftmost nonzero entry of the row above it.

Row Reduction and Echelon Forms

Any matrix satisfying properties P1 and P2 is said to be in row echelon form (REF). In REF, the leftmost nonzero entry in a row is called a leading entry:

$$\begin{bmatrix} \mathbf{1} & 5 & 0 & -2 & -1 & 7 & -4 \\ 0 & \mathbf{2} & -2 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & \mathbf{-9} & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & \mathbf{5} & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A consequence of property P2 is that every entry below a leading entry is zero:

$$\begin{bmatrix} \mathbf{1} & 5 & 0 & -2 & -4 & -1 & -7 \\ \mathbf{0} & \mathbf{2} & -2 & 0 & 0 & 3 & 0 \\ \mathbf{0} & \mathbf{0} & 0 & \mathbf{-9} & -1 & 1 & -1 \\ \mathbf{0} & \mathbf{0} & 0 & \mathbf{0} & \mathbf{5} & 1 & 5 \\ \mathbf{0} & \mathbf{0} & 0 & \mathbf{0} & \mathbf{0} & 0 & 0 \end{bmatrix}$$

We can perform elementary row operations, or row reduction, to transform a matrix into REF.

Row Reduction and Echelon Forms

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$$\begin{bmatrix} \mathbf{1} & 5 & 0 & -2 & -1 & 7 & -4 \\ 0 & \mathbf{2} & -2 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & \mathbf{-9} & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & \mathbf{5} & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A consequence of property P2 is that every entry below a leading entry is zero:

$$\begin{bmatrix} \mathbf{1} & 5 & 0 & -2 & -4 & -1 & -7 \\ \mathbf{0} & \mathbf{2} & -2 & 0 & 0 & 3 & 0 \\ \mathbf{0} & \mathbf{0} & 0 & \mathbf{-9} & -1 & 1 & -1 \\ \mathbf{0} & \mathbf{0} & 0 & \mathbf{0} & \mathbf{5} & 1 & 5 \\ \mathbf{0} & \mathbf{0} & 0 & \mathbf{0} & \mathbf{0} & 0 & 0 \end{bmatrix}$$

We can perform elementary row operations, or row reduction, to transform a matrix into REF.

Examples: REF

Example. Explain why the following matrices are not in REF. Use elementary row operations to put them in REF.

Solution. Matrix $\mathbf{M}$ fails property P1. To put $\mathbf{M}$ in REF we interchange R_2 with R_3:

$$\mathbf{M} = \begin{bmatrix} 3 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 3 & -1 & 0 & 3 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix $\mathbf{N}$ fails property P2. To put $\mathbf{N}$ in REF we perform the operation $-2R_2 + R_3 \to R_3$:

$$\begin{bmatrix} 7 & 5 & 0 & -3 \\ 0 & 3 & -1 & 1 \\ 0 & 6 & -5 & 2 \end{bmatrix} \xrightarrow{-2R_2+R_3} \begin{bmatrix} 7 & 5 & 0 & -3 \\ 0 & 3 & -1 & 1 \\ 0 & 0 & -3 & 0 \end{bmatrix}$$

Why is REF useful? Certain properties of a matrix can be easily deduced if it is in REF. For now, REF is useful to us for solving a linear system of equations.

Reduced row echelon form (RREF)

Although REF simplifies the problem of solving a linear system, later on in the course we will need to completely row reduce matrices into what is called reduced row echelon form (RREF).

A matrix is in RREF if it is in REF (so it satisfies properties P1 and P2) and in addition satisfies the following properties:

P1. All nonzero rows are above any rows of all zeros.
P2. The leftmost nonzero entry of a row is to the right of the leftmost nonzero entry of the row above it.

P3. The leading entry in each nonzero row is a 1.
P4. All the entries above (and below) a leading 1 are all zero.

A leading 1 in the RREF of a matrix is called a pivot. For example, the following matrix in RREF:

$$\begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

Reduced row echelon form (RREF)

Example. Use row reduction to transform the matrix into RREF.

Create zeros under the newly created leading 1:

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \xrightarrow{-3R_1+R_2} \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Create a leading 1 in the second row:

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix}$$

Create zeros under the newly created leading 1:

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{bmatrix} \xrightarrow{-3R_2+R_3} \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Reduced row echelon form (RREF)

We have now completed the top-to-bottom phase of the row reduction algorithm

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{-R_3+R_2} \begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 2 & 5 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{-2R_3+R_1} \begin{bmatrix} 1 & -3 & 4 & -3 & 0 & -3 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Create zeros above the leading 1 in the second row:

$$\begin{bmatrix} 1 & -3 & 4 & -3 & 0 & -3 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \xrightarrow{3R_2+R_1} \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

This completes the row reduction algorithm and the matrix is in RREF.

Question

Use row reduction to solve the linear system.

$$
\begin{aligned}
2x_1 + 4x_2 + 6x_3 &= 8\\
x_1 + 2x_2 + 4x_3 &= 8\\
3x_1 + 6x_2 + 9x_3 &= 12
\end{aligned}
$$

Solution

Create a leading 1 in the first row:

$$\begin{bmatrix} 2 & 4 & 6 & 8 \\ 1 & 2 & 4 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}$$

Create zeros under the first leading 1:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix} \xrightarrow{-R_1+R_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 4 \\ 3 & 6 & 9 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 4 \\ 3 & 6 & 9 & 12 \end{bmatrix} \xrightarrow{-3R_1+R_3} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Question

Find the inverse of $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}$ if it exists.

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Solution

Thus, $\mathtt{rref}(\mathbf{A}) = \mathbf{I}_2$, and therefore $\mathbf{A}$ is invertible. The inverse is

$$\mathbf{A}^{-1} = \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix}$$

Verify:

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Recap

Identity Matrix (I): A square matrix with 1s on the main diagonal and 0s elsewhere.
Elementary Row Operations: These operations don't change the solution of a system of linear equations represented by the matrix.
Reduced Row Echelon Form: The identity matrix is a specific form of a matrix called reduced row echelon form.
This method works for any square matrix. To avoid errors, it's crucial to be organized and accurate with your row operations.
Solving Linear Systems: Many machine learning problems involve solving systems of linear equations. For example, linear regression can be expressed as a matrix equation. Finding the inverse of the coefficient matrix allows you to solve for the unknown variables. Row reduction is a fundamental way to solve those linear systems.

Coming up next................

Module 3: Concepts related to matrix inverse:

Minors: Finding the largest non-zero determinant of a square submatrix.

Using linear independence: directly determining the number of independent rows or columns.

Eigenvalues and Eigenvectors

Module 3.1: Matrix minor

In linear algebra, a **"minor"** of a matrix is fundamental, especially when dealing with determinants and matrix inverses. Here's a breakdown:

Definition:

- A minor of a matrix is the determinant of a smaller square matrix derived from the original matrix.
- Specifically, the (i, j) minor of a matrix A, denoted as M_{ij}, is the determinant of the submatrix formed by removing the i-th row and the j-th column of A.

Key points of the concept minor of a Matrix

Key Points:

- Minors are primarily defined for square matrices, although the concept can be extended.
- Minors are essential for calculating cofactors, which, in turn, are used to compute determinants and adjoints of matrices.
- The process of finding a minor involves:
  - Identifying the element for which you want to find the minor.
  - Removing the row and column containing that element.
  - Calculating the determinant of the remaining submatrix.

Matrix minor explained with examples

If **A** is an n x n matrix and one row and one column are deleted, the resulting matrix is an (n-1) x (n-1) submatrix of **A**.

The determinant of such a submatrix is called a minor of **A** and is designated by m_{ij} , where *i* and *j* correspond to the deleted row and column, respectively.

m_{ij} is the minor of the element a_{ij} in **A**.

Determinant of a submatrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Each element in **A** has a minor

Delete first row and column from **A** .

The determinant of the remaining 2 x 2 submatrix is the minor of a_{11}

$$m_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Determining other minors

Therefore the minor of a_{12} is:

$$m_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

And the minor for a_{13} is:

$$m_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Cofactors of a matrix

The cofactor C_{ij} of an element a_{ij} is defined as:

$$C_{ij} = (-1)^{i+j} m_{ij}$$

When the sum of a row number *i* and column *j* is even, $c_{ij} = m_{ij}$ and when *i*+*j* is odd, $c_{ij} = -m_{ij}$

$$c_{11}(i=1, j=1) = (-1)^{1+1} m_{11} = +m_{11}$$

$$c_{12}(i=1, j=2) = (-1)^{1+2} m_{12} = -m_{12}$$

$$c_{13}(i=1, j=3) = (-1)^{1+3} m_{13} = +m_{13}$$

Cofactors method for a 2x2 matrix

Therefore the 2 x 2 matrix :

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Has cofactors :

$$c_{11} = m_{11} = |a_{22}| = a_{22}$$

And:

$$c_{12} = -m_{12} = -|a_{21}| = -a_{21}$$

And the determinant of **A** is:

$$|A| = a_{11}c_{11} + a_{12}c_{12} = a_{11}a_{22} - a_{12}a_{21}$$

Adjoint of a matrix

The adjoint matrix of A, denoted by adj A, is the transpose of its cofactor matrix

$$adjA = C^T$$

It can be shown that:

$$\mathbf{A}(\text{adj } \mathbf{A}) = (\text{adj}\mathbf{A})\ \mathbf{A} = |\mathbf{A}|\ \mathbf{I}$$

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

$$|A| = (1)(4) - (2)(-3) = 10$$

$$adjA = C^T = \begin{bmatrix} 4 & -2 \\ 3 & 1 \end{bmatrix}$$

Using the Adjoint matrix in matrix inversion

Since

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\,\mathbf{A} = \mathbf{I}$$

and

$$\mathbf{A}(\text{adj }\mathbf{A}) = (\text{adj}\mathbf{A})\,\mathbf{A} = |\mathbf{A}|\,\mathbf{I}$$

then

$$A^{-1} = \frac{adjA}{|A|}$$

Cofactors of a 3x3 matrix

For a 3 x 3 matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The cofactors of the first row are:

$$c_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$$

$$c_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = -(a_{21}a_{33} - a_{23}a_{31})$$

$$c_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{22}a_{31}$$

Cofactors of a matrix compact

$$\mathrm{Cof}(\mathbf{A}) = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix} = \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \vdots \\ \mathbf{c}_n \end{bmatrix}.$$

Cofactors method and Cramer's rule

The Cofactor method can be used to give an explicit formula for the solution of a linear system where the coefficient matrix is invertible. The formula is known as Cramer's Rule. To derive this formula, recall that if $\mathbf{A}$ is invertible then the solution to $\mathbf{Ax} = \mathbf{b}$ is $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. Using the Cofactor method, $\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}(\mathrm{Cof}(\mathbf{A}))^T$, and therefore

$$\mathbf{x} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Question

A cofactor matrix **C** of a matrix **A** is the square matrix of the same order as **A.** Estimate the cofactor matrix **C** of a matrix **A** shown the example below

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

Coming up next................

Module 3: Methods of Finding eigenvalues and linear independence

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Using linear independence: directly determining the number of independent rows or columns.

Eigenvalues and Eigenvectors

Thank you

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