BS./BSc.

in Applied AI and Data Science

Linear algebra and numerical analysis

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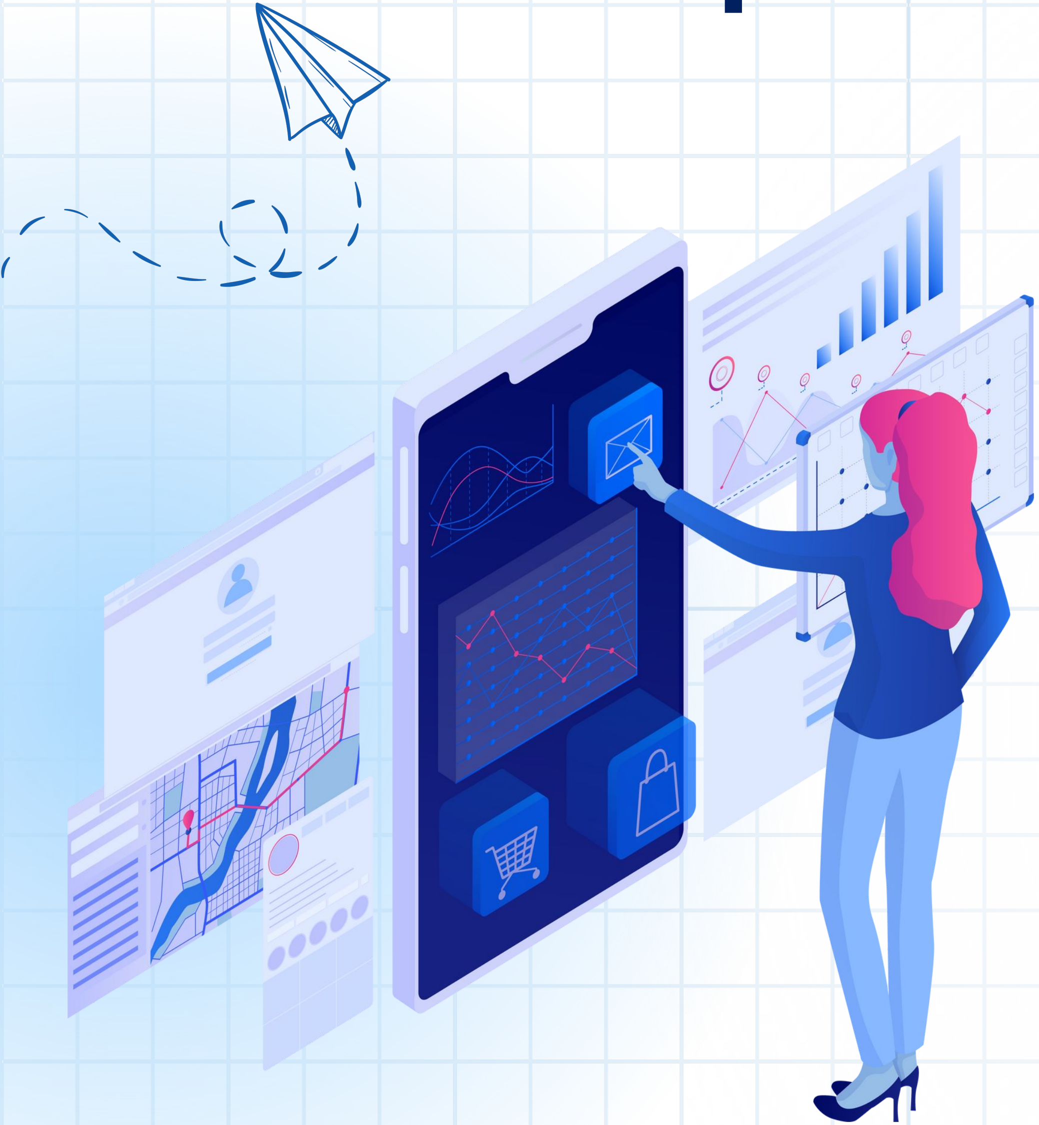
Module 3.4: Recap Vector space and linear independence



1. Review of Vector space and linear independence
2. Geometric interpretation of Projection in vector space
3. Orthogonal matrices
4. Gram-Schmidt procedure for orthogonalization

Review Vector space and linear independence

Vector space, linear independence and span

- The vectors coming from the vector form of the solution of a matrix equation Ax = 0 are linearly independent

Example

- Vectors related to x_2 and x_3 are linear independent.
- Columns of A related to to x_2 and x_3 are linear dependent.
- Span$\{A_1, A_2, A_3\} = Span\{A_1\}$

$$\mathrm{A} = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

Review of basis vectors

A basis is the combination of span and independence: A set of vectors **{v1, v2, ..., vn}** forms a basis for some subspace of $\mathbf{R}^N$ if it

(1) spans that subspace, and (2) is an independent set of vectors.

Geometrically, a basis is like a ruler for a space. The basis vectors tell you the fundamental units (length and direction) to measure the space they describe.

For example, the most common basis set Vector spaces is the familiar Cartesian axis basis vectors, which contain only 0s and 1s:

$$\mathbb{R}^2 : \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \qquad \mathbb{R}^3 : \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Vector spaces in real world

This basis set is so widely used because of its simplicity: Each basis vector has unit length and all vectors in the set are mutually orthogonal (that is, the dot product of any vector with any other vector is zero).

These sets fulfill the definition of basis because they (1) are linearly independent sets that (2) span R^2 or R^3.

Consider the spanning set, S, with five 4×1 vectors. $S = \{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4, \mathbf{s}_5\}$. The span of S is a subspace of R^4.

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 6 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -2 \\ 3 \end{bmatrix} \right\}$$

Affine combination

The vectors of S are linearly dependent. Writing the augmented matrix for the relationship $a_1\mathbf{s}_1 + a_2\mathbf{s}_2 + a_3\mathbf{s}_3 + a_4\mathbf{s}_4 + a_5\mathbf{s}_5 = 0$ and reducing to echelon form (I don't show all the steps — just the beginning and end), you get

$$\left[\begin{array}{ccccc|c} 1 & 1 & 2 & -1 & 0 & 0 \\ 1 & 2 & 1 & 6 & 3 & 0 \\ -2 & 1 & 4 & 1 & -2 & 0 \\ 1 & 1 & -1 & 5 & 3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 5 & -2 & 0 \\ 0 & 0 & 1 & -2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

- **From the echelon form, you get the following relationships between the variables: a1 = 2a4, a2 = –5a4 + 2a5, and a3 = 2a4 + a5. If the vectors were linearly independent, then the only solutions making the equations in the system equal to 0 would be for each ai to be 0.**
- **As you see, with the relationships I found in the echelon form, you could make a3 = 0 if a4 = 1 and a5 = –2. Other such combinations arise, too.**
- **In any case, the vectors are linearly dependent and don't form a basis.**

Affine combination

So, how do we eliminate enough vectors to do away with linear dependency but keep enough to form a basis?

Here's the plan:

1. **Write the vectors in a 4x5 matrix.**
2. **Change the matrix to its transpose.**
3. **Row-reduce the transposed matrix until it's triangular.**
4. **Transpose back to a 4x5 matrix.**
5. **Collect the nonzero vectors as the basis.**

Python examples of vector spaces and subspaces

$$\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = -3, \quad \mathbf{v_1} = \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}, \ \mathbf{v_2} = \begin{bmatrix} -4 \\ 0 \\ -4 \end{bmatrix}, \ \mathbf{v_3} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

$$\mathrm{w} = \lambda_1 \mathbf{v_1} + \lambda_2 \mathbf{v_2} + \lambda_3 \mathbf{v_3} = \begin{bmatrix} -7 \\ -4 \\ -13 \end{bmatrix}$$

```python
import numpy as np
l1 = 1
l2 = 2
l3 = -3
v1 = np.array([4,5,1])
v2 = np.array([-4,0,-4])
v3 = np.array([1,3,2])
l1*v1 + l2*v2 + l3*v3
```

Algebraic definition of a subspace

$$\forall\, \mathbf{v}, \mathbf{w} \in V, \quad \forall\, \lambda, \alpha \in \mathbb{R}; \quad \lambda \mathbf{v} + \alpha \mathbf{w} \in V$$

Collinearity in vector spaces

Below is an example of a 1D subspace V defined by all linear combinations of a row vector in R^3.

$$V = \{\lambda \begin{bmatrix} 1 & 3 & 4 \end{bmatrix}, \lambda \in \mathbb{R}\}$$

Note that this example is not a line in R1 but a 1D subspace embedded in a 3D ambient space R3. Now consider the following specific vectors.

$$\begin{array}{cc} \begin{bmatrix} 3 & 9 & 12 \end{bmatrix} & \in & V \\ \begin{bmatrix} -2 & -6 & -8 \end{bmatrix} & \in & V \\ \begin{bmatrix} 1 & 3 & 5 \end{bmatrix} & \notin & V \end{array}$$

The first two vectors are contained in the subspace V . Algebraically that's the case because you can find some such that [1 3 4] = [3 9 12]; same for the second vector. Geometrically, the first two vectors are collinear with the original vector [1 3 4]; they're on the same infinitely long time, just scaled by some factor.

Linearly Independent Sets versus Spanning Sets

Span	Linearly Independent
Want many vectors in small space	Want few vectors in big space
Adding vectors to list only helps	Deleting vectors from list only helps
Suppose that $v_1, \ldots, v_k$ are columns of A, now we have: AX= b has solution $\Leftrightarrow b \in span\{v_1, \ldots, v_k\}$	Suppose that $v_1, \ldots, v_k$ are columns of A, now we have: AX = 0 has only trivial solution(X=0) $\Leftrightarrow v_1, \ldots, v_k$ are linearly independent.

Question 1

Solution

Recap

- An affine combination is a special type of linear combination where the sum of the scalar coefficients is equal to 1.
- For points (or vectors) p1,p2,...,pn and scalars a1,a2,...,an, the affine combination is: a1p1+a2p2+...+anpn, where a1+a2+...+an=1.
- **Affine Spaces:** Affine spaces are closely related to vector spaces, but they don't have a fixed origin.
- They focus on the relationships between points, rather than absolute positions relative to an origin.
- Affine combinations are used to describe points within an affine space.

Coming up next...............

- Projections in R^2
- Orthogonal Projections
- Orthogonal matrices

Thank you