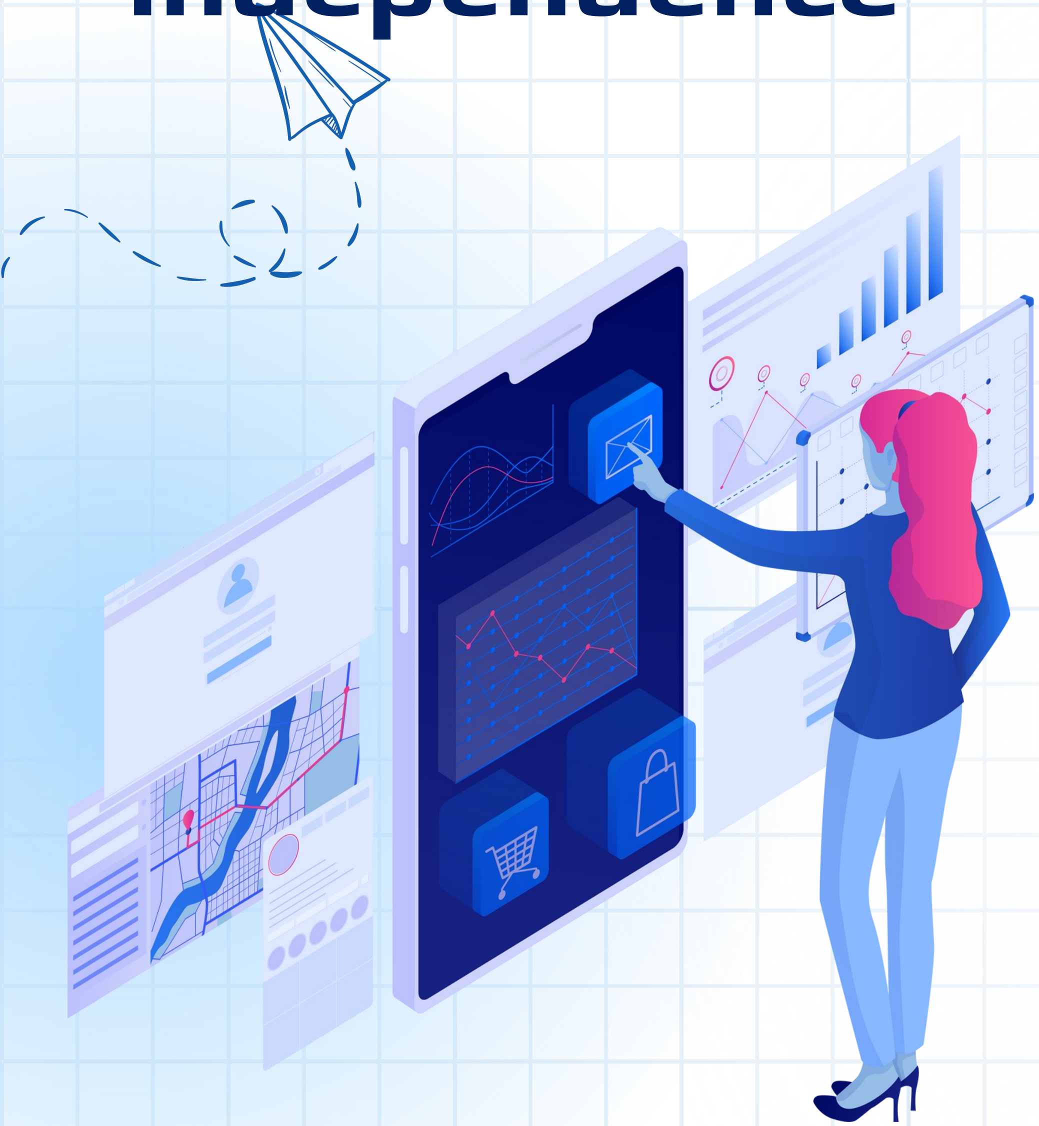
BS./BSc.

in Applied AI and Data Science

Linear algebra and numerical analysis

Module 3.4 Vector space and linear independence



1. Vector space, basis, and span
2. Linear Independence
3. Orthogonality, Inner product
4. Orthogonal projections and matrix derivatives

Why Vector space and linear independence

Why are we interested in Vector space and linear independence?

- Since our ultimate goal is to fully characterize the behavior of linear systems of equations in terms of the existence and uniqueness of solutions, we have to introduce new fundamental linear algebraic concepts such as vector space.
- Foremost among these concepts are linear independence and matrix rank.
- Keep in mind that these concepts are intimately linked with the important Gauss elimination method and how it works.

Vector spaces in real world

- **Mechanics:** Vectors represent forces, velocities, and accelerations. Analyzing the motion of objects, whether a projectile or a complex mechanical system, relies heavily on vector calculations and concepts of vector space.

  - In **structural engineering**, vector spaces help analyze stresses and strains within materials.
- **Computer Graphics:** Vectors represent points, directions, and transformations in 3D space.
- This is essential for rendering **images**, **animations**, and virtual environments.
- **Economics:** Vector spaces can be used to model economic data, such as market trends and resource allocation.

Vector spaces and axiom

A vector space refers to objects for which addition and scalar multiplication are defined. Addition and scalar multiplication obey the following axioms; these should all be sensible requirements based on your knowledge of arithmetic:

$$\textit{Additive inverse}: \mathbf{v} + (-\mathbf{v}) = \mathbf{0}$$

$$\textit{Associativity}: (\mathbf{v} + \mathbf{w}) + \mathbf{u} = \mathbf{v} + (\mathbf{w} + \mathbf{u})$$

$$\textit{Commutativity}: \mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$$

$$\textit{Additive identity}: \mathbf{v} + \mathbf{0} = \mathbf{v}$$

$$\textit{Multiplicative identity}: \mathbf{v}1 = \mathbf{v}$$

$$\textit{Distributivity}: (\alpha + \beta)(\mathbf{v} + \mathbf{w}) = \alpha\mathbf{v} + \alpha\mathbf{w} + \beta\mathbf{v} + \beta\mathbf{w}$$

An axiom is a statement that is taken to be true without requiring formal proof.

Vector space examples

$$A = \begin{bmatrix} 3 \\ -2 \end{bmatrix}, B = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

$$A \oplus B = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \oplus \begin{bmatrix} 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 3+4 \\ -2+8 \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} = C$$

$$6 \otimes C = 6 \otimes \begin{bmatrix} 7 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \cdot 7 \\ 6 \cdot 6 \end{bmatrix} = \begin{bmatrix} 42 \\ 36 \end{bmatrix}$$

Yes, you're probably wondering about these strange $\oplus$ and $\otimes$ symbols and asking yourself if I couldn't just have used $+$ and $\times$. The answer is a simple "No." The operations of $\oplus$ and $\otimes$ sometimes act in a manner similar to our

The above operations are different from usual addition and multiplication. The rules are very strict and a bit different in vector spaces. But one bright note is that sets containing all the matrices of a particular dimension, m x n, will form a vector space.

Letting vector spaces grow with vector addition

A vector space is tied to the operation of vector addition. Depending on the objects involved, the operation itself may vary, but in all cases, vector addition is performed on two vectors (unlike vector multiplication). For example, consider the vector space, V, consisting of all ordered pairs (x,y) representing a 2 x1 vector in standard position.

In the case of these ordered pairs, (x,y), vector addition is defined as (x1, y1) + (x2, y2) = (x1 + x2, y1+ y2). You add the respective coordinates together to get a new set of coordinates.

So, vector addition has some adding in it, but the format is just a tad different from adding 2 + 3.

Question 1

A vector space that uses a familiar method of vector addition is the set of all 2 x 2 matrices with a trace equal to zero. Write down the result of vector addition of the following two 2 x 2 matrices and the trace equation?

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \oplus \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Solution

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \oplus \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}$$

and $(a_{11} + b_{11}) + (a_{22} + b_{22}) = 0.$

Stepping through basis and span

- Suppose vectors $v_1, \dots, v_n$ are linearly dependent:

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0$$

with $c_1 \neq 0$. Then:

$$span\{v_1, \dots, v_n\} = span\{v_2, \dots, v_n\}$$

- When we write a vector space as the space of a list of vectors, we would like that list to be as short as possible. This can achieved by iterating.

- *A linearly independent set in V consisting of the maximum possible number of vectors in V is called a basis for V.*
- *In other words, any largest possible set of independent vectors in V forms the basis for V.*
- *That means if we add one or more vectors to that set, the set will be linearly dependent.*

The set of all linear combinations of given vectors with the same number of components is called the span of these vectors.

Stepping through basis and span

In the span? A frequent question in linear algebra is whether one vector is "in the span" of another vector or set of vectors. This is just some fancy math-speak for asking whether you can create some vector **w** by scalar-multiplying and adding vectors from set **S**.

Thus, a vector **w** is in the span of the vector set **S** if **w** can be exactly created by some linear combination of vectors in set **S**. For example, consider the following two vectors **w** and **v** and set **S**. The question at hand is whether either of these vectors is in the span of **S**.

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}. \quad S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix} \right\}$$

Example leading to span of vector space

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \ \mathbf{w} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}. \quad S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix} \right\}$$

Let's start with vector **v**. We have a positive answer here: **v** is in the span of **S**. Written formally:

$$\mathbf{v} \in span(S) \text{ because } \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \frac{5}{6} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix}$$

In contrast to the difficulty of finding whether v was in the span of S, it should be obvious why w is not in the span of S: The third element in w is nonzero, whereas all vectors in set S have a third element of zero. There is no linear combination of zeros that will produce a nonzero value. Therefore, w cannot possibly be in the span of S.

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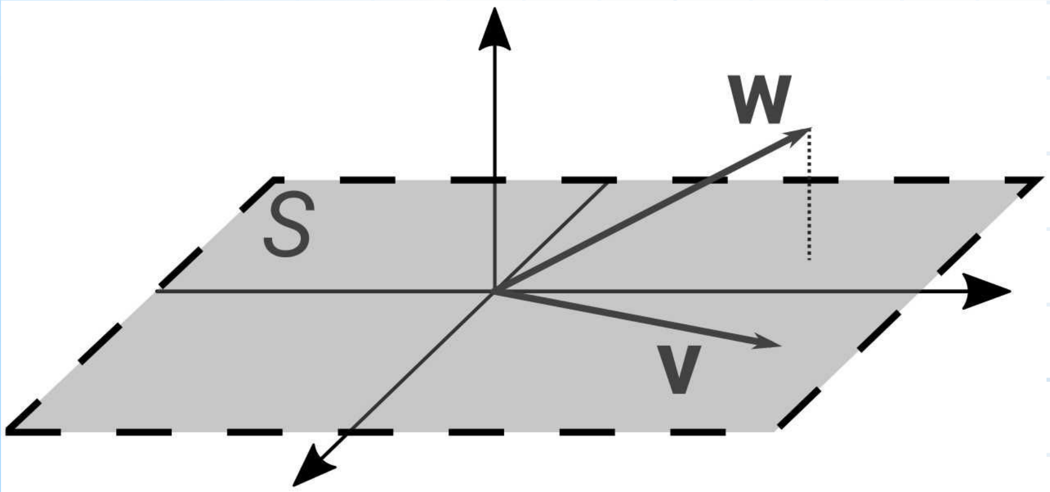
Geometric interpretation of span of vector space

There is a geometric interpretation of the previous example



The span of S is a 2D plane embedded in a 3D ambient space; vector v is a line in the plane, whereas vector w pops out of that plane. We could also describe v as being coplanar to set S.

Question 2

Determine whether each vector is in the span of the associated set.

$$\mathbf{m} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ 4 \\ 0 \end{bmatrix}. \quad U = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Solution: Invalid dimensions

Subspace of Vector Spaces

Let V be a vector space. A subset W of V is called a subspace of V

if it satisfies the following properties:

(1) The zero vector of **V** is also in **W**.

(2) **W** is closed under addition, that is, if **u** and **v** are in **W** then **u + v** is in **W**.

(3) **W** is closed under scalar multiplication, that is, if **u** is in **W** and **α** is a scalar then αu is in **W**.

Creation of subspace of any vector space

Let W be the graph of the function f(x) = 2x:

W = {(x, y) ∈ R^2 | y = 2x}.

Is W a subspace of V = R^2?

Solution. If $x = 0$ then $y = 2 \cdot 0 = 0$ and therefore $\mathbf{0} = (0, 0)$ is in W. Let $\mathbf{u} = (a, 2a)$ and $\mathbf{v} = (b, 2b)$ be elements of W. Then

$$\mathbf{u} + \mathbf{v} = (a, 2a) + (b, 2b) = (a + b, 2a + 2b) = (\underbrace{a + b}_{x}, 2\underbrace{(a + b)}_{x}).$$

Because the x and y components of $\mathbf{u} + \mathbf{v}$ satisfy $y = 2x$ then $\mathbf{u} + \mathbf{v}$ is inside in W. Thus, W is closed under addition. Let α be any scalar and let $\mathbf{u} = (a, 2a)$ be an element of W. Then

$$\alpha\mathbf{u} = (\alpha a, \alpha 2a) = (\underbrace{\alpha a}_{x}, 2\underbrace{(\alpha a)}_{x})$$

Because the x and y components of $\alpha\mathbf{u}$ satisfy $y = 2x$ then $\alpha\mathbf{u}$ is an element of W, and thus W is closed under scalar multiplication. All three conditions of a subspace are satisfied for W and therefore W is a subspace of V. □

Creation of subspace of any vector space

Vector Space R^n

The vector space R^n consisting of all vectors with n components (n real numbers) has dimension n.

For example, the subspace defined as all of $\mathbb{R}^2$ can be created by the span of the vectors [0 1] and [1 0]. That is to say, all of $\mathbb{R}^2$ can be reached by some linear weighted combination of those two vectors.

Another example: The vector [0 1] spans a 1D subspace that is embedded inside $\mathbb{R}^2$ (not $\mathbb{R}^1$! It's in $\mathbb{R}^2$ because the vector has two elements). The vector [1 2] also spans a 1D subspace, but it's a different 1D subspace from that spanned by [0 1].

PROOF A basis of n vectors is $\mathbf{a}_{(1)} = [1 \quad 0 \quad \cdots \quad 0]$, $\mathbf{a}_{(2)} = [0 \quad 1 \quad 0 \quad \cdots \quad 0]$, $\cdots$, $\mathbf{a}_{(n)} = [0 \quad \cdots \quad 0 \quad 1]$. ■

For a matrix **A**, we call the span of the row vectors the **row space** of **A**. Similarly, the span of the column vectors of **A** is called the **column space** of **A**.

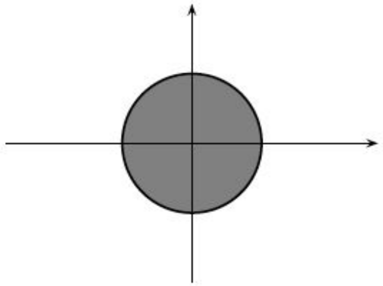
Closure under scalar multiplication

The scalar multiple of v by α, denoted αv, is in V. (closure under scalar multiplication)

Example 14.2. Let V be the unit disc in $\mathbb{R}^2$:

$$V = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$$

Is V a vector space?



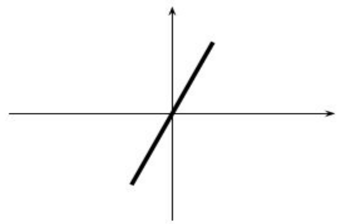
Solution. The circle is not closed under scalar multiplication. For example, take $\mathbf{u} = (1, 0) \in$ V and multiply by say $\alpha = 2$. Then $\alpha\mathbf{u} = (2, 0)$ is not in V. Therefore, property (6) of the definition of a vector space fails, and consequently the unit disc is not a vector space. $\square$

Vector space closure under addition

Example 14.4. Let V be the graph of the function $f(x) = 2x$:

$$\mathsf{V} = \{(x, y) \in \mathbb{R}^2 \mid y = 2x\}.$$

Is V a vector space?



Solution. We will show that V is a vector space. First, we verify that V is closed under addition. We first note that an arbitrary point in V can be written as $\mathbf{u} = (x, 2x)$. Let then $\mathbf{u} = (a, 2a)$ and $\mathbf{v} = (b, 2b)$ be points in V. Then

$$\mathbf{u} + \mathbf{v} = (a + b, 2a + 2b) = (a + b, 2(a + b)).$$

Therefore V is closed under addition. Verify that V is closed under scalar multiplication:

$$\alpha\mathbf{u} = \alpha(a, 2a) = (\alpha a, \alpha 2a) = (\alpha a, 2(\alpha a)).$$

Therefore V is closed under scalar multiplication. There is a zero vector $\mathbf{0} = (0, 0)$ in V:

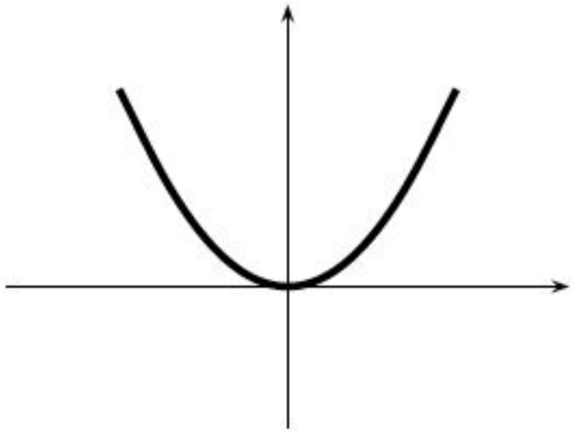
$$\mathbf{u} + \mathbf{0} = (a, 2a) + (0, 0) = (a, 2a).$$

All the other properties of a vector space can be verified to hold; for example, addition is commutative and associative in V because addition in $\mathbb{R}^2$ is commutative/associative, etc. Therefore, the graph of the function $f(x) = 2x$ is a vector space. $\square$

Question 3

V be the graph of the quadratic function $f(x) = x^2$:

$$V = \left\{(x, y) \in \mathbb{R}^2 \mid y = x^2\right\}.$$



Is V a vector space?

Solution

Solution. The set V is not closed under scalar multiplication. For example, $\mathbf{u} = (1, 1)$ is a point in V but $2\mathbf{u} = (2, 2)$ is not. You may also notice that V is not closed under addition either. For example, both $\mathbf{u} = (1, 1)$ and $\mathbf{v} = (2, 4)$ are in V but $\mathbf{u} + \mathbf{v} = (3, 5)$ and $(3, 5)$ is not a point on the parabola V. Therefore, the graph of $f(x) = x^2$ is not a vector space. $\square$

Recap

- A vector space is a fundamental mathematical structure consisting of a set of vectors that can be added together and multiplied by scalars.
- These operations must adhere to specific axioms, ensuring consistent and predictable behavior.
- Vector spaces generalize the concept of Euclidean vectors and are essential for representing and manipulating quantities with both magnitude and direction.
- They provide a framework for linear algebra, enabling the analysis of linear equations and transformations.

Coming up next...............

- Concepts of Linear Independence.
- Conditions on the linear dependency of a set
- Linear Independence and Dependence of functions

Thank you