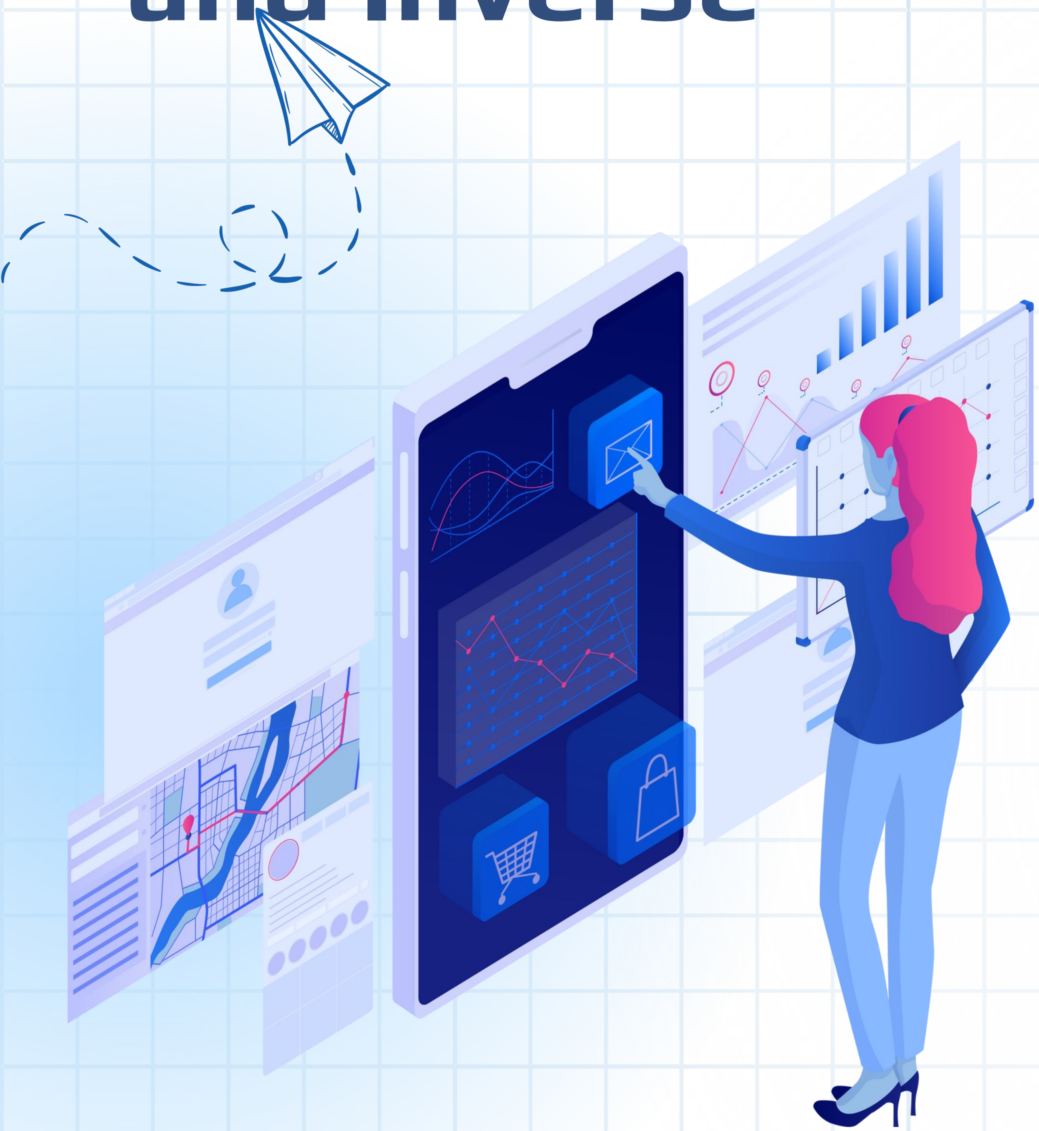
BS./BSc.

in Applied AI and Data Science

Linear algebra and numerical analysis

Module 2.2 Symmetry, Determinants, and Inverse



1. Symmetric and Square matrix
2. The determinant of a matrix
3. Inverse of a matrix
4. Rank of a matrix

Symmetric matrix operations in daily life

Symmetric matrices have many special properties that make them great to work with in numerous real-world applications.

Please see Examples below:

Image Processing: Covariance matrices, which are always symmetric, describe the relationships between image pixel values.

Covariance Matrices: In statistics and machine learning, symmetric covariance matrices are used to describe the relationships between different variables in a dataset. They are fundamental in techniques like principal component analysis (PCA).

Kernel Methods: Kernel matrices, which are often symmetric, are used in support vector machines (SVMs) and other kernel-based learning algorithms. These matrices capture the similarity between data points.

Network Analysis: Adjacency matrices representing undirected graphs are symmetric. These matrices analyze social networks, web graphs, and other interconnected systems.

What does it mean for a matrix to be symmetric?

- It means that the corresponding rows and columns are equal. And that means that nothing happens to the matrix when you swap the rows and columns.

A Square matrix is symmetric if it is equal to its transpose:

$$A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

$$\mathbf{A} = \mathbf{A}^{\mathrm{T}}$$

$$A^T = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

Creating Symmetric Matrices from Nonsymmetric Matrices

This may be surprising at first, but multiplying any matrix—even a nonsquare and nonsymmetric matrix—by its transpose will produce a square symmetric matrix.

The proof is obtained simply by considering the matrix sizes: if A is **$M \times N$**, then **A^TA** is **$(N \times M)(M \times N)$**, which means the product matrix is of size $N \times N$. You can work through the same logic for AA^T.

Now, to prove symmetry. Recall that the definition of a symmetric matrix equals its transpose. So, let's transpose **A^TA**, do some algebra, and see what happens. Make sure you can follow each step here; the proof relies on the **LIVE EVIL** rule:

Symmetric Matrix operation in Python

In this exercise, you will write a Python function that checks whether a matrix is symmetric. **Here's how you can do it:**

It should take a matrix as input and output a boolean True if the matrix is symmetric or False if the matrix is nonsymmetric.

Code for symmetric matrix creation

- **create_symmetric_matrix(size):** It initializes a square matrix of the given size with zeros using **np.zeros().**

- It iterates through the upper triangle of the matrix (including the main diagonal) using nested loops.

- Inside the loops, it generates a random integer using **np.random.randint()** and assigns it to the current element **matrix[i, j].**

- To ensure symmetry, it also assigns the same value to the corresponding element **matrix[j, i].**

- It returns the generated symmetric matrix.

- **is_symmetric(matrix):** It takes a NumPy array as input.

- It uses np.array_**equal()** to compare the matrix with its transpose **(matrix.T).**

- It returns **True** if the matrix is symmetric (equal to its transpose), and **False** otherwise.

Question

Show that the Product of The following Matrix and Its Transpose Is Symmetric

Let A be the 2×3 matrix

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix}$$

Solution

$$A^TA = \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 10 & -2 & -11 \\ -2 & 4 & -8 \\ -11 & -8 & 41 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 0 & -5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 0 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 21 & -17 \\ -17 & 34 \end{bmatrix}$$

Observe that A^TA and AA^T are symmetric as expected.

The inverse of a Matrix and real-world applications

Solving Linear Systems: A fundamental use of matrix inverses is to solve systems of linear equations.
In **data engineering**, this is relevant when dealing with linear models that represent relationships between variables. For example, in certain types of statistical modeling, you might need to solve for unknown parameters in a linear system.

Computational Cost: Calculating the inverse of large matrices can be computationally expensive. In data engineering, where datasets can be massive, alternative methods for solving linear systems (like decomposition techniques) are often preferred.

Computing inverse of a Matrix

The determinant is required to compute the inverse of a matrix. Each **square matrix A** has a unit scalar value called the determinant of **A**, denoted by **det A or |A|**

If $$A = \begin{bmatrix} 1 & 2 \\ 6 & 5 \end{bmatrix}$$

then $$|A| = \begin{vmatrix} 1 & 2 \\ 6 & 5 \end{vmatrix}$$

Properties of inverse Matrix Example

If $A = [A]$ is a single element (1x1), then the determinant is defined as the value of the element

Then $|A| = \det A = a_{11}$

If A is (n x n), its determinant may be defined in terms of order (n-1) or less.

Challenges in determining determinants

Once you get to 4x4 matrices, the determinant becomes a hassle to compute unless the matrix has many carefully placed zeros. But I know you're curious, so the Equation below shows the determinant of a 4x4 matrix. ***Yeah, good luck with that***

$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} = \begin{array}{c} afkp - aflo - agjp + agln + ahjo - ahkn - bekp + belo \\ + bgip - bglm - bhio + bhkm + cejp - celn - cfip + cflm \\ + chin - chjm - dejo + dekn + dfio - dfkm - dgin + dgjm \end{array}$$

Anyway, the point is that if you ever need to compute the determinant, you use np.linalg.det() or scipy.linalg.det().

What is the determinant? What does it mean and how do we interpret it?

Computing the determinant is time-consuming and tedious.

Example: **Computing the determinant of a 2x2 matrix**

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

You can see from this equation that the determinant is not limited to integers or positive values. Depending on the numerical values in the matrix, the determinant can be −1223729358 or +.00000002 or any other number. (The determinant will always be a real number for a real-valued matrix.)

Question

Find the determinant of a 3x3 matrix given below

Find the determinant of the matrix $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 4 \\ 3 & 1 & 4 \end{pmatrix}$.

Solution

$$|\mathbf{A}| = 1\begin{vmatrix} 3 & 4 \\ 1 & 4 \end{vmatrix} - 2\begin{vmatrix} 0 & 4 \\ 3 & 4 \end{vmatrix} + 1\begin{vmatrix} 0 & 3 \\ 3 & 1 \end{vmatrix}$$

$$\begin{aligned} &= 1(3 \times 4 - 4 \times 1) - 2(0 \times 4 - 4 \times 3) + 1(0 \times 1 - 3 \times 3) \\ &= 1(12 - 4) - 2(0 - 12) + 1(0 - 9) \\ &= 8 + 24 - 9 \\ &= 23 \end{aligned}$$

Recap

- **Square Matrices:** Matrices with equal rows and columns (n x n).
- Essential for operations like finding determinants and inverses.
- **Symmetric Matrices:** Square matrices equal to their transpose ($A = A^T$).
- Elements are mirrored across the main diagonal.
- Have real eigenvalues and are important in various applications.
- All symmetric matrixes are square matrixes.
- **Determinants:** Scalar values calculated only for square matrices.
- Indicate invertibility (non-zero determinant) and are used in various calculations.
- The determinant is a property of a square matrix.

Coming up next................

Matrix Algebra: Inverse and Rank of a Matrix and Linear independence

Thank you